On the dimension group of unimodular $\mathcal{S}$-adic subshifts

Valérie Berthé, Paulina Cecchi Bernales, Fabien Durand, Dominique Perrin, J. Leroy, Samuel Petite

ON THE DIMENSION GROUP OF UNIMODULAR $\mathcal{S}$-ADIC SUBSHIFTS

V. BERTHÉ, P. CECCHI BERNALES, F. DURAND, J. LEROY, D. PERRIN, AND S. PETITE

Abstract. Dimension groups are complete invariants of strong orbit equivalence for minimal Cantor systems. This paper studies a natural family of minimal Cantor systems having a finitely generated dimension group, namely the primitive unimodular proper $\mathcal{S}$-adic subshifts. They are generated by iterating sequences of substitutions. Proper substitutions are such that the images of letters start with a same letter, and similarly end with a same letter. This family includes various classes of subshifts such as Brun subshifts or dendric subshifts, that in turn include Arnoux-Rauzy subshifts and natural coding of interval exchange transformations. We compute their dimension group and investigate the relation between the triviality of the infinitesimal subgroup and rational independence of letter measures. We also introduce the notion of balanced functions and provide a topological characterization of balancedness for primitive unimodular proper S-adic subshifts.

1. Introduction

Two dynamical systems are topologically orbit equivalent if there is a homeomorphism between them preserving the orbits. Originally, the notion of orbit equivalence was studied in the measurable context (see for instance [33, 59]), motivated by the classification of von Neumann algebras. In contrast with the measurable case, Giordano, Putnam and Skau showed that, in the topological setting, uncountably many classes appear by providing a dimension group as a complete invariant of strong orbit equivalence [46]. Dimension groups are ordered direct limit groups defined by sequences of positive homomorphisms $(\theta_n : \mathbb{Z}^{d_n} \to \mathbb{Z}^{d_{n+1}})_n$, where $\mathbb{Z}^d$ is given the standard or simplicial order, and were defined by Elliott [39] to study approximately finite dimensional C^*-algebras. In fact, an ordered group is a dimension group if and only if it is a Riesz group [35]. They have been widely

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studied in the late 70's and at the beginning of the 80's [34], in particular when the dimension group is a direct limit given by unimodular matrices [36, 37, 38, 65, 64].

The present paper studies dynamical and ergodic properties of subshifts having dimension groups with a group part of the form $\mathbb{Z}^d$. We focus on the class of primitive unimodular proper $\mathcal{S}$-adic subshifts. Such subshifts are generated by iterating sequences of substitutions. They have recently attracted much attention in symbolic dynamics [11] and in tiling theory [44, 43]. Proper substitutions are such that images of letters start with a same letter and also end with a same letter. Proper minimal proper $\mathcal{S}$-adic systems have played an important role for the characterization of linearly recurrent subshifts [27, 28]. The term unimodular refers to the unimodularity of the incidence matrices of the associated substitutions.

Sturmian subshifts, subshifts generated by natural codings of interval exchange transformations or Arnoux-Rauzy subshifts are prominent examples of unimodular proper $\mathcal{S}$-adic subshifts. They also belong to a recently defined family of subshifts, called dendric subshifts, and considered in [8, 9, 10, 7, 12] (see also Section 3.2). In this series of papers, their elements have been studied under the name of tree words. We have chosen to use the terminology dendric subshift in order to avoid any ambiguity with respect to shifts defined on trees (see, e.g., [3]) and also to avoid the puzzling term "tree word".

Minimal dendric subshifts are defined with respect to combinatorial properties of their language expressed in terms of extension graphs. For precise definitions, see Section 3.2. In particular, they have linear factor complexity. Focusing on extension properties of factors is a combinatorial viewpoint that allows to highlight the common features shared by dendric subshifts, even if the corresponding symbolic systems have very distinct dynamical, ergodic and spectral properties. For instance, a coding of a generic interval exchange is topologically weakly mixing for irreducible permutations not of rotation class [56], whereas an Arnoux-Rauzy subshift is generically not topologically weakly mixing [20, 15]. Even though one can disprove, in some cases, whether two given minimal dendric subshifts are topologically conjugate by using e.g. asymptotic pairs (see for instance Section 6.4), the question of orbit equivalence is more subtle and is one of the motivations for the present work.

The aim of this paper is to study topological orbit equivalence and strong orbit equivalence for minimal unimodular proper $\mathcal{S}$-adic subshifts. Let (X, S) be such a subshift over a d-letter alphabet $\mathcal{A}$ and let $\mathcal{M}(X, S)$ stand for its set of shift-invariant probability measures. One of our main results states that any continuous integer-valued function defined on X is cohomologous to some integer linear combination of characteristic functions of letter cylinders (Theorem 4.1). This relies on the fact that such subshifts being aperiodic (see Proposition 3.5) and recognizable by [14], they have a sequence of Kakutani-Rohlin tower partitions with suitable topological properties. We then deduce an explicit computation of their dimension group (Theorem 4.5). Indeed, the dimension group $K^0(X,S)$ with ordered unit is isomorphic to $\big(\mathbb{Z}^d, \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \boldsymbol{\mu}\rangle > 0 \text{ for all } \mu \in \mathcal{M}(X,S)\} \cup \{\mathbf{0}\}, \mathbf{1}\big)$, where $\boldsymbol{\mu}$ denotes the vector of measures of letter cylinders.

In other words, strong orbit equivalence can be characterized by means of letter measures, i.e., by measures of letter cylinders. In particular, two shift-invariant probability measures on (X,S) coinciding on the letter cylinders are proved to be equal (see Corollary 4.2). This result extends a statement initially proved for interval exchanges in [41]; see also [60, 61, 47] and [4, 5]. Moreover, two primitive unimodular proper $\mathcal{S}$-adic subshifts are proved to be strong orbit equivalent if and only if their simplexes of letter measures coincide up to a unimodular matrix (see Corollary 4.7), with the simplex of letter measures being the d-simplex generated by the vectors $(\nu[a])_{a\in\mathcal{A}}$, for ν in $\mathcal{M}(X,S)$.

We also investigate in Section 5 the triviality of the infinitesimal subgroup and relate it to the notion of balance. We provide a characterization of the triviality of the infinitesimal subgroup for minimal unimodular proper $\mathcal{S}$-adic subshifts in terms of rational independence of measures of letters (see Proposition 5.1). Inspired by the classical notion of balance in word combinatorics (see e.g. references in [6]), we also introduce the notion of balanced functions and provide a topological characterization of this balance property for primitive unimodular proper $\mathcal{S}$-adic subshifts (see Corollary 5.5).

We briefly describe the contents of this paper. Definitions and basic notions are recalled in Section 2, including, in particular, the notions of dimension group and orbit equivalence in Section 2.3, and of image subgroup in Section 2.4. Primitive unimodular $\mathcal{S}$-adic subshifts are introduced in Section 3, with dendric subshifts being discussed in more details in Section 3.2. Their dimension groups are studied in Section 4. Section 5 is devoted to the study of infinitesimals and their connections with the notion of balance. Some examples are handled in Section 6.

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2. First definitions and background

2.1. **Topological dynamical systems.** By a *topological dynamical system*, we mean a pair (X,T) where X is a compact metric space and $T : X \to X$ is a homeomorphism. It is a *Cantor system* when X is a Cantor space, that is, X has a countable basis of its topology which consists of closed and open sets (clopen sets) and does not have isolated points. This system is *aperiodic* if it does not have periodic points, i.e., points x such that $T^n(x) = x$ for some $n > 0$. It is *minimal* if it does not contain any non-empty proper closed T-invariant subset. Any minimal Cantor system is aperiodic. Two topological dynamical systems (X_1, T_1), (X_2, T_2) are *conjugate* when there is a *conjugacy* between them, i.e., a homeomorphism $\varphi : X_1 \to X_2$ such that $\varphi \circ T_1 = T_2 \circ \varphi$.

A complex number λ is a *continuous eigenvalue* of (X, T) if there exists a non-zero continuous function $f \colon X \to \mathbb{C}$ such that $f \circ T = \lambda f$. An *additive eigenvalue* is a real number α such that $\exp(2i\pi\alpha)$ is a continuous eigenvalue. Let $E(X, T)$ be the (additive) group of additive eigenvalues of (X, T). We consider its rank over $\mathbb{Q}$, i.e., the maximal number of rationally independent elements of $E(X, T)$. Note that 1 is always an additive eigenvalue and thus $\mathbb{Z}$ is included in $E(X, T)$.

A probability measure μ on X is said to be *T-invariant* if $\mu(T^{-1}A) = \mu(A)$ for every measurable subset A of X. Let $\mathcal{M}(X, T)$ be the set of all T-invariant probability measures on (X, T). It is a convex set and any extremal point is called an *ergodic* T-invariant measure. It is well known that any topological dynamical system admits an ergodic invariant measure. The set of ergodic T-invariant probability measures is denoted $\mathcal{M}_e(X, T)$. Observe that if (X, T) is a minimal Cantor system, then for all clopen E and all T-invariant probability measures μ, one has $\mu(E) > 0$. The topological dynamical system (X, T) is *uniquely ergodic* if there exists a unique T-invariant probability measure on X. It is said to be *strictly ergodic* if it is minimal and uniquely ergodic.

The notation χ_E stands for the characteristic function of E; $\mathbb{N}$ stands for the set of non-negative integers ($0 \in \mathbb{N}$).

2.2. **Subshifts.** Let $\mathcal{A}$ be a finite alphabet of cardinality $d \geq 2$. Let us denote by ε the empty word of the free monoid $\mathcal{A}^*$ (endowed with concatenation), and by $\mathcal{A}^{\mathbb{Z}}$ the set of bi-infinite words over $\mathcal{A}$. For a bi-infinite word $x \in \mathcal{A}^{\mathbb{Z}}$, and for $i, j \in \mathbb{Z}$ with $i \leq j$, the notation $x_{[i,j)}$ (resp., $x_{[i,j]}$) stands for $x_i \cdots x_{j-1}$ (resp., $x_i \cdots x_j$) with the convention $x_{[i,i)} = \varepsilon$. For a word $w = w_1 \cdots w_\ell \in \mathcal{A}^\ell$, its *length* is denoted $|w|$ and equals ℓ. We say that a word u is a *factor* of a word w if there exist words p, s such that $w = pus$. If $p = \varepsilon$ (resp., $s = \varepsilon$) we say that u is a *prefix* (resp., *suffix*) of w. For a word $u \in \mathcal{A}^*$, an index $1 \leq j \leq \ell$ such that $w_j \cdots w_{j+|u|-1} = u$ is called an *occurence* of u in w and we use the same term for bi-infinite word in $\mathcal{A}^{\mathbb{Z}}$. The number of occurrences of a word $u \in \mathcal{A}^*$ in a finite word w is denoted as $|w|_u$.

The set $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$ is topologically a Cantor set. The *shift map* S defined by $S\left((x_n)_{n\in\mathbb{Z}}\right) = (x_{n+1})_{n\in\mathbb{Z}}$ is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$. A *subshift* is a pair (X, S) where X is a closed shift-invariant subset of some $\mathcal{A}^{\mathbb{Z}}$. It is thus a *topological dynamical system*. Observe that a minimal subshift is aperiodic whenever it is infinite.

The set of factors of a sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is denoted $\mathcal{L}(x)$. For a subshift (X, S) its *language* $\mathcal{L}(X)$ is $\cup_{x \in X} \mathcal{L}(x)$. The *factor complexity* p_X of the subshift (X, S) is the function that with $n \in \mathbb{N}$ associates the number $p_X(n)$ of factors of length n in $\mathcal{L}(X)$.

Let w^-, w^+ be two words. The cylinder $[w^-.w^+]$ is defined as the set $\{x \in X \mid x_{[-|w^-|,|w^+|)} = w^- w^+\}$. It is a clopen set. When w^- is the empty word ε, we set $[\varepsilon.w^+] = [w^+]$.

For μ a S-invariant probability measure, the measure of a factor $w \in \mathcal{L}(X)$ is defined as the measure of the cylinder $[w]$. The notation $\boldsymbol{\mu}$ stands for the vector $(\mu([a])_{a \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. The *simplex of letter measures* is defined as the d-simplex consisting in all the vectors $\boldsymbol{\mu}$ with $\mu \in \mathcal{M}(X, S)$, i.e., it consists of all the convex combinations of the vectors $\boldsymbol{\mu}$ with $\mu \in \mathcal{M}_e(X, S)$.

2.3. **Dimension groups and orbit equivalence.** Two minimal Cantor systems (X_1, T_1) and (X_2, T_2) are *orbit equivalent* if there exists a homeomorphism $\Phi \colon X_1 \to X_2$ mapping orbits onto orbits, i.e., for all $x \in X_1$, one has

$$\Phi(\{T_1^n x \mid n \in \mathbb{Z}\}) = \{T_2^n \Phi(x) \mid n \in \mathbb{Z}\}.$$

This implies that there exist two maps $n_1 \colon X_1 \to \mathbb{Z}$ and $n_2 \colon X_2 \to \mathbb{Z}$ (uniquely defined by aperiodicity) such that, for all $x \in X_1$,

$$\Phi \circ T_1(x) = T_2^{n_1(x)} \circ \Phi(x) \quad \text{and} \quad \Phi \circ T_1^{n_2(x)}(x) = T_2 \circ \Phi(x).$$

The minimal Cantor systems (X_1, T_1) and (X_2, T_2) are *strongly orbit equivalent* if n_1 and n_2 both have at most one point of discontinuity. For more on the subject, see e.g. [46].

There is a powerful and convenient way to characterize orbit and strong orbit equivalence in terms of ordered groups and dimension groups due to [46]. An *ordered group* is a pair (G, G^+), where G is a countable abelian group and G^+ is a subset of G, called the *positive cone*, satisfying

$$G^+ + G^+ \subset G^+, \quad G^+ \cap (-G^+) = \{0\}, \quad G^+ - G^+ = G.$$

We write $a \leq b$ if $b - a \in G^+$, and $a < b$ if $b - a \in G^+$ and $b \neq a$. An *order ideal* J of an ordered group (G, G^+) is a subgroup J of G such that $J = J^+ - J^+$ (with $J^+ = J \cap G^+$) and such that $0 \leq a \leq b \in J$ implies $a \in J$. An ordered group is *simple* if it has no nonzero proper order ideals.

An element u in G^+ such that, for all a in G, there exists some nonnegative integer n with $a \leq nu$ is called an *order unit* for (G, G^+). Two ordered groups with order unit (G_1, G_1^+, u_1) and (G_2, G_2^+, u_2) are *isomorphic* when there exists a group isomorphism $\phi \colon G_1 \to G_2$ such that $\phi(G_1^+) = G_2^+$ and $\phi(u_1) = u_2$.

We say that an ordered group is *unperforated* if for all $a \in G$, if $na \in G^+$ for some $n \in \mathbb{N} \setminus \{0\}$, then $a \in G^+$. Observe that this implies in particular that G has no torsion element. A *dimension group* is an unperforated ordered group with order unit (G, G^+, u) satisfying the *Riesz interpolation property*: given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ with $a_i \leq c \leq b_j$.

Most examples of dimension groups we will deal with in this paper are of the following type: $(G, G^+, u) = (\mathbb{Z}^d, \{\mathbf{x} \in \mathbb{Z}^d \mid \theta_i(\mathbf{x}) > 0, 1 \leq i \leq e\} \cup \{\mathbf{0}\}, u)$, where the θ_i's are independent linear forms such that $\theta_i(u) = 1$.

Let (X, T) be a Cantor system. Let $C(X, \mathbb{R})$ and $C(X, \mathbb{Z})$ respectively stand for the group of continuous functions from X to $\mathbb{R}$ and $\mathbb{Z}$, and let $C(X, \mathbb{N})$ stand for the monoid of continuous functions from X to $\mathbb{N}$, with the group and monoid operation being the addition. Let

$$\begin{array}{rccc} \beta\colon & C(X,\mathbb{Z}) & \to & C(X,\mathbb{Z}) \\ & f & \mapsto & f \circ T - f. \end{array}$$

A map f is called a *coboundary* (resp., a *real coboundary*) if there exists a map g in $C(X, \mathbb{Z})$ (resp. in $C(X, \mathbb{R})$) such that $f = g \circ T - g$. Two maps $f, g \in C(X, \mathbb{Z})$ are said to be *cohomologous* whenever $f - g$ is a coboundary.

We consider the quotient group $H(X, T) = C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$. We denote $[f]$ the class of a function f in H, and π the natural projection $\pi\colon C(X, \mathbb{Z}) \to H(X, T)$. We define $H^+(X, T) = \pi(C(X, \mathbb{N}))$ as the set of classes of functions in $C(X, \mathbb{N})$. The ordered group with order unit

$$K^0(X, T) := (H(X, T), H^+(X, T), [1]),$$

where 1 stands for the one constant valued function, is a dimension group according to [60], called the *dynamical dimension group* of (X, T). We will use in this paper the abbreviated terminology *dimension group of* (X, T). The next result shows that any simple dimension group can be realized as the dimension group of a minimal Cantor system.

Theorem 2.1. [50, Theorem 5.4 and Corollary 6.3] *Let (G, G^+, u) be a dimension group with order unit. It is simple if and only if there exists a minimal Cantor system (X, T) such that (G, G^+, u) is isomorphic to $K^0(X, T)$.*

We also define the set of *infinitesimals* of $K^0(X, T)$ as

$$\mathrm{Inf}(K^0(X, T)) = \left\{ [f] \in H(X, T) : \int f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X, T) \right\}.$$

Note that $H(X, T)/\mathrm{Inf}(K^0(X, T))$ with the induced order also determines a dimension group [46]. We denote it $K^0(X, T)/\mathrm{Inf}(K^0(X, T))$.

The dimension groups $K^0(X, T)$ and $K^0(X, T)/\mathrm{Inf}(K^0(X, T))$ are complete invariants of strong orbit equivalence and orbit equivalence, respectively.

Theorem 2.2. [46] *Let (X_1, T_1) and (X_2, T_2) be two minimal Cantor systems. The following are equivalent:*

- *(X_1, T_1) and (X_2, T_2) are strong orbit equivalent;*
- *$K^0(X_1, T_1)$ and $K^0(X_2, T_2)$ are isomorphic.*

Similarly, the following are equivalent:

- (X_1, T_1) *and* (X_2, T_2) *are orbit equivalent;*
- $K^0(X_1, T_1)/\mathrm{Inf}(K^0(X_1, T_1))$ *and* $K^0(X_2, T_2)/\mathrm{Inf}(K^0(X_2, T_2))$ *are isomorphic.*

2.4. **Image subgroup.** A *trace* (also called state) of a dimension group (G, G^+, u) is a group homomorphism $p : G \to \mathbb{R}$ such that p is non-negative ($p(G^+) \geq 0$) and $p(u) = 1$. The collection of all traces of (G, G^+, u) is denoted by $\mathcal{T}(G, G^+, u)$. It is known [34] that $\mathcal{T}(G, G^+, u)$ completely determines the order on G, if the dimension group is simple. In fact, one has

$$G^+ = \{a \in G : p(a) > 0, \forall p \in \mathcal{T}(G, G^+, u)\} \cup \{0\}.$$

For more on the subject, see e.g. [34].

Let (X, T) be a minimal Cantor system. Given $\mu \in \mathcal{M}(X, T)$, we define the trace τ_μ on $K^0(X, T)$ as $\tau_\mu([f]) := \int f d\mu$. It is shown in [50] that the correspondence $\mu \mapsto \tau_\mu$ is an affine isomorphism from $\mathcal{M}(X, T)$ onto $\mathcal{T}(K^0(X, T))$. Thus it sends the extremal points of $\mathcal{M}(X, T)$, i.e., the ergodic measures, to the extremal points of $\mathcal{T}(K^0(X, T))$, called *pure traces*.

The *image subgroup* of $K^0(X, T)$ is defined as the ordered group with order unit

$$(I(X, T), I(X, T) \cap \mathbb{R}^+, 1),$$

where

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu \, : \, f \in C(X, \mathbb{Z}) \right\}.$$

Actually, $E(X, T)$ is a subgroup of $I(X, T)$ (see [22, Proposition 11] and also [45, Corollary 3.7].

If (X, T) is uniquely ergodic with unique T-invariant probability measure μ, then $K^0(X, T)/\mathrm{Inf}(K^0(X, T))$ is isomorphic to $(I(X, T), I(X, T) \cap \mathbb{R}^+, 1)$, via the correspondence

$$[f] + \mathrm{Inf}(K^0(X, T)) \mapsto \int f d\mu.$$

Let us recall the following description of $I(X, T)$.

Proposition 2.3. [45, Corollary 2.6], [22, Lemma 12]. *Let (X, T) be a minimal Cantor system. Then*

$$I(X, T) = \left\{ \alpha : \exists g \in C(X, \mathbb{Z}), \alpha = \int g d\mu \ \forall \mu \in \mathcal{M}(X, T) \right\}.$$

We give an other description of $I(X, T)$ using the following well known lemma.

Lemma 2.4. [48, Lemma 2.4] *Let (X, T) be a minimal Cantor system. Let $f \in C(X, \mathbb{Z})$ such that $\int_X f d\mu$ belongs to $]0, 1[$ for every $\mu \in \mathcal{M}(X, T)$. Then, there exists a clopen set U such that $\int_X f d\mu = \mu(U)$ for every $\mu \in \mathcal{M}(X, T)$.*

For a family of real numbers $N = \{\alpha_i : i \in J\}$, we let $\langle N \rangle$ denote the abelian additive group generated by these real numbers.

Proposition 2.5. *Let (X,T) be a minimal Cantor system. Then,*

$$I(X,T) = \langle \{\alpha : \exists \text{ clopen set } U \subset X,\ \alpha = \mu(U)\ \forall \mu \in \mathcal{M}(X,T)\} \rangle . \tag{1}$$

Proof. There is just one inclusion to prove. Let β be in $I(X,T)$. If β is an integer then using that $\beta = \beta\mu(X)$ for all $\mu \in \mathcal{M}(X,T)$, it implies that β belongs to the right member of the equality in (1). Otherwise, let $n \in \mathbb{Z}$ be such that $\beta - n$ belongs to $]0,1[$. From Proposition 2.3 and Lemma 2.4 there exists a clopen set U such that $\mu(U) = \beta - n$ for all $\mu \in \mathcal{M}(X,T)$. It follows $\mu(U)$ and n belong to $\langle \{\alpha : \exists \text{ clopen set } U \subset X,\ \alpha = \mu(U)\ \forall \mu \in \mathcal{M}(X,T)\} \rangle$. So it is also the case for β. □

One can obtain a more explicit description of the set $I(X,S)$ for minimal subshifts.

Proposition 2.6. *Let (X,S) be a minimal subshift. Then,*

$$I(X,S) \quad = \bigcap_{\mu \in \mathcal{M}(X,S)} \langle \{\mu([w]) : w \in \mathcal{L}(X)\} \rangle .$$

In particular, if (X,S) is uniquely ergodic with μ its unique S-invariant probability measure, then $I(X,S) = \langle \{\mu([w]) : w \in \mathcal{L}(X)\} \rangle$.

Proof. The proof of the first equality is a direct consequence of the fact that every function belonging to $C(X,\mathbb{Z})$ is cohomologous to some cylinder function in $C(X,\mathbb{Z})$, i.e., to some function h in $C(X,\mathbb{Z})$ for which there exists $n > 0$ such that for all $x \in X$, $h(x)$ depends only on $x_{[0,n)}$. Indeed, let $f \in C(X,\mathbb{Z})$. Since f is integer-valued, it is locally constant, and by compactness of X, there exists $k \in \mathbb{N}$ such that for all $x \in X$, $f(x)$ depends only on $x_{[-k,k]}$. Therefore, $g(x) = f \circ S^k(x)$ belongs to $C(X,\mathbb{Z})$ and depends only on $x_{[0,2k]}$ for all $x \in X$. Finally, $f - g = f - f \circ S^k(x) = f - f \circ S + f \circ S - f \circ S^2 + \cdots + f \circ S^{k-1}(x) + f \circ S^k(x)$ is a coboundary. Hence, $\int f d\mu = \int g d\mu$. Since g is a cylinder function, g can be written as a finite sum of the form $g = \sum \ell_u \chi_{[u]}$, $u \in \mathcal{L}(X)$ and $\ell_u \in \mathbb{Z}$. Thus, $\int f d\mu = \sum \ell_u \mu([u]) \in \langle \{\mu([w]) : w \in \mathcal{L}(X)\} \rangle$. □

3. Primitive unimodular proper $\mathcal{S}$-adic subshifts

In this section we first recall the notion of primitive unimodular proper $\mathcal{S}$-adic subshift in Section 3.1. We then illustrate it with the class of minimal dendric subshifts in Section 3.2.

3.1. **$\mathcal{S}$-adic subshifts.** Let $\mathcal{A}$, $\mathcal{B}$ be finite alphabets and let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a *non-erasing* morphism (also called a *substitution* if $\mathcal{A} = \mathcal{B}$). Let us note that a morphism is uniquely determined by its values on the alphabet $\mathcal{A}$ and this will be the way we will define them (see e.g. Example 3.6). By non-erasing, we mean that the image of any letter is a non-empty word. We stress the fact that all morphisms are assumed to be non-erasing in the following. Using concatenation, we extend σ to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$. With a morphism $\tau : \mathcal{A}^* \to \mathcal{B}^*$, where $\mathcal{A}$ and $\mathcal{B}$ are finite alphabets, we classically associate an *incidence matrix* M_τ indexed by $\mathcal{B} \times \mathcal{A}$ such that for every $(b, a) \in \mathcal{B} \times \mathcal{A}$, its entry at position (b, a) is the number of occurrences of b in $\tau(a)$. Alphabets are always assumed to have cardinality at least 2. The morphism τ is said to be *left proper* (resp. *right proper*) when there exist a letter $b \in \mathcal{B}$ such that for all $a \in \mathcal{A}$, $\tau(a)$ starts with b (resp., ends with b). It is said to be *proper* if it is both left and right proper.

We recall the definition of an $\mathcal{S}$-adic subshift as stated in [14], see also [11] for more on $\mathcal{S}$-adic subshifts. Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 1}$ be a sequence of morphisms such that $\max_{a\in\mathcal{A}_n} |\tau_1 \circ \cdots \circ \tau_{n-1}(a)|$ goes to infinity when n increases. By non-erasing, we mean that the image of any letter is a non-empty word. For $1 \leq n < N$, we define $\tau_{[n,N)} = \tau_n \circ \tau_{n+1} \circ \cdots \circ \tau_{N-1}$ and $\tau_{[n,N]} = \tau_n \circ \tau_{n+1} \circ \cdots \circ \tau_N$. For $n \geq 1$, the *language* $\mathcal{L}^{(n)}(\boldsymbol{\tau})$ *of level* n *associated with* $\boldsymbol{\tau}$ is defined by

$$\mathcal{L}^{(n)}(\boldsymbol{\tau}) = \left\{ w \in \mathcal{A}_n^* \mid w \text{ occurs in } \tau_{[n,N)}(a) \text{ for some } a \in \mathcal{A}_N \text{ and } N > n \right\}.$$

As $\max_{a\in\mathcal{A}_n} |\tau_{[1,n)}(a)|$ goes to infinity when n increases, $\mathcal{L}^{(n)}(\boldsymbol{\tau})$ defines a non-empty subshift $X_{\boldsymbol{\tau}}^{(n)}$ that we call the *subshift generated by* $\mathcal{L}^{(n)}(\boldsymbol{\tau})$. More precisely, $X_{\boldsymbol{\tau}}^{(n)}$ is the set of points $x \in \mathcal{A}_n^{\mathbb{Z}}$ such that $\mathcal{L}(x) \subseteq \mathcal{L}^{(n)}(\boldsymbol{\tau})$. Note that it may happen that $\mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ is strictly contained in $\mathcal{L}^{(n)}(\boldsymbol{\tau})$. We set $\mathcal{L}(\boldsymbol{\tau}) = \mathcal{L}^{(1)}(\boldsymbol{\tau})$,$X_{\boldsymbol{\tau}} = X_{\boldsymbol{\tau}}^{(1)}$ and call $(X_{\boldsymbol{\tau}}, S)$ the $\mathcal{S}$*-adic subshift* generated by the *directive sequence* $\boldsymbol{\tau}$.

We say that $\boldsymbol{\tau}$ is *primitive* if, for any $n \geq 1$, there exists $N > n$ such that $M_{\tau_{[n,N)}} > 0$, *i.e.*, for all $a \in \mathcal{A}_N$, $\tau_{[n,N)}(a)$ contains occurrences of all letters of $\mathcal{A}_n$. Of course, $M_{\tau_{[n,N)}}$ is equal to $M_{\tau_n} M_{\tau_{n+1}} \cdots M_{\tau_{N-1}}$. Observe that if $\boldsymbol{\tau}$ is primitive, then $\min_{a\in\mathcal{A}_n} |\tau_{[1,n)}(a)|$ goes to infinity when n increases. In the primitive case $\mathcal{L}(X_{\boldsymbol{\tau}}^{(n)}) = \mathcal{L}^{(n)}(\boldsymbol{\tau})$, and $X_{\boldsymbol{\tau}}^{(n)}$ is a minimal subshift (see for instance [27, Lemma 7]).

We say that $\boldsymbol{\tau}$ is (*left, right*) *proper* whenever each morphism τ_n is (left, right) proper. We also say that $\boldsymbol{\tau}$ is *unimodular* whenever, for all $n \geq 1$, $\mathcal{A}_{n+1} = \mathcal{A}_n$ and the matrix M_{τ_n} has determinant of absolute value 1. By abuse of language, we say that a subshift is a (unimodular, left or right proper, primitive) $\mathcal{S}$-adic subshift if there exists a (unimodular, left or right proper, primitive) sequence of morphisms $\boldsymbol{\tau}$ such that $X = X_{\boldsymbol{\tau}}$.

Let us give another way to define $X_{\boldsymbol{\tau}}$ when $\boldsymbol{\tau}$ is primitive and proper. For an endomorphism τ of $\mathcal{A}^*$, let $\Omega(\tau) = \overline{\bigcup_{k\in\mathbb{Z}} S^k \tau(\mathcal{A}^{\mathbb{Z}})}$.

Lemma 3.1. *Let* $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 1}$ *be a sequence of morphisms such that* $\min_{a\in\mathcal{A}_n} |\tau_{[1,n)}(a)|$ *goes to infinity when* n *increases. Then,*

$$X_{\boldsymbol{\tau}} \subset \bigcap_{n\in\mathbb{N}} \Omega(\tau_{[1,n]}).$$

Furthermore, when $\boldsymbol{\tau}$ is primitive and proper, then the equality $X_{\boldsymbol{\tau}} = \bigcap_{n\in\mathbb{N}} \Omega(\tau_{[1,n]})$ holds.

Proof. The proof is left to the reader. □

With a left proper morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ such that $b \in \mathcal{B}$ is the first letter of all images $\sigma(a)$, $a \in \mathcal{A}$, we associate the right proper morphism $\overline{\sigma} : \mathcal{A}^* \to \mathcal{B}^*$ defined by $b\overline{\sigma}(a) = \sigma(a)b$ for all $a \in \mathcal{A}$. For all $x \in A^{\mathbb{Z}}$, we thus have $\bar{\sigma}(x) = S\sigma(x)$. The next result is a weaker version of [31, Corollary 2.3].

Lemma 3.2. *Let (X, S) be an $\mathcal{S}$-adic subshift generated by the primitive and left proper directive sequence $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 1}$. Then (X, S) is also generated by the primitive and proper directive sequence $\tilde{\boldsymbol{\tau}} = (\tilde{\tau}_n)_{n\geq 1}$, where for all n, $\tilde{\tau}_n = \tau_{2n-1}\overline{\tau}_{2n}$. In particular, if $\boldsymbol{\tau}$ is unimodular, then so is $\tilde{\boldsymbol{\tau}}$.*

Proof. Each morphism $\tilde{\tau}_n$ is trivially proper. It is also clear that the unimodularity and the primitiveness of $\boldsymbol{\tau}$ are preserved in this process. Using the relation $\bar{\sigma}(x) = S\sigma(x)$ and Lemma 3.1, we then get

$$X_{\boldsymbol{\tau}} \subset \bigcap_{n\in\mathbb{N}} \Omega(\tau_{[1,n]}) = \bigcap_{n\in\mathbb{N}} \Omega(\tilde{\tau}_{[1,n]}) = X_{\tilde{\boldsymbol{\tau}}}.$$

Since both $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ are primitive, the subshifts $X_{\boldsymbol{\tau}}$ and $X_{\tilde{\boldsymbol{\tau}}}$ are minimal, hence they are equal. □

Lemma 3.3. *All primitive unimodular proper $\mathcal{S}$-adic subshifts are aperiodic.*

Proof. Let $\boldsymbol{\tau}$ be a primitive unimodular proper directive sequence on the alphabet $\mathcal{A}$ of cardinality $d \geq 2$. Suppose that it has a periodic point x of period p, where p is the smallest period of x ($p > 0$). By primitiveness, all letters of $\mathcal{A}$ occur in x, so $p \geq d$. We have $x = \cdots uu.uu \cdots$ for some word u with $|u| = p$. There exists some n such that, for all a, one has $\tau_{[1,n]}(a) = s(a)u^{q(a)}p(a)$, where $s(a), p(a)$ are a strict suffix and a strict prefix of u and $q(a) > 1$. Let $b \in \mathcal{A}$ and set $\tau_{n+1}(b) = b_0 b_1 \cdots b_k$. As the directive sequence $\boldsymbol{\tau}$ is proper, $b_0 b_1 \cdots b_k b_0$ is also a word in $\mathcal{L}^{(n+1)}(\boldsymbol{\tau})$. By a classical argument due to Fine and Wilf [42], one has $p(b_0)s(b_1) = p(b_i)s(b_{i+1}) = p(b_k)s(b_0) = u$ for $1 \leq i \leq k-1$. Hence

$$|\tau_1 \cdots \tau_{n+1}(b)| \equiv |s(c)p(c)s(b_1)p(b_1) \cdots s(b_k)p(b_k)| \equiv 0 \text{ modulo } |u|,$$

which contradicts the unimodularity of $\boldsymbol{\tau}$. □

The next two results will be important for the computation of the dimension group of primitive unimodular proper $\mathcal{S}$-adic subshifs. The first one is a weaker version of [14, Theorem 3.1].

Theorem 3.4 ([14])**.** *Let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be such that its incidence matrix M_τ is unimodular. Then, for any aperiodic $y \in \mathcal{B}^{\mathbb{Z}}$, there exists at most one $(k, x) \in \mathbb{N} \times A^{\mathbb{Z}}$ such that $y = S^k\tau(x)$, with $0 \leq k < |\tau(x_0)|$.*

Proposition 3.5. *Let* $\boldsymbol{\tau} = (\tau_n : \mathcal{A}^* \to \mathcal{A}^*)_{n\geq 1}$ *be a unimodular proper sequence of morphisms such that* $\max_{a\in\mathcal{A}} |\tau_{[1,n)}(a)|$ *goes to infinity when* n *increases. Then* $(X_{\boldsymbol{\tau}}, S)$ *is aperiodic and minimal if and only if* $\boldsymbol{\tau}$ *is primitive.*

Proof. Recall that any $\mathcal{S}$-adic subshift with a primitive directive sequence is minimal (see, e.g. [27, Lemma 7]) and that aperiodicity is proved in Lemma 3.3.

We only have to show that the condition is necessary. We assume that $(X_{\boldsymbol{\tau}}, S)$ is aperiodic and minimal. For all $n \geq 1$, $(X_{\boldsymbol{\tau}}^{(n)}, S)$ is trivially aperiodic. Let us show that it is minimal.

Assume by contradiction that for some $n \geq 1$, $(X_{\boldsymbol{\tau}}^{(n)}, S)$ is minimal, but not $(X_{\boldsymbol{\tau}}^{(n+1)}, S)$. There exist $u \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n+1)})$ and $x \in X_{\boldsymbol{\tau}}^{(n+1)}$ such that u does not occur in x. By Theorem 3.4, $\{\tau_n([v]) \mid v \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n+1)}) \cap \mathcal{A}^{|u|}\}$ is a finite clopen partition of $\tau_n(X_{\boldsymbol{\tau}}^{(n+1)})$. Thus, considering $y = \tau_n(x)$, by minimality of $(X_{\boldsymbol{\tau}}^{(n)}, S)$, there exists $k \geq 0$ such that $S^k y$ is in $\tau_n([u])$. Take $z \in [u]$ such that $S^k y = \tau_n(z)$. Since y is aperiodic and since we also have $S^k y = S^{k'}\tau_n(S^\ell x)$ for some $\ell \in \mathbb{N}$ and $0 \leq k' < |\tau_n(x_\ell)|$, we obtain that $\tau_n(z) = S^{k'}\tau_n(S^\ell x)$ with $z \in [u]$, $S^\ell x \notin [u]$ and $0 \leq k' < |\tau_n(x_\ell)|$; this contradicts Theorem 3.4.

We now show that $\lim_{n\to+\infty} \min_{a\in\mathcal{A}} |\tau_{[1,n)}(a)| = +\infty$. We again proceed by contradiction, assuming that $(\min_{a\in\mathcal{A}} |\tau_{[1,n)}(a)|)_{n\geq 1}$ is bounded. Then there exists $N > 0$ and a sequence $(a_n)_{n\geq N}$ of letters in $\mathcal{A}$ such that for all $n \geq N$, $\tau_n(a_{n+1}) = a_n$. We claim that there are arbitrary long words of the form a_N^k in $\mathcal{L}(X_{\boldsymbol{\tau}}^{(N)})$ which contradicts the fact that $(X_{\boldsymbol{\tau}}^{(N)}, S)$ is minimal and aperiodic. Since $\boldsymbol{\tau}$ is proper, for all $n \geq N$ and all $b \in \mathcal{A}$, $\tau_n(b)$ starts and ends with a_n. As $\max_{a\in\mathcal{A}} |\tau_{[1,n)}(a)|$ goes to infinity, there exists a sequence $(b_n)_{n\geq N}$ of letters in $\mathcal{A}$ such that $|\tau_{[N,n)}(b_n)|$ goes to infinity and for all $n \geq N$, b_n occurs in $\tau_n(b_{n+1})$. This implies that there exists $M \geq N$ such that for all $n \geq M$, $b_n \neq a_n$ and, consequently, that $\tau_n(b_{n+1}) = a_n u_n$ for some word u_n containing b_n. It is then easily seen that, for all $k \geq 1$, a_M^k is a prefix of $\tau_{[M,M+k)}(b_{M+k})$, which proves the claim.

We finally show that $\boldsymbol{\tau}$ is primitive. If not, there exist $N \geq 1$ and a sequence $(a_n)_{n\geq N}$ of letters in $\mathcal{A}$ such that for all $n > N$, a_N does not occur in $\tau_{[N,n)}(a_n)$. As $(|\tau_{[N,n)}(a_n)|)_n$ goes to infinity, this shows that there are arbitrarily long words in $\mathcal{L}(X_{\boldsymbol{\tau}}^{(N)})$ in which a_N does not occur. Since $\boldsymbol{\tau}$ is unimodular, there is also a sequence $(a'_n)_{n\geq N}$ of letters in $\mathcal{A}$ such that $a_N = a'_N$ and for all $n \geq N$, a'_n occurs in $\tau_n(a'_{n+1})$. Again using the fact that $|\tau_{[N,n)}(a'_n)|$ goes to infinity, this shows that a_N belongs to $\mathcal{L}(X_{\boldsymbol{\tau}}^{(N)})$. We conclude that $(X_{\boldsymbol{\tau}}^{(N)}, S)$ is not minimal, a contradiction. □

3.2. **Dendric subshifts.** We now describe a subclass of the family of primitive unimodular proper $\mathcal{S}$-adic subshifts, namely the class of dendric subshifts, that encompasses Sturmian subshifts, Arnoux-Rauzy subshifts (see

Section 6.2), as well as subshifts generated by interval exchanges (see [8]). The ternary words generated by the Cassaigne-Selmer multidimensional continued fraction algorithm also provide dendric subshifts [2, 21].

Dendric subshifts are defined with respect to combinatorial properties of their language expressed in terms of extension graphs. We recall the notion of dendric words and subshifts, studied in [8, 9, 10, 7, 12]. Let (X, S) be a minimal subshift defined on the alphabet $\mathcal{A}$. For $w \in \mathcal{L}_X$, let

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \mid aw \in \mathcal{L}_X\}, &\quad \ell(w) &= \mathrm{Card}(L(w)), \\ R(w) &= \{a \in \mathcal{A} \mid wa \in \mathcal{L}_X\}, &\quad r(w) &= \mathrm{Card}(R(w)). \end{aligned}$$

A word $w \in \mathcal{L}_X$ is said to be *right special* (resp. *left special*) if $r(w) \geq 2$ (resp. $\ell(w) \geq 2$). It is *bispecial* if it is both left and right special.

For a word $w \in \mathcal{L}(X)$, we consider the undirected bipartite graph $\mathcal{E}(w)$ called its *extension graph* with respect to X and defined as follows: its set of vertices is the disjoint union of $L(w)$ and $R(w)$ and its edges are the pairs $(a, b) \in L(w) \times R(w)$ such that $awb \in \mathcal{L}(X)$. For an illustration, see Example 3.6 below. We then say that a subshift X is a *dendric subshift* if, for every word $w \in \mathcal{L}(X)$, the graph $\mathcal{E}(w)$ is a tree. Note that the extension graph associated with every non-bispecial word is trivially a tree. We will consider here only minimal dendric subshifts.

The factor complexity of a dendric subshift over a d-letter alphabet is $(d-1)n + 1$ (see [7]), and on a two-letter alphabet, the minimal dendric subshifts are the Sturmian subshifts. Thus minimal dendric subshift are aperiodic when d is greater or equal to 2.

Example 3.6. Let σ be the Fibonacci substitution defined over the alphabet $\{a, b\}$ by $\sigma\colon a \mapsto ab, b \mapsto a$ and consider the subshift generated by σ (i.e., the set of bi-infinite words over $\mathcal{A}$ whose factors belong to some $\sigma^n(a)$). The extension graphs of the empty word and of the two letters a and b are represented in Figure 1.

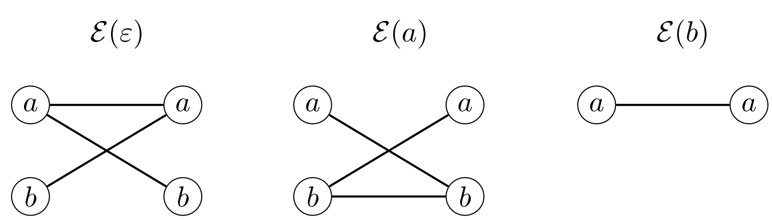

FIGURE 1. The extension graphs of ε (on the left), a (on the center) and b (on the right) are trees.

The following theorem states a structural property for return words of minimal dendric subshifts, from which a description as primitive unimodular proper $\mathcal{S}$-adic subshifts can be deduced (Proposition 3.8 below). Let (X, S) be a minimal subshift over the alphabet $\mathcal{A}$ and let $w \in \mathcal{L}(X)$. A *return* *word* to w is a word v in $\mathcal{L}(X)$ such that w is a prefix of vw and vw contains exactly two occurrences of w. We recall below a corollary of [7, Theorem 4.5].

Theorem 3.7. *Let (X,S) be a minimal dendric subshift defined on the alphabet $\mathcal{A}$. Then, for any $w \in \mathcal{L}(X)$, the set of return words to w is a basis of the free group on $\mathcal{A}$.*

In particular, dendric subshifts have bounded topological rank. The next result shows that minimal dendric subshifts are primitive unimodular proper $\mathcal{S}$-adic subshifts. Similar results are proved with the same method in [10, 12, 14] but not highlighting all the properties stated below, so we provide a proof for the sake of self-containedness. It relies on $\mathcal{S}$-adic representations built from return words [27, 28] together with the remarkable property of return words of dendric subshifts stated in Theorem 3.7. We also provide in Section 6.5 an example of a primitive unimodular proper subshift which is not dendric and whose strong orbit equivalence class contains no dendric subshift.

Proposition 3.8. *Minimal dendric subshifts are primitive unimodular proper $\mathcal{S}$-adic subshifts.*

Proof. Let (X,S) be a minimal dendric subshift over the alphabet $\mathcal{A} = \{1,2,\ldots,d\}$ and take any $x \in X$. For every $n \geq 1$, let $V_n(x) := \{v_{1,n},\cdots,v_{d,n}\}$ be the set of return words to $x_{[0,n)}$ and $V_0(x) = \mathcal{A}$. We stress the fact that $V_n(x)$ has cardinality d for all n, according to Theorem 3.7. Let $(n_i)_{i\geq 1}$ be a strictly increasing integer sequence such that $n_1 = 1$ and such that each $v_{j,n_i}x_{[0,n_i)}$ occurs in $x_{[0,n_{i+1})}$ and in each $v_{k,n_{i+1}}$. Let θ_i be an endomorphism of $\mathcal{A}^*$ such that $\theta_i(\mathcal{A}) = V_{n_i}(x)$. Since $x_{[0,n_i)}$ is a prefix of $x_{[0,n_{i+1})}$, any $v_{j,n_{i+1}} \in V_{n_{i+1}}(x)$ has a unique decomposition as a concatenation of elements $v_{k,n_i} \in V_{n_i}(x)$. More precisely, for any $v_{j,n_{i+1}} \in V_{n_{i+1}}(x)$, there is a unique sequence $(v_{k_j(1),n_i}, v_{k_j(2),n_i},\ldots,v_{k_j(\ell_j),n_i})$ of elements of $V_{n_i}(x)$ such that $v_{k_j(1),n_i}\cdots v_{k_j(\ell_j),n_i} = v_{j,n_{i+1}}$ and for all $m \in \{1,\ldots,\ell_j\}$, $v_{k_j(1),n_i}\cdots v_{k_j(m),n_i}x_{[0,n_i)}$ is a prefix of $v_{j,n_{i+1}}x_{[0,n_{i+1})}$. This induces a unique endomorphism λ_i of $\mathcal{A}^*$ defined by $\theta_{i+1} = \theta_i \circ \lambda_i$. From the choice of the sequence $(n_i)_{i\geq 1}$, the matrices M_{λ_i} have positive coefficients, so the sequence of morphisms $(\lambda_i)_{i\geq 1}$ is primitive. Furthermore, as $x_{[0,n_{i+1}]}$ is prefix of each $v_{j,n_{i+1}}x_{[0,n_{i+1})}$, there exists some $v \in V_{n_i}(x)$ such that $v_{k_j(1)} = v$ for all j. In other words, the morphisms λ_i are left proper. Finally, from Theorem 3.7, the matrices M_{λ_i} are unimodular. Hence (X,S) is $\mathcal{S}$-adic generated by the primitive directive sequence of unimodular left proper endomorphisms $\boldsymbol{\lambda} = (\lambda_i)_{i\geq 1}$. We deduce from Lemma 3.2 that minimal dendric subshifts are primitive unimodular proper $\mathcal{S}$-adic subshifts. □

Observe that using Lemma 3.3 we recover that minimal dendric subshifts on at least two letters are aperiodic.

4. Dimension groups of primitive unimodular proper $\mathcal{S}$-adic subshifts

In this section we first prove a key result of this paper, namely Theorem 4.1, which states that $H(X,T) = C(X,\mathbb{Z})/\beta C(X,\mathbb{Z})$ is generated as an additive group by the classes of the characteristic functions of letter cylinders. We then deduce a simple expression for the dimension group of primitive unimodular proper $\mathcal{S}$-adic subshifts.

4.1. **From letters to factors.** We recall that χ_U stands for the characteristic function of the set U.

Theorem 4.1. *Let (X,S) be a primitive unimodular proper $\mathcal{S}$-adic subshift. Any function $f \in C(X,\mathbb{Z})$ is cohomologuous to some integer linear combination of the form $\sum_{a\in\mathcal{A}} \alpha_a \chi_{[a]} \in C(X,\mathbb{Z})$. Moreover, the classes $[\chi_{[a]}]$, $a \in \mathcal{A}$, are $\mathbb{Q}$-independent.*

Proof. Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}^* \to \mathcal{A}^*)_{n\geq 1}$ be a primitive unimodular proper directive sequence of (X,S), hence $X = X_{\boldsymbol{\tau}}$. Using Proposition 3.5, all subshifts $(X_{\boldsymbol{\tau}}^{(n)}, S)$ are minimal and aperiodic and $\min_{a\in\mathcal{A}} |\tau_{[1,n)}(a)|$ goes to infinity when n increases.

Let us show that the group $H(X,S) = C(X,\mathbb{Z})/\beta C(X,\mathbb{Z})$ is spanned by the set of classes of characteristic functions of letter cylinders $\{[\chi_{[a]}] \mid a \in \mathcal{A}\}$. From Theorem 3.4 and using the fact that (X,S) is minimal and aperiodic, one has, for all positive integer n, that

$$\mathcal{P}_n = \{S^k \tau_{[1,n]}([a]) \mid 0 \leq k < |\tau_{[1,n]}(a)|, a \in \mathcal{A}\}$$

is a finite partition of X into clopen sets. This provides a family of nested Kakutani-Rohlin tower partitions. The morphisms of the directive sequence $\boldsymbol{\tau}$ being proper, for all n, there are letters a_n, b_n such that all images $\tau_n(c)$, $c \in \mathcal{A}$, start with a_n and end with b_n. From this, it is classical to check that $(\mathcal{P}_n)_n$ generates the topology of X (the proof is the same as [30, Proposition 14] that is concerned with the particular case $\tau_{n+1} = \tau_n$ for all n).

We first claim that $H(X,S)$ is spanned by the set of classes $\cup_n \Omega_n$, where

$$\Omega_n = \{[\chi_{\tau_{[1,n]}([a])}] \mid a \in \mathcal{A}\} \quad n \geq 1.$$

In other words, $H(X,S)$ is spanned by the set of classes of characteristic functions of bases of the sequence of partitions $(\mathcal{P})_n$. It suffices to check that, for all $u^-u^+ \in \mathcal{L}(X)$, the class $[\chi_{[u^-.u^+]}]$ is a linear integer combination of elements belonging to some Ω_n.

Let us check this assertion. Let u^-u^+ belong to $\mathcal{L}(X)$. Since $\min_{a\in\mathcal{A}} |\tau_{[1,n)}(a)|$ goes to infinity, there exists n such that $|u^-|, |u^+| < \min_{a\in\mathcal{A}} |\tau_{[1,n)}(a)|$. The directive sequence $\boldsymbol{\tau}$ being proper, there exist words w, w' with respective lengths $|w| = |u^-|$ and $|w'| = |u^+|$ such that all images $\tau_{[1,n]}(a)$ start with w and end with w'.

Let $x \in [u^-.u^+]$. Let $a \in \mathcal{A}$ and $k \in \mathbb{N}$, $0 \leq k < |\tau_{[1,n]}(a)|$, such that x belongs to the atom $S^k\tau_{[1,n]}([a])$. Observing that $\tau_{[1,n]}([a])$ is included in $[w'.\tau_{[1,n]}(a)w]$, this implies that the full atom $S^k\tau_{[1,n]}([a])$ is included in $[u^-.u^+]$. Consequently $[u^-.u^+]$ is a finite union of atoms in $\mathcal{P}_n$. But each characteristic function of an atom of the form $S^k\tau_{[1,n]}([a])$ is cohomologous to $\chi_{\tau_{[1,n]}([a])}$. The proof works as in the proof of Proposition 2.6. This thus proves the claim.

Now we claim that each element of Ω_n is a linear integer combination of elements in $\{[\chi_{[a]}] \mid a \in \mathcal{A}\}$. More precisely, let us show that $\chi_{\tau_{[1,n]}([b])}$ is cohomologous to

$$\sum_{a\in\mathcal{A}} (M^{-1}_{\tau_{[1,n]}})_{b,a}\chi_{[a]}.$$

Let $a \in \mathcal{A}$ and $n \geq 1$. One has $[a] = \cup_{B\in\mathcal{P}_n}(B \cap [a])$ and thus $\chi_{[a]}$ is cohomologous to the map

$$\sum_{b\in\mathcal{A}} (M_{\tau_{[1,n]}})_{a,b}\chi_{\tau_{[1,n]}([b])},$$

by using the fact that the maps $\chi_{S^k\tau_{[1,n]}([a])}$ are cohomologous to $\chi_{\tau_{[1,n]}([a])}$. This means that for $U = ([\chi_{[a]}])_{a\in\mathcal{A}} \in H(X,S)^{\mathcal{A}}$ and $V = ([\chi_{\tau_{[1,n]}([a])}])_{a\in\mathcal{A}} \in H(X,S)^{\mathcal{A}}$, one has

$$U = M_{\tau_{[1,n]}}V$$

and as a consequence $V = M^{-1}_{\tau_{[1,n]}}U$. This proves the claim and the first part of the theorem.

To show the independence, suppose that there exists some row vector $\alpha = (\alpha_a)_{a\in\mathcal{A}} \in \mathbb{Z}^{\mathcal{A}}$ such that $\sum_a \alpha_a[\chi_{[a]}] = 0$. Hence there is some $f \in C(X,\mathbb{Z})$ such that $\sum_a \alpha_a\chi_{[a]} = f\circ S - f$. We now fix some n for which f is constant on each atom of $\mathcal{P}_n$. Observe that for all $x \in X$ and all $k \in \mathbb{N}$, one has $f(S^kx) - f(x) = \sum_{j=0}^{k-1}\alpha_{x_j}$. Let $c \in \mathcal{A}$ and $x \in \tau_{[1,n+1]}([c])$. Then, x and $S^{|\tau_{[1,n+1]}(c)|}(x)$ belong to $\tau_{[1,n]}([a_{n+1}])$. Hence, $f(S^{|\tau_{[1,n+1]}(c)|}x) - f(x) = 0$, and thus

$$(\alpha M_{\tau_{[1,n]}})_c = \sum_{j=0}^{|\tau_{[1,n+1]}(c)|-1} \alpha_{x_j} = 0.$$

This holds for all c, hence $\alpha M_{\tau_{[1,n]}} = 0$, which yields $\alpha = 0$, by invertibility of the matrix $M_{\tau_{[1,n]}}$. □

Observe that in the previous result, we can relax the assumption of minimality. Indeed, one checks that the same proof works if we assume that (X,S) is aperiodic (recognizability then holds by [14]) and that $\min_{a\in\mathcal{A}}|\tau_{[1,n)}(a)|$ goes to infinity.

We now derive two corollaries from Theorem 4.1 dealing respectively with invariant measures and with the image subgroup. Note that Corollary 4.2 extends a statement initially proved for interval exchanges [41]. See also [4] for a similar result in the framework of automorphisms of the free group and [5] for subshifts with finite rank.

Corollary 4.2. *Let (X,S) be a primitive unimodular proper $\mathcal{S}$-adic subshift over the alphabet $\mathcal{A}$ and let $\mu,\mu' \in \mathcal{M}(X,S)$. If μ and μ' coincide on the letters, then they are equal, that is, if $\mu([a]) = \mu'([a])$ for all a in $\mathcal{A}$, then $\mu(U) = \mu'(U)$, for any clopen subset U of X.*

Corollary 4.3. *Let (X,S) be a primitive unimodular proper $\mathcal{S}$-adic subshift over the alphabet $\mathcal{A}$. The image subgroup of (X,S) satisfies*

$$\begin{aligned} I(X,S) &= \bigcap_{\mu\in\mathcal{M}(X,S)} \langle \mu([a]) : a \in A\rangle \\ &= \left\{ \alpha : \exists (\alpha_a)_{a\in\mathcal{A}} \in \mathbb{Z}^{\mathcal{A}}, \alpha = \sum_{a\in\mathcal{A}} \alpha_a \mu([a]) \ \forall \mu \in \mathcal{M}(X,S) \right\}. \end{aligned}$$

The proof of Corollary 4.3 uses the two descriptions of the image subgroup given in Proposition 2.3 and Proposition 2.6.

In both corollaries, the assumption of being proper can be dropped. The proof then uses the measure-theoretical Bratteli-Vershik representation of the primitive unimodular $\mathcal{S}$-adic subshift given in [14, Theorem 6.5].

4.2. **An explicit description of the dimension group.** Theorem 4.1 allows a precise description of the dimension group of primitive unimodular proper $\mathcal{S}$-adic subshifts. Note that in the case of interval exchanges, one recovers the results obtained in [60]; see also [61, 47].

We first need the following Gottschalk-Hedlund type statement [49, Theorem 14.11].

Lemma 4.4 ([51], Lemma 2, [32], Theorem 4.2.3)**.** *Let (X,T) be a minimal Cantor system and let $f \in C(X,\mathbb{Z})$. There exists $g \in C(X,\mathbb{N})$ that is cohomologous to f if and only if for every $x \in X$, the sequence $\left(\sum_{k=0}^{n} f\circ T^k(x)\right)_{n\geq 0}$ is bounded from below.*

We now can deduce one of our main statements.

Theorem 4.5. *Let (X,S) be a primitive unimodular proper $\mathcal{S}$-adic subshift over a d-letter alphabet. The linear map $\Phi : H(X,S) \to \mathbb{Z}^d$ defined by $\Phi([\chi_{[a]}]) = e_a$, where $\{e_a \mid a \in \mathcal{A}\}$ is the canonical base of $\mathbb{Z}^d$, defines an isomorphism of dimension groups from $K^0(X,S)$ onto*

$$\left(\mathbb{Z}^d,\ \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \boldsymbol{\mu}\rangle > 0 \textit{ for all } \mu \in \mathcal{M}(X,S)\} \cup \{\mathbf{0}\},\ \mathbf{1}\right), \tag{2}$$

where the entries of $\mathbf{1}$ are equal to 1.

Proof. From Theorem 4.1, Φ is well defined and is a group isomorphism from $H(X,S)$ onto $\mathbb{Z}^d$. We obviously have $\Phi([1]) = \Phi(\sum_{a\in\mathcal{A}}[\chi_{[a]}]) = \mathbf{1}$ and it remains to show that

$$\Phi(H^+(X,S)) = \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \boldsymbol{\mu}\rangle > 0 \text{ for all } \mu \in \mathcal{M}(X,S)\} \cup \{\mathbf{0}\}.$$

Any element of $H^+(X,S)$ is of the form $[f]$ for some $f \in C(X,\mathbb{N})$. From Theorem 4.1, there exists a unique vector $\mathbf{x} = (x_a)_{a\in\mathcal{A}}$ such that $[f] = \sum_{a\in\mathcal{A}} x_a[\chi_{[a]}]$. As f is non-negative, we have, for any $\mu \in \mathcal{M}(X,S)$,

$$\langle \Phi([f]), \boldsymbol{\mu}\rangle = \sum_{a\in\mathcal{A}} x_a \mu([a]) = \int f d\mu \geq 0,$$

with equality if and only if $f = 0$ (in which case $\mathbf{x} = \mathbf{0}$).

For the other inclusion, assume that $\mathbf{x} = (x_a)_{a\in\mathcal{A}} \in \mathbb{Z}^d$ satisfies $\langle \mathbf{x}, \boldsymbol{\mu}\rangle > 0$ for all $\mu \in \mathcal{M}(X,S)$ (the case $\mathbf{x} = \mathbf{0}$ is trivial). We consider the function $f = \sum_{a\in\mathcal{A}} x_a \chi_{[a]}$. According to Lemma 4.4, the existence of $f' \in [f]$ such that f' is non-negative is equivalent to the existence of a lower bound for ergodic sums. Assume by contradiction that there exists a point $x \in X$ such that the sequence $\left(\sum_{k=0}^n f \circ S^k(x)\right)_{n\geq 0}$ is not bounded from below. Thus there is a an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that

$$\lim_{i\to+\infty} \sum_{k=0}^{n_i-1} f \circ S^k(x) = -\infty.$$

Passing to a subsequence $(m_i)_{i\geq 0}$ of $(n_i)_{i\geq 0}$ if necessary, there exists $\mu \in \mathcal{M}(X,S)$ satisfying

$$\langle \mathbf{x}, \boldsymbol{\mu}\rangle = \int f d\mu = \lim_{i\to+\infty} \frac{1}{m_i} \sum_{k=0}^{m_i-1} f \circ S^k(x) \leq 0,$$

which contradicts our hypothesis. The sequence $\left(\sum_{k=0}^n f \circ S^k(x)\right)_{n\geq 0}$ is thus bounded from below and we conclude by using Lemma 4.4. □

Remark 4.6. *We cannot remove the hypothesis of being left or right proper in Theorem 4.5. Consider indeed the subshift* (X,S) *defined by the primitive unimodular non-proper substitution* τ *defined over* $\{a,b\}^*$ *as* $\tau\colon a \mapsto aab, b \mapsto ba$. *According to* [26, p.114], *the dimension group of* (X,S) *is isomorphic to* $\left(\mathbb{Z}^3, \left\{\mathbf{x} \in \mathbb{Z}^3 : \langle \mathbf{x}, \mathbf{v}\rangle > 0\right\}, (2,0,-1)\right)$ *where* $\mathbf{v} = (\frac{1+\sqrt{5}}{2}, 2, 1)$.

4.3. **Ergodic measures.** We now focus on further consequences of Theorem 4.1 for invariant measures of primitive unimodular proper $\mathcal{S}$-adic subshifts.

Corollary 4.7. *Two primitive unimodular proper* $\mathcal{S}$*-adic subshifts* (X_1,S) *and* (X_2,S) *are strong orbit equivalent if and only if there is a unimodular matrix* M *such that* $M\mathbf{1} = \mathbf{1}$ *and*

$$\{\boldsymbol{\nu} \mid \nu \in \mathcal{M}(X_2,S)\} = \{M^{\mathrm{T}}\boldsymbol{\mu} \mid \mu \in \mathcal{M}(X_1,S)\}.$$

In particular, (X_1,S) *and* (X_2,S) *are defined on alphabets with the same cardinality.*

Proof. For $i = 1,2$, let $\Phi_i : H(X_i,S) \to \mathbb{Z}^{d_i}$ be the map given in Theorem 4.5, where d_i is the cardinality of the alphabets $\mathcal{A}_i$ of X_i. Let us also write $\mathbf{1}_i$ the vector of dimension d_i only consisting in 1's and

$$C_i = \Phi_i(H^+(X_i,S)) = \{\mathbf{x} \in \mathbb{Z}^{d_i} \mid \langle \mathbf{x}, \boldsymbol{\mu} \rangle > 0 \text{ for all } \mu \in \mathcal{M}(X_i,S)\} \cup \{\mathbf{0}\},$$

so that Φ_i defines an isomorphism of dimension groups from $K^0(X_i,S)$ onto $(\mathbb{Z}^{d_i}, C_i, \mathbf{1}_i)$.

First assume that (X_1,S) and (X_2,S) are strong orbit equivalent. Theorem 2.2 implies that there is an isomorphism of dimension group from $(\mathbb{Z}^{d_2}, C_2, \mathbf{1}_2)$ onto $(\mathbb{Z}^{d_1}, C_1, \mathbf{1}_1)$. Hence $d_1 = d_2 = d$ and this isomorphism is given by a unimodular matrix M of dimension d satisfying $M\mathbf{1} = \mathbf{1}$ (where $\mathbf{1} = \mathbf{1}_1 = \mathbf{1}_2$) and $MC_2 = C_1$. We also denote by M the map $\mathbf{x} \in \mathbb{Z}^d \mapsto M\mathbf{x}$.

Recall from Section 2.4 that the map

$$\mu \in \mathcal{M}(X_i,S) \mapsto \left(\tau_\mu : [f] \in H(X_i,S) \mapsto \int f d\mu\right)$$

is an affine isomorphism from $\mathcal{M}(X_i,S)$ to $\mathcal{T}(K^0(X_i,S))$. Observing that for all $\mu \in \mathcal{M}(X_1,S)$, $\tau_\mu \circ \Phi_1^{-1} \circ M \circ \Phi_2$ is a trace of $K^0(X_2,S)$, it defines an affine isomorphism $\mu \in \mathcal{M}(X_1,S) \mapsto \nu \in \mathcal{M}(X_2,S)$, where ν is such that $\tau_\nu = \tau_\mu \circ \Phi_1^{-1} \circ M \circ \Phi_2$. Since $\boldsymbol{\mu} = (\tau_\mu([\chi_{[a]}]))_{a \in \mathcal{A}_1}$, we have, for all $a \in \mathcal{A}_2$,

$$\nu([a]) = \tau_\nu([\chi_{[a]}]) = \tau_\mu \circ \Phi_1^{-1} \circ M \circ \Phi_2([\chi_{[a]}]) = \boldsymbol{\mu}^{\mathrm{T}} M e_a = e_a^{\mathrm{T}} M^{\mathrm{T}} \boldsymbol{\mu},$$

so that $\boldsymbol{\nu} = M^{\mathrm{T}}\boldsymbol{\mu}$.

Now assume that we are given a unimodular matrix M satisfying $M\mathbf{1} = \mathbf{1}$ and

$$\{\boldsymbol{\nu} \mid \nu \in \mathcal{M}(X_2,S)\} = \{M^{\mathrm{T}}\boldsymbol{\mu} \mid \mu \in \mathcal{M}(X_1,S)\}.$$

In particular, this implies that $d_1 = d_2 = d$. Let us show that the map $M : \mathbf{x} \in \mathbb{Z}^d \mapsto M\mathbf{x}$ defines an isomorphism of dimension groups from $(\mathbb{Z}^d, C_1, \mathbf{1})$ to $(\mathbb{Z}^d, C_2, \mathbf{1})$. We only need to show that $MC_1 = C_2$. The matrix M being unimodular, we have $M\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. For $\mathbf{x} \neq \mathbf{0}$, we have

$$\begin{aligned} \mathbf{x} \in C_2 &\Leftrightarrow \langle \mathbf{x}, \boldsymbol{\nu} \rangle > 0 \text{ for all } \nu \in \mathcal{M}(X_2,S) \\ &\Leftrightarrow \langle \mathbf{x}, M^{\mathrm{T}}\boldsymbol{\mu} \rangle > 0 \text{ for all } \mu \in \mathcal{M}(X_1,S) \\ &\Leftrightarrow \langle M\mathbf{x}, \boldsymbol{\mu} \rangle > 0 \text{ for all } \mu \in \mathcal{M}(X_1,S) \\ &\Leftrightarrow M\mathbf{x} \in C_1, \end{aligned}$$

which ends the proof. □

According to Theorem 4.5, dimension groups of primitive unimodular proper subshifts have rank d. This implies that the number e of ergodic measures satisfies $e \leq d$. In fact, we have even more from the following result.

Proposition 4.8. [38, Proposition 2.4] *Finitely generated simple dimension groups of rank d have at most $d - 1$ pure traces.*

Dimension groups of minimal Cantor systems (X, T) are simple dimension groups (Theorem 2.1) and, since the Choquet simplex of traces is affinely isomorphic to the simplex of ergodic measures, we derive the following.

Corollary 4.9. *Primitive unimodular proper $\mathcal{S}$-adic subshifts over a d-letter alphabet have at most $d-1$ ergodic measures.*

If the primitive unimodular proper $\mathcal{S}$-adic subshift (X, S) has some extra combinatorial properties, then the number of ergodic measures can be smaller. Suppose indeed that (X, S) is a minimal dendric subshift on a d-letter alphabet. As its factor complexity equals $(d-1)n+1$, one has a priori $e \leq d-2$ for $d \geq 3$ according to [13, Theorem 7.3.4]. One can even have more as a direct consequence of [23] and [25]. Note that this statement encompasses the case of interval exchanges handled in [52, 67].

Theorem 4.10. *Let (X, S) be a minimal dendric subshift over a d-letter alphabet. One has*

$$\mathrm{Card}(\mathcal{M}_e(X, S)) \leq \frac{d}{2}.$$

Proof. According to [23], a minimal subshift is said to satisfy the regular bispecial condition if any large enough bispecial word w has only one left extension $aw \in \mathcal{L}(X)$, $a \in \mathcal{A}$, that is right special and only one right extension $wa \in \mathcal{L}(X)$, $a \in \mathcal{A}$, that is left special. Now we use the fact that minimal dendric subshifts satisfy the regular bispecial condition according to [25]. We conclude by using the upper bound on the number ergodic measures from [23]. □

5. Infinitesimals and balance property

When the infinitesimal subgroup $\mathrm{Inf}(K^0(X, T))$ of a minimal Cantor system (X, T) is trivial, the system is called *saturated*. This property is proved in [1] to hold for primitive, aperiodic, irreducible substitutions for which images of letters have a common prefix. At the opposite, an example of a dendric subshift with non-trivial infinitesimal subgroup is provided in Example 6.3. A formulation of saturation in terms of the topological full group is given in [17]. Recall also that for saturated systems, the quotient group $I(X, T)/E(X, T)$ is torsion-free by [22, Theorem 1] (see also [45]).

We first state a characterization of the triviality of the infinitesimal subgroup $\mathrm{Inf}(K^0(X, S))$ for minimal unimodular proper $\mathcal{S}$-adic subshifts (see Proposition 5.1). We then relate the saturation property with a combinatorial notion called *balance property* and we provide a topological characterization of primitive unimodular proper $\mathcal{S}$-adic subshifts that are balanced (see Corollary 5.5).

Proposition 5.1. *Let (X, S) be a minimal unimodular proper $\mathcal{S}$-adic subshift on a d-letter alphabet A. The infinitesimal subgroup Inf$(K^0(X, S))$ is non-trivial if and only if there is a non-zero vector $\mathbf{x} \in \mathbb{Z}^d$ orthogonal to any element of the simplex of letter measures.*

In particular, if there exists some invariant measure $\mu \in \mathcal{M}(X,S)$ for which the frequencies of letters $\mu([a])$, $a \in A$, are rationally independent, then the infinitesimal subgroup Inf$(K^0(X,S))$ is trivial.

Proof. According to Theorem 4.5, the elements of $\mathrm{Inf}(K^0(X,S))$ are the classes of functions that are represented by vectors $\mathbf{x} \in \mathbb{Z}^d$ such that $\langle \mathbf{x}, \boldsymbol{\mu} \rangle = 0$ for every $\mu \in \mathcal{M}(X,S)$. Recall also that coboundaries are represented by the vector $\mathbf{0}$. Hence $\mathrm{Inf}(K^0(X,S))$ is not trivial if and only if there exists $\mathbf{x} \in \mathbb{Z}^d$, with $\mathbf{x} \neq \mathbf{0}$, such that $\langle \mathbf{x}, \boldsymbol{\mu} \rangle = 0$, for every $\mu \in \mathcal{M}(X,S)$.

Assume now that there exists some invariant measure $\mu \in \mathcal{M}(X,S)$ for which the frequencies of letters are rationally independent. Hence, for any vector $\mathbf{x} \in \mathbb{Z}^d$, $\langle \mathbf{x}, \boldsymbol{\mu} \rangle = 0$ implies that $\mathbf{x} = \mathbf{0}$. From above, this implies that $\mathrm{Inf}(K^0(X,S))$ is trivial. □

See Example 6.3 for an example of a dendric subshift with non-trivial infinitesimals.

We now introduce a notion of balance for functions. Let (X,T) be a minimal Cantor system. We say that $f \in C(X,\mathbb{R})$ is *balanced* for (X,T) whenever there exists a constant $C_f > 0$ such that

$$|\sum_{i=0}^{n} f(T^i x) - f(T^i y)| \leq C_f \text{ for all } x, y \in X \text{ and for all } n.$$

Balance property is usually expressed for letters and factors (see for instance [6]). Indeed a minimal subshift (X,S) is said to be *balanced on the factor* $v \in \mathcal{L}(X)$ if $\chi_{[v]} : X \to \{0,1\}$ is balanced, or, equivalently, if there exists a constant C_v such that for all w, w' in $\mathcal{L}_X$ with $|w| = |w'|$, then $||w|_v - |w'|_v| \leq C_v$. It is *balanced on letters* if it is balanced on each letter, and it is *balanced on factors* if it is balanced on all its factors.

More generally, we say that a system (X,T) is balanced on a subset $H \subset C(X,\mathbb{R})$ whenever it is balanced for all f in H. It is standard to check that any system (X,T) is balanced on the (real) coboundaries. Of course, a subshift (X,S) is balanced on a generating set of $C(X,\mathbb{Z})$ if and only if it is balanced on factors or, equivalently, if every $f \in C(X,\mathbb{Z})$ is balanced.

One can observe that the balance property on letters is not necessarily preserved under topological conjugacy whereas the balance property on factors is. Indeed, consider the shift generated by the Thue–Morse substitution $\sigma\colon a \mapsto ab, b \mapsto ba$. It is clearly balanced on letters. It is conjugate to the shift generated by the substitution $\tau\colon a \mapsto bb$, $b \mapsto bd$, $c \mapsto ca$, $d \mapsto cb$ via the sliding block code $00 \mapsto a$, $01 \mapsto b$, $10 \mapsto c$, $11 \mapsto d$ (see [62, p.149]). The subshift generated by τ is not balanced on letters (see [6]).

Next proposition will be useful to characterize balanced functions of a system (X,T).

Proposition 5.2. *Let (X,T) be a minimal dynamical system. An integer valued continuous function $f \in C(X,\mathbb{Z})$ is balanced for (X,T) if and only* *if there exists $\alpha \in \mathbb{R}$ such that the map $f - \alpha$ is a real coboundary. In this case, $\alpha = \int f d\mu$, for any T-invariant probability measure μ in X.*

Proof. If the function $f - \alpha$ is a real coboundary, one easily checks that f is balanced. Moreover, the integral with respect to any T-invariant probability measure is zero, providing the last claim.

Assume that $f \in C(X, \mathbb{R})$ is balanced for (X, T). Let $C > 0$ be a constant such that $|\sum_{i=0}^{n} f \circ T^i(x) - f \circ T^i(y)| \leq C$ holds uniformly in $x, y \in X$ for all $n \geq 0$. Thus, for any non-negative integer $p \in \mathbb{N}$, there exists N_p such that, for any $x \in X$, one has the following inequalities:

$$N_p \leq \sum_{i=0}^{p} f \circ T^i(x) \leq N_p + C.$$

Moreover, one checks that, for any $p, q \in \mathbb{N}$:

$$qN_p \leq \sum_{i=0}^{pq} f \circ T^i(x) \leq qN_p + qC \text{ and } pN_q \leq \sum_{i=0}^{pq} f \circ T^i(x) \leq pN_q + pC.$$

It follows that $-qC \leq qN_p - pN_q \leq pC$ and thus $-C/p \leq N_p/p - N_q/q \leq C/q$. Hence the sequence $(N_p/p)_p$ is a Cauchy sequence. Let $\alpha = \lim_{p\to\infty} N_p/p$. By letting q going to infinity, we get $-C \leq N_p - p\alpha \leq 0$, so that $-C \leq \sum_{i=0}^{p} f \circ T^i(x) - p\alpha \leq C$ for any $x \in X$. By the classical Gottschalk–Hedlund's Theorem [49], the function $f - \alpha$ is a real coboundary. □

As a corollary, we deduce that a minimal Cantor system (X, T) balanced on $C(X, \mathbb{Z})$ is uniquely ergodic. It also follows that for a minimal subshift (X, S) balanced on the factor v, the frequency $\mu_v \in \mathbb{R}_+$ of v exists, i.e., for any $x \in X$, $\lim_{n\to\infty} \frac{|x_{-n}\cdots x_0 \cdots x_n|_v}{2n+1} = \mu_v$, and even, the quantity $\sup_{n\in\mathbb{N}} ||x_{-n} \cdots x_0 \cdots x_n|_v - (2n + 1)\mu_v|$ is finite (see also [16]).

Actually, integer-valued continuous functions that are balanced for a minimal Cantor system (X, T) are related to the continuous eigenvalues of the system as illustrated by the following folklore lemma. We recall that $E(X, T)$ stands for the set of additive continuous eigenvalues.

Lemma 5.3. *Let (X, T) be a minimal Cantor system and let μ be a T-invariant measure. If $f \in C(X, \mathbb{Z})$ is balanced for (X, T), then $\int f d\mu$ belongs to $E(X, T)$.*

Proof. If $f \in C(X, \mathbb{Z})$ is balanced for (X, T), then so is $-f$ and there exists $g \in C(X, \mathbb{R})$ such that $-f + \int f d\mu = g \circ T - g$ (by Proposition 5.2). This yields $\exp(2i\pi g \circ T) = \exp(2i\pi \int f d\mu) \exp(2i\pi g)$ by noticing that $\exp(-2i\pi f(x)) = 1$ for any $x \in X$. Hence $\exp(2i\pi g)$ is a continuous eigenfunction associated with the additive eigenvalue $\int f d\mu$. □

We first give a statement valid for any minimal Cantor system that will then be applied below to primitive unimodular proper $\mathcal{S}$-adic subshifts. We recall from [45, Theorem 3.2, Corollary 3.6] that there exists a one-to-one homomorphism Θ from $I(X,T)$ to $K^0(X,T)$ such that, for $\alpha \in (0,1) \cap E(X,T)$, $\Theta(\alpha) = [\chi_{U_\alpha}]$ where U_α is a clopen set, sucht that $\mu(U_\alpha) = \alpha$ for every invariant measure μ, and $\chi_{U_\alpha} - \mu(U_\alpha)$ is a real coboundary. Hence χ_{U_α} is balanced for (X,T) (by Proposition 5.2).

Proposition 5.4. *Let (X,T) be a minimal Cantor system. The following are equivalent:*

(1) (X,T) is balanced on some $H \subset C(X,\mathbb{Z})$ and $\{[h] : h \in H\}$ generates $K^0(X,T)$,
(2) (X,T) is balanced on $C(X,\mathbb{Z})$,
(3) $\Theta(E(X,T))$ generates $K^0(X,T)$.

In this case (X,T) is uniquely ergodic, then $I(X,T) = E(X,T)$ and $\mathrm{Inf}(K^0(X,T))$ is trivial.

Proof. Let us prove that (1) implies (2). Let $f \in C(X,\mathbb{Z})$. One has $[f] = \sum_{i=1}^n z_i[h_i]$ for some integers z_i and some functions $h_i \in H$. Hence $f = g \circ T - g + \sum_{i=1}^n z_i h_i$ for some $g \in C(X,\mathbb{Z})$ and f is balanced. Consequently, (X,T) is balanced on $C(X,\mathbb{Z})$. It is immediate that (3) implies (1).

Let us show that (2) implies (3). Unique ergodicity holds by Proposition 5.2. Let μ be the unique shift invariant probability measure of (X,T). For any $f \in C(X,\mathbb{Z})$, there are clopen sets U_i and integers z_i such that $f = \sum_{i=1}^n z_i \chi_{U_i}$. From Lemma 5.3 the values $\mu(U_i) \in I(X,T)$ are additive continuous eigenvalues in $E(X,T)$. We get $[\chi_{U_i}] = \Theta(\mu(U_i))$ and $[f] = \sum_{i=1}^n z_i \Theta(\mu(U_i))$. This shows the claim (3).

Assume that one of the three equivalent conditions holds. Let μ denote the unique shift invariant probability measure. Then, any map $f - \int f d\mu$, with $f \in C(X,\mathbb{Z})$, is a real coboundary (by Proposition 5.2). Hence, since any integer valued continuous function that is a real coboundary is a coboundary ([57, Proposition 4.1]), the infinitesimal subgroup $\mathrm{Inf}(K^0(X,T))$ is trivial. Moreover, Lemma 5.3 implies that $I(X,T) \subset E(X,T)$. The reverse implication $E(X,T) \subset I(X,T)$ comes from [22, Proposition 11], see also [45, Corollary 3.7]. □

Observe that when (X,S) is a minimal subshift, by taking H to be the set of classes of characteristic functions of cylinder sets, the balance property is equivalent to the algebraic condition (3) of Proposition 5.4.

We now provide a topological proof of the fact that the balance property on letters implies the balance property on factors for primitive unimodular proper $\mathcal{S}$-adic subshifts. For minimal dendric subshifts, this was already proved in [6, Theorem 1.1] using a combinatorial proof.

Corollary 5.5. *Let (X,S) be a primitive unimodular proper $\mathcal{S}$-adic subshift on a d-letter alphabet. The following are equivalent:*

(1) (X,S) is balanced for all integer valued continuous maps in $C(X,\mathbb{Z})$,
(2) (X,S) is balanced on factors,
(3) (X,S) is balanced on letters,

(4) $\mathrm{rank}(E(X,S)) = d$,

and in this case (X,S) *is uniquely ergodic,* $I(X,S) = E(X,S)$ *and* $\mathrm{Inf}(X,S)$ *is trivial.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate. Let us prove the implication (3) ⇒ (4). We deduce from Theorem 4.1, by taking H to be the set of classes of characteristic functions of cylinder sets, that the conditions of Proposition 5.4 hold. We deduce from Proposition 5.1 that (4) holds.

It remains to prove the implication (4) ⇒ (1). Suppose that $E(X,S)$ has rank d. Let $\alpha_1,\dots,\alpha_d \in E(X,S)$ be rationally independent. There is no restriction to assume that they are all in $(0,1)$. Consider, for $i = 1\cdots,d$, $\Theta(\alpha_i) = [\chi_{U_{\alpha_i}}]$ where U_{α_i} is a clopen set such that $\mu(U_{\alpha_i}) = \alpha_i$ for any S-invariant measure $\mu \in \mathcal{M}(X,S)$ and $\chi_{U_{\alpha_i}}$ is balanced for (X,S). The classes $\Theta(\alpha_i)$'s are rationally independent because the image of $\Theta(\alpha_i)$ by any trace is α_i and these values are assumed to be rationally independent. As $K^0(X,S)$ has rank d, by Theorem 4.1, and since it has no torsion (as recalled in Section 2.3), any element $[f] \in K^0(X,S)$ is a rational linear combination of the $\Theta(\alpha_i)$'s. By Proposition 5.4, any $f \in C(X,\mathbb{Z})$ is balanced for (X,S). □

Remark 5.6. *We deduce that primitive unimodular proper* $\mathcal{S}$*-adic subshifts that are balanced on letters have the maximal continuous eigenvalue group property, as defined in* [29]*, i.e.,* $E(X,S) = I(X,S)$*. This implies in particular that non-trivial additive eigenvalues are irrational. Indeed, by Corollary 5.5, Inf(*$K^0(X,S)$*) is trivial and thus, by Proposition 5.1, the frequencies of letters (for the unique shift-invariant measure) are rationally independent, which yields that* $I(X,S)$ *and thus* $E(X,S)$ *contain no rational non-trivial elements.*

More generally, the fact that non-trivial additive eigenvalues are irrational hold for minimal dendric subshifts (even without the balance property) [12]*. Note also that the triviality of Inf(*$K^0(X,S)$*) says nothing about the balance property (see Example 6.4), but the existence of non-trivial infinitesimals indicates that some letter is not balanced. Lastly, the Thue–Morse substitution* $\sigma\colon a \mapsto ab, b \mapsto ba$ *generates a subshift that is balanced on letters but not on factors* [6]*. This substitution is neither unimodular, nor proper.*

6. Examples and observations

6.1. **Brun subshifts.** We provide a family of primitive unimodular proper $\mathcal{S}$-adic subshifts which are not dendric. We consider the set of endomorphisms $S_{\mathrm{Br}} = \{\beta_{ab} \mid a \in \mathcal{A},\, b \in \mathcal{A}\setminus\{a\}\}$ over d letters defined by

$$\beta_{ab}:\ b \mapsto ab,\ c \mapsto c \text{ for } c \in \mathcal{A}\setminus\{b\}.$$

A subshift (X,S) is a *Brun subshift* if it is generated by a primitive directive sequence $\boldsymbol{\tau} = (\tau_n)_n \in S_{\mathrm{Br}}^{\mathbb{N}}$ such that for all n the endomorphism $\tau_n\tau_{n+1}$ belongs to

$$\big\{\beta_{ab}\beta_{ab} \mid a \in \mathcal{A},\, b \in \mathcal{A}\setminus\{a\}\big\} \cup \big\{\beta_{ab}\beta_{bc} \mid a \in \mathcal{A},\, b \in \mathcal{A}\setminus\{a\},\, c \in \mathcal{A}\setminus\{b\}\big\}.$$

Observe that primitiveness of $\boldsymbol{\tau}$ is equivalent to the fact that for each $a \in \mathcal{A}$ there is $b \in \mathcal{A}$ such that β_{ab} occurs infinitely often in $\boldsymbol{\tau}$. Brun subshifts are not dendric in general: on a three-letter alphabet, they may contain strong and weak bispecial factors, hence that have an extension graph which is not a tree [55]. However, we show below that they are primitive unimodular proper $\mathcal{S}$-adic subshifts.

Lemma 6.1. *Let $\mathcal{A}$ be a finite alphabet and $\gamma_{ab} : \mathcal{A} \to \mathcal{A}$, $a \neq b$, be the letter-to-letter map defined by $\gamma_{ab}(a) = \gamma_{ab}(b) = a$ and $\gamma_{ab}(c) = c$ for $c \in \mathcal{A} \setminus \{a, b\}$. Let $(a_n)_{1 \le n \le N}$ be such that $\{a_n \mid 1 \le n \le N\} = \mathcal{A}$. Then, $\gamma_{a_1 a_2} \gamma_{a_2 a_3} \cdots \gamma_{a_{N-1} a_N}$ is constant.*

Proof. It suffices to observe that $\gamma_{a_m a_{m+1}} \gamma_{a_{m+1} a_{m+2}} \cdots \gamma_{a_{n-1} a_n}$ identifies the letters $a_m, a_{m+1}, a_{m+2} \ldots, a_n$ to a_m. □

Lemma 6.2. *Brun subshifts are primitive unimodular proper $\mathcal{S}$-adic subshifts.*

Proof. Let (X, S) be a Brun subshift over the alphabet $\mathcal{A}$, generated by the directive sequence $\boldsymbol{\beta} = (\beta_{a_n b_n})_{n \ge 1} \in S_{\mathrm{Br}}^{\mathbb{N}}$. With each endomorphism β_{ab} one can associate the map $\gamma : \mathcal{A} \to \mathcal{A}$ defined by $\gamma(c) = \beta(c)_0$. Clearly γ is equal to γ_{ab}.

By primitiveness, there exists an increasing sequence of integers $(n_k)_k$, with $n_0 = 0$, such that $\{a_i \mid n_k \le i < n_{k+1}\} = \mathcal{A}$ for all k. Hence from Lemma 6.1 the morphisms $\boldsymbol{\beta}_{[n_k, n_{k+1})}$ are left proper. We conclude by using Lemma 3.2. □

As a corollary, we recover the following result (which also follows from [11, Theorem 5.7]).

Proposition 6.3. *Brun subshifts are uniquely ergodic.*

Proof. This follows from Corollary 4.2 and from the fact that Brun subshifts have a simplex of letter measures generated by a single vector (see [19, Theorem 3.5]). □

Brun subshifts have been introduced in [15] in order to provide symbolic models for two-dimensional toral translations. In particular, they are proved to have generically pure discrete spectrum in [15].

6.2. **Arnoux-Rauzy subshifts.** A minimal subshift (X, S) over $\mathcal{A} = \{1, 2, \ldots, d\}$ is an *Arnoux-Rauzy subshift* if for all n it has $(d-1)n + 1$ factors of length n, with exactly one left special and one right special factor of length n. Consider the following set of endomorphisms defined on the alphabet $\mathcal{A} = \{1, \ldots, d\}$, namely $S_{\mathrm{AR}} = \{\alpha_a \mid a \in \mathcal{A}\}$ with

$$\alpha_a : \ a \mapsto a, \ b \mapsto ab \text{ for } b \in \mathcal{A} \setminus \{a\}.$$

A subshift (X, S) generated by a primitive directive sequence $\boldsymbol{\tau} \in S_{\mathrm{AR}}^{\mathbb{N}}$ is called an *Arnoux-Rauzy subshift*. It is standard to check that primitiveness of $\boldsymbol{\alpha}$ is equivalent to the fact that each morphism α_a occurs infinitely often in $\boldsymbol{\alpha}$. Arnoux-Rauzy subshifts being dendric subshifts, they are in particular primitive unimodular proper $\mathcal{S}$-adic subshifts. We similarly recover, as in Proposition 6.3, that Arnoux-Rauzy subshifts are uniquely ergodic (see [24, Lemma 2] for the fact that Arnoux-Rauzy subshifts have a simplex of letter measures generated by a single vector).

6.3. **A dendric subshift with non-trivial infinitesimals.** Let us provide an example of a minimal dendric subshift with non-trivial infinitesimal subgroup, and thus with rationally dependent letter measures according to Proposition 5.1. We take the interval exchange T with permutation $(1,3,2)$ with intervals $[0,1-2\alpha),[1-2\alpha,1-\alpha)$, and $[1-\alpha,1)$, with $\alpha=(3-\sqrt{5})/2$. The transformation T is represented in Figure 2, with $I_1=[0,1-2\alpha)$, $I_2=[1-2\alpha,1-\alpha)$, $I_3=[1-\alpha,1)$ and $J_1=[0,\alpha)$, $J_2=[\alpha,2\alpha)$, $J_3=[2\alpha,1)$.

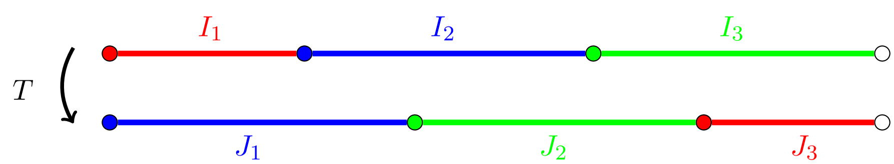

FIGURE 2. The transformation T.

Measures of letters are rationally dependent and the natural coding of this interval exchange is a strictly ergodic dendric subshift (X,S,μ) by Theorem 4.10. It is actually a representation on 3 intervals of the rotation of angle 2α (the point $1-\alpha$ is a separation point which is not a singularity of this interval exchange). One has $\mu([2])=\mu([3])$. The class of the function $\chi_{[2]}-\chi_{[3]}$ is thus a non-trivial infinitesimal, according to Theorem 4.1.

6.4. **Dendric subshifts having the same dimension group and different spectral properties.** It is well known that within any given class of strong orbit equivalence (i.e., by Theorem 2.2, within any family of minimal Cantor systems sharing the same dimension group (G,G^+,u)), all minimal Cantor systems share the same set of rational additive continuous eigenvalues $E(X,T)\cap\mathbb{Q}$ [58]. When this set is reduced to $\{0\}$, then, in the strong orbit equivalence class of (X,T), there are many weakly mixing systems, see [58, Theorem 6.1], [45, Theorem 5.4] or [29, Corollary 23].

We provide here an example of a strong orbit equivalence class that contains two minimal dendric subshifts, one being weakly mixing and the other one having pure discrete spectrum. Both systems are saturated (they have no non-trivial infinitesimals) but they have different balance properties. They are defined on a three-letter alphabet and have factor complexity $2n+1$. According to Theorem 4.10, they are uniquely ergodic. From Corollary 4.7, two minimal dendric subshifts on a three-letter alphabet are strong orbit equivalent if and only if there is a unimodular row-stochastic matrix M sending the vector of letter measures of one subshift to the vector of letter measures of the other. In particular, any Arnoux-Rauzy subshift is strong orbit equivalent to any natural coding of an i.d.o.c. exchange of three intervals for which the length of the intervals are given by the letter measures of the Arnoux-Rauzy subshift (recall that an interval exchange transformation satisfies the *infinite distinct orbit condition*, i.d.o.c. for short, if the negative trajectories of the discontinuity points are infinite disjoint sets; this condition implies minimality [54]). We thus consider the subshift (X, S) generated by the Tribonacci substitution $\sigma\colon a \mapsto ab, b \mapsto ac, c \mapsto a$ which is uniquely ergodic, dendric, balanced and has discrete spectrum [63]. Let μ be its unique invariant measure. We also consider the natural coding (Y, S) of the three-letter interval exchange defined on intervals of length $\mu[a]$, $\mu[b]$, $\mu[c]$ with permutation (13)(2). It is uniquely ergodic, topologically weakly mixing [53, 40] and strong orbit equivalent to (X, S) by Proposition 3.8. Hence, for spectral reasons, (X, S) and (Y, S) are not topologically conjugate, even if they are strong orbit equivalent.

We provide a further proof of non-conjugacy for the systems (X, S) and (Y, S) based on asymptotic pairs. We first recall a few definitions. Two points x, y in a given subshift are said to be *right asymptotic* if they have a common tail, i.e., there exists n such that $(x_k)_{k\geq n} = (y_k)_{k\geq n}$. This defines an equivalence relation on the collection of orbits: two S-orbits $\mathcal{O}_S(x) = \{S^n x \mid n \in \mathbb{Z}\}$ and $\mathcal{O}_S(y)$ are asymptotically equivalent if for any $x' \in \mathcal{O}_S(x)$, there is $y' \in \mathcal{O}_S(y)$ that is right asymptotic to x'. We call *asymptotic component* any equivalence class under the asymptotic equivalence. We say that it is *non-trivial* whenever it is not reduced to one orbit.

An Arnoux-Rauzy subshift (X, S) has a unique non-trivial asymptotic component formed of three distincts orbits as, for all n, there is a unique word w of length n such that $\ell(w) \geq 2$ and this word is such that $\ell(w) = 3$ (see Section 3.2 for the notation). On the other side, any i.d.o.c. exchange of three-intervals (Y, S) has 2 asymptotic components and thus cannot be conjugate to (X, S). Indeed, suppose that it has a unique non-trivial asymptotic component. As a natural coding of an i.d.o.c interval exchange transformation has two left special factors for each large enough length, this component should contain three sequences $x'x$, $x''ux$ and $x'''ux$ belonging to Y where u is a non-empty word. This would imply that the interval exchange transformation is not i.d.o.c.

Next statement illustrates the variety of spectral behaviours within strong orbit equivalence classes of dendric subshifts.

Proposition 6.4. *For Lebesgue a.e. probability vector $\boldsymbol{\mu}$ in $\mathbb{R}^3_+$, there exist two strictly ergodic proper unimodular $\mathcal{S}$-adic subshifts, one with pure discrete spectrum and another one which is weakly mixing, both having the same dimension group* $\left(\mathbb{Z}^3, \{\mathbf{x} \in \mathbb{Z}^3 \mid \langle \mathbf{x}, \boldsymbol{\mu}\rangle > 0\} \cup \{\mathbf{0}\}, \mathbf{1}\right)$.

Proof. Brun subshifts such as introduced in Section 6.1 are proved to have generically pure discrete spectrum in [15]. See [53, 40] for the genericity of weak mixing for subshifts generated by three-letter interval exchanges. □

6.5. **Dendric vs. primitive unimodular proper $\mathcal{S}$-adic subshifts.** In this section, we give an example of a primitive unimodular proper $\mathcal{S}$-adic subshift whose strong orbit equivalence class contains no minimal dendric subshift. Theorem 4.5 provides a description of the dimension group of any primitive unimodular proper $\mathcal{S}$-adic subshift. It is natural to ask whether a strong orbit equivalence class represented by such a dimension group includes a primitive unimodular proper $\mathcal{S}$-adic subshift. This was conjectured in different terms in [36]. It was shown to be true when the dimension group has a unique trace [64] (or, equivalently, when all minimal systems in this class are uniquely ergodic) but shown to be false in general [65]. In the same spirit, one may ask if the strong orbit equivalence class of any primitive unimodular proper $\mathcal{S}$-adic subshift contains a dendric subshift. Inspired from [36], we negatively answer to that question below.

Indeed, this example provides a family of examples of primitive unimodular $\mathcal{S}$-adic subshifts on a three-letter alphabet with two ergodic invariant probability measures. They thus cannot be dendric by Theorem 4.10 and their strong orbit equivalence class contains no minimal dendric subshift.

Let $\mathcal{A} = \{1,2,3\}$ and consider the directive sequence $\boldsymbol{\tau} = (\tau_n : \mathcal{A}^* \to \mathcal{A}^*)_{n\geq 1}$ defined by

$$\tau_{2n} : 1 \mapsto 2^{a_n}3, \quad 2 \mapsto 1, \quad 3 \mapsto 2$$
$$\tau_{2n+1} : 1 \mapsto 32^{a_n}, \quad 2 \mapsto 1, \quad 3 \mapsto 2$$

where $(a_n)_{n\geq 1}$ is an increasing sequence of positive integers satisfying $\sum_{n\geq 1} 1/a_n < 1$. The incidence matrix of each morphism τ_n is the unimodular matrix

$$A_n = \begin{pmatrix} 0 & 1 & 0 \\ a_n & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easily checked that any morphism $\tau_{[n,n+5)}$ is proper and has an incidence matrix with positive entries. Therefore, the $\mathcal{S}$-adic subshift $(X_{\boldsymbol{\tau}}, S)$ is primitive, unimodular and proper. Let us show that $(X_{\boldsymbol{\tau}}, S)$ has two ergodic measures. In fact, we prove that $(X_{\boldsymbol{\tau}}, S)$ has at least two ergodic measures. This will imply that it has exactly two ergodic measures by Theorem 4.8.

For all $n \geq 1$, let C_n be the first column vector of $A_{[1,n]} = A_1 \cdots A_n$ and $J_n = C_n/\|C_n\|_1$ where $\|\cdot\|_1$ stands for the L^1-norm. If $(X_{\boldsymbol{\tau}}, S)$ had only one ergodic measure μ, then $(J_n)_{n\geq 1}$ would converge to $\boldsymbol{\mu}$. Hence it suffices to show that $(J_n)_{n\geq 1}$ does not converge.

Observe that, using the shape of the matrices A_n, that for $n \geq 1$, and setting $C_0 = e_1, C_{-1} = e_2, C_{-2} = e_3$, one has

$$C_n = a_n C_{n-2} + C_{n-3},$$

where e_1, e_2, e_3 are the canonical vectors. Hence we have $J_n = b_n J_{n-2} + c_n J_{n-3}$ with $b_n = a_n \frac{\|C_{n-2}\|_1}{\|C_n\|_1}$ and $c_n = \frac{\|C_{n-3}\|_1}{\|C_n\|_1}$. In particular, $b_n + c_n = 1$. As $(\|C_n\|_1)_n$ is non-decreasing, we have $1 \geq b_n \geq a_n c_n$ and thus $c_n \leq a_n^{-1}$.

Moreover, we have

$$\|J_n - J_{n-2}\|_1 = \|(b_n - 1)J_{n-2} - c_n J_{n-3}\|_1 = c_n \|J_{n-2} - J_{n-3}\|_1 \leq \frac{2}{a_n},$$

hence, for $0 \leq m \leq n$,

$$\|J_{2n} - J_{2m}\|_1 \leq 2 \sum_{k=m+1}^{n} \frac{1}{a_{2k}}.$$

This shows that $(J_{2n})_{n\geq 1}$ is a Cauchy sequence. Let β stand for its limit. For $n \geq m = 0$, we obtain $\|J_{2n} - e_1\|_1 \leq 2 \sum_{k=1}^{n} \frac{1}{a_{2k}}$ and thus $\|\beta - e_1\|_1 \leq 2 \sum_{k=1}^{\infty} \frac{1}{a_{2k}}$.

We similarly show that $(J_{2n+1})_{n\geq 1}$ is a Cauchy sequence. Let α stand for its limit. We have $\|\alpha - e_2\|_1 \leq 2 \sum_{k=0}^{\infty} \frac{1}{a_{2k+1}}$. Consequently,

$$\|\alpha - \beta\|_1 = \|(\alpha - e_2) + (e_1 - \beta) + e_2 - e_1\|_1 \geq 2 - 2 \sum_{k=1}^{\infty} \frac{1}{a_k}$$

and $(J_n)_{n\geq 1}$ does not converge. Consequently, $(X_{\boldsymbol{\tau}}, S)$ has exactly two ergodic measures.

7. Questions and further works

According to [65] (see also Section 6.5), not all strong orbit equivalence classes represented by dimension groups of the type (2) in Theorem 4.5 contain primitive unimodular proper $\mathcal{S}$-adic subshifts. The description of the dynamical dimension group in Theorem 4.5 is not precise enough to explain the restrictions that occur for instance for the measures, so that a complete characterization of the dynamical dimension groups of primitive unimodular proper $\mathcal{S}$-adic subshifts is still missing.

Similarly, we address the question of characterizing the strong orbit equivalence classes containing minimal dendric subshifts. The combinatorial properties of these subshifts imply constraints, especially for the invariant measures, such as stated in Theorem 4.10. For example, the question arises as to whether dimension groups of rank d having at most $d/2$ extremal traces are dimension groups of minimal dendric subshifts.

Another question is about the properness assumption. For dendric or Brun subshifts, we were able to find a primitive unimodular proper $\mathcal{S}$-adic representation. One can easily define $\mathcal{S}$-adic subshifts by a primitive unimodular directive sequence that is not proper. The question now is whether a primitive unimodular proper $\mathcal{S}$-adic representation (up to conjugacy) of this subshift can be found. Even in the substitutive case, we do not know whether such a representation exists.

The factor complexity of dendric subshifts is affine. It is well known [18, 58, 66] that inside the strong orbit equivalence class of any minimal Cantor system one can find another minimal Cantor systems with any other prescribed topological entropy. Primitive unimodular proper $\mathcal{S}$-adic subshifts being of finite topological rank, they have zero topological entropy. It would be interesting to exhibit a variety of asymptotic behaviours for complexity functions within a strong orbit equivalence class.

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Université de Paris, IRIF, CNRS, F-75013 Paris, France
E-mail address: berthe@irif.fr

Centro de Modelamiento Matemático, Universidad de Chile, Chile
E-mail address: pcecchi@ing.uchile.cl

LAMFA, UMR 7352 CNRS, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens, France
E-mail address: fabien.durand@u-picardie.fr

Département de mathématique, Université de Liège, 12, Allée de la découverte (B37), 4000 Liège, Belgique
E-mail address: j.leroy@uliege.be

Laboratoire d'Informatique Gaspard-Monge, Université Paris-Est, France
E-mail address: Dominique.Perrin@u-pem.fr

LAMFA, UMR 7352 CNRS, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens, France
E-mail address: samuel.petite@u-picardie.fr