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TWISTED PRE-LIE ALGEBRAS OF FINITE TOPOLOGICAL SPACES

MOHAMED AYADI

ABSTRACT. In this paper, we first study the species of finite topological spaces recently considered by F. Fauvet, L. Foissy, and D. Manchon. Then, we construct a twisted pre-Lie structure on the species of connected finite topological spaces. The underlying pre-Lie structure defines a coproduct on the species of finite topological spaces different from those already defined by the Authors above. In the end, we illustrate the link between the Grossman-Larson product and the proposed coproduct.

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1. INTRODUCTION

A finite topological space is a finite set E endowed with a preorder \leq . The study of finite topological spaces was initiated by Alexandroff in 1937 [2], and revived at several periods since then, using the following well-known bijection [7, 10]. Any topology \mathcal{T} on X defines a quasi-order (i.e. a reflexive transitive relation) denoted by $\leq_{\mathcal{T}}$ on X :

$$(1.1) \quad x \leq_{\mathcal{T}} y \iff \text{any open subset containing } x \text{ also contains } y.$$

Conversely, any quasi-order \leq on X defines a topology \mathcal{T}_{\leq} given by its upper ideals, i.e. subsets $Y \subset X$ such that $(y \in Y \text{ and } y \leq z) \implies z \in Y$. Both operations are inverse to each other:

$$(1.2) \quad \leq_{\mathcal{T}_{\leq}} = \leq, \quad \mathcal{T}_{\leq_{\mathcal{T}}} = \mathcal{T}.$$

Hence there is a natural bijection between topologies and quasi-orders on a finite set X . Any quasi-order (hence any topology \mathcal{T}) on X gives rise to an equivalence relation:

$$(1.3) \quad x \sim_{\mathcal{T}} y \iff (x \leq_{\mathcal{T}} y \text{ and } y \leq_{\mathcal{T}} x).$$

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Let \mathcal{T} and \mathcal{T}' be two topologies on a finite set X . We say that \mathcal{T}' is finer than \mathcal{T} , and we write $\mathcal{T}' < \mathcal{T}$, when any open subset for \mathcal{T} is an open subset for \mathcal{T}' . This is equivalent to the fact that for any $x, y \in X$, $x \leq_{\mathcal{T}'} y \Rightarrow x \leq_{\mathcal{T}} y$.

The *quotient* \mathcal{T}/\mathcal{T}' of two topologies \mathcal{T} and \mathcal{T}' with $\mathcal{T}' < \mathcal{T}$ is defined as follows ([8, Paragraph 2.2]): The associated quasi-order $\leq_{\mathcal{T}/\mathcal{T}'}$ is the transitive closure of the relation \mathcal{R} defined by:

$$(1.4) \quad x\mathcal{R}y \iff (x \leq_{\mathcal{T}} y \text{ or } y \leq_{\mathcal{T}'} x).$$

More on finite topological spaces can be found in [1, 3, 5, 15, 20].

Recall that a linear (tensor) species is a contravariant functor from the category of finite sets **Fin** with bijections into the category **Vect** of vector spaces (on some field \mathbf{k}). The tensor product of two species \mathbb{E} and \mathbb{F} is given by

$$(1.5) \quad (\mathbb{E} \otimes \mathbb{F})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{E}_Y \otimes \mathbb{F}_Z,$$

where the notation \sqcup stands for disjoint union. The unit of the tensor product denoted by $\mathbf{1}$ is defined by $\mathbf{1}_{\emptyset} = \mathbf{k}$ and $\mathbf{1}_X = \{0\}$, if $X \neq \emptyset$.

We write $x \in \mathbb{E}$ if there exists a finite set X such that $x \in \mathbb{E}_X$.

A twisted algebra [12] is an algebra in the linear symmetric monoidal category of linear species. See [4, 18, 19] for further details on and references to Joyal's theory of twisted algebras. Concretely, a twisted algebra is a linear species \mathbb{E} provided with a product map (which is a map of linear species: $\mathbb{E} \otimes \mathbb{E} \rightarrow \mathbb{E}$). Associative algebras, commutative algebras, Lie algebras, pre-Lie algebras and so on, are defined accordingly.

The species \mathbb{T} of finite topological spaces is defined as follows: For any finite set X , \mathbb{T}_X is the vector space freely generated by the topologies on X . For any bijection $\varphi : X \rightarrow X'$, the isomorphism $\mathbb{T}_{\varphi} : \mathbb{T}_{X'} \rightarrow \mathbb{T}_X$ is defined by the obvious relabelling:

$$\mathbb{T}_{\varphi}(\mathcal{T}) = \{\varphi^{-1}(Y), Y \in \mathcal{T}\},$$

for any topology \mathcal{T} on X' .

A unital associative algebra ([8, Paragraph 2.3]) on the species of finite topologies is defined as follows: for any pair X_1, X_2 of finite sets we introduce

$$m : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} \longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\ \mathcal{T}_1 \otimes \mathcal{T}_2 \longmapsto \mathcal{T}_1 \mathcal{T}_2,$$

where $\mathcal{T}_1 \mathcal{T}_2$ is the disjoint union topology characterised by $Y \in \mathcal{T}_1 \mathcal{T}_2$ if and only if $Y \cap X_1 \in \mathcal{T}_1$ and $Y \cap X_2 \in \mathcal{T}_2$. The unit is given by the unique topology on the empty set.

For any topology \mathcal{T} on a finite set X and for any subset $Y \subset X$, we denote by $\mathcal{T}|_Y$ the restriction of \mathcal{T} to Y . It is defined by:

$$\mathcal{T}|_Y = \{Z \cap Y, Z \in \mathcal{T}\}.$$

The external coproduct Δ on \mathbb{T} is defined as follows:

$$\begin{aligned} \Delta : \mathbb{T}_X &\longrightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\ \mathcal{T} &\longmapsto \sum_{Y \in \mathcal{T}} \mathcal{T}_{|X \setminus Y} \otimes \mathcal{T}_{|Y}. \end{aligned}$$

The species \mathbb{T} is this way endowed with a twisted bialgebra structure in [8].

Now consider the graded vector space:

$$(1.6) \quad \mathcal{H} = \overline{\mathcal{K}}(\mathbb{T}) = \bigoplus_{n \geq 0} \mathcal{H}_n$$

where $\mathcal{H}_0 = \mathbf{k}.1$, and where \mathcal{H}_n is the linear span of topologies on $\{1, \dots, n\}$ when $n \geq 1$, modulo the action of the symmetric group S_n . The vector space \mathcal{H} can be seen as the quotient of the species \mathbb{T} by the "forget the labels" equivalence relation: $\mathcal{T} \sim \mathcal{T}'$ if \mathcal{T} (resp. \mathcal{T}') is a topology on a finite set X (resp. X'), such that there is a bijection from X onto X' which is a homeomorphism with respect to both topologies. The functor $\overline{\mathcal{K}}$ from linear species to graded vector spaces thus obtained is intensively studied in ([1, chapter 15]) under the name "bosonic Fock functor". The twisted Hopf algebra structure on \mathbb{T} [8] naturally leads to the following:

(\mathcal{H}, m, Δ) is a commutative connected Hopf algebra, graded by the number of elements.

L. Foissy, C. Malvenuto and F. Patras in [9, section 6] were the first to prove that the finite topological spaces can be organized in a graded commutative Hopf algebra. The latter can be recovered by applying the $\overline{\mathcal{K}}$ functor to the twisted Hopf algebra structure on \mathbb{T} described in [8]. The coproduct Δ defined therein is however not built from a pre-Lie structure. We define in the present work two twisted pre-Lie structures \searrow and \nearrow on the species of connected finite topological spaces, giving rise to two more coproducts Δ_{\searrow} and Δ_{\nearrow} , hence two more twisted Hopf algebra structures. We expect that this will contribute to a better understanding of the finite topological spaces considered as a whole.

In section 2, we recall the method of D. Guin and J.-M. Oudom [16] to describe the enveloping algebra of a pre-Lie algebra, and we adapt it to the twisted context, following indications in [22].

In Section 3 of this paper, we define the enveloping algebra of the grafting twisted pre-Lie algebra of connected finite topological spaces, as well as its enveloping algebra using the Guin-Oudom method. Denoting by \mathbb{V} the species of connected finite topological spaces, we consider the Hopf symmetric algebra $\mathcal{H}' = S(\mathbb{V})$ of the pre-Lie twisted algebra (\mathbb{V}, \searrow) , equipped with its usual unshuffling coproduct Δ_{unsh} and a product \star defined on \mathbb{T} by: For any pair X_1, X_2 of finite sets

$$\begin{aligned} \star : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\ (\mathcal{T}_1, \mathcal{T}_2) &\longmapsto \sum_{(\mathcal{J}_1)} \mathcal{T}_1^{(1)}(\mathcal{T}_1^{(2)} \searrow \mathcal{T}_2). \end{aligned}$$

In section 4 we prove that there exists a twisted bialgebra structure on \mathbb{T} , where the external coproduct is defined by

$$\begin{aligned} \Delta_{\searrow} : \mathbb{T}_X &\longrightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\ \mathcal{T} &\longmapsto \sum_{Y \in \mathcal{T}} \mathcal{T}_{|Y} \otimes \mathcal{T}_{|X \setminus Y} \end{aligned}$$

where $Y \in \mathcal{T}$ stands for

- $Y \in \mathcal{T}$,
- $\mathcal{T}_{|Y} = \mathcal{T}_1 \dots \mathcal{T}_n$, such that for all $i \in \{1, \dots, n\}$, \mathcal{T}_i connected and ($\min \mathcal{T}_i = (\min \mathcal{T}) \cap \mathcal{T}_i$, or there is a single common ancestor $x_i \in \overline{X \setminus Y}$ to $\min \mathcal{T}_i$), where $\overline{X \setminus Y} = (X \setminus Y) / \sim_{\mathcal{T}_{|X \setminus Y}}$.

We moreover give a relation between the two structures Δ_{\searrow} and \star .

Finally we define in section 5 a new pre-Lie law \nearrow on the species of connected finite topological spaces by: For all $\mathcal{T} = (X, \leq_{\mathcal{T}})$ and $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ be two finite topological spaces,

$$\mathcal{T} \nearrow \mathcal{S} := j(j(\mathcal{T}) \searrow j(\mathcal{S})),$$

where j is the involution which transforms \leq into \geq . This law \nearrow gives rise to a coproduct denoted Δ_{\nearrow} defined by $\Delta_{\nearrow} = (j \otimes j) \Delta_{\searrow} \circ j$.

For any finite set A and for any pair of parts A_1, A_2 of A with $A_1 \cap A_2 = \emptyset$, we define

$$\Psi_{A_1, A_2} : \mathbb{T}_A \rightarrow \mathbb{T}_A,$$

as follows: for any topology $\mathcal{T} \in \mathbb{T}_A$, the topology $\Psi_{A_1, A_2}(\mathcal{T})$ is associated with the following pre-order \leq defined by:

- If $a \in A_1$, and $b \in A_2$ then a and b are incomparable,
- If not, we have: $a \leq b$ if and only if $a \leq_{\mathcal{T}} b$.

In this section, we provide a relation between both pre-Lie structures, by proving that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{V}_X \otimes \mathbb{V}_Y \otimes \mathbb{V}_Z & \xrightarrow{id \otimes \searrow u} & \mathbb{V}_X \otimes \mathbb{V}_{Y \sqcup Z} \\ \searrow^s \otimes id \downarrow & & \downarrow \nearrow^s \\ \mathbb{V}_{X \sqcup Y} \otimes \mathbb{V}_Z & & \\ \searrow u \downarrow & & \downarrow \\ \mathbb{V}_{X \sqcup Y \sqcup Z} & \xrightarrow{\Psi_{X, Z}} & \mathbb{V}_{X \sqcup Y \sqcup Z} \end{array}$$

2. THE ENVELOPING ALGEBRA OF PRE-LIE ALGEBRAS AND TWISTED PRE-LIE ALGEBRAS

In this section, we recall the method of D. Guin and J.-M. Oudom [16] to describe the enveloping algebra of a pre-Lie algebra. We also recall how T. Schedler in [22] generalizes this method to twisted pre-Lie algebras.

Definition 2.1. A Lie algebra over a field \mathbf{k} is a vector space V endowed with a bilinear bracket $[\cdot, \cdot]$ satisfying:

(1) The antisymmetry:

$$[x, y] = -[y, x], \forall x, y \in V.$$

(2) *The Jacobi identity:*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \forall x, y, z \in V.$$

Definition 2.2. [5, 14] *A left pre-Lie algebra over a field \mathbf{k} is a \mathbf{k} -vector space A with a binary composition \triangleright that satisfies the left pre-Lie identity:*

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),$$

for all $x, y, z \in A$. The left pre-Lie identity rewrites as:

$$(2.1) \quad L_{[x,y]} = [L_x, L_y],$$

where $L_x : A \rightarrow A$ is defined by $L_x y = x \triangleright y$, and where the bracket on the left-hand side is defined by $[x, y] = x \triangleright y - y \triangleright x$. As a consequence this bracket satisfies the Jacobi identity.

The pre-Lie product is extended to the symmetric algebra as follows [17]. Let (A, \triangleright) be a pre-Lie algebra. We consider the Hopf symmetric algebra $S(A)$ equipped with its usual unshuffle coproduct denoted Δ_{unsh} . We will use without restraint the classical Sweedler notation: $\Delta_{unsh}(a) = \sum_a a^{(1)} \otimes a^{(2)}$.

We extend the product \triangleright to $S(A)$. Let a, b and $c \in S(A)$, and $x \in A$. We put:

- $1 \triangleright a = a$
- $a \triangleright 1 = \varepsilon(a)1$
- $(xa) \triangleright b = x \triangleright (a \triangleright b) - (x \triangleright a) \triangleright b$
- $a \triangleright (bc) = \sum_a (a^{(1)} \triangleright b)(a^{(2)} \triangleright c)$.

Definition 2.3. *We define the following \star product on $S(A)$ by:*

$$(2.2) \quad a \star b = \sum_a a^{(1)}(a^{(2)} \triangleright b).$$

Theorem 2.1. [13, 16] *The triple $(S(A), \star, \Delta_{unsh})$ is a Hopf algebra which is isomorphic to the enveloping Hopf algebra $\mathcal{U}(A_{Lie})$ of the Lie algebra A_{Lie} .*

Proof. This theorem was proved by D. Guin and J.-M. Oudom in [16] (Lemma 2.10 and Theorem 2.12). \square

Definition 2.4. [4] *A twisted Lie algebra over a field \mathbf{k} , is a species \mathbb{E} endowed with a bilinear bracket $[,] : \mathbb{E} \otimes \mathbb{E} \rightarrow \mathbb{E}$, satisfying:*

$$(i) [,] + [,]\tau = 0,$$

$$(ii) [, [,]] + [, [,]]\Sigma + [, [,]]\Sigma^2 = 0,$$

where $\tau : \mathbb{E} \otimes \mathbb{E} \rightarrow \mathbb{E} \otimes \mathbb{E}$ is the flip, and $\Sigma : \mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E} \rightarrow \mathbb{E} \otimes \mathbb{E} \otimes \mathbb{E}$ is the cyclic permutation of factors.

Definition 2.5. *A left twisted pre-Lie algebra over a field \mathbf{k} , is a species \mathbb{E} with a binary composition $\circ : \mathbb{E} \otimes \mathbb{E} \rightarrow \mathbb{E}$, satisfying the left twisted pre-Lie algebra identity*

$$\circ(\circ \otimes Id) - \circ(Id \otimes \circ) = (\circ(\circ \otimes Id) - \circ(Id \otimes \circ))(\tau \otimes Id).$$

T. Schedler in [22] shows that the properties of D. Guin and J.-M. Oudom above also work for the linear species, i.e:

Let (\mathbb{E}, \circ) be a twisted pre-Lie algebra. We consider the twisted Hopf symmetric algebra $S(\mathbb{E})$ equipped with its usual unshuffle coproduct denoted Δ_{unsh} . We extend the product \circ to $S(\mathbb{E})$ as follows. Let a, b and $c \in S(\mathbb{E})$, and $x \in \mathbb{E}$. We put:

- $1 \circ a = a$
- $(xa) \circ b = x \circ (a \circ b) - (x \circ a) \circ b$
- $a \circ (bc) = \sum_a (a^{(1)} \circ b)(a^{(2)} \circ c)$,

and if we define the product \star on $S(\mathbb{E})$ by:

$$(2.3) \quad a \star b = \sum_a a^{(1)}(a^{(2)} \circ b),$$

then $(S(\mathbb{E}), \star, \Delta_{unsh})$ is isomorphic to the enveloping Hopf algebra $\mathcal{U}(\mathbb{E}_{Lie})$ of the twisted Lie algebra \mathbb{E}_{Lie} .

3. THE ENVELOPING ALGEBRA OF THE TWISTED PRE-LIE ALGEBRA OF FINITE TOPOLOGICAL SPACES

3.1. The pre-Lie algebra of rooted trees. Let T the vector space spanned by the set of isomorphism classes of rooted trees and $\mathcal{H} = S(T)$. Grafting pre-Lie algebras of rooted trees were studied for the first time by F. Chapoton and M. Livernet [6], see also D. Manchon and A. Saidi [15]. The grafting product is given, for all $t, s \in T$, by:

$$(3.1) \quad t \rightarrow s = \sum_{s' \text{ vertex of } s} t \rightarrow_{s'} s,$$

where $t \rightarrow_{s'} s$ is the tree obtained by grafting the root of t on the vertex s' of s . More explicitly, the operation $t \rightarrow s$ consists of grafting the root of t on every vertex of s and summing up.

Theorem 3.1. [6] *Equipped by \rightarrow , the space T is the free pre-Lie algebra with one generator.*

Now, we can use the method of D. Guin and J.-M. Oudom [16] to find the enveloping algebra of the grafting pre-Lie algebra of rooted trees. We consider the Hopf symmetric algebra $\mathcal{H} = S(T)$ of the pre-Lie algebra (T, \rightarrow) , equipped with its usual unshuffling coproduct Δ_{unsh} . We extend the product \rightarrow to \mathcal{H} by the same method used in (3.1), and we define the Grossman-Larson product [11] \star on \mathcal{H} by:

$$t \star t' = \sum_t t^{(1)}(t^{(2)} \rightarrow t').$$

By construction, the space $(\mathcal{H}, \star, \Delta_{unsh})$ is a Hopf algebra.

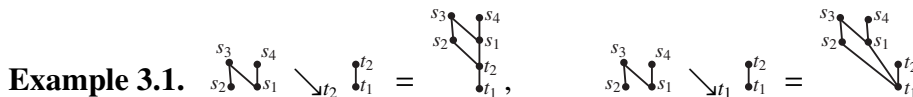
3.2. Twisted pre-Lie algebra of the finite topological spaces. Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ be two finite topological spaces, and let $v \in X_2$. We define:

$$\mathcal{T}_1 \searrow_v \mathcal{T}_2 := (X_1 \sqcup X_2, \leq),$$

where \leq is obtained from $\leq_{\mathcal{T}_1}$ and $\leq_{\mathcal{T}_2}$ as follows: compare any pair in X_2 (resp. X_1) by using $\leq_{\mathcal{T}_2}$ (resp. $\leq_{\mathcal{T}_1}$), and compare any element $y \in X_2$ with any element $x \in X_1$.

To sum up, for any $x, y \in X_1 \sqcup X_2$, $x \leq y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x \in X_2, y \in X_1$ and $x \leq_{\mathcal{T}_2} v$.



Proposition 3.1. *Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ be two connected finite topological spaces, and let $v \in X_2$. Then $\mathcal{T}_1 \searrow_v \mathcal{T}_2 := (X_1 \sqcup X_2, \leq)$, is a connected finite topological space.*

Proof. Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ be two connected finite topological spaces, and let $v \in X_2$.

We must show that \leq is a preorder relation on $X_1 \sqcup X_2$:

Reflexivity; Let $x \in X_1 \sqcup X_2$, then $x \in X_1$ or $x \in X_2$.

If $x \in X_1$, we have $x \leq_{\mathcal{T}_1} x$, then $x \leq x$.

If $x \in X_2$, we have $x \leq_{\mathcal{T}_2} x$, then $x \leq x$.

Transitivity; Let $x, y, z \in X_1 \sqcup X_2$ such that $x \leq y$ and $y \leq z$. So we have four possible cases:

- First case; $x, y, z \in X_1$, and $(x \leq_{\mathcal{T}_1} y$ and $y \leq_{\mathcal{T}_1} z)$.
Since $\leq_{\mathcal{T}_1}$ is transitive, then $x \leq_{\mathcal{T}_1} z$, then $x \leq z$.
- Second case; $x, y, z \in X_2$, and $(x \leq_{\mathcal{T}_2} y$ and $y \leq_{\mathcal{T}_2} z)$.
Since $\leq_{\mathcal{T}_2}$ is transitive, then $x \leq_{\mathcal{T}_2} z$, then $x \leq z$.
- Third case; $x, y \in X_2, z \in X_1$, and $(x \leq_{\mathcal{T}_2} y$ and $y \leq_{\mathcal{T}_2} v)$.
Since $\leq_{\mathcal{T}_2}$ is transitive, then $x \leq_{\mathcal{T}_2} v$, and since $x \in X_2$ and $z \in X_1$ therefore $x \leq z$.
- Fourth case; $x \in X_2, y, z \in X_1$, and $(x \leq_{\mathcal{T}_2} v$ and $y \leq_{\mathcal{T}_1} z)$.
In this case we have $x \in X_2, z \in X_1$, and $x \leq_{\mathcal{T}_2} v$, then $x \leq z$.

□

Proposition 3.2. *Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$, $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ and $\mathcal{T}_3 = (X_3, \leq_{\mathcal{T}_3})$ be three finite connected topological spaces, and let $u \in X_2, v, w \in X_3$. Then*

1) $(\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_w \mathcal{T}_3)$.

2) $\mathcal{T}_1 \searrow_v (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = \mathcal{T}_2 \searrow_w (\mathcal{T}_1 \searrow_v \mathcal{T}_3)$.

Proof. 1) Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$, $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ and $\mathcal{T}_3 = (X_3, \leq_{\mathcal{T}_3})$ be three finite connected topologies, and let $u \in X_2, w \in X_3$. We denote $\mathcal{T}'_3 = (\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = (X_1 \sqcup X_2, \leq_3)$, with \leq_3 defined on $X_1 \sqcup X_2$ by:

$x, y \in X_1 \sqcup X_2$ et $x \leq_3 y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x \in X_2, y \in X_1$ and $x \leq_{\mathcal{T}_2} u$,

and we denote $\mathcal{T} = (\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = (X_1 \sqcup X_2 \sqcup X_3, \leq)$, with \leq defined on $X_1 \sqcup X_2 \sqcup X_3$ by:

$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_1 \sqcup X_2$ and $x \leq_3 y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_1 \sqcup X_2$ and $x \leq_{\mathcal{T}_3} w$,

then

$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x \in X_2, y \in X_1$ and $x \leq_{\mathcal{T}_2} u$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_1 \sqcup X_2$ and $x \leq_{\mathcal{T}_3} w$.

On the other hand,

we denote $\mathcal{T}'_1 = \mathcal{T}_2 \searrow_w \mathcal{T}_3 = (X_2 \sqcup X_3, \leq_1)$, with \leq_1 defined on $X_2 \sqcup X_3$ by:

$x, y \in X_2 \sqcup X_3$ et $x \leq_1 y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,

and we denote $\mathcal{T}' = \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = (X_1 \sqcup X_2 \sqcup X_3, \leq')$, with \leq' defined on $X_1 \sqcup X_2 \sqcup X_3$ by:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2 \sqcup X_3$ and $x \leq_1 y$,
- or $x \in X_2 \sqcup X_3, y \in X_1$ and $x \leq_1 u$,

then

$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,
- or $x \in X_2, y \in X_1$ and $x \leq_{\mathcal{T}_2} u$,
- or $x \in X_3, y \in X_1$ and $x \leq_{\mathcal{T}_3} w$,

then $\leq = \leq'$ on $X_1 \sqcup X_2 \sqcup X_3$.

Then

$$(\mathcal{T}_1 \searrow_u \mathcal{T}_2) \searrow_w \mathcal{T}_3 = \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_w \mathcal{T}_3).$$

2) Let $v, w \in X_3$, we denote $\mathcal{T}'_2 = (\mathcal{T}_1 \searrow_v \mathcal{T}_3) = (X_1 \sqcup X_3, \leq_2)$, with \leq_2 defined on $X_1 \sqcup X_3$ by:

$x, y \in X_1 \sqcup X_3$ et $x \leq_2 y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_1$ and $x \leq_{\mathcal{T}_3} v$,

and we denote

$\mathcal{T} = \mathcal{T}_2 \searrow_w (\mathcal{T}_1 \searrow_v \mathcal{T}_3) = (X_1 \sqcup X_2 \sqcup X_3, \leq)$, with \leq defined on $X_1 \sqcup X_2 \sqcup X_3$ by:

$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_1 \sqcup X_3$ and $x \leq_2 y$,
- or $x \in X_1 \sqcup X_3, y \in X_2$ and $x \leq_2 w$,

then

$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_1$ and $x \leq_{\mathcal{T}_3} v$,
- or $x \in X_3, y \in X_2$ and $x \leq_{\mathcal{T}_3} w$.

On the other hand,

we denote $\mathcal{T}'_1 = (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = (X_2 \sqcup X_3, \leq_1)$, with \leq_1 defined on $X_2 \sqcup X_3$ by:

$x, y \in X_2 \sqcup X_3$ et $x \leq_1 y$ if and only if:

- Either $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,

and we denote

$\mathcal{T}' = \mathcal{T}_1 \searrow_v (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = (X_1 \sqcup X_2 \sqcup X_3, \leq')$, with \leq' defined on $X_1 \sqcup X_2 \sqcup X_3$ by:
 $x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2 \sqcup X_3$ and $x \leq_1 y$,
- or $x \in X_2 \sqcup X_3, y \in X_1$ and $x \leq_1 v$,

then

$x, y \in X_1 \sqcup X_2 \sqcup X_3$ et $x \leq' y$ if and only if:

- Either $x, y \in X_1$ and $x \leq_{\mathcal{T}_1} y$,
- or $x, y \in X_2$ and $x \leq_{\mathcal{T}_2} y$,
- or $x, y \in X_3$ and $x \leq_{\mathcal{T}_3} y$,
- or $x \in X_3, y \in X_2$ and $x \leq_{\mathcal{T}_3} w$,
- or $x \in X_3, y \in X_1$ and $x \leq_{\mathcal{T}_3} v$,

then $\leq = \leq'$ on $X_1 \sqcup X_2 \sqcup X_3$.

Then

$$\mathcal{T}_1 \searrow_v (\mathcal{T}_2 \searrow_w \mathcal{T}_3) = \mathcal{T}_2 \searrow_w (\mathcal{T}_1 \searrow_v \mathcal{T}_3).$$

□

We then define the grafting law in the species of connected finite topological spaces by:

For all $\mathcal{T}_1 \in \mathbb{T}_{X_1}, \mathcal{T}_2 \in \mathbb{T}_{X_2}$

$$\mathcal{T}_1 \searrow \mathcal{T}_2 = \sum_{v \in X_2} \mathcal{T}_1 \searrow_v \mathcal{T}_2 \in \mathbb{T}_{X_1 \sqcup X_2}.$$

Theorem 3.2. (\mathbb{V}, \searrow) is a twisted pre-Lie algebra.

Proof. Let $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$, $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ and $\mathcal{T}_3 = (X_3, \leq_{\mathcal{T}_3})$ three finite topological spaces, we have:

$$\begin{aligned} \mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow \mathcal{T}_3) &= \sum_{v \in X_3} \mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow_v \mathcal{T}_3) \\ &= \sum_{u \in X_2 \sqcup X_3} \sum_{v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3) \\ &= \sum_{u \in X_2} \sum_{v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3) \\ &\quad + \sum_{u \in X_3} \sum_{v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3). \end{aligned}$$

On the other hand we have:

$$\begin{aligned} (\mathcal{T}_1 \searrow \mathcal{T}_2) \searrow \mathcal{T}_3 &= \sum_{r \in X_2} (\mathcal{T}_1 \searrow_r \mathcal{T}_2) \searrow \mathcal{T}_3 \\ &= \sum_{s \in X_3} \sum_{r \in X_2} (\mathcal{T}_1 \searrow_r \mathcal{T}_2) \searrow_s \mathcal{T}_3. \end{aligned}$$

Then

$$\mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow \mathcal{T}_3) - (\mathcal{T}_1 \searrow \mathcal{T}_2) \searrow \mathcal{T}_3 = \sum_{u, v \in X_3} \mathcal{T}_1 \searrow_u (\mathcal{T}_2 \searrow_v \mathcal{T}_3).$$

Which is symmetric on \mathcal{T}_1 and \mathcal{T}_2 . Then we obtain:

$$\mathcal{T}_1 \searrow (\mathcal{T}_2 \searrow \mathcal{T}_3) - (\mathcal{T}_1 \searrow \mathcal{T}_2) \searrow \mathcal{T}_3 = \mathcal{T}_2 \searrow (\mathcal{T}_1 \searrow \mathcal{T}_3) - (\mathcal{T}_2 \searrow \mathcal{T}_1) \searrow \mathcal{T}_3.$$

Consequently, (\mathbb{V}, \searrow) is a twisted pre-Lie algebra, thus yielding a pre-Lie algebra structure on $\overline{\mathcal{K}(\mathbb{T})}$. \square

We showed that (\mathbb{V}, \searrow) is a twisted pre-Lie algebra, so we consider the Hopf symmetric algebra $\mathcal{H}' = S(\mathbb{V})$ equipped with its usual unshuffling coproduct Δ_{unsh} . We extend the product \searrow to \mathbb{T} by using Definition 2.3 and we define a product \star on \mathbb{T} by: For any pair X_1, X_2 of finite sets

$$\begin{aligned} \star : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\ (\mathcal{T}_1, \mathcal{T}_2) &\longmapsto \sum_{\mathcal{T}_1} \mathcal{T}_1^{(1)} (\mathcal{T}_1^{(2)} \searrow \mathcal{T}_2). \end{aligned}$$

By construction, the space $(\mathcal{H}', \star, \Delta_{unsh})$ is a cocommutative twisted Hopf algebra.

Remark 3.1. *The species of finite connected posets (i.e. finite connected T_0 topological spaces) is a twisted pre-Lie subalgebra of (\mathbb{V}, \searrow) , and the species of finite posets is a Hopf subalgebra of \mathcal{H}' .*

Example 3.2.

$$\begin{aligned} (\text{hook}) \searrow \text{!} &= \text{hook} \searrow (\text{!} \searrow \text{!}) - (\text{hook} \searrow \text{!}) \searrow \text{!} \\ &= \text{hook} \searrow (\text{!} + \text{V}) - \text{hook} \searrow \text{!} \\ &= \text{hook} \searrow \text{!} + \text{hook} \searrow \text{V} + \text{hook} \searrow \text{!} + \text{hook} \searrow \text{!} + 2 \text{hook} \searrow \text{!} \\ &\quad - \text{hook} \searrow \text{!} - \text{hook} \searrow \text{!} \\ &= \text{hook} \searrow \text{!} + \text{hook} \searrow \text{V} + \text{hook} \searrow \text{!} + \text{hook} \searrow \text{!} \end{aligned}$$

$$\begin{aligned}
 (\wedge \bullet) \star \downarrow &= (\wedge \bullet) \searrow \downarrow + \wedge \bullet \downarrow + \wedge (\bullet \searrow \downarrow) + \bullet (\wedge \searrow \downarrow) \\
 &= \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \wedge \bullet \downarrow + \wedge (\bullet \searrow \downarrow) + \bullet (\wedge \searrow \downarrow) \\
 &\quad + \bullet (\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array}) \\
 &= \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \wedge \bullet \downarrow + \wedge (\bullet \searrow \downarrow) + \bullet (\wedge \searrow \downarrow) \\
 &\quad + \bullet (\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array})
 \end{aligned}$$

4. BIALGEBRAS OF FINITE TOPOLOGICAL SPACES

4.1. A twisted bialgebra of finite topological spaces. Let X be any finite set, we define the coproduct Δ_{\searrow} by:

$$\begin{aligned}
 \Delta_{\searrow} : \mathbb{T}_X &\longrightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\
 \mathcal{J} &\longmapsto \sum_{Y \in \overline{\mathcal{J}}} \mathcal{J}_{|Y} \otimes \mathcal{J}_{|X \setminus Y}.
 \end{aligned}$$

Where $Y \in \overline{\mathcal{J}}$, stands for

- $Y \in \mathcal{J}$,
- $\mathcal{J}_{|Y} = \mathcal{J}_1 \dots \mathcal{J}_n$, such that for all $i \in \{1, \dots, n\}$, \mathcal{J}_i connected and $(\min \mathcal{J}_i = (\min \mathcal{J}) \cap \mathcal{J}_i)$, or there is a single common ancestor $x_i \in \overline{X \setminus Y}$ to $\min \mathcal{J}_i$, where $\overline{X \setminus Y} = (X \setminus Y) / \sim_{\mathcal{J}_{|X \setminus Y}}$.

Example 4.1. $\Delta_{\searrow}(\wedge) = \wedge \otimes \mathbf{1} + \mathbf{1} \otimes \wedge$

$$\Delta_{\searrow}(\vee) = \bullet \otimes \bullet + \vee \otimes \mathbf{1} + \mathbf{1} \otimes \vee$$

Theorem 4.1. $(\mathbb{T}, m, \Delta_{\searrow})$ is a commutative connected twisted bialgebra, and $\mathcal{H} = \overline{\mathcal{K}}(\mathbb{T})$ is a commutative graded bialgebra.

Proof. To show that \mathbb{T} is a twisted bialgebra [1], it is necessary to show that Δ_{\searrow} is coassociative, and that the species coproduct Δ_{\searrow} and the product defined by:

$$\begin{aligned}
 m : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\
 \mathcal{J}_1 \otimes \mathcal{J}_2 &\longmapsto \mathcal{J}_1 \mathcal{J}_2,
 \end{aligned}$$

are compatible. The unit $\mathbf{1}$ is identified to the empty topology. Coassociativity is checked by a careful, but straightforward computation. We have

$$\begin{aligned} (\Delta_{\searrow} \otimes id)\Delta_{\searrow}(\mathcal{T}) &= (\Delta_{\searrow} \otimes id) \left(\sum_{Y \in \overline{\mathcal{T}}} \mathcal{T}_{|Y} \otimes \mathcal{T}_{|X \setminus Y} \right) \\ &= \sum_{Z \in \overline{\mathcal{T}}_Y, Y \in \overline{\mathcal{T}}} \mathcal{T}_{|Z} \otimes \mathcal{T}_{|Y \setminus Z} \otimes \mathcal{T}_{|X \setminus Y}. \end{aligned}$$

On the other hand

$$\begin{aligned} (id \otimes \Delta_{\searrow})\Delta_{\searrow}(\mathcal{T}) &= (id \otimes \Delta_{\searrow}) \left(\sum_{U \in \overline{\mathcal{T}}} \mathcal{T}_{|U} \otimes \mathcal{T}_{|X \setminus U} \right) \\ &= \sum_{W \in \overline{\mathcal{T}}_{|X \setminus U}, U \in \overline{\mathcal{T}}} \mathcal{T}_{|U} \otimes \mathcal{T}_{|W} \otimes \mathcal{T}_{|X \setminus (U \sqcup W)}. \end{aligned}$$

Coassociativity will come from the fact that $(Z, Y) \mapsto (Z, Y \setminus Z)$ is a bijection from the set of pairs (Z, Y) with $Y \in \overline{\mathcal{T}}$ and $Z \in \overline{\mathcal{T}}_{|Y}$ and, onto the set of pairs (U, W) with $U \in \overline{\mathcal{T}}$ and $W \in \overline{\mathcal{T}}_{|X \setminus U}$. The inverse map is given by $(U, W) \mapsto (U, U \sqcup W)$.

Let $A = \{(Z, Y), Y \in \overline{\mathcal{T}} \text{ and } Z \in \overline{\mathcal{T}}_{|Y}\}$, and $B = \{(U, W), U \in \overline{\mathcal{T}} \text{ and } W \in \overline{\mathcal{T}}_{|X \setminus U}\}$.

We define

$$\begin{aligned} f : A &\longrightarrow B & g : B &\longrightarrow A \\ (Z, Y) &\longmapsto (Z, Y \setminus Z) & (U, W) &\longmapsto (U, U \sqcup W) \end{aligned}$$

Let us prove that f and g are well defined.

Let $(Z, Y) \in A$, i.e

- $Y \in \mathcal{T}$ and $\mathcal{T}_{|Y} = \mathcal{T}_1 \dots \mathcal{T}_n$, such that for all $i \in \{1, \dots, n\}$, \mathcal{T}_i connected component and ($\min \mathcal{T}_i = (\min \mathcal{T}) \cap \mathcal{T}_i$ or there is a unique common ancestor $x_i \in \overline{X \setminus Y}$ to $\min \mathcal{T}_i$), and

- $Z \in \mathcal{T}_{|Y}$ and $\mathcal{T}_{|Z} = \mathcal{T}_{|1} \dots \mathcal{T}_{|n} \setminus Z = \mathcal{T}_{1,1} \mathcal{T}_{1,2} \dots \mathcal{T}_{1,i_1} \mathcal{T}_{2,1} \mathcal{T}_{2,2} \dots \mathcal{T}_{2,i_2} \dots \mathcal{T}_{n,1} \mathcal{T}_{n,2} \dots \mathcal{T}_{n,i_n}$, such that for all $i \in \{1, \dots, n\}$, $j \in \{i_1, \dots, i_n\}$, $\mathcal{T}_{i,j}$ connected component and ($\min \mathcal{T}_{i,j} = \min \mathcal{T}_{|Y} \cap \mathcal{T}_{i,j}$ or there is a unique common ancestor $x_{i,j} \in \overline{Y \setminus Z}$ to $\min \mathcal{T}_{i,j}$).

Then we can visualise \mathcal{T} by the graph illustrated below in figure 1:

Graphically it is clear that $Z \in \mathcal{T}$ and $Y \setminus Z \in \mathcal{T}_{|X \setminus Z}$. Then $(Z, Y \setminus Z) \in B$.

Then f is well defined.

Let $(U, W) \in B$, i.e

- $U \in \mathcal{T}$ and $\mathcal{T}_{|U} = \mathcal{T}_{|U_1} \dots \mathcal{T}_{|U_p}$, such that for all $i \in \{1, \dots, p\}$, $\mathcal{T}_{|U_i}$ connected component and ($\min \mathcal{T}_{|U_i} = (\min \mathcal{T}) \cap \mathcal{T}_{|U_i}$ or there is a unique $x_i \in \overline{X \setminus U}$ common ancestor to $\min \mathcal{T}_{|U_i}$). and

- $W \in \mathcal{T}_{|X \setminus U}$ and $\mathcal{T}_{|W} = \mathcal{T}^1 \dots \mathcal{T}^q$, such that for all $j \in \{1, \dots, q\}$, \mathcal{T}^j connected component and ($\min \mathcal{T}^j = \min \mathcal{T}_{|X \setminus U} \cap \mathcal{T}^j$ or there is a unique $x_j \in \overline{X \setminus (U \sqcup W)}$ common ancestor to $\min \mathcal{T}^j$).

For all $k \in \{1, \dots, q\}$, we notice $U^k = \bigsqcup_{0 \leq n \leq p} U_i$, where U_i verifies the existence of $x_i \in \overline{W_k} = \overline{\mathcal{V}(\mathcal{T}^k)}$

common ancestor to $\min \mathcal{T}_{|U_i}$, where $v \in \mathcal{V}(\mathcal{T}^k)$ denotes that v is a element of the topological space \mathcal{T}^k .

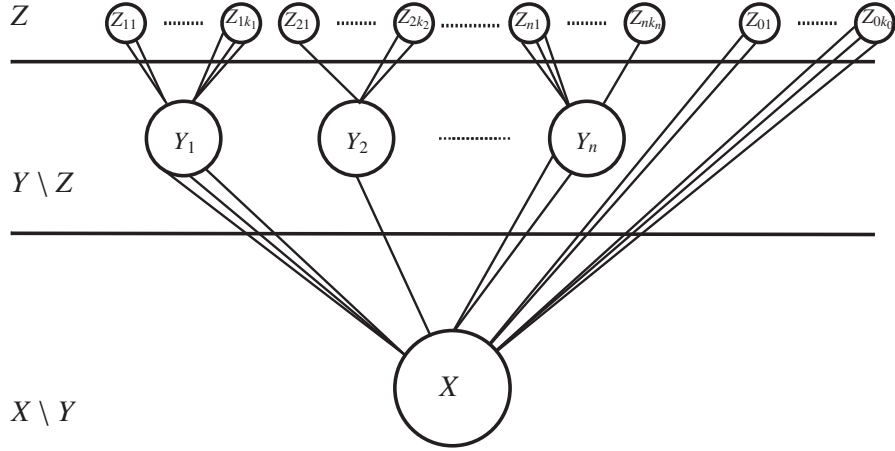


FIGURE 1

We notice $U^0 = \bigsqcup_{0 \leq n \leq p} U_i$, where U_i verifies the existence of a unique $x_i \in \overline{X \setminus (U \sqcup W)}$ common ancestor to $\min \mathcal{T}_{|U_i}$.

We notice $W_0 \subset W$, where W_0 verify that for all $x \in W_0$, there is no $y \in U$ such that $x \leq_{\mathcal{T}} y$. Then we can visualise \mathcal{T} by the graph illustrated below in figure 2 below:

Graphically it is clear that $U \in \mathcal{T}_{|W \sqcup U}$ and $W \sqcup U \in \mathcal{T}$. Then $(U, U \sqcup W) \in A$.

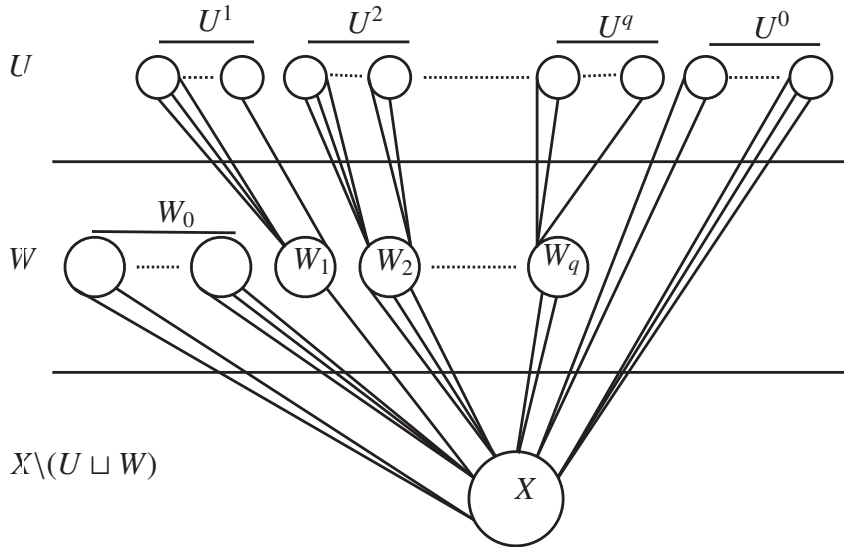


FIGURE 2

Then g is well defined.

We have for all $(Z, Y) \in A$ then $(Z, Y \setminus Z) \in B$, and for all $(U, W) \in B$ then $(U, U \sqcup W) \in A$. Then

$|A| = |B|$.

Let $(Z_1, Y_1), (Z_2, Y_2) \in A$ such that $f(Z_1, Y_1) = f(Z_2, Y_2)$, then $Z_1 = Z_2$, then $Y_1 \setminus Z_1 = Y_2 \setminus Z_1$, then f is injective, then f is bijective.

In the same way we show that g is bijective, and $g \circ f = f \circ g = Id$.

Then Δ is coassociative.

Finally, we show immediately that

$$\Delta_{\setminus}(\mathcal{T}_1 \mathcal{T}_2) = \Delta_{\setminus}(\mathcal{T}_1) \Delta_{\setminus}(\mathcal{T}_2).$$

□

Remark 4.1. For any finite set X , let us recall from [8] the internal coproduct Γ on \mathbb{T}_X :

$$(4.1) \quad \Gamma(\mathcal{T}) = \sum_{\mathcal{T}' \otimes \mathcal{T}} \mathcal{T}' \otimes \mathcal{T} / \mathcal{T}'.$$

The sum runs over topologies \mathcal{T}' which are \mathcal{T} -admissible, i.e

- finer than \mathcal{T} ,
- such that $\mathcal{T}'_Y = \mathcal{T}_Y$ for any subset $Y \subset X$ connected for the topology \mathcal{T}' ,
- such that for any $x, y \in X$,

$$(4.2) \quad x \sim_{\mathcal{T}/\mathcal{T}'} y \iff x \sim_{\mathcal{T}'/\mathcal{T}'} y.$$

F. Fauvet, L. Foissy, and D. Manchon in [8] show that Γ and Δ are compatible. On the other hand, we notice that Γ and Δ_{\setminus} are not compatible. In fact:

$$\begin{aligned} \Delta_{\setminus}(\diamond) &= \wedge \otimes \bullet + \mathbf{1} \otimes \diamond + \diamond \otimes \mathbf{1} \\ \Gamma(\diamond) &= \vee \bullet \otimes \odot + \wedge \bullet \otimes \odot + 2 \parallel \otimes \odot + \dots \otimes \diamond \\ &\quad + \diamond \otimes \odot \end{aligned}$$

then

$$\begin{aligned} (Id \otimes \Delta_{\setminus})\Gamma(\diamond) &= \vee \bullet \otimes [\bullet \otimes \odot + \mathbf{1} \otimes \odot + \odot \otimes \mathbf{1}] \\ &\quad + \wedge \bullet \otimes [\odot \otimes \bullet + \mathbf{1} \otimes \odot + \odot \otimes \mathbf{1}] \\ &\quad + 2 \parallel \otimes [\odot \otimes \odot + \mathbf{1} \otimes \odot + \odot \otimes \mathbf{1}] \\ &\quad + \dots \otimes [\wedge \bullet \otimes \bullet + \mathbf{1} \otimes \diamond + \diamond \otimes \mathbf{1}] \\ &\quad + \diamond \otimes [\odot \otimes \mathbf{1} + \mathbf{1} \otimes \odot] \end{aligned}$$

On the other hand

$$\begin{aligned}
 m^{13}(\Gamma \otimes \Gamma)\Delta_{\searrow}(\diamond) &= \vee \cdot \otimes [1 \otimes \textcircled{\bullet} + \textcircled{\bullet} \otimes 1] \\
 &+ \wedge \cdot \otimes [\textcircled{\bullet} \otimes \cdot + 1 \otimes \textcircled{\bullet} + \textcircled{\bullet} \otimes 1] \\
 &+ 2 \textcircled{\bullet} \otimes [1 \otimes \textcircled{\bullet} + \textcircled{\bullet} \otimes 1] \\
 &+ \dots \otimes [\wedge \cdot \otimes \cdot + 1 \otimes \diamond + \diamond \otimes 1] \\
 &+ \diamond \otimes [\textcircled{\bullet} \otimes 1 + 1 \otimes \textcircled{\bullet}] \\
 &+ \textcircled{\bullet} \cdot \otimes [\textcircled{\bullet} \otimes \cdot + \textcircled{\bullet} \otimes \cdot]
 \end{aligned}$$

then $(Id \otimes \Delta_{\searrow})\Gamma(\diamond) \neq m^{13}(\Gamma \otimes \Gamma)\Delta_{\searrow}(\diamond)$. Then Γ and Δ_{\searrow} are not compatible.

4.2. Relation between \star and Δ_{\searrow} . In this subsection, we prove that there exist relations between the Grossman-Larson product \star and the coproduct Δ_{\searrow} .

Let G be a group acting on X . For every $x \in X$, we denote by $G \cdot x$ the orbit of x and we denote by G_x the stabilizer subgroup of G with respect to x . The group action is transitive if and only if it has exactly one orbit, that is if there exists x in X with $G \cdot x = X$ (i.e. X is non-empty and if for each pair $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$). This is the case if and only if $G \cdot x = X$, for all x in X .

If G and X is finite, then the orbit-stabilizer theorem, together with Lagrange's theorem [21](theorem 3.9), gives

$$(4.3) \quad |G \cdot x| = [G_x : G] = \frac{|G|}{|G_x|},$$

that implies that the cardinal of the orbit is a divisor of the group order.

Definition 4.1. For any topology \mathcal{T} on a finite set X , we denote by $Aut(\mathcal{T})$ the subgroup of permutations of X which are homeomorphisms with respect to \mathcal{T} . The symmetry factor is defined by $\sigma(\mathcal{T}) = |Aut(\mathcal{T})|$. We define the linear map $e_{\mathcal{T}} : \mathbb{T}_X \rightarrow \mathbb{K}$ by:

$$e_{\mathcal{T}}(\mathcal{T}') = \sigma(\mathcal{T}), \text{ if } \mathcal{T} = \mathcal{T}', \text{ and } 0 \text{ if not.}$$

Definition 4.2. We define the graft operator $\mathcal{B} : \mathbb{T} \rightarrow \mathbb{T}$ by, $\mathcal{B}(\mathcal{T}) = \mathcal{T} \searrow \{*\}$, for any topology \mathcal{T} on X , this is the topology on $X \sqcup \{*\}$ obtained by keeping the preorder and X and by putting $* < x$ for any $x \in X$.

Theorem 4.2. Let $\mathcal{T}_1 \in \mathbb{T}_{X_1}$, $\mathcal{T}_2 \in \mathbb{T}_{X_2}$ and $\mathcal{T}' \in \mathbb{T}_X$, then

$$\langle e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' \rangle = \langle e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') \rangle.$$

Proof. Let $\mathcal{T}_1 \in \mathbb{T}_{X_1}$, $\mathcal{T}_2 \in \mathbb{T}_{X_2}$ and $\mathcal{T}' \in \mathbb{T}_X$.

Case 1; \mathcal{T}_1 is connected, we have

$$\begin{aligned} \langle e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' \rangle &= \langle e_{\mathcal{T}_1 \searrow_{\mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}'} \rangle \\ &= \sum_{v \in X_2 \sqcup \{*\}} \langle e_{\mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2}, \mathcal{B}\mathcal{T}' \rangle \\ &= \sum_{\substack{v \in X_2 \sqcup \{*\} \\ \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2}} \sigma(\mathcal{B}\mathcal{T}'). \end{aligned}$$

Let us consider the set $B = \{v \in X_2 \sqcup \{*\}, \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2\}$, we show that $\text{Aut}(\mathcal{B}\mathcal{T}_2)$ acts transitively on B . We define the map

$$\begin{aligned} \Phi_1 : \text{Aut}(\mathcal{B}\mathcal{T}_2) \times B &\longrightarrow B \\ (\varphi, v) &\longmapsto \varphi(v). \end{aligned}$$

Let $v \in B$, if $v = *$, then $\varphi(v) = v$. If not then $\mathcal{T}_1 \searrow_{\varphi(v)} \mathcal{B}\mathcal{T}_2 = \mathcal{T}_1 \searrow_{\varphi(v)} \varphi(\mathcal{B}\mathcal{T}_2) = h(\mathcal{T}_1 \searrow_v \mathcal{B}\mathcal{T}_2)$, where $h|_{X_1} = Id$ and if $v \in X_2 \sqcup \{*\}$, $h(v) = \varphi(v)$, where $\varphi \in \text{Aut}(\mathcal{B}\mathcal{T}_2)$. It is clear that $h \in \text{Aut}(\mathcal{B}\mathcal{T}')$, then $\mathcal{T}_1 \searrow_{\varphi(v)} \mathcal{B}\mathcal{T}_2 = h(\mathcal{B}\mathcal{T}') = \mathcal{B}\mathcal{T}'$. Then Φ_1 is well defined.

Moreover for all $v \in B$, $Id(v) = v$, and for all $\varphi, \varphi' \in \text{Aut}(\mathcal{B}\mathcal{T}_2)$, $\Phi_1(\varphi, \varphi'(v)) = \varphi(\varphi'(v)) = (\varphi\varphi')(v)$. Then Φ_1 is an action.

Now to show that Φ_1 is transitive, let $u, v \in B$, and let us define $f : X \sqcup \{*\} \longrightarrow X \sqcup \{*\}$ by $f(u) = v$, $f(v) = u$, and for all $w \in X \sqcup \{*\} \setminus \{u, v\}$, $f(w) = w$, it is clear that $f \in \text{Aut}(\mathcal{B}\mathcal{T}')$.

If we take $\varphi : X_2 \sqcup \{*\} \longrightarrow X_2 \sqcup \{*\}$, defined by $\varphi = f|_{X_2 \sqcup \{*\}}$, so we have, $\varphi \in \text{Aut}(\mathcal{B}\mathcal{T}_2)$, and $\varphi(u) = v$, then Φ_1 is transitive. Then $B = \text{Aut}(\mathcal{B}\mathcal{T}_2) \cdot v$, for all $v \in B$.

For all $v \in B$, we call the stabilizer of v the set:

$$\text{Aut}(\mathcal{B}\mathcal{T}_2)_v = \{\varphi \in \text{Aut}(\mathcal{B}\mathcal{T}_2), \varphi(v) = v\}.$$

And since $|\text{Aut}(\mathcal{B}\mathcal{T}_2)|$ is finite, then $|B| = |\text{Aut}(\mathcal{B}\mathcal{T}_2) \cdot v| = \frac{|\text{Aut}(\mathcal{B}\mathcal{T}_2)|}{|\text{Aut}(\mathcal{B}\mathcal{T}_2)_v|}$, for all $v \in B$.

Then

$$|B| = \frac{|\sigma(\mathcal{T}_2)|}{|\text{Aut}(\mathcal{T}_2)_v|}, \text{ for all } v \in B \setminus \{*\}.$$

On the other hand

$$\begin{aligned} \langle e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') \rangle &= \sum_{Y \in \mathcal{T}'} \langle e_{\mathcal{T}_1}, \mathcal{T}'_Y \rangle \langle e_{\mathcal{T}_2}, \mathcal{T}'_{X \setminus Y} \rangle \\ &= \sum_{\substack{Y \in \mathcal{T}' \\ \mathcal{T}_1 = \mathcal{T}'_Y, \mathcal{T}_2 = \mathcal{T}'_{X \setminus Y}}} \sigma(\mathcal{T}_1) \sigma(\mathcal{T}_2). \end{aligned}$$

Let us consider the set

$$A = \{v \in X, \text{ the cut above } v \text{ give the term of } \Delta_{\searrow}(\mathcal{T}') \text{ isomorphic to } \mathcal{T}_1 \otimes \mathcal{T}_2\},$$

we notice that $A \cap B \setminus \{*\} \neq \emptyset$.

We show that $\text{Aut}(\mathcal{T}')$ acts transitively on A . We define the map

$$\begin{aligned} \Phi_2 : \text{Aut}(\mathcal{T}') \times A &\longrightarrow A \\ (\varphi, v) &\longmapsto \varphi(v). \end{aligned}$$

Let $v \in A$ then $\mathcal{T}'_{|X_1}$ isomorphic to \mathcal{T}_1 and $\mathcal{T}'_{|X_2}$ isomorphic to \mathcal{T}_2 , then for all $\varphi \in \text{Aut}(\mathcal{T}')$, $\varphi(\mathcal{T}'_{|X_i})$ isomorph to \mathcal{T}_i , $i \in \{1, 2\}$, and $\Delta_{\setminus}(\varphi(\mathcal{T}'))$ isomorphic to $\Delta_{\setminus}(\mathcal{T}')$, then for all $\varphi \in \text{Aut}(\mathcal{T}')$, the cut above $\varphi(v)$ give the term of $\Delta_{\setminus}(\mathcal{T}')$ isomorphic to $\mathcal{T}_1 \otimes \mathcal{T}_2$, then $\varphi(v) \in A$. Then Φ_2 is well defined.

Let $v \in A$, and $\varphi, \varphi' \in \text{Aut}(\mathcal{T}')$, then $Id(v) = v$ and $\Phi_2(\varphi, \varphi'(v)) = \varphi(\varphi'(v)) = (\varphi\varphi')(v)$. Then Φ_2 is an action.

Let $u, v \in A$, we defined $f : X \longrightarrow X$ by: $f(u) = v$, $f(v) = u$, and for all $w \notin \{u, v\}$, $f(w) = w$, it is clear that $f \in \text{Aut}(\mathcal{T}')$, then Φ_2 is transitive. And since $|\text{Aut}(\mathcal{T}')|$ is finite, then

$$|A| = \frac{|\text{Aut}(\mathcal{T}')|}{|\text{Aut}(\mathcal{T}')_v|}, \text{ for all } v \in A.$$

Let $v \in A$, then $|\text{Aut}(\mathcal{T}')_v| = |\text{Aut}(\mathcal{T}_1)\text{Aut}(\mathcal{T}_2)_v| = |\text{Aut}(\mathcal{T}_1)||\text{Aut}(\mathcal{T}_2)_v|$. Then

$$|\text{Aut}(\mathcal{T}_2)_v| = \frac{|\text{Aut}(\mathcal{T}')|}{|A||\text{Aut}(\mathcal{T}_1)|}, \text{ for all } v \in A,$$

that since $A \cap B \setminus \{*\} \neq \emptyset$, there exists $v \in A \cap B \setminus \{*\}$ such that

$$|\text{Aut}(\mathcal{T}_2)_v| = \frac{|\text{Aut}(\mathcal{T}')|}{|A||\text{Aut}(\mathcal{T}_1)|} = \frac{|\text{Aut}(\mathcal{T}_2)|}{|B|}$$

then

$$\frac{\sigma(\mathcal{T}')}{|A|\sigma(\mathcal{T}_1)} = \frac{\sigma(\mathcal{T}_2)}{|B|},$$

then

$$|B|\sigma(\mathcal{T}') = |A|\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2).$$

We define $A' = \{Y, Y \in \bar{\mathcal{T}}', \mathcal{T}_1 = \mathcal{T}'_Y, \mathcal{T}_2 = \mathcal{T}'_{|X \setminus Y}\}$, we notice that $|A| = |A'|$. Then

$$\begin{aligned} \langle e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' \rangle &= \sum_{\substack{v \in X_2 \setminus \{*\} \\ \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \setminus_v \mathcal{B}\mathcal{T}_2}} \sigma(\mathcal{B}\mathcal{T}') \\ &= |B|\sigma(\mathcal{B}\mathcal{T}') \\ &= |B|\sigma(\mathcal{T}') \\ &= |A'|\sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2) \\ &= \sum_{\substack{Y \in \bar{\mathcal{T}}' \\ \mathcal{T}_1 = \mathcal{T}'_Y, \mathcal{T}_2 = \mathcal{T}'_{|X \setminus Y}}} \sigma(\mathcal{T}_1)\sigma(\mathcal{T}_2) \\ &= \langle e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\setminus}(\mathcal{T}') \rangle. \end{aligned}$$

Case2; \mathcal{T}_1 not connected.

Let $\mathcal{T}_1 = \mathcal{T}_{1,1} \dots \mathcal{T}_{1,n} \in \mathbb{T}_{X_1}$, where $\mathcal{T}_{1,i}$ is connected for all $i \in [n]$. And let $\mathcal{T}_2 \in \mathbb{T}_{X_2}$, $\mathcal{T}' \in \mathbb{T}_X$, we

have

$$\begin{aligned}
\langle e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' \rangle &= \langle e_{\mathcal{T}_1 \searrow_{\underline{B}} \mathcal{T}_2}, \mathcal{B}\mathcal{T}' \rangle \\
&= \sum_{\underline{v}=(v_1, \dots, v_n) \in X_2} \langle e_{\mathcal{T}_1 \searrow_{\underline{v}} \mathcal{T}_2}, \mathcal{B}\mathcal{T}' \rangle \\
&= \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in X_2 \\ \mathcal{B}\mathcal{T}' = \mathcal{T}_{1,1} \searrow_{v_1} (\mathcal{T}_{1,2} \searrow_{v_2} \dots (\mathcal{T}_{1,n} \searrow_{v_n} \mathcal{B}\mathcal{T}_2) \dots)}} \sigma(\mathcal{B}\mathcal{T}').
\end{aligned}$$

Let us consider the set $\underline{B} = \{\underline{v} = (v_1, \dots, v_n) \in X_2 \sqcup \{*\}, \mathcal{B}\mathcal{T}' = \mathcal{T}_1 \searrow_{\underline{v}} \mathcal{B}\mathcal{T}_2\}$.

$\text{Aut}(\mathcal{T}_2)$ acts transitively on \underline{B} by the action. (In the same way that we used to show that Φ_1 is a transitive action.)

$$\begin{aligned}
\Phi_3 : \text{Aut}(\mathcal{T}_2) \times \underline{B} &\longrightarrow \underline{B} \\
(\varphi, \underline{v}) &\longmapsto \varphi(\underline{v}).
\end{aligned}$$

And since $|\text{Aut}(\mathcal{T}_2)|$ is finite, then $|\underline{B}| = \frac{\sigma(\mathcal{T}_2)}{|\text{Aut}(\mathcal{T}_2)_{\underline{v}}|}$, where $\underline{v} \in \underline{B}$.

On the other hand

$$\begin{aligned}
\langle e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') \rangle &= \sum_{Y \in \mathcal{T}'} \langle e_{\mathcal{T}_1}, \mathcal{T}'_Y \rangle \langle e_{\mathcal{T}_2}, \mathcal{T}'_{X \setminus Y} \rangle \\
&= \sum_{\substack{Y \in \mathcal{T}' \\ \mathcal{T}_1 = \mathcal{T}'_Y, \mathcal{T}_2 = \mathcal{T}'_{X \setminus Y}}} \sigma(\mathcal{T}_1) \sigma(\mathcal{T}_2).
\end{aligned}$$

Let us consider the set

$\underline{A} = \{\underline{v} = (v_1, \dots, v_n) \in X, \text{ the cut above } \underline{v} \text{ give the term of } \Delta_{\searrow}(\mathcal{T}') \text{ isomorphic to } \mathcal{T}_1 \otimes \mathcal{T}_2\}$.

We notice that $\underline{A} \cap \underline{B}_{X_2} \neq \emptyset$.

$\text{Aut}(\mathcal{T}')$ acts transitively on \underline{A} by the action. (In the same way that we used to show that Φ_2 is a transitive action.)

$$\begin{aligned}
\Phi_4 : \text{Aut}(\mathcal{T}') \times \underline{A} &\longrightarrow \underline{A} \\
(\varphi, \underline{v}) &\longmapsto \varphi(\underline{v}).
\end{aligned}$$

And since $|\text{Aut}(\mathcal{T}')|$ is finite, then $|\underline{A}| = \frac{\sigma(\mathcal{T}')}{|\text{Aut}(\mathcal{T}')_{\underline{v}}|}$, where $\underline{v} \in \underline{A}$.

If $\underline{v} \in \underline{A}$, then $|\text{Aut}(\mathcal{T}')_{\underline{v}}| = |\text{Aut}(\mathcal{T}_1) \text{Aut}(\mathcal{T}_2)_{\underline{v}}| = |\text{Aut}(\mathcal{T}_1)| |\text{Aut}(\mathcal{T}_2)_{\underline{v}}|$,

then $|\text{Aut}(\mathcal{T}_2)_{\underline{v}}| = \frac{|\text{Aut}(\mathcal{T}')|}{|\underline{A}| |\text{Aut}(\mathcal{T}_1)|}$ for all $\underline{v} \in \underline{A}$,

then

$$|\text{Aut}(\mathcal{T}_2)_{\underline{v}}| = \frac{|\text{Aut}(\mathcal{T}')|}{|\underline{A}| |\text{Aut}(\mathcal{T}_1)|} = \frac{|\text{Aut}(\mathcal{T}_2)|}{|\underline{B}|}, \text{ for all } \underline{v} \in \underline{A} \cap \underline{B} \cap X_2,$$

then $\frac{\sigma(\mathcal{T}')}{|\underline{A}| \sigma(\mathcal{T}_1)} = \frac{\sigma(\mathcal{T}_2)}{|\underline{B}|}$. Then $|\underline{B}| \sigma(\mathcal{T}') = |\underline{A}| \sigma(\mathcal{T}_1) \sigma(\mathcal{T}_2)$.

We define $\underline{A}' = \{Y, Y \in \mathcal{T}', \mathcal{T}_1 = \mathcal{T}'_{|Y} \text{ and } \mathcal{T}_2 = \mathcal{T}'_{|X \setminus Y}\}$, we notice that $|\underline{A}| = |\underline{A}'|$. Then

$$\begin{aligned}
 \langle e_{\mathcal{T}_1} \star e_{\mathcal{T}_2}, \mathcal{T}' \rangle &= \langle e_{\mathcal{T}_1 \searrow_{\underline{B}} \mathcal{T}_2}, \mathcal{B}\mathcal{T}' \rangle \\
 &= \sum_{\underline{v}=(v_1, \dots, v_n) \in X_2} \langle e_{\mathcal{T}_1 \searrow_{\underline{v}} \mathcal{T}_2}, \mathcal{B}\mathcal{T}' \rangle \\
 &= \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in X_2 \\ \mathcal{B}\mathcal{T}' = \mathcal{T}_{1,1} \searrow_{v_1} (\mathcal{T}_{1,2} \searrow_{v_2} \dots (\mathcal{T}_{1,n} \searrow_{v_n} \mathcal{B}\mathcal{T}_2) \dots)}} \sigma(\mathcal{B}\mathcal{T}') \\
 &= |\underline{B}| \sigma(\mathcal{T}') \\
 &= |\underline{A}'| \sigma(\mathcal{T}_1) \sigma(\mathcal{T}_2) \\
 &= \sum_{\substack{Y \in \mathcal{T}' \\ \mathcal{T}_1 = \mathcal{T}'_{|Y}, \mathcal{T}_2 = \mathcal{T}'_{|X \setminus Y}}} \sigma(\mathcal{T}_1) \sigma(\mathcal{T}_2) \\
 &= \sum_{Y \in \mathcal{T}'} \langle e_{\mathcal{T}_1}, \mathcal{T}'_{|Y} \rangle \langle e_{\mathcal{T}_2}, \mathcal{T}'_{|X \setminus Y} \rangle \\
 &= \langle e_{\mathcal{T}_1} \otimes e_{\mathcal{T}_2}, \Delta_{\searrow}(\mathcal{T}') \rangle.
 \end{aligned}$$

□

5. RELATION BETWEEN \searrow AND \nearrow

In this part we define the law \nearrow on \mathbb{V} by: For all $\mathcal{T} = (X, \leq_{\mathcal{T}})$ and $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ be two finite connected topological spaces,

$$\mathcal{T} \nearrow \mathcal{S} := j(j(\mathcal{T}) \searrow j(\mathcal{S})),$$

where j is the involution which transforms \leq into \geq . In other words, open subsets in \mathcal{T} are closed subsets in $j(\mathcal{T})$ and vice-versa. In particular, it is obvious that (\mathbb{V}, \nearrow) is a twisted pre-Lie algebra due to the fact that (\mathbb{V}, \searrow) is a twisted pre-Lie algebra.

Definition 5.1. For any finite set A and for any pair of parts A_1, A_2 of A with $A_1 \cap A_2 = \emptyset$, we define $\Psi_{A_1, A_2} : \mathbb{T}_A \rightarrow \mathbb{T}_A$, as follows: for any topology $\mathcal{T} \in \mathbb{T}_A$, $\Psi_{A_1, A_2}(\mathcal{T}) = (A, \leq)$, where \leq is defined by

- If $a \in A_1$, and $b \in A_2$ then a and b are incomparable,
- otherwise, we have $a \leq b$ if and only if $a \leq_{\mathcal{T}} b$.

Proposition 5.1. For any finite set A and for any pair of parts A_1, A_2 of A with $A_1 \cap A_2 = \emptyset$, and let $\mathcal{T} \in \mathbb{T}_A$ then

- 1) $\Psi_{A_1, A_2}(\mathcal{T}) = (A, \leq)$ is a finite topological space.
- 2) Ψ_{A_1, A_2} is a projector.

Proof. 1) Let $A = A_1 \sqcup A_2$, with $A_1 \cap A_2 = \emptyset$, and let $\mathcal{T} \in \mathbb{T}_A$, we must show that \leq is a preorder relation on A :

Reflexivity; Let $x \in A$, then $x \in A_1$ or $x \in A_2$.

If $x \in A_1$, we have $x \leq_{\mathcal{T}} x$ then $x \leq x$, same thing if $x \in A_2$.

Transitivity; Let $x, y, z \in A$ such that $x \leq y$ and $y \leq z$. So we have two possible cases:

- First case; $x, y, z \in A_1$, and $(x \leq y \text{ and } y \leq z)$, then $(x \leq_{\mathcal{T}} y \text{ and } y \leq_{\mathcal{T}} z)$. Since $\leq_{\mathcal{T}}$ is transitive, then $x \leq_{\mathcal{T}} z$, then $x \leq z$.
- Second case; $x, y, z \in A_2$, likewise the first case.

2) Let $\mathcal{T} \in \mathbb{T}_A$, we must show that $\Psi_{A_1, A_2}(\mathcal{T}) = \Psi_{A_1, A_2}^2(\mathcal{T})$.

If not $\mathcal{T}' = (A, \leq') = \Psi_{A_1, A_2}(\mathcal{T})$, then $\Psi_{A_1, A_2}^2(\mathcal{T}) = \Psi_{A_1, A_2}(\mathcal{T}') = (A, \leq)$, where \leq defined by

- If $a \in A_1$, and $b \in A_2$ then a and b are uncomparable,
- otherwise, we have $a \leq b$ if and only if $a \leq' b$.

And since we have, $a \leq' b$ if and only if $a \leq_{\mathcal{T}} b$, then \leq defined by

- If $a \in A_1$, and $b \in A_2$ then a and b are uncomparable,
- otherwise, we have $a \leq b$ if and only if $a \leq_{\mathcal{T}} b$.

Then $\Psi_{A_1, A_2} = \Psi_{A_1, A_2}^2$. □

Theorem 5.1. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, and let $s \in Y$, $u \in Z$. The following diagram is commutative:

$$\begin{array}{ccc}
 \mathbb{V}_X \otimes \mathbb{V}_Y \otimes \mathbb{V}_Z & \xrightarrow{id \otimes \searrow u} & \mathbb{V}_X \otimes \mathbb{V}_{Y \sqcup Z} \\
 \nearrow^s \otimes id \downarrow & & \downarrow \nearrow^s \\
 \mathbb{V}_{X \sqcup Y} \otimes \mathbb{V}_Z & & \\
 \searrow u \downarrow & & \downarrow \\
 \mathbb{V}_{X \sqcup Y \sqcup Z} & \xrightarrow{\Psi_{X, Z}} & \mathbb{V}_{X \sqcup Y \sqcup Z}
 \end{array}$$

Proof. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, and let $s \in Y$, $u \in Z$, then for $\mathcal{W} = (X \sqcup Y \sqcup Z, \leq_{\mathcal{W}}) = (\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}$, we have

- for all $x \in X$, $x \leq_{\mathcal{W}} s$,
- and for all $y \in X \sqcup Y$, $u \leq_{\mathcal{W}} y$,

then

- for all $x \in X$, $x \leq_{\Psi_{X, Z}(\mathcal{W})} s$,
- and for all $y \in Y$, $u \leq_{\Psi_{X, Z}(\mathcal{W})} y$,

then $\Psi_{X, Z}(\mathcal{W})$ is connected.

moreover we have $\mathcal{T} \nearrow^s \mathcal{S} = (X \sqcup Y, \leq'_1)$, with \leq'_1 defined on $X \sqcup Y$ by:

$x, y \in X \sqcup Y$ et $x \leq'_1 y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y$ and $x \leq_{\mathcal{S}} y$,
- or $x \in X$, $y \in Y$ and $s \leq_{\mathcal{S}} y$,

then $(\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U} = (X \sqcup Y \sqcup Z, \leq_{\mathcal{W}})$, with $\leq_{\mathcal{W}}$ defined on $X \sqcup Y \sqcup Z$ by:

$x, y \in X \sqcup Y \sqcup Z$ et $x \leq_{\mathcal{W}} y$ if and only if:

- Either $x, y \in X \sqcup Y$ and $x \leq'_1 y$,
- or $x, y \in Z$ and $x \leq_{\mathcal{U}} y$,
- or $x \in Z$, $y \in X \sqcup Y$ and $x \leq_{\mathcal{U}} u$,

then $x, y \in X \sqcup Y \sqcup Z$ et $x \leq_{\mathcal{W}} y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y$ and $x \leq_{\mathcal{S}} y$,
- or $x \in X$, $y \in Y$ and $s \leq_{\mathcal{S}} y$,
- or $x, y \in Z$ and $x \leq_{\mathcal{U}} y$,
- or $x \in Z$, $y \in Y$ and $x \leq_{\mathcal{U}} u$,
- or $x \in Z$, $y \in X$ and $x \leq_{\mathcal{U}} u$,

if we apply $\Psi_{X,Z}$ to $(\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}$, we can eliminate the cases: $x \in Z, y \in X$ and $x \leq_u u$.

On the other hand

$\mathcal{S} \searrow_u \mathcal{U} = (Y \sqcup U, \leq_1)$, with \leq_1 defined on $Y \sqcup Z$ by:

$x, y \in Y \sqcup Z$ et $x \leq_1 y$ if and only if:

- Either $x, y \in Y$ and $x \leq_s y$,
- or $x, y \in Z$ and $x \leq_u y$,
- or $x \in Z, y \in Y$ and $x \leq_u u$,

then $\mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) = (X \sqcup Y \sqcup Z, \leq)$, with \leq defined on $X \sqcup Y \sqcup Z$ by:

$x, y \in X \sqcup Y \sqcup Z$ et $x \leq y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y \sqcup Z$ and $x \leq_1 y$,
- or $x \in X, y \in Y \sqcup Z$ and $s \leq_1 y$,

then $x, y \in X \sqcup Y \sqcup Z$ et $x \leq y$ if and only if:

- Either $x, y \in X$ and $x \leq_{\mathcal{T}} y$,
- or $x, y \in Y$ and $x \leq_s y$,
- or $x, y \in Z$ and $x \leq_u y$,
- or $x \in Z, y \in Y$ and $x \leq_u u$,
- or $x \in X, y \in Y$ and $s \leq_s y$.

Then the equality between

$$\mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) = \Psi_{X,Z}((\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}).$$

□

Corollary 5.1. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_s)$ and $\mathcal{U} = (Z, \leq_u)$ be three finite connected topological spaces, and let $s \in Y, u \in Z$. Then

$$\mathcal{T} \searrow_s (\mathcal{S} \nearrow^u \mathcal{U}) = \Psi_{X,Z}((\mathcal{T} \searrow_s \mathcal{S}) \nearrow^u \mathcal{U}).$$

i.e, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{V}_X \otimes \mathbb{V}_Y \otimes \mathbb{V}_Z & \xrightarrow{id \otimes \nearrow^u} & \mathbb{V}_X \otimes \mathbb{V}_{Y \sqcup Z} \\ \searrow_s \otimes id \downarrow & & \downarrow \searrow_s \\ \mathbb{V}_{X \sqcup Y} \otimes \mathbb{V}_Z & & \\ \nearrow^u \downarrow & & \downarrow \\ \mathbb{V}_{X \sqcup Y \sqcup Z} & \xrightarrow{\Psi_{X,Z}} & \mathbb{V}_{X \sqcup Y \sqcup Z} \end{array}$$

Proof. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_s)$ and $\mathcal{U} = (Z, \leq_u)$ be three finite connected topological spaces, and let $s \in Y, u \in Z$.

We notice $\mathcal{T}' = j(\mathcal{T})$, $\mathcal{S}' = j(\mathcal{S})$ and $\mathcal{U}' = j(\mathcal{U})$, according to theorem 5.1, we have:

$$\mathcal{T}' \nearrow^s (\mathcal{S}' \searrow_u \mathcal{U}') = \Psi_{X,Z}((\mathcal{T}' \nearrow^s \mathcal{S}') \searrow_u \mathcal{U}'),$$

then

$$j(\mathcal{T}' \nearrow^s (\mathcal{S}' \searrow_u \mathcal{U}')) = j[\Psi_{X,Z}((\mathcal{T}' \nearrow^s \mathcal{S}') \searrow_u \mathcal{U}')],$$

then

$$j(\mathcal{T}') \searrow_s j((\mathcal{S}' \searrow_u \mathcal{U}')) = \Psi_{X,Z}[j((\mathcal{T}' \nearrow^s \mathcal{S}')) \nearrow^u j(\mathcal{U}')],$$

then

$$j(\mathcal{T}') \searrow_s (j(\mathcal{S}') \nearrow^u j(\mathcal{U}')) = \Psi_{X,Z}[(j(\mathcal{T}') \searrow_s j(\mathcal{S}')) \nearrow^u j(\mathcal{U}')].$$

Then

$$\mathcal{T} \searrow_s (\mathcal{S} \nearrow^u \mathcal{U}) = \Psi_{X,Z}((\mathcal{T} \searrow_s \mathcal{S}) \nearrow^u \mathcal{U}).$$

□

Proposition 5.2. *Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, then*

$$\mathcal{T} \nearrow (\mathcal{S} \searrow \mathcal{U}) - \Psi_{X,Z}((\mathcal{T} \nearrow \mathcal{S}) \searrow \mathcal{U}) = \mathcal{S} \searrow (\mathcal{T} \nearrow \mathcal{U}) - \Psi_{Y,Z}((\mathcal{S} \searrow \mathcal{T}) \nearrow \mathcal{U}).$$

Proof. Let $\mathcal{T} = (X, \leq_{\mathcal{T}})$, $\mathcal{S} = (Y, \leq_{\mathcal{S}})$ and $\mathcal{U} = (Z, \leq_{\mathcal{U}})$ be three finite connected topological spaces, then

$$\begin{aligned} \mathcal{T} \nearrow (\mathcal{S} \searrow \mathcal{U}) - \Psi_{X,Z}((\mathcal{T} \nearrow \mathcal{S}) \searrow \mathcal{U}) &= \sum_{u \in Z, s \in Y \sqcup Z} \mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) - \sum_{u \in Z, s \in Y} \Psi_{X,Z}((\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U}) \\ &= \sum_{u, s \in Z} \mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) + \sum_{u \in Z, s \in Y} [\mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) \\ &\quad - \Psi_{X,Z}((\mathcal{T} \nearrow^s \mathcal{S}) \searrow_u \mathcal{U})] \\ &= \sum_{u, s \in Z} \mathcal{T} \nearrow^s (\mathcal{S} \searrow_u \mathcal{U}) \\ &= \sum_{u, s \in Z} \mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) \\ &= \sum_{u, s \in Z} \mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) + \sum_{s \in Z, u \in X} [\mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) \\ &\quad - \Psi_{Y,Z}((\mathcal{S} \searrow_u \mathcal{T}) \nearrow^s \mathcal{U})] \\ &= \sum_{s \in Z, u \in X \sqcup Z} \mathcal{S} \searrow_u (\mathcal{T} \nearrow^s \mathcal{U}) - \sum_{u \in X, s \in Z} \Psi_{Y,Z}((\mathcal{S} \searrow_u \mathcal{T}) \nearrow^s \mathcal{U}) \\ &= \mathcal{S} \searrow (\mathcal{T} \nearrow \mathcal{U}) - \Psi_{Y,Z}((\mathcal{S} \searrow \mathcal{T}) \nearrow \mathcal{U}). \end{aligned}$$

□

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