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Dan Goreac, Ionut Munteanu

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Dan Goreac[a,b,*], Ionut Munteanu[c,d]

[a]School of Mathematics and Statistics, Shandong University, Weihai, Weihai 264209, PR China
[b] LAMA, Univ Gustave Eiffel, UPEM, Univ Paris Est Creteil, CNRS, F-77447 Marne-la-Valleée, France
[c]Faculty of Mathematics, Al. I. Cuza University, Bd. Carol I, 11, Iasi-700506, Romania
[d] O. Mayer Institute of Mathematics, Romanian Academy, Bd. Carol I, 8, Iasi-700505, Romania

Abstract

This paper is motivated by the asymptotic stabilization of abstract SPDEs of linear type. As a first step, it proposes an abstract contribution to the exact controllability (in a general $\mathbb{L}^p$-sense, $p > 1$) of a class of linear SDEs with general time-invariant rank control coefficient in the diffusion term. From this point of view, our paper generalizes some of the results in [13] where full and null rank were considered. Necessary conditions and sufficient ones are discussed and their hierarchy and connections with the approximate controllability are illustrated. Second, our paper illustrates, on relevant frameworks of linear SPDEs, a way to drive exactly to 0 their unstable part of dimension $n \geq 1$ by using M internal, respectively N boundary controls such that $\max\{M, N\} < n$. Extensive examples are presented as is the minimal gain for judicious control pairs.

Keywords: Stochastic control; Exact controllability; (Stochastic) Partial differential equations; Asymptotic stability

1. Introduction

Our paper is motivated by the stabilization as the time parameter increases to ∞ of a class of stochastic partial differential equations driven by Brownian motion W of type

$$\begin{cases} dy_t = \left(-\mathcal{A}y_t + \beta_t^2\right) dt + \left(\mathcal{C}y_t + \beta_t^1\right) dW_t, \ \forall x \in \mathcal{O}, \ \forall t > 0, \\ y_t = \alpha_t, \ \text{on } \Gamma_1, \ y_t = 0, \text{on } \Gamma_2, \ y(0) = y_0, \ \text{in } \mathcal{O}. \end{cases}$$

The exact assumptions and the differential structure of the operator $\mathcal{A}$ will be made available later on. In order to achieve stabilization as $t \to \infty$, one can use either internal controls of type $\beta_t = \mathbf{1}_{\mathcal{O}_0}\tilde{\beta}_t$, on some open subset $\mathcal{O}_0 \subset \mathcal{O}$, $\mathbf{1}_{\mathcal{O}_0}$ being the characteristic function of the set $\mathcal{O}_0$), or frontier control of type $y_t = \alpha_t$, on Γ_1 where $\Gamma_1 \cup \Gamma_2 = \partial\mathcal{O}$, the boundary of $\mathcal{O}$. Roughly speaking, assuming that $-\mathcal{A}$ has a countable family of eigen-functions, denoted by $(\phi_k)_{k\geq 1}$, one fixes $M, N \geq 1$ and constructs these controls

1. either from internal controls of type $\beta_t := \sum_{1\leq k\leq M} \beta_k(t)\mathbf{1}_{\mathcal{O}_0}\phi_k$ (other formulations are possible cf. [11, (9.11)-(9.13)]).
2. or constructed from frontier controls of type $\alpha_t := \sum_{1\leq k\leq N} \alpha_k(t)D^\diamond\phi_k$ (the operator $D^\diamond$ will become clear later on but it follows [11, (2.16)].

Concerning the problem of stabilization of SPDEs, the existing literature is highly rich. We mention here the results of Krstic [7], where, via the backstepping technique, a deterministic boundary stabilizing controller is designed for the stochastic Burgers equation. Using the Lyapunov function technique, Mao and his co-workers propose in [4] a stabilizing controller for the stochastic heat equation. The stabilizing effects of the noise for a class of stochastic reaction-diffusion equations is studied in [5], while for the cubic nonlinear Chafee-Infante, we mention the work [6]. Other results concerning noise stabilization can be found in

*Corresponding author
Email addresses: dan.goreac@univ-eiffel.fr (Dan Goreac), ionut.munteanu@uaic.ro (Ionut Munteanu)

[3, 8]. The stabilizability problem for two-dimensional hydrodynamical type systems is also considered in [1]. These systems usually study the stand-alone effect of either boundary or internal controls and are based on techniques different than the one exhibited below.

To fix the ideas, let χ stand for one of the values M or N., depending on which type of control one is using. The stabilization procedure is generally divided into two actions dealing with the projection of y onto the eigen-space of $\mathcal{A}$ spanned by the first χ eigenvalues, respectively with the complementary space. These first χ eigenvalues will be hereafter denoted by λ_i, for $1 \leq i \leq \chi$ and referred to as "unstable". The eigen-space associated to these eigenvalues will also be referred to as unstable. The stabilizing controls are used for the unstable part in order to drive this projection in a neighbourhood of 0. The remaining stable part benefits from the bounds on the eigenvalues and gives an exponential decrease with some parameter ρ. In each of the two situations referring to internal or boundary control, whenever the controls can be reasonably designed, it has been shown that the stabilization speed is of order $\rho := \lambda_{\chi+1}$ as exhibited in [11, Theorem 2.1] for non-stochastic setting or in [2, Eq. (7) and Theorem 1], [11, Theorem 7.1] for a stochastic one. Roughly said, it all depends on the dimension χ of the unstable space that can be controlled. Furthermore, the asymptotic estimates also involve the error in approximating 0. From this point of view, it would appear natural that mixing the two types of control (internal/boundary) led, perhaps, to a better approximation of 0 but would still involve $\lambda_{\chi+1}$ in the speed.

We aim at disproving this intuition. To achieve this goal, we use stochastic exact controllability techniques for the unstable part in order to drive this component *exactly* to 0; for a deterministic result in this sense, the reader is referred to [12]). Moreover, we show that the "unstable" space to which controllability applies is of dimension n in the interval $[\max\{M, N\}, M+N]$. This leads to an improvement of the stabilization speed to λ_{n+1}, thus decoupling the power of the control.
The paper has two novelties.

1. From the finite-dimensional SDE point of view, we generalize the results in [13] where the authors deal with either full/surjective control on the noise or no control on the noise. Our result, stated and proven in a general non-homogeneous setting for the sake of completion, allows an arbitrary, time-constant rank for the control coefficient in the noise term. Necessary conditions and sufficient ones are discussed and their hierarchy and connections with the approximate controllability illustrated.

2. From the SPDE point of view, we design, starting from the first $\max\{M, N\}$ eigen-functions a way to drive to 0 an n-dimensional part of the SPDE, where n strictly exceeds the number of eigen-functions employed. The method is illustrated on several multi-dimensional examples that do not fit, to our best knowledge, previously known frameworks. We emphasize that the stabilization is a stochastic one as opposed to analytic methods lifted to a stochastic setting.

The paper is organized as follows. The remaining of the section is devoted to notations as well as some details on the method to be applied to the SPDEs. In particular, we explain the qualitative nature of the finite-dimensional controlled SDE resulting by projecting the initial SPDE onto the unstable space. Section 2 presents the main contributions to the controllability of linear SDEs with general rank for the coefficient acting on the control in the diffusion term. We give a necessary Grammian-related condition for controllability in Subsection 2.4 and discuss its (non-)sufficiency in Subsection 2.5. The main result is given in Theorem 18 and discussions are continued in Subsection 2.6. Finally, the implications on the stabilization of SPDE of linear type and extensive examples make the object of Section 3.

1.1. Notations

For our readers' comfort, we gather here some of the notations used throughout the paper.

1. $n, m, m', d, N, M \in \mathbb{N}^*$ (i.e. $n \geq 1$, etc.) are dimensions for the state-space(s) or the control space;
2. $\mathbb{R}^n$ stands for the standard n-dimensional Euclidean space endowed with the usual product $\langle \cdot, \cdot \rangle$ end the induced norm $|\cdot|$;
3. $\mathbb{R}^{n\times m}$ stands for the space of $n \times m$-matrices with real entries; then $\mathbb{R}^n$ is identified with $\mathbb{R}^{n\times 1}$. The transposition of a matrix is denoted by $\cdot^*$ (t, T being reserved for the time parameters). The kernel of a matrix is denoted by ker and the image (or range) by $\mathcal{R}$;

4. For integers $r \leq \min\{m, n\}$,
$\overline{\pi}_r^{n,m} := \begin{pmatrix} I_r & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{pmatrix} \in \mathbb{R}^{n\times m}$ and $\underline{\pi}_r^{n,m} := \begin{pmatrix} 0_{(n-r)\times(m-r)} & 0_{n-r\times r} \\ 0_{r\times(m-r)} & I_r \end{pmatrix} \in \mathbb{R}^{n\times m}$;
5. Given a finite time horizon $T > 0$, a Hilbert space $(H, \|\cdot\|)$, a complete filtered probability space $\left(\Omega, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}\right)$ and $p, q \in [1, \infty)$, we define
   - $C([0,T]; H)$ be the space of continuous H-valued functions on $[0, T]$.
   - $\mathbb{L}^p_{\mathcal{F}_T}(H)$ stands for the space of $H-$valued, $\mathcal{F}_T$-measurable random variables ξ defined on Ω s.t. $\mathbb{E}[\|\xi\|^p] < \infty$. The norm of this space will be referred to as $|\cdot|_p$. Whenever there is no confusion at risk, the dependence on H will be dropped.
   - The spaces

$$\mathbb{L}^{1,p,q}_{\mathbb{F}}(H) := \mathbb{L}^p_{\mathbb{F}}(\Omega; \mathbb{L}^q(0,T;H)) = \left\{ \begin{array}{r} \varphi : [0,T]\times\Omega \to H: \ \varphi \text{ prog. measurable,} \\ \mathbb{E}\left[\|\varphi(\cdot,\omega)\|^p_{\mathbb{L}^q([0,T];H)}\right] < \infty \end{array} \right\},$$

$$\mathbb{L}^{2,p,q}_{\mathbb{F}}(H) := \mathbb{L}^q_{\mathbb{F}}([0,T]; \mathbb{L}^p(\Omega; H)) = \left\{ \begin{array}{r} \varphi : [0,T]\times\Omega \to H: \ \varphi \text{ prog. measurable,} \\ \int_0^T \left(\mathbb{E}\left[\|\varphi(t)\|^p_H\right]\right)^{\frac{q}{p}} dt < \infty \end{array} \right\}.$$

   - We denote by $\mathcal{U}(\mathbb{R}^d)$ the space of all progressively measurable control processes $u : [0,T]\times\Omega \to \mathbb{R}^d$ such that (4) admits a unique strong solution $X^{x,u} \in \mathbb{L}^1(\Omega; C([0,T]; \mathbb{R}^n))$.
   - Let $B, D \in \mathbb{R}^{n\times d}$. The family of all controls $u \in \mathcal{U}(\mathbb{R}^d)$ for which $Bu \in \mathbb{L}^{1,p,1}_{\mathbb{F}}(\mathbb{R}^n)$, $Du \in \mathbb{L}^{1,p,2}_{\mathbb{F}}(\mathbb{R}^n)$ (s.t. $X^{x,u} \in \mathbb{L}^p(\Omega; C([0,T]; \mathbb{R}^n))$) will be denoted by $\mathcal{U}^p(\mathbb{R}^d)$.
   - If the coefficients B, D are transformed (appearing under the form $\widehat{\cdot}$), the corresponding space $\mathcal{U}^p$ will become $\widehat{\mathcal{U}}^p$.
6. The (state) space for our SPDE will be a bounded open set $\mathcal{O} \subset \mathbb{R}^n$. Its frontier $\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$, where Γ_1 is assumed to have nonzero surface measure.
7. $\mathbb{L}^2(\mathcal{O})$ stands for the usual space of square-integrable elements.
8. If $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times m'}$ are two matrices, we denote by $(A|B) \in \mathbb{R}^{n\times(m+m')}$ their joint matrix.

1.2. Setting for the SPDE and Projected Finite-Dimensional SDEs

In order to explain the systems of finite nature treated shortly after, let us give a few hints on the procedure applied to the unstable part of the SPDE. To this purpose, we consider an open bounded domain $\mathcal{O} \subset \mathbb{R}^m$ with smooth boundary $\partial\mathcal{O}$ split into two sets $\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$ of which Γ_1 is assumed to have nonzero (surface) measure. We consider a linear self-adjoint differential operator $\mathcal{A}$ assumed to be closed, densely defined on $\mathbb{L}^2(\mathcal{O})$. For simplicity, its domain is denoted by $\mathcal{D}(\mathcal{A})$. Moreover, we assume that $-\mathcal{A}$ satisfies the following.

- It has a countable family of eigenfunctions $(\phi_j)_{j\geq 1}$ associated to the eigenvalues $(\lambda_j)_{j\geq 1}$;
- The family $(\lambda_j)_{j\geq 1}$ is assumed to be ordered increasingly.

In the spirit of [11, (2.16)], we introduce the lifting operator $\mathbb{D}_\mu$ as follows: let $\mu > 0$ be sufficiently large, then given $\chi \in \mathbb{R}$, we denote by $\mathbb{D}_\mu\chi := y$ the unique solution to

$$\mathcal{A}y + \mu y = 0, \quad \forall x \in \mathcal{O}, \ y = \chi \text{ on } \Gamma_1, \ y = 0, \text{ on } \Gamma_2. \tag{1}$$

The adjoint operator $\mathbb{D}^*_\mu$ can be written with respect to a μ-independent (universal) operator $\mathbb{D}^\diamond$ via

$$\mathbb{D}^\diamond\phi_j = (\mu - \lambda_j)\,\mathbb{D}^*_\mu\phi_j,$$

(cf. [11, Section 2.2]). From now on, unless stated otherwise, we make the following.

Assumption 1. None of the functions $(\mathbb{D}^\diamond\phi_j)_{1\leq j\leq N}$ is trivially 0 on Γ_1.

For the boundary control, one envisages the use of controls of type $\alpha(t) := \sum\limits_{1\leq k\leq N} u_k(t)\mathbb{D}^{\diamond}\phi_k$. For the internal control, one uses $\beta^i(t) := \sum\limits_{1\leq k\leq M} v_k^i(t)\mathbf{1}_{\mathcal{O}_0}\phi_k$, $i=1$ standing for the control on the noise and $i=2$ for the control on the drift. Whenever possible, we will try to set as many v^i as possible to 0 either in the drift or in the noise; see Remark 2. With these consideration, we consider the mixed boundary/internal control system

$$\begin{cases} dy_t = \left(\mathcal{A}y_t + \beta_t^2\right)dt + \left(\mathcal{C}y_t + \beta_t^1\right)dW_t, \ \forall x\in\mathcal{O}, \ \forall t>0, \\ y_t = 0, \text{on } \Gamma_2, y_t = \alpha_t, \text{on } \Gamma_1. \end{cases} \tag{2}$$

(Above, by $\mathcal{A}$, we understand its differential form, in fact. Here, $\mathcal{C}$ is a bounded operator. Using the operator $\mathbb{D}_\mu$ (see [11, (2.29)], we lift the boundary control into an internal one and (2) modifies the dynamics to

$$dy_t = \left(\mathcal{A}y_t + \mathcal{A}\mathbb{D}\alpha_t + \mu\mathbb{D}\alpha_t + \beta_t^2\right)dt + \left(Cy_t + \beta_t^1\right)dW_t, \ t>0. \tag{3}$$

Here, by $\mathcal{A}$ we understand its extension to the whole $L^2(\mathcal{O})$. For details, see [11].

We will only give the details on the treatment of the unstable part, the second step described in our introduction being similar to the cited references.
Let n be an integer belonging to the interval $n\in[\max\{M,N\}, M+N]$. Throughout the section, we assume that the eigen-space generated by the first n eigen-functions of $\mathcal{A}$ is $\mathcal{C}$-invariant. By projecting the equation (3) onto the n-dimensional space generated by the eigenfunctions $(\phi_i)_{1\leq i\leq n}$, we get a system of type

$$dX_t^{x,\bar{u}} = \left(AX_t^{x,\bar{u}} + B\bar{u}_t\right)dt + \left(CX_t^{x,\bar{u}} + D\bar{u}_t\right)dW_t, \ t\geq 0; \ X_0^{x,\bar{u}} = x\in\mathbb{R}^n,$$

as follows:

1. A is a diagonal matrix of type $n\times n$ containing the first n eigenvalues of $\mathcal{A}$;
2. C is got by setting $C := \langle\mathcal{C}\phi_i,\phi_j\rangle_{1\leq i,j\leq n}$;
3. We choose to write our controls as column vector in the order $\left(\beta^1,\alpha,\beta^2\right)$. The dimension of the control $\bar{u}$ whose components are v^1, u, v^2 is $d = M+N+M$;
4. D is the control coefficient in the noise. It only contains non-null entries for the β^1 component
$$\left(\bar{D}v^1\right)_i = \sum_{1\leq k\leq M}\langle\phi_i,\phi_k\rangle_{\mathcal{O}_0} v_k^1, \ \forall 1\leq i\leq n.$$
Then, $D = \begin{pmatrix}\bar{D} & 0_{n\times(N+M)}\end{pmatrix}$.
5. B is the control coefficient in the drift. The matrix B consists of contributions of the boundary control α and of the internal control β^2. We set $\bar{B} := \left(\langle\mathbf{D}^{\diamond}\phi_i,\mathbf{D}^{\diamond}\phi_k\rangle\right)$, $\forall 1\leq i\leq n$, $\forall 1\leq k\leq N$, then $B = \begin{pmatrix}0_{m\times M} & \bar{B} & \bar{D}'\end{pmatrix}$. The nature of $\bar{D}'$ is similar to that of $\bar{D}$.

Remark 2. 1. As it will be detailed in Section 3, the matrix $\bar{D}$ has a positive rank (for the previous choice, it is M). However, it is not of full rank n.
2. A simple choice is $\bar{D}' = \bar{D}$ (same coefficient in noise and drift).
3. Alternatively, whenever possible, **we will replace, in B as many columns of $\bar{D}$ as possible by** $0_{m\times 1}$. Indeed, a careful look at the proof of Theorem 18 shows that v^1 elements can be explicitly obtained by solving the associated BSDE and are of class $\mathbb{L}_{\mathbb{F}}^{1,p,2}$. However, the remaining elements u and v^2 are given via the representation result in [10], can only be approximated and, furthermore, they have less regularity, belonging to $\mathbb{L}_{\mathbb{F}}^{1,p,1}$.
4. Finally, one can turn to the control of the equation **without acting on the noise**, i.e. by picking $D := D^0 = 0_{n\times(N+2M)}$.

2. The SDE Abstract Result

2.1. Controllability Notions

We consider, on a complete probability space satisfying the usual conditions of completeness and right-continuity, a one-dimensional Brownian motion W and the controlled linear stochastic systems of type

$$dX_t^{x,u} = \left(A_tX_t^{x,u} + B_tu_t\right)dt + \left(C_tX_t^{x,u} + D_tu_t\right)dW_t, \ t\geq 0; \ X_0^{x,u} = x\in\mathbb{R}^n. \tag{4}$$

Remark 3. The reader will note that this system contains random non-homogeneous coefficients. Such generality is not directly needed in order to consider the SPDEs we envisage. However, we find it convenient both for further developments and in order to illustrate the relations with [13] to treat this rather general setting.

Assumption 4. The following stands.

$$A, C \in \mathbb{L}_{\mathbb{F}}^{\infty}\left(\Omega \times [0,T]; \mathbb{R}^{n\times n}\right); \ B, D : \Omega \times [0,T] \longrightarrow \mathbb{R}^{n\times d}, \ \mathbb{F}\text{-progressive.} \tag{5}$$

Let us briefly recall some controllability concepts.

Definition 5.
1. The system (4) is said to be $\mathbb{L}^p$ exactly controllable (with some $p > 1$ and generic $\mathcal{U}$-controls) if, for every initial datum $x \in \mathbb{R}^n$ and every target $\xi \in \mathbb{L}^p_{\mathcal{F}_T}(\mathbb{R}^n)$, there exists a control $u \in \mathcal{U}$ such that $X^{x,u} \in \mathbb{L}^p_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}^n))$ satisfies $X^{x,u} = \xi, \ \mathbb{P} - a.s.$;
2. The system (4) is said to be $\mathbb{L}^p$ approximately controllable (with some $p > 1$ and generic $\mathcal{U}$-controls) if, for every initial datum $x \in \mathbb{R}^n$ and every target $\xi \in \mathbb{L}^p_{\mathcal{F}_T}(\mathbb{R}^n)$ and every $\varepsilon > 0$, there exists a control $u^\varepsilon \in \mathcal{U}$ such that the associated trajectory $X^{x,u^\varepsilon} \in \mathbb{L}^p_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}^n))$ satisfies $\left|X_T^{x,u^\varepsilon} - \xi\right|_p \leq \varepsilon$;
3. The concepts of null exact and approximate controllability are defined for $\xi = 0$.

2.2. A class of time-constant rank D coefficients

We begin with a small remark concerning a (time independent) $D \in \mathbb{R}^{n\times d}$. Let us assume that, for some $r \leq \min(d,n)$, $rank(D) = r$. Then, using equivalence of matrices, there exist two invertible matrices $G \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{d\times d}$ such that $GDF = \begin{pmatrix} I_r & 0_{r\times(d-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(d-r)} \end{pmatrix} \left(= \overline{\pi}_r^{n,d}\right)$. Of course, whenever D changes, so does G (and F). To simplify arguments on admissibility of controls, we assume the following regularity of F.

Assumption 6. There exists a time independent invertible matrix $G \in \mathbb{R}^{n\times n}$ and regular matrix-valued processes F and $F^{-1} \in \mathbb{L}_{\mathbb{F}}^{\infty}\left(\Omega \times [0,T]; \mathbb{R}^{d\times d}\right)$ such that $GD_tF_t = \overline{\pi}_r^{n,d}$.

Remark 7.
1. The regularity of F is equivalent to the existence of a positive constant $\delta_F > 0$ such that $\delta_F \leq \inf_{0\leq t\leq T} Tr\left(F_tF_t^*\right) \leq \sup_{0\leq t\leq T} Tr\left(F_tF_t^*\right) \leq \delta_F^{-1}$.
2. When $rank(D_t) = n$, one recalls that $D_t^*\left(D_tD_t^*\right)^{-1}$ is the Moore-Penrose pseudoinverse of D_t. Then, one simply takes the linearly independent rows of $D_t^*\left(D_tD_t^*\right)^{-1}$ (forming a basis of $\mathcal{R}(D_t^*)$) and completes them into a basis of $\mathbb{R}^d$ by adding $\overline{F}_t \in \mathbb{R}^{d\times(d-n)}$ (a basis of $\ker(D)$). The matrix $F_t := \left(D_t^*\left(D_tD_t^*\right)^{-1} \mid \overline{F}_t\right)$ satisfies $D_tF_t = \begin{pmatrix} I_n & \mid 0_{n\times(d-n)} \end{pmatrix}$ such that G may be taken as time-constant.

2.3. Equivalent Equation

As it is standard by now, we will simplify the equation (4) by getting rid of the matrix D and decoupling the control u following the action of D (thus an r-dimensional component), respectively the complementary part. To this purpose, we introduce the transformed coefficients.

$$\widehat{B}_t^1 := GB_tF_t\overline{\pi}_r^{d,r}, \ \widehat{B}_t^2 := GB_tF_t\underline{\pi}_{d-r}^{d,d-r}, \ \widehat{C}_t := GC_tG^{-1}, \ \widehat{A}_t := GA_tG^{-1} - GB_tF_t\overline{\pi}_r^{d,n}\widehat{C}_t.$$

Next, we consider the associated controlled stochastic system

$$\begin{cases} d\widehat{X}_t^{x,u^1,u^2} = \left(\widehat{A}_t\widehat{X}_t^{x,u^1,u^2} + \widehat{B}_t^1u_t^1 + \widehat{B}_t^2u_t^2\right)dt + \left(\underline{\pi}_{n-r}^{n,n}\widehat{C}_t\widehat{X}_t^{x,u^1,u^2} + \begin{pmatrix} u_t^1 \\ 0_{(n-r)\times 1}\end{pmatrix}\right)dW_t, \\ \text{for } 0 \leq t \leq T, \ \widehat{X}_0^{x,u^1,u^2} = x \in \mathbb{R}^n. \end{cases} \tag{6}$$

Proposition 8. *Let us assume that* (5) *holds true and there exists* $\delta > 1$ *such that*

$$\widehat{A} \in \mathbb{L}_{\mathbb{F}}^{1,\infty,\delta}\left(\mathbb{R}^{n\times n}\right), \ \widehat{B}^1 \in \mathbb{L}_{\mathbb{F}}^{1,\infty,2}\left(\mathbb{R}^{n\times r}\right), \ \widehat{B}^2 \in \mathbb{L}_{\mathbb{F}}^{1,\infty,2}\left(\mathbb{R}^{n\times(d-r)}\right). \tag{7}$$

Then

(i) The $\mathbb{L}^p$-exact controllability of the initial system using $\mathcal{U}^p\left(\mathbb{R}^d\right)$-controls is equivalent to the $\mathbb{L}^p$-exact controllability of the transformed system (6) using $\mathbb{L}_{\mathbb{F}}^{1,p,2}\left(\mathbb{R}^r\right)\times\widehat{\mathcal{U}}^p\left(\mathbb{R}^{d-r}\right)$-controls.

(ii) The $\mathbb{L}^p$-approximate (full, resp. null) controllability of the initial system using $\mathbb{L}_{\mathbb{F}}^{1,\infty,\infty}\left(\mathbb{R}^d\right)$-controls is equivalent to the $\mathbb{L}^p$-approximate (full, resp. null) controllability of the transformed system using $\mathbb{L}_{\mathbb{F}}^{1,\infty,\infty}\left(\mathbb{R}^r\right)\times\mathbb{L}_{\mathbb{F}}^{1,\infty,\infty}\left(\mathbb{R}^{d-r}\right)$-controls.

The arguments are very much like those in [13, Theorem 3.4] and the only effort is to be made in order to prove admissibility of controls. As a consequence, we will skip it.

2.4. A Gramian-like Necessary Condition for (Approximate) Null-Controllability

From now on, we assume the following.

Assumption 9. (5) holds true and there exists $\delta>1$ such that

$$\widehat{A}\in\mathbb{L}_{\mathbb{F}}^{1,\infty,\delta}\left(\mathbb{R}^{n\times n}\right),\ \widehat{B}^1\in\mathbb{L}_{\mathbb{F}}^{1,\infty,2}\left(\mathbb{R}^{n\times r}\right),\ \widehat{B}^2\in\mathbb{L}_{\mathbb{F}}^{1,\infty,2\delta}\left(\mathbb{R}^{n\times(d-r)}\right). \tag{8}$$

Remark 10. This assumption is not optimal from the integrability point of view of $\widehat{B}^1$. Whenever $p>1$ is fixed, one can, for instance, ask that $\widehat{B}^1\in\mathbb{L}_{\mathbb{F}}^{1,\delta\max\{p,2\},2}\left(\mathbb{R}^{n\times r}\right)$.

A necessary condition for null-controllability (both exact and approximate) can be obtained using the SDE

$$d\mathcal{Y}_t=-\mathcal{Y}_t\widehat{A}_tdt-\mathcal{Y}_t\widehat{B}_t^1\overline{\pi}_r^{r,n}dW_t,\ t\geq 0,\ \mathcal{Y}_0=I_n. \tag{9}$$

whose solution belongs to all spaces $\mathbb{L}_{\mathbb{F}}^p\left(\Omega;C\left([0,T];\mathbb{R}^{n\times n}\right)\right)$, for all $p>1$. The arguments are rather standard in the spirit of [9] and will be omitted here.

Proposition 11. *A necessary condition for the approximate (hence, a fortiori exact!) null-controllability of (6) (using $\mathbb{L}_{\mathbb{F}}^{1,p,2}\left(\mathbb{R}^r\right)\times\widehat{\mathcal{U}}^p\left(\mathbb{R}^{d-r}\right)$-controls) is that the Gramian-like matrix*

$$\mathcal{G}:=\mathbb{E}\left[\int_0^T\mathcal{Y}_t\widehat{B}_t^2\left(\widehat{B}_t^2\right)^*\mathcal{Y}_t^*dt\right]\text{ be invertible.} \tag{10}$$

Remark 12. In fact, due to [13, Theorem 3.6], whenever D is of full rank $r=n$ (and the Assumption 9 holds true), the approximate null-controllability and the exact controllability (in $\mathbb{L}^p$) are equivalent.

2.5. Counterexamples on Sufficiency

We will give two examples concerning the case when $r<n$ as follows:

- this condition on the Grammian may fail to imply approximate null-controllability of a particular system;
- even when null-controllability can be achieved, invertibility of $\mathcal{G}$ may fail to imply exact controllability (to general targets).

Let us begin with an example of system satisfying the previous necessary Grammian-like condition and failing to be approximately null-controllable.

Example 13. *We consider $d=2$ and $n=3$, $r=1$ and the coefficients* $A=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}$, $C=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$, $B:=\begin{pmatrix}1&1\\0&1\\0&0\end{pmatrix}$, $D=\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$. *Then* $\widehat{B}^1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ *and* $\widehat{B}^2=\begin{pmatrix}1\\1\\0\end{pmatrix}$. *We get* $\mathcal{Y}_t=\begin{pmatrix}\exp\left(-W_t-\frac{3t}{2}\right)&0&0\\0&1&0\\0&-t&1\end{pmatrix}$, *such that* $\mathcal{G}=\begin{pmatrix}1-e^{-T}&1-e^{-T}&(T+1)\,e^{-T}-1\\1-e^{-T}&T&-\frac{T^2}{2}\\(T+1)e^{-T}-1&-\frac{T^2}{2}&\frac{T^3}{3}\end{pmatrix}$. *One easily notes (by simply computing the* *determinant) that $\mathcal{G}$* ***is always invertible*** *if $T > 0$. However, if $X = (x\ y\ z)^*$ is the solution of the initial system (6), one gets $y_t^{y,u,v} = y + \int_0^t v_s ds$ and $z_t^{z,u,v} = z + \int_0^t y_s^{y,u,v} ds + \int_0^t y_s^{y,u,v} dW_s$. In particular $\mathbb{E}\left[z_T^{z,u,v} e^{-W_T - \frac{T}{2}}\right] = z$ which shows that* ***one cannot control the z component in $\mathbb{L}^p_{\mathcal{F}_T}$-neighborhoods of*** *$0$ with any $p > 1$.*

The second example shows that even when the null-controllability condition is added to the previous condition on $\mathcal{G}$, this may still fail to imply exact controllability.

Example 14. *We consider $d = 2 = n$ and $r = 1$, $C := 0_2$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\widehat{B}^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\widehat{B}^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One easily notes $\mathcal{Y}_t = \begin{pmatrix} 1 & 0 \\ -t - W_t & 1 \end{pmatrix}$ and, thus, $\mathcal{G} = \begin{pmatrix} T & -\frac{T^2}{2} \\ -\frac{T^2}{2} & \frac{T^3}{3} + \frac{T^2}{2} \end{pmatrix}$ is invertible (for every $T > 0$). On the other hand, $X_t^{x,u} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1 + \int_0^t \left(u_s^2 ds + u_s^1 dW_s\right) \\ x_2 + \int_0^t \left(u_s^1 + x_1(s)\right) ds \end{pmatrix}$. We make the notation $v_\cdot := u_\cdot^1 + x_1(\cdot) \in \mathbb{L}_{\mathbb{F}}^{1,p,2}(\mathbb{R})$ and we get $dx_2(t) = v_t dt$ which* ***is not exactly controllable with $\mathbb{L}_{\mathbb{F}}^{1,p,2}(\mathbb{R})$-controls***. *However,* ***the system is exactly null-controllable*** *(using $u_s^1 := -\frac{x_2}{T} - x_1(s)$).*

Remark 15. This example shows that as soon as D is not null, neither the Gramian-like condition nor the full rank of B suffice to guarantee exact controllability.

2.6. Exact Controllability

With regards to the decomposition of the matrix D, if one reasons in the simplest framework in which the first r and the last $n-r$ component are independent and the coefficients are bloc-diagonal, the last $n-r$ component should benefit of "full rank" associated control coefficient ($\widehat{B}_t^2$). Motivated by this remark, let us assume the following (similar) condition $\mathcal{R}\left(\widehat{A}_t \underline{\pi}_{n-r}^{n,n}\right) \cup \mathcal{R}\left(\underline{\pi}_{n-r}^{n,n}\right) \subset \mathcal{R}\left(\widehat{B}_t^2\right)$. In other words (using, for example, Douglas' Theorem), one gets $\left(\widehat{A}_t \underline{\pi}_{n-r}^{n,n} \mid \underline{\pi}_{n-r}^{n,n}\right) = \widehat{B}_t^2 H_t$, for some $H_t \in \mathbb{R}^{(d-r)\times 2n}$. We are actually going to assume a little more, namely

Assumption 16. The following range inclusion holds true$\mathcal{R}\left(\widehat{A}_t \underline{\pi}_{n-r}^{n,n}\right) \cup \mathcal{R}\left(\underline{\pi}_{n-r}^{n,n}\right) \subset \mathcal{R}\left(\widehat{B}_t^2\right)$ and the feedback matrix-valued process H such that $\left(\widehat{A}_t \underline{\pi}_{n-r}^{n,n} \mid \underline{\pi}_{n-r}^{n,n}\right) = \widehat{B}_t^2 H_t$ can be chosen regular enough $H \in \mathbb{L}_{\mathbb{F}}^{1,\infty,2}\left(\mathbb{R}^{(d-r)\times 2n}\right)$.

We denote by $H^1 := H\overline{\pi}_n^{2n,n}$ and $H^2 := H\underline{\pi}_n^{2n,n}$ and note that $\widehat{A}_t \underline{\pi}_{n-r}^{n,n} = \widehat{B}_t^2 H_t^1$ and $\underline{\pi}_{n-r}^{n,n} = \widehat{B}_t^2 H_t^2$.

Remark 17.

1. Since one assumes $\mathcal{R}\left(\underline{\pi}_{n-r}^{n,n}\right) \subset \mathcal{R}\left(\widehat{B}_t^2\right)$, the respective dimensions should satisfy $n-r \leq \min(n, d-r)$ which amounts to asking $n \leq d$. This is hardly surprising since the condition is also present in [13, Theorem 3.2] when $r = 0$.
2. In the case when $r = 0$ ($D = 0$), it follows that $\underline{\pi}_{n-r}^{n,n} = I_n$. This implies that B should have full rank (the condition on A is redundant). In this case, a natural choice is $H_t^2 := \left(B_t^2\right)^* \left[B_t^2 \left(B_t^2\right)^*\right]^{-1}$ (there is no point in using $\widehat{\cdot}$) and the regularity can be obtained by asking$\left[B_\cdot^2 \left(B_\cdot^2\right)^*\right]^{-1} \in \mathbb{L}_{\mathbb{F}}^{\infty}\left(\Omega \times [0,T]; \mathbb{R}^{n\times n}\right)$.
3. This condition can equivalently be stated as $\mathcal{R}\left(\underline{\pi}_{n-r}^{n,n}\right) \subset \mathcal{R}\left(\widehat{B}_t^2\right)$ and $\mathcal{R}\left(\underline{\pi}_{n-r}^{n,n}\right)$ is $\left(\widehat{A}_t; \widehat{B}_t^2\right)$-(strictly) invariant.

Theorem 18. *We let Assumptions 9 and 16 hold true and the Gramian $\mathcal{G} = \mathbb{E}\left[\int_0^T \mathcal{Y}_t \widehat{B}_t^2 \left(\widehat{B}_t^2\right)^* \mathcal{Y}_t^* dt\right]$ to be invertible. Then, for every $q \in (1,p)$, the system (6) is $\mathbb{L}^p$-exactly controllable with $\mathbb{L}_{\mathbb{F}}^{1,p,2}(\mathbb{R}^r) \times \widehat{\mathcal{U}}^q\left(\mathbb{R}^{d-r}\right)$-controls.*

Proof. Let us consider arbitrary inital data $x \in \mathbb{R}^n$ and final random variable $\xi \in \mathbb{L}^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. First, let us consider the BSDE

$$dY_t^1 = \left(\widehat{A}_t Y_t^1 + \left(\widehat{B}_t^1 \mid 0_{n\times(n-r)}\right) Z_t^1\right) dt + \left(\underline{\pi}^{n,n}_{n-r}\widehat{C}_t Y_t^1 + Z_t^1\right) dW_t,\ 0 \le t \le T;\ \ Y_T^1 = \xi. \tag{11}$$

This equation admits a unique solution $\left(Y_\cdot^{T,\xi}, Z_\cdot\right) \in \mathbb{L}^p_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times \mathbb{L}^{1,p,2}_{\mathbb{F}}(\mathbb{R}^n)$.
Second, for every $x \in \mathbb{R}^n$, consider the feedback control $u_t^{gram,2} := -\left(\widehat{B}_t^2\right)^* \mathcal{Y}_t^* \mathcal{G}^{-1}\left(x - Y_0^1\right)$ and the BSDE

$$dY_t^2 = \left(\widehat{A}_t Y_t^2 + \widehat{B}_t^2 u_t^{gram,2} + \widehat{B}_t^1 \overline{\pi}^{r,n}_r Z_t^2\right) dt + \left(\underline{\pi}^{n,n}_{n-r}\widehat{C}_t Y_t^2 + Z_t^2\right) dW_t,\ 0 \le t \le T,\ \ Y_T^2 = 0. \tag{12}$$

Itô's formula applied to $\mathcal{Y}Y^2$ on $[0,T]$,with $\mathcal{Y}$ obeying (9), yields

$$Y_0^2 = -\mathbb{E}\left[\int_0^T \mathcal{Y}_t \widehat{B}_t^2 u_t^{gram,2} dt\right] = x - Y_0^1. \tag{13}$$

Finally, we consider the $\mathbb{R}^{n\times n}$-valued fundamental system $d\overline{\theta}_t = \underline{\pi}^{n,n}_{n-r}\widehat{C}_t\overline{\theta}_t dW_t$, for $0 \le t \le T$, $\theta_0 = I_n$. Due to the $\mathbb{L}^\infty$-regularity of $\widehat{C} := GCG^{-1}$, these solutions are in $\mathbb{L}^{p'}_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^{n\times n}))$, for every $p' > 1$. One easily notes that $\overline{\theta}_t = \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ \alpha_t & \theta_t \end{pmatrix}$, where$d\theta_t = \underline{\pi}^{n-r,n}_{n-r}\widehat{C}_t\underline{\pi}^{n,n-r}_{n-r}\theta_t dW_t$, for $0 \le t \le T$, $\theta_0 = I_{n-r}$, and $\alpha \in \mathbb{R}^{(n-r)\times r}$. The inverse is given (for $t \ge 0$) by $d\overline{\theta}_t^{-1} = -\overline{\theta}_t^{-1}\underline{\pi}^{n,n}_{n-r}\widehat{C}_t dW_t + \overline{\theta}_t^{-1}\underline{\pi}^{n,n}_{n-r}\widehat{C}_t\underline{\pi}^{n,n}_{n-r}\widehat{C}_t dt$, with initial condition $\overline{\theta}_0^{-1} = I_n$ or, in explicit form, $\overline{\theta}_t^{-1} = \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ -\theta_t^{-1}\alpha_t & \theta_t^{-1}\end{pmatrix}$, where $\theta_0^{-1} = I_{n-r}$ and

$$d\theta_t^{-1} = -\theta_t^{-1}\underline{\pi}^{n-r,n}_{n-r}\widehat{C}_t\underline{\pi}^{n,n-r}_{n-r} dW_t + \theta_t^{-1}\underline{\pi}^{n-r,n}_{n-r}\widehat{C}_t\underline{\pi}^{n,n}_{n-r}\widehat{C}_t\underline{\pi}^{n,n-r}_{n-r} dt,\ \text{for } 0 \le t \le T.$$

Then, for $1 < q < p$ and $\widehat{\mathcal{U}}^q\left(\mathbb{R}^{d-r}\right)$-controls u^2, the solution of the system

$$dX_t = \left(\underline{\pi}^{n,n}_{n-r}\left(\widehat{C}_t X_t - Z_t^1 - Z_t^2\right)\right) dW_t + \widehat{B}_t^2 u_t^2 dt,\ 0 \le t \le T,\ X_0 = x \in \mathbb{R}^n.$$

is explicitely given, for $0 \le t \le T$, as

$$\begin{aligned} X_t = \overline{\theta}_t\Big\{x &+ \int_0^t \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ -\theta_s^{-1}\alpha_s & \theta_s^{-1}\end{pmatrix}\widehat{B}_s^2 u_s^2 ds - \int_0^t \begin{pmatrix} 0_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & \theta_s^{-1}\end{pmatrix}\left(Z_s^1 + Z_s^2\right) dW_s \\ &+ \int_0^t \begin{pmatrix} 0_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & \theta_s^{-1}\end{pmatrix}\widehat{C}_s\underline{\pi}^{n,n}_{n-r}\left(Z_s^1 + Z_s^2\right) ds\Big\}. \end{aligned}$$

We wish to emphasize that whenever $x \in \in \mathcal{R}\left(\underline{\pi}^{n,n}_{n-r}\right)$ and u^2 is such that $\widehat{B}_s^2 u_s^2 \in \mathcal{R}\left(\underline{\pi}^{n,n}_{n-r}\right)$, $\mathbb{P}\otimes\mathcal{L}eb$-a.s. on $\Omega\times[0,T]$, the solution $X \in \mathcal{R}\left(\underline{\pi}^{n,n}_{n-r}\right)$. Using the regularity of $\theta^{-1} \in \mathbb{L}^{p'}_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^{n\times n}))$, for every $p' > 1$, respectively of $Z^i \in \mathbb{L}^{1,p,2}_{\mathbb{F}}(\mathbb{R}^n)$ (for $i \in \{1,2\}$), for every $1 < q < p$, owing to [10, Theorem 3.1], there exists $v^2 \in \mathbb{L}^{2,q,1}_{\mathbb{F}}(\mathbb{R}^{n-r}) \subset \mathbb{L}^{1,q,1}_{\mathbb{F}}(\mathbb{R}^{n-r})$ such that

$$\int_0^T v_s^2 ds = \int_0^T \theta_s^{-1}\underline{\pi}^{n-r,n}_{n-r}\left(Z_s^1 + Z_s^2\right) dW_s - \int_0^T \theta_s^{-1}\left(\underline{\pi}^{n-r,n}_{n-r}\widehat{C}_s\underline{\pi}^{n,n-r}_{n-r}\right)\underline{\pi}^{n-r,n}_{n-r}\left(Z_s^1 + Z_s^2\right) ds,$$

$\mathbb{P} - a.s.$. Let us now define the feedback control $\overline{u}_s^2(x) := H_s^2\begin{pmatrix} 0_r \\ \theta_s v_s^2\end{pmatrix} - H_s^1 x$ and note that the solution of the equation

$$dX_t = \left[\widehat{A}_t X_t + \widehat{B}_s^2\overline{u}_t^2(X_t)\right] dt + \left(\underline{\pi}^{n,n}_{n-r}\left(\widehat{C}_t X_t - Z_t^1 - Z_t^2\right)\right) dW_t,\ t \ge 0;\ X_0 = 0. \tag{14}$$

which belongs to $\mathbb{L}^q_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$ remains in $\mathcal{R}\left(\underline{\pi}^{n,n}_{n-r}\right)$. Furthermore, it also obeys

$$dX_t = \left(\underline{\pi}^{n,n}_{n-r}\left(\widehat{C}_t X_t - Z_t^1 - Z_t^2\right)\right) dW_t + \begin{pmatrix} 0_r \\ \theta_t v_t^2\end{pmatrix} dt,\ 0 \le t \le T,\ \ X_0 = 0. \tag{15}$$

Due to the choice of v^2, this solution satisfies $X_T = 0,\ \mathbb{P}-a.s.$ Then, owing to (11),(12),(13),(14) and (15), $\mathcal{X}_t :=Y_t^1 + Y_t^2 + X_t$ satisfies

$$d\mathcal{X}_t = \left(\widehat{A}_t\mathcal{X}_t + \widehat{B}_t^1 u_t^1 + \widehat{B}_t^2 u_t^2\right) dt + \left(\underline{\pi}_{n-r}^{n,n}\widehat{C}_t\mathcal{X}_t + \begin{pmatrix} u_t^1 \\ 0 \end{pmatrix}\right) dW_t,\ \mathcal{X}_0 = x, \mathcal{X}_T = \xi.$$

where $u_t^1 := \overline{\pi}_r^{r,n}(Z_1 + Z_2)$, $u_t^2 := u_t^{gram,2} + \overline{u}_t^2(X_t)$. Our theorem is now complete. □

Remark 19. A careful look at the previous proof shows that special care (asking for $q < p$) is needed because of the a priori $\mathbb{L}^p$ regularity of Z^1 (coming from the arbitrariness of the target ξ). Whenever one aims at null-controllability (i.e. $\xi = 0$), then, under the assumptions of the previous theorem, this can be achieved with $\mathbb{L}_{\mathbb{F}}^{1,p,2}(\mathbb{R}^r) \times \widehat{\mathcal{U}}^p(\mathbb{R}^{d-r})$-controls.

We give an easily verifiable criterion and infer a condition for the case in which A, B, C, D are deterministic and no longer depend on the time parameter t.

Corollary 20.
1. *If rank $\left(\widehat{B}^2\right) = n$ and the coefficients $\widehat{A}, \widehat{B}^i,\ i \in \{1,2\}$ are $\mathbb{L}^\infty$-regular, then, for every $p > q > 1$, the system (4) is $\mathbb{L}^p$-exactly controllable using $\mathbb{L}_{\mathbb{F}}^{1,p,2}\left(\mathbb{R}^{rank(D)}\right) \times \widehat{\mathcal{U}}^q\left(\mathbb{R}^{d-rank(D)}\right)$-controls.*
2. *In the deterministic, time-invariant framework, if rank $\left(\begin{pmatrix} B \\ D \end{pmatrix}\right) \geq n + rank(D)$, then, for every $p > q > 1$, the system (4) is $\mathbb{L}^p$-exactly controllable using $\mathbb{L}_{\mathbb{F}}^{1,p,2}\left(\mathbb{R}^{rank(D)}\right) \times \widehat{\mathcal{U}}^q\left(\mathbb{R}^{d-rank(D)}\right)$-controls.*

Proof.
1. The first assertion is straightforward.
2. Note that, in the deterministic, time-invariant case, if $r = rank(D)$ and G and F are as before, $\begin{pmatrix} G & 0_n \\ 0_n & G \end{pmatrix}\begin{pmatrix} D \\ B \end{pmatrix} F = \begin{pmatrix} I_r & 0_{r\times(d-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(d-r)} \\ \widehat{B}^1 & \widehat{B}^2 \end{pmatrix}$ .As a consequence, $rank\left(\widehat{B}^2\right) = n$ and we conclude by using the first assertion.

□

We stress out that the condition on $\widehat{B}^2$ being of n-rank is only sufficient, without being necessary. The reader is referred to Example 23.

3. Applications to SPDEs

3.1. The Coefficients and Consequences of the Abstract Result

We come back to the notations and considerations in Section 1.2. One knows that, due to the linear independence of the eigenfunctions system in $L^2(\mathcal{O}_0)$ (for any choice of $\mathcal{O}_0$), the matrix $\bar{D}_1 := \left(\bar{D}_{i,j}\right)_{1\leq i,j\leq M}$ is invertible. For simplicity, we set $\bar{D}_2 := \left(\bar{D}_{i,j}\right)_{M+1\leq i\leq n, 1\leq j\leq M}$. Then, by setting $G := \begin{pmatrix} \bar{D}_1^{-1} & 0_{M\times(n-M)} \\ \bar{D}_2\bar{D}_1^{-1} & -I_{n-M} \end{pmatrix}$, one gets $G\bar{D} = \overline{\pi}_M^{n,M}$. As a by-product (and independently of C),

$$\widehat{B}^1 = 0_{n\times M},\ \widehat{B}^2 = \left(G\bar{B} \quad | \ G\bar{D}'\right),\ \widehat{A} = GAG^{-1}.$$

Moreover, by setting A_1 to be the diagonal matrix consisting of the M first eigenvalues and A_2 the diagonal matrix consisting of the remaining $n - M$ eigenvalues, one has $\widehat{A} = \begin{pmatrix} \bar{D}_1^{-1}A_1\bar{D}_1 & 0_{M,n-M} \\ \bar{D}_2\bar{D}_1^{-1}A_1\bar{D}_1 - A_2\bar{D}_2 & A_2 \end{pmatrix}$. As a by product, $\widehat{A}\underline{\pi}_{n-M}^{n,n} = \begin{pmatrix} 0_{M\times M} & 0_{M\times(n-M)} \\ 0_{(n-M)\times M} & A_2 \end{pmatrix}$. The invariance Assumption 16 now reduces to

$$\mathcal{R}\left(\underline{\pi}_{n-M}^{n,n}\right) \subset \mathcal{R}\left(\left(G\bar{B} \quad | \ G\bar{D}'\right)\right). \tag{16}$$

Theorem 21.
1. *An explicit (sufficient) condition for (16) to hold true is rank $\left(\left(\bar{B} \mid \bar{D}'\right)\right) = n$.*

2. *Moreover, if rank* $\left(\left(\bar{B} \mid \bar{D}'\right)\right) = n$*, the projection of* (2) *onto the first* n *components is controllable with the null coefficient* D^0 *on noise.*
3. *If* (16) *holds true, then the projection of* (2) *onto the first* n *components is exactly controllable to* 0 *(with coefficient* D *on the noise) if and only if the largest subspace of* $\ker\left(\begin{pmatrix} \bar{B}^* \\ (\bar{D}')^* \end{pmatrix}\right)$ *which is* A^**-invariant is reduced to* 0.

Proof. The first assertion is straight-forward.
The second assertion follows from corollary 20 assertion (1).
Finally, in general, the Grammian (10) is purely deterministic (recall that, in this setting, $\widehat{B}^1 = 0_{n\times M}$ such that $\mathcal{Y}$ is a deterministic exponential). Then, the invertibility of the Grammian is equivalent to Kalman's condition on $\left(\widehat{A}, \widehat{B}^2 := \left(G\bar{B} \quad \mid G\bar{D}'\right)\right)$ i.e.

$$rank\left(\left(\widehat{B}^2 \quad \mid \widehat{A}\widehat{B}^2 \quad \mid \cdots \mid \left(\widehat{A}\right)^{n-1}\widehat{B}^2\right)\right) = n.$$

Equivalently, by recalling that $\widehat{A} = GAG^{-1}$, it follows that the couple A and $\left(\bar{B} \mid \bar{D}'\right)$ satisfies Kalman's criterion. Writing this condition from the observability (dual) point of view, this is equivalent to the largest subspace of $\ker\left(\begin{pmatrix} \bar{B}^* \\ (\bar{D}')^* \end{pmatrix}\right)$ which is A^*-invariant being reduced to $\{0\}$. □

3.2. Examples and Comments

We begin with some examples for $\mathcal{A} = -\Delta$. In this case, the lifting operator, $\mathbb{D}^\diamond$, is exactly the trace of the normal derivative on Γ_1. We first take dimension 2 in order to emphasize the following aspects:

1. there exist choices of M, N such that the use of M internal controls acting on the drift and N border controls improves the controlled space to $n > \max(M, N)$;
2. there exist choices of M, N for which the dimension of the internal controls on the drift (of low regularity $\mathbb{L}_{\mathbb{F}}^{1,p,1}$) can be reduced (i.e. systems for which *rank* $\left(\left(\bar{B}\ \bar{D}'\right)\right) < n$ fails to hold, yet the system is **stochastically** controllable to 0 although it may not be deterministically controllable);
3. the minimal improvement $n - \max\{N, M\}$ is not limited by the dimension of the underlying space of our SPDE m;
4. the improvement depends on the choice of $\mathcal{O}_0$.

For all (but last of) the following examples, we take $m = 2$, $\mathcal{O} = (0,\pi) \times (0,\pi)$, $\Gamma_1 = \{0\} \times (0,\pi)$. The eigen-functions are of type $\phi(x,y) = \frac{2}{\pi}\sin(ix)\sin(jy)$, where $(i,j) \in \mathbb{N}^* \times \mathbb{N}^*$ are ordered accordingly to the distance to origin, then according to the first component (i). The eigenvalues are $i^2 + j^2$.

Example 22. *We consider the case when,* $N = M = 2$*,* $n = 4$ *and* $\mathcal{O}_0 = (0,\pi) \times (0,\pi/2)$*. The eigen-functions correspond to* $(i,j) \in \{(1,1)(1,2),(2,1),(2,2)\}$*. The family* $\mathbb{D}^\diamond\phi$ *is given by* $\frac{2i}{\pi}\sin(jy)$ *i.e.* $\left\{\frac{2}{\pi}\sin(y), \frac{2}{\pi}\sin(2y), \frac{4}{\pi}\sin(y), \frac{4}{\pi}\sin(2y)\right\}$*. We explicitely compute the matrices* $\mathbb{R}^{4\times 2} \ni \bar{D}' = \bar{D} = \begin{pmatrix} \frac{1}{2} & \frac{4}{3\pi} \\ \frac{4}{3\pi} & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{R}^{4\times 2} \ni \bar{B} = \begin{pmatrix} \frac{2}{\pi} & 0 \\ 0 & \frac{2}{\pi} \\ \frac{4}{\pi} & 0 \\ 0 & \frac{4}{\pi} \end{pmatrix}$*. A simple glance shows that rank* $\left(\left(\bar{B} \quad \mid \bar{D}'\right)\right) = 4 = n$*. The conclusion follows from the second assertion in Theorem 21.*

Example 23. *We consider the case when* $N = 2$*,* $M = 5$*,* $n = 6$ *and* $\mathcal{O}_0 = (0,\pi)^2$*. This particular choice of* $\mathcal{O}_0$ *guarantees that* $\bar{D} = \pi_5^{6,5}$*. The family* $\mathbb{D}^\diamond\phi$ *is computed as in the previous example from* $(i,j) \in \{(1,1)(1,2),(2,1),(2,2),(1,3)(3,1)\}$*.*
1) We have $\mathbb{R}^{2\times 6} \ni \bar{B}^* = \begin{pmatrix} \frac{2}{\pi} & 0 & \frac{4}{\pi} & 0 & 0 & \frac{6}{\pi} \\ 0 & \frac{2}{\pi} & 0 & \frac{4}{\pi} & 0 & 0 \end{pmatrix}$*. Due to the form of* $\bar{D}$ *and* $\bar{B}$*, it is clear that* *rank* $\left(\left(\bar{B} \quad \mid \bar{D}\right)\right) = 6 = n$*. The associated system is controllable with* 2 *boundary controls and* 5 *internal controls and one can even consider all the internal controls acting on the drift (deterministic controllability). Of course, when* C *is non-null, the system is* $\mathbb{L}^p$*-exactly controllable to* 0 *with* $\widehat{\mathcal{U}}^q$*-regular controls but these*

may not be in $\mathbb{L}_{\mathbb{F}}^{1,p,2}$.
2) Now, if we try to control the system without the terms $\mathbf{1}_{\mathcal{O}_0}\phi_2$ *and* $\mathbf{1}_{\mathcal{O}_0}\phi_4$ *in the drift, i.e. by considering* $\bar{D}' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, *one has* $rank\left(\left(\bar{B} \mid \bar{D}'\right)\right) = 5 < n = 6$. *We cannot apply our Theorem 21 assertion 2. However, the equation* $\left(\bar{B} \mid \bar{D}'\right)x = \underline{\pi}_1^{6,1}$ *admits the obvious solution* $x = \left(\frac{\pi}{6},\ 0,\ -\frac{1}{3},\ 0,\ -\frac{2}{3},\ 0,\ 0\right)^*$, *which guarantees that the condition* (16) *holds true. Furthermore,* $\ker\left(\left(\bar{B} \mid \bar{D}'\right)^*\right)$ *is generated by* $e_0 := \left(0,\ 0,\ 0,\ 1,\ 0,\ 0\right)^*$. *We recall that the diagonal of* A *is given by* $-(2,5,5,8,10,10)$ *such that* $A^*e_0 = -\left(0,\ -10,\ 0,\ 8,\ 0,\ 0\right)^* \notin \ker\left(\left(\bar{B} \mid \bar{D}'\right)^*\right)$. *As a consequence, we can apply Theorem 21 assertion (3) to deduce the (stochastic!) exact controllability to* 0 *even without allowing* $\mathbf{1}_{\mathcal{O}_0}\phi_2$ *and* $\mathbf{1}_{\mathcal{O}_0}\phi_4$ *in the drift.*

The simplest algorithm (whose efficiency is guaranteed by Theorem (21), assertion (2)) giving a hint on the improvement when using controls of size (N, M) is described in **Algorithm 1** below.

Algorithm 1 Minimal-guaranteed improvement algorithm

function DIMENSION(M, N)
    $n \leftarrow M + N;\ n_{guess} \leftarrow n$
    initialize $\bar{B},\ \bar{D}$ with n_{guess} lines; $found \leftarrow 0$
    while $found \neq 1$ **do**
        $\bar{B} \leftarrow \bar{B}\left(1 : n_{guess}, :\right);\ \bar{D} \leftarrow \bar{D}\left(1 : n_{guess}, :\right);\ n_{guess} \leftarrow rank\left(\left(\bar{B} \mid \bar{D}\right)\right);$
        if $n_{guess} = n$ **then** $found \leftarrow 1$
        else $n \leftarrow n_{guess}$
        end if
    end while
    return n_{guess}
end function

Example 24. *We list hereafter the progressive (incremental) minimally guaranteed gain using various combinations* M, N. *Moreover, we show the differences when using various internal controls support* $\mathcal{O}_0$.

- *for* $\mathcal{O}_0 = (0, \pi) \times (0, \pi)$,

N	*1*	*2*	*5*	*9*	*23*	*31*	*55*
M	*2*	*2*	*5*	*9*	*23*	*31*	*55*
n	*3*	*4*	*8*	*13*	*28*	*37*	*62*

- *for* $\mathcal{O}_0 = (0, \pi) \times \left(0, \frac{\pi}{2}\right)$,

N	*1*	*2*	*5*	*9*	*16*	*23*	*31*
M	*1*	*2*	*5*	*9*	*18*	*27*	*38*
n	*2*	*4*	*8*	*13*	*23*	*33*	*45*

- *Of course, the reader will immediately note that, in the second case, the choice of* $N = 16$ *and* $M = 18$ *leads to a gain of* 5 *dimensions but this still holds true by enlarging* N *(e.g. to* M*). The data are the first increment in gain using the (sufficient) algorithm described before.*

To end this section and for completeness, we equally consider a three-dimensional example (i.e. $m = 3$).

Example 25. *For the case* $m = 3$, *we consider* $\mathcal{O} = (0, \pi)^3$, $\mathcal{O}_0 = (0, \pi) \times \left(0, \frac{\pi}{2}\right)^2$ *and* $\Gamma_1 = \{0\} \times (0, \pi)^2$. *Again, the eigen-functions are generically given by a triple* $(i, j, k) \in (\mathbb{N}^*)^3$ *by setting* $\phi(x, y, z) = \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \sin(ix)\sin(jy)\sin(kz)$ *(ordered first according to the distance to origin, then according to the first, then the second component). Moreover, to each triple we associate the generic expression of* $\mathbb{D}^{\diamond}\phi(y, z) = i\left(\frac{2}{\pi}\right)^{\frac{3}{2}} \sin(jy)\sin(kz)$.

1. *We explicitly compute, for $M = N = 3, n = 6$, the coefficients in* $\mathbb{R}^{6\times 3}$

$$\bar{B} := \begin{pmatrix} \frac{2}{\pi} & 0 & 0 \\ 0 & \frac{2}{\pi} & 0 \\ 0 & 0 & \frac{2}{\pi} \\ \frac{4}{\pi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{4}{\pi} \end{pmatrix}, \quad \bar{D} := \begin{pmatrix} \frac{1}{4} & \frac{2}{3\pi} & \frac{2}{3\pi} \\ \frac{2}{3\pi} & \frac{1}{4} & \frac{16}{9\pi^2} \\ \frac{2}{3\pi} & \frac{16}{9\pi^2} & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{16}{9\pi^2} & \frac{2}{3\pi} & \frac{2}{3\pi} \\ 0 & 0 & 0 \end{pmatrix}$$

The extended matrix $\left(\bar{B} \mid \bar{D}\right)$ *has full* (6) *rank such that, owing to Theorem 21, assertion (2), the associated system is controllable with coefficient* $\left(\bar{B} \mid \bar{D}\right)$ *in the drift and one can even consider no action on the noise.*

2. *Finally, the numerically computed gain (up to* 10 *units) is given by*

N	*1*	*2*	*3*	*5*	*8*	*9*	*12*	*13*	*21*	*22*
M	*1*	*3*	*3*	*7*	*19*	*9*	*15*	*13*	*22*	*22*
n	*2*	*5*	*6*	*11*	*24*	*15*	*22*	*21*	*31*	*32*

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