TWISTED PRE-LIE ALGEBRAS OF FINITE TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we first study the species of finite topological spaces recently considered by F. Fauvet, L. Foissy, and D. Manchon. Then, we construct a twisted pre-Lie structure on the species of connected finite topological spaces. The underlying pre-Lie structure defines a coproduct on the species of finite topological spaces different from those already defined by the Authors above. In the end, we illustrate the link between the Grossman-Larson product and the proposed coproduct.

CONTENTS

1. INTRODUCTION

A finite topological space is a finite set E endowed with a preorder $\leq$. The study of finite topological spaces was initiated by Alexandroff in 1937 [2], and revived at several periods since then, using the following well-known bijection [7,10]. Any topology $\mathcal{T}$ on X defines a quasi-order (i.e. a reflexive transitive relation) denoted by $\leq_{\mathcal{T}}$ on X:

$$x \leq_{\mathcal{T}} y \Longleftrightarrow \text{ any open subset containing } x \text{ also contains } y. \tag{1.1}$$

Conversely, any quasi-order $\leq$ on X defines a topology $\mathcal{T}_{\leq}$ given by its upper ideals, i.e. subsets $Y \subset X$ such that $(y \in Y$ and $y \leq z) \implies z \in Y$. Both operations are inverse to each other:

$$\leq_{\mathcal{T}_{\leq}} = \leq, \qquad \mathcal{T}_{\leq_{\mathcal{T}}} = \mathcal{T}. \tag{1.2}$$

Hence there is a natural bijection between topologies and quasi-orders on a finite set X. Any quasi-order (hence any topology $\mathcal{T}$) on X gives rise to an equivalence relation:

$$x \sim_{\mathcal{T}} y \Longleftrightarrow (x \leq_{\mathcal{T}} y \text{ and } y \leq_{\mathcal{T}} x)\,. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 16T05, 16T10, 16T15, 16T30, 06A11.
Key words and phrases. Finite topological spaces, Species, Bimonoids, Bialgebras, Hopf algebras.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a finite set X. We say that $\mathcal{T}'$ is finer than $\mathcal{T}$, and we write $\mathcal{T}' \prec \mathcal{T}$, when any open subset for $\mathcal{T}$ is an open subset for $\mathcal{T}'$. This is equivalent to the fact that for any $x, y \in X$, $x \leq_{\mathcal{T}'} y \Rightarrow x \leq_{\mathcal{T}} y$.

The *quotient* $\mathcal{T}/\mathcal{T}'$ of two topologies $\mathcal{T}$ and $\mathcal{T}'$ with $\mathcal{T}' \prec \mathcal{T}$ is defined as follows ([8, Paragraph 2.2]): The associated quasi-order $\leq_{\mathcal{T}/\mathcal{T}'}$ is the transitive closure of the relation $\mathcal{R}$ defined by:

$$x\mathcal{R}y \Longleftrightarrow (x \leq_{\mathcal{T}} y \text{ or } y \leq_{\mathcal{T}'} x). \tag{1.4}$$

More on finite topological spaces can be found in [1, 3, 5, 15, 20].

Recall that a linear (tensor) species is a contravariant functor from the category of finite sets **Fin** with bijections into the category **Vect** of vector spaces (on some field $\mathbf{k}$). The tensor product of two species $\mathbb{E}$ and $\mathbb{F}$ is given by

$$(\mathbb{E} \otimes \mathbb{F})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{E}_Y \otimes \mathbb{F}_Z, \tag{1.5}$$

where the notation $\sqcup$ stands for disjoint union. The unit of the tensor product denoted by $\mathbf{1}$ is defined by $\mathbf{1}_\emptyset = \mathbf{k}$ and $\mathbf{1}_X = \{0\}$, if $X \neq \emptyset$.
We write $x \in \mathbb{E}$ if there exists a finite set X such that $x \in \mathbb{E}_X$.

A twisted algebra [12] is an algebra in the linear symmetric monoidal category of linear species. See [4, 18, 19] for further details on and references to Joyal's theory of twisted algebras. Concretely, a twisted algebra is a linear species $\mathbb{E}$ provided with a product map (which is a map of linear species: $\mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$). Associative algebras, commutative algebras, Lie algebras, pre-Lie algebras and so on, are defined accordingly.

The species $\mathbb{T}$ of finite topological spaces is defined as follows: For any finite set X, $\mathbb{T}_X$ is the vector space freely generated by the topologies on X. For any bijection $\varphi : X \longrightarrow X'$, the isomorphism $\mathbb{T}_\varphi : \mathbb{T}_{X'} \longrightarrow \mathbb{T}_X$ is defined by the obvious relabelling:

$$\mathbb{T}_\varphi(\mathcal{T}) = \{\varphi^{-1}(Y), Y \in \mathcal{T}\},$$

for any topology $\mathcal{T}$ on X'.

A unital associative algebra ([8, Paragraph 2.3]) on the species of finite topologies is defined as follows: for any pair X_1, X_2 of finite sets we introduce

$$\begin{aligned} m : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\ \mathcal{T}_1 \otimes \mathcal{T}_2 &\longmapsto \mathcal{T}_1\mathcal{T}_2, \end{aligned}$$

where $\mathcal{T}_1\mathcal{T}_2$ is the disjoint union topology characterised by $Y \in \mathcal{T}_1\mathcal{T}_2$ if and only if $Y \cap X_1 \in \mathcal{T}_1$ and $Y \cap X_2 \in \mathcal{T}_2$. The unit is given by the unique topology on the empty set.

For any topology $\mathcal{T}$ on a finite set X and for any subset $Y \subset X$, we denote by $\mathcal{T}_{|Y}$ the restriction of $\mathcal{T}$ to Y. It is defined by:

$$\mathcal{T}_{|Y} = \{Z \cap Y, Z \in \mathcal{T}\}.$$

The external coproduct Δ on $\mathbb{T}$ is defined as follows:

$$\begin{aligned}\Delta : \mathbb{T}_X &\longrightarrow (\mathbb{T}\otimes\mathbb{T})_X = \bigoplus_{Y\sqcup Z=X} \mathbb{T}_Y \otimes \mathbb{T}_Z\\ \mathcal{T} &\longmapsto \sum_{Y\in\mathcal{T}} \mathcal{T}_{|X\setminus Y} \otimes \mathcal{T}_{|Y}.\end{aligned}$$

The species $\mathbb{T}$ is this way endowed with a twisted bialgebra structure in [8].

Now consider the graded vector space:

$$\mathcal{H} = \overline{\mathcal{K}}(\mathbb{T}) = \bigoplus_{n\geq 0} \mathcal{H}_n \tag{1.6}$$

where $\mathcal{H}_0 = \mathbf{k}.1$, and where $\mathcal{H}_n$ is the linear span of topologies on $\{1, ..., n\}$ when $n \geq 1$, modulo the action of the symmetric group S_n. The vector space $\mathcal{H}$ can be seen as the quotient of the species $\mathbb{T}$ by the "forget the labels" equivalence relation: $\mathcal{T} \sim \mathcal{T}'$ if $\mathcal{T}$ (resp.$\mathcal{T}'$) is a topology on a finite set X (resp. X'), such that there is a bijection from X onto X' which is a homeomorphism with respect to both topologies. The functor $\overline{\mathcal{K}}$ from linear species to graded vector spaces thus obtained is intensively studied in ([1, chapter 15]) under the name "bosonic Fock functor". The twisted Hopf algebra structure on $\mathbb{T}$ [8] naturally leads to the following:

$(\mathcal{H}, m, \Delta)$ is a commutative connected Hopf algebra, graded by the number of elements.

L. Foissy, C. Malvenuto and F. Patras in [9, section 6] were the first to prove that the finite topological spaces can be organized in a graded commutative Hopf algebra. The latter can be recovered by applying the $\overline{\mathcal{K}}$ functor to the twisted Hopf algebra structure on $\mathbb{T}$ described in [8]. The coproduct Δ defined therein is however not built from a pre-Lie structure. We define in the present work two twisted pre-Lie structures $\searrow$ and $\nearrow$ on the species of connected finite topological spaces, giving rise to two more coproducts $\Delta_{\searrow}$ and $\Delta_{\nearrow}$, hence two more twisted Hopf algebra structures. We expect that this will contribute to a better understanding of the finite topological spaces considered as a whole.

In section 2, we recall the method of D. Guin and J.-M. Oudom [16] to describe the enveloping algebra of a pre-Lie algebra, and we adapt it to the twisted context, following indications in [22].

In Section 3 of this paper, we define the enveloping algebra of the grafting twisted pre-Lie algebra of connected finite topological spaces, as well as its enveloping algebra using the Guin-Oudom method. Denoting by $\mathbb{V}$ the species of connected finite topological spaces, we consider the Hopf symmetric algebra $\mathcal{H}' = S(\mathbb{V})$ of the pre-Lie twisted algebra $(\mathbb{V}, \searrow)$, equipped with its usual unshuffling coproduct Δ_{unsh} and a product $\star$ defined on $\mathbb{T}$ by: For any pair X_1, X_2 of finite sets

$$\begin{aligned}\star : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1\sqcup X_2}\\ (\mathcal{T}_1, \mathcal{T}_2) &\longmapsto \sum_{(\mathcal{T}_1)} \mathcal{T}_1^{(1)}(\mathcal{T}_1^{(2)} \searrow \mathcal{T}_2).\end{aligned}$$

In section 4 we prove that there exists a twisted bialgebra structure on $\mathbb{T}$, where the external coproduct is defined by

$$\begin{array}{rcl}\Delta_{\searrow}:\mathbb{T}_X & \longrightarrow & (\mathbb{T}\otimes\mathbb{T})_X=\bigoplus_{Y\sqcup Z=X}\mathbb{T}_Y\otimes\mathbb{T}_Z\\ \mathcal{T} & \longmapsto & \sum_{Y\overline{\in}\mathcal{T}}\mathcal{T}_{|Y}\otimes\mathcal{T}_{|X\setminus Y}\end{array}$$

where $Y\overline{\in}\mathcal{T}$ stands for

- $Y\in\mathcal{T}$,
- $\mathcal{T}_{|Y}=\mathcal{T}_1...\mathcal{T}_n$, such that for all $i\in\{1,...,n\}$, $\mathcal{T}_i$ connected and ($\min\mathcal{T}_i=(\min\mathcal{T})\cap\mathcal{T}_i$, or there is a single common ancestor $x_i\in\overline{X\setminus Y}$ to $\min\mathcal{T}_i$), where $\overline{X\setminus Y}=(X\setminus Y)/\sim_{\mathcal{T}_{|X\setminus Y}}$.

We moreover give a relation between the two structures $\Delta_{\searrow}$ and $\star$.

Finally we define in section 5 a new pre-Lie law $\nearrow$ on the species of connected finite topological spaces by: For all $\mathcal{T}=(X,\leq_{\mathcal{T}})$ and $\mathcal{S}=(Y,\leq_{\mathcal{S}})$ be two finite topological spaces,

$$\mathcal{T}\nearrow\mathcal{S}:=j(j(\mathcal{T})\searrow j(\mathcal{S})),$$

where j is the involution which transforms $\leq$ into $\geq$. This law $\nearrow$ gives rise to a coproduct denoted $\Delta_{\nearrow}$ defined by $\Delta_{\nearrow}=(j\otimes j)\Delta_{\searrow}\circ j$.
For any finite set A and for any pair of parts A_1,A_2 of A with $A_1\cap A_2=\varnothing$, we define

$$\Psi_{A_1,A_2}:\mathbb{T}_A\to\mathbb{T}_A,$$

as follows: for any topology $\mathcal{T}\in\mathbb{T}_A$, the topology $\Psi_{A_1,A_2}(\mathcal{T})$ is associated with the following pre-order $\leq$ defined by:

- If $a\in A_1$, and $b\in A_2$ then a and b are incomparable,
- If not, we have: $a\leq b$ if and only if $a\leq_{\mathcal{T}}b$.

In this section, we provide a relation between both pre-Lie structures, by proving that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{V}_X\otimes\mathbb{V}_Y\otimes\mathbb{V}_Z \arrow[r, "id\otimes\searrow_u"] \arrow[d, "\nearrow^s\otimes id"'] & \mathbb{V}_X\otimes\mathbb{V}_{Y\sqcup Z} \arrow[dd, "\nearrow^s"] \\
\mathbb{V}_{X\sqcup Y}\otimes\mathbb{V}_Z \arrow[d, "\searrow_u"'] & \\
\mathbb{V}_{X\sqcup Y\sqcup Z} \arrow[r, "\Psi_{X,Z}"'] & \mathbb{V}_{X\sqcup Y\sqcup Z}
\end{tikzcd}$$

2. The enveloping algebra of pre-Lie algebras and twisted pre-Lie algebras

In this section, we recall the method of D. Guin and J.-M. Oudom [16] to describe the enveloping algebra of a pre-Lie algebra. We also recall how T. Schedler in [22] generalizes this method to twisted pre-Lie algebras.

Definition 2.1. *A Lie algebra over a field* $\mathbf{k}$ *is a vector space* V *endowed with a bilinear bracket* $[.,.]$ *satisfying:*
(1) The antisymmetry:

$$[x,y]=-[y,x],\forall x,y\in V.$$

(2) The Jacobi identity:

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0, \forall x,y,z \in V.$$

Definition 2.2. *[5, 14] A left pre-Lie algebra over a field* $\mathbf{k}$ *is a* $\mathbf{k}$*-vector space* A *with a binary composition* $\rhd$ *that satisfies the left pre-Lie identity:*

$$(x \rhd y) \rhd z - x \rhd (y \rhd z) = (y \rhd x) \rhd z - y \rhd (x \rhd z),$$

for all $x, y, z \in A$*. The left pre-Lie identity rewrites as:*

$$L_{[x,y]} = [L_x, L_y], \tag{2.1}$$

where $L_x : A \longrightarrow A$ *is defined by* $L_x y = x \rhd y$*, and where the bracket on the left-hand side is defined by* $[x,y] = x \rhd y - y \rhd x$*. As a consequence this bracket satisfies the Jacobi identity.*

The pre-Lie product is extended to the symmetric algebra as follows [17]. Let $(A, \rhd)$ be a pre-Lie algebra. We consider the Hopf symmetric algebra $S(A)$ equipped with its usual unshuffle coproduct denoted Δ_{unsh}. We will use without restraint the classical Sweedler notation: $\Delta_{unsh}(a) = \sum\limits_a a^{(1)} \otimes a^{(2)}$.

We extend the product $\rhd$ to $S(A)$. Let a, b and $c \in S(A)$, and $x \in A$. We put:

- $1 \rhd a = a$
- $a \rhd 1 = \varepsilon(a)1$
- $(xa) \rhd b = x \rhd (a \rhd b) - (x \rhd a) \rhd b$
- $a \rhd (bc) = \sum\limits_a (a^{(1)} \rhd b)(a^{(2)} \rhd c)$.

Definition 2.3. *We define the following* $\star$ *product on* $S(A)$ *by:*

$$a \star b = \sum_a a^{(1)}(a^{(2)} \rhd b). \tag{2.2}$$

Theorem 2.1. *[13, 16] The triple* $(S(A), \star, \Delta_{unsh})$ *is a Hopf algebra which is isomorphic to the enveloping Hopf algebra* $\mathcal{U}(A_{Lie})$ *of the Lie algebra* A_{Lie}*.*

Proof. This theorem was proved by D. Guin and J.-M. Oudom in [16] (Lemma 2.10 and Theorem 2.12). □

Definition 2.4. *[4] A twisted Lie algebra over a field* $\mathbf{k}$*, is a species* $\mathbb{E}$ *endowed with a bilinear bracket* $[\,,\,] : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$*, satisfying:*
(i) $[\,,\,] + [\,,\,]\tau = 0$,
(ii) $[\,,[\,,\,]] + [\,,[\,,\,]]\Sigma + [\,,[\,,\,]]\Sigma^2 = 0$,
where $\tau : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E} \otimes \mathbb{E}$ *is the flip, and* $\Sigma : \mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E} \to \mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E}$ *is the cyclic permutation of factors.*

Definition 2.5. *A left twisted pre-Lie algebra over a field* $\mathbf{k}$*, is a species* $\mathbb{E}$ *with a binary composition* $\circ : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$*, satisfing the left twisted pre-Lie algebra identity*

$$\circ(\circ \otimes Id) - \circ(Id \otimes \circ) = (\circ(\circ \otimes Id) - \circ(Id \otimes \circ))(\tau \otimes Id).$$

T. Schedler in [22] shows that the properties of D. Guin and J.-M. Oudom above also work for the linear species, i.e:
Let $(\mathbb{E}, \circ)$ be a twisted pre-Lie algebra. We consider the twisted Hopf symmetric algebra $S(\mathbb{E})$ equipped with its usual unshuffle coproduct denoted Δ_{unsh}. We extend the product $\circ$ to $S(\mathbb{E})$ as follows. Let a, b and $c \in S(\mathbb{E})$, and $x \in \mathbb{E}$. We put:

- $1 \circ a = a$
- $(xa) \circ b = x \circ (a \circ b) - (x \circ a) \circ b$
- $a \circ (bc) = \sum_{a} (a^{(1)} \circ b)(a^{(2)} \circ c)$,

and if we define the product $\star$ on $S(\mathbb{E})$ by:

$$a \star b = \sum_{a} a^{(1)}(a^{(2)} \circ b), \tag{2.3}$$

then $(S(\mathbb{E}), \star, \Delta_{unsh})$ is isomorphic to the enveloping Hopf algebra $\mathcal{U}(\mathbb{E}_{Lie})$ of the twisted Lie algebra $\mathbb{E}_{Lie}$.

3. The enveloping algebra of the twisted pre-Lie algebra of finite topological spaces

3.1. The pre-Lie algebra of rooted trees. Let T the vector space spanned by the set of isomorphism classes of rooted trees and $\mathcal{H} = S(T)$. Grafting pre-Lie algebras of rooted trees were studied for the first time by F. Chapoton and M. Livernet [6], see also D. Manchon and A. Saidi [15]. The grafting product is given, for all $t, s \in T$, by:

$$t \to s = \sum_{s' \text{ vertex of } s} t \to_{s'} s, \tag{3.1}$$

where $t \to_{s'} s$ is the tree obtained by grafting the root of t on the vertex s' of s. More explicitly, the operation $t \to s$ consists of grafting the root of t on every vertex of s and summing up.

Theorem 3.1. *[6] Equipped by $\to$, the space T is the free pre-Lie algebra with one generator.*

Now, we can use the method of D. Guin and J.-M. Oudom [16] to find the enveloping algebra of the grafting pre-Lie algebra of rooted trees. We consider the Hopf symmetric algebra $\mathcal{H} = S(T)$ of the pre-Lie algebra $(T, \to)$, equipped with its usual unshuffling coproduct Δ_{unsh}. We extend the product $\to$ to $\mathcal{H}$ by the same method used in (3.1), and we define the Grossman-Larson product [11] $\star$ on $\mathcal{H}$ by:

$$t \star t' = \sum_{t} t^{(1)}(t^{(2)} \to t').$$

By construction, the space $(\mathcal{H}, \star, \Delta_{unsh})$ is a Hopf algebra.

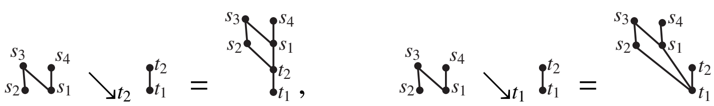
3.2. Twisted pre-Lie algebra of the finite topological spaces. Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ be two finite topological spaces, and let $v \in X_2$. We define:

$$\mathcal{T}_1 \searrow_v \mathcal{T}_2 := (X_1 \sqcup X_2, \leq),$$

where $\leq$ is obtained from $\leq_{\mathcal{T}_1}$ and $\leq_{\mathcal{T}_2}$ as follows: compare any pair in X_2 (resp. X_1) by using $\leq_{\mathcal{T}_2}$ (resp. $\leq_{\mathcal{T}_1}$), and compare any element $y \in X_2$ with any element $x \in X_1$.

To sum up, for any $x, y \in X_1 \sqcup X_2$, $x \leq y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x \in X_2$, $y \in X_1$ and $x \leq_{\mathcal{T}_2} v$.

Example 3.1. 

Proposition 3.1. *Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ be two connected finite topological spaces, and let $v \in X_2$. Then $\mathcal{T}_1 \searrow_v \mathcal{T}_2 := (X_1 \sqcup X_2, \leq)$, is a connected finite topological space.*

Proof. Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ be two connected finite topological spaces, and let $v \in X_2$.
We must show that $\leq$ is a preorder relation on $X_1 \sqcup X_2$:
Reflexivity; Let $x \in X_1 \sqcup X_2$, then $x \in X_1$ or $x \in X_2$.
If $x \in X_1$, we have $x \leq_{\mathcal{T}_1} x$, then $x \leq x$.
If $x \in X_2$, we have $x \leq_{\mathcal{T}_2} x$, then $x \leq x$.
Transitivity; Let $x, y, z \in X_1 \sqcup X_2$ such that $x \leq y$ and $y \leq z$. So we have four possible cases:

- First case; $x, y, z \in X_1$, and ($x \leq_{\mathcal{T}_1} y$ and $y \leq_{\mathcal{T}_1} z$).
  Since $\leq_{\mathcal{T}_1}$ is transitive, then $x \leq_{\mathcal{T}_1} z$, then $x \leq z$.
- Second case; $x, y, z \in X_2$, and ($x \leq_{\mathcal{T}_2} y$ and $y \leq_{\mathcal{T}_2} z$).
  Since $\leq_{\mathcal{T}_2}$ is transitive, then $x \leq_{\mathcal{T}_2} z$, then $x \leq z$.
- Third case; $x, y \in X_2$, $z \in X_1$, and ($x \leq_{\mathcal{T}_2} y$ and $y \leq_{\mathcal{T}_2} v$).
  Since $\leq_{\mathcal{T}_2}$ is transitive, then $x \leq_{\mathcal{T}_2} v$, and since $x \in X_2$ and $z \in X_1$ therefore $x \leq z$.
- Fourth case; $x \in X_2$, $y, z \in X_1$, and ($x \leq_{\mathcal{T}_2} v$ and $y \leq_{\mathcal{T}_1} z$).
  In this case we have $x \in X_2$, $z \in X_1$, and $x \leq_{\mathcal{T}_2} v$, then $x \leq z$.

□

Proposition 3.2. *Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$, $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ and $\mathcal{T}_3 = (X_3, \leq_{\mathcal{T}_3})$ be three finite connected topological spaces, and let $u \in X_2$, $v, w \in X_3$. Then*
1) $(\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_w \mathcal{T}_3)$.
2) $\mathcal{T}_1 \searrow_v (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = \mathcal{T}_2 \searrow_w (\mathcal{T}_1 \searrow_v \mathcal{T}_3)$.

Proof. 1) Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$, $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ and $\mathcal{T}_3 = (X_3, \leq_{\mathcal{T}_3})$ be three finite connected topologies, and let $u \in X_2$, $w \in X_3$. We denote $\mathcal{T}'_3 = (\mathcal{T}_1 \searrow_u \mathcal{T}_2) = (X_1 \sqcup X_2, \leq_3)$, with $\leq_3$ defined on $X_1 \sqcup X_2$ by:
$x, y \in X_1 \sqcup X_2$ et $x \leq_3 y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x \in X_2$, $y \in X_1$ and $x \leq_{\mathcal{T}_2} u$,

and we denote $\mathcal{T} = (\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = (X_1 \sqcup X_2 \sqcup X_3, \leq)$, with $\leq$ defined on $X_1 \sqcup X_2 \sqcup X_3$ by:
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_1 \sqcup X_2$ and $x \leq_3 y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_1 \sqcup X_2$ and $x \leq_{\mathcal{T}_3} w$,

then
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x \in X_2$, $y \in X_1$ and $x \leq_{\mathcal{T}_2} u$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_1 \sqcup X_2$ and $x \leq_{\mathcal{T}_3} w$.

On the other hand,
we denote $\mathcal{T}'_1 = \mathcal{T}_2 \searrow_w \mathcal{T}_3 = (X_2 \sqcup X_3, \leq_1)$, with $\leq_1$ defined on $X_2 \sqcup X_3$ by:
$x, y \in X_2 \sqcup X_3$ et $x \leq_1 y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,

and we denote $\mathcal{T}' = \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = (X_1 \sqcup X_2 \sqcup X_3, \leq')$, with $\leq'$ defined on $X_1 \sqcup X_2 \sqcup X_3$ by:
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2 \sqcup X_3$ and $x \leq_1 y$,
- or $x \in X_2 \sqcup X_3$, $y \in X_1$ and $x \leq_1 u$,

then
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,
- or $x \in X_2$, $y \in X_1$ and $x \leq_{\mathcal{T}_2} u$,
- or $x \in X_3$, $y \in X_1$ and $x \leq_{\mathcal{T}_3} w$,

then $\leq = \leq'$ on $X_1 \sqcup X_2 \sqcup X_3$.
Then

$$(\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_w \mathcal{T}_3).$$

2) Let $v, w \in X_3$, we denote $\mathcal{T}_2' = (\mathcal{T}_1 \searrow_v \mathcal{T}_3) = (X_1 \sqcup X_3, \leq_2)$, with $\leq_2$ defined on $X_1 \sqcup X_3$ by:
$x, y \in X_1 \sqcup X_3$ et $x \leq_2 y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_1$ and $x \leq_{\mathcal{T}_3} v$,

and we denote
$\mathcal{T} = \mathcal{T}_2 \searrow_w (\mathcal{T}_1 \searrow_v \mathcal{T}_3) = (X_1 \sqcup X_2 \sqcup X_3, \leq)$, with $\leq$ defined on $X_1 \sqcup X_2 \sqcup X_3$ by:
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_1 \sqcup X_3$ and $x \leq_2 y$,
- or $x \in X_1 \sqcup X_3$, $y \in X_2$ and $x \leq_2 w$,

then
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_1$ and $x \leq_{\mathcal{T}_3} v$,
- or $x \in X_3$, $y \in X_2$ and $x \leq_{\mathcal{T}_3} w$.

On the other hand,
we denote $\mathcal{T}_1' = (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = (X_2 \sqcup X_3, \leq_1)$, with $\leq_1$ defined on $X_2 \sqcup X_3$ by:
$x, y \in X_2 \sqcup X_3$ et $x \leq_1 y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,

and we denote
$\mathcal{T}' = \mathcal{T}_1 \searrow_v (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = (X_1 \sqcup X_2 \sqcup X_3, \leq')$, with $\leq'$ defined on $X_1 \sqcup X_2 \sqcup X_3$ by:
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2 \sqcup X_3$ and $x \leq_1 y$,
- or $x \in X_2 \sqcup X_3$, $y \in X_1$ and $x \leq_1 v$,

then
$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3$, $y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,
- or $x \in X_3$, $y \in X_1$ and $x \leq_{\mathcal{T}_3} v$,

then $\leq = \leq'$ on $X_1 \sqcup X_2 \sqcup X_3$.
Then

$$\mathcal{T}_1 \searrow_v (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = \mathcal{T}_2 \searrow_w (\mathcal{T}_1 \searrow_v \mathcal{T}_3).$$

□

We then define the grafting law in the species of connected finite topological spaces by:
For all $\mathcal{T}_1 \in \mathbb{T}_{X_1}, \mathcal{T}_2 \in \mathbb{T}_{X_2}$

$$\mathcal{T}_1 \searrow \mathcal{T}_2 = \sum_{v \in X_2} \mathcal{T}_1 \searrow_v \mathcal{T}_2 \in \mathbb{T}_{X_1 \sqcup X_2}.$$

Theorem 3.2. $(\mathbb{V}, \searrow)$ *is a twisted pre-Lie algebra.*

Proof. Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$, $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ and $\mathcal{T}_3 = (X_3, \leq_{\mathcal{T}_3})$ three finite topological spaces, we have:

$$\begin{aligned}
\mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow \mathcal{T}_3) &= \sum_{v \in X_3} \mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow_v \mathcal{T}_3) \\
&= \sum_{u \in X_2 \sqcup X_3} \sum_{v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3) \\
&= \sum_{u \in X_2} \sum_{v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3) \\
&\quad + \sum_{u \in X_3} \sum_{v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3).
\end{aligned}$$

On the other hand we have:

$$\begin{aligned}
(\mathcal{T}_1 \searrow \mathcal{T}_2) \searrow \mathcal{T}_3 &= \sum_{r \in X_2} (\mathcal{T}_1 \searrow_r \mathcal{T}_2) \searrow \mathcal{T}_3 \\
&= \sum_{s \in X_3} \sum_{r \in X_2} (\mathcal{T}_1 \searrow_r \mathcal{T}_2) \searrow_s \mathcal{T}_3.
\end{aligned}$$

Then

$$\mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow \mathcal{T}_3) - (\mathcal{T}_1 \searrow \mathcal{T}_2) \searrow \mathcal{T}_3 = \sum_{u, v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3).$$

Which is symmetric on $\mathcal{T}_1$ and $\mathcal{T}_2$. Then we obtain:

$$\mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow \mathcal{T}_3) - (\mathcal{T}_1 \searrow \mathcal{T}_2) \searrow \mathcal{T}_3 = \mathcal{T}_2 \searrow (\mathcal{T}_1 \searrow \mathcal{T}_3) - (\mathcal{T}_2 \searrow \mathcal{T}_1) \searrow \mathcal{T}_3.$$

Consequently, $(\mathbb{V}, \searrow)$ is a twisted pre-Lie algebra, thus yielding a pre-Lie algebra structure on $\overline{\mathcal{K}}(\mathbb{T})$. □

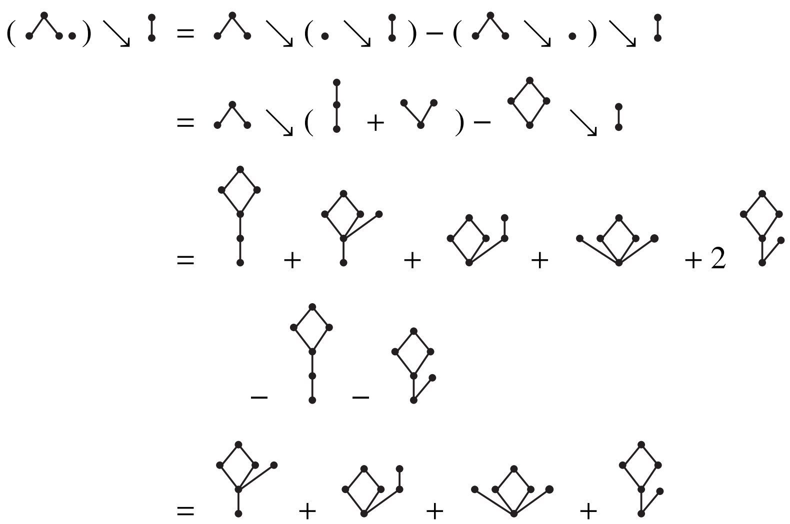
We showed that $(\mathbb{V}, \searrow)$ is a twisted pre-Lie algebra, so we consider the Hopf symmetric algebra $\mathcal{H}' = S(\mathbb{V})$ equipped with its usual unshuffling coproduct Δ_{unsh}. We extend the product $\searrow$ to $\mathbb{T}$ by using Definition 2.3 and we define a product $\star$ on $\mathbb{T}$ by: For any pair X_1, X_2 of finite sets

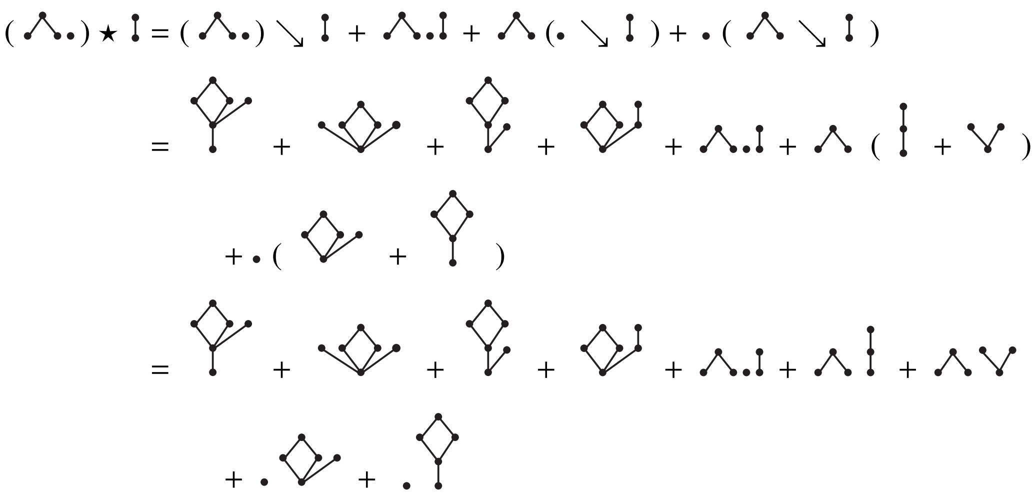
$$\begin{array}{rcl} \star : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} & \longrightarrow & \mathbb{T}_{X_1 \sqcup X_2} \\ (\mathcal{T}_1, \mathcal{T}_2) & \longmapsto & \sum_{\mathcal{T}_1} \mathcal{T}_1^{(1)}(\mathcal{T}_1^{(2)} \searrow \mathcal{T}_2). \end{array}$$

By construction, the space $(\mathcal{H}', \star, \Delta_{unsh})$ is a cocommutative twisted Hopf algebra.

Remark 3.1. *The species of finite connected posets (i.e. finite connected T_0 topological spaces) is a twisted pre-Lie subalgebra of* $(\mathbb{V}, \searrow)$*, and the species of finite posets is a Hopf subalgebra of* $\mathcal{H}'$*.*

Example 3.2.





4. Bialgebras of finite topological spaces

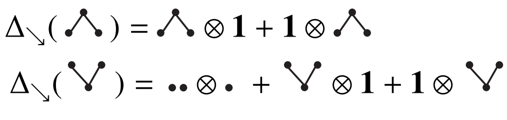
4.1. **A twisted bialgebra of finite topological spaces.** Let X be any finite set, we define the coproduct $\Delta_{\searrow}$ by:

$$\begin{array}{rcl} \Delta_{\searrow} : \mathbb{T}_X & \longrightarrow & (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus\limits_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\ \mathcal{T} & \longmapsto & \sum\limits_{Y \overline{\in} \mathcal{T}} \mathcal{T}_{|Y} \otimes \mathcal{T}_{|X \backslash Y}. \end{array}$$

Where $Y \overline{\in} \mathcal{T}$, stands for

- $Y \in \mathcal{T}$,
- $\mathcal{T}_{|Y} = \mathcal{T}_1 ... \mathcal{T}_n$, such that for all $i \in \{1, ..., n\}, \mathcal{T}_i$ connected and (min$\mathcal{T}_i$ = (min$\mathcal{T}$) $\cap \mathcal{T}_i$, or there is a single common ancestor $x_i \in \overline{X \backslash Y}$ to min$\mathcal{T}_i$), where $\overline{X \backslash Y} = (X \backslash Y)/ \sim_{\mathcal{T}_{|X \backslash Y}}$.

Example 4.1.



Theorem 4.1. *$(\mathbb{T}, m, \Delta_{\searrow})$ is a commutative connected twisted bialgebra, and $\mathcal{H} = \overline{\mathcal{K}}(\mathbb{T})$ is a commutative graded bialgebra.*

Proof. To show that $\mathbb{T}$ is a twisted bialgebra [1], it is necessary to show that $\Delta_{\searrow}$ is coassociative, and that the species coproduct $\Delta_{\searrow}$ and the product defined by:

$$\begin{array}{rl} m : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} \longrightarrow & \mathbb{T}_{X_1 \sqcup X_2} \\ \mathcal{T}_1 \otimes \mathcal{T}_2 \longmapsto & \mathcal{T}_1 \mathcal{T}_2, \end{array}$$

are compatible. The unit **1** is identified to the empty topology. Coassociativity is checked by a careful, but straighforward computation. We have

$$\begin{aligned}(\Delta_{\searrow} \otimes id)\Delta_{\searrow}(\mathfrak{T}) &= (\Delta_{\searrow} \otimes id)\left(\sum_{Y \Subset \mathfrak{T}} \mathfrak{T}_{|Y} \otimes \mathfrak{T}_{|X\backslash Y}\right) \\ &= \sum_{Z \Subset \mathfrak{T}_{|Y},\, Y \Subset \mathfrak{T}} \mathfrak{T}_{|Z} \otimes \mathfrak{T}_{|Y\backslash Z} \otimes \mathfrak{T}_{|X\backslash Y}.\end{aligned}$$

On the other hand

$$\begin{aligned}(id \otimes \Delta_{\searrow})\Delta_{\searrow}(\mathfrak{T}) &= (id \otimes \Delta_{\searrow})\left(\sum_{U \Subset \mathfrak{T}} \mathfrak{T}_{|U} \otimes \mathfrak{T}_{|X\backslash U}\right) \\ &= \sum_{W \Subset \mathfrak{T}_{|X\backslash U},\, U \Subset \mathfrak{T}} \mathfrak{T}_{|U} \otimes \mathfrak{T}_{|W} \otimes \mathfrak{T}_{|X\backslash (U \sqcup W)}.\end{aligned}$$

Coassociativity will come from the fact that $(Z, Y) \longmapsto (Z, Y\backslash Z)$ is a bijection from the set of pairs (Z, Y) with $Y \Subset \mathfrak{T}$ and $Z \Subset \mathfrak{T}_{|Y}$ and, onto the set of pairs (U, W) with $U \Subset \mathfrak{T}$ and $W \Subset \mathfrak{T}_{|X\backslash U}$. The inverse map is given by $(U, W) \longmapsto (U, U \sqcup W)$.

Let $A = \{(Z, Y), Y \Subset \mathfrak{T} \text{ and } Z \Subset \mathfrak{T}_{|Y}\}$, and $B = \{(U, W), U \Subset \mathfrak{T} \text{ and } W \Subset \mathfrak{T}_{|X\backslash U}\}$.
We define

$$\begin{aligned} f : A &\longrightarrow B & \qquad g : B &\longrightarrow A \\ (Z, Y) &\longmapsto (Z, Y\backslash Z) & (U, W) &\longmapsto (U, U \sqcup W)\end{aligned}$$

Let us prove that f and g are well defined.
Let $(Z, Y) \in A$, i.e

- $Y \in \mathfrak{T}$ and $\mathfrak{T}_{|Y} = \mathfrak{T}_1 ... \mathfrak{T}_n$, such that for all $i \in \{1, ..., n\}$, $\mathfrak{T}_i$ connected component and $(\min\mathfrak{T}_i = (\min\mathfrak{T}) \cap \mathfrak{T}_i$ or there is a unique common ancestor $x_i \in \overline{X\backslash Y}$ to $\min\mathfrak{T}_i)$,

and

- $Z \in \mathfrak{T}_{|Y}$ and $\mathfrak{T}_{|Z} = \mathfrak{T}_{1|Z} ... \mathfrak{T}_{n|Z} = \mathfrak{T}_{1,1}\mathfrak{T}_{1,2} ... \mathfrak{T}_{1,i_1}\mathfrak{T}_{2,1}\mathfrak{T}_{2,2} ... \mathfrak{T}_{2,i_2} ... \mathfrak{T}_{n,1}\mathfrak{T}_{n,2} ... \mathfrak{T}_{n,i_n}$, such that for all $i \in \{1, ..., n\}$, $j \in \{i_1, ..., i_n\}$, $\mathfrak{T}_{i,j}$ connected component and $(\min\mathfrak{T}_{i,j} = \min\mathfrak{T}_{|Y} \cap \mathfrak{T}_{i,j}$ or there is a unique common ancestor $x_{i,j} \in \overline{Y\backslash Z}$ to $\min\mathfrak{T}_{i,j})$.

Then we can visualise $\mathfrak{T}$ by the graph illustrated below in figure 1:
Graphically it is clear that $Z \in \mathfrak{T}$ and $Y\backslash Z \in \mathfrak{T}_{|X\backslash Z}$. Then $(Z, Y\backslash Z) \in B$.
Then f is well defined.

Let $(U, W) \in B$, i.e

- $U \in \mathfrak{T}$ and $\mathfrak{T}_{|U} = \mathfrak{T}_{|U_1} ... \mathfrak{T}_{|U_p}$, such that for all $i \in \{1, ..., p\}$, $\mathfrak{T}_{|U_i}$ connected component and $(\min\mathfrak{T}_{|U_i} = (\min\mathfrak{T}) \cap \mathfrak{T}_{|U_i}$ or there is a unique $x_i \in \overline{X\backslash U}$ common ancestor to $\min\mathfrak{T}_{|U_i})$.

and

- $W \in \mathfrak{T}_{|X\backslash U}$ and $\mathfrak{T}_{|W} = \mathfrak{T}^1 ... \mathfrak{T}^q$, such that for all $j \in \{1, ..., q\}$, $\mathfrak{T}^j$ connected component and $(\min\mathfrak{T}^j = \min\mathfrak{T}_{|X\backslash U} \cap \mathfrak{T}^j$ or there is a unique $x_j \in \overline{X\backslash (U \sqcup W)}$ common ancestor to $\min\mathfrak{T}^j)$.

For all $k \in \{1, ..., q\}$, we notice $U^k = \bigsqcup_{0 \le n \le p} U_i$, where U_i verifies the existence of $x_i \in \overline{W_k} = \overline{\mathcal{V}(\mathfrak{T}^k)}$ common ancestor to $\min\mathfrak{T}_{|U_i}$, where $v \in \mathcal{V}(\mathfrak{T}^k)$ denotes that v is a element of the topological space $\mathfrak{T}^k$.

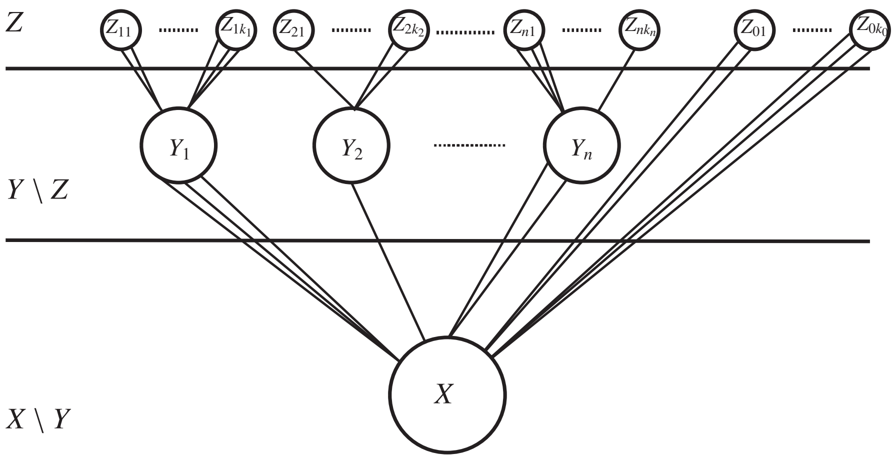


FIGURE 1

We notice $U^0 = \bigsqcup_{0 \leq n \leq p} U_i$, where U_i verifies the existence of a unique $x_i \in \overline{X \backslash (U \sqcup W)}$ common ancestor to $\min \mathcal{T}_{|U_i}$.
We notice $W_0 \subset W$, where W_0 verifie thet for all $x \in W_0$, there is no $y \in U$ such that $x \leq_{\mathcal{T}} y$.
Then we can visualise $\mathcal{T}$ by the graph illustrated below in figure 2 below:
Graphically it is clear that $U \in \mathcal{T}_{|W \sqcup U}$ and $W \sqcup U \in \mathcal{T}$. Then $(U, U \sqcup W) \in A$.

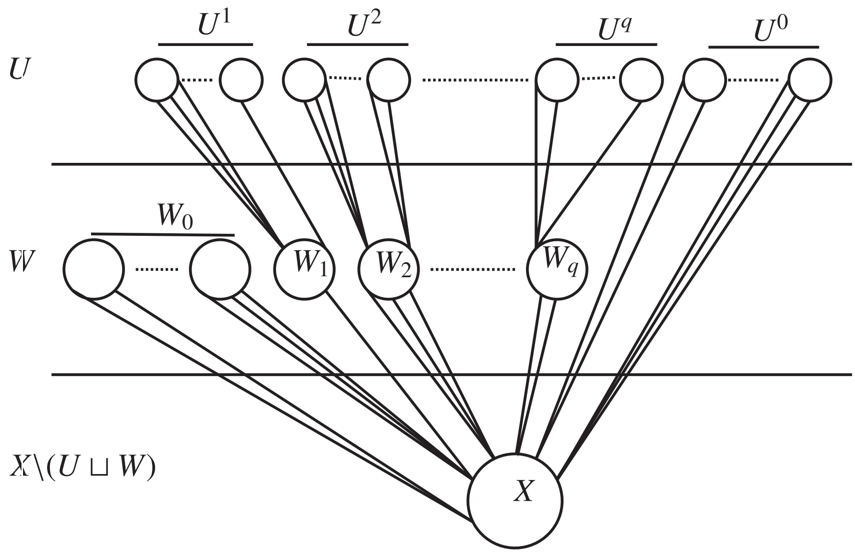


FIGURE 2

Then g is well defined.
We have for all $(Z, Y) \in A$ then $(Z, Y \backslash Z) \in B$, and for all $(U, W) \in B$ then $(U, U \sqcup W) \in A$. Then

$|A| = |B|$.
Let $(Z_1, Y_1), (Z_2, Y_2) \in A$ such that $f(Z_1, Y_1) = f(Z_2, Y_2)$, then $Z_1 = Z_2$, then $Y_1 \backslash Z_1 = Y_2 \backslash Z_1$, then f is injective, then f is bijective.
In the same way we show that g is bijective, and $g \circ f = f \circ g = Id$.
Then Δ is coassociative.
Finally, we show immediately that

$$\Delta_{\searrow}(\mathcal{T}_1\mathcal{T}_2) = \Delta_{\searrow}(\mathcal{T}_1)\Delta_{\searrow}(\mathcal{T}_2).$$

$\square$

Remark 4.1. *For any finite set X, let us recall from [8] the internal coproduct Γ on $\mathbb{T}_X$:*

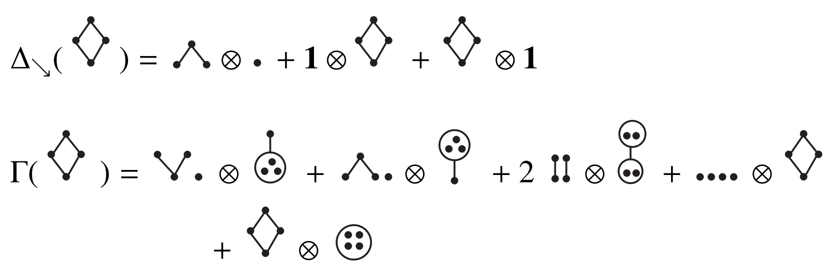
$$\Gamma(\mathcal{T}) = \sum_{\mathcal{T}' \circledcirc \mathcal{T}} \mathcal{T}' \otimes \mathcal{T}/\mathcal{T}'. \tag{4.1}$$

The sum runs over topologies $\mathcal{T}'$ which are $\mathcal{T}$-admissible, i.e

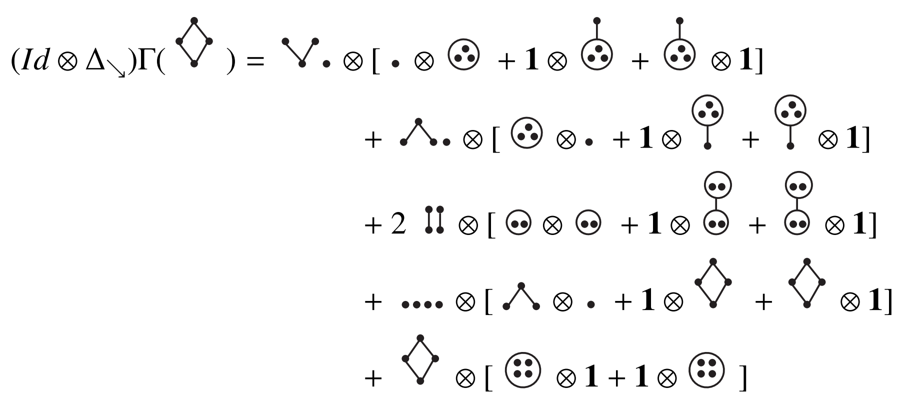
- *finer than $\mathcal{T}$,*
- *such that $\mathcal{T}'_{|Y} = \mathcal{T}_{|Y}$ for any subset $Y \subset X$ connected for the topology $\mathcal{T}'$,*
- *such that for any $x, y \in X$,*

$$x \sim_{\mathcal{T}/\mathcal{T}'} y \iff x \sim_{\mathcal{T}'/\mathcal{T}'} y. \tag{4.2}$$

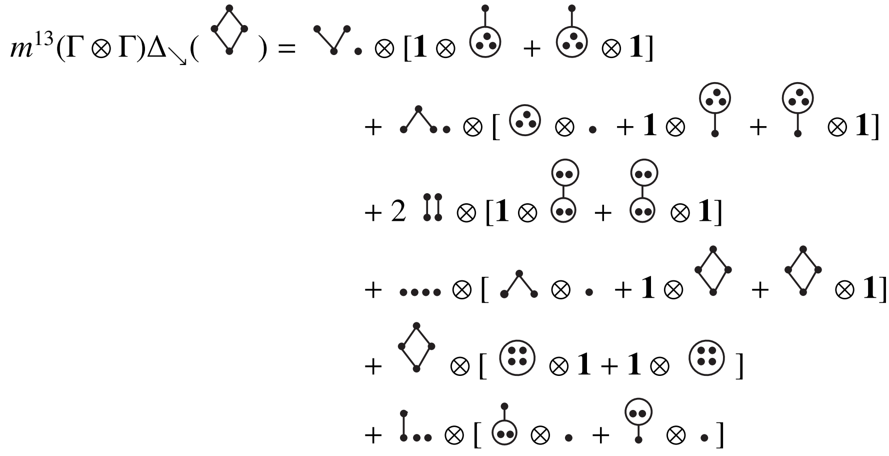
F. Fauvet, L. Foissy, and D. Manchon in [8] show that Γ and Δ are compatible.
On the other hand, we notice that Γ and $\Delta_{\searrow}$ are not compatible. In fact:



then



On the other hand



then $(Id \otimes \Delta_{\searrow})\Gamma($  $) \neq m^{13}(\Gamma \otimes \Gamma)\Delta_{\searrow}($  $)$. *Then* Γ *and* $\Delta_{\searrow}$ *are not compatible.*

4.2. **Relation between $\star$ and $\Delta_{\searrow}$.** In this subsection, we prove that there exist relations between the Grossman-Larson product $\star$ and the coproduct $\Delta_{\searrow}$.
Let G be a group acting on X. For every $x \in X$, we denote by $G \cdot x$ the orbit of x and we denote by G_x the stabilizer subgroup of G with respect to x. The group action is transitive if and only if it has exactly one orbit, that is if there exists x in X with $G \cdot x = X$ (i.e. X is non-empty and if for each pair $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$). This is the case if and only if $G \cdot x = X$, for all x in X.
If G and X is finite, then the orbit-stabilizer theorem, together with Lagrange's theorem [21](theorem 3.9), gives

$$|G \cdot x| = [G_X \,:\, G] = \frac{|G|}{|G_x|}, \tag{4.3}$$

that implies that the cardinal of the orbit is a divisor of the group order.

Definition 4.1. *For any topology $\mathcal{T}$ on a finite set X, we denote by $Aut(\mathcal{T})$ the subgroup of permutations of X which are homeomorphisms with respect to $\mathcal{T}$. The symmetry factor is defined by $\sigma(\mathcal{T}) = |Aut(\mathcal{T})|$. We define the linear map $e_{\mathcal{T}} : \mathbb{T}_X \longrightarrow \mathbb{K}$ by:*

$$e_{\mathcal{T}}(\mathcal{T}') = \sigma(\mathcal{T}), \text{ if } \mathcal{T} = \mathcal{T}', \text{ and } 0 \text{ if not.}$$

Definition 4.2. *We define the graft operator $\mathcal{B} : \mathbb{T} \to \mathbb{T}$ by, $\mathcal{B}(\mathcal{T}) = \mathcal{T} \searrow \{*\}$, for any topology $\mathcal{T}$ on X, this is the topology on $X \sqcup \{*\}$ obtained by keeping the preorder and X and by putting $* < x$ for any $x \in X$.*

Theorem 4.2. *Let $\mathcal{T}_1 \in \mathbb{T}_{X_1}$, $\mathcal{T}_2 \in \mathbb{T}_{X_2}$ and $\mathcal{T}' \in \mathbb{T}_X$, then*

$$< e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' >=< e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') > .$$

Proof. Let $\mathcal{T}_1 \in \mathbb{T}_{X_1}$, $\mathcal{T}_2 \in \mathbb{T}_{X_2}$ and $\mathcal{T}' \in \mathbb{T}_X$.
Case 1; $\mathcal{T}_1$ is connected, we have

$$\begin{aligned} < e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' > &=< e_{\mathcal{T}_1 \searrow \mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}' > \\ &= \sum_{v \in X_2 \sqcup \{*\}} < e_{\mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}' > \\ &= \sum_{\substack{v \in X_2 \sqcup \{*\} \\ \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2}} \sigma(\mathcal{B}\mathcal{T}'). \end{aligned}$$

Let us consider the set $B = \{v \in X_2 \sqcup \{*\}, \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2\}$, we show that $\mathrm{Aut}(\mathcal{B}\mathcal{T}_2)$ acts transitively on B. We define the map

$$\begin{aligned} \Phi_1 : \mathrm{Aut}(\mathcal{B}\mathcal{T}_2) \times B &\longrightarrow B \\ (\varphi, v) &\longmapsto \varphi(v). \end{aligned}$$

Let $v \in B$, if $v = *$, then $\varphi(v) = v$. If not then $\mathcal{T}_1 \searrow_{\varphi(v)} \mathcal{B}\mathcal{T}_2 = \mathcal{T}_1 \searrow_{\varphi(v)} \varphi(\mathcal{B}\mathcal{T}_2) = h(\mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2)$, where $h_{|X_1} = Id$ and if $v \in X_2 \sqcup \{*\}$, $h(v) = \varphi(v)$, where $\varphi \in \mathrm{Aut}(\mathcal{B}\mathcal{T}_2)$. It is clear that $h \in \mathrm{Aut}(\mathcal{B}\mathcal{T}')$, then $\mathcal{T}_1 \searrow_{\varphi(v)} \mathcal{B}\mathcal{T}_2 = h(\mathcal{B}\mathcal{T}') = \mathcal{B}\mathcal{T}'$. Then Φ_1 is well defined.
Moreover for all $v \in B$, $Id(v) = v$, and for all $\varphi, \varphi' \in \mathrm{Aut}(\mathcal{B}\mathcal{T}_2)$, $\Phi_1(\varphi, \varphi'(v)) = \varphi(\varphi'(v)) = (\varphi\varphi')(v)$. Then Φ_1 is an action.
Now to show that Φ_1 is transitive, let $u, v \in B$, and let us define $f : X \sqcup \{*\} \longrightarrow X \sqcup \{*\}$ by $f(u) = v$, $f(v) = u$, and for all $w \in X \sqcup \{*\} \backslash \{u, v\}$, $f(w) = w$, it is clear that $f \in \mathrm{Aut}(\mathcal{B}\mathcal{T}')$.
If we take $\varphi : X_2 \sqcup \{*\} \longrightarrow X_2 \sqcup \{*\}$, defined by $\varphi = f_{|X_2 \sqcup \{*\}}$, so we have, $\varphi \in \mathrm{Aut}(\mathcal{B}\mathcal{T}_2)$, and $\varphi(u) = v$, then Φ_1 is transitive. Then $B = \mathrm{Aut}(\mathcal{B}\mathcal{T}_2) \cdot v$, for all $v \in B$.
For all $v \in B$, we call the stabilizer of v the set:

$$\mathrm{Aut}(\mathcal{B}\mathcal{T}_2)_v = \{\varphi \in \mathrm{Aut}(\mathcal{B}\mathcal{T}_2), \varphi(v) = v\}.$$

And since $|(\mathrm{Aut}(\mathcal{B}\mathcal{T}_2)|$ is finite, then $|B| = |\mathrm{Aut}(\mathcal{B}\mathcal{T}_2) \cdot v| = \dfrac{|\mathrm{Aut}(\mathcal{B}\mathcal{T}_2)|}{|\mathrm{Aut}(\mathcal{B}\mathcal{T}_2)_v|}$, for all $v \in B$.
Then

$$|B| = \frac{|\sigma(\mathcal{T}_2)|}{|\mathrm{Aut}(\mathcal{T}_2)_v|}, \text{ for all } v \in B \backslash \{*\}.$$

On the other hand

$$\begin{aligned} < e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') > &= \sum_{Y \Subset \mathcal{T}'} < e_{\mathcal{T}_1}, \mathcal{T}'_{|Y} >< e_{\mathcal{T}_2}, \mathcal{T}'_{|X \backslash Y} > \\ &= \sum_{\substack{Y \Subset \mathcal{T}' \\ \mathcal{T}_1 = \mathcal{T}'_{|Y},\, \mathcal{T}_2 = \mathcal{T}'_{|X \backslash Y}}} \sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2). \end{aligned}$$

Let us consider the set

$$A = \{v \in X, \text{ the cut above } v \text{ give the term of } \Delta_{\searrow}(\mathcal{T}') \text{ isomorphic to } \mathcal{T}_1 \otimes \mathcal{T}_2\},$$

we notice that $A \cap B \backslash \{*\} \neq \emptyset$.
We show that $\mathrm{Aut}(\mathcal{T}')$ acts transitively on A. We define the map

$$\begin{aligned} \Phi_2 : \mathrm{Aut}(\mathcal{T}') \times A &\longrightarrow A \\ (\varphi, v) &\longmapsto \varphi(v). \end{aligned}$$

Let $v \in A$ then $\mathcal{T}'_{|X_1}$ isomorphic to $\mathcal{T}_1$ and $\mathcal{T}'_{|X_2}$ isomorphic to $\mathcal{T}_2$, then for all $\varphi \in \mathrm{Aut}(\mathcal{T}')$, $\varphi(\mathcal{T}'_{|X_i})$ isomorph to $\mathcal{T}_i$, $i \in \{1,2\}$, and $\Delta_{\searrow}(\varphi(\mathcal{T}'))$ isomorphic to $\Delta_{\searrow}(\mathcal{T}')$,
then for all $\varphi \in \mathrm{Aut}(\mathcal{T}')$, the cut above $\varphi(v)$ give the term of $\Delta_{\searrow}(\mathcal{T}')$ isomorphic to $\mathcal{T}_1 \otimes \mathcal{T}_2$, then $\varphi(v) \in A$. Then Φ_2 is well defined.
Let $v \in A$, and $\varphi, \varphi' \in \mathrm{Aut}(\mathcal{T}')$, then $Id(v) = v$ and $\Phi_2(\varphi, \varphi'(v)) = \varphi(\varphi'(v)) = (\varphi\varphi')(v)$. Then Φ_2 is an action.
Let $u, v \in A$, we defined $f : X \longrightarrow X$ by: $f(u) = v$, $f(v) = u$, and for all $w \notin \{u, v\}$, $f(w) = w$, it is clear that $f \in \mathrm{Aut}(\mathcal{T}')$, then Φ_2 is transitive. And since $|\mathrm{Aut}(\mathcal{T}')|$ is finite, then

$$|A| = \frac{|\mathrm{Aut}(\mathcal{T}')|}{|\mathrm{Aut}(\mathcal{T}')_v|}, \text{ for all } v \in A.$$

Let $v \in A$, then $|\mathrm{Aut}(\mathcal{T}')_v| = |\mathrm{Aut}(\mathcal{T}_1)\mathrm{Aut}(\mathcal{T}_2)_v| = |\mathrm{Aut}(\mathcal{T}_1)||\mathrm{Aut}(\mathcal{T}_2)_v|$. Then

$$|\mathrm{Aut}(\mathcal{T}_2)_v| = \frac{|\mathrm{Aut}(\mathcal{T}')|}{|A||\mathrm{Aut}(\mathcal{T}_1)|}, \text{ for all } v \in A,$$

that since $A \cap B \backslash \{*\} \neq \emptyset$, there exists $v \in A \cap B \backslash \{*\}$ such that

$$|\mathrm{Aut}(\mathcal{T}_2)_v| = \frac{|\mathrm{Aut}(\mathcal{T}')|}{|A||\mathrm{Aut}(\mathcal{T}_1)|} = \frac{|\mathrm{Aut}(\mathcal{T}_2)|}{|B|}$$

then

$$\frac{\sigma(\mathcal{T}')}{|A|\sigma(\mathcal{T}_1)} = \frac{\sigma(\mathcal{T}_2)}{|B|},$$

then

$$|B|\sigma(\mathcal{T}') = |A|\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2).$$

We define $A' = \{Y, Y \overline{\in} \mathcal{T}', \mathcal{T}_1 = \mathcal{T}'_Y, \mathcal{T}_2 = \mathcal{T}'_{|X \backslash Y}\}$, we notice that $|A| = |A'|$. Then

$$\begin{aligned} < e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' > &= \sum_{\substack{v \in X_2 \sqcup \{*\} \\ \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2}} \sigma(\mathcal{B}\mathcal{T}') \\ &= |B|\sigma(\mathcal{B}\mathcal{T}') \\ &= |B|\sigma(\mathcal{T}') \\ &= |A'|\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2) \\ &= \sum_{\substack{Y \overline{\in} \mathcal{T}' \\ \mathcal{T}_1 = \mathcal{T}'_{|Y}, \mathcal{T}_2 = \mathcal{T}'_{|X \backslash Y}}} \sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2) \\ &=< e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') > . \end{aligned}$$

Case2; $\mathcal{T}_1$ not connected.
Let $\mathcal{T}_1 = \mathcal{T}_{1,1}...\mathcal{T}_{1,n} \in \mathbb{T}_{X_1}$, where $\mathcal{T}_{1,i}$ is connected for all $i \in [n]$. And let $\mathcal{T}_2 \in \mathbb{T}_{X_2}$, $\mathcal{T}' \in \mathbb{T}_X$, we

have

$$\begin{aligned}
< e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' > &=< e_{\mathcal{T}_1 \searrow \mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}' > \\
&= \sum_{\underline{v}=(v_1,\dots,v_n)\in X_2} < e_{\mathcal{T}_1 \searrow_{\underline{v}} \mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}' > \\
&= \sum_{\substack{\underline{v}=(v_1,\dots,v_n)\in X_2 \\ \mathcal{B}\mathcal{T}'=\mathcal{T}_{1,1}\searrow_{v_1}(\mathcal{T}_{1,2}\searrow_{v_2}\dots(\mathcal{T}_{1,n}\searrow_{v_n}\mathcal{B}\mathcal{T}_2)\dots)}} \sigma(\mathcal{B}\mathcal{T}').
\end{aligned}$$

Let us consider the set $\underline{B} = \{\underline{v} = (v_1, \dots, v_n) \in X_2 \sqcup \{*\}, \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_{\underline{v}} \mathcal{B}\mathcal{T}_2\}$.
$\mathrm{Aut}(\mathcal{T}_2)$ acts transitively on $\underline{B}$ by the action. (In the same way that we used to show that Φ_1 is a transitive action.)

$$\begin{aligned}
\Phi_3 : \mathrm{Aut}(\mathcal{T}_2) \times \underline{B} &\longrightarrow \underline{B} \\
(\varphi, \underline{v}) &\longmapsto \varphi(\underline{v}).
\end{aligned}$$

And since $|\mathrm{Aut}(\mathcal{T}_2)|$ is finite, then $|\underline{B}| = \dfrac{\sigma(\mathcal{T}_2)}{|\mathrm{Aut}(\mathcal{T}_2)_{\underline{v}}|}$, where $\underline{v} \in \underline{B}$.
On the other hand

$$\begin{aligned}
< e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') > &= \sum_{Y \in \mathcal{T}'} < e_{\mathcal{T}_1}, \mathcal{T}'_{|Y} >< e_{\mathcal{T}_2}, \mathcal{T}'_{|X\setminus Y} > \\
&= \sum_{\substack{Y\in\mathcal{T}' \\ \mathcal{T}_1=\mathcal{T}'_{|Y},\, \mathcal{T}_2=\mathcal{T}'_{|X\setminus Y}}} \sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2).
\end{aligned}$$

Let us consider the set
$\underline{A} = \{\underline{v} = (v_1, \dots, v_n) \in X,$ the cut above $\underline{v}$ give the term of $\Delta_{\searrow}(\mathcal{T}')$ isomorphic to $\mathcal{T}_1 \otimes \mathcal{T}_2\}$.
We notice that $\underline{A} \cap \underline{B}_{|X_2} \neq \emptyset$.
$\mathrm{Aut}(\mathcal{T}')$ acts transitively on $\underline{A}$ by the action. (In the same way that we used to show that Φ_2 is a transitive action.)

$$\begin{aligned}
\Phi_4 : \mathrm{Aut}(\mathcal{T}') \times \underline{A} &\longrightarrow \underline{A} \\
(\varphi, \underline{v}) &\longmapsto \varphi(\underline{v}).
\end{aligned}$$

And since $|\mathrm{Aut}(\mathcal{T}')|$ is finite, then $|\underline{A}| = \dfrac{\sigma(\mathcal{T}')}{|\mathrm{Aut}(\mathcal{T}')_{\underline{v}}|}$, where $\underline{v} \in \underline{A}$.
If $\underline{v} \in \underline{A}$, then $|\mathrm{Aut}(\mathcal{T}')_{\underline{v}}| = |\mathrm{Aut}(\mathcal{T}_1)\mathrm{Aut}(\mathcal{T}_2)_{\underline{v}}| = |\mathrm{Aut}(\mathcal{T}_1)||\mathrm{Aut}(\mathcal{T}_2)_{\underline{v}}|$,
then $|\mathrm{Aut}(\mathcal{T}_2)_{\underline{v}}| = \dfrac{|\mathrm{Aut}(\mathcal{T}')|}{|\underline{A}||\mathrm{Aut}(\mathcal{T}_1)|}$ for all $v \in \underline{A}$,
then

$$|\mathrm{Aut}(\mathcal{T}_2)_{\underline{v}}| = \frac{|\mathrm{Aut}(\mathcal{T}')|}{|\underline{A}||\mathrm{Aut}(\mathcal{T}_1)|} = \frac{|\mathrm{Aut}(\mathcal{T}_2)|}{|\underline{B}|}, \text{ for all } v \in \underline{A} \cap \underline{B} \cap X_2,$$

then $\dfrac{\sigma(\mathcal{T}')}{|\underline{A}|\sigma(\mathcal{T}_1)} = \dfrac{\sigma(\mathcal{T}_2)}{|\underline{B}|}$. Then $|\underline{B}|\sigma(\mathcal{T}') = |\underline{A}|\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2)$.

We define $\underline{A}' = \{Y, Y \bar{\in} \mathcal{T}', \mathcal{T}_1 = \mathcal{T}'_{|Y}$ and $\mathcal{T}_2 = \mathcal{T}'_{|X\setminus Y}\}$, we notice that $|\underline{A}| = |\underline{A}'|$. Then

$$\begin{aligned}
< e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' > &=< e_{\mathcal{T}_1 \searrow_{\mathcal{B}} \mathcal{T}_2}, \mathcal{B}\mathcal{T}' > \\
&= \sum_{\underline{v}=(v_1,\dots,v_n)\in X_2} < e_{\mathcal{T}_1 \searrow_{\underline{v}} \mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}' > \\
&= \sum_{\substack{\underline{v}=(v_1,\dots,v_n)\in X_2 \\ \mathcal{B}\mathcal{T}'=\mathcal{T}_{1,1}\searrow_{v_1}(\mathcal{T}_{1,2}\searrow_{v_2}\dots(\mathcal{T}_{1,n}\searrow_{v_n}\mathcal{B}\mathcal{T}_2)\dots)}} \sigma(\mathcal{B}\mathcal{T}') \\
&= |\underline{B}|\sigma(\mathcal{T}') \\
&= |\underline{A}'|\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2) \\
&= \sum_{\substack{Y \bar{\in} \mathcal{T}' \\ \mathcal{T}_1=\mathcal{T}'_{|Y},\, \mathcal{T}_2=\mathcal{T}'_{|X\setminus Y}}} \sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2) \\
&= \sum_{Y \bar{\in} \mathcal{T}'} < e_{\mathcal{T}_1}, \mathcal{T}'_{|Y} >< e_{\mathcal{T}_2}, \mathcal{T}'_{|X\setminus Y} > \\
&=< e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') > .
\end{aligned}$$

□

5. Relation between $\searrow$ and $\nearrow$

In this part we define the law $\nearrow$ on $\mathbb{V}$ by: For all $\mathcal{T} = (X, \leq_{\mathcal{T}})$ and $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ be two finite connected topological spaces,

$$\mathcal{T} \nearrow \mathcal{S} := j(j(\mathcal{T}) \searrow j(\mathcal{S})),$$

where j is the involution which transforms $\leq$ into $\geq$. In other words, open subsets in $\mathcal{T}$ are closed subsets in $j(\mathcal{T})$ and vice-versa. In particular, it is obvious that $(\mathbb{V}, \nearrow)$ is a twisted pre-Lie algebra due to the fact that $(\mathbb{V}, \searrow)$ is a twisted pre-Lie algebra.

Definition 5.1. *For any finite set A and for any pair of parts A_1, A_2 of A with $A_1 \cap A_2 = \emptyset$, we define $\Psi_{A_1,A_2} : \mathbb{T}_A \to \mathbb{T}_A$, as follows: for any topology $\mathcal{T} \in \mathbb{T}_A$, $\Psi_{A_1,A_2}(\mathcal{T}) = (A, \leq)$, where $\leq$ defined by*

- *If $a \in A_1$, and $b \in A_2$ then a and b are uncomparable,*
- *otherwise, we have $a \leq b$ if and only if $a \leq_{\mathcal{T}} b$.*

Proposition 5.1. *For any finite set A and for any pair of parts A_1, A_2 of A with $A_1 \cap A_2 = \emptyset$, and let $\mathcal{T} \in \mathbb{T}_A$ then*
1) $\Psi_{A_1,A_2}(\mathcal{T}) = (A, \leq)$ is a finite topological space.
2) Ψ_{A_1,A_2} is a projector.

Proof. 1) Let $A = A_1 \sqcup A_2$, with $A_1 \cap A_2 = \emptyset$, and let $\mathcal{T} \in \mathbb{T}_A$, we must show that $\leq$ is a preorder relation on A:
Reflexivity; Let $x \in A$, then $x \in A_1$ or $x \in A_2$.
If $x \in A_1$, we have $x \leq_{\mathcal{T}} x$ then $x \leq x$, same thing if $x \in A_2$.
Transitivity; Let $x, y, z \in A$ such that $x \leq y$ and $y \leq z$. So we have two possible cases:

- First case; $x, y, z \in A_1$, and ($x \leq y$ and $y \leq z$), then ($x \leq_{\mathcal{T}} y$ and $y \leq_{\mathcal{T}} z$).
  Since $\leq_{\mathcal{T}}$ is transitive, then $x \leq_{\mathcal{T}} z$, then $x \leq z$.
- Second case; $x, y, z \in A_2$, likewise the first case.

2) Let $\mathcal{T} \in \mathbb{T}_A$, we must show that $\Psi_{A_1,A_2}(\mathcal{T}) = \Psi^2_{A_1,A_2}(\mathcal{T})$.
If not $\mathcal{T}' = (A, \leq') = \Psi_{A_1,A_2}(\mathcal{T})$, then $\Psi^2_{A_1,A_2}(\mathcal{T}) = \Psi_{A_1,A_2}(\mathcal{T}') = (A, \leq)$, where $\leq$ defined by

- If $a \in A_1$, and $b \in A_2$ then a and b are uncomparable,
- otherwise, we have $a \leq b$ if and only if $a \leq' b$.

And since we have, $a \leq' b$ if and only if $a \leq_{\mathcal{T}} b$, then $\leq$ defined by

- If $a \in A_1$, and $b \in A_2$ then a and b are uncomparable,
- otherwise, we have $a \leq b$ if and only if $a \leq_{\mathcal{T}} b$.

Then $\Psi_{A_1,A_2} = \Psi^2_{A_1,A_2}$. □

Theorem 5.1. *Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, and let $s \in Y$, $u \in Z$. The following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{V}_X \otimes \mathbb{V}_Y \otimes \mathbb{V}_Z \arrow[r, "id\otimes\searrow_u"] \arrow[d, "\nearrow^s\otimes id"'] & \mathbb{V}_X \otimes \mathbb{V}_{Y\sqcup Z} \arrow[dd, "\nearrow^s"] \\
\mathbb{V}_{X\sqcup Y} \otimes \mathbb{V}_Z \arrow[d, "\searrow_u"'] & \\
\mathbb{V}_{X\sqcup Y\sqcup Z} \arrow[r, "\Psi_{X,Z}"'] & \mathbb{V}_{X\sqcup Y\sqcup Z}
\end{tikzcd}$$

Proof. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, and let $s \in Y$, $u \in Z$, then for $\mathcal{W} = (X \sqcup Y \sqcup Z, \leq_{\mathcal{W}}) = (\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}$, we have

- for all $x \in X$, $x \leq_{\mathcal{W}} s$,
- and for all $y \in X \sqcup Y$, $u \leq_{\mathcal{W}} y$,

then

- for all $x \in X$, $x \leq_{\Psi_{X,Z}(\mathcal{W})} s$,
- and for all $y \in Y$, $u \leq_{\Psi_{X,Z}(\mathcal{W})} y$,

then $\Psi_{X,Z}(\mathcal{W})$ is connected.
moreover we have $\mathcal{T} \nearrow^s \mathcal{S} = (X \sqcup Y, \leq_1^{'})$, with $\leq_1^{'}$ defined on $X \sqcup Y$ by:
$x, y \in X \sqcup Y$ et $x \leq_1^{'} y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y$ and $x \leq_{\mathcal{S}} y$,
- or $x \in X$, $y \in Y$ and $s \leq_{\mathcal{S}} y$,

then $(\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U} = (X \sqcup Y \sqcup Z, \leq_{\mathcal{W}})$, with $\leq_{\mathcal{W}}$ defined on $X \sqcup Y \sqcup Z$ by:
$x, y \in X \sqcup Y \sqcup Z$ et $x \leq_{\mathcal{W}} y$ if and only if:

- Either $x, y \in X \sqcup Y$ and $x \leq_1' y$,
- or $x, y \in Z$ and $x \leq_{\mathcal{U}} y$,
- or $x \in Z$, $y \in X \sqcup Y$ and $x \leq_{\mathcal{U}} u$,

then $x, y \in X \sqcup Y \sqcup Z$ et $x \leq_{\mathcal{W}} y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y$ and $x \leq_{\mathcal{S}} y$,
- or $x \in X$, $y \in Y$ and $s \leq_{\mathcal{S}} y$,
- or $x, y \in Z$ and $x \leq_{\mathcal{U}} y$,
- or $x \in Z$, $y \in Y$ and $x \leq_{\mathcal{U}} u$,
- or $x \in Z$, $y \in X$ and $x \leq_{\mathcal{U}} u$,

if we apply $\Psi_{X,Z}$ to $(\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}$, we can eliminate the cases: $x \in Z$, $y \in X$ and $x \leq_{\mathcal{U}} u$.
On the other hond
$\mathcal{S} \searrow_u \mathcal{U} = (Y \sqcup U, \leq_1)$, with $\leq_1$ defined on $Y \sqcup Z$ by:
$x, y \in Y \sqcup Z$ et $x \leq_1 y$ if and only if:

- Either $x, y \in Y$ and $x \leq_{\mathcal{S}} y$,
- or $x, y \in Z$ and $x \leq_{\mathcal{U}} y$,
- or $x \in Z$, $y \in Y$ and $x \leq_{\mathcal{U}} u$,

then $\mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) = (X \sqcup Y \sqcup Z, \leq)$, with $\leq$ defined on $X \sqcup Y \sqcup Z$ by:
$x, y \in X \sqcup Y \sqcup Z$ et $x \leq y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y \sqcup Z$ and $x \leq_1 y$,
- or $x \in X$, $y \in Y \sqcup Z$ and $s \leq_1 y$,

then $x, y \in X \sqcup Y \sqcup Z$ et $x \leq y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y$ and $x \leq_{\mathcal{S}} y$,
- or $x, y \in Z$ and $x \leq_{\mathcal{U}} y$,
- or $x \in Z$, $y \in Y$ and $x \leq_{\mathcal{U}} u$,
- or $x \in X$, $y \in Y$ and $s \leq_{\mathcal{S}} y$.

Then the equality between

$$\mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) = \Psi_{X,Z}((\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}).$$

□

Corollary 5.1. *Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, and let $s \in Y$, $u \in Z$. Then*

$$\mathcal{T} \searrow_s (\mathcal{S} \nearrow^u \mathcal{U}) = \Psi_{X,Z}((\mathcal{T} \searrow_s \mathcal{S}) \nearrow^u \mathcal{U}).$$

i.e, the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{V}_X \otimes \mathbb{V}_Y \otimes \mathbb{V}_Z \arrow[r, "id \otimes \nearrow^u"] \arrow[d, "\searrow_s \otimes id"'] & \mathbb{V}_X \otimes \mathbb{V}_{Y \sqcup Z} \arrow[dd, "\searrow_s"] \\
\mathbb{V}_{X \sqcup Y} \otimes \mathbb{V}_Z \arrow[d, "\nearrow^u"'] & \\
\mathbb{V}_{X \sqcup Y \sqcup Z} \arrow[r, "\Psi_{X,Z}"'] & \mathbb{V}_{X \sqcup Y \sqcup Z}
\end{tikzcd}$$

Proof. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, and let $s \in Y$, $u \in Z$.
We notice $\mathcal{T}' = j(\mathcal{T})$, $\mathcal{S}' = j(\mathcal{S})$ and $\mathcal{U}' = j(\mathcal{U})$, according to theorem 5.1, we have:

$$\mathcal{T}' \nearrow^s (\mathcal{S}' \searrow_u \mathcal{U}') = \Psi_{X,Z}((\mathcal{T}' \nearrow^s \mathcal{S}') \searrow_u \mathcal{U}'),$$

then $\quad j(\mathcal{T}' \nearrow^s (\mathcal{S}' \searrow_u \mathcal{U}')) = j[\Psi_{X,Z}((\mathcal{T}' \nearrow^s \mathcal{S}') \searrow_u \mathcal{U}')]$,
then $\quad j(\mathcal{T}') \searrow_s j((\mathcal{S}' \searrow_u \mathcal{U}')) = \Psi_{X,Z}[j((\mathcal{T}' \nearrow^s \mathcal{S}')) \nearrow^u j(\mathcal{U}')]$,
then $\quad j(\mathcal{T}') \searrow_s (j(\mathcal{S}') \nearrow^u j(\mathcal{U}')) = \Psi_{X,Z}[(j(\mathcal{T}') \searrow_s j(\mathcal{S}')) \nearrow^u j(\mathcal{U}')]$.
Then

$$\mathcal{T} \searrow_s (\mathcal{S} \nearrow^u \mathcal{U}) = \Psi_{X,Z}((\mathcal{T} \searrow_s \mathcal{S}) \nearrow^u \mathcal{U}).$$

□

Proposition 5.2. *Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, then*

$$\mathcal{T} \nearrow (\mathcal{S} \searrow \mathcal{U}) - \Psi_{X,Z}((\mathcal{T} \nearrow \mathcal{S}) \searrow \mathcal{U}) = \mathcal{S} \searrow (\mathcal{T} \nearrow \mathcal{U}) - \Psi_{Y,Z}((\mathcal{S} \searrow \mathcal{T}) \nearrow \mathcal{U}).$$

Proof. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, then

$$\begin{aligned}
\mathcal{T} \nearrow (\mathcal{S} \searrow \mathcal{U}) - \Psi_{X,Z}((\mathcal{T} \nearrow \mathcal{S}) \searrow \mathcal{U}) &= \sum_{u\in Z,\, s\in Y\sqcup Z} \mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) - \sum_{u\in Z,\, s\in Y} \Psi_{X,Z}((\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}) \\
&= \sum_{u,s\in Z} \mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) + \sum_{u\in Z,\, s\in Y} [\mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) \\
&\qquad - \Psi_{X,Z}((\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U})] \\
&= \sum_{u,s\in Z} \mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) \\
&= \sum_{u,s\in Z} \mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) \\
&= \sum_{u,s\in Z} \mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) + \sum_{s\in Z,\, u\in X} [\mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) \\
&\qquad - \Psi_{Y,Z}((\mathcal{S} \searrow_u \mathcal{T}) \nearrow^s \mathcal{U})] \\
&= \sum_{s\in Z,\, u\in X\sqcup Z} \mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) - \sum_{u\in X,\, s\in Z} \Psi_{Y,Z}((\mathcal{S} \searrow_u \mathcal{T}) \nearrow^s \mathcal{U}) \\
&= \mathcal{S} \searrow (\mathcal{T} \nearrow \mathcal{U}) - \Psi_{Y,Z}((\mathcal{S} \searrow \mathcal{T}) \nearrow \mathcal{U}).
\end{aligned}$$

□

Acknowledgements: I would like to thank my Ph.D. thesis supervisors: Professor Dominique Manchon and Professor Ali Baklouti for their advice, guidance and constant support.

Funding: This work was supported by the University of Sfax, Tunisia.

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