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SIR Epidemics With State-Dependent Costs and ICU Constraints: A Hamilton-Jacobi Verification Argument and Dual LP Algorithms

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Abstract

The aim of this paper is twofold. On one hand, we strive to give a simpler proof of the optimality of greedy controls when the cost of interventions is control-affine and the dynamics follow a state-constrained controlled SIR model. This is achieved using the Hamilton-Jacobi characterization of the value function, via the verification argument and explicit trajectory-based computations. Aside from providing an alternative to the Pontryagin complex arguments in [6] (see also [5]), this method allows one to consider more general classes of costs; in particular state-dependent ones. On the other hand, the paper is completed by linear programming methods allowing one to deal with possibly discontinuous costs. In particular, we propose a brief exposition of classes of linearized dynamic programming principles based on our previous work and ensuing dual linear programming algorithms. We emphasize the particularities of our state space and possible generations of forward scenarios using the description of reachable sets.

Keywords: Optimal control, State Constraints,Hamilton-Jacobi, Epidemics, Modelling, Linear programming;
MSC2020: 49L12, 49L25, 92D30, 65K15.

1 Introduction

We deal here with a classical SIR model (cf. [25]) in which the total population is considered to be time-independent and the contact rate is controlled via an external parameter (linked to, say, confinement). Such models are far from being new as witnessed by the rich literature on the subject [3, 8, 22, 31] and the recent pandemic has stimulated an increasing number of contributions, e.g., [1, 27, 26, 11]. For us, the aim is to maintain the number of infected individuals (or, equivalently,

the proportion of infection from the total population) upper-bounded by a chosen parameter I^* (resp., a proportion i^*). The parameter is to be thought in connection with the intensive-care unit capacities (see [24, 32, 6]). Motivated by the recent paper [6], we recall the *green* maximal zone in which no control is needed to stabilize the infections and the *yellow* zone in which policies allowing the trajectory to be kept under the level i^* do exist. The explicit treatment of the yellow zone (see also [4] for geometrical intuitions) is achieved in [6] by using normal cone (viability) conditions; see also the recent papers [14], [36] for analogous treatments of such sets. We give here some further geometrical insights into the splitting of this yellow zone following the need for *strict confinement* and some conditional viability properties of such zones (see Proposition 3 in the Appendix).

In [6], we have considered a state-independent affine-type cost and established the optimality of the greedy policy acting only when the trajectory is about to exit the manageable yellow set. The optimality has been investigated by using an extensive analysis of the Pontryagin extremals of the underlying control problem. We have chosen here to give a different proof of the optimality of such greedy policies. This is achieved by giving explicit worked-out formulae for the associated cost function, based on computations on the trajectories. It turns out that the regularity of this candidate function allows one to check the naturally-associated Hamilton-Jacobi equation (in an almost sure sense). Furthermore, the method allows to treat considerably more general costs, the price to pay being essentially linked to the usual regularity (cf. Assumption 1 and Eq. (8)) and an inequality (15). A general class of examples connected with [6], is presented. We should emphasize that while several authors (e.g. [2], [14]) look into such greedy policies, in connection with the "safety" issues, the conditions on the cost guaranteeing their optimality are not made explicit.

A technical point concerns some hints into the actual regularity of the value function without (15). Since we are focused on a particular type of verification with a specific candidate in mind, we will only present these results in the Appendix A1. The first discussion, in Section 5.1.1, concerns the uniqueness of the solution of the associated Hamilton-Jacobi equation. We follow the outward qualification and backward invariance arguments in [16]. The second discussion (cf. Section 5.1.2) concerns the inward qualification arguments in [38].

The second part of the paper (Section 4) presents a method allowing one to deal with possibly discontinuous costs. We recall here the embeddings of the state-constrained controlled trajectories into occupation measures, in the spirit of [18] or [20], [17], [12], etc. The state constraints imply a support condition for such measures and the differential formula applied to regular functions of the dynamics imply a linear constraint, where the linearity is in the spirit of functionals of regular measures. Minimization over such sets of measures is paired with convenient dual formulations (e.g. [18], [35], [13], [20]). Such formulations are naturally connected with the (subsolutions of the) Hamilton-Jacobi equation satisfied by the value function (e.g., [18, EQ. 3.1], see also our formulation in (19)).

Since we work here under state constraints, we present, in the spirit of [20], a linearized dynamical programming principle (DPP) in (20). This is complemented by a modified DPP, in (21). For our readers' sake, we sketch the proof, following Krylov's *shaking of coefficients* method in the Appendix 5.2. This formulation (in (21)) is very much in the spirit of the dual dynamic programming algorithms introduced in [33] and [37], also known as Benders decomposition (cf. [9]). Of course, in essence, such algorithms are not completely new, their connections with occupation measures-formulations appearing, for example in [29] or, more recently, in [23]. The fundamentals consist in the use of polynomial bases (see [30] for connections with Nachbin's theorem) and Putinar's results on positiveness and moments coherence (see [34]). However, the presence of state constraints, the various formulations extending to semi-continuous settings and the Krylov-based duality arguments justify our interest. Furthermore, we provide, in the SIR setting, a detailed description of the state-induced constraints on semi-algebraic sets (based on geometrical/viability descriptions) as well as

reachability considerations with an impact on the generation of scenarios in the forward stage(s). To the best of our knowledge, such arguments are new.

Let us briefly explain the structure of the paper. We begin, in Section 2 by recalling the controlled SIR dynamics and basic explicit computations when using constant control policies. We proceed with a description of green (self-regulating, mathematically invariant) and yellow (manageable, mathematically viability kernels) areas with respect to ICU capabilities-induced constraints. The arguments are illustrated on a toy example where the data correspond more or less to the most recent decision of strict confinement in France retaining an incidence of 400 cases per 100.000 inhabitants, and complemented by the description of some further curves of interest. The control problem is presented in Section 3. We provide the Hamilton-Jacobi equation associated to the value function as well as a verification argument as to the optimality of a greedy policy. Explicit computations in connection to such closed-loop policies are provided in Lemma 1 which, in turn, allows us to obtain the main verification result, Theorem 2. The consistency of the assumptions are illustrated in Example 2. Section 4 deals with the linear formulation of control problems. Relevant results are recalled (set of constraints, primal and dual formulations, equivalence) and they are employed to provide dynamic programming principles (cf. Proposition 1). The different formulations of two-stage problems are presented and followed by an extensive description of tools and procedures. These are thought in tight connection with the SIR system under analysis.

Throughout the paper, we denote by $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{R}_+^* := (0, +\infty)$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Given an interval $I \subset \mathbb{R}$ and a (subspace of a) metric space B, $\mathbb{L}^0[I; B]$ will stand for the family of Borel-measurable B-valued functions whose domain is I. The usual $0/1$-valued indicator function of sets will be denoted by $\mathbf{1}$, while the $0/\infty$-version is denoted by χ.

2 Controlled SIR systems: geometrical considerations

2.1 The Dynamics

We consider the following deterministic evolution of a SI(R) model

$$\begin{cases} ds^{s_0,i_0,a}(t) = -\beta\,(1-a(t))\,s^{s_0,i_0,a}(t) i^{s_0,i_0,a}(t)\,dt \\ di^{s_0,i_0,a}(t) = \big(\beta(1-a(t)) s^{s_0,i_0,a}(t) - \gamma\big)\, i^{s_0,i_0,a}(t)\,dt \\ dr^{s_0,i_0,a}(t) = \gamma i^{s_0,i_0,a}(t)\,dt. \end{cases} \tag{1}$$

The control policy a takes its values in a set $[0, \overline{a}]$, for some parameter $\overline{a} \in (0, 1)$. For the construction of trajectories, one uses Borel measurable inputs $a \in \mathbb{L}^0(\mathbb{R}; [0, \overline{a}])$. The contact rate $\beta > 0$ and the recovery rate γ are assumed to satisfy the consistency condition $0 < \gamma < \beta(1 - \overline{a})$. As it will be clear afterwards, the proportion $\frac{\gamma}{\beta(1-\overline{a})}$ is the one after which strict confinements are to be envisaged. When the spontaneous recovery parameter γ is large, the epidemics tend to disappear without strict interventions.

Remark 1. *1. As it is usually the case, we deal with a time-constant population size. The components s_0 and i_0 are regarded as proportions i.e. $s_0 + i_0 \leq 1$.*
2. A simple look at the dynamics shows that $\mathbb{R}_+ \times \mathbb{R}_+$ is time-invariant in the sense that, independently of the control used, starting with $s_0 \geq 0$ and $i_0 \geq 0$, neither $s^{s_0,i_0,a}(t)$ nor $i^{s_0,i_0,a}(t)$ becomes negative.
3. Actually, in a normalized framework, if $s_0 + i_0 \leq 1$ (and both non-negative), then $s^{s_0,i_0,a}(t) + i^{s_0,i_0,a}(t) \leq s_0 + i_0 \leq 1$, for every $t \geq 0$. This yields the time-invariance of the triangle $\mathbb{T} := \big\{(s_0, i_0) \in \mathbb{R}_+^2 :\ s_0 + i_0 \leq 1\big\}$ with respect to the control system (1).

One notes that, for every $t \in \mathbb{R}$, and for constant controls $a \in [0, \overline{a}]$,

$$s^{s_0,i_0,a}(t) + i^{s_0,i_0,a}(t) = s_0 + i_0 - \gamma \int_0^t i^{s_0,i_0,a}(u)\, du = s_0 + i_0 + \frac{\gamma}{\beta(1-a)} \log \frac{s^{s_0,i_0,a}(t)}{s_0}. \quad (2)$$

2.2 Colour zones

Definition 1. *1. We call green zone the set $\mathcal{G}$ of initial configurations (s_0, i_0) belonging to the set $\mathbb{T} := \left\{(s_0, i_0) \in \mathbb{R}^2_+ :\ s_0 + i_0 \leq 1\right\}$ for which the infections remain under the level i^* even without any intervention, i.e.,*

$$\mathcal{G} := \left\{(s_0, i_0) \in \mathbb{T} :\ \forall a \in \mathbb{L}^0\left(\mathbb{R}; [0, \overline{a}]\right),\ \forall t \geq 0, i^{s_0,i_0,a}(t) \leq i^*\right\}.$$

2. We call yellow zone the set $\mathcal{Y}$ of initial configurations $(s_0, i_0) \in \mathbb{T}$ for which the infections can be kept under i^* with some Borel measurable, $[0, \overline{a}]$-valued, intervention $a \in \mathbb{L}^0\left(\mathbb{R}_+; [0, \overline{a}]\right)$, i.e.,

$$\mathcal{Y} := \left\{(s_0, i_0) \in \mathbb{T} :\ \exists\, a \in \mathbb{L}^0\left(\mathbb{R}; [0, \overline{a}]\right)\ \ s.t.\ \ i^{s_0,i_0,a}(t) \leq i^*,\ \forall t \geq 0\right\}.$$

The strategies a keeping the trajectory under ICU capability will be referred to as admissible*, i.e., describing a set $\mathcal{A}d(s_0, i_0)$; when there is no confusion at risk, we will drop the dependency on the initial data.*

The following result is taken from the recent paper [6]. Its proof is based on viability (state-constrained evolution) theory and will be omitted here.

Theorem 1 (Theorem 2.3 in [6])**.** *1. The green zone is explicitly given by*

$$\mathcal{G} = \left\{(s_0, i_0) \in \mathbb{T} :\ s_0 \leq \phi(i_0)\right\}, \quad (3)$$

where $x := \phi(i)$ is the unique solution of the equation $-x + \frac{\gamma}{\beta} \log x = i - i^* + \frac{\gamma}{\beta} \log \frac{\gamma}{\beta} - \frac{\gamma}{\beta}$ satisfying $x \geq \frac{\gamma}{\beta}$.[1]
2. The yellow zone is explicitly given by

$$\mathcal{Y} = \left\{(s_0, i_0) \in \mathbb{T} :\ s_0 \leq \psi(i_0)\right\}, \quad (4)$$

where $x := \psi(i)$ is the unique solution of the equation $-x + \frac{\gamma}{\beta(1-\overline{a})} \log x = i - i^* + \frac{\gamma}{\beta(1-\overline{a})} \log \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}$ satisfying $x \geq \frac{\gamma}{\beta(1-\overline{a})}$.[2]

Example 1. Let us illustrate this on the example of a total population of $N = 67,000,000$ individuals with a maximal tolerance of 400 cases per 100,000 habitants (rolling 7 days average) and an incubation-to-recovery time of $t_0 = 14$ days. This leads to $I^* = t_0 \times \frac{400}{100000} \times N = 3,752,000$,[3] i.e., a proportion $i^* = 0.056$. We pick $\beta = 1/3$ (one infection out of three contacts approximately) and a recovery parameter $\gamma = \frac{1}{14}$ (recovery intervenes in an average of 14 days). With these data, cf. Figure 1,

1. a control (confinement) range in $[0, 60]$ percent is applied, i.e., $\overline{a} = 0.6$;

2. for the green zone,

[1] Another solution of this equation exists but it does not exceed $\frac{\gamma}{\beta}$.
[2] Another solution of this equation exists but it does not exceed $\frac{\gamma}{\beta(1-\overline{a})}$.
[3] We are talking here of a level of infected individuals at some moment and not a cumulated number of infections.

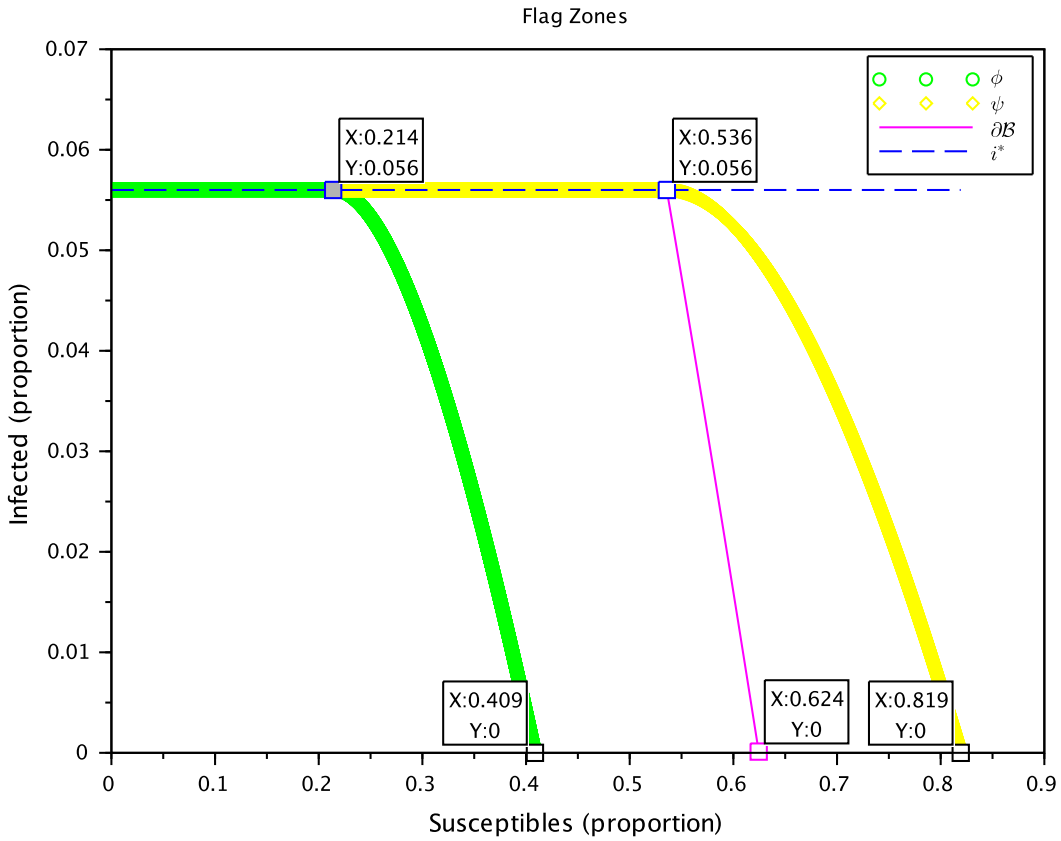


Figure 1: Flag zones for Example 1: $N = 67M$, $I^* = 3.75M$, $\beta = \frac{1}{3}$, $1 - \overline{a} = 40\%$, $\gamma = \frac{1}{14}$.

   (a) one can start with I^* infections for every $S_0 \leq N\frac{\gamma}{\beta} = 14,357,143$, that is, a proportion $s_0 \leq \frac{3}{14} \simeq 0.214$, leading to the upper-left data tip $(0.214, 0.056)$;

   (b) as $I_0 \to 0$, the green zone extends until $\overline{S}_{max} := 27,374,986$; equivalently, a proportion $\overline{s}_{max} \simeq 0.4085819$ leading to the lower-left data tip $(0.409, 0+)$;

3. for the yellow zone,

   (a) one can start with I^* infections for every $S_0 \leq N\frac{\gamma}{\beta(1-\overline{a})} = 35,892,857$, that is, a proportion $s_0 \leq \frac{\gamma}{\beta(1-\overline{a})} \simeq 0.5357143$, i.e., the upper-right label $(0.536, 0.056)$;

   (b) as $I_0 \to 0$, the yellow zone extends until $\underline{S}_{max} \simeq 54,895,452$; equivalently, a proportion $\underline{s}_{max} \simeq 0.8193351$ leading to the lower-right data tip $(0.819, 0+)$;

4. finally, with a slightly over-unitary contagion parameter $\beta = 1.01$, we get the following dramatic decrease on safe (green) and yellow (feasible) zones (cf. Figure 2)

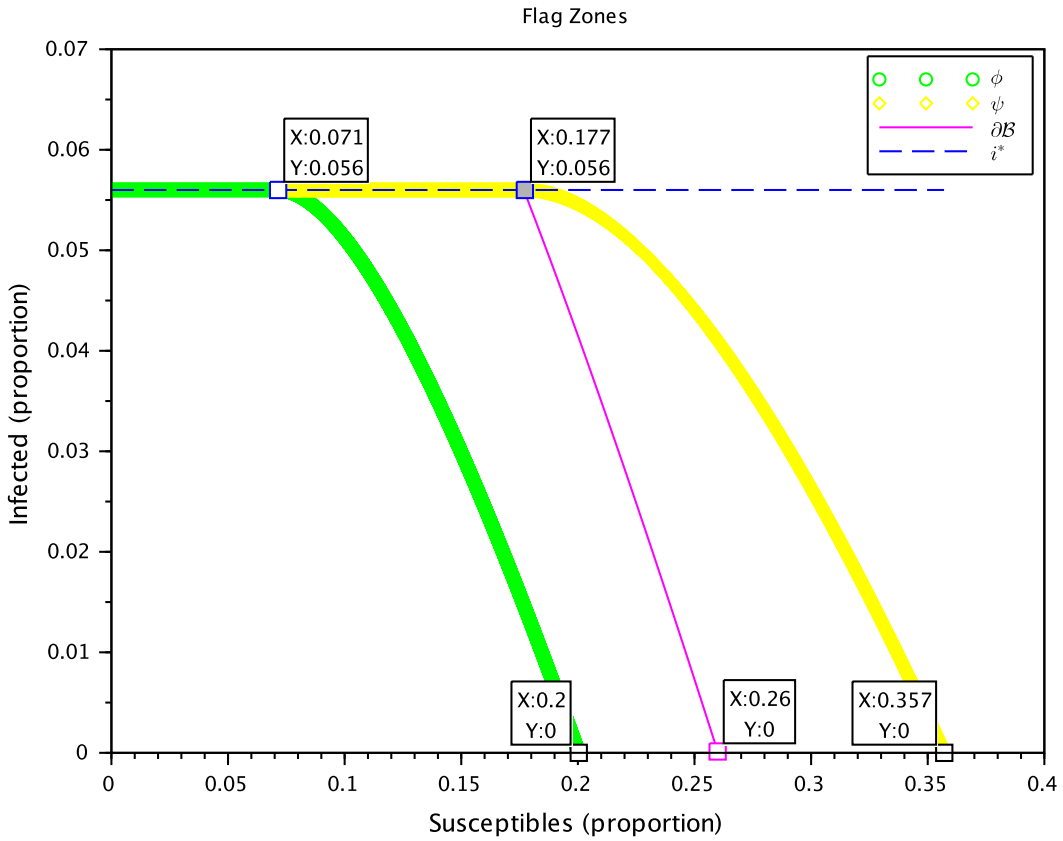


Figure 2: Flag zones for Example 1: $N = 67M$, $I^* = 3.75M$, $\beta = 1.01$, $1 - \overline{a} = 40\%$, $\gamma = \frac{1}{14}$.

2.3 Further geometric considerations

At this point, the reader may be intrigued by the magenta curve in Figure 1. As it turns out, it corresponds to

$$\mathbb{R}_+ \ni t \mapsto \left(s^{\frac{\gamma}{\beta(1-\overline{a})}, i^*, 0}(-t), i^{\frac{\gamma}{\beta(1-\overline{a})}, i^*, 0}(-t) \right),$$

i.e., initial positions from which the 0-controlled trajectory of (1) reaches the point $\left(\frac{\gamma}{\beta(1-\overline{a})}, i^* \right)$. Since (2) holds for every $t \in \mathbb{R}$, by reversing time we obtain that an equivalent description of this curve is

$$\{(\varphi(i), i) : \ i \in (0, i^*]\},$$

where $x := \varphi(i) \geq \frac{\gamma}{\beta(1-\overline{a})} \geq \frac{\gamma}{\beta}$ satisfies $-x + \frac{\gamma}{\beta} \log x = i - i^* + \frac{\gamma}{\beta} \log \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}$.
In general, let us define the set

$$\mathcal{B}^{\overline{s}} := \left\{ (s, i) : \ s \geq 0, \ i \in [0, i^*], \ s \leq \varphi^{\overline{s}}(i) \right\}, \tag{5}$$

where $x := \varphi^{\overline{s}}(i)$ satisfies $-x + \frac{\gamma}{\beta} \log x = i - i^* + \frac{\gamma}{\beta} \log \overline{s} - \overline{s}$, and $\operatorname{sign}\left(x - \frac{\gamma}{\beta} \right) = \operatorname{sign}\left(\overline{s} - \frac{\gamma}{\beta} \right)$ with $\operatorname{sign}(t) := \frac{t}{|t|}$, if $t \neq 0$, $\operatorname{sign}(0) := 1$. In other words, $\varphi^{\overline{s}}$ is taken on the same side of $\frac{\gamma}{\beta}$ as $\overline{s}$.

Remark 2. *1. The reader will have noticed that $\mathcal{B}^{\frac{\gamma}{\beta}} = \mathcal{G}$.*

2. *Furthermore, to avoid complicated notations, we drop the superscript when* $\overline{s} = \frac{\gamma}{\beta(1-\overline{a})}$, *i.e., we let* $\mathcal{B} := \mathcal{B}^{\frac{\gamma}{\beta(1-\overline{a})}}$. *This equally explains the notation employed in Figure 1. As it will be clear in the verification arguments, the boundary of* $\mathcal{B}$ *(represented in magenta in Figure 1) delimits the zone in which strict confinement is required.*

3. *It is easy to show (please take a look at Proposition 3) that if* $(s_0, i_0) \in \mathcal{B} \setminus \mathcal{G}$, *then, prior to hitting* i^*, *one has* $\left(s^{s_0,i_0,0}(t), i^{s_0,i_0,0}(t)\right) \in \mathcal{B}$. *Furthermore, due to the maximality of* $\mathcal{G}$ *(as invariant set), it follows that there exists* $t_0 \in \mathbb{R}_+$ *such that* $\left(s^{s_0,i_0,0}(t_0), i^{s_0,i_0,0}(t_0)\right) \in \left(\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\overline{a})}\right] \times \{i^*\}$.

3 Optimality of greedy policies

3.1 The control problem

We now consider the following control problem.

Problem 1. *For every controllable initial configuration* $(s_0, i_0) \in \mathcal{Y}$, *i.e., subject to* $s_0 \leq \psi(i_0)$, *minimize*

$$J(s_0, i_0, a) := \int_0^{\tau_{green}^{s_0,i_0,a}} l_1\left(s^{s_0,i_0,a}(t), i^{s_0,i_0,a}(t), a(t)\right) dt = \int_0^{\infty} l_1\left(s(t), i(t), a(t)\right) \mathbf{1}_{s(t)\geq\phi(i(t))}\, dt, \quad (6)$$

(in the second equation, we have dropped the superscript $(s, i) := \left(s^{s_0,i_0,a}, i^{s_0,i_0,a}\right)$*;)*

- *under the* state constraint
$$i^{s_0,i_0,a}(t) \leq i^*, \text{ a.s. on } \mathbb{R}_+, \quad (7)$$
- *until reaching the* target $\mathcal{G}$, *i.e., until*
$$\tau_{green}^{s_0,i_0,a} := \inf\left\{t \geq 0 : \ s^{s_0,i_0,a}(t) < \phi\left(i^{s_0,i_0,a}(t)\right)\right\}.$$

3.2 The Hamilton-Jacobi equation

Before proceeding, let us explain the standing assumptions.

Assumption 1. *Throughout the rest of the paper, unless otherwise stated, the following assumptions hold true:*

i. the cost l_1 *is non-negative and lower-semicontinuous;*

ii. for every $(s, i) \in \mathbb{R}^2$, *the map* $[0, \overline{a}] \ni a \mapsto l_1(s, i, a)$ *is non-decreasing;*

iii. the set $\{l_1(s, i, a) : \ a \in [0, \overline{a}]\}$ *is convex, for every* $(s, i) \in \mathbb{R}^2$.

Remark 3. *1. The reader is invited to note that the second formulation in* (6) *is independent of* τ_{green}.
2. Please note that the main case of interest is the one in which l_1 *is control-affine for which Assumption 1 iii. is trivially satisfied.*
3. Sometimes it is convenient to assume that

$$l_1(s, i, 0) = 0 \text{ on } \mathcal{G}. \quad (8)$$

Then, owing to $a = 0$ being optimal as soon as it is admissible (due to Assumption 1 ii.), in such framework, minimizing J is equivalent to minimizing

$$\tilde{J}(s_0, i_0, a) := \int_0^\infty l_1\left(s^{s_0,i_0,a}(t), i^{s_0,i_0,a}(t), a(t)\right) dt.$$

The function $\tilde{J}$ is to be regarded as an infinite horizon discounted value function with discount parameter $q = 0$. The arguments developed hereafter equally apply to $q \geq 0$.

Heuristically speaking, the control Problem 1[4] is connected with the Hamilton-Jacobi equation

$$\sup_{a\in[0,\overline{a}]} \left\{\partial_s V(s,i)\beta(1-a)si - \partial_i V(s,i)\left(\beta(1-a)s-\gamma\right)i - l_1(s,i,a)\mathbf{1}_{s\geq\phi(i)}\right\} = 0, \tag{9}$$

i.e., provided that l_1 is lower-semicontinuous (l.s.c.) and bounded from below, $\mathcal{Y} \ni (s,i) \mapsto V(s,i) := \inf_{a\in\mathcal{A}d} J(s,i,a)$ is the smallest l.s.c. viscosity super-solution of (9) (with various meanings). For semicontinuity results and connections with the Hamilton-Jacobi equation, the reader is referred to [21], [16] (see also [10]). For the continuity of V, we refer the reader to [6, Remark 5.7 assertion 5]. For the connection with [16] and the inward pointing condition from [38], we provide some elements in the Appendix 5.1.

3.3 A verification argument

We now consider the "greedy" closed-loop feedback control

$$a^*(s,i) := \left[\left(1 - \frac{\gamma}{\beta s}\right)^+ \wedge \overline{a}\right] \mathbf{1}_{\partial\mathcal{Y}^+}(s,i), \tag{10}$$

where, for rigour, we set $\partial\mathcal{Y}^+$ to be the "active" boundary of the viable set $\mathcal{Y}$ i.e.

$$\partial\mathcal{Y}^+ = \left(\left[\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\overline{a})}\right] \times \{i^*\}\right) \cup \{(\psi(i), i) :\ 0 < i \leq i^*\}. \tag{11}$$

Remark 4. *This control is called greedy as it takes no action until forced (about to exit the yellow set) and, at this time, one takes minimal action guaranteeing that $\mathcal{Y}$ is not exited. We recall that $a^+ = \max\{a, 0\}$ stands for the positive part and $a \wedge b = \min\{a, b\}$ is the minimum operator. between two real quantities In the interior of $\mathcal{Y}$ and on the set $\mathcal{G}$, no action ($a = 0$) is taken. When the current position is at $(\psi(i), i)$, the trajectory is kept on this boundary $\partial\mathcal{Y}^+$ with the only control available i.e. $\overline{a}$ until the number of susceptibles reaches $\frac{\gamma}{\beta(1-\overline{a})}$. When the constraint on i is saturated (only manageable for $s \in \left[\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\overline{a})}\right]$), the deconfinement is progressive ($a = 1 - \frac{\gamma}{\beta s}$).*

Lemma 1. *Under the assumption* (8)*, the function $\mathcal{Y} \cap \left(\mathbb{R}_+^*\right)^2 \ni (s_0, i_0) \mapsto W(s_0, i_0) := \tilde{J}(s_0, i_0, a^*)$ is explicitly given by*

1) $$0, \ \textit{if } (s_0, i_0) \in \mathcal{G},$$

2) $$\frac{1}{\gamma i^*} \int_{\frac{\gamma}{\beta}}^{s_1(s_0,i_0)} l_1\left(u, i^*, 1 - \frac{\gamma}{\beta u}\right) du, \ \textit{if } (s_0, i_0) \in \mathcal{B} \setminus \mathcal{G},$$

3) $$\frac{1}{\beta(1-\overline{a})} \int_{\frac{\gamma}{\beta(1-\overline{a})}}^{s_2(s_0,i_0)} \frac{l_1\left(s, \theta^* - s + \frac{\gamma}{\beta(1-\overline{a})}\log s, \overline{a}\right)}{s\left(\theta^* - s + \frac{\gamma}{\beta(1-\overline{a})}\log s\right)} ds + W\left(\frac{\gamma}{\beta(1-\overline{a})}, i^*\right), \ \textit{if } (s_0, i_0) \in \mathcal{Y} \setminus \mathcal{B},$$

[4] seen as an infinite horizon 0-discounted control problem

where

$$\begin{cases} \text{(i)} & s_1(s_0,i_0) > \frac{\gamma}{\beta} \text{ is the unique solution to } s_1 - s_0 - i_0 + i^* - \frac{\gamma}{\beta}\log\frac{s_1}{s_0} = 0, \\ \text{(ii)} & s_2(s_0,i_0) := \exp\left(\frac{\beta(1-\overline{a})}{\gamma\overline{a}}\left(s_0 + i_0 - \frac{\gamma}{\beta}\log s_0 - \theta^*\right)\right), \\ \text{(iii)} & \theta^* := i^* + \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}\log\frac{\gamma}{\beta(1-\overline{a})}. \end{cases} \tag{12}$$

Proof. On $\mathcal{G}$, there is nothing to prove.
We first deal with the case when $(s_0,i_0) \in \mathcal{B}\setminus\mathcal{G}$. The 0 controlled trajectory reaches $\partial\mathcal{Y}^+$ at some point (s_1,i^*) (see Remark 2, item 3). Using the computations on the constant controlled trajectories (2), it follows that

$$s_1 = s_0 + i_0 - i^* + \frac{\gamma}{\beta}\log\frac{s_1}{s_0}.$$

On the other hand, one explictly computes $s^{s_1,i^*,a^*}(t) = s_1 - \gamma i^* t$ prior to reaching $\frac{\gamma}{\beta}$ (at which point $\mathcal{G}$ has been reached). This leads to the explicit computation (see also Remark 1, 2.)

$$W(s_0,i_0) = W(s_1,i^*) = \tilde{J}(s_1,i^*,a^*) = \int_0^{\frac{s_1-\frac{\gamma}{\beta}}{\gamma i^*}} l_1\left(s_1 - \gamma i^* t, i^*, 1 - \frac{\gamma}{\beta\left(s_1 - \gamma i^* t\right)}\right) dt.$$

The conclusion follows by using the change of variable $r(t) := s_1 - \gamma i^* t$.
Let us now consider $(s_0,i_0) \in \mathcal{Y}\setminus\mathcal{B}$. It is clear that one uses 0 control to reach $\partial\mathcal{Y}$ at some point $(s_2 = \psi(i_2), i_2)$ after which, one uses $\overline{a}$ until reaching $\mathcal{B}$ through the entry point $\left(\frac{\gamma}{\beta(1-\overline{a})}, i^*\right)$.
As in the previous case, using the computations on the constant controlled trajectories (2), we have

$$s_2 + i_2 = s_0 + i_0 + \frac{\gamma}{\beta}\log\frac{s_2}{s_0}.$$

Using the definition of $s_2 = \psi(i_2)$(cf. Theorem 1), it follows that

$$s_2 + i_2 = \frac{\gamma}{\beta(1-\overline{a})}\log s_2 + i^* + \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}\log\frac{\gamma}{\beta(1-\overline{a})} = \frac{\gamma}{\beta(1-\overline{a})}\log s_2 + \theta^*.$$

Combining the two, we get

$$\log s_2 = \frac{\beta(1-\overline{a})}{\gamma\overline{a}}\big(s_0 + i_0 - \frac{\gamma}{\beta}\log s_0 - \theta^*\big). \tag{13}$$

The a^*-controlled trajectory stays on $\partial\mathcal{Y}^+$ and reaches $\left(\frac{\gamma}{\beta(1-\overline{a})}, i^*\right)$ in time t. On this set both s and i are monotone. It follows that

$$J := \int_0^t l_1\left(s^{s_2,i_2,\overline{a}}(u), i^{s_2,i_2,\overline{a}}(u), \overline{a}\right) du = \frac{1}{\beta(1-\overline{a})}\int_{\frac{\gamma}{\beta(1-\overline{a})}}^{s_2} \frac{l_1\left(s,\psi^{-1}(s),\overline{a}\right)}{s\psi^{-1}(s)}\, ds.$$

The definition of ψ gives a simple definition for

$$\psi^{-1}(s) = i^* - s + \frac{\gamma}{\beta(1-\overline{a})}\log s + \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}\log\frac{\gamma}{\beta(1-\overline{a})} = \theta^* - s + \frac{\gamma}{\beta(1-\overline{a})}\log s,$$

so that

$$J = \frac{1}{\beta(1-\overline{a})}\int_{\frac{\gamma}{\beta(1-\overline{a})}}^{s_2} \frac{l_1\left(s, \theta^* - s + \frac{\gamma}{\beta(1-\overline{a})}\log s, \overline{a}\right)}{s\left(\theta^* - s + \frac{\gamma}{\beta(1-\overline{a})}\log s\right)}\, ds.$$

The final expression for $W(s_0, i_0)$ follows then by observing that

$$W(s_0, i_0) = J + W\left(\frac{\gamma}{\beta(1-\overline{a})}, i^*\right).$$

The lemma is now proved owing to (13). □

Remark 5. *The reader is invited to note that, by definition, $s_2 = s^{s_0,i_0,a^*}(t_0)$, for some $t_0 \geq 0$ such that $s_2(t) \in [\frac{\gamma}{\beta(1-\overline{a})}, s_0]$ for every $t \in [0, t_0]$. As a consequence, $i^{s_0,i_0,a^*}(t) \geq i_0$ for every $t \in [0, t_0]$. In particular, at t_0,*

$$\theta^* - s_2 + \frac{\gamma}{\beta(1-\overline{a})} \log s_2 = \psi^{-1}(s_2) = i_2 \geq i_0. \tag{14}$$

Theorem 2. *The following propositions hold true.*

1. $\lim\limits_{\mathcal{B}\setminus\mathcal{G}\ni(s_0,i_0)\to(s',i')\in\mathcal{G}} s_1(s_0, i_0) = \frac{\gamma}{\beta}$*; moreover, for $(s_0, i_0) \in \mathcal{B} \setminus \mathcal{G}$ we have*

$$\partial_s s_1(s_0, i_0) = \frac{(\beta s_0 - \gamma)\, s_1}{(\beta s_1 - \gamma)\, s_0}, \quad \partial_i s_1(s_0, i_0) = \frac{\beta s_1}{\beta s_1 - \gamma}.$$

2. $\lim\limits_{\mathcal{Y}\setminus\mathcal{B}\ni(s_0,i_0)\to(s',i')\in\mathcal{B}} s_2(s_0, i_0) = \frac{\gamma}{\beta(1-\overline{a})}$*; moreover, for $(s_0, i_0) \in \mathcal{Y} \setminus \mathcal{B}$ we have*

$$\partial_s s_2(s_0, i_0) = \frac{\beta(1-\overline{a})}{\gamma\overline{a}}\left(1 - \frac{\gamma}{\beta s_0}\right) s_2, \ \partial_i s_2(s_0, i_0) = \frac{\beta(1-\overline{a})}{\gamma\overline{a}} s_2.$$

3. *Under the Assumption 1 and assuming* (8) *to hold true and l_1 to be continuous, the function W is continuous and differentiable almost everywhere on $\mathcal{Y}$.*[5]

4. *Let us introduce the function $\left(\mathbb{R}_+^*\right)^2 \times (0, \overline{a}] \ni (s, i, a) \mapsto \tilde{l}_1(s, i, a) := \frac{l_1(s,i,a)}{\gamma i a}$ and assume that*

$$\begin{cases} \tilde{l}_1\left(s_1(s_0, i_0), i^*, 1 - \frac{\gamma}{\beta s_1(s_0,i_0)}\right) & \leq \min\limits_{a\in(0,\overline{a}]} \tilde{l}_1(s_0, i_0, a), \ \textit{if } (s_0, i_0) \in \mathcal{B} \setminus \mathcal{G}, \\ \tilde{l}_1\left(s_2, \theta^* - s_2 + \frac{\gamma}{\beta(1-\overline{a})} \log s_2, \overline{a}\right) & \leq \min\limits_{a\in(0,\overline{a}]} \tilde{l}_1(s_0, i_0, a), \ \textit{if } (s_0, i_0) \in \mathcal{Y} \setminus \mathcal{B}, \\ & \text{where } s_2 = s_2(s_0, i_0). \end{cases} \tag{15}$$

Then W satisfies the Hamilton-Jacobi equation (9) *at all points where it is differentiable (hence almost everywhere).*

Proof. The first two assertions are simple consequences of the definitions in (12) and those of the sets $\mathcal{G}$ and $\mathcal{B}$.
Let us sketch the proof of the third assertion concerning W. The continuity is a mere consequence of the limiting behaviour of s_1 and s_2. We wish to point out that the boundaries where differentiability issues may appear ($\{(\phi(i), i) : i \in [0, i^*]\}$, resp. $\{(\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i), i) : i \in [0, i^*]\}$), are Lebesgue-null sets in $\mathbb{R}^2$. This type of consideration is also valid for $\partial\mathcal{Y}$ (which is 1-dimensional) on which differentiability may only fail at $\frac{\gamma}{\beta}$, resp. $\frac{\gamma}{\beta(1-\overline{a})}$. From now on, we leave aside these sets and prove

[5] Since the equation (9) usually concerns an equality condition on the interior $\mathring{\mathcal{Y}}$ and an inequality on the boundary $\partial\mathcal{Y}$, the "almost everywhere" here is intended with respect to the Lebesgue measure on $\mathbb{R}^2$ for the interior, respectively a suitable 1-dimensional measure on the regular curve describing $\partial\mathcal{Y}$.

assertion 4. On $\overset{\circ}{\mathcal{G}}$, there is nothing to prove (as W is null). On $\mathcal{B}\setminus\Big(\{(\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i),i):\ i\in(0,i^*]\}\cup\mathcal{G}\Big)$, one easily computes

$$\begin{aligned}\partial_s W(s_0,i_0) &= \frac{1}{\gamma i^*} l_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)\frac{(\beta s_0-\gamma)\,s_1}{(\beta s_1-\gamma)\,s_0} = \tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)\frac{\beta s_0-\gamma}{\beta s_0},\\ \partial_i W(s_0,i_0) &= \frac{1}{\gamma i^*} l_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)\frac{\beta s_1}{\beta s_1-\gamma} = \tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right).\end{aligned}$$

On $\mathcal{Y}\setminus\mathcal{B}$, one easily computes

$$\begin{aligned}\partial_s W(s_0,i_0) &= \frac{1}{\gamma\overline{a}}\frac{l_1\left(s_2,\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2,\overline{a}\right)}{\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2}\left(1-\frac{\gamma}{\beta s_0}\right)\\ &= \tilde{l}_1\left(s_2,\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2,\overline{a}\right)\frac{\beta s_0-\gamma}{\beta s_0},\\ \partial_i W(s_0,i_0) &= \frac{1}{\gamma\overline{a}}\frac{l_1\left(s_2,\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2,\overline{a}\right)}{\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2} = \tilde{l}_1\left(s_2,\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2,\overline{a}\right).\end{aligned}$$

One deduces the following.

1. On $\mathcal{B}\setminus\Big(\{(\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i),i):\ i\in(0,i^*]\}\cup\mathcal{G}\Big)$,

$$\begin{aligned}&\partial_s W(s_0,i_0)\beta(1-a)s_0 i_0-\partial_i W(s_0,i_0)\left(\beta(1-a)s_0-\gamma\right)i_0-l_1(s_0,i_0,a)=\\ =&i_0 a\left[-\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)(\beta s_0-\gamma)+\beta s_0\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)-\gamma\tilde{l}_1(s_0,i_0,a)\right]\\ &+i_0\left[\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)(\beta s_0-\gamma)-\beta s_0\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)+\gamma\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)\right]\\ =&\gamma i_0 a\left[\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)-\tilde{l}_1(s_0,i_0,a)\right].\end{aligned}$$

   The conclusion follows using the assumption on $\tilde{l}_1$ (the coefficient of a is non-positive such that the max is attained for $a=0$ and this maximum is 0).

2. In the remaining case when $(s_0,i_0)\in\mathcal{Y}\setminus\mathcal{B}$, the conclusion is obtained in exactly the same way; the only difference is that $\tilde{l}_1\left(s_1,i^*,1-\frac{\gamma}{\beta s_1}\right)$ is replaced with $\tilde{l}_1\left(s_2,\theta^*-s_2+\frac{\gamma}{\beta(1-\overline{a})}\log s_2,\overline{a}\right)$ and one concludes using the second inequality in (15).

□

Remark 6. *1. As a consequence, a^* is (an) optimal control for Problem 1 provided the uniqueness result in [16, Theorem 11] holds true; please refer to the remarks in the Appendix 5.1.1. Roughly speaking, with the exception of $\left\{\left(\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i),i\right):\ 0<i\leq i^*\right\}$, W is a regular (sub)solution of (9) and, thus, it does not exceed the supersolution V. On the other hand, since a^* is admissible, the reverse inequality is valid, hence providing the optimality of a^*.*

2. The condition (15) is not only sufficient but actually necessary (among the costs l_1 satisfying Assumption 1 and (8)) in order for a^* to be optimal (or, equivalently, for W to be a solution of (9)).

Example 2. 1. The typical example one has in mind is a multiplicative one

$$l_1(s,i,a) = \lambda(s,i)a,$$

for some non-negative, uniformly continuous function $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ such that

- for every $i \in \mathbb{R}_+^*$, the map $\mathbb{R}_+^* \ni s \mapsto \lambda(s,i)$ is non-decreasing (the higher the number of susceptibles, the higher the price);
- for every $s \in \mathbb{R}_+^*$, the map $\mathbb{R}_+^* \ni i \mapsto \frac{\lambda(s,i)}{i}$ is non-increasing.

In this case, $\tilde{l}_1(s,i,a) = \frac{\lambda(s,i)}{\gamma i}$. Recalling that $s_1 \leq s_0$ (s_1 is on a trajectory starting at (s_0,i_0) and the s component of such trajectories are non-increasing in time) and that $i_0 \leq i^*$ for all $i_0 \in \mathcal{Y}$, it follows that $\tilde{l}_1\left(s_1, i^*, 1 - \frac{\gamma}{\beta s_1}\right) = \frac{\lambda(s_1,i^*)}{\gamma i^*} \leq \frac{\lambda(s_0,i^*)}{\gamma i^*} \leq \frac{\lambda(s_0,i_0)}{\gamma i_0}$, thus implying the first condition in (15). The second condition in (15) is similar and relies on (14).
We emphasize, for completeness sake, that the Assumption 1 and the condition (8) hold true for such functions.

2. The case $l_1(s,i,a) := \lambda a$ studied in [6] is a special case of the previous one. In fact, one can consider general contributions of type $\lambda i^\eta a$, for every $0 \leq \eta \leq 1$ (eventually by slightly altering them around 0 if Lipschitz continuity is desired).

Remark 7. *The non-increasing character of functions $i \mapsto \frac{\lambda(i)}{i}$ amounts to a sub-homogeneity requirement $\lambda(\alpha i) \leq \alpha\lambda(i)$, $\forall \alpha \geq 1$. Concave functions λ for which $\lambda(0) = 0$ satisfy the above-mentioned condition at $i \geq 0$.*

4 A short journey into discontinuous costs

Although the previous arguments give enough novelty with respect to our previous paper [6] and with respect to the current literature, let us explain the dual programming approach allowing one to compute the value function in semi-continuous settings. Numerical illustrations and analysis of the algorithm are left for a future work.
We wish to point out that, throughout the section, the cost l_1 is assumed to be lower semi-continuous, bounded from below and control independent, i.e., of the form $l_1(s,i)$. This is a sufficient condition in order to guarantee the equality between strict and relaxed formulations, see, for instance, [18, Assumption I, Page 2494] and discussions following for an idea of more general settings.[6]

4.1 The linear programming (LP) approach

As we have already mentioned before, we deal with an infinite-horizon control problem for which we have chosen the discount parameter $q = 0$. Hereafter, we consider a general discount $q > 0$ (the case $q = 0$ can be obtained as a limit). To every initial data $(s,i) \in \mathcal{Y}$ and every control $a_0 \in \mathcal{A}d$, one associates

- the so-called (q-discounted) occupation probability

$$\mu^{s_0,i_0,a_0}(ds,di,da) = q\int_0^\infty e^{-qt}\mathbf{1}_{s^{s_0,i_0,a_0}(t)\in ds, i^{s_0,i_0,a_0}(t)\in di, a_0(t)\in da}\,dt;$$

[6] The algorithm developed hereafter can be used for the relaxed problem (where an optimal measure exists). The assumptions are only sufficient to guarantee that these optimal measures correspond to a control policy. Of course, convexity in control of l_1 is also sufficient.

- the $T>0$ finite-horizon couple of occupation probability measures

$$\mu_1^{s_0,i_0,a_0}(dt,ds,di,da) := \frac{1}{T}\int_{dt\cap[0,T]} \mathbf{1}_{s^{s_0,i_0,a_0}(t)\in ds, i^{s_0,i_0,a_0}(t)\in di, a_0(t)\in da}\, dt,$$
$$\mu_2^{s_0,i_0,a_0}(ds,di) := \delta_{\left(s^{s_0,i_0,a_0}(T), i^{s_0,i_0,a_0}(T)\right)}(ds,di).$$

The viability condition of $\mathcal{Y}$ and the dynamic formula for test functions $f\in C^1\left(\mathbb{R}^2\right)$ applied to the flow $\left(s^{s_0,i_0,a_0}, i^{s_0,i_0,a_0}\right)$ yields the following definition of the measure set of constraints

$$\Theta\left(s_0,i_0\right) := \left\{\begin{array}{l}\mu\in\mathcal{P}\left(\mathcal{Y}\times[0,\overline{a}]\right):\ \forall f\in C^1\left(\mathbb{R}^2\right)\\ qf(s_0,i_0)+\int_{\mathcal{Y}\times[0,\overline{a}]}\left\{\begin{array}{l}\left(-\partial_s f(s,i)\right)\beta(1-a)si\\ +\partial_i f(s,i)\left(\beta(1-a)s-\gamma\right)i - qf(s,i)\end{array}\right\}\mu(ds,di,da)=0\end{array}\right\}, \tag{16}$$

respectively

$$\Theta_{t_0,T}\left(s_0,i_0\right) := \left\{\begin{array}{l}\mu:=(\mu_1,\mu_2)\in\mathcal{P}\left([t_0,T]\times\mathcal{Y}\times[0,\overline{a}]\right)\times\mathcal{P}\left(\mathcal{Y}\right):\ \forall f\in C^1\left(\mathbb{R}^3\right)\\ \int_{\mathcal{Y}} f(T,s,i)\mu_2(ds,di) - f(t_0,s_0,i_0) =\\ = T\int_{[t_0,T]\times\mathcal{Y}\times[0,\overline{a}]}\left\{\begin{array}{l}\partial_t f(t,s,i)-\partial_s f(t,s,i)\beta(1-a)si\\ +\partial_i f(t,s,i)\left(\beta(1-a)s-\gamma\right)i\end{array}\right\}\mu_1(dt,ds,di,da)\end{array}\right\}. \tag{17}$$

As usual, given a (subset of a) metric space S, $\mathcal{P}(S)$ stands for the family of probability measures on S endowed with the Borel σ-field. The reader is referred to [18] (see also [20] for the finite horizon setting or [13]) for the structure of such sets. In particular (e.g. [18, Eq. (4.20)], see also [20, Proposition 1]), $\Theta\left(s_0,i_0\right)$ is the closure of the convex hull of occupation measures as described before and, for every continuous cost l_1,

$$\begin{aligned} qV(s_0,i_0) &= q\inf_{a\in\mathcal{A}d}\int_{\mathbb{R}_+} e^{-qt} l_1\left(s^{s_0,i_0,a}(t), i^{s_0,i_0,a}(t)\right)dt\\ &= \Lambda(s_0,i_0) := \inf_{\mu\in\Theta(s_0,i_0)}\int_{\mathcal{Y}} l_1(s,i)\mu(ds,di,[0,\overline{a}]).\end{aligned}$$

A similar property holds true for $\Theta_{t_0,T}\left(s_0,i_0\right)$ but, for our readers' sake, we will skip unnecessary details. The dual value (obtained by standard duality algorithms e.g. in [13] or Hamilton-Jacobi methods e.g. in [18, Lemma 3.2], [20, Appendix]) can be written in numerous ways:

1. (cf. [18, Eq (3.1)])

$$\begin{aligned}\Lambda(s_0,i_0) &= \Lambda^*(s_0,i_0)\\ &= \sup\left\{\begin{array}{l}\eta\in\mathbb{R}:\ \exists f\in C^1\ s.t.\ \forall\left(s,i,a\right)\in\mathcal{Y}\times[0,\overline{a}]\\ \eta\le -\partial_s f(s,i)\beta(1-a)si+\partial_i f(s,i)\left(\beta(1-a)s-\gamma\right)i\\ \qquad +q\left(f(s_0,i_0)-f(s,i)\right)+l_1(s,y)\end{array}\right\}.\end{aligned}$$

Let us point out that the first equality (see the cited reference) gives the no-gap property.

2. (cf. [35, Proposition 12] rewriting the original [7, Lemmas 2.6, 2.7])

$$\Lambda^*(s_0, i_0) = \sup \{qf(s_0, i_0) : \ f \in \mathbf{\Phi}\}, \tag{18}$$

where

$$\mathbf{\Phi} := \left\{ \begin{array}{l} f \in C^1 \ : \ \forall (s,i,a) \in \mathcal{Y} \times [0, \overline{a}] \\ -\partial_s f(s,i)\beta(1-a)si + \partial_i f(s,i)\left(\beta(1-a)s - \gamma\right) i - qf(s,i) + l_1(s,i) \leq 0 \end{array} \right\}. \tag{19}$$

Using the dynamic programming principle for lower semi-continuous functions (given as in [20, Theorem 14 2.] [7]), one gets

Proposition 1. *1. If l_1 is lower semi-continuous, then the following dynamic programming principle holds true:*

$$\Lambda^*(s_0, i_0) = \inf_{\mu=(\mu_1,\mu_2) \in \Theta_{0,T}(s_0,i_0)} \left\{ \begin{array}{l} Tq \displaystyle\int_{[0,T]\times\mathbb{R}^2\times[0,\overline{a}]} e^{-qt} l_1(s,i)\, \mu_1(dt, ds, di, da) \\ + e^{-qT} \displaystyle\int_{\mathcal{Y}} \Lambda^*(s,i) \mu_2(ds, di) \end{array} \right\}. \tag{20}$$

2. If l_1 is continuous, the following (modified) Dual DPP holds true

$$\Lambda^*(s_0, i_0) = \inf_{\mu=(\mu_1,\mu_2) \in \Theta_{0,T}(s_0,i_0)} \left\{ \begin{array}{l} Tq \displaystyle\int_{[0,T]\times\mathbb{R}^2\times[0,\overline{a}]} e^{-qt} l_1(s,i)\, \mu_1(dt, ds, di, da) \\ + e^{-qT} q \displaystyle\sup_{f \in \mathbf{\Phi}} \int_{\mathcal{Y}} f(s,i) \mu_2(ds, di) \end{array} \right\}. \tag{21}$$

The proof is forwardly inspired by the arguments in [20, Theorem 14 2.] and the characterizations of Θ and $\Theta_{t_0,T}$. We provide some elements of proof regarding particularly the second formulation in the appendix 5.2.

Furthermore, the compactness of the set $\Theta_{0,T}(s_0, i_0)$ with respect to the weak* convergence of probability measures, and the weak* lower semi-continuity of the functionals

$$\mu \mapsto \theta(\mu) := Tq \int_{[0,T]\times\mathbb{R}^2\times[0,\overline{a}]} e^{-qt} l_1(s,i)\, \mu_1(dt, ds, di, da) + e^{-qT} \int_{\mathcal{Y}} \Lambda^*(s,i) \mu_2(ds, di),$$
$$\mu \mapsto \tilde{\theta}(\mu) := Tq \int_{[0,T]\times\mathbb{R}^2\times[0,\overline{a}]} e^{-qt} l_1(s,i)\, \mu_1(dt, ds, di, da) + e^{-qT} q \sup_{f \in \mathbf{\Phi}} \int_{\mathcal{Y}} f(s,i) \mu_2(ds, di),$$

show that the infimum in problems (20) and (21) is actually a minimum.

4.2 The dual LP algorithm

We are now equipped with all the elements allowing one to state the problem as a two-stage linear problem (in the spirit of [33] or [37]).

Problem 2. *Solve the two stage problem*

[7] The original reference concerns the finite time horizon, but it is easily adaptable to this discounted setting.

$$\overrightarrow{\mathbb{F}}(s_0,i_0) := \min\left\{e^{-qt}\left\langle Tql_1(s,i),\mu_1\right\rangle + \left\langle e^{-qT}\overleftarrow{\mathbb{B}}(s,i),\mu_2\right\rangle\right\}$$

where the minimum is taken over all measures $\mu=(\mu_1,\mu_2)$ *with* $\mathrm{supp}(\mu)\subseteq[t_0,T]\times\mathcal{Y}\times[0,\overline{a}]\times\mathcal{Y}$ *and such that*

$$\begin{aligned}&\left\langle f(T,s,i),\mu_2\right\rangle - f(0,s_0,i_0)\\ &= T\left\langle \partial_t f(t,s,i) - \partial_s f(t,s,i)\beta(1-a)si + \partial_i f(t,s,i)\left(\beta(1-a)s-\gamma\right)i,\mu_1\right\rangle,\end{aligned}$$

for every $f\in C^1\left(\mathbb{R}^3\right)$*;*

$$\overleftarrow{\mathbb{B}}(s,i) := \sup_{f\in C^1}\left\{qf(s,i) + \inf_{(s',i')\in\mathcal{Y}}\begin{bmatrix}-\partial_s f(s',i')\beta(1-a)s'i' + \partial_i f(s',i')\left(\beta(1-a)s'-\gamma\right)i'\\ -qf(s',i') + l_1(s',i')\end{bmatrix}\right\}.$$

Remark 8. *1. The spaces over which one integrates and the integration parameters have been dropped to simplify the presentation. Furthermore,* $\langle\cdot,\cdot\rangle$ *denotes the usual duality pairing between measurable functions and measures.*

2. Provided a regular candidate $\hat{b}$ *for* $\overleftarrow{\mathbb{B}}$*, the forward formulation can be transformed into a dual one*

$$\begin{aligned}&\overleftarrow{\mathbb{B}}^0(s_0,i_0) :=\\ &\sup\left\{\eta\in\mathbb{R}:\ \exists\, f\in C^1\left(\mathbb{R}^2\right)\ \ s.t.\ \forall(t,s',i',s,i,a)\in[0,T]\times\mathcal{Y}^2\times[0,\overline{a}]\right.\\ &\left.\eta\le\left\{\begin{array}{l}Te^{-qt}\left[-\partial_s f(s,i)\beta(1-a)si + \partial_i f(s,i)\left(\beta(1-a)s-\gamma\right)i + ql_1(s,i)\right]\\ -qTe^{-qt}f(s,i) + f(s_0,i_0) - e^{-qT}f(s',i') + e^{-qT}\hat{b}(s',i')\end{array}\right\}\right\}.\end{aligned}$$

or, again,

$$\overleftarrow{\mathbb{B}}^{00}(s_0,i_0) := \sup_{f\in\mathbf{\Phi},\ qf\ge\hat{b}}\left\langle qf(s,i),\delta_{(s_0,i_0)}\right\rangle.$$

As it is usually the case, one projects the problem onto some convenient space that is dense in C^1 and for which computations are simplified. Since we are looking for optimal measures that are traditionally best fitted by their moments, we will work with polynomial functions $\mathbb{R}_k[t,s,i]$, where $k\in\mathbb{N}^*$ large enough. For the density of such functions, the reader uses Nachbin's theorem applied to monomials (see [30, Chapter 1]). As such, we are morally dealing with continuous functions, for which one will use the second form of the dynamic programming principle in Proposition 1 leading to

Problem 3. *Solve the two stage problem*

$$\overrightarrow{\mathbb{F}}(s_0,i_0) := \min\left\{Tq\langle e^{-qt}l_1(s,i),\mu_1\rangle + e^{-qT}\overleftarrow{\mathbb{B}}_1(\mu_2)\right\}$$

where the minimum is taken over all measures $\mu=(\mu_1,\mu_2)$ *with* $\mathrm{supp}(\mu)\subseteq[t_0,T]\times\mathcal{Y}\times[0,\overline{a}]\times\mathcal{Y}$ *and such that*

$$\begin{aligned}&\left\langle f(T,s,i),\mu_2\right\rangle - f(0,s_0,i_0)\\ &= T\left\langle \partial_t f(t,s,i) - \partial_s f(t,s,i)\beta(1-a)si + \partial_i f(t,s,i)\left(\beta(1-a)s-\gamma\right)i,\mu_1\right\rangle,\end{aligned}$$

for every $f\in C^1\left(\mathbb{R}^3\right)$*;*

$$\overleftarrow{\mathbb{B}}_1(\mu_2) := \sup_{f\in\mathbf{\Phi}} \langle qf(s,i), \mu_2\rangle\,.$$

Remark 9. *The reader is invited to note the formulation* $\overleftarrow{\mathbb{B}}_1$ *and the relation with* $\overleftarrow{\mathbb{B}}^{00}$ *in Remark 8, especially for multi-stage problems when the backward step can be repeated in order to provide a lower estimate for the value function at* (s_0, i_0).

Let us now describe an approximation scheme for Problem 3.

1. One begins with recalling the class of polynomials with multi-index of length at most $r \geq 1$, $\mathbb{R}_r[s,i,a] \subset \mathbb{R}[s,i,a]$. We recall that the monomial $s^{\alpha_1} i^{\alpha_2} a^{\alpha_3}$ has the multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ with length $\|\alpha\| = \alpha_1 + \alpha_2 + \alpha_3$.

2. The cost function l_1 has been assumed to be bounded from below and, without loss of generality, one can assume $\inf_{(s,i)\in\mathcal{Y}} l_1(s,i) > 0$. As such, both the value functions and the polynomials candidate to approximate this value function should be non-negative.

3. The reader is reminded that the boundary of the set of constraints $\mathcal{Y}$ is characterized by a region (s, i^*) with $s \leq \frac{\gamma}{\beta(1-\overline{a})}$ and a region $-s + \frac{\gamma}{\beta(1-\overline{a})} \log s = i - i^* + \frac{\gamma}{\beta(1-\overline{a})} \log \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}$, with $s \geq \frac{\gamma}{\beta(1-\overline{a})}$. For the second region, by adding the interior, one gets $-s + \frac{\gamma}{\beta(1-\overline{a})} \log s \geq i - i^* + \frac{\gamma}{\beta(1-\overline{a})} \log \frac{\gamma}{\beta(1-\overline{a})} - \frac{\gamma}{\beta(1-\overline{a})}$. It is perfectly possible to approximate the log with polynomials[8], in order to have semi-algebraic spaces. However, to simplify arguments, we focus on

$$\left\{ (s,i) : 0 \leq s \leq \frac{\gamma}{\beta(1-\overline{a})},\ 0 \leq i \leq i^* \right\}.$$

   The control component is already in an interval $[0, \overline{a}]$. The time component needs no approximation (functions to integrate are always exponential in t and it can be easily shown that the μ_1 measures have $\mathcal{L}eb(dt)$ marginals).

4. Following [34, Theorem 1.3], we consider the (truncated) quadratic modules

$$Q_r := \left\{ \begin{array}{l} p_0^2(s,i,a) + p_s^2(s,i,a) s \left(\dfrac{\gamma}{\beta(1-\overline{a})} - s \right) + p_i^2(s,i,a)(i^* - i)i + p_a^2(s,i,a)(\overline{a} - a)a : \\ p_0 \in \mathbb{R}_r[s,i,a],\ p_s, p_i, p_a \in \mathbb{R}_{r-1}[s,i,a] \end{array} \right\}.$$

   Note that we imposed that the overall index in Q_r does not exceed $2r$. Approximating l_1 comes to picking some $l_1^r \in Q_r$ that is assumed to be fixed throughout the developments. Hereafter, we will just write l_1 but we actually use the modifications l_1^r.

5. By abuse of notation, we will still write Q_r (instead of $\tilde{Q}_r$) for polynomials $p(s,i)$ (that do not depend on a). The reason is that one easily obtains $\tilde{Q}_r = \{p(s,i,0) :\ p \in Q_r\}$ thus getting a simple identification.

6. For the *backward step*: given an estimate $\hat{\mu}^{r,N-1}$ obtained from the previous step, solve the problem

$$\begin{array}{l} \overleftarrow{\mathbb{B}}_1^{r,N} := \\ \max \Big\{ \langle qp(s,i), \hat{\mu}_2^{r,N-1} \rangle \ :\ p \in Q_r, \\ \qquad \partial_s p(s,i)\beta(1-a)si - \partial_i p(s,i) \left(\beta(1-a)s - \gamma\right) i + qp(s,i) - l_1(s,i) \in Q_r \Big\}. \end{array} \tag{22}$$

[8] Please note that $i^* < 1$ implies $\frac{\gamma}{\beta(1-\overline{a})} \leq \psi \leq \lim_{i\to 0+} \psi(i) < 1 + \frac{\gamma}{\beta(1-\overline{a})}$. As a consequence, $\psi(i) = 1 + l$, with $l \in (-1, 1)$.

7. Add the optimal $\hat{p}^{r,N}$ to the set of vertices $\Pi^{r,N-1}$ i.e.

$$\Pi^{r,N} := \Pi^{r,N-1} \cup \{\hat{p}^{r,N}\}.$$

We emphasize that $\hat{p}^{r,N}$ consists of a finite number of coefficients describing elements of $Q_r \subset \mathbb{R}_{2r}[s,i,a]$. Since $\Pi^{r,N} \subset \mathbf{\Phi}$ (the identification of elements $\Pi^{r,N}$ with either vectors of coefficients or polynomials being made with a slight abuse of notation, but cannonically), it follows that

$$\hat{b}^{r,N}(\mu_2) := \sup_{p \in \Pi^{r,N}} \langle qp(s,i), \mu_2 \rangle \leq \overleftarrow{\mathbb{B}}_1(\mu_2). \tag{23}$$

8. Since $p \in Q_r$, computing $\langle qp(s,i), \mu_2 \rangle$ amounts to compute the moments m_α for a multi-index α whose length does not exceed $2r$. As such, "remembering" a measure μ amounts to "remember" the moments

$$m_1(\alpha_1,\alpha_2,\alpha_3) := q \int_{\mathbb{R}_+ \times \mathcal{Y} \times [0,\overline{a}]} e^{-qt} s^{\alpha_1} i^{\alpha_2} a^{\alpha_3} \mu_1(dt,ds,dy,da),$$
$$m_2(\alpha_1,\alpha_2) := \int_{\mathcal{Y}} s^{\alpha_1} i^{\alpha_2} \mu_2(ds,da),$$

for $0 \leq \alpha_1 + \alpha_2 \leq 2r$ and $\alpha_3 \in \{0,1\}$. In general, if l_1 depends on the control a, one adapts to $\|\alpha\| \leq 2r$ for μ_1 moments.

9. We wish to emphasize that, in our problems, t only appears in a q-exponential form. By further assuming that the optimal answer a is a positional strategy $a(s,i)$ (but independent of the time), one can replace μ_1 with the marginal $\overline{\mu}_1(ds,di,da) := \int_{\mathbb{R}_+} qe^{-qt}\mu_1(dt,ds,di,da)$ in the forward problem.

10. For the *forward step*: given $\hat{b}^{r,N}$ as in (23), find the arguments $(\hat{\overline{\mu}}_1^{r,N}, \hat{\mu}_2^{r,N})$ realizing the minimum (with respect to some generated scenarios that will be explained below)

$$\min \Bigg\{ T\langle l_1(s,i), \overline{\mu}_1 \rangle + e^{-qT}\hat{b}^{r,N}(\mu_2) \ : \ \mathrm{supp}(\overline{\mu}_1,\mu_2) \subset \mathcal{Y} \times [0,\overline{a}] \times \mathcal{Y} \ s.t. \ \forall\, \alpha_1+\alpha_2 \leq 2r$$
$$e^{-qT} m_2(\alpha_1,\alpha_2) - s_0^{\alpha_1} i_0^{\alpha_2} =$$
$$= \frac{T}{q} \begin{pmatrix} -qm_1(\alpha_1,\alpha_2,0) - \alpha_1\beta m_1(\alpha_1,\alpha_2+1,0) + \alpha_1\beta m_1(\alpha_1,\alpha_2+1,1) \\ +\alpha_2\beta m_1(\alpha_1+1,\alpha_2,0) - \alpha_2\beta m_1(\alpha_1+1,\alpha_2,1) - \alpha_2\gamma m_1(\alpha_1,\alpha_2,0) \end{pmatrix} \Bigg\}. \tag{24}$$

Please note that the above equality on moments is nothing else than the equality constraint on measures in $\Theta_{0,T}$ written for the function $f(t,s,i) := e^{-qt}s^{\alpha_1}i^{\alpha_2}$.

The value obtained $\underline{\Gamma}^{r,N}$ cannot exceed $\overline{\Gamma}^{r,N} := T\langle l_1(s,i), \overline{\mu}_1^{r,N} \rangle + e^{-qT}\overleftarrow{\mathbb{B}}_1^{r,N+1}$ (the second-stage being computed as in (22) with respect to $\hat{\mu}_2^{r,N}$). The difference between these quantities is used as a stopping criterion.

11. The measures μ_2 can be generated by an a-priori discretization of the compact state space. To limit the generation of μ_2, one may want to note that the (convex) reachable set starting from $(s_0,i_0) \in \mathcal{Y}$ with viable controls is described by the (trace over $\mathcal{Y}$ of) convex combinations of

extremal trajectories (controlled with 0 for the "upper" branch, respectively $\overline{a}$ for the "lower one") i.e.

$$\mathcal{R}(s_0,i_0):=$$
$$\left\{\begin{array}{l}\alpha\,(s_1,i_1)+(1-\alpha)\,(s_2,i_2)\ :\ 0\leq\alpha\leq 1,\ 0\leq i_1,i_2\leq i^*,\ s_1\leq\psi(i_1),\ s_2\leq\psi(i_2),\\ s_1+i_1=s_0+i_0+\dfrac{\gamma}{\beta}\log\dfrac{s_1}{s_0},\ s_2+i_2=s_0+i_0+\dfrac{\gamma}{\beta(1-\overline{a})}\log\dfrac{s_2}{s_0}\end{array}\right\}.$$

For the two equalities in $\mathcal{R}$, the reader is invited to recall the computations in (2) (with minimal control 0 and maximal control $\overline{a}$). Further precision can be added by computing, for $T>0$, $\underline{s_1}^T:=s^{s_0,i_0,0}(T)$, $\underline{s_2}^T:=s^{s_0,i_0,\overline{a}}(T)$ and defining $\mathcal{R}^T(s_0,i_0)$ by further imposing $\underline{s_i}^T\leq s_i,\ i\in\{1,2\}$. This will provide the positions reachable prior to time $T>0$.

12. Reduction of reachable sets starting from the support of μ_2 is possible when one uses several stages. Randomization of T such that one fully takes advantage of $\mathcal{R}^T$ is also possible (see [21]).

13. Finally, for each scenario of μ_2 generated, the moments m_2 are computed and they provide the family m_1 (satisfying the relations in (24)). Compatibility with the measure structure is then checked for the functionals m_2 (cf. [34, Eq. (16)] and the comments following this condition).

Finally, let us mention that target problems can be obtained using the same procedure as in [21, Section 5].

5 Appendix

5.1 A1

As mentioned before, the regularity of the value function V (with an impact on the definition of a solution to (9)) makes the object of several papers, usually under some inward pointing qualification assumption (cf. [38]) or some outward qualification ones (cf. [16]). None of these results apply directly to our problem: either because the analysis partially covers the space (as it is the case when using [38]), or because the condition per se is not satisfied everywhere on the boundary (see the analysis in connection with [16]) but it can be generalized.

To understand the importance of these considerations, the reader is reminded the following. We have hinted that the value function is a "solution" to (9). Under reasonable controllability (or qualification) conditions, the solution satisfies (in a viscosity sense), the solution equality in (9) on the interior of the set of constraints $\mathcal{Y}$ and a super-solution ($\leq$ inequality) on the boundary; see [38, Definition 2.1]. Roughly said, the existing results are for either inward pointing conditions (cf. [38]) or outward pointing conditions (cf. [16]). Our system is of "hybrid" nature boundary. It turns out that, up till $\mathcal{G}$, the boundary $\partial\mathcal{Y}$ will comply with [16] and on $\mathcal{G}$ (and actually on the larger boundary $\partial\mathcal{Y}\cap\partial\mathcal{B}$), the inward pointing condition is satisfied. It appears in no way surprising that theoretical results can be obtained, but we choose to only hint the connections.

5.1.1 Connection with [16]

Let us point out some connections with [16] from which we borrow the notations (D, M, etc.) in order to facilitate referral.

Proposition 2. *The locally compact set* $D := \mathcal{Y} \setminus \left\{(s_0, i^*) : \ \frac{\gamma}{\beta} \leq s_0 \leq \frac{\gamma}{\beta(1-\overline{a})}\right\}$ *is locally-in-time backward invariant for* $[0, \overline{a}]$*-valued policies, i.e., for every* $(s_0, i_0) \in D$*, and every* $a \in \mathbb{L}^0\left(\mathbb{R}; [0, \overline{a}]\right)$*, there exists* $t_0 > 0$ *such that* $\left(s^{s_0,i_0,a}(-t), i^{s_0,i_0,a}(-t)\right) \in D$ *for every* $t \in [0, t_0]$*. Moreover,* $\overline{D} = \mathcal{Y}$*.*

Proof. We only need to show something for initial conditions on the active boundary of $\mathcal{Y}$, defined in (11), that is if $(s_0, i_0) \in \partial\mathcal{Y}^+ \cap D$; otherwise, using the continuous dependency of the initial data, and the explicit form of $\mathcal{Y}$, the conclusion is obvious. In this setting, $s_0 \geq \frac{\gamma}{\beta(1-\overline{a})}$. Then, this inequality is preserved for $t \mapsto s^{s_0,i_0,a}(-t)$. As a consequence,

$$d\left[i^{s_0,i_0,a}(-t)\right] = -\left(\beta(1-a)s^{s_0,i_0,a}(-t) - \gamma\right) i^{s_0,i_0,a}(-t) \leq 0$$

which implies $i^{s_0,i_0,a}(-t) \leq i_0 < i^*$ and, thus, it follows that $\left(s^{s_0,i_0,a}(-t), i^{s_0,i_0,a}(-t)\right) \in \mathcal{Y}$. Indeed, setting $(s_1, i_1) = (s^{s_0,i_0,a}(-t), i^{s_0,i_0,a}(-t))$, by taking the control $\tilde{a}(r) := a(-t + r)$, $r \geq 0$, $(s^{s_1,i_1,\tilde{a}}(r), i^{s_1,i_1,\tilde{a}}(r)) = \left(s^{s_0,i_0,a}(r-t), i^{s_0,i_0,\tilde{a}}(r-t)\right)$ satisfies $i \leq i^*$ for every $r \in [0, t]$,. At time t, the trajectory reaches (s_0, i_0) after which a control can be found to stay in $\mathcal{Y}$. Thus, for (s_1, i_1), we have found a control keeping $i \leq i^*$ (which is the definition of $\mathcal{Y}$). On the other hand, $i^{s_0,i_0,a}(-t) < i^*$, thus, $\left(s^{s_0,i_0,a}(-t), i^{s_0,i_0,a}(-t)\right) \in D$. □

Let us set $M := \mathcal{Y} \setminus D = \left\{(s_0, i^*) : \ \frac{\gamma}{\beta} \leq s_0 \leq \frac{\gamma}{\beta(1-\overline{a})}\right\}$. If $\frac{\gamma}{\beta} < s_0 < \frac{\gamma}{\beta(1-\overline{a})}$, then $\partial\mathcal{Y}^+$ is regular and the outward normals are of type $(0, 1)$. Then $\langle(-\beta s_0 i^*, (\beta s_0 - \gamma) i^*), (0, 1)\rangle > 0$ and it follows that the 0-controlled trajectory does not belong to the paratingent set $P^M_{\mathcal{Y}}(s_0, i^*)$ (for the definition and the relevance of these computations, please refer to [16, Page 822 and the Remark (2) on page 823]). This is easily extended at $\left(\frac{\gamma}{\beta(1-\overline{a})}, i^*\right)$. However, the same argument no longer works at $\left(\frac{\gamma}{\beta}, i^*\right)$. It follows that one cannot directly apply [16, Theorem 11] to infer the connection with (9). The attentive reader will have noted that the litigious point $\left(\frac{\gamma}{\beta}, i^*\right)$ belongs to $\mathcal{G}$ which is the invariance domain with respect to our control system (1) and that the remaining of $\partial\mathcal{Y}$ (i.e. $\left[0, \frac{\gamma}{\beta}\right] \times \{i^*\}$ satisfies the inward pointing qualification condition with every control but 0).

Remark 10.

1. *The generalization of [16, Theorem 11] to our present setting can be obtained, under Assumption 1 and the hypotheses* (8) *and* (15)*, as follows. First, note that our function* W *in* (1) *is differentiable almost everywhere and continuous. At the points of differentiability, the subdifferential set is a singleton and, using the computation of the derivatives of* W *done in the proof of Theorem 2, it is easily seen that the conditions [16, Eq. (26), (27)] are satisfied. This yields [16, Step 1 in Theorem 11], first at every point where* W *is differentiable, then, using the continuity of* W*, also on the curve where* W *may not be differentiable. The inequality* $W \leq V$ *can be proven following the arguments in [16, Proposition 9] due to the previous considerations.*

2. *[16, Proposition 9] uses a boundary condition at the final time. In our case, this condition is replaced with* $V = 0$*, on* $\mathcal{G}$*. Furthermore, in order to guarantee that accumulations do not happen outside* $\mathcal{G}$*, one requires*

$$l_1(s, 0, 0) > 0, \ \forall (s, 0) \notin \mathcal{G}. \tag{25}$$

The need for boundary conditions in order to guarantee comparison results is also illustrated in [10].

The assumptions in [16, Proposition 9] can be directly enforced by slightly modifying the yellow (admissible) set in such a way that $\mathcal{G}$ is the only one affected. The reader will note that the value function is kept the same if one further assumes

$$l_1(s,i,a)=0, \text{ for all } a\in[0,\overline{a}] \text{ whenever } (s,i)\in\mathcal{G}. \tag{26}$$

We consider

$$\tilde{\mathcal{G}}=\left\{(s,i)\in\mathcal{G}:\ s\geq\tilde{\psi}(i)\right\}, \text{ and } \tilde{Y}=(Y\setminus\mathcal{G})\cup\tilde{\mathcal{G}},$$

where $\left\{\tilde{\psi}(i),i\right)$ describes the curve $\left\{\left(s^{\frac{\gamma}{\beta},i^*,\overline{a}}(t),i^{\frac{\gamma}{\beta},i^*,\overline{a}}(t)\right):\ t\geq 0\right\}$ (please see Figure 3). It is easy to see that $\tilde{\psi}(i)=x$ is the solution of $-x+\frac{\gamma}{\beta(1-\overline{a})}\log x=i-i^*+\frac{\gamma}{\beta(1-\overline{a})}\log\frac{\gamma}{\beta}-\frac{\gamma}{\beta}$, but, this time asking that $x\leq\frac{\gamma}{\beta}$. By computing derivatives in the above definition, one gets

$$\tilde{\psi}'(i)\left[-1+\frac{\gamma}{\beta(1-\overline{a})\tilde{\psi}(i)}\right]=1 \iff \tilde{\psi}'(i)=\frac{1}{-1+\frac{\gamma}{\beta(1-\overline{a})\tilde{\psi}(i)}}.$$

Please note that this derivative is positive. It is easy to see that

1. The 0-controlled trajectory starting from $\left(\frac{\gamma}{\beta},i^*\right)$ leaves $\partial\mathcal{Y}$ (thus completing the paratingent assumption at $\left(\frac{\gamma}{\beta},i^*\right)\in M$). The reader is referred to the red line in Figure 3.

2. The set $\tilde{D}:=\tilde{\mathcal{Y}}\setminus M$ is locally-in-time backward invariant. To see this, we only need to show that every (backward) trajectory starting at $\left(\tilde{\psi}(i),i\right)$ with $i<i^*$ satisfies $s^{\tilde{\psi}(i),i,a}(-t)-\tilde{\psi}\left(i^{\tilde{\psi}(i),i,a}(-t)\right)\geq 0$. To this purpose, we drop the superscript and compute

$$\begin{aligned}
&d\left[s(-t)-\tilde{\psi}\left(i(-t)\right)\right]\\
=&\beta(1-a(-t))s(-t)i(-t)+\frac{1}{-1+\frac{\gamma}{\beta(1-\overline{a})\tilde{\psi}(i(-t))}}\left(\beta(1-a(-t))s(-t)-\gamma\right)i(-t)\\
=&\frac{\gamma i(-t)}{-1+\frac{\gamma}{\beta(1-\overline{a})\tilde{\psi}(i(-t))}}\left(\frac{1-a(-t)}{1-\overline{a}}\frac{s(-t)}{\tilde{\psi}\left(i(-t)\right)}-1\right)\geq\frac{\gamma i(-t)}{-1+\frac{\gamma}{\beta(1-\overline{a})\tilde{\psi}(i(-t))}}\frac{s(-t)-\tilde{\psi}\left(i(-t)\right)}{\tilde{\psi}\left(i(-t)\right)}.
\end{aligned}$$

   It follows that $s(-t)-\tilde{\psi}\left(i(-t)\right)$ is non-decreasing, hence it stays non-negative.

We do not actually deal with the equation (9) as written, but merely check for this equation in order to avoid the presence of the discount factor q. We focus on the analysis on $\tilde{\mathcal{Y}}$ with the aforementioned assumptions and properties. To be more rigorous, one should define

$$\begin{cases}\tilde{J}_q\left(s_0,i_0,a\right):=\int_0^\infty e^{-qt}l_1\left(s^{s_0,i_0,a}(t),i^{s_0,i_0,a}(t),a(t)\right)dt;\\ V_q\left(s_0,i_0\right):=\inf\limits_{a\in\mathcal{A}d}\tilde{J}_q\left(s_0,i_0,a\right).\end{cases} \tag{27}$$

The associated equation is

$$qV_q(s,i)+\sup_{a\in[0,\overline{a}]}\left\{\beta si(1-a)\partial_sV_q\left(s,i\right)-\left(\beta(1-a)s-\gamma\right)i\partial_iV_q(s,i)-l_1(s,i,a)\right\}=0, \text{ on } \tilde{\mathcal{G}}. \tag{28}$$

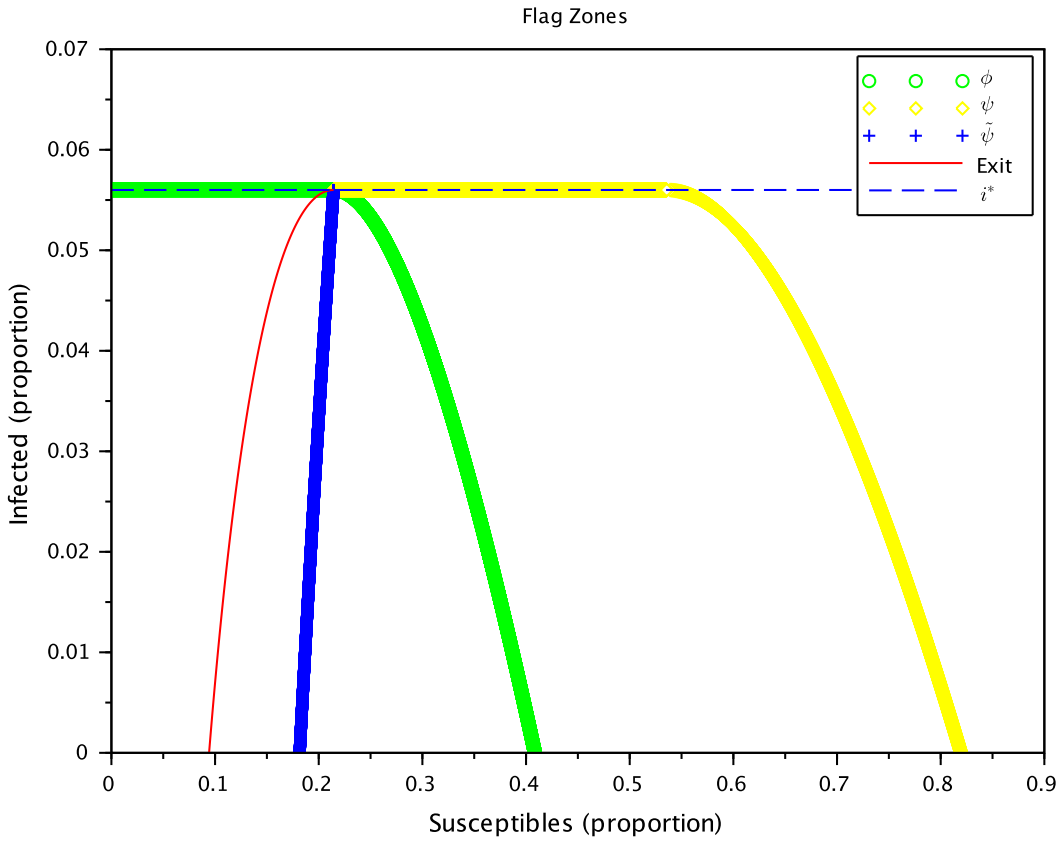


Figure 3: Modified $\tilde{\mathcal{Y}}$ and the outer pointing trajectory at $\frac{\gamma}{\beta}$.

To simplify notations, we set, for $(s,i,p_s,p_i) \in \mathbb{R}^4$,

$$\begin{cases} H(s,i,p_s,p_i) := & \sup\limits_{a\in[0,\overline{a}]} \{-\beta si(1-a)p_s + (\beta(1-a)s-\gamma)\, ip_i - l_1(s,i,a)\}\,; \\ H_{out}\,(s,i,p_s,p_i) := & \sup\Big\{ -\beta si(1-a)p_s + (\beta(1-a)s-\gamma)\, ip_i - l_1(s,i,a) \ : \\ & \qquad a\in[0,\overline{a}]\,,\ (-\beta si(1-a),(\beta s(1-a)-\gamma)i) \notin P^M_{\tilde{\mathcal{G}}}\Big\},\ \forall\,(s,i)\in M. \end{cases} \tag{29}$$

The reader is invited to note that the paratingent condition defining H_{out} and the continuity of the vector field defining the dynamics lead to

$$H_{out}\,(s,i,p_s,p_i) := \sup_{a\in\left[0,1-\frac{\gamma}{\beta s}\right]} \{-\beta si(1-a)p_s + (\beta(1-a)s-\gamma)\, ip_i - l_1(s,i,a)\}\,,$$

for all $\frac{\gamma}{\beta} < s_0 < \frac{\gamma}{\beta(1-\overline{a})}$, $i = i^*$. At the corners, all but one control $\overline{a}$ take the trajectory out of $\tilde{\mathcal{Y}}$ such that $H_{out} = H_{in}$. With these in mind, the precise understanding of the solution (see [15, Theorem 6, c)]) is the following.

Definition 2. *W_q is a (viscosity) solution to (28) if*

1. *W_q is a lower semi-continuous bounded function;*

2. *for every* $(s,i) \in \tilde{D}$, *and for all* $n \in \partial_- W_q(s,i)$, $qW_q(s,i) + H(s,i,-n) = 0$;

3. *for every* $(s,i) = (s,i^*) \in M$, *and every* $n \in \partial_- W_q(s,i)$, *one has* $qW_q(s,i) + H(s,i,-n) \geq 0$ *and* $qW_q(s,i) + H_{out}(s,i,-n) \leq 0$.

Remark 11. 1. *We do not insist here neither on the definitions of subdifferentials nor on their link with negative polar cones to the epigraph or to proximal or Mordukovich cones. Instead, the interested reader can take a further look at [15, Section 2].*

2. *The condition on* H_{out} *is only needed (see [15, Lemma 15]) to guarantee continuity at points in* M. *Our candidates are shown to satisfy such continuity properties such that* H_{out} *can be left aside (please see [15, Proposition 16] providing the comparison result without the supersolution-like condition on* H_{out}*).*

3. *Subdiferentials at* (s,i) *are either approximated by (convex combinations) of gradients at points of differentiability or computed on the lines* $\{(\phi(i),i) \ : \ i \in [0,i^*]\}$, *respectively on* $\left\{(\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i),i) \ : \ i \in [0,i^*]\right\}$ *or, eventually, on* M.

4. *Along the aforementioned line* $(\phi(i),i)$, *the function* $\tilde{J}_q(s_0,i_0,a^*) = 0$ *is constant such that the contribution to the subdifferentials is* $n = (0,0)$ *and there is nothing to prove. On the contrary, the definition of* s_1 *shows that there is no particular problem of differentiability on* $\left\{(\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i),i) \ : \ i \in [0,i^*]\right\}$. *The same holds true on* M, *where* $\tilde{J}_q(s_0,i^*,a^*) = \frac{1}{\gamma i^*} e^{-\frac{qs_0}{\gamma i^*}} \int_{\frac{\gamma}{\beta}}^{s_0} e^{\frac{qu}{\gamma i^*}} l_1\left(u,i^*,1-\frac{\gamma}{\beta u}\right) du$.

Following similar arguments as for Lemma 1, it follows that $\tilde{J}_q(s_0,i_0,a^*)$ is a (continuous) solution to (28). Since all the assumptions of [15, Theorem 6] are satisfied on $\tilde{\mathcal{G}}$, it follows that $V_q(\cdot) = \tilde{J}_q(\cdot,a^*)$, hence a^* is optimal for V_q.
As a by-product, a^* is optimal for V (with $q = 0$). Indeed, since l_1 is non-negative, it is obvious that $V_q \leq V$. Using Lebesgue's dominated convergence, it is obvious that $\tilde{J}(s_0,i_0,a^*) = \lim_{q\to 0+} \tilde{J}_q(s_0,i_0,a^*)$. Then $V(s_0,i_0) \geq \tilde{J}(s_0,i_0,a^*)$ and the optimality follows.
When we say that "heuristically" V satisfies the equation, it is because, under the standing assumptions on l_1, it is obtained as the limit of continuous solutions to (28).

5.1.2 Connection with [38]

Let us assume that, in addition to the Assumption 1, the condition (8) holds true.

Proposition 3. *Whenever* $(s_0,i_0) \in \mathcal{B}^{\overline{s}}$, *with* $\overline{s} \in \left[\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\overline{a})}\right]$ *and* $a \in \mathcal{A}d$ *such that* $\left(s^{s_0,i_0,a}, i^{s_0,i_0,a}\right) \in \mathcal{Y}$ *on* $\mathbb{R}_+$, *it holds true that* $\left(s^{s_0,i_0,a}, i^{s_0,i_0,a}\right) \in \mathcal{B}^{\overline{s}}$ *on* $\mathbb{R}_+$.

Proof. The argument is quite simple. Nonetheless, we provide the elements for our readers' sake.
First, the conclusion is obvious for $\overline{s} = \frac{\gamma}{\beta}$ as $\mathcal{B}^{\frac{\gamma}{\beta}} = \mathcal{G}$ which is invariant.
For $\overline{s} > \frac{\gamma}{\beta}$, by computing derivatives in the definition of $\varphi^{\overline{s}}$, it follows that

$$\left(\varphi^{\overline{s}}\right)'(i)\left(-1+\frac{\gamma}{\beta\varphi^{\overline{s}}(i)}\right) = 1 \Leftrightarrow \left(\varphi^{\overline{s}}\right)'(i) = \frac{\beta\varphi^{\overline{s}}(i)}{\gamma - \beta\varphi^{\overline{s}}(i)}. \tag{30}$$

Thus, $\varphi^{\overline{s}\,\prime}(i) < 0$ as long as $\varphi^{\overline{s}}(i) > \frac{\gamma}{\beta}$ (which is always true due to the inequality $\overline{s} > \frac{\gamma}{\beta}$). Computing the derivative in time of $t \mapsto s^{s_0,i_0,a}(t) - \varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right)$ one gets

$$\begin{aligned}\frac{d}{dt}\left(s^{s_0,i_0,a}(t) - \varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right)\right) &= \frac{-\gamma i^{s_0,i_0,a}(t)}{\gamma - \beta\varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right)}\left(\beta(1-a(t))s^{s_0,i_0,a}(t) - \beta\varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right)\right)\\ &\leq \frac{-\gamma\beta i^{s_0,i_0,a}(t)}{\gamma - \beta\varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right)}\left(s^{s_0,i_0,a}(t) - \varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right)\right).\end{aligned}$$

The conclusion follows from Gronwall's inequality and the initial condition $(s_0, i_0) \in \mathcal{B}^{\overline{s}}$ (which implies $s_0 \leq \varphi^{\overline{s}}(i_0)$). □

Since, on the other hand, every control $a \in \mathcal{A}d$ keeping the trajectory in $\mathcal{B}^{\overline{s}}$ also keeps it in $\mathcal{Y}$, it follows that Problem 1 restricted to $\mathcal{B}^{\overline{s}}$ (with $\overline{s} \in \left[\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\overline{a})}\right]$) is equivalent with

Problem 4. *For every initial configuration $(s_0, i_0) \in \mathcal{B}^{\overline{s}} \subset \mathcal{Y}$,*

- *minimize* (6),
- *under the state constraint*

$$\left(s^{s_0,i_0,a}(t), i^{s_0,i_0,a}(t)\right) \in \mathcal{B}^{\overline{s}} \iff s^{s_0,i_0,a}(t) \leq \varphi^{\overline{s}}\left(i^{s_0,i_0,a}(t)\right), \text{ a.s. on } \mathbb{R}_+. \tag{31}$$

Let us note that the outward normal unit vector to the "active" part of the boundary of $\mathcal{B}^{\overline{s}}$, that is $\left\{\left(\varphi^{\overline{s}}(i), i\right) : i > 0\right\} \cup ((0, \overline{s}] \times \{i^*\})$, with the exception of the singular point $(\overline{s}, i^*)$ in which one could instead consider the normal cone, is

$$n_{(s,i)} = \begin{cases} (0,1), & \text{if } (s,i) \in (0,\overline{s}) \times \{i^*\}, \\ \dfrac{\left(1, -\left(\varphi^{\overline{s}}\right)'(i)\right)}{\sqrt{1 + |\left(\varphi^{\overline{s}}\right)'(i)|^2}}, & \text{if } i \in (0, i^*) \text{ and } s = \varphi^{\overline{s}}(i). \end{cases}$$

Using (30), we have

$$\begin{aligned}&\left\langle n_{(s,i)}, \left(-\beta(1-\overline{a})si, \left(\beta(1-\overline{a})s - \gamma\right)i\right)\right\rangle \\ &= \begin{cases} \left(\beta(1-\overline{a})s - \gamma\right)i^*, & \text{if } (s,i) \in (0,\overline{s}) \times \{i^*\} \\ \frac{1}{\sqrt{1+|(\varphi^{\overline{s}})'(i)|^2}}\overline{a}\gamma i \frac{\beta\varphi^{\overline{s}}(i)}{\gamma - \beta\varphi^{\overline{s}}(i)}, & \text{if } i \in (0, i^*),\ s = \varphi^{\overline{s}}(i). \end{cases}\end{aligned} \tag{32}$$

It is easy to note that we always have $\left\langle n_{(s,i)}, \left(-\beta(1-\overline{a})si, \left(\beta(1-\overline{a})s - \gamma\right)i\right)\right\rangle \leq 0$. Moreover, on the set $(0, \overline{s}) \times \{i^*\}$ we have

$$\left(\beta(1-\overline{a})s - \gamma\right)i^* \leq \left(\beta(1-\overline{a})\overline{s} - \gamma\right)i^* < 0 \iff \overline{s} < \frac{\gamma}{\beta(1-\overline{a})}.$$

On the remaining part of the boundary, that is $\{(\varphi^{\overline{s}}(i), i) : i \in (0, i^*)\}$, due to the monotonicity of the function $f(x) = \frac{x}{\gamma - x}$ on $(\gamma, +\infty)$ and of $\varphi^{\overline{s}}$ on $(0, i^*)$, we have

$$\frac{\beta\varphi^{\overline{s}}(i)}{\gamma - \beta\varphi^{\overline{s}}(i)} \leq \frac{\beta\varphi^{\overline{s}}(0^+)}{\gamma - \beta\varphi^{\overline{s}}(0^+)} < 0,$$

where $\varphi^{\overline{s}}(0^+) := \lim_{i\to 0+} \varphi^{\overline{s}}(i)$ which is finite and larger than $\frac{\gamma}{\beta}$; hence, for $i \geq \delta > 0$ we have

$$\begin{aligned}&\left\langle n_{(s,i)}, \left(-\beta(1-\overline{a})si, \left(\beta(1-\overline{a})s-\gamma\right)i\right)\right\rangle\\ &= \frac{1}{\sqrt{1+\left|\left(\varphi^{\overline{s}}\right)'(i)\right|^2}}\overline{a}\gamma i\frac{\beta\varphi^{\overline{s}}(i)}{\gamma-\beta\varphi^{\overline{s}}(i)} \leq \overline{a}\gamma\delta\frac{\beta\varphi^{\overline{s}}(0^+)}{\gamma-\beta\varphi^{\overline{s}}(0^+)},\end{aligned}$$

and the right hand side is a strictly negative constant.

Second, the reader is invited to note that $\lim_{t\to\infty} s^{\frac{\gamma}{\beta(1-\overline{a})},i^*,0}(-t) = \lim_{i\to 0+}\varphi^{\frac{\gamma}{\beta(1-\overline{a})}}(i) > \frac{\gamma}{\beta(1-\overline{a})}$. In particular, if $\overline{s}$ is close to $\frac{\gamma}{\beta(1-\overline{a})}$, one has $\lim_{i\to 0+}\varphi^{\overline{s}}(i) > \frac{\gamma}{\beta(1-\overline{a})}$. It follows that, locally around the boundary, $i^{s_0,i_0,b} > i_0$, so that, by reducing the study to such (viable) sets, i.e., $\mathcal{B}^{\overline{s}}\setminus\left(\left[\frac{\gamma}{\beta(1-\overline{a})},+\infty\right]\times[0,\delta]\right)$, the estimates in (32) give an actual inward pointing qualification condition on these sets (see [38, A3 Page 558] for details). As a conclusion, on the reunion of these sets first over $\delta > 0$ then over $\overline{s} \nearrow \frac{\gamma}{\beta(1-\overline{a})}$, containing the interior of $\mathcal{B}^{\frac{\gamma}{\beta(1-\overline{a})}}$, the value function is bounded and uniformly continuous.

5.2 A2

Sketch of the proof of Proposition 1. The first equality follows the same reasoning as in [20, Theorem 14] (with the only difference that we are dealing here with a discounted problem) and its proof will be omitted. Let us just mention that, following [20, Theorem 14], the problem can be reduced to continuous l_1 (via inf-convolutions) and without state constraints by adding a penalty term involving ndy (the distance function) and allowing $n \to \infty$.
If l_1 is continuous, one considers a state penalty

$$\begin{cases} l_1^n(s,i) & := l_1(s,i) + n\left(d_{\mathcal{Y}}(s,i)\wedge 1\right),\\ V_n(s_0,i_0) & := \inf\limits_{a\in\mathbb{L}^0(\mathbb{R};[0,\overline{a}])}\displaystyle\int_0^\infty e^{-qt}l_1^n\left(s^{s_0,i_0,a}(t), i^{s_0,i_0,a}(t)\right)dt.\end{cases}$$

As in [20, Theorem 14],

$$\Lambda^* = q\ \sup_{n\geq 1} V_n,\ \text{on } \mathcal{Y}. \tag{33}$$

For this kind of (state unconstrained) problems, with $n \geq 1$ fixed, using [28] (see also [7, Lemmas 2.6 and 2.7]), it follows that for every $n \geq 1$, for some $C_n > 0$ and for every $\varepsilon > 0$, there exists $v_n^\varepsilon \in C_b^1\left(\mathbb{R}^2\right)$ (continuously differentiable real functions that are bounded and have bounded first-order derivatives) such that

$$\begin{cases} \sup\limits_{(s,i)\in\mathbb{R}^2}\left|v_n^\varepsilon(s,i) - V_n(s,i)\right| \leq C_n\varepsilon,\\ \sup\limits_{(s,i,a)\in\mathbb{R}^2\times[0,\overline{a}]}\left\{\begin{array}{l} -\beta(1-a)si\partial_s v_n^\varepsilon(s,i) + \left(\beta(1-a)s-\gamma\right)i\partial_i v_n^\varepsilon(s,i)\\ -qv^\varepsilon(s,i) + l_1^n(s,i)\end{array}\right\} \leq 0,\\ C_n \leq C(1+n),\end{cases} \tag{34}$$

where $C > 0$ is a constant independent of n and ε. The last inequality, merely stating that C_n is chosen depending on the modulus of continuity of V_n (i.e. the Lipschitz constants of the dynamics

and coefficients) can be found, for instance, in [19, Propositions 18, 19] in a much more general setting. In particular, since $l_1^n \geq l_1$, it follows that

$$v_n^\varepsilon \in \mathbf{\Phi},\ \forall n \geq 1, \forall \varepsilon > 0.$$

[9] A glance at (18) and (20) shows that one only needs to prove the inequality $\leq$ in (21). More precisely, it suffices to prove that for all measures $\mu \in \Theta_{0,T}(s_0, i_0)$,

$$\int_{\mathcal{Y}} \Lambda^*(s,i)\mu_2(ds,di) \leq q \sup_{f\in\mathbf{\Phi}} \int_{\mathcal{Y}} f(s,i)\mu_2(ds,di).$$

For every $n \geq 1$, by considering $\mathbf{\Phi} \ni f_n(\cdot) := v_n^{\frac{1}{n^2}}$, it follows from (34) that, on $\mathcal{Y}$, $V_n(s,i) \leq f_n(s,i) + C\frac{1+n}{n^2}$. By integrating with respect to μ_2, it follows that

$$\int_{\mathcal{Y}} V_n(s,i)\mu_2(ds,di) \leq \sup_{f\in\mathbf{\Phi}} \int_{\mathcal{Y}} f(s,i)\mu_2(ds,di) + C\frac{1+n}{n+1}.$$

The conclusion follows by passing to the limit as $n \to \infty$ and using (33), the boundedness of Λ^* and recalling that μ_2 is a probability measure. □

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