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Cantor equicontinuous factors of the Coven cellular automaton of three neighbours

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Abstract

We prove a sufficient condition for the non-existence of a nontrivial Cantor equicontinuous factor in dynamical systems. We study the Coven cellular automaton of three neighbours to show that it does not have a nontrivial Cantor equicontinuous factor. Through this study, we show that the blocking words in this cellular automaton are all of the same form.

Keywords : Dynamical systems, equicontinuous factor, cellular automata.

1 Introduction

Cellular automata (CA) are particular (topological) dynamical systems (DS)[10], their (topological) weak mixing is equivalent to their transitivity[12]. Every DS admits a maximal equicontinuous factor[7] and every equicontinuous factor (EF) of a weakly mixing DS is trivial[4]. Although the study of EF is classic, the main focus is on minimal DS for which their weak mixing is equivalent to the triviality of their EF[1]. A natural question is whether there are DS that are neither minimal nor weakly mixing, but do not have a nontrivial EF. If we consider that a nilpotent DS is trivial, a CA that has a finite generic limit set does not have a non-nilpotent factor, even if it is almost equicontinuous[5],[6]. The following question therefore concerns surjective DS. Since they have full generic limit set, this theory is useless for this case. In this paper, we are interested in the Coven CA of three neighbours which is a particular case of the Coven CA that was introduced in[3]. This CA is chain transitive, but do not have the shadowing property[2], while any Cantor equicontinuous DS has the shadowing property[11]. We establish a condition for a DS to admit no nontrivial Cantor EF and we prove that this CA satisfies this condition.

2 Preliminaries

Dynamical systems. A **(topological) dynamical system** (DS) is a pair $(X, \mathcal{F})$, where X is a compact metric space and $\mathcal{F} : X \to X$ is a continuous map. If X is the Cantor space, $(X, \mathcal{F})$ is called a **Cantor system**. A morphism $\Phi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ between two DS is a continuous map $\Phi : X \to Y$ satisfying $\Phi \circ \mathcal{F} = \mathcal{G} \circ \Phi$. If Φ is surjective, we say that Φ is a factor map and $(Y, \mathcal{G})$ is a **factor** of $(X, \mathcal{F})$. If $Z \subseteq X$ is a closed invariant subset, then $(Z, \mathcal{F})$ is a **subsystem** of $(X, \mathcal{F})$. We say that some subset $U \subseteq X$ is **strongly $\mathcal{F}$-invariant** if $\mathcal{F}^{-1}(U) = U$. For $\varepsilon > 0$, a point $x \in X$ is **ε-stable** if there exists $\delta > 0$ such that $\forall y \in B_\delta(x), \forall t \in \mathbb{N}, d(\mathcal{F}^t(x), \mathcal{F}^t(y)) < \varepsilon$. The set $\mathcal{E}_\mathcal{F} \subseteq X$ of **equicontinuous** points for $\mathcal{F}$ is the set of points that are ε-stable for every $\varepsilon > 0$. If $\mathcal{E}_\mathcal{F}$ is comeager, then we say that $\mathcal{F}$ is **almost equicontinuous** (a subset is

comeager if it includes a countable intersection of dense open sets). If $\mathcal{E}_{\mathcal{F}} = X$, then we say that $\mathcal{F}$ is **equicontinuous**. A DS $(X, \mathcal{F})$ is **weakly mixing**, if for any nonempty open sets $U, V, U', V' \subseteq X$, $\exists t \in \mathbb{N}, \mathcal{F}^t(U) \cap U' \neq \emptyset$ and $\mathcal{F}^t(V) \cap V' \neq \emptyset$. The **limit set** of $U \subseteq X$ is the set $\Omega_{\mathcal{F}}(U) = \bigcap_{T \in \mathbb{N}} \overline{\bigcup_{t \geq T} \mathcal{F}^t(U)}$. The **asymptotic set** of $U \subseteq X$ is $\omega_{\mathcal{F}}(U) = \bigcup_{x \in U} \Omega_{\mathcal{F}}(\{x\})$.

Symbolic dynamics. Let A be a finite set called the **alphabet**. A word is any finite sequence of elements of A. Denote $A^* = \bigcup_{n \in \mathbb{N}} A^n$ the set of all finite words $u = u_{[\![0,n-1]\!]}$; $|u| = n$ is the length of u. We say that v is a **subword** of u and write $v \sqsubseteq u$, if there are $k, l < |u|$ with $k \leq l$ such that $v = u_{[\![k,l[\![} = u_k u_{k+1} \dots u_{l-1}$. $A^{\mathbb{Z}}$ is the **space of configurations**, equipped with the metric: $d(x,y) := 2^{-n}$, where $n = \min \{ i \in \mathbb{N} |\, x_i \neq y_i \text{ or } x_{-i} \neq y_{-i} \}$. $A^{\mathbb{Z}}$ is a Cantor space. The **cylinder** of $u \in A^*$ in position i is $[u]_i = \left\{ x \in A^{\mathbb{Z}} \middle|\, x_{[\![i,i+|u|[\![} = u \right\}$. Cylinders are clopen (closed and open). The **shift** is the DS σ over $A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ for $i \in \mathbb{Z}$ and $x \in A^{\mathbb{Z}}$. A **subshift** is any subsystem of the full shift $A^{\mathbb{Z}}$. Let Σ be a subshift. Then $\mathcal{L}(\Sigma) = \{ u \in A^* |\, \exists x \in \Sigma, u \sqsubseteq x \}$ is the **language** of Σ.

Trace. [8] If $\mathcal{P}$ is a partition of some space X and $x \in X$ a point, then we denote $\mathcal{P}(x) \in \mathcal{P}$ the unique subset such that $x \in \mathcal{P}(x)$. The **trace** of some Cantor system $(X, \mathcal{F})$ with respect to some clopen partition $\mathcal{P}$ is $\begin{array}{cccc} T_{\mathcal{F}}^{\mathcal{P}} & : & X & \to & \mathcal{P}^{\mathbb{N}} \\ & & x & \mapsto & (\mathcal{P}(\mathcal{F}^j(x)))_{j \in \mathbb{N}} \end{array}$. It is a factor map of the system $(X, \mathcal{F})$ into the trace subshift $(\tau_{\mathcal{F}}^{\mathcal{P}} = T_{\mathcal{F}}^{\mathcal{P}}(X), \sigma)$. Every factor subshift of a Cantor system is a factor of some of its trace subshifts. A Cantor system is essentially the inverse limit of its sequence of (wider and wider) trace subshifts.

Theorem 1. *[9] A Cantor system is equicontinuous iff all of its trace subshifts are finite.*

Cellular automata. $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a **cellular automaton** (CA) if there exist integers $r_- \leq r_+$ and a local rule $f : A^{r_+ - r_- + 1} \to A$ such that for any $x \in A^{\mathbb{Z}}$ and any $i \in \mathbb{Z}, F(x)_i = f(x_{[\![i+r_-, i+r_+]\!]})$. $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a CA if and only if it is continuous and commutes with the shift. Thus, a CA is a DS over $A^{\mathbb{Z}}$. Let $s > 0$. A word $u \in A^+$ with $|u| \geq s$ is s-**blocking** for $(A^{\mathbb{Z}}, F)$, if there exists an offset $p \in [0, |u| - s]$ such that $\forall x, y \in [u]_0, \forall t \geq 0, F^t(x)_{[\![p,p+s[\![} = F^t(y)_{[\![p,p+s[\![}$.

3 A criterion for absence of Cantor equicontinuous factors

The following result shows the relation between Cantor equicontinuous factors and finite factors.

Proposition 1. *A DS $\mathcal{F}$ admits a nontrivial Cantor equicontinuous factor if and only if $\mathcal{F}$ admits a nontrivial finite factor.*
The latter has nontrivial period if the former was not the identity.

Note that a nontrivial finite factor may correspond to the identity over a nontrivial (that is, not a singleton) space.

Proof.
- Let $\mathcal{G}$ be a nontrivial Cantor equicontinuous factor of $\mathcal{F}$. Then, all trace subshifts of $\mathcal{G}$ are finite by Theorem 1 and $\mathcal{G}$ admits a nontrivial trace subshift (if every traces were trivial, then this means that the system itself was trivial) that is a finite factor of $\mathcal{G}$ (see the definition of trace), therefore of $\mathcal{F}$.
- Conversely, every finite space is a subspace of the Cantor space $A^{\mathbb{Z}}$, so $\mathcal{F}$ admits a nontrivial Cantor factor. Moreover, every finite system is equicontinuous. □

The following result gives a sufficient condition for the non-existence of finite factors.

Proposition 2. *Let $\mathcal{F}$ be a surjective DS. If there exists a weakly mixing subsystem that intersects every nonempty strongly $\mathcal{F}$-invariant clopen set, then $\mathcal{F}$ admits no nontrivial Cantor equicontinuous factor.*

The word "Cantor" is necessary in this statement: think about a rotation of a disk: it is equicontinuous, the only nonempty clopen set is the whole disk (by connectedness), and it contains a weakly mixing subsystem: the (invariant) center.

Proof of Proposition 2. Let $\mathcal{F}_{|W}$ be a weakly mixing subsystem. Suppose that $\mathcal{F}_{|W}$ intersects every nonempty strongly $\mathcal{F}$-invariant clopen set. If Φ is a factor map from $\mathcal{F}$ onto a finite system $\mathcal{G}$, then $\mathcal{G}_{|\Phi(W)}$ is an equicontinuous factor of $\mathcal{F}_{|W}$, so it is a singleton. $\Phi^{-1}(\bigcup_{n\in\mathbb{Z}}\mathcal{G}^n\Phi(W)^C)$ is a strongly $\mathcal{F}$-invariant clopen set (since Φ has finite image, all preimage sets are clopen). By definition, it does not intersect W. So by hypothesis, this clopen set is empty. This means that every orbit of $\mathcal{G}$ gets into $\Phi(W)$. Since $\mathcal{F}$ is surjective, then so is $\mathcal{G}$, so that $\mathcal{G}$ is actually the identity over a singleton. By Proposition 1, every equicontinuous Cantor factor is trivial. □

4 Case of the Coven CA of three neighbours

Let $\mathbb{2} = \{0,1\}$. The Coven CA of three neighbours is $F : \mathbb{2}^{\mathbb{Z}} \to \mathbb{2}^{\mathbb{Z}}$ defined by $f : \mathbb{2}^3 \to \mathbb{2}$ such that $f(x_{[\![i,i+2]\!]}) = x_i + x_{i+1}(x_{i+2}+1) \bmod 2 = \begin{cases} x_i + 1 \bmod 2 & if \ \ x_{[\![i+1,i+2]\!]} = 10 \\ x_i & otherwise \end{cases}$.
It is surjective and almost equicontinuous (see [2] and [11]). It is not hard to show that $(\{{}^\infty 1^\infty\}, F)$ is a (trivial) weakly mixing subsystem. We will prove that $(\{{}^\infty 1^\infty\}, F)$ intersects every invariant clopen set and we will need the following remark and definition.

Remark 1. *Let* $\Sigma_k = \left\{x \in \mathbb{2}^{\mathbb{Z}} \middle| \forall i \in 2\mathbb{Z}+k, x_i = 1\right\}$.
$\forall x \in \mathbb{2}^{\mathbb{Z}}, x \notin \Sigma_0 \cup \Sigma_1 \iff \exists k \in \mathbb{N}, 01^{2k}0 \sqsubseteq x.$

Proof. $x \notin \Sigma_0 \cup \Sigma_1 \iff x \notin \left\{x \in \mathbb{2}^{\mathbb{Z}} \middle| \forall i \in 2\mathbb{Z}, x_i = 1\right\}$ and $x \notin \left\{x \in \mathbb{2}^{\mathbb{Z}} \middle| \forall i \in 2\mathbb{Z}+1, x_i = 1\right\}$
$\iff \exists i \in 2\mathbb{Z}, x_i = 0$ and $\exists i \in 2\mathbb{Z}+1, x_i = 0 \iff \exists k \in \mathbb{N}, 01^{2k}0 \sqsubseteq x.$ □

Definition 1. *Let w be a word. We define the following two generalized cylinders by*
$[(\mathbb{2}1)^n]_i = \{x \in \mathbb{2}^{\mathbb{Z}} / x_{[\![i,i+2n[\![} = w \in \mathcal{L}(\Sigma_0)$ *such that w ends in* 1 *and* $|w| = 2n\}$.
$[(1\mathbb{2})^n]_i = \{x \in \mathbb{2}^{\mathbb{Z}} / x_{[\![i,i+2n[\![} = w \in \mathcal{L}(\Sigma_0)$ *such that w begins with* 1 *and* $|w| = 2n\}$.

4.1 Minimal blocking words of the Coven CA of three neighbours

We will use the following lemma to show Proposition 3 and Lemma 2.

Lemma 1. *Let $n, k \geq 1$ and $a, b \in \mathbb{2}$. Then,*

1. $F^k([a1^{2k-1}b]) \subseteq \begin{cases} [1] & if \ a = b \\ [0] & if \ a \neq b \end{cases}$. *Hence,* $\begin{cases} F^k([1^{4k}]) \subseteq [1^{2k}] \\ F^k([01^{4k}]) \subseteq [01^{2k}] \end{cases}$.
2. $F^{2^{n-1}}([(\mathbb{2}1)^{2^n}]) \subseteq [(\mathbb{2}1)^{2^{n-1}}]$ *and* $F^{2^{n-1}}([(1\mathbb{2})^{2^n}]) \subseteq [(1\mathbb{2})^{2^{n-1}}]$.
3. $F^{2^{n-1}}([a1(\mathbb{2}1)^{2^{n-1}-1}b]) \subseteq \begin{cases} [1] & if \ a = b \\ [0] & if \ a \neq b \end{cases}$ *and* $F^{2^{n-1}}([(1\mathbb{2})^{2^{n-1}}1]) \subseteq [1]$.

Proof. 1. When $k = 1$: $F([a1b]) \subseteq \begin{cases} [1] \ if \ a = b \\ [0] \ if \ a \neq b \end{cases}$. Assume that, for some $k \geq 1$,
$F^k([a1^{2k-1}b]) \subseteq \begin{cases} [1] \ if \ a = b \\ [0] \ if \ a \neq b \end{cases}$. We show that $F^{k+1}([a1^{2k+1}b]) \subseteq \begin{cases} [1] \ if \ a = b \\ [0] \ if \ a \neq b \end{cases}$.

By Induction hypothesis, $F^k([a1^{2k+1}b]) \subseteq [a1b]$ (we apply the hypothesis in position -1, 0 and 1). Hence, $F^{k+1}([a1^{2k+1}b]) \subseteq F([a1b]) \subseteq \begin{cases} [1]\ if\ a=b \\ [0]\ if\ a \neq b \end{cases}$.

2. - When $n=1$: $F([\mathbb{2}1\mathbb{2}1]) \subseteq [\mathbb{2}1]$. Assume that $F^{2^{n-1}}([(\mathbb{2}1)^{2^n}]) \subseteq [(\mathbb{2}1)^{2^{n-1}}]$ for some $n \geq 1$. We show that $F^{2^n}([(\mathbb{2}1)^{2^{n+1}}]) \subseteq [(\mathbb{2}1)^{2^n}]$.
By Induction hypothesis, $F^{2^{n-1}}([(1\mathbb{2})^{2^{n+1}}]) \subseteq [(\mathbb{2}1)^{2^{n-1}}(\mathbb{2}1)^{2^{n-1}}(\mathbb{2}1)^{2^{n-1}}]$.
Hence, $F^{2^n}([(\mathbb{2}1)^{2^{n+1}}]) \subseteq F^{2^{n-1}}([(\mathbb{2}1)^{2^{n-1}}(\mathbb{2}1)^{2^{n-1}}(\mathbb{2}1)^{2^{n-1}}]) \subseteq [(\mathbb{2}1)^{2^n}]$.
   - When $n=1$: $F([1\mathbb{2}1\mathbb{2}]) \subseteq [1\mathbb{2}]$. Assume that $F^{2^{n-1}}([(1\mathbb{2})^{2^n}]) \subseteq [(1\mathbb{2})^{2^{n-1}}]$ for some $n \geq 1$. We show that $F^{2^n}([(1\mathbb{2})^{2^{n+1}}]) \subseteq [(1\mathbb{2})^{2^n}]$.
By Induction hypothesis, $F^{2^{n-1}}([(1\mathbb{2})^{2^{n+1}}]) \subseteq [(1\mathbb{2})^{2^{n-1}}(1\mathbb{2})^{2^{n-1}}(1\mathbb{2})^{2^{n-1}}]$.
Hence, $F^{2^n}([(1\mathbb{2})^{2^{n+1}}]) \subseteq F^{2^{n-1}}([(1\mathbb{2})^{2^{n-1}}(1\mathbb{2})^{2^{n-1}}(1\mathbb{2})^{2^{n-1}}]) \subseteq [(1\mathbb{2})^{2^n}]$.

3. - $a1(\mathbb{2}1)^{2^{n-1}-1}b$ is of the form $(\mathbb{2}1)^{2^{n-1}}b$ such that the first $\mathbb{2}$, on the left, is a.
When $n=1$: $F([a1b]) \subseteq \begin{cases} [1]\ if\ a=b \\ [0]\ if\ a \neq b \end{cases}$. Assume that
$F^{2^{n-1}}([(\mathbb{2}1)^{2^{n-1}}b]) \subseteq \begin{cases} [1]\ if\ a=b \\ [0]\ if\ a \neq b \end{cases}$ for some $n \geq 1$. We show that
$F^{2^n}([(\mathbb{2}1)^{2^n}b]) \subseteq \begin{cases} [1]\ if\ a=b \\ [0]\ if\ a \neq b \end{cases}$. By Induction hypothesis and by Point 2,
$F^{2^{n-1}}([a1(\mathbb{2}1)^{2^{n-1}-1}a1(\mathbb{2}1)^{2^{n-1}-1}b]) \subseteq \begin{cases} [11(\mathbb{2}1)^{2^{n-1}-1}1]\ if\ a=b \\ [11(\mathbb{2}1)^{2^{n-1}-1}0]\ if\ a \neq b \end{cases}$.
Hence, $F^{2^n}([a1(\mathbb{2}1)^{2^{n-1}-1}a1(\mathbb{2}1)^{2^{n-1}-1}b]) \subseteq \begin{cases} F^{2^{n-1}}([11(\mathbb{2}1)^{2^{n-1}-1}1]) \subseteq [1]\ if\ a=b \\ F^{2^{n-1}}([11(\mathbb{2}1)^{2^{n-1}-1}0]) \subseteq [0]\ if\ a \neq b \end{cases}$.
   - When $n=1$: $F([1\mathbb{2}1]) \subseteq [1]$. Assume that $F^{2^{n-1}}([(1\mathbb{2})^{2^{n-1}}1]) \subseteq [1]$ for some $n \geq 1$. We show that $F^{2^n}([(1\mathbb{2})^{2^n}1]) \subseteq [1]$. By Induction hypothesis and Point 2, $F^{2^{n-1}}([(1\mathbb{2})^{2^n}1]) \subseteq [(1\mathbb{2})^{2^{n-1}}1]$. Hence, $F^{2^n}([(1\mathbb{2})^{2^n}1]) \subseteq F^{2^{n-1}}([(1\mathbb{2})^{2^{n-1}}1]) \subseteq [1]$.

$\square$

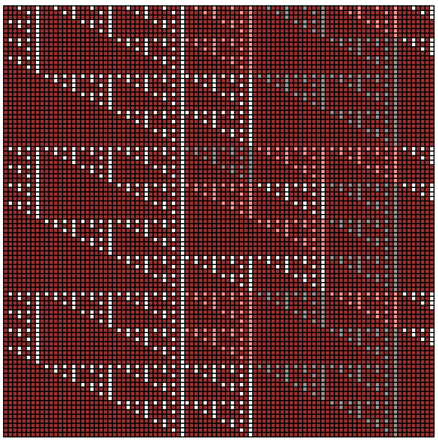

Figure 1: Two superimposed diagrams whose two initial configurations share the blocking word $01^{14}0$ and the left part. 0s are represented by white squares and 1s are represented by dark red squares when the two diagrams agree; gray squares and light red squares correspond to where they do not. Time evolves upwards.

In the following proposition, we show that the minimal blocking words are all of the same form (a minimal blocking word means that any strict subword is not blocking).

Proposition 3. *Let $k \in \mathbb{N}$. Then,*

1. *$01^{2k}0$ is a 1-blocking word with offset 0. Moreover, $\forall t \in \mathbb{N}, F^t([01^{2k}0]) \subseteq [0]$.*
2. *$\forall t \in \mathbb{N}, \exists k' \geq 0, F^t([01^{2k}0]) \subseteq [01^{2k'}0]$ (see Figure 1).*
3. *The minimal blocking words, with offset 0, are all of the form $01^{2k}0$.*

Proof of Proposition 3. 1. By Point 1 of Lemma 1, $F^k([01^{2k}0]) \subseteq [00]$. Moreover, 00 is a 1-blocking word with offset 0 and $\forall t \geq k, F^t([01^{2k}0]) \subseteq [0]$ (see [11]). Hence, $01^{2k}0$ is 1-blocking word with offset 0 and $\forall t \in \mathbb{N}, F^t([01^{2k}0]) \subseteq [0]$.

2. Assume that $\exists k \geq 0,\ t \in \mathbb{N},\ \forall k' \geq 0,\ F^t([01^{2k}0]) \subseteq [01^{2k'-1}0]$. By Point 1 of Lemma 1, $F^{t+k'}([01^{2k}0]) \subseteq F^{k'}([01^{2k'-1}0]) \subseteq [1]$. But $\forall t \in \mathbb{N}, F^t([01^{2k}0]) \subseteq [0]$, by Point 1. Then $\forall k \geq 0, t \in \mathbb{N},\ \exists k' \geq 0,\ F^t([01^{2k}0]) \subseteq [01^{2k'}0]$.

3. By Point 1, $01^{2k}0$ is a 1-blocking word with offset 0 and by Remark 1, $w \in \mathcal{L}(\Sigma_0)$ if and only if $\forall k \geq 0, 01^{2k}0 \not\sqsubseteq w$. Moreover, if we take $|w| = 2^n - 1$ such that w ends in 1 so that $awb \in \mathcal{L}(\Sigma_0),\ a, b \in \mathbb{2}$, then $F^{2^{n-1}}([awb]) \subseteq \begin{cases} [1]\ if\ a = b \\ [0]\ if\ a \neq b \end{cases}$, by Point 3 of Lemma 1. In other words, the dynamics to the right and to the left of w are not independant. Hence, w cannot have a blocking word. Then, the minimal blocking words are all of the form $01^{2k}0$. □

According to Remark 1 and Proposition 3, every point without blocking word is in $\Sigma_0 \cup \Sigma_1$, and, every point of $\Sigma_0 \cup \Sigma_1$ is without blocking word.

4.2 Cantor equicontinuous factor of the Coven CA of three neighbours

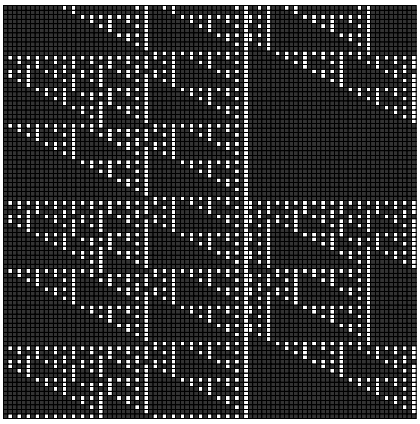

Figure 2: $F^{2^{n-1}}([w01^{10}w01^{2^n-1}]) \subseteq [1^{2^n}w]$ for $n = 5$ and $w = 101010101010101010101$, 0s are represented by white squares and 1s are represented by black squares.

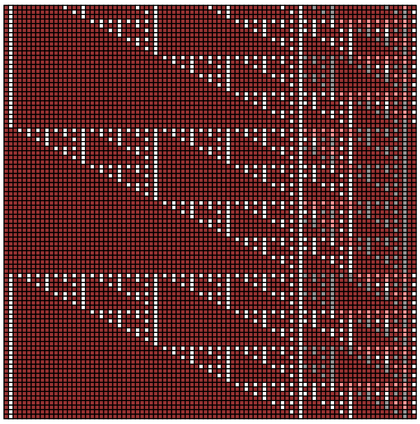

Figure 3: $F^{2^{n-1}}([01^{2^n-1}01^{10}0]) \subseteq [1^{2^n}0]$ for $n = 6$, they are two superimposed diagrams whose two initial configurations share the blocking word $01^{10}0$ and the left part.

We will use the following lemma to show Lemma 3, Lemma 4, and Proposition 4.

Lemma 2. *Let $n \geq 1$ and $k \geq 0$. Then,*

1. *Let* $w' \in 1(\mathbb{2}1)^{2^{n-1}-1}$ *and* $w \sqsubseteq w'$ *such that* w *ends in* 1 *and* $k+|w| = 2^n - 1$*. Then,*
   - $F^{2^{n-1}}([w01^{2^n-1}]) \subseteq [w]$.
   - $F^{2^{n-1}}([w01^k w01^k]) \subseteq [1^{2^n}]$*. Hence,* $F^{2^{n-1}}([w01^k w01^{2^n-1}]) \subseteq [1^{2^n} w]$.
   - $F^{2^{n-1}}([1^{2^n} w0]) \subseteq [w0]$.

2. $F^{2^{n-1}}([a1^{2^n-1}01^{2^n-1}b]) \subseteq \begin{cases} [1^{2^n}0] \ \textit{if}\ a=0, b=1 \\ [01^{2^n}] \ \textit{if}\ a=1, b=0 \end{cases}$ *. Hence,* $F^{2^{n-1}}([1^{2^n-1}01^{2^n}]) \subseteq [1^{2^n-1}0]$.

3. $F^{2^{n-1}}([1^{2^n}01^{2k}0]) \subseteq [01^{2^n-1}0]$ *and* $F^{2^{n-1}}([(\mathbb{2}1)^{2^{n-1}}01^{2k}0]) \subseteq [(\mathbb{2}1)^{2^{n-1}}0]$ *(see Figure 3).*

Proof. 1. $w \sqsubseteq w' \in 1(\mathbb{2}1)^{2^{n-1}-1}$, w ends in 1 and $k+|w| = 2^n - 1$. By Point 3 of Lemma 1,

- Since if we take any $w'' \sqsubseteq w01^{2^n-1}$ of size 2^n-1, the letter that is just to the left of w'' is a letter of w and the letter that is to the right of w'' is 1, $F^{2^{n-1}}([w01^{2^n-1}]) \subseteq [w]$.
- Since if we take any $w'' \sqsubseteq w01^k w01^k$ of size 2^n-1, the letter that is just to the left of w'' is the same as the letter that is just to the right of w'', $F^{2^{n-1}}([w01^k w01^k]) \subseteq [1^{2^n}]$.
- Since if we take any $w'' \sqsubseteq 1^{2^n} w0$ of size 2^n-1, the letter that is just to the left of w'' is 1 and the letter that is just to the right of w'' is a letter of $w0$, $F^{2^{n-1}}([1^{2^n} w0]) \subseteq [w0]$.

2. By Point 1, where $w = 1^{2^n-1}$ and Point 1 of Lemma 1, where $k = 2^{n-1}$.

3.
   - When $n=1$: $F([1101^{2k}0]) \subseteq [010]$. Assume that $F^{2^{n-1}}([1^{2^n}01^{2k}0]) \subseteq [01^{2^n-1}0]$ for some $n \geq 1$. We show that $F^{2^n}([1^{2^{n+1}}01^{2k}0]) \subseteq [01^{2^{n+1}-1}0]$. By Point 1 of Lemma1, where $k = 2^{n-1}$, Induction hypothesis and Point 2 of Proposition 3, there exists $k' \geq 0$ such that $F^{2^{n-1}}([1^{2^{n+1}}01^{2k}0]) \subseteq [1^{2^n}01^{2^n-1}01^{2k'}0]$. By Point 2 and Induction hypothesis, $F^{2^n}([1^{2^{n+1}}01^{2k}0]) \subseteq F^{2^{n-1}}([1^{2^n}01^{2^n-1}01^{2k'}0]) \subseteq [01^{2^{n+1}-1}0]$.
   - When $n=1$: $F([\mathbb{2}101^{2k}0]) \subseteq [\mathbb{2}10]$. Assume that $F^{2^{n-1}}([(\mathbb{2}1)^{2^{n-1}}01^{2k}0]) \subseteq [(\mathbb{2}1)^{2^{n-1}}0]$ for some $n \geq 1$. We show that $F^{2^n}([(\mathbb{2}1)^{2^n}01^{2k}0]) \subseteq [(\mathbb{2}1)^{2^n}0]$. By Point 2 of Lemma1, Induction hypothesis, and Point 2 of Proposition 3, there exists $k' \geq 0$ such that $F^{2^{n-1}}([(\mathbb{2}1)^{2^n}01^{2k}0]) \subseteq [(\mathbb{2}1)^{2^n}01^{2k'}0]$. By Point 2 of Lemma 1 and Induction hypothesis, $F^{2^n}([(\mathbb{2}1)^{2^n}01^{2k}0])) \subseteq F^{2^{n-1}}([(\mathbb{2}1)^{2^n}01^{2k'}0]) \subseteq [(\mathbb{2}1)^{2^n}0]$.

□

We will show that every invariant clopen set intersects $\Sigma_0 \cup \Sigma_1$ contains ${}^\infty 1^\infty$.

Lemma 3. *Let* U *be a strongly invariant clopen set. If* U *intersects* $\Sigma_0 \cup \Sigma_1$*, then it contains* ${}^\infty 1^\infty$*.*

Proof. Let $j \in \mathbb{Z}$. If U contains a cylinder $[u_0]_j$ such that $u_0 \in \mathcal{L}(\Sigma_0)$ and u_0 contains a single zero. Then u_0 is of the form $1^{k_1}01^{k_2}$, where $k_1, k_2 \geq 0$. Let $n > 1$ and $x \in [u_0]_j \subseteq U$ such that $x = 1^\infty 1^{2^n} 1^{k_0} 1^{k_1} 01^{k_2} 1^{k_3} 01^{2^n} 1^{2^n} 1^\infty$, where $|1^{k_0}1^{k_1}| = 2^n$ and $|1^{k_2}1^{k_3}| = 2^n - 1$. Then, $F^{2^{n-1}}(x) = F^{2^{n-1}}(1^\infty \underbrace{1^{2^n}}_{A} \underbrace{1^{2^n}}_{B} \underbrace{0}_{C} \underbrace{1^{2^n-1}0}_{D} \underbrace{1^{2^n}}_{E} \underbrace{1^{2^n}}_{F} 1^\infty) = 1^\infty \underbrace{1^{2^n}}_{A'} \underbrace{01^{2^n}}_{B'} \underbrace{1^{2^n-1}0}_{C'} \underbrace{1^{2^n}}_{D'} 1^\infty$.

- A and B give A', and, E and F give D', by Point 1 of Lemma 1, where $k = 2^{n-1}$.
- B, C and D give B', and, D and E give C', by Point 2 of Lemma 2.

Then, $F^{2^{n-1}}(x) = 1^\infty 01^{2^{n+1}-1}01^\infty$. When $n \to \infty$, $F^{2^{n-1}}(x) \to {}^\infty 1^\infty$. So, we can find a configuration in $[u_0]_j$ whose orbit has a subsequence which converges to the configuration ${}^\infty 1^\infty$. Since U is a strongly F-invariant clopen set, and, $[u_0]_j \subseteq U$, $\omega([u_0]_j) \subseteq U$, hence U contains ${}^\infty 1^\infty$.
Induction hypothesis : Assume that for some $N \geq 1$, if U contains a cylinder $[u_1]_j$ such that

$u_1 \in \mathcal{L}(\Sigma_0)$ and contains at most N zeros, then U contains ${}^\infty 1^\infty$. We show that, if U contains a cylinder $[u]_j$ such that $u \in \mathcal{L}(\Sigma_0)$ and contains $N+1$ zeros, then U contains ${}^\infty 1^\infty$.
If U contains a cylinder $[u]_j$ such that $u \in \mathcal{L}(\Sigma_0)$ and contains $N+1$ zeros. Then u is of the form $v01^{k_1}$, where $k_1 \geq 0$ and v contains N zeros. Let $n > 1$ and $x \in [u]_j$ such that
$x = 1^\infty 1^{2^n} 1^{2^n} 1^{k_0} v 0 1^{k_1} 1^{k_2} 0 1^{2^n} 1^{2^n} 1^\infty$, where $\left|1^{k_0} v\right| = 2^n - 1$ and $\left|1^{k_1} 1^{k_2}\right| = 2^n - 1$.

$$\begin{aligned}\text{Then, } F^{2^{n-1}}(x) &= F^{2^{n-1}}(1^\infty \underbrace{1^{2^n}}_{A} \underbrace{1^{2^n}}_{B} \underbrace{1^{2^n-1-|v|}v}_{C} \underbrace{0}_{D} \underbrace{1^{2^n-1}}_{E} \underbrace{0}_{F} \underbrace{1^{2^n}}_{G} \underbrace{1^{2^n}}_{H} 1^\infty) \\ &= 1^\infty \underbrace{1^{2^n}}_{A'} \underbrace{1^{2^n-1-|v|}v0}_{B'} \underbrace{1^{2^n-1-|v|}v}_{C'} \underbrace{1^{2^n}0}_{D'} \underbrace{1^{2^n}}_{E'} 1^\infty.\end{aligned}$$

- A and B give A', and, G and H give E', by Point 1 of Lemma 1, where $k = 2^{n-1}$.
- Since (B, C, and D) is of the form $1^{2^n}w0$ such that $w = 1^{2^n-1-|v|}v \in 1(\mathbb{2}1)^{2^{n-1}-1}$ and v ends in 1, B, C, and D give B' that is of the form $w0$, and, since (C, D, and E) is of the form $w01^{2^n-1}$ such that $w = 1^{2^n-1-|v|}v \in 1(\mathbb{2}1)^{2^{n-1}-1}$ and v ends in 1, C, D, and E give C' that is of the form w, by Point 1 of Lemma 2.
- D, E, F, and G give D', by Point 2 of Lemma 2.

Then, $F^{2^{n-1}}(x) = 1^\infty 1^{2^n-1-|v|} v 0 1^{2^n-1-|v|} v 1^{2^n} 0 1^\infty$. When $n \to \infty, F^{2^{n-1}}(x) \to 1^\infty v 1^\infty$. So, we can find a configuration in $[u]_j$ whose orbit has a subsequence which converges to a configuration in $\Sigma_0 \cup \Sigma_1$ and contains N zeros, because v contains N zeros. Since U is a strongly F-invariant clopen set and $[u]_j \subseteq U$, $\omega([u]_j) \subseteq U$, hence U contains a cylinder $[u_1]_j$ such that $u_1 \in \mathcal{L}(\Sigma_0)$ and contains at most N zeros. By Induction hypothesis, U contains ${}^\infty 1^\infty$. □

The following lemma shows that the asymptotic set of every cylinder containing a single blocking word intersects $\Sigma_0 \cup \Sigma_1$.

Lemma 4. *Let $j \in \mathbb{Z}$ and $[u]_j$ be a cylinder such that u contains a single minimal 1-blocking word. Then $\omega([u]_j)$ intersects $\Sigma_0 \cup \Sigma_1$.*

Proof. Since u contains a single minimal 1-blocking word, there exists $v \in 01(\mathbb{2}1)^{k_1'} 01^{2k_1} 01(\mathbb{2}1)^{k_1''} 0$, where $k_1, k_1', k_1'' \geq 0$ (which also contains a single minimal 1-blocking word), say $[v]_m \subseteq [u]_j$, $m \in \mathbb{Z}$. Let $n > 1$ such that $|v| < 2^n$. Let $x \in [v]_m$ such that

$$x \in 1^\infty 1^k 01(\mathbb{2}1)^{k_1'} 01^{2k_1} 01(\mathbb{2}1)^{k_1''} 01^{k'} 1^{2k_1} 01(\mathbb{2}1)^{k_1''} 01^\infty ,$$

where the length of words in $1^k 01(\mathbb{2}1)^{k_1'}$ is 2^n and in $1^{k'} 1^{2k_1} 01(\mathbb{2}1)^{k_1''}$ is $2^n - 1$. Then,

$$F^{2^{n-1}}(x) \in F^{2^{n-1}}(1^\infty \underbrace{1^{2^n}}_{A} \underbrace{1^{2^n}}_{B} \underbrace{1^k 01(\mathbb{2}1)^{k_1'}}_{C} \underbrace{0}_{D} \underbrace{1^{2k_1}0}_{E} \underbrace{1(\mathbb{2}1)^{k_1''} 01^{k'}}_{F} \underbrace{1^{2k_1} 01(\mathbb{2}1)^{k_1''}}_{G} \underbrace{01^{2^n-1}}_{H} \underbrace{1^{2^n+1}}_{I} 1^\infty).$$

$$\text{Hence, } F^{2^{n-1}}(x) \in 1^\infty \underbrace{1^{2^n}}_{A'} \underbrace{(\mathbb{2}1)^{2^{n-1}}}_{B'} \underbrace{(\mathbb{2}1)^{2^{n-1}}0}_{C'} \underbrace{1^{2^n} 1^{2k_1} 01(\mathbb{2}1)^{k_1''}}_{D'} \underbrace{0\,1^{2^n}}_{E'} 1^\infty .$$

- A and B give A', and, H and I give E', by Point 1 of Lemma 1, where $k = 2^{n-1}$.
- 1^{2^n} and $1^k 01(\mathbb{2}1)^{k_1'}$ are of the form $(\mathbb{2}1)^{2^{n-1}}$, B and C give B', by Point 2 of Lemma 1.
- Since $1^k 01(\mathbb{2}1)^{k_1'}$ is of the form $(\mathbb{2}1)^{2^{n-1}}$, C, D, and E give C', by Point 3 of Lemma 2.
- Since (E, F, G, and H) is of the form $w01^{k'}w01^{2^n-1}$ such that $w \in 1^{2k_1} 01(\mathbb{2}1)^{k_1''}$ and $1^{k'}w \in 1(\mathbb{2}1)^{2^{n-1}-1}$, E, F, G and H give D', by Point 1 of Lemma 2.

Then, $F^{2^{n-1}}(x) \in 1^\infty (\mathbb{2}1)^{2^n} 01^{2^n} 1^{2k_1} 01(\mathbb{2}1)^{k_1''} 01^\infty$. Hence, $F^{2^{n-1}}(x)$ contains a single minimal 1-blocking word $01^{2^n} 1^{2k_1} 0$. When $n \to \infty, F^{2^{n-1}}(x) \in (\mathbb{2}1)^\infty 01^\infty$. So, we can find a configuration in $[v]_m$ whose orbit has a subsequence which converges to a configuration without minimal 1-blocking word. Hence, $\omega([v]_m)$ intersects $\Sigma_0 \cup \Sigma_1$. Since $[v]_m \subseteq [u]_j$, $\omega([u]_j) \cap (\Sigma_0 \cup \Sigma_1) \neq \emptyset$. □

The following proposition shows that $^{\infty}1^{\infty}$ is contained in every invariant clopen set.

Proposition 4. *Let U be a strongly F-invariant clopen set. Then, U contains $^{\infty}1^{\infty}$.*

Proof. Let $j, m \in \mathbb{Z}$. If U contains a cylinder $[u_0]_j$ such that u_0 contains a single minimal 1-blocking word, $\omega([u_0]_j)$ intersects $\Sigma_0 \cup \Sigma_1$, by Lemma 4. Since U is a strongly F-invariant clopen set and $[u_0]_j \subseteq U$, $\omega([u_0]_j) \subseteq U$. Hence, U intersects $\Sigma_0 \cup \Sigma_1$.
Induction hypothesis: Assume that for some $N \geq 1$, if U contains a cylinder $[u_1]_j$ such that u_1 contains N minimal 1-blocking words, then $U \cap (\Sigma_0 \cup \Sigma_1) \neq \emptyset$. We show that, if U contains a cylinder $[u]_j$ such that u contains $N+1$ minimal 1-blocking words, then $U \cap (\Sigma_0 \cup \Sigma_1) \neq \emptyset$.
Since u contains $N+1$ minimal 1-blocking words, there exists $v = 0v_1 0v_2 0 \ldots 0v_N 0v_{N+1} 0$ such that $[v]_m \subseteq [u]_j$, $v_1 \in 1(21)^{k'_1}01^{2k_1}$, $v_{N+1} \in 1^{2k_{N+1}}01(21)^{k''_{N+1}}$, $v_i = 1^{2k_i}$ or $v_i \in 1(21)^{k'_i}01^{2k_i}$

and $v_N \in \begin{cases} 1^{2k_N} \quad or \\ 1^{2k_N}01(21)^{k''_N} \quad or \\ 1(21)^{k'_N}01^{2k_N} \quad or \\ 1(21)^{k'_N}01^{2k_N}01(21)^{k''_N} \end{cases}$, where $k_i, k'_i, k''_i \geq 0$, $i = \overline{2, N-1}$.

Let $n \geq 1$ such that $|v| \leq 2^n$. Let $x \in [v]_m$ such that

$$x \in 1^{\infty}1^k01(21)^{k'_1}01^{2k_1}0v_20\ldots0v_N01^{2k_{N+1}}01(21)^{k''_{N+1}}01^{k'}1^{2k_{N+1}}01(21)^{k''_{N+1}}01^{2^n-1}1^{\infty} ,$$

where the length of words in $1^k01(21)^{k'_1}$ is 2^n and in $1^{k'}1^{2k_{N+1}}01(21)^{k''_{N+1}}$ is $2^n - 1$. Then,
$F^{2^{n-1}}(1^{\infty}\underbrace{1^{2^n}}_{A}\underbrace{1^{2^n}}_{B}\underbrace{1^k01(21)^{k'_1}}_{C}\underbrace{0}_{D}\underbrace{1^{2k_1}0}_{E}\underbrace{v_20\ldots0v_N0}_{F}\underbrace{1^{2k_{N+1}}0}_{G}\underbrace{1(21)^{k''_{N+1}}01^{k'}}_{H}\underbrace{v_{N+1}}_{I}\underbrace{01^{2^n-1}}_{J}\underbrace{1^{2^n+1}}_{K}1^{\infty})$.
Hence, $F^{2^{n-1}}(x) \in 1^{\infty}\underbrace{1^{2^n}}_{A'}\underbrace{(21)^{2^{n-1}}}_{B'}\underbrace{(21)^{2^{n-1}}0}_{C'}\ \underbrace{(21)^{\alpha_1}0\ldots0(21)^{\alpha_N}0}_{D'}\ \underbrace{1^{2^n}v_{N+1}}_{E'}\underbrace{01^{2^n}}_{F'}1^{\infty}$,
where $\alpha_i \geq 1, i = \overline{1, N}$ and $v_{N+1} \in 1^{2k_{N+1}}01(21)^{k''_{N+1}}$.

- A and B give A', and, J and K give F', by Point 1 of Lemma 1, where $k = 2^{n-1}$.
- 1^{2^n} and $1^k01(21)^{k'_1}$ are of the form $(21)^{2^{n-1}}$, B and C give B', by Point 2 of Lemma 1.
- Since $1^k01(21)^{k_1} \subseteq (21)^{2^{n-1}}$, C, D, and E give C', by Point 3 of Lemma 2.
- Let $i = \overline{1, N-1}$. By Point 3 of Lemma 2,
  We have $1^{2k_i}0v_{i+1}0 = 1^{2k_i}01^{2k_{i+1}}0$ or $1^{2k_i}0v_{i+1}0 \in 1^{2k_i}01(21)^{k'_{i+1}}01^{2k_{i+1}}0$. Since $|1^{2k_i}|$ and the length of word in $1^{2k_i}01(21)^{k'_{i+1}}$ are $< 2^n$, $F^{2^{n-1}}([1^{2k_i}0v_{i+1}0]) \subseteq [(21)^{\alpha_i}0]$. Since $|1^{2k_N}|$ and the length of word in $1^{2k_N}01(21)^{k''_N}$ are $< 2^n$, $F^{2^{n-1}}([1^{2k_N}0v_{N+1}0]) \subseteq [(21)^{\alpha_N}0]$, with $\alpha_i = k_i$ *or* $k_i + 1 + k'_{i+1}$ and $\alpha_N = k_N$ *or* $k_N + 1 + k''_N$. Hence, E, F, and G give D'.
- Since (G, H, I, and J) is of the form $w01^{k'}w01^{2^n-1}$ such that $k' + |w| = 2^n - 1$ and $1^{k'}w = 1^{k'}v_{N+1} \in 1(21)^{2^{n-1}-1}$, G, H, I, and J give E', by Point 1 of Lemma 2.

Then, $F^{2^{n-1}}(x) \in 1^{\infty}(21)^{2^n}0v'_10\ldots0v'_N01^{2^n}v_{N+1}01^{\infty}$,
where $v_{N+1} \in 1^{2k_{N+1}}01(21)^{k''_{N+1}}$ and $v'_i \in (21)^{\alpha_i}$ such that $\alpha_i \geq 1$ and $i = \overline{1, N}$. Hence, $F^{2^{n-1}}(x)$ contains $N+1$ minimal 1-blocking words. When $n \to \infty$, $F^{2^{n-1}}(x) \in (21)^{\infty}0v'_10\ldots0v'_N01^{\infty}$. So, the orbit of x has a subsequence which converges to a configuration with N minimal 1-blocking words. In other words, $\omega(x)$ contains a configuration with N minimal 1-blocking words. Since, U is a strongly F-invariant clopen set and $x \in [v]_m \subseteq [u]_j \subseteq U$, $\omega(x) \subseteq U$. Hence, U contains a cylinder $[u_1]_j$ such that u_1 contains at most N minimal 1-blocking words. By Induction hypothesis, $U \cap (\Sigma_0 \cup \Sigma_1) \neq \emptyset$. In particular, U contains $^{\infty}1^{\infty}$, by Lemma 3. $\square$

Thus, the Coven CA of three neighbours has no nontrivial Cantor equicontinuous factor, by Proposition 2.

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