Some results on ramifications

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Abstract. We investigate ramifications, which are simplicial complexes defined by a very simple inductive property: if two complexes X and Y are ramifications, then the union of X and Y is a ramification whenever the intersection of X and Y is a ramification. We show that the collection of all ramifications properly contains the collection of all collapsible complexes, and that it is properly contained in the collection of all contractible complexes. We introduce the notion of a ramification pair, which is a couple of complexes satisfying also an inductive property. We establish a strong relation between ramification pairs and ramifications.

Keywords: Combinatorial topology, ramifications, contractibility, collapse, completions.

1 Introduction

Simple homotopy, introduced by J. H. C. Whitehead in the early 1930's, may be seen as a refinement of the concept of homotopy [1]. Two complexes are simple homotopy equivalent if one of them may be obtained from the other by a sequence of elementary collapses and expansions.

In this paper, we investigate ramifications, which are complexes defined by a very simple inductive property: if two complexes X and Y are ramifications, then the union of X and Y is a ramification whenever the intersection of X and Y is a ramification.

It could be seen that the collection of all trees satisfies the above property. Also, any complex of arbitrary dimension is a ramification whenever it is collapsible, *i.e.*, whenever it reduces to a single vertex with a sequence composed solely of collapses.

Our main results include the following:
- We show that the collection $\mathbb{R}$ of all ramifications properly contains the collection $\mathbb{E}$ of all collapsible complexes. Also we show that $\mathbb{R}$ is properly contained in the collection $\mathbb{H}$ of all contractible complexes, *i.e.*, all complexes that are homotopy equivalent to a single vertex.
- We introduce the notion of a ramification pair, which is a couple of complexes satisfying also an inductive property. We show there is a strong relation between the collection of all ramification pairs $\ddot{\mathbb{R}}$ and $\mathbb{R}$. In particular, $\ddot{\mathbb{R}}$ is uniquely determined by $\mathbb{R}$.

The paper is organized as follows. First, we give some basic definitions for simplicial complexes (Sec. 2) and simple homotopy (Sec. 3). Then, we recall some facts relative to completions, which allow us to formulate inductive properties (Sec. 4). We investigate the containment relations between the collections $\mathbb{E}$, $\mathbb{R}$, and $\mathbb{H}$ in Sec. 5. Then, we introduce the collection $\ddot{\mathbb{R}}$ of ramification pairs and give the fundamental relation between $\ddot{\mathbb{R}}$ and $\mathbb{R}$ (Sec. 6).
Note that the paper is self contained. Other results on ramifications will be given in a forthcoming paper [8].

2 Basic definitions for simplicial complexes

Let X be a finite family composed of finite sets. The *simplicial closure of* X is the complex $X^- = \{y \subseteq x \mid x \in X\}$. The family X is a *(simplicial) complex* if $X = X^-$. We write $\mathbb{S}$ for the collection of all finite simplicial complexes. Note that $\emptyset \in \mathbb{S}$ and $\{\emptyset\} \in \mathbb{S}$, $\emptyset$ is the *void complex*, and $\{\emptyset\}$ is the *empty complex*.

Let $X \in \mathbb{S}$. An element of X is *a simplex of* X or *a face of* X. A *facet of* X is a simplex of X that is maximal for inclusion. For example, the family $X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is a simplicial complex with four faces and one facet. Note that the empty set is necessarily a face of X whenever $X \neq \emptyset$.

A *simplicial subcomplex* of $X \in \mathbb{S}$ is any subset Y of X that is a simplicial complex. If Y is a subcomplex of X, we write $Y \preceq X$.

Let $X \in \mathbb{S}$. The *dimension* of $x \in X$, written $dim(x)$, is the number of its elements minus one. The *dimension of* X, written $dim(X)$, is the largest dimension of its simplices, the *dimension of* $\emptyset$, the void complex, being defined to be -1. Observe that the dimension of the empty complex $\{\emptyset\}$ is also -1.

A complex $A \in \mathbb{S}$ is *a cell* if $A = \emptyset$ or if A has precisely one non-empty facet x. We set $A^\circ = A \setminus \{x\}$ and $\emptyset^\circ = \emptyset$. We write $\mathbb{C}$ for the collection of all cells. A cell $\alpha \in \mathbb{C}$ is *a vertex* if $dim(\alpha) = 0$.

The *ground set* of $X \in \mathbb{S}$ is the set $\underline{X} = \cup\{x \in X \mid dim(x) = 0\}$. Thus, if $A \in \mathbb{C}$, with $A \neq \emptyset$, then $\underline{A}$ is precisely the unique facet of A. In particular, if α is a vertex, we have $\alpha = \{\emptyset, \underline{\alpha}\}$.
We say that $X \in \mathbb{S}$ and $Y \in \mathbb{S}$ are *disjoint*, or that X is *disjoint from* Y, if $\underline{X} \cap \underline{Y} = \emptyset$. Thus, X and Y are disjoint if and only if $X \cap Y = \emptyset$ or $X \cap Y = \{\emptyset\}$.

If $X \in \mathbb{S}$ and $Y \in \mathbb{S}$ are disjoint, the *join of* X *and* Y is the simplicial complex XY such that $XY = \{x \cup y \mid x \in X, y \in Y\}$. Thus, $XY = \emptyset$ if $Y = \emptyset$ and $XY = X$ if $Y = \{\emptyset\}$. The join αX of a vertex α and a complex $X \in \mathbb{S}$ is a *cone*.

Let $X \in \mathbb{S}$. We set $X^\diamond = \{x \in X \mid x \neq \emptyset\}$. Let $\pi = \langle x_0, ..., x_k \rangle$ be a sequence of faces in $X^\diamond$. The sequence π is a *path in* X *(from* x_0 *to* x_k*)* if $x_i \subseteq x_{i-1}$ or $x_{i-1} \subseteq x_i$, with $i \in [1, k]$. We say that X is *path-connected* if $X \neq \{\emptyset\}$ and for any x, $y \in X^\diamond$, there exists a path in X from x to y.
Note that the void complex is path-connected, but the empty complex is not.

3 Simple homotopy

We recall some basic definitions related to the collapse operator [1].

Let $X \in \mathbb{S}$ and let x, y be two distinct faces of X. The couple (x, y) is a *free pair for* X if y is the only face of X that contains x. Thus, the face y is necessarily a facet of X. If (x, y) is a free pair for X, then $Y = X \setminus \{x, y\}$ is *an elementary collapse of* X, and X is *an elementary expansion of* Y.
We say that X *collapses onto* Y, or that Y *expands onto* X, if there exists a sequence $\langle X_0, ..., X_k \rangle$ such that $X_0 = X$, $X_k = Y$, and X_i is an elementary collapse of X_{i-1}, $i \in [1, k]$. The complex X is *collapsible* if X collapses onto $\emptyset$.
We say that X is *(simply) homotopic to* Y, or that X and Y are *(simply) homotopic*, if there exists a sequence $\langle X_0, ..., X_k \rangle$ such that $X_0 = X$, $X_k = Y$, and X_i is an elementary collapse or an elementary expansion of X_{i-1}, $i \in [1, k]$. The complex X is *(simply) contractible* if X is simply homotopic to $\emptyset$.

Remark 1. We observe that a complex X, $X \neq \emptyset$, is an elementary collapse of a complex Z if and only if we have $Z = X \cup \gamma D$ and $X \cap \gamma D = \gamma D^{\circ}$, where D, $D \neq \emptyset$, is a cell, and γ is a vertex disjoint from D. In fact, the first definition of an elementary collapse was formulated in this way (see [1], p. 247).

Let $X, Y \in \mathbb{S}$, we observe that:
1) If $x, y \in X \setminus Y$, then (x, y) is a free pair for X if and only if (x, y) is a free pair for $X \cup Y$.
Furthermore, if α is a vertex disjoint from $X \cup Y$:
2) If $x \in X \setminus Y$ is a facet of X, then $(x, \underline{\alpha} \cup x)$ is a free pair for $\alpha X \cup Y$.
3) If $x, y \in X$, then the couple (x, y) is a free pair for X if and only if $(\underline{\alpha} \cup x, \underline{\alpha} \cup y)$ is a free pair for $\alpha X \cup Y$.

By induction, we have the following results which will be used in this paper.

Proposition 1. *Let $X, Y \in \mathbb{S}$. The complex X collapses onto $X \cap Y$ if and only if $X \cup Y$ collapses onto Y.*

Proposition 2. *Let $X, Y \in \mathbb{S}$, and let α be a vertex disjoint from $X \cup Y$.
The complex $\alpha X \cup Y$ collapses onto $\alpha(X \cap Y) \cup Y$.
In particular, the complex αX collapses onto $\emptyset$. Thus any cone is collapsible.*

Proposition 3. *Let $X, Y \in \mathbb{S}$, and let α be a vertex disjoint from $X \cup Y$.
The complex X collapses onto Z if and only if $\alpha X \cup Y$ collapses onto $\alpha Z \cup X \cup Y$.
In particular, if X is collapsible, then $\alpha X \cup Y$ collapses onto $X \cup Y$.*

4 Completions

We give some basic definitions for completions. A completion may be seen as a rewriting rule that permits to derive collections of sets. See [4] for more details.

Let $\mathbf{S}$ be a given collection and let $\mathcal{K}$ be an arbitrary subcollection of $\mathbf{S}$. Thus, we have $\mathcal{K} \subseteq \mathbf{S}$. In the sequel of the paper, the symbol $\mathcal{K}$, with possible superscripts, will be a dedicated symbol (a kind of variable).

Let $\boldsymbol{\kappa}$ be a binary relation on $2^{\mathbf{S}}$, thus $\boldsymbol{\kappa} \subseteq 2^{\mathbf{S}} \times 2^{\mathbf{S}}$. We say that $\boldsymbol{\kappa}$ is *finitary*, if $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{G}) \in \boldsymbol{\kappa}$.

Let $\langle \mathrm{K} \rangle$ be a property that depends on $\mathcal{K}$. We say that $\langle \mathrm{K} \rangle$ is a *completion (on* $\mathbf{S}$*)* if $\langle \mathrm{K} \rangle$ may be expressed as the following property:
-> If $\mathbf{F} \subseteq \mathcal{K}$, then $\mathbf{G} \subseteq \mathcal{K}$ whenever $(\mathbf{F}, \mathbf{G}) \in \boldsymbol{\kappa}$. $\langle \boldsymbol{\kappa} \rangle$
where $\boldsymbol{\kappa}$ is a finitary binary relation on $2^{\mathbf{S}}$.

If $\langle \mathrm{K} \rangle$ is a property that depends on $\mathcal{K}$, we say that a given collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle \mathrm{K} \rangle$ if the property $\langle \mathrm{K} \rangle$ is true for $\mathcal{K} = \mathbf{X}$.

Theorem 1 (from [4]). *Let* $\langle \mathrm{K} \rangle$ *be a completion on* $\mathbf{S}$ *and let* $\mathbf{X} \subseteq \mathbf{S}$*. There exists, under the subset ordering, a unique minimal collection that contains* $\mathbf{X}$ *and that satisfies* $\langle \mathrm{K} \rangle$*.*

If $\langle \mathrm{K} \rangle$ is a completion on $\mathbf{S}$ and if $\mathbf{X} \subseteq \mathbf{S}$, we write $\langle \mathbf{X}; \mathrm{K} \rangle$ for the unique minimal collection that contains $\mathbf{X}$ and that satisfies $\langle \mathrm{K} \rangle$.

Let $\langle \mathrm{K}_1 \rangle, \langle \mathrm{K}_2 \rangle, ..., \langle \mathrm{K}_k \rangle$ be completions on $\mathbf{S}$. We write $\wedge$ for the logical "and". It may be seen that $\langle \mathrm{K} \rangle = \langle \mathrm{K}_1 \rangle \wedge \langle \mathrm{K}_2 \rangle ... \wedge \langle \mathrm{K}_k \rangle$ is a completion. In the sequel, we write $\langle \mathrm{K}_1, \mathrm{K}_2, ..., \mathrm{K}_k \rangle$ for $\langle \mathrm{K} \rangle$. Thus, if $\mathbf{X} \subseteq \mathbf{S}$, the notation $\langle \mathbf{X}; \mathrm{K}_1, \mathrm{K}_2, ..., \mathrm{K}_k \rangle$ stands for the smallest collection that contains $\mathbf{X}$ and that satisfies each of the properties $\langle \mathrm{K}_1 \rangle, \langle \mathrm{K}_2 \rangle, ..., \langle \mathrm{K}_k \rangle$.

We observe that, if $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are two completions on $\mathbf{S}$, then we have $\langle \mathbf{X}; \mathrm{K} \rangle \subseteq \langle \mathbf{X}; \mathrm{K}, \mathrm{Q} \rangle$ whenever $\mathbf{X} \subseteq \mathbf{S}$.

Furthermore, if $\langle \mathrm{K} \rangle$ is a completion on $\mathbf{S}$ and $\mathbf{X} \subseteq \mathbf{S}$, it could be seen that:
- We have $\langle \mathbf{X}; \mathrm{K} \rangle \subseteq \langle \mathbf{Y}; \mathrm{K} \rangle$ whenever $\mathbf{X} \subseteq \mathbf{Y} \subseteq \mathbf{S}$.
- If $\mathbf{Z} = \langle \mathbf{X}; \mathrm{K} \rangle$, then $\langle \mathbf{Z}; \mathrm{K} \rangle = \mathbf{Z}$.
- We have $\langle \mathbf{X}; \mathrm{K} \rangle = \langle \mathbf{Y}; \mathrm{K} \rangle$ whenever $\mathbf{X} \subseteq \mathbf{Y} \subseteq \langle \mathbf{X}; \mathrm{K} \rangle$.

5 Ramifications

5.1 Definition

The notion of a dendrite was introduced in [4] as a way for defining a collection made of acyclic complexes. Let us consider the collection $\mathbf{S} = \mathbb{S}$, and let $\mathcal{K}$ denote an arbitrary collection of simplicial complexes.

We define the two completions $\langle \mathrm{R} \rangle$ and $\langle \mathrm{D} \rangle$ on $\mathbb{S}$: For any $S, T \in \mathbb{S}$,
-> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{R} \rangle$
-> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D} \rangle$

Let $\mathbb{D} = \langle \mathbb{C}; \mathrm{R}, \mathrm{D} \rangle$. Each element of $\mathbb{D}$ is a *dendrite* or an *acyclic complex.* We have the general result [4]:

A complex is a dendrite if and only if it is acyclic in the sense of homology.

We set $\mathbb{R} = \langle \mathbb{C}; \mathrm{R} \rangle$. Each element of $\mathbb{R}$ is a *ramification.* Thus, the collection $\mathbb{R}$ is the unique minimal collection that contains $\mathbb{C}$ and that satisfies the property $\langle \mathrm{R} \rangle$. Also, the collection $\mathbb{R}$ is the very collection that may be obtained by starting from $\mathcal{K} = \mathbb{C}$, and by iteratively adding to $\mathcal{K}$ all the sets $S \cup T$ such that S, $T \in \mathcal{K}$ and $S \cap T \in \mathcal{K}$.

Note that the notion of a ramification corresponds to the buildable complexes introduced by J. Jonsson [3]. Here, we have a formulation in terms of completions.

The collection of all cones provides a basic example of ramifications. If a cone αZ has more than one facet, then it may be split in two distinct cones αX and αY such that $\alpha Z = \alpha X \cup \alpha Y$. Since $\alpha X \cap \alpha Y$ is a cone, and αZ is a cell whenever αZ has a single facet, it follows by induction that any cone is a ramification.

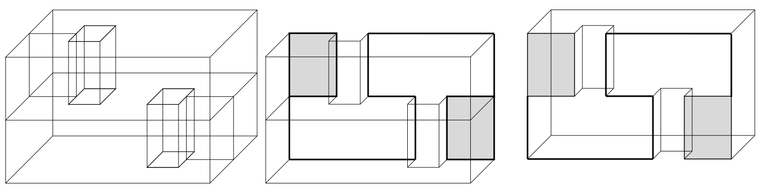
By using also an inductive argument, it is easy to prove that a ramification is necessarily path-connected.

5.2 Ramifications and collapsible complexes

Let us denote by $\mathbb{E}$ the collection of all complexes X such that $\emptyset$ expands onto X, *i.e.*, such that X is collapsible. This collection may be described by completions. See Sec. 6 of [4] and Sec. 8 of [7]. Now let us consider the alternative definition of an elementary collapse given in Remark 1. If X is an elementary collapse of Z, we have $Z = X \cup Y$, where Y and $X \cap Y$ are cones. Since cones are ramifications, and since the void complex is a ramification, we can again prove by induction that any collapsible complex is a ramification. Thus, we have $\mathbb{E} \subseteq \mathbb{R}$. See [4] and [7] (Sec. 8). See also [3] (Def. 3.14 and Prop. 5.17) where a slightly different definition of a collapsible complex is used.

The Bing's house [10] is a classical example of an object that is contractible but not collapsible, see Fig. 1 (a). This two dimensional object is made of two rooms. Two tunnels allow to enter to the upper room by the lower face, and to the lower room by the upper face. Two small walls are attached to the tunnels in order to make this object acyclic.

In [4], it was noticed that the Bing's house B is a ramification. Let us consider the two complexes B_1 and B_2 of Fig. 1 (b) and (c). We have $B = B_1 \cup B_2$. If B is correctly triangulated, then we can see that B_1, B_2, and $B_1 \cap B_2$ are all collapsible. Since $\mathbb{E} \subseteq \mathbb{R}$, these three complexes are ramifications. Thus, the Bing's house B is a ramification. But the Bing's house is not collapsible, in fact there is nowhere we can start a collapse sequence. In consequence, the inclusion $\mathbb{E} \subseteq \mathbb{R}$ is strict.



(a) (b) (c)

Fig. 1. (a): A Bing's house B with two rooms, (b): An object $B_1 \subseteq B$, (c): An object $B_2 \subseteq B$. We have $B = B_1 \cup B_2$, the object $B_1 \cap B_2$ is outlined in (b) and (c).

5.3 Ramifications and contractible complexes

Now, let us consider the collection $\mathbb{H}$ made of all contractible complexes. We have $\mathbb{H} \subseteq \mathbb{D}$, this inclusion is strict (see [7]).

It was shown [3] (Prop. 5.17) that any buildable complex (or any ramification) is contractible. The arguments given for the proof are based on the Hurewicz theorem (Th. 4.32 of [12]). It follows that these arguments do not allow to explicit a sequence of collapses and expansions that transform any ramification into the void complex. In fact, it is undecidable to determine whether a finite simplicial complex is contractible or not (for example see [13], Appendix). Thus, such an explicit sequence cannot, in general, be given.

In Appendix A, we provide a direct proof that permits to build such a sequence. We illustrate an aspect of this proof with the decomposition $B = B_1 \cup B_2$ of the Bing's house given Fig. 1.

Let α be a vertex disjoint from B. Since B_1 is collapsible, B expands onto the complex $C = \alpha B_1 \cup B_2$ (Prop. 3, by replacing collapses by expansions). Now, we can collapse C onto the complex $D = \alpha(B_1 \cap B_2) \cup B_2$ (Prop. 2). Since $B_1 \cap B_2$ is collapsible, the complex D collapses onto B_2 (Prop. 2), which is collapsible. Thus, the sequence $B \nearrow \alpha B_1 \cup B_2 \searrow \alpha(B_1 \cap B_2) \cup B_2 \searrow B_2 \searrow \emptyset$ gives an homotopic deformation between B and $\emptyset$; the symbol $\nearrow$ stands for expansions and the symbol $\searrow$ for collapses. Now, let us consider a complex $B' = B'_1 \cup B'_2$ where B'_1 and B'_2 are two copies of B such that $B'_1 \cap B'_2$ is a ramification. The complex B' is a ramification but, since B'_1 and B'_2 are not collapsible, the above sequence is no longer valid. Furthermore, this process may be iterated by considering two copies of B', and so on. In Appendix A, we handle this problem by proposing an inductive construction which allows us to perform iteratively the above sequence.

Thus, we have $\mathbb{R} \subseteq \mathbb{H}$. Are there contractible complexes that are not ramification? This question corresponds to a conjecture formulated by J. Jonsson [3] (Problem 5.21). In Appendix B, we give a positive answer to this question. The counter-example is given by the dunce hat [11], which is another classical example of an object that is contractible but not collapsible. Note that we only proved that a specific triangulation of the dunce hat is not a ramification. This leaves open the question for any triangulation of this complex.

The following proposition summarizes the facts given in this section.

Proposition 4. *We have $\mathbb{E} \subseteq \mathbb{R} \subseteq \mathbb{H} \subseteq \mathbb{D}$, all these inclusions are strict.*

6 Ramification pairs

In order to achieve a better understanding of the collection $\mathbb{R}$ of all ramifications, we will consider an extension of $\mathbb{R}$. This leads to a collection $\ddot{\mathbb{R}}$, which is composed of couples of complexes. It should be noted that the following completion $\langle \widetilde{\mathrm{R}} \rangle$ has already been introduced in a previous paper [4]. Nevertheless, it was always associated with another completion (its dual), so that all the following results are new.

We set $\ddot{\mathbb{S}} = \{(X,Y) \mid X,Y \in \mathbb{S}, X \preceq Y\}$ and $\ddot{\mathbb{C}} = \{(A,B) \in \ddot{\mathbb{S}} \mid A,B \in \mathbb{C}\}$. The notation $\ddot{\mathcal{K}}$ stands for an arbitrary subcollection of $\ddot{\mathbb{S}}$.
We define the completion $\langle \widetilde{\mathrm{R}} \rangle$ on $\ddot{\mathbb{S}}$: For any (S,T), (S',T') in $\ddot{\mathbb{S}}$,
-> If (S,T), (S',T'), $(S \cap S', T \cap T') \in \ddot{\mathcal{K}}$, then $(S \cup S', T \cup T') \in \ddot{\mathcal{K}}$. $\langle \widetilde{\mathrm{R}} \rangle$
We set $\ddot{\mathbb{R}} = \langle \ddot{\mathbb{C}} \cup \ddot{\mathbb{I}}; \widetilde{\mathrm{R}} \rangle$, where $\ddot{\mathbb{I}} = \{(X,X) \mid X \in \mathbb{S}\}$.
Each couple of $\ddot{\mathbb{R}}$ is a *ramification pair*.
In fact, the collection $\ddot{\mathbb{R}}$ may be generated with a smaller starting collection.

Proposition 5. *We have* $\ddot{\mathbb{R}} = \langle \ddot{\mathbb{C}}^{\#}; \widetilde{\mathrm{R}} \rangle$, *where* $\ddot{\mathbb{C}}^{\#} = \ddot{\mathbb{C}} \cup \{(\{\emptyset\}, \{\emptyset\})\}$.

Proof. Since $\ddot{\mathbb{C}}^{\#} \subseteq \ddot{\mathbb{C}} \cup \ddot{\mathbb{I}}$, we have $\langle \ddot{\mathbb{C}}^{\#}; \widetilde{\mathrm{R}} \rangle \subseteq \ddot{\mathbb{R}}$.
Since $\ddot{\mathbb{C}} \subseteq \ddot{\mathbb{C}}^{\#}$, it is sufficient to show that $\ddot{\mathbb{I}} \subseteq \langle \ddot{\mathbb{C}}^{\#}; \widetilde{\mathrm{R}} \rangle$.
If $X = \emptyset$, then we have $(X,X) \in \ddot{\mathbb{C}}$. Suppose that, for any X with $Card(X) \leq k$, we have $(X,X) \in \langle \ddot{\mathbb{C}}^{\#}; \widetilde{\mathrm{R}} \rangle$. Let X such that $Card(X) = k+1$. If X has a single facet, then we have either $X = \{\emptyset\}$ or $X \in \mathbb{C}$. In both cases, we have $(X,X) \in \ddot{\mathbb{C}}^{\#}$. If X has more that one facet, then there exist two complexes Y and Z, with $Card(Y) \leq k$ and $Card(Z) \leq k$, such that $X = Y \cup Z$. By the induction hypothesis, the couples (Y,Y), (Z,Z), and $(Y \cap Z, Y \cap Z)$ are in $\langle \ddot{\mathbb{C}}^{\#}; \widetilde{\mathrm{R}} \rangle$. By $\langle \widetilde{\mathrm{R}} \rangle$, it means that (X,X) is in $\langle \ddot{\mathbb{C}}^{\#}; \widetilde{\mathrm{R}} \rangle$. $\square$

Remark 2. Let $(X,Y) \in \ddot{\mathbb{S}}$. We can check that we have $(X,Y) \in \langle \ddot{\mathbb{C}}; \widetilde{\mathrm{R}} \rangle$ if and only if $X \in \mathbb{R}$ and $Y \in \mathbb{R}$. Choosing $\ddot{\mathbb{C}} \cup \ddot{\mathbb{I}}$ in the definition of a ramification pair will allow us to build a much wider collection.

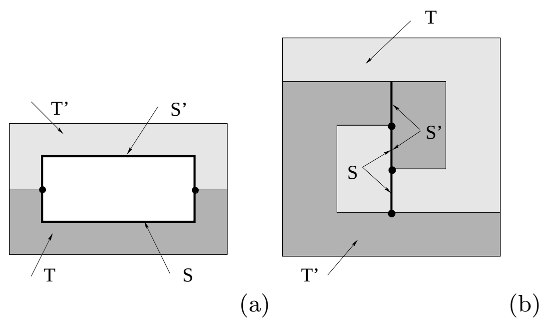


Fig. 2. Four couples (S,T), (S',T'), $(S \cap S', T \cap T')$, $(S \cup S', T \cup T')$, which are ramification pairs; (a): S and S' are two simple open curves, $S \cap S'$ is made of two vertices. (b): S and S' are also two simple open curves, but $S \cap S'$ is a segment.

The four couples given in Fig. 2 correspond to the four couples appearing in the definition of the completion $\langle \widetilde{\mathrm{R}} \rangle$. In this specific illustration, we observe that, if (X,Y) is one of these four couples, then Y collapses onto X (under an appropriate triangulation).

We introduce the notion of a Δ-form, the symbol Δ corresponds to a binary relation over $\ddot{\mathbb{S}}$ and $\mathbb{S}$.

Let $(X, Y) \in \ddot{\mathbb{S}}$ and $Z \in \mathbb{S}$. We write $\Delta(X, Y, Z)$ if there exists a vertex α, disjoint from Y, such that $Z = \alpha X \cup Y$. In this case, we write $\alpha(X, Y)$ for the complex Z, and we say that $\alpha(X, Y)$ is a *Δ-form*. We also say that $\alpha(X, Y)$ is a *Δ-form of* (X, Y) or a *Δ-form of* Z.

If $Z \in \mathbb{S}$ and α is an arbitrary vertex, it may be seen that there exists a unique couple $(X, Y) \in \ddot{\mathbb{S}}$ such that $Z = \alpha(X, Y)$. We have:

$$X = \{x \in Z \mid x \cap \underline{\alpha} = \emptyset \text{ and } x \cup \underline{\alpha} \in Z\} \text{ and } Y = \{x \in Z \mid x \cap \underline{\alpha} = \emptyset\}.$$

The complex X is the so-called link of the face $\underline{\alpha}$ in Z, and Y is the so-called deletion of $\underline{\alpha}$ from Z, see [3]. Thus, we have:

$$\alpha(X, Y) = \alpha(X', Y') \text{ if and only if } (X, Y) = (X', Y').$$

Note that we have $X = \emptyset$ and $Y = Z$ whenever α is disjoint from Z.

We now clarify the correspondence between $\ddot{\mathbb{R}}$ and $\mathbb{R}$ induced by Δ-forms:
1) If $(X, Y) \in \ddot{\mathbb{S}}$, then, up to a renaming of the vertex α, the couple (X, Y) has a unique Δ-form $Z = \alpha(X, Y)$. Thus, up to this renaming, there is a unique complex in $\mathbb{S}$ which is the Δ-form of a couple in $\ddot{\mathbb{S}}$.
2) If $Z \in \mathbb{S}$, and for a given α, there is a unique couple (X, Y) such that $Z = \alpha(X, Y)$. Now, for all possible choices of α, we observe that there are precisely $k + 1$ different such couples, where k is the number of vertices included in Z (we have to consider the case where α is disjoint from Z). Thus, in general, there are several different couples in $\ddot{\mathbb{S}}$ which are the Δ-forms of a complex in $\mathbb{S}$.

Proposition 6. *Let $\alpha(X', Y')$ and $\alpha(X'', Y'')$ be two Δ-forms.*
1) We have $\alpha(X', Y') \cup \alpha(X'', Y'') = \alpha(X' \cup X'', Y' \cup Y'')$.
2) We have $\alpha(X', Y') \cap \alpha(X'', Y'') = \alpha(X' \cap X'', Y' \cap Y'')$.

Proof. Let $\alpha(X', Y')$ and $\alpha(X'', Y'')$ be two Δ-forms. Then $X' \cup X'' \preceq Y' \cup Y''$, $X' \cap X'' \preceq Y' \cap Y''$, and α is disjoint from $Y' \cup Y''$ and $Y' \cap Y''$. Thus, both $\alpha(X' \cup X'', Y' \cup Y'')$ and $\alpha(X' \cap X'', Y' \cap Y'')$ are Δ-forms.
1) Let $Z = \alpha(X', Y') \cup \alpha(X'', Y'')$. We have $Z = (\alpha X' \cup Y') \cup (\alpha X'' \cup Y'') = \alpha(X' \cup X'') \cup (Y' \cup Y'') = \alpha(X' \cup X'', Y' \cup Y'')$.
2) Let $Z = \alpha(X', Y') \cap \alpha(X'', Y'')$. Thus $Z = (\alpha X' \cup Y') \cap (\alpha X'' \cup Y'')$.
Therefore $Z = \alpha(X' \cap X'') \cup (Y' \cap Y'') \cup (\alpha X' \cap Y'') \cup (\alpha X'' \cap Y')$.
Since $(\alpha X' \cap Y'') \cup (\alpha X'' \cap Y') \subseteq Y' \cap Y''$, we have $Z = \alpha(X' \cap X'') \cup (Y' \cap Y'')$.
We obtain $Z = \alpha(X' \cap X'', Y' \cap Y'')$. $\square$

By induction on $\mathbb{R}$ and $\ddot{\mathbb{R}}$, Prop. 6 leads to the following relation between these two collections.

Theorem 2. *Let $(X, Y) \in \ddot{\mathbb{S}}$ and let $Z \in \mathbb{S}$ such that $\Delta(X, Y, Z)$.*
We have $(X, Y) \in \ddot{\mathbb{R}}$ if and only if $Z \in \mathbb{R}$.

Proof. Let $Z = \alpha(X, Y)$.
1) Suppose (X, Y) is a ramification pair. If $(X, Y) \in \ddot{\mathbb{C}} \cup \ddot{\mathbb{I}}$, we easily check that $Z = \alpha(X, Y)$ is a ramification. If $(X, Y) \notin \ddot{\mathbb{C}} \cup \ddot{\mathbb{I}}$, then there exist $(X', Y') \in \ddot{\mathbb{R}}$ and $(X'', Y'') \in \ddot{\mathbb{R}}$, with $(X' \cap X'', Y' \cap Y'') \in \ddot{\mathbb{R}}$, such that $X = X' \cup X''$,

$Y = Y' \cup Y''$. We observe that $\alpha(X', Y')$ and $\alpha(X'', Y'')$ are Δ-forms.
By Prop. 6, we have $Z = \alpha(X' \cup X'', Y' \cup Y'') = \alpha(X', Y') \cup \alpha(X'', Y'')$.
Furthermore, we have $\alpha(X', Y') \cap \alpha(X'', Y'') = \alpha(X' \cap X'', Y' \cap Y'')$.
Thus, if $\alpha(X', Y')$, $\alpha(X'', Y'')$, and $\alpha(X' \cap X'', Y' \cap Y'')$ are ramifications, then Z is a ramification. By induction on $\ddot{\mathbb{R}} = \langle \ddot{\mathbb{C}} \cup \ddot{\mathbb{I}}; \widetilde{\mathrm{R}} \rangle$, it follows that the complex Z is a ramification whenever (X, Y) is a ramification pair.
2) Suppose Z is a ramification.
i) Suppose $Z \in \mathbb{C}$. If $X = \emptyset$, we have $(X, Y) \in \ddot{\mathbb{C}}$. If $X = \{\emptyset\}$, we must have $Y = \{\emptyset\}$, otherwise Z would not be path-connected. Thus $(X, Y) \in \ddot{\mathbb{I}}$. If $X \neq \emptyset$ and $X \neq \{\emptyset\}$, it may be seen that we must have $X \in \mathbb{C}$ and $Y = X$, thus $(X, Y) \in \ddot{\mathbb{C}}$.
ii) If $Z \notin \mathbb{C}$, then there exists $Z' \in \mathbb{R}$ and $Z'' \in \mathbb{R}$, with $Z' \cap Z'' \in \mathbb{R}$, such that $Z = Z' \cup Z''$. Let us consider the couples (X', Y') and (X'', Y'') such that $Z' = \alpha(X', Y')$ and $Z'' = \alpha(X'', Y'')$. By Prop. 6, we have $Z' \cap Z'' = \alpha(X' \cap X'', Y' \cap Y'')$. We also have $Z = \alpha(X, Y) = \alpha(X' \cup X'', Y' \cup Y'')$.
Since there exists a unique couple (X, Y) such that $Z = \alpha(X, Y)$, we have $X = X' \cup X''$ and $Y = Y' \cup Y''$. If (X', Y'), (X'', Y''), $(X' \cap X'', Y' \cap Y'')$ are ramification pairs, then $(X' \cup X'', Y' \cup Y'')$ is a ramification pair (by $\langle \widetilde{\mathrm{R}} \rangle$), which means that (X, Y) is a ramification pair .
By i) and ii), we may affirm, by induction on $\mathbb{R} = \langle \mathbb{C}; \mathrm{R} \rangle$, that (X, Y) is a ramification pair whenever $\alpha(X, Y)$ is a ramification. $\square$

References

1. Whitehead, J.H.C.: Simplicial spaces, nuclei, and m-groups, Proc. London Math. Soc. (2), **45**, 243–327 (1939)
2. Björner, A.: Topological methods, Handbook of Combinatorics, R. Graham, M. Grötschel and L. Lovász (eds), North-Holland, Amsterdam, 1819–1872 (1995)
3. Jonsson, J.: Simplicial Complexes of Graphs, Springer Verlag (2008)
4. Bertrand, G.: Completions and simplicial complexes, HAL-00761162 (2012)
5. Bertrand, G.: New structures based on completions. In: Discrete Geometry for Computer Imagery. Volume 7749 of LNCS, Springer 83–94 (2013)
6. Bertrand, G.: Completions and simple homotopy. In: Discrete Geometry for Computer Imagery. Volume 8668 of LNCS, Springer 63–74 (2014)
7. Bertrand, G.: Completions, perforations and fillings. In: Discrete Geometry and Mathematical Morphology. Volume 12708 of LNCS, Springer 137–151 (2021)
8. Bertrand, G.: Completions and ramifications, Submitted for publication (2022)
9. Giblin, P.: Graphs, surfaces and homology, Chapman and Hall (1981)
10. Bing, R.H.: Some aspects of the topology of 3-manifolds related to the Poincaré Conjecture, Lectures on Modern Mathematics II, Wiley, 93–128 (1964)
11. Zeeman, E.C.: On the dunce hat, Topology, Vol. 2, 341–358 (1964)
12. Hatcher, A.: Algebraic topology, Cambridge University Press (2001)
13. Tancer, M.: Recognition of collapsible complexes is NP-complete, arXiv 1211.6254 (2012)

A Ramifications and contractibility

In this section, we show that any ramification is contractible. We present a construction that provides a sequence of expansion and collapse operations, which transform any ramification into a single vertex (equivalently into the void complex).

Let $X, Y \in \mathbb{S}$ with $X \preceq Y$, and let α be a vertex disjoint from Y. If X is collapsible, then we say that the complex $Z = \alpha X \cup Y$ is an *α-expansion of Y*. We observe that:
1) If $Y \preceq T$ and α is disjoint from T, then Z is an α-expansion of T whenever Z is an α-expansion of Y.
2) The complex Z collapses onto Y, *i.e.*, Y expands onto Z, whenever Z is an α-expansion of Y. This fact is a direct consequence of Prop. 3.

We say that Y is a *conic expansion of X*, or that (X, Y) is a *CE-pair*, if there exist a sequence $\langle X_0, ..., X_k \rangle$, with $X_0 = X$ and $X_k = Y$, and a sequence of distinct vertices $\langle \alpha_1, ..., \alpha_k \rangle$ such that X_i is an α_i-expansion of X_{i-1}, $i \in [1, k]$. Thus, by 2), the complex Y collapses onto X whenever (X, Y) is a CE-pair.

Let $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ be two CE-pairs. We say that $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ are *independent* if $\widetilde{X} \cap \widetilde{Y} = X \cap Y$.
Thus, two CE-pairs $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ are independent if the vertices involved in the sequences relative to $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ are all distinct and if these vertices are disjoint from $X \cup Y$. If two CE-pairs are not independent, it is sufficient to rename the vertices of the sequences in order to obtain independent pairs. In the following, we will consider only independent CE-pairs.

The two following facts are a direct consequence of independence.
3) Let $X, Y \in \mathbb{S}$ and let $(X, \widetilde{X})$, $(Y, \widetilde{Y})$ be two independent CE-pairs. Then $(X \cup Y, \widetilde{X} \cup \widetilde{Y})$ is a CE-pair. Furthermore $\widetilde{X} \cup \widetilde{Y}$ collapses onto $\widetilde{X} \cup Y$ and $\widetilde{X} \cup Y$ collapses onto $X \cup Y$.
4) Let $X, Y \in \mathbb{S}$. Let $(X, \widetilde{X})$, $(Y, \widetilde{Y})$ and $(X \cap Y, \widetilde{T})$ be three mutually independent CE-pairs. Let $\widetilde{X}^+ = \widetilde{X} \cup \widetilde{T}$ and $\widetilde{Y}^+ = \widetilde{Y} \cup \widetilde{T}$. Then $(X, \widetilde{X}^+)$, $(Y, \widetilde{Y}^+)$ are two CE-pairs such that $\widetilde{X}^+ \cap \widetilde{Y}^+ = \widetilde{T}$. Furthermore, $(X \cup Y, \widetilde{X}^+ \cup \widetilde{Y}^+)$ is a CE-pair.

Proposition 7. *Let $X, Y \in \mathbb{S}$ and let $(X, \widetilde{X})$, $(Y, \widetilde{Y})$, $(X \cap Y, \widetilde{T})$, be three mutually independent CE-pairs such that $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{T}$ are collapsible. We set $\widetilde{X}^+ = \widetilde{X} \cup \widetilde{T}$ and $\widetilde{Y}^+ = \widetilde{Y} \cup \widetilde{T}$. We consider the complexes $Z = X \cup Y$ and $\widetilde{Z} = \alpha \widetilde{X}^+ \cup \widetilde{Y}^+$, where α is a vertex disjoint from $\widetilde{X}^+ \cup \widetilde{Y}^+$.*
Then $(Z, \widetilde{Z})$ is a CE-pair. Furthermore the complex $\widetilde{Z}$ is collapsible.

Proof. By 3) the complex $\widetilde{X}^+ = \widetilde{X} \cup \widetilde{T}$ collapses onto $\widetilde{X} \cup (X \cap Y) = \widetilde{X}$. Thus, the complex $\widetilde{X}^+$ is collapsible. Similarly $\widetilde{Y}^+$ is collapsible.
By 4) the couple $(X \cup Y, \widetilde{X}^+ \cup \widetilde{Y}^+)$ is a CE-pair and we have $\widetilde{X}^+ \cap \widetilde{Y}^+ = \widetilde{T}$. The complex $\widetilde{Z} = \alpha \widetilde{X}^+ \cup \widetilde{Y}^+$ is an α-expansion of $\widetilde{X}^+ \cup \widetilde{Y}^+$. Thus $(X \cup Y, \widetilde{Z})$ is

a CE-pair. By Prop. 2, the complex $\alpha\widetilde{X}^+ \cup \widetilde{Y}^+$ collapses onto $\alpha(\widetilde{X}^+ \cap \widetilde{Y}^+) \cup \widetilde{Y}^+$. Thus, $\widetilde{Z}$ collapses onto $\alpha\widetilde{T} \cup \widetilde{Y}^+$. By Prop. 3, the complex $\alpha\widetilde{T} \cup \widetilde{Y}^+$ collapses onto $\widetilde{Y}^+$. Therefore, the complex $\widetilde{Z}$ is collapsible. □

By induction, one can immediately deduce that any ramification is contractible. The base step is obtained noting that any cell is collapsible. The inductive step is a direct consequence of Prop. 7 and the completion ⟨R⟩ that defines a ramification. For any ramification X, there exists a collapsible complex $\widetilde{X}$ such that $(X, \widetilde{X})$ is a CE-pair. Since $\widetilde{X}$ also collapses onto X, this shows that X is contractible.

B Ramifications and the dunce hat

In this section we show that a certain triangulation of the dunce hat is not a ramification. Our arguments are based on homology modulo 2, see [9].

Let $Z \in \mathbb{S}$. If $x \in Z$ and $dim(x) = p$, we say that x is a p-simplex. We denote by $Z[p]$ the set composed of all p-simplexes of Z.
If $A \subseteq Z[p+1]$, the *boundary of* A is the set $\partial(A) \subseteq Z[p]$ such that a p-simplex x is in $\partial(A)$ iff the number of $(p+1)$-simplexes of A containing x is odd.
Let $B \subseteq Z[p]$. The set B is a *boundary in* Z if there exists $A \subseteq Z[p+1]$ such that $\partial(A) = B$. The set B is a *cycle in* Z if $\partial(B) = \emptyset$.
It is well-known that every boundary in Z is a cycle in Z. The converse is, in general, not true. We say that Z is *acyclic* if each cycle in Z is a boundary in Z.

We introduce a tool that will allow us to check boundaries in the dunce hat. Let $Z \in \mathbb{S}$ and let $B \subseteq Z[p]$ be a cycle in Z. We define the binary relation $\mathbf{R} \subseteq Z[p+1] \times Z[p+1]$ which is such that $(x, y) \in \mathbf{R}$ iff:
1) We have $x \cap y \in Z[p] \setminus B$; and
2) The simplexes x and y are the only simplexes of $Z[p+1]$ containing $x \cap y$.
We denote by $\widehat{\mathbf{R}}$ the reflexive and transitive closure of $\mathbf{R}$. Since $\mathbf{R}$ is symmetric, the relation $\widehat{\mathbf{R}}$ is an equivalence relation. The equivalent class of $x \in Z[p+1]$ under $\widehat{\mathbf{R}}$ is *the extension of* x *for* B. If $x \in B$ and $y \in Z[p+1]$ are such that $x \subseteq y$, then we say that the extension of y for B is *a* B*-extension of* x.

Let $Z \in \mathbb{S}$ and let $B \subseteq Z[p]$ be a cycle in Z. Let $A \subseteq Z[p+1]$ such that $\partial(A) = B$. Let $x \in Z[p+1]$ and let $E \subseteq Z[p+1]$ be the extension of x for B. From the above definitions, we can see that we have $E \subseteq A$ whenever $x \in A$.
Furthermore, if $x \in B$, then there exists a B-extension of x that is a subset of A.

In the following the complex Z corresponds to the triangulation of the dunce hat represented Fig. 3 (a).

We first illustrate the preceding property through an example on Z, see Fig. 3 (b). We consider the set $B \subseteq Z[1]$ such that $B = \{\{1,5\},\{5,4\},\{4,3\},\{3,1\}\}$. The set B is a cycle in Z. Let $x = \{4,5,6\}$. Suppose there exists $A \subseteq Z[2]$ such that $\partial(A) = B$ and such that $x \in A$. Let us consider the simplex $y = \{4,6,8\}$. We have $x \cap y = \{4,6\}$. Since $x \cap y$ is not in B and since x, y are the only simplexes of $Z[2]$ containing $x \cap y$, we have $(x, y) \in \mathbf{R}$. We see that we must have $y \in A$ otherwise the simplex $x \cap y$ would be in $\partial(A)$. By induction, the extension of x for B must be a subset of A. Observe that this extension E is also a B-extension of the simplex $\{4, 5\} \in B$. The set E is highlighted Fig. 3 (c). In fact, in this example, it turns out that we have $\partial(E) = B$.

In the following we consider a decomposition of Z. We set $Z = X \cup Y$, where X and Y are proper subcomplexes of Z. We will show it is not possible that X and Y are both acyclic. Since a ramification is necessarily acyclic, we get the desired result.

First let us consider the set $B \subseteq Z[1]$ such that $B = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$, see Fig. 3 (a). The set B is a cycle in Z. Furthermore we observe that $\partial(Z) = B$. Now suppose X is acyclic and suppose B is a subset of X, *i.e.*, we have $B \subseteq X[1]$. There must exists at least one set A in $X[2]$ such that $\partial(A) = B$. Let us consider the three simplexes $x_1 = \{1, 3, 6\}$, $x_2 = \{1, 3, 5\}$, and $x_3 = \{1, 3, 4\}$. Since $\{1, 3\}$ is in B, the set A must contain at least one of the simplexes x_1, x_2, x_3. We can check that, for each $i = 1, 2, 3$, the extension of x_i for B is precisely the complex Z. In other words, any B-extension of $\{1, 3\}$ is equal to Z. It follows that Z is the unique subcomplex of Z such that $\partial(Z) = B$. Thus, it is not possible that X is a proper subcomplex of Z.
Therefore, we have three possible cases up to a renaming of X and Y.

1. Suppose $\{1, 3\} \in X \setminus Y$ and $\{2, 1\} \in Y \setminus X$. It implies that the three 2-simplexes containing $\{1, 3\}$ must be in $X \setminus Y$ and that the three 2-simplexes containing $\{2, 1\}$ must be in $Y \setminus X$, see Fig. 4 (a).
- Suppose $\{4, 8\}$ is in Y. Thus the cycle $B = \{\{2, 4\}, \{4, 8\}, \{8, 2\}\}$ is in Y. In this case the cycle B is not a boundary in Y. In other words, it is not possible there exists a set $A \subseteq Y[2]$ such that $\partial(A) = B$. To see this, we observe that, since $\{4, 8\}$ is in B, such a set A would contain one of the two simplexes $x_1 = \{3, 4, 8\}$ or $x_2 = \{4, 6, 8\}$. But we see that the two extensions of x_1 and x_2 contain some simplexes of $X \setminus Y$. Therefore $\{4, 8\}$ is not in Y and the simplexes $\{3, 4, 8\}$ and $\{4, 6, 8\}$ are in $X \setminus Y$. We obtain the configuration of Fig. 4 (b)
- Now let us consider the cycle $B = \{\{3, 6\}, \{6, 4\}, \{4, 3\}\}$. We have $B \subseteq X$. Again, it is not possible there exists a set $A \subseteq X[2]$ such that $\partial(A) = B$. To see this, we observe that, since $\{4, 6\}$ is in B, such a set A would contain one of the two simplexes $x_1 = \{4, 5, 6\}$ or $x_2 = \{4, 6, 8\}$. But the two extensions of x_1 and x_2 contain some simplexes of $Y \setminus X$. Thus, this case leads to a contradiction.

The two other cases may be analyzed in the same way.

2. Suppose $\{1, 3\} \in X \setminus Y$ and $\{3, 2\} \in Y \setminus X$, we obtain the configuration depicted Fig. 4 (c).
- We must have $\{4, 5\} \notin X$. Otherwise $B = \{\{3, 5\}, \{5, 4\}, \{4, 3\}\}$ would be a cycle in X but not a boundary in X (the two B-extensions of $\{4, 5\}$ contain some simplexes of $Y \setminus X$). We obtain the configuration of Fig. 4 (d).

- We must have $\{4,8\} \notin Y$. Otherwise $B = \{\{2,4\},\{4,8\},\{8,2\}\}$ would be a cycle but not a boundary in Y (the two B-extensions of $\{4,8\}$ contain some simplexes of $X \setminus Y$). We obtain the configuration of Fig. 4 (e).
- Now let us consider the cycle $B = \{\{3,6\},\{6,4\},\{4,3\}\}$. We have $B \subseteq X$. The cycle B is not a boundary in X since the two extensions of $\{4,6,8\}$ and $\{4,5,6\}$ contain some simplexes of $Y \setminus X$.

3. Suppose $\{2,3\} \in X \setminus Y$ and $\{1,2\} \in Y \setminus X$, we obtain the configuration depicted Fig. 4 (f).
- We must have $\{7,8\} \notin X$. Otherwise $B = \{\{2,7\},\{7,8\},\{8,2\}\}$ would be a cycle but not a boundary in X (the two B-extensions of $\{7,8\}$ contain some simplexes of $Y \setminus X$). We obtain the configuration of Fig. 4 (g).
- We must have $\{4,8\} \notin Y$. Otherwise $B = \{\{2,4\},\{4,8\},\{8,2\}\}$ would be a cycle but not a boundary in Y (the B-two extensions of $\{4,8\}$ contain some simplexes of $X \setminus Y$). We obtain the configuration of Fig. 4 (h).
- We must have $\{4,5\} \notin X$. Otherwise $B = \{\{3,5\},\{5,4\},\{4,3\}\}$ would be a cycle but not a boundary in X (the two B-extensions of $\{4,5\}$ contain some simplexes of $Y \setminus X$). We obtain the configuration of Fig. 4 (i).
- Now let us consider the cycle $B = \{\{2,4\},\{4,6\},\{6,7\}\{7,2\}\}$. We have $B \subseteq Y$. The cycle B is not a boundary in Y since the two extensions of $\{4,6,8\}$ and $\{4,5,6\}$ contain some simplexes of $X \setminus Y$.

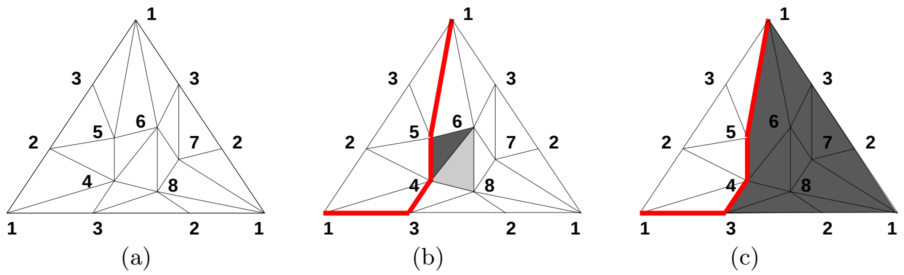


Fig. 3. (a): a triangulation of the dunce hat; (b): a cycle B (in red) and two simplexes x (dark grey) and y (light grey) such that $(x, y) \in \mathbf{R}$; (c) the extension of x for B (dark grey).

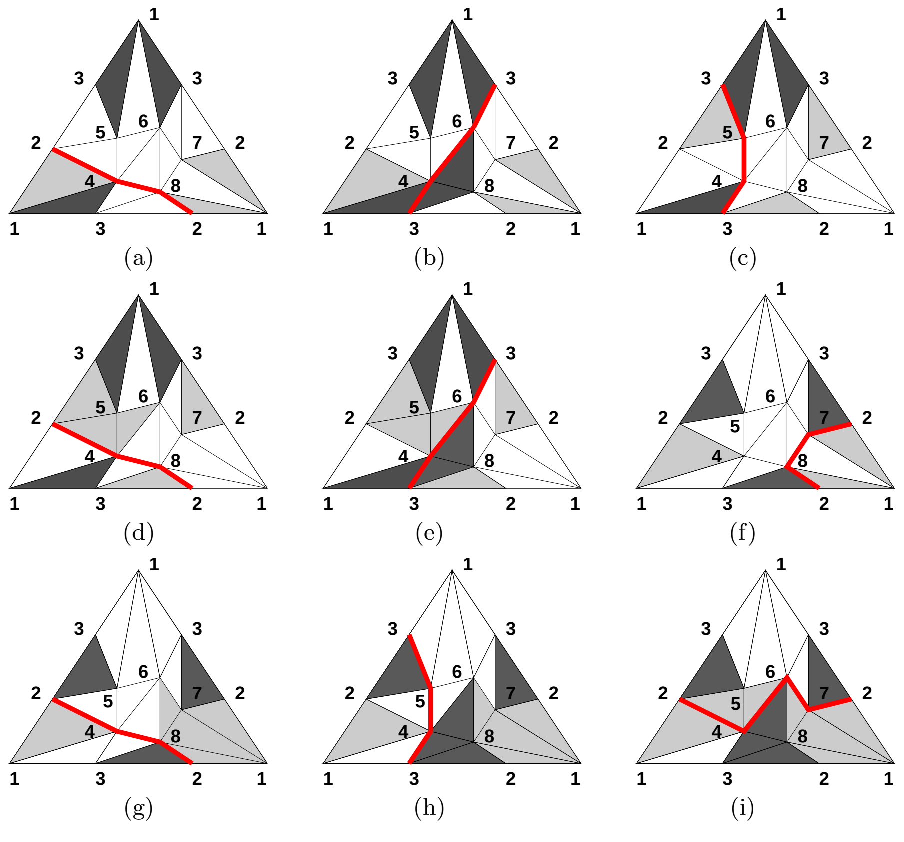


Fig. 4. The highlighted 2-simplexes are the simplexes that belong to $X \setminus Y$ (dark grey) and to $Y \setminus X$ (light grey). The 1-simplexes highlighted in red correspond to cycles.