EXTRINSIC EIGENVALUES UPPER BOUNDS FOR SUBMANIFOLDS IN WEIGHTED MANIFOLDS

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ABSTRACT. We prove Reilly-type upper bounds for divergence-type operators of the second order as well as for Steklov problems on submanifolds of Riemannian manifolds of bounded sectional curvature endowed with a weighted measure.

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Key words: *Submanifolds, Reilly-type upper bounds, eigenvalues estimates, divergence-type operators, Steklov problems.*

1. Introduction

Let (M^n, g) be an n-dimensional compact, connected, oriented manifold without boundary, and consider an isometric immersion $X : M^n \to \mathbb{R}^{n+1}$ in the Euclidean space. The spectrum of Laplacian of (M, g) is an increasing sequence of real numbers

$$0 = \lambda_0(\Delta) < \lambda_1(\Delta) \leqslant \lambda_2(\Delta) \leqslant \cdots \leqslant \lambda_k(\Delta) \leqslant \cdots \longrightarrow +\infty.$$

The eigenvalue 0 (corresponding to constant functions) is simple and $\lambda_1(\Delta)$ is the first positive eigenvalue. In [12], Reilly proved the following well-known upper bound for $\lambda_1(\Delta)$

$$\lambda_1(\Delta) \leqslant \frac{n}{\mathrm{V}(M)} \int_M H^2 dv_g, \tag{1}$$

where H is the mean curvature of the immersion. He also proved an analogous inequality involving the higher order mean curvatures. Namely, for $r \in \{1, \cdots, n\}$

$$\lambda_1(\Delta) \left(\int_M H_{r-1} dv_g \right)^2 \leqslant \mathrm{V}(M) \int_M H_r^2 dv_g, \tag{2}$$

where H_r is the r-th mean curvature, defined by the r-th symmetric polynomial of the principal curvatures. Moreover, Reilly studied the equality cases and proved that equality in (1) or (2) is attained if and only if $X(M)$ is a geodesic sphere.

In the case of higher codimension, Reilly also proved that

$$\lambda_1(\Delta) \leqslant \frac{n}{\mathrm{V}(M)} \int_M \|\mathbf{H}\|^2 dv_g, \tag{3}$$

where $\mathbf{H}$ is here the mean curvature vector, with equality if and only if M is minimally immersed in a geodesic sphere.

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The Reilly inequality can be easily extended to submanifolds of the sphere $\mathbb{S}^n$ using the canonical embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$:

$$\lambda_1(\Delta) \leqslant \frac{n}{\mathrm{V}(M)} \int_M (\|\mathbf{H}\|^2 + 1) dv_g. \tag{4}$$

Moreover, El Soufi and Ilias [7] proved an analogue for submanifold of the hyperbolic space as

$$\lambda_1(\Delta) \leqslant \frac{n}{\mathrm{V}(M)} \int_M (\|\mathbf{H}\|^2 - 1) dv_g. \tag{5}$$

In case the ambient space has non-constant sectional curvature, Heintze [10] proved the following weaker inequality

$$\lambda_1(\Delta) \leqslant n(\|\mathbf{H}\|^2 + \delta), \tag{6}$$

where the ambient sectional curvature is bounded above by δ.

On the other hand, more recently, in [14], the second author prove the following general inequality

$$\lambda_1(L_{T,f}) \left(\int_M \mathrm{tr}\,(S) \mu_f \right)^2 \leqslant \left(\int_M \mathrm{tr}\,(T) \mu_f \right) \int_M \left(\|H_S\|^2 + \|S\nabla f\|^2 \right) \mu_f, \tag{7}$$

where $\mu_f = e^{-f} dv_g$ is the weighted measure of (M, g) endowed with the density e^{-f}, T, S are two symmetric, free-divergence $(1,1)$-tensors with T positive definite, and $L_{T,f}$ is the second order differential operator defined for any smooth function u on M by

$$L_{T,f} = -\mathrm{div}(T\nabla u) + \langle \nabla f, T\nabla u \rangle.$$

When $f = 0$ and the tensor S and T are associated with higher order mean curvatures H_s and H_r, we recover the inequality of Alias and Malacarné [2] and, in particular, Reilly's inequality (2) if $r = 0$.

The first result of this paper gives upper bounds for the first eigenvalue of the operator $L_{T,f}$ for submanifolds of Riemannian manifolds with sectional curvature bounded by above which generalizes inequality (7) in the non-constant curvature case. Namely, we prove the following.

Theorem 1.1. *Let $(\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)$ be a weighted Riemannian manifold with sectional curvature $\mathrm{sect}_{\bar{M}} \leqslant \delta$ and $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$. Let (M, g) be a closed Riemannian manifold isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$ by X. We endow M with the weighted measure $\mu_f = e^{-f} dv_g$. Let T be a positive definite $(1,1)$-tensor on M and denote by λ_1 the first positive eigenvalue of the operator $L_{T,f}$.*

(1) *If $\delta \leqslant 0$, then*

$$\lambda_1 \leqslant \sup_M \left[\delta \mathrm{tr}\,(T) + \sup_M \left(\frac{\|H_T - T\nabla f\|}{\mathrm{tr}\,(S)} \right) \|H_S - S\nabla f\| \right].$$

(2) *If $\delta > 0$ and $X(M)$ is contained in a geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$,*

$$\lambda_1 \leqslant \frac{\left(\int_M \mathrm{tr}\,(T) \mu_f \right)}{V_f(M)} \left(\delta + \frac{\int_M \|H_S - S(\nabla f)\|^2 \mu_f}{V_f(M) \inf\left(\mathrm{tr}\,(S)^2\right)} \right).$$

The second eigenvalue problem that we consider in this paper is the Steklov problem associated with the operator $L_{T,f}$ on a submanifold Ω with non-empty boundary $\partial\Omega = M$ of a Riemannian manifold with sectional curvature bounded by above. We can consider the following generalized weighted Steklov problem

$$\begin{cases} L_{T,f} u = 0 \quad \text{on } \Omega, \\ \dfrac{\partial u}{\partial \nu_T} = \sigma u \quad \text{on } M = \partial\Omega, \end{cases} \tag{8}$$

where $\frac{\partial u}{\partial \nu_T} = \langle T(\widetilde{\nabla} u), \nu\rangle$. In the case where f is constant, the operator $L_{T,f}$ is of particular interest for the study of r-stability when $T = T_r$ is the tensor associated with r-th mean curvature (see [1] for instance). More precisely, from [3], we know that this problem (8) has a discrete nonnegative spectrum and we denote by σ_1 its first eigenvalue. In [13], the second author has obtained upper bounds for this problem for domains of a manifold lying in a Euclidean space. Namely, he has proved

$$\sigma_1 \left(\int_M \mathrm{tr}\,(S) \widetilde{\mu}_f \right)^2 \leqslant \left(\int_\Omega \mathrm{tr}\,(T) \mu_f \right) \int_M \left(||H_S||^2 + ||S\nabla f||^2 \right) \widetilde{\mu}_f,$$

where $\mu_f = e^{-f} dv_g$ and $\widetilde{\mu}_f = e^{-f} dv_{\widetilde{g}}$ are respectively, the weighted measures of (Ω, g) and $(M, \widetilde{g})$ endowed with the density e^{-f} and where T and S are two symmetric, free-divergence $(1,1)$-tensors with T positive definite. Note that without density and for $T = \mathrm{Id}$, this inequality has been proven by Ilias and Makhoul [11].

The second result of the present paper gives a generalization of this estimate when the manifold with boundary (M, g) is immersed into an ambient Riemannian manifold of sectional curvature bounded by above. Namely, we prove the following.

Theorem 1.2. *Let $(\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)$ be a weighted Riemannian manifold with sectional curvature $\mathrm{sect}_{\bar{M}} \leqslant \delta$ and $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$. Let (Ω, g) be a compact Riemannian manifold with non-empty boundary M isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$ by X. We endow Ω and M, respectively with the weighted measure $\mu_f = e^{-f} dv_g$ and $\tilde{\mu}_f = e^{-f} dv_{\tilde{g}}$, where $\tilde{g}$ is the induced metric on M. Let T, S be a symmetric, divergence-free and positive definite $(1,1)$-tensors on Ω and M, respectively, and denote by σ_1 the first eigenvalue of the Steklov problem (8).*

1. *If $\delta \leqslant 0$ and $X(\Omega)$ is contained in the geodesic ball $B(p, R)$ or radius R, where p is the center of mass of M for the measure $\widetilde{\mu}_f$, then*

$$\sigma_1 \leqslant \sup_\Omega \left[\delta \mathrm{tr}\,(T) + \sup_\Omega \left(\frac{\|H_T - T(\overline{\nabla} f)\|}{\mathrm{tr}\,(T)} \right) \|H_T - T(\overline{\nabla} f)\| \right] \times \left[\delta + \frac{\sup_M \|H_S - S(\nabla f)\|^2}{\inf_M (\mathrm{tr}\,(S))^2} \right] \frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R).$$

2. *If $\delta > 0$ and $X(\Omega)$ is contained in a geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$, then*

$$\sigma_1 \leqslant \frac{\displaystyle\int_\Omega \mathrm{tr}\,(T) \mu_f}{V_f(M)} \left(\delta + \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2 \widetilde{\mu}_f}{V_f(M) \inf (\mathrm{tr}\,(S)^2)} \right).$$

Finally, we will consider the so-called eigenvalue problem for Wentzell boundary conditions

$$\text{(SW)} \qquad \begin{cases} \Delta u &= 0 \quad \text{in } \Omega \\ -b\widetilde{\Delta} u - \dfrac{\partial u}{\partial \nu} &= \alpha u \quad \text{on } M \end{cases},$$

where b is a given positive constant, Ω is a submanifold with non-empty boundary $\partial\Omega = M$ of a Riemannian manifold M with sectional curvature bounded by above, and Δ, $\widetilde{\Delta}$ denote the Laplacians on Ω and M, respectively. It is clear that if $b = 0$, then we recover the classical Steklov problem. The spectrum of this problem is an increasing sequence (see [5]) with 0 as first eigenvalue which is simple and the corresponding eigenfunctions are the constant ones. We denote by α_1 the first positive eigenvalue. In [14], the second author proved the following estimate when Ω is a submanifold of the Euclidean space $\mathbb{R}^n$

$$\alpha_1 \left(\int_{\partial M} \operatorname{tr}(S) dv_g \right)^2 \leqslant \Big(nV(M) + b(n-1)V(\partial M) \Big) \left(\int_{\partial M} \|H_S\|^2 dv_g \right).$$

In the following theorem, we obtain a comparable estimate when the ambient space is of bounded sectional curvature. Namely, we prove

Theorem 1.3. *Let $(\bar{M}^{n+p}, \bar{g})$ be a Riemannian manifold with sectional curvature $sect_{\bar{M}} \leqslant \delta$. Let (Ω, g) be a compact Riemannian manifold with non-empty boundary M isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$ by X. We denote by $\widetilde{g}$ the induced metric on M. Let S be a symmetric, divergence-free and positive definite $(1,1)$-tensor on M and denote by α_1 the first eigenvalue of the Steklov-Wentzell problem* (SW)*.*

(1) *If $\delta \leqslant 0$ and $X(\Omega)$ is contained in the geodesic ball $B(p, R)$ or radius R, where p is the center of mass of $\overline{M}$, then*

$$\alpha_1 \leqslant \left[n\frac{V(\Omega)}{V(M)} + b(n-1) - \delta s_\delta^2(R) \left(\frac{V(\Omega)}{V(M)} + b \right) \right] \left(\delta + \frac{\sup_M \|H_S\|^2}{\inf_M (\operatorname{tr}(S))^2} \right).$$

(2) *If $\delta > 0$ and so $X(\Omega)$ is contained in a geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$, then*

$$\alpha_1 \leqslant \left(n\frac{V(\Omega)}{V(M)} + b(n-1) \right) \left(\delta + \frac{\displaystyle\int_M \|H_S\|^2 dv_{\widetilde{g}}}{V(M) \inf(\operatorname{tr}(S)^2)} \right).$$

2. Preliminaries

Let $(\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)$ be a weighted Riemannian manifold with sectional curvature $sect_{\bar{M}} \leqslant \delta$ and weighted measure $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$. Let p a fixed point in $\bar{M}$, we denote by $r(x)$ the geodesic distance between x and p. Moreover, we define the vector field X by $X(x) := s_\delta(r(x))(\bar{\nabla} r)(x)$, s_δ is the function defined by

$$s_\delta(r) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta} r) & \text{if } \delta > 0 \\ r & \text{if } \delta = 0 \\ \frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|} r) & \text{if } \delta < 0. \end{cases}$$

We also define

$$c_\delta(r) = \begin{cases} \cos(\sqrt{\delta} r) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \\ \cosh(\sqrt{|\delta|} r) & \text{if } \delta < 0. \end{cases}$$

Hence, we have

$$c_\delta^2 + \delta s_\delta^2 = 1, \quad s'_\delta = c_\delta \quad \text{and} \quad c'_\delta = -\delta s_\delta.$$

In addition, let (M^n, g) be a closed Riemannian manifold isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$ by ϕ. If $\delta > 0$, then we assume that $\phi(M)$ is contained in a geodesic ball of radius $\frac{\pi}{2\sqrt{\delta}}$. We endow M with the weighted measure $\mu_f = e^{-f} dv_g$. We can define on M a divergence associated with the volume form $\mu_f = e^{-f} dv_g$ by

$$\mathrm{div}_f Y = \mathrm{div} Y - \langle \nabla f, Y \rangle$$

or, equivalently,

$$d(\iota_Y \mu_f) = \mathrm{div}_f(Y)\mu_f,$$

where ∇ is the gradient on Σ, that is, the projection on $T\Sigma$ of the gradient $\bar{\nabla}$ on $\bar{M}$. We call it the f-divergence. We recall briefly some basic facts about the f-divergence. In the case where Σ is closed, we first have the weighted version of the divergence theorem:

$$\int_\Sigma \mathrm{div}_f Y \ \mu_f = 0, \tag{9}$$

for any vector field Y on Σ. From this, we deduce easily the integration by parts formula

$$\int_\Sigma u \ \mathrm{div}_f Y \ \mu_f = -\int_\Sigma \langle \nabla u, Y \rangle \ \mu_f, \tag{10}$$

for any smooth function u and any vector field Y on Σ. First, we prove the following elementary lemma which generalize in non constant curvature the classical Hsiung-Minkowski formula (see [8, 13] for instance).

Lemma 2.1. *Let T be a symmetric divergence-free positive $(1,1)$-tensor on M. Then the following hold*

(1) $\mathrm{div}_f \left(TX^\top\right) \geqslant \mathrm{tr}\,(T)c_\delta + \langle X, H_T - T(\nabla f)\rangle$.
(2) $\displaystyle\int_M \mathrm{tr}\,(T)c_\delta\mu_f \leqslant -\int_M \langle X, H_T - T(\nabla f)\rangle\mu_f$.
(3) $\displaystyle\delta\int_M \langle TX^\top, X^\top\rangle\mu_f \geqslant \int_M \mathrm{tr}\,(T)c_\delta^2\mu_f - \int_M \|H_T - T(\nabla f)\| s_\delta c_\delta\mu_f$.

Proof. The proof is a straightforward consequence of the analogue non-weighted result proven by Grosjean. Namely, in [8], the author has shown that

$$\mathrm{div}_M(TX^\top) \geqslant \mathrm{tr}\,(T)c_\delta(r) + \langle X, H_T\rangle.$$

Hence, from the definition of the f-divergence, we have

$$\begin{aligned}\mathrm{div}_f(TX^\top) &= \mathrm{div}(TX^\top) - \langle \nabla f, TX^\top\rangle \\ &\geqslant \mathrm{tr}\,(T)c_\delta(r) + \langle X, H_T - T(\nabla f)\rangle,\end{aligned}$$

and this proves part (1). For the second part, we integrate the last inequality with respect to the measure μ_f and we get immediately

$$\int_M \mathrm{tr}\,(T)c_\delta\mu_f \leqslant -\int_M \langle X, H_T - T(\nabla f)\rangle\mu_f,$$

since

$$\int_M \mathrm{div}_f(TX^\top)\mu_f = 0.$$

Finally, for the last part, if $\delta = 0$, then $c_\delta = 1$ and we get directly the conclusion from the second one by using

$$|\langle X, H_T - T(\nabla f)\rangle| \leqslant ||X|| \cdot ||H_T - T(\nabla f)|| = s_\delta ||H_T - T(\nabla f)||.$$

If $\delta \neq 0$, since $X^\top = s_\delta(r)\nabla r = -\delta \nabla c_\delta(r)$, then we have

$$\begin{aligned}
\delta \int_M \langle TX^\top, X^\top\rangle \mu_f &= \frac{1}{\delta}\int_M \langle T(\nabla c_\delta, \nabla c_\delta)\mu_f \\
&= -\frac{1}{\delta}\int_M \mathrm{div}_f(T\nabla c_\delta) c_\delta \mu_f \\
&= \int_M \mathrm{div}_f(X^\top) c_\delta \mu_f \\
&\geqslant \int_M \mathrm{tr}\,(T) c_\delta^2 \mu_f - \int_M \|H_T - T(\nabla f)\| s_\delta c_\delta \mu_f,
\end{aligned}$$

where we have used the first part of the lemma and the well-known Cuachy-Schwarz inequality. □

3. Two key lemma

In this section we will prove two basic key lemma that will be used throughout the paper.

Lemma 3.1. *Let $(\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)$ be a weighted Riemannian manifold with sectional curvature $sect_{\bar{M}} \leqslant \delta \leqslant 0$, and $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$. Let (M, g) be a closed Riemannian manifold isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$, and we endow M with the weighted measure $\mu_f = e^{-f} dv_g$. Let S be a symmetric, divergence-free and positive definite $(1,1)$-tensor on M. Then, we have*

$$\frac{\int_M \|X\|^2 \mu_f}{V_f(M)} \geqslant \frac{1}{\delta + \dfrac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\mathrm{tr}\,(S))^2}}.$$

Proof. We have

$$\begin{aligned}
&\int_M \left(\mathrm{tr}\,(S) - \delta\langle SX^\top, X^\top\rangle\right)\mu_f = \int_M \left(\mathrm{tr}\,(S) - \mathrm{div}(SX^\top) c_\delta(r)\right)\mu_f \\
&\leqslant \int_M \left(\mathrm{tr}\,(S) - \mathrm{tr}\,(S) c_\delta^2(r) - \langle H_S - S(\nabla f), X\rangle c_\delta(r)\right)\mu_f \\
&\leqslant \int_M \left(\delta \mathrm{tr}\,(S) s_\delta^2(r) - \langle H_S - S(\nabla f), X\rangle c_\delta(r)\right)\mu_f \\
&\leqslant \int_M \left(\delta \mathrm{tr}\,(S) s_\delta^2(r) + \frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\mathrm{tr}\,(S))}\int_M \mathrm{tr}\,(S) s_\delta(r) c_\delta(r)\right)\mu_f \\
&\leqslant \frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\mathrm{tr}\,(S))}\int_M \left(s_\delta(r)\mathrm{div}(SX^\top) - s_\delta(r)\langle H_S - S(\nabla f), X\rangle\right)\mu_f \\
&\quad + \delta \inf(\mathrm{tr}\,(S)) \textstyle\int_M s_\delta^2(r)\mu_f \\
&\leqslant -\frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\mathrm{tr}\,(S))}\int_M \left(c_\delta(r) s_\delta(r)\langle S\nabla^M r, \nabla^M r\rangle + s_\delta^2(r)\langle H_S - S(\nabla f), \nabla^M r\rangle\right)\mu_f \\
&\quad + \delta \inf(\mathrm{tr}\,(S)) \textstyle\int_M s_\delta^2(r)\mu_f
\end{aligned}$$

$$\leqslant \left(\delta \inf(\mathrm{tr}\,(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\mathrm{tr}\,(S))}\right)\int_M s_\delta^2(r)\mu_f$$
$$-\frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\mathrm{tr}\,(S))}\int_M c_\delta(r)s_\delta(r)\langle S\nabla^M r, \nabla^M r\rangle\mu_f.$$

Hence, we get

$$\begin{aligned}
\int_M \mathrm{tr}\,(S)\mu_f &\leqslant \left(\delta \inf(\mathrm{tr}\,(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\mathrm{tr}\,(S))}\right)\int_M s_\delta^2(r)\mu_f \\
&\quad -\frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\mathrm{tr}\,(S))}\int_M c_\delta(r)s_\delta(r)\langle S\nabla^M r, \nabla^M r\rangle\mu_f \\
&\quad +\delta\int_M s_\delta^2(r)\langle S\nabla^M r, \nabla^M r\rangle\mu_f \\
&\leqslant \left(\delta \inf(\mathrm{tr}\,(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\mathrm{tr}\,(S))}\right)\int_M s_\delta^2(r)\mu_f \\
&\quad +\int_M \left(\delta s_\delta^2(r) - c_\delta(r)s_\delta(r)\frac{\|H_S - S(\nabla f)\|_\infty}{\inf(\mathrm{tr}\,(S))}\right)\langle S\nabla^M r, \nabla^M r\rangle\mu_f.
\end{aligned}$$

Since $\delta \leqslant 0$, the second term of the right hand side is nonpositive, and thus we get

$$\begin{aligned}
\inf(\mathrm{tr}\,(S))V_f(M) &\leqslant \int_M \mathrm{tr}\,(S)\mu_f \\
&\leqslant \left(\delta \inf(\mathrm{tr}\,(S)) + \frac{\|H_S - S(\nabla f)\|_\infty^2}{\inf(\mathrm{tr}\,(S))}\right)\int_M s_\delta^2(r)\mu_f,
\end{aligned}$$

which gives immediately the result since $\|X\| = s_\delta(r)$. □

Lemma 3.2. *Let $(\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)$ be a weighted Riemannian manifold with sectional curvature $\mathrm{sect}_{\bar{M}} \leqslant \delta$, with $\delta > 0$, and $\bar{\mu}_f = e^{-f}dv_{\bar{g}}$. Let (M, g) be a closed Riemannian manifold isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$ by X so that $X(M)$ is contained in a geodesic ball of radius $\frac{\pi}{2\sqrt{\delta}}$. We endow M with the weighted measure $\mu_f = e^{-f}dv_g$. Let S be a symmetric, divergence-free and positive definite $(1,1)$-tensor on M. Then, we have*

$$1 - \left(\frac{\int_M c_\delta(r)\mu_f}{V_f(M)}\right)^2 \geqslant \frac{1}{1 + \dfrac{\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta \inf(\mathrm{tr}\,(S))^2 V_f(M)}}.$$

Proof. For a sake of compactness, we will write

$$\alpha = \frac{\int_M c_\delta(r)\mu_f}{V_f(M)} \quad \text{and} \quad \beta = 1 + \frac{\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta \inf(\mathrm{tr}\,(S))^2 V_f(M)}.$$

We thus have to show that $(1-\alpha^2)\beta \geqslant 1$. We have

$$\begin{aligned}
(1-\alpha^2)\beta &= \beta - \left(\frac{\displaystyle\int_M c_\delta(r)\mu_f}{V_f(M)}\right)^2 - \left(\frac{\displaystyle\int_M c_\delta(r)\mu_f}{V_f(M)}\right)^2 \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta\inf(\operatorname{tr}(S))^2 V_f(M)} \\
&\geqslant \beta - \left(\frac{\displaystyle\int_M \operatorname{tr}(S) c_\delta(r)\mu_f}{\inf(\operatorname{tr}(S))V_f(M)}\right)^2 - \left(\frac{\displaystyle\int_M c_\delta(r)^2\mu_f}{V_f(M)}\right) \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta\inf(\operatorname{tr}(S))^2 V_f(M)} \\
&\geqslant \beta - \left(\frac{\displaystyle\int_M s_\delta(r)\|H_S - S(\nabla f)\|\mu_f}{\inf(\operatorname{tr}(S))V_f(M)}\right)^2 - \left(\frac{\displaystyle\int_M c_\delta(r)^2\mu_f}{V_f(M)}\right) \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta\inf(\operatorname{tr}(S))^2 V_f(M)} \\
&\geqslant \beta - \frac{\left(\displaystyle\int_M s_\delta^2(r)\mu_f\right)\left(\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f\right)}{\inf(\operatorname{tr}(S))^2 V_f(M)^2} \\
&\quad - \left(\frac{\displaystyle\int_M c_\delta(r)^2\mu_f}{V_f(M)}\right) \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta\inf(\operatorname{tr}(S))^2 V_f(M)} \\
&\geqslant \beta - \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta\inf(\operatorname{tr}(S))^2 V_f(M)^2}\left(\int_M (s_\delta^2(r) + \delta^2 c_\delta^2(r))\mu_f\right) \\
&= \beta - \frac{\displaystyle\int_M \|H_S - S(\nabla f)\|^2\mu_f}{\delta\inf(\operatorname{tr}(S))^2 V_f(M)} \\
&= 1,
\end{aligned}$$

and this concludes the proof. □

4. Proofs of the Theorems

Proof of Theorem 1.1. Case $\delta \leqslant 0$. Let $p \in \overline{M}$ be a fixed point and let $\{x_1, \cdots, x_N\}$ be the normal coordinates of $\overline{M}$ centered at p. For any $x \in \overline{M}$, $r(x)$ is the geodesic distance between p and x over $\overline{M}$. We want to use as test functions the functions

$$\frac{s_\delta(r)}{r}x_i,$$

for $1 \leqslant i \leqslant N$. For this purpose, we will choose p as the center of mass of M with respect for the measure μ_f, that is, p is the only point in M so that

$$\int_M \frac{s_\delta(r)}{r}x_i\mu_f = 0,$$

for any $i \in \{1, \cdots, N\}$. Note at that point that we assume that M is contained in a ball of radius $\dfrac{\pi}{4\sqrt{\delta}}$ when δ is positive allows us to ensure that M is contained in a ball of radius $\dfrac{\pi}{4\sqrt{\delta}}$ centered at p. This holds also for Theorems 1.2 and 1.3. These functions are candidates for test functions since they are $L^2(\mu_f)$-orthogonal to the constant functions which are the eigenfunctions for the first eigenvalue $\lambda_0 = 0$. Thus, we have

$$\lambda_1 \int_M \sum_{i=1}^N \frac{s_\delta^2(r)}{r^2} x_i^2 \mu_f \leqslant \int_M \sum_{i=1}^N \left\langle T\nabla \left(\frac{s_\delta(r)}{r} x_i \right), \nabla \left(\frac{s_\delta(r)}{r} x_i \right) \right\rangle \mu_f. \tag{11}$$

We recall that Grosjean proved in [8, Lemma 2.1] that

$$\sum_{i=1}^N \left\langle T\nabla \left(\frac{s_\delta(r)}{r} x_i \right), \nabla \left(\frac{s_\delta(r)}{r} x_i \right) \right\rangle \leqslant \mathrm{tr}\,(T) - \delta \left\langle TX^\top, X^\top \right\rangle. \tag{12}$$

Moreover, by Lemma 2.1, item (3), we have

$$\delta \int_M \langle TX^\top, X^\top \rangle \mu_f \geqslant \int_M \mathrm{tr}\,(T) c_\delta^2(r) \mu_f - \int_M \|H_T - T(\nabla f)\| s_\delta(r) c_\delta(r) \mu_f$$

which together with (11) and (12) gives

$$\begin{aligned} \lambda_1 \int_M s_\delta^2(r) \mu_f &\leqslant \int_M \Big(\mathrm{tr}\,(T) - c_\delta^2(r) \mathrm{tr}\,(T) + \|H_T - T(\nabla f)\| s_\delta(r) c_\delta(r) \Big) \mu_f \\ &\leqslant \delta \int_M s_\delta^2(r) \mathrm{tr}\,(T) \mu_f + \int_M \|H_T - T(\nabla f)\| s_\delta(r) c_\delta(r) \mu_f, \end{aligned}$$

where we have used $c_\delta^2 + \delta s_\delta^2 = 1$. Hence, we obtain

$$\begin{aligned} \lambda_1 \int_M s_\delta^2(r) \mu_f &\leqslant \delta \int_M s_\delta^2(r) \mathrm{tr}\,(T) \mu_f \\ &\quad + \sup_M \left(\frac{\|H_T - T(\nabla f)\|}{\mathrm{tr}\,(S)} \right) \int_M \mathrm{tr}\,(S) s_\delta(r) c_\delta(r) \mu_f. \end{aligned} \tag{13}$$

Now, we claim the following

Lemma 4.1. *We have*

$$\int_M \mathrm{tr}\,(S) s_\delta(r) c_\delta(r) \mu_f \leqslant \int_M \|H_S - S(\nabla f)\| s_\delta^2(r) \mu_f.$$

Reporting this into (13), we get

$$\lambda_1 \int_M s_\delta^2(r) \mu_f \leqslant \int_M \left[\delta \mathrm{tr}\,(T) + \sup_M \left(\frac{\|H_T - T(\nabla f)\|}{\mathrm{tr}\,(S)} \right) \|H_S - S(\nabla f)\| \right] s_\delta^2(r) \mu_f.$$

which gives immediately the desired estimate

$$\lambda_1 \leqslant \sup_M \left[\delta \mathrm{tr}\,(T) + \sup_M \left(\frac{\|H_T - T\nabla f\|}{\mathrm{tr}\,(S)} \right) \|H_S - S\nabla f\| \right].$$

This concludes the proof for the case $\delta < 0$, up to the proof of the lemma that we give now.

Proof of Lemma 4.1. Multiplying the first part of Lemma 2.1 for the tensor S by $s_\delta(r)$, we get

$$\mathrm{div}_f(SX^\top) s_\delta(r) \geqslant \mathrm{tr}\,(S) c_\delta(r) s_\delta(r) - \langle X, H_S - S(\nabla f) \rangle,$$

and thus, by the Cauchy-Schwarz inequality, we have

$$\mathrm{div}_f(SX^\top) s_\delta(r) \geqslant \mathrm{tr}\,(S) c_\delta(r) s_\delta(r) - \|H_S - S(\nabla f)\| s_\delta(r).$$

Now, integrating this relation and using the integration by parts in the formula (10), we get

$$\begin{aligned}\int_M \operatorname{tr}(S)c_\delta(r)s_\delta(r)\mu_f - \int_M \|H_S - S(\nabla f)\| s_\delta(r)\mu_f &\leqslant -\int_M \langle \nabla s_\delta(r), SX^\top\rangle \mu_f \\ &\leqslant -\int_M c_\delta(r)s_\delta(r)\langle \nabla r, S\nabla\rangle \\ &\leqslant 0,\end{aligned}$$

since S is positive. This concludes the proof of Lemma 4.1. □

Case $\delta > 0$. Like in the case $\delta < 0$, we use $\frac{s_\delta(r)}{r}x_i$, $1 \leqslant i \leqslant N$, as test functions. Using (12) again, we get

$$\lambda_1 \int_M s_\delta^2(r)\mu_f \leqslant \int_M \left(\operatorname{tr}(T) - \delta\left\langle TX^\top, X^\top\right\rangle\right)\mu_f. \tag{14}$$

On the other, we use another test function in this case, namely $c_\delta(r) - \overline{c_\delta}$ where for more convenience, we have denoted by $\overline{c_\delta}$ the mean value of $c_\delta(r)$, that is $\overline{c_\delta} = \dfrac{1}{V_f(M)}\int_M c_\delta(r)\mu_f$. Here again, this function is $L^2(\mu_f)$-orthogonal to the constant functions, so it is a candidate for being a test function. Hence, we have

$$\begin{aligned}\lambda_1 \int_M \left(c_\delta(r) - \overline{c_\delta}\right)^2\mu_f &\leqslant \int_M \left\langle T\nabla\left(c_\delta(r) - \overline{c_\delta}\right), \nabla\left(c_\delta(r) - \overline{c_\delta}\right)\right\rangle\mu_f \\ &\leqslant \int_M \left\langle T\nabla c_\delta(r), \nabla c_\delta(r)\right\rangle\mu_f \\ &\leqslant \delta^2\int_M s_\delta^2(r)\left\langle T\nabla r, \nabla r\right\rangle\mu_f \\ &\leqslant \delta^2\int_M \left\langle TX^\top, X^\top\right\rangle\mu_f.\end{aligned}$$

From this, we deduce immediately that

$$\lambda_1 \int_M c_\delta^2(r)\mu_f \leqslant \delta^2\int_M \left\langle TX^\top, X^\top\right\rangle + \frac{\lambda_1}{V_f(M)}\left(\int_M c_\delta(r)\mu_f\right)^2. \tag{15}$$

Now, using the fact that $c_\delta^2 + \delta s_\delta^2 = 1$, (15) plus δ times (14) gives

$$\lambda_1 V_f(M) \leqslant \delta\int_M \operatorname{tr}(T)\mu_f + \left(\frac{\int_M c_\delta(r)\mu_f}{V_f(M)}\right)^2$$

and thus

$$\lambda_1 V_f(M)\left(1 - \left(\frac{\int_M c_\delta(r)\mu_f}{V_f(M)}\right)^2\right) \leqslant \delta\int_M \operatorname{tr}(T)\mu_f.$$

Now, we conclude by using Lemma 3.2 to get the desired upper bound

$$\lambda_1 \leqslant \frac{\left(\int_M \operatorname{tr}(T)\mu_f\right)}{V_f(M)}\left(\delta + \frac{\int_M \|H_S - S(\nabla f)\|^2\mu_f}{V_f(M)\inf\left(\operatorname{tr}(S)^2\right)}\right),$$

and this concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Case $\delta \leqslant 0$. Like in the proof of Theorem 1.1, we will consider $p \in \overline{M}$ as the center of mass of Ω, $\{x_1, \ldots, x_N\}$ the normal coordinates of $\overline{M}$ centered at p and by $r(x)$ the geodesic distance (on $\overline{M}$) between x and p, for any $x \in \overline{M}$. By the choice of p, we have

$$\int_M \frac{s_\delta(r)}{r} x_i \widetilde{\mu}_f = 0,$$

for any $i \in \{1, \ldots, N\}$, and we can use the functions $\frac{s_\delta(r)}{r} x_i$ as test functions in the variational characterization of σ_1. Thus, we have

$$\begin{aligned}\sigma_1 \int_M \sum_{i=1}^N \frac{s_\delta^2(r)}{r^2} x_i^2 \widetilde{\mu}_f &\leqslant \int_\Omega \sum_{i=1}^N \left\langle T\nabla \left(\frac{s_\delta(r)}{r} x_i\right), \nabla \left(\frac{s_\delta(r)}{r} x_i\right)\right\rangle \mu_f \\ &\leqslant \int_\Omega \left(\operatorname{tr}(T) - \delta \left\langle TX^\top, X^\top\right\rangle\right) \mu_f,\end{aligned} \tag{16}$$

where we have used inequality (12) to get the second line. Here, $X = s_\delta(r)\overline{\nabla} r$ and $X^\top = s_\delta(r)\nabla r$ is the tangent component of X to Ω. On the other hand, by item (3) of Lemma 2.1 applied to $\overline{M}$, we have

$$\delta \int_\Omega \langle TX^\top, X^\top\rangle \mu_f \geqslant \int_\Omega \operatorname{tr}(T) c_\delta^2 \mu_f - \int_\Omega \|H_T - T(\nabla f)\| s_\delta c_\delta \mu_f.$$

Reporting into (16), we have

$$\begin{aligned}\sigma_1 \int_M s_\delta^2(r)\widetilde{\mu}_f &\leqslant \int_\Omega \Big(\operatorname{tr}(T) - c_\delta^2(r)\operatorname{tr}(T) + \|H_T - T(\nabla f)\| s_\delta(r) c_\delta(r)\Big)\mu_f \\ &\leqslant \delta \int_\Omega s_\delta^2(r)\operatorname{tr}(T)\mu_f + \int_\Omega \|H_T - T(\overline{\nabla} f)\| s_\delta(r) c_\delta(r)\mu_f, \\ &\leqslant \delta \int_\Omega s_\delta^2(r)\operatorname{tr}(T)\mu_f + \sup_\Omega \left(\frac{\|H_T - T(\overline{\nabla} f)\|}{\operatorname{tr}(T)}\right) \int_\Omega \operatorname{tr}(T) s_\delta(r) c_\delta(r)\mu_f.\end{aligned}$$

From Lemma 4.1 applied on Ω for the tensor T, we have

$$\int_\Omega \operatorname{tr}(T) s_\delta(r) c_\delta(r)\mu_f \leqslant \int_\Omega \|H_T - T(\nabla f)\| s_\delta^2(r)\mu_f,$$

which yields to

$$\sigma_1 \int_M s_\delta^2(r)\widetilde{\mu}_f \leqslant \int_\Omega s_\delta^2(r) \left[\delta \operatorname{tr}(T) + \sup_\Omega \left(\frac{\|H_T - T(\nabla f)\|}{\operatorname{tr}(T)}\right) \|H_T - T(\nabla f)\|\right] \mu_f.$$

Moreover, from the assumption that Ω is contained in the ball of radius $B(p, R)$, we get that $s_\delta(r) \leqslant s_\delta(R)$ and thus

$$\sigma_1 \int_M s_\delta^2(r)\widetilde{\mu}_f \leqslant s_\delta^2(R) V_f(\Omega) \sup_\Omega \left[\delta \operatorname{tr}(T) + \sup_\Omega \left(\frac{\|H_T - T(\nabla f)\|}{\operatorname{tr}(T)}\right) t\|H_T - T(\nabla f)\|\right].$$

Finally, by Lemma 3.2, we have

$$\frac{\int_M s_\delta(r)^2 \widetilde{\mu}_f}{V_f(M)} \geqslant \frac{1}{\delta + \dfrac{\sup_M \|H_S - S(\widetilde{\nabla} f)\|^2}{\inf_M (\operatorname{tr}(S))^2}},$$

which give the desired estimate when $\delta \leqslant 0$:

$$\sigma_1 \leqslant \sup_\Omega \left[\delta \mathrm{tr}\,(T) + \sup_\Omega \left(\frac{\|H_T - T(\nabla f)\|}{\mathrm{tr}\,(T)}\right) \|H_T - T(\overline{\nabla} f)\|\right] \times \left[\delta + \frac{\sup_M \|H_S - S(\widetilde{\nabla} f)\|^2}{\inf_M (\mathrm{tr}\,(S))^2}\right] \frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R).$$

Case $\delta > 0$. Like in the case $\delta \leqslant 0$, we have

$$\sigma_1 \int_M s_\delta^2(r)\widetilde{\mu}_f \leqslant \int_\Omega \left(\mathrm{tr}\,(T) - \delta \left\langle TX^\top, X^\top\right\rangle\right) \mu_f \tag{17}$$

by using $\frac{s_\delta(r)}{r} x_i$, $1 \leqslant i \leqslant N$ as test functions. In addition, we also use another test function, $c_\delta(r) - \widetilde{c_\delta}$, with $\widetilde{c_\delta} = \dfrac{1}{V_f(M)} \int_M c_\delta(r)\widetilde{\mu}_f$. By a computation analogue to the proof of Theorem 1.1, we have

$$\begin{aligned}
\sigma_1 \int_M \left(c_\delta(r) - \widetilde{c_\delta}\right)^2 \widetilde{\mu}_f &\leqslant \int_\Omega \left\langle T\nabla\left(c_\delta(r) - \widetilde{c_\delta}\right), \nabla\left(c_\delta(r) - \widetilde{c_\delta}\right)\right\rangle \mu_f \\
&\leqslant \int_\Omega \langle T\nabla c_\delta(r), \nabla c_\delta(r)\rangle\, \overline{\mu}_f \\
&\leqslant \delta^2 \int_\Omega s_\delta^2(r) \langle T\nabla r, \nabla r\rangle \mu_f \\
&\leqslant \delta^2 \int_\Omega \left\langle TX^\top, X^\top\right\rangle \mu_f.
\end{aligned}$$

From this, we deduce

$$\sigma_1 \int_M c_\delta^2(r)\widetilde{\mu}_f \leqslant \delta^2 \int_\Omega \left\langle TX^\top, X^\top\right\rangle \mu_f + \frac{\sigma_1}{V_f(M)} \left(\int_M c_\delta(r)\widetilde{\mu}_f\right)^2. \tag{18}$$

Now, using the fact that $c_\delta^2 + \delta s_\delta^2 = 1$, (18) plus δ times (17) gives

$$\sigma_1 V_f(M) \leqslant \delta \int_\Omega \mathrm{tr}\,(T)\mu_f + \left(\frac{\int_M c_\delta(r)\widetilde{\mu}_f}{V_f(M)}\right)^2$$

and so

$$\sigma_1 V_f(M) \left(1 - \left(\frac{\int_M c_\delta(r)\widetilde{\mu}_f}{V_f(M)}\right)^2\right) \leqslant \delta \int_\Omega \mathrm{tr}\,(T)\mu_f.$$

Finally, since we have

$$1 - \left(\frac{\int_M c_\delta(r)\widetilde{\mu}_f}{V_f(M)}\right)^2 \geqslant \frac{1}{1 + \dfrac{\int_M \|H_S - S(\widetilde{\nabla} f)\|^2 \widetilde{\mu}_f}{\delta \inf(\mathrm{tr}\,(S))^2 V_f(M)}},$$

by Lemma 3.2, we get

$$\sigma_1 \leqslant \frac{\int_\Omega \mathrm{tr}\,(T)\mu_f}{V_f(M)} \left(\delta + \frac{\int_M \|H_S - S(\widetilde{\nabla} f)\|^2 \widetilde{\mu}_f}{V_f(M) \inf\left(\mathrm{tr}\,(S)\right)^2}\right),$$

and this concludes the proof. □

Proof of Theorem 1.3. Here again we consider differently the two cases. *Case* $\delta \leqslant 0$. First, we recall the variational characterization of α_1 (see [5])

$$\alpha_1 = \inf \left\{ \frac{\displaystyle\int_\Omega \|\overline{\nabla} u\|^2 dv_{\overline{g}} + b \int_M \|\nabla u\|^2 dv_{\widetilde{g}}}{\displaystyle\int_M u^2 dv_{\widetilde{g}}} \;\middle|\; \int_M u dv_{\widetilde{g}} = 0 \right\}. \tag{19}$$

As in the proof of Theorem 1.2, we use $\dfrac{s_\delta(r)}{r} x_i$ as test functions, where r is the geodesic distance to the center of mass p of Ω and $\{x_1, \cdots, x_N\}$ the normal coordinates of $\overline{M}$ centered at p. Hence, we have

$$\alpha_1 \int_M \sum_{i=1}^N \frac{s_\delta^2(r)}{r^2} x_i^2 dv_{\widetilde{g}} \leqslant \int_\Omega \sum_{i=1}^N \left\| \overline{\nabla} \left(\frac{s_\delta(r)}{r} x_i \right) \right\| dv_{\overline{g}} + b \int_M \sum_{i=1}^N \left\| \nabla \left(\frac{s_\delta(r)}{r} x_i \right) \right\| dv_{\widetilde{g}}$$

that is,

$$\begin{aligned} \alpha_1 \int_M s_\delta(r)^2 dv_{\widetilde{g}} &\leqslant \int_\Omega \left(n - \delta \|X^\top\|^2\right) dv_{\overline{g}} \\ &\quad + b \int_M \left((n-1) - \delta \|X^{\widetilde{\top}}\|^2\right) dv_{\widetilde{g}}, \end{aligned} \tag{20}$$

where we use use inequality (12) twice, once on Ω for the first term and once on M for the second term. Moreover we have, $\|X^{\widetilde{\top}}\| \leqslant \|X^\top\| \leqslant \|X\| = s_\delta(r)$ and since δ is nonpositive, s_δ is an increasing function. So, by the assumption that Ω is contained in the ball $B(p, R)$, we have

$$\begin{aligned} \alpha_1 \int_M s_\delta^2(r) dv_{\widetilde{g}} &\leqslant \left(n - \delta s_\delta(R)^2\right) V(\Omega) + b\,(n-1) - \delta s_\delta(R))\, V(M) \\ &\leqslant \Big(nV(\Omega) + b(n-1)V(M)\Big) - \delta s_\delta(R)^2 \Big(V(\Omega) + bV(M)\Big). \end{aligned} \tag{21}$$

We conclude by applying Lemma 3.2, which says for $f = 0$,

$$\frac{\displaystyle\int_M s_\delta(r)^2 dv_{\widetilde{g}}}{V(M)} \geqslant \frac{1}{\delta + \dfrac{\sup_M \|H_S\|}{\inf_M (\operatorname{tr}(S))^2}},$$

to obtain the desired estimate when $\delta \leqslant 0$, that is,

$$\alpha_1 \leqslant \left[n\frac{V(\Omega)}{V(M)} + b(n-1) - \delta s_\delta^2(R) \left(\frac{V(\Omega)}{V(M)} + b \right) \right] \left(\delta + \frac{\sup_M \|H_S\|^2}{\inf_M (\operatorname{tr}(S))^2} \right).$$

Case $\delta > 0$. Like in the cas $\delta \leqslant 0$, using the functions $\dfrac{s_\delta(r)}{r} x_i$ as test functions in the variational characterization of α_1, we get

$$\alpha_1 \int_M s_\delta^2(r) dv_{\widetilde{g}} \leqslant \int_\Omega \left(n - \delta \|X^\top\|^2\right) dv_{\overline{g}} + b \int_M \left((n-1) - \delta \|X^{\widetilde{\top}}\|^2\right) dv_{\widetilde{g}}. \tag{22}$$

Moreover, we use another test function, $c_\delta(r)-\widetilde{c_\delta}$ with $\widetilde{c_\delta}=\dfrac{1}{V(M)}\int_M c_\delta(r)dv_{\widetilde{g}}$.

$$\begin{aligned}
\alpha_1\int_M \left(c_\delta(r)-\widetilde{c_\delta}\right)^2 dv_{\widetilde{g}} &\leqslant \int_\Omega \left\langle \nabla\left(c_\delta(r)-\widetilde{c_\delta}\right), \nabla\left(c_\delta(r)-\widetilde{c_\delta}\right)\right\rangle dv_g \\
&\quad +b\int_M \left\langle \widetilde{\nabla}\left(c_\delta(r)-\widetilde{c_\delta}\right), \widetilde{\nabla}\left(c_\delta(r)-\widetilde{c_\delta}\right)\right\rangle dv_{\widetilde{g}} \\
&\leqslant \int_\Omega \langle \nabla c_\delta(r), \nabla c_\delta(r)\rangle dv_g + b\int_M \left\langle \widetilde{\nabla} c_\delta(r), \widetilde{\nabla} c_\delta(r)\right\rangle dv_{\widetilde{g}} \\
&\leqslant \delta^2\int_\Omega s_\delta^2(r)\langle \nabla r, \nabla r\rangle dv_g + b\delta^2\int_M s_\delta^2(r)\left\langle \widetilde{\nabla} r, \widetilde{\nabla} r\right\rangle dv_{\widetilde{g}} \\
&\leqslant \delta^2\int_\Omega \|X^\top\|^2 dv_g + b\delta^2\int_M \|X^{\widetilde{\top}}\|^2 dv_{\widetilde{g}},
\end{aligned}$$

where $X^\top = s_\delta(r)\nabla r$ is the tangent component of X to Ω and $X^{\widetilde{\top}} = s_\delta(r)\widetilde{\nabla} r$ is the part of X tangent to M. From this, we get

$$\begin{aligned}
\alpha_1\int_M c_\delta(r)^2 dv_{\widetilde{g}} \leqslant &\delta^2\int_\Omega \|X^\top\|^2 dv_g + b\delta^2\int_M \|X^{\widetilde{\top}}\|^2 dv_{\widetilde{g}} \\
&+\frac{\alpha_1}{V(M)}\left(\int_M c_\delta(r)dv_{\widetilde{g}}\right)^2.
\end{aligned} \tag{23}$$

Hence summing (23) and δ times (22), using the fact that $c_\delta^2+\delta s_\delta^2=1$, gives

$$\alpha_1 V(M) \leqslant \delta\Big(nV(\Omega)+b(n-1)V(M)\Big) + \frac{\alpha_1}{V(M)}\left(\int_M c_\delta(r)dv_{\widetilde{g}}\right)^2,$$

and so

$$\alpha_1 V(M)\left(1-\left(\frac{1}{V(M)}\int_M c_\delta(r)dv_{\widetilde{g}}\right)^2\right) \leqslant \delta\Big(nV(\Omega)+b(n-1)V(M)\Big).$$

Moreover, from we have Lemma 3.2 (with $f=0$), we have

$$1-\left(\frac{\displaystyle\int_M c_\delta(r)dv_{\widetilde{g}}}{V(M)}\right)^2 \geqslant \frac{1}{1+\dfrac{\displaystyle\int_M \|H_S\|^2 dv_{\widetilde{g}}}{\delta\inf(\operatorname{tr}(S))^2 V(M)}}$$

which gives

$$\alpha_1 V(M) \leqslant \delta\Big(nV(\Omega)+b(n-1)V(M)\Big)\left(\delta+\frac{\displaystyle\int_M \|H_S\|^2 dv_{\widetilde{g}}}{V(M)\inf\left(\operatorname{tr}(S)\right)^2}\right)$$

and finally the desired estimate

$$\alpha_1 \leqslant \left(n\frac{V(\Omega)}{V(M)}+b(n-1)\right)\left(\delta+\frac{\displaystyle\int_M \|H_S\|^2 dv_{\widetilde{g}}}{V(M)\inf\left(\operatorname{tr}(S)^2\right)}\right).$$

This conclues the proof. □

5. A remark about the case $\delta > 0$

The aim of this section is to compare the estimates in both case $\delta > 0$ and $\delta \leqslant 0$. Indeed, in the estimates of Theorems 1.2, the radius R of a ball containing Ω appears when $\delta \leqslant 0$ but not for the estimates for $\delta > 0$. It turns out that when $\delta > 0$, we can bound the radius R from above in terms of H_T and $\operatorname{tr} T$ and so obtain upper bounds comparable to those obtained for $\delta \leqslant 0$. First, we have the following

Proposition 5.1. *Let $(\bar{M}^{n+p}, \bar{g}, \bar{\mu}_f)$ be a weighted Riemannian manifold with sectional curvature $\mathrm{sect}_{\bar{M}} \leqslant \delta$, with $\delta > 0$, and $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$. Let (Ω, g) be a compact Riemannian manifold with non-empty boundary M isometrically immersed into $(\bar{M}^{n+p}, \bar{g})$ by X. We endow Ω with the weighted measure $\mu_f = e^{-f} dv_g$. Let T be a symmetric, divergence-free and positive definite $(1,1)$-tensor on Ω. If Ω is contained a ball of radius R, then*

$$s_\delta^2(R)\left(\frac{\|H_T - T\nabla f\|_\infty^2}{\inf\limits_\Omega (\operatorname{tr}(T))^2} + \delta\right) \geqslant 1.$$

Proof. From the second part of Lemma 2.1 applied on Ω, we have

$$\int_\Omega c_\delta(r) \operatorname{tr}(T)\mu_f \leqslant -\int_\Omega \langle X, H_T - T\nabla f\rangle \mu_f.$$

We recall that $X = s_\delta(r)\nabla r$, which gives

$$\int_\Omega c_\delta(r) \operatorname{tr}(T)\mu_f \leqslant \int_\Omega \|H_T - T\nabla f\| s_\delta(r)\mu_f.$$

We are in the case where $\delta > 0$, so c_δ and s_δ are respectively decreasing and increasing on $[0, \frac{\pi}{2\sqrt{\delta}}]$. Moreover, Ω is contained in a ball of radius $R < \frac{\pi}{4\sqrt{\delta}}$ which implies that Ω is contained in a ball of radius $\frac{\pi}{2\sqrt{\delta}}$ centered at p. Hence, $r < R$ and so $cd(r) \geqslant c_\delta(R)$ and $s_\delta(r) \leqslant s_\delta(R)$ on Ω. Thus, we get

$$c_\delta(R)\inf_\Omega(\operatorname{tr}(T)) \leqslant s_\delta(R)\sup_\Omega \|H_T - T\nabla f\|.$$

We deduce easily from this and the fact that $c_\delta^2 + \delta s_\delta^2 = 1$ that

$$s_\delta^2(R)\left(\frac{\|H_T - T\nabla f\|_\infty^2}{\inf\limits_\Omega (\operatorname{tr}(T))^2} + \delta\right) \geqslant 1,$$

which concludes the proof of the proposition. □

Now, using the above proposition together with the estimate of Theorem 1.2 in the case $\delta > 0$, we get the following estimate

$$\begin{aligned}
\sigma_1 &\leqslant \left(\frac{\|H_T - T\nabla f\|_\infty^2}{\inf\limits_\Omega (\operatorname{tr}(T))^2} + \delta\right)\left(\delta + \frac{\int_M \|H_S - S(\nabla f)\|^2 \widetilde{\mu}_f}{V_f(M)\inf(\operatorname{tr}(S)^2)}\right)\frac{\int_\Omega \operatorname{tr}(T)\mu_f}{V_f(M)} s_\delta^2(R) \\
&\leqslant \sup_\Omega(\operatorname{tr}(T))\left(\frac{\|H_T - T\nabla f\|_\infty^2}{\inf\limits_\Omega (\operatorname{tr}(T))^2} + \delta\right)\left(\delta + \frac{\int_M \|H_S - S(\nabla f)\|^2 \widetilde{\mu}_f}{V_f(M)\inf(\operatorname{tr}(S)^2)}\right)\frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R)
\end{aligned}$$

which is comparable to the estimate for the case $\delta \leqslant 0$:

$$\sigma_1 \leqslant \sup_\Omega \left[\delta \mathrm{tr}\,(T) + \sup_\Omega \left(\frac{\|H_T - T(\overline{\nabla} f)\|}{\mathrm{tr}\,(T)}\right) \|H_T - T(\overline{\nabla} f)\|\right] \times \left[\delta + \frac{\sup_M \|H_S - S(\nabla f)\|^2}{\inf_M (\mathrm{tr}\,(S))^2}\right] \frac{V_f(\Omega)}{V_f(M)} s_\delta^2(R).$$

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