# The functional form of Mahler conjecture for even log-concave functions in dimension 2. 

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#### Abstract

Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be an even convex function and $\mathcal{L} \varphi$ be its Legendre transform. We prove the functional form of Mahler conjecture concerning the functional volume product $P(\varphi)=\int e^{-\varphi} \int e^{-\mathcal{L} \varphi}$ in dimension 2: we give the sharp lower bound of this quantity and characterize the equality case. The proof uses the computation of the derivative in $t$ of $P(t \varphi)$ and ideas due to Meyer $[\mathrm{M}]$ for unconditional convex bodies, adapted to the functional case by Fradelizi-Meyer [FM2] and extended for symmetric convex bodies in dimension 3 by IriyehShibata [IS] (see also [FHMRZ]).


## 1 Introduction

In the theory of convex bodies, many geometric inequalities can be generalized to functional inequalities. This is the case of Prékopa-Leindler, which is the functional form of Brunn-Minkowski inequality. Let us mention also Blaschke-Santaló inequality [San], which states in the symmetric case that if $K$ is a symmetric convex body in $\mathbb{R}^{n}$ (in this paper, $K$ symmetric means $K=-K$ ) and

$$
P(K)=|K|\left|K^{*}\right|,
$$

where $K^{*}=\left\{y \in \mathbb{R}^{n} ;\langle y, x\rangle \leq 1\right.$, for all $\left.x \in K\right\}$ is the polar body of K then

$$
P(K) \leq P\left(B_{2}^{n}\right),
$$

with equality if and only if $K$ is an ellipsoid, where $B_{2}^{n}$ is the Euclidean ball associated to the standard scalar product in $\mathbb{R}^{n}$ and $|B|$ stands for the Lebesgue measure of a Borel subset $B$ of $\mathbb{R}^{n}$ ( $[\mathrm{P}]$, see $[\mathrm{MP}]$ or also $[\mathrm{MR}]$ for a simple proof of both the inequality and the case of equality).

Mahler conjectured an inverse form of the Blaschke-Santaló inequality for symmetric convex bodies in $\mathbb{R}^{n}$ in [Mah1]. He asked if for every symmetric convex body $K$,

$$
P(K) \geq P\left([-1,1]^{n}\right)=\frac{4^{n}}{n!} .
$$

It was later conjectured that the equality case occurs if and only if $K$ is a Hanner polytope (see [RZ] for the definition). The inequality was proved by Mahler for $n=2$ [Mah1] (see also [Me] and $[\mathrm{S}]$ Section 10.7, for other proofs and the characterization of the equality case). This conjecture has been proved also in a number of particular cases, for zonoids by Reisner [Re] (see also [GMR]) and for unconditional convex bodies by Saint Raymond [SR] (see also [M]), for hyperplane sections of $B_{p}^{n}=\left\{x \in \mathbb{R}^{n} ; \sum\left|x_{i}\right|^{p} \leq 1\right\}$ and Hanner polytopes by Karasev [K]. The 3-dimensional case of the conjecture was proved by Iriyeh and Shibata [IS] (see [FHMRZ] for a shorter proof).

Some functional versions of the previous inequalities were proposed with convex bodies replaced with log-concave functions, and polarity with Legendre transform. Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be an even convex function. Then the Legendre transform $\mathcal{L} \varphi$ of $\varphi$ is defined for $y \in \mathbb{R}^{n}$ by

$$
\mathcal{L} \varphi(y)=\sup _{x \in \mathbb{R}^{n}}(\langle x, y\rangle-\varphi(x)) .
$$

We define the functional volume product of an even convex function to be

$$
P(\varphi)=\int e^{-\varphi(x)} d x \int e^{-\mathcal{L} \varphi(y)} d y
$$

The functional version of the Blaschke-Santaló inequality for even convex functions states that

$$
P(\varphi) \leq P\left(\frac{|\cdot|^{2}}{2}\right)=(2 \pi)^{n}
$$

where $|$.$| stands here for the Euclidean norm in \mathbb{R}^{n}$, with equality if and only if $\varphi$ is a positive quadratic form. This statement was proved by Ball [B] (see also Artstein-Klartag-Milman [AKM], Fradelizi-Meyer [FM1] and Lehec [L1, L2] for more general results). For $\varphi(x)=\|x\|_{K}^{2} / 2$, one has $\mathcal{L} \varphi(y)=\|y\|_{K^{*}}^{2} / 2$, thus it is not difficult to see that

$$
P(\varphi)=\frac{(2 \pi)^{n}}{\left|B_{2}^{n}\right|^{2}} P(K) .
$$

This shows that the functional form indeed implies the geometric form of the inequality. We deal in this article with a functional version of Mahler conjecture for even convex functions. The following conjecture was stated in [FM2]. We prove it for $n=2$.

Conjecture 1. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be an even convex function such that $0<$ $\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Then

$$
\int e^{-\varphi} \int e^{-\mathcal{L} \varphi} \geq 4^{n}
$$

with equality if and only if there exists $c \in \mathbb{R}$ and two Hanner polytopes $K_{1} \subset F_{1}$ and $K_{2} \subset F_{2}$, where $F_{1}$ and $F_{2}$ are two complementary subspaces in $\mathbb{R}^{n}$, such that for all $\left(x_{1}, x_{2}\right) \in F_{1} \times F_{2}$

$$
\varphi\left(x_{1}+x_{2}\right)=c+\left\|x_{1}\right\|_{K_{1}}+I_{K_{2}}\left(x_{2}\right)
$$

where $I_{K}$ is the function defined by $I_{K}(x)=0$ if $x \in K$ and $I_{K}(x)=+\infty$ if $x \notin K$.
For unconditional functions (and in particular if $n=1$ ) the inequality in conjecture 1 was proved in [FM2, FM3] and the equality case was proved in [FGMR]. For a symmetric convex body $K$ of $\mathbb{R}^{n}$ and for any $y \in \mathbb{R}^{n}$ one has

$$
\mathcal{L} I_{K}(y)=\sup _{x \in \mathbb{R}^{n}}\left(\langle x, y\rangle-I_{K}(x)\right)=\sup _{x \in K}\langle x, y\rangle=h_{K}(y)=\|y\|_{K^{*}},
$$

where $h_{K}$ denotes the support function of $K$. In addition, using Fubini, we have

$$
\int_{\mathbb{R}^{n}} e^{-\|y\|_{K^{*}}} d y=\int_{\mathbb{R}^{n}} \int_{\|y\|_{K^{*}}}^{+\infty} e^{-t} d t d y=\int_{0}^{+\infty} \int_{\left\{\|y\|_{K^{*}} \leq t\right\}} e^{-t} d y d t=\int_{0}^{+\infty}\left|t K^{*}\right| e^{-t} d t=n!\left|K^{*}\right| .
$$

Thus if $\varphi=I_{K}$ we get

$$
P(\varphi)=\int_{\mathbb{R}^{n}} e^{-I_{K}} \int_{\mathbb{R}^{n}} e^{-\|y\|_{K^{*}}}=|K| \int_{\mathbb{R}^{n}} e^{-\|y\|_{K^{*}}}=n!\left|K \| K^{*}\right|=n!P(K)
$$

Hence Conjecture 1 implies Mahler conjecture for symmetric convex bodies. Notice that recently Gozlan [G] established precise relationships between the functional form of Mahler conjecture and the deficit in the Gaussian log-Sobolev inequality, thus our results implies better bounds in dimension 2 for these deficits.
In the proof, as in [IS, FHMRZ], we use the notion of equipartition. Denote by $\left(e_{1}, \ldots, e_{n}\right)$ the canonical basis of $\mathbb{R}^{n}$. A function $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is equipartioned if

$$
\int_{\mathbb{R}_{\varepsilon}^{n}} \varphi e^{-\varphi}=\frac{1}{2^{n}} \int_{\mathbb{R}^{n}} \varphi e^{-\varphi} \quad \text { and } \quad \int_{\mathbb{R}_{\varepsilon}^{n}} e^{-\varphi}=\frac{1}{2^{n}} \int_{\mathbb{R}^{n}} e^{-\varphi},
$$

where $\forall \varepsilon \in\{-1,1\}^{n}, \mathbb{R}_{\varepsilon}^{n}=\left\{x \in \mathbb{R}^{n} ; \varepsilon_{i} x_{i} \geq 0, \forall i \in\{1, \ldots, n\}\right\}$. We use that in dimension $n \leq 2$, for any even convex function $\varphi$ there exists a "position" of $\varphi$ which is equipartioned. Moreover we also prove that if one has an even convex function $\varphi$ on $\mathbb{R}^{n}$ such that $\varphi$ and $\varphi_{i}=\varphi_{\mid e_{i}^{\perp}}$ are equipartioned for all $1 \leq i \leq n$ and $\varphi_{i}$ satisfy the inequality of the conjecture in dimension $n-1$ then $\varphi$ satisfies the conjectured inequality in dimension $n$.

This paper is organized in the following way. In section 2 , we present some general results on the Legendre transform. In section 3, we establish some properties of the functional volume product. In section 4 , we apply the results of sections 2 and 3 to prove the inequality and the case of equality of the functional volume product of $\varphi$ in dimension 2 . Finally, in section 5 we prove the inequality in dimension $n$ for "strongly" equipartioned convex functions.

## 2 General results on convex functions and Legendre transform

Let us recall some useful facts about convex functions and the Legendre duality that can be found in the part I and V of the book of Rockafellar $[\mathrm{R}]$. Recall that if $K$ is convex, closed and contains 0 then $\left(K^{*}\right)^{*}=K$. Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a convex function. We denote the domain of $\varphi$ by

$$
\operatorname{dom}(\varphi)=\left\{x \in \mathbb{R}^{n} ; \varphi(x)<+\infty\right\}
$$

It is a convex set. If $\varphi$ is moreover lower semi-continuous and $\operatorname{dom}(\varphi) \neq \emptyset$ then $\mathcal{L} \mathcal{L} \varphi=\varphi$. The following lemma recalls some standard facts that can be found for example in Lemma 4 of $[\mathrm{G}]$.

Lemma 2. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a convex function such that $\min \varphi=\varphi(0)=0$. Then the following are equivalent.

1. One has $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$.
2. The set $K_{\varphi}:=\left\{x \in \mathbb{R}^{n}, \varphi(x) \leq 1\right\}$ is convex bounded and contains 0 in its interior.
3. There exists $a, b>0$ such that for every $x \in \mathbb{R}^{n}$ one has $a|x|-1 \leq \varphi(x) \leq I_{b B_{2}^{n}}(x)+1$.

Notice that for every $x \in \mathbb{R}^{n}$ one has $a|x|-1 \leq \varphi(x) \leq I_{b B_{2}^{n}}(x)+1$ if and only if for every $y \in \mathbb{R}^{n}$ one has $b|y|-1 \leq \mathcal{L} \varphi(y) \leq I_{a B_{2}^{n}}(y)+1$. Thus $0<\int_{\mathbb{R}^{n}} e^{-\varphi}<+\infty$ is equivalent to $0<\int_{\mathbb{R}^{n}} e^{-\mathcal{L} \varphi}<+\infty$.

We define the analogue of sections and projections of convex sets for convex functions. The section of $\varphi$ by an affine subspace $F$ is simply the restriction of $\varphi$ to this subspace and is denoted by $\varphi_{\mid F}$. The projection $P_{F} \varphi: F \rightarrow \mathbb{R} \cup\{+\infty\}$ of $\varphi$ onto a linear subspace $F$ is defined for $x \in F$ by

$$
P_{F} \varphi(x)=\inf _{z \in F^{\perp}} \varphi(x+z) .
$$

The term projection comes from the fact that if $\tilde{P}_{F}: \mathbb{R}^{n} \times \mathbb{R} \rightarrow F \times \mathbb{R}$ denotes the orthogonal projection on $F \times \mathbb{R}$ parallel to $F^{\perp}$ then $\tilde{P}_{F}(\operatorname{Epi}(\varphi))=\operatorname{Epi}\left(P_{F} \varphi\right)$ where $\operatorname{Epi}(\varphi)=\left\{(x, t) \in \mathbb{R}^{n} \times\right.$ $\mathbb{R} ; \varphi(x) \leq t\}$. The infimal convolution of two convex functions $\varphi, \psi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is

$$
\varphi \square \psi(x)=\inf _{z \in \mathbb{R}^{n}}(\varphi(x-z)+\psi(z)) .
$$

The infimal convolution is a convex function and it interacts with the Legendre transform in the following way: $\mathcal{L}(\varphi \square \psi)=\mathcal{L} \varphi+\mathcal{L} \psi$. One can also define the projection using infimal convolution by noticing that for any $x \in \mathbb{R}^{n}$

$$
\varphi \square I_{F^{\perp}}(x)=\inf _{z \in \mathbb{R}^{n}}\left(\varphi(x-z)+I_{F^{\perp}}(z)\right)=\inf _{z \in F^{\perp}} \varphi(x-z) .
$$

Hence for $x \in F$ one has $\varphi \square I_{F^{\perp}}(x)=P_{F} \varphi(x)$. Thus the same nice duality relationship between sections and projections that holds for convex sets holds also for convex functions, for any $y \in F$

$$
\begin{aligned}
P_{F}(\mathcal{L} \varphi)(y) & =\mathcal{L} \varphi \square I_{F^{\perp}}(y)=\mathcal{L} \varphi \square \mathcal{L} I_{F}(y)=\mathcal{L}\left(\varphi+I_{F}\right)(y)=\sup _{x \in \mathbb{R}^{n}}\left(\langle x, y\rangle-\varphi(x)-I_{F}(x)\right) \\
& =\sup _{x \in F}(\langle x, y\rangle-\varphi(x))=\mathcal{L}\left(\varphi_{F}\right)(y),
\end{aligned}
$$

where, by an abuse of notation, we have denoted in the same way by $\mathcal{L}$ the Legendre transform applied to a function defined on $\mathbb{R}^{n}$ or on a subspace $F$. In each situation the supremum in the Legendre transform should be understood as taken in the subspace where the function is defined.

For $1 \leq i \leq n$ we denote the restriction of $\varphi$ to $e_{i}^{\perp}$ by $\varphi_{i}=\varphi_{\mid e_{i}^{\perp}}$ and we define the analogue of the projection onto $e_{i}^{\perp}$ to be the function $P_{i} \varphi: e_{i}^{\perp} \rightarrow \mathbb{R} \cup\{+\infty\}$ defined for $x \in e_{i}^{\perp}$ by

$$
P_{i} \varphi(x)=P_{e_{i}^{\perp}} \varphi(x)=\inf _{t \in \mathbb{R}} \varphi\left(x+t e_{i}\right) .
$$

From the preceding, for every $y \in e_{i}^{\perp}$ one has

$$
\mathcal{L} \varphi_{i}(y)=\sup _{x \in e_{i}^{+}}(\langle x, y\rangle-\varphi(x))=P_{i} \mathcal{L} \varphi(y)=\inf _{t \in \mathbb{R}} \mathcal{L} \varphi\left(y+t e_{i}\right) .
$$

Lemma 3. Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be differentiable and strictly convex such that $0<\int e^{-\varphi}<+\infty$, and for $1 \leq i \leq n$ let $\varphi_{i}=\varphi_{\mid e_{i}^{\perp}}$ then

1. The function $\nabla \varphi$ is a bijection from $\mathbb{R}^{n}$ to $\operatorname{dom}(\mathcal{L} \varphi)$.
2. One has $\nabla \varphi\left(e_{i}^{\perp}\right)=\left\{y+t_{i}(y) e_{i} ; y \in \operatorname{dom}\left(\mathcal{L} \varphi_{i}\right)\right\}$ where $t_{i}(y)=\left\langle\nabla \varphi \circ\left(\nabla \varphi_{i}\right)^{-1}(y), e_{i}\right\rangle$.

Proof. Since $\varphi$ is differentiable, we deduce from [R] Theorem 26.3 that $\mathcal{L} \varphi$ is strictly convex.

1. The fact that $\nabla \varphi$ is a bijection can be found in Corollary 26.3.1 in $[\mathrm{R}]$.
2. Since the supremum of $\mathcal{L} \varphi(y)=\sup (\langle x, y\rangle-\varphi(x))$ is reached at $x=(\nabla \varphi)^{-1}(y)$ one has

$$
\mathcal{L} \varphi(\nabla \varphi(x))=\langle x, \nabla \varphi(x)\rangle-\varphi(x)
$$

and one can conclude from Corollary 23.5 .1 of $[\mathrm{R}]$ that $(\nabla \varphi)^{-1}=\nabla(\mathcal{L} \varphi)$. Let now $y \in \operatorname{dom}\left(\mathcal{L} \varphi_{i}\right)$ be fixed and $g_{y}(t)=\mathcal{L} \varphi\left(y+t e_{i}\right)$. The function $g_{y}$ is strictly convex and tends to infinity at infinity so there exists a unique $t_{i}(y) \in \mathbb{R}$ at which the function $g_{y}$ reaches its infimum and it satisfies $g_{y}^{\prime}\left(t_{i}(y)\right)=$ 0 , i.e. $\left\langle\nabla \mathcal{L} \varphi\left(y+t_{i}(y) e_{i}\right), e_{i}\right\rangle=0$ which means that $(\nabla \varphi)^{-1}\left(y+t_{i}(y) e_{i}\right)=\nabla \mathcal{L} \varphi\left(y+t_{i}(y) e_{i}\right) \in e_{i}^{\perp}$. This also means equivalently that $t_{i}(y)$ is the unique $t \in \mathbb{R}$ such that $y+t e_{i} \in \nabla \varphi\left(e_{i}^{\perp}\right)$. Hence
$\nabla \varphi\left(e_{i}^{\perp}\right)=\left\{y+t_{i}(y) e_{i} ; y \in \operatorname{dom}\left(\mathcal{L} \varphi_{i}\right)\right\}$ and the orthogonal projection $P_{i}$ onto $e_{i}^{\perp}$ is a bijection from $\nabla \varphi\left(e_{i}^{\perp}\right)$ onto $e_{i}^{\perp}$. Moreover one has

$$
P_{i} \mathcal{L} \varphi(y)=\inf _{t \in \mathbb{R}} \mathcal{L} \varphi\left(y+t e_{i}\right)=\inf _{t \in \mathbb{R}} g_{y}(t)=g_{y}\left(t_{i}(y)\right)=\mathcal{L} \varphi\left(y+t_{i}(y) e_{i}\right) .
$$

Thus

$$
P_{i} \mathcal{L} \varphi(y)=\sup _{x \in \mathbb{R}^{n}}\left(\left\langle x, y+t_{i}(y) e_{i}\right\rangle-\varphi(x)\right) \geq \sup _{x \in e_{i}^{\perp}}\left(\left\langle x, y+t_{i}(y) e_{i}\right\rangle-\varphi(x)\right)=\sup _{x \in e_{i}^{\perp}}(\langle x, y\rangle-\varphi(x))=\mathcal{L}\left(\varphi_{i}\right)(y) .
$$

But in fact, we know that in the above, the left hand side supremum is reached at

$$
x=(\nabla \varphi)^{-1}\left(y+t_{i}(y) e_{i}\right)=\nabla \mathcal{L} \varphi\left(y+t_{i}(y) e_{i}\right) \in e_{i}^{\perp}
$$

hence the above inequality is an equality. But the right hand side supremum is reached at $x=$ $\left(\nabla \varphi_{i}\right)^{-1}(y)$. Since they are reached at the same point this implies that $(\nabla \varphi)^{-1}\left(y+t_{i}(y) e_{i}\right)=$ $\left(\nabla \varphi_{i}\right)^{-1}(y)$, and $y+t_{i}(y) e_{i}=\nabla \varphi \circ\left(\nabla \varphi_{i}\right)^{-1}(y)$, thus $t_{i}(y)=\left\langle\nabla \varphi \circ\left(\nabla \varphi_{i}\right)^{-1}(y), e_{i}\right\rangle$, where $\left(\nabla \varphi_{i}\right)^{-1}$ is a bijection from $e_{i}^{\perp}$ to $\operatorname{dom}\left(\mathcal{L} \varphi_{i}\right)=\operatorname{dom} P_{i}(\mathcal{L} \varphi)$.

## 3 General results on the functional volume product

Let $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be an invertible linear map. Then putting $z=T x$ we get for every $y \in \mathbb{R}^{n}$

$$
\mathcal{L}(\varphi \circ T)(y)=\sup _{x}(\langle x, y\rangle-\varphi(T x))=\sup _{z}\left(\left\langle T^{-1} z, y\right\rangle-\varphi(z)\right)=(\mathcal{L} \varphi)\left(\left(T^{-1}\right)^{*}(y)\right) .
$$

Therefore $\mathcal{L}(\varphi \circ T)=(\mathcal{L} \varphi) \circ\left(T^{-1}\right)^{*}$. Hence changing variables, we get $P(\varphi \circ T)=P(\varphi)$. The functional $P$ admits another invariance: for any $c \in \mathbb{R}$ one has $\mathcal{L}(\varphi+c)(y)=\mathcal{L} \varphi(y)-c$. Thus

$$
P(\varphi+c)=\int e^{-(\varphi+c)} \int e^{-\mathcal{L} \varphi+c}=P(\varphi) .
$$

Hence one may assume in the following that $\varphi(0)=0$. Since we are dealing with even functions one has also $\varphi(0)=\min \varphi$ and $\mathcal{L} \varphi(0)=-\inf \varphi=-\varphi(0)$ thus if $\varphi(0)=0$ then $\mathcal{L} \varphi(0)=0$. On the opposite, when $\varphi$ is replaced by $t \varphi$, for $t>0$, the functional $P$ is not invariant. We shall take advantage of this. For every $y \in \mathbb{R}^{n}$ and $t>0$, one has

$$
\mathcal{L}(t \varphi)(y)=\sup _{x}(\langle x, y\rangle-t \varphi(x))=t \sup _{x}\left(\left\langle x, \frac{y}{t}\right\rangle-\varphi(x)\right)=t \mathcal{L} \varphi\left(\frac{y}{t}\right) .
$$

Hence, changing variables, we get

$$
P(t \varphi)=\int_{\mathbb{R}^{n}} e^{-t \varphi(x)} d x \int_{\mathbb{R}^{n}} e^{-\mathcal{L}(t \varphi)(y)} d y=t^{n} \int_{\mathbb{R}^{n}} e^{-t \varphi(x)} d x \int_{\mathbb{R}^{n}} e^{-t \mathcal{L} \varphi(y)} d y
$$

In the following we denote by $\mu_{\varphi}$ the measure on $\mathbb{R}^{n}$ with density $e^{-\varphi}$ with respect to the Lebesgue measure and for an oriented hypersurface $S$ of $\mathbb{R}^{n}$ whose normal is defined a.e., we denote

$$
V_{S}(\varphi)=\int_{S} n_{S}(y) e^{-\varphi(y)} d y \quad \text { and } \quad Q_{S}(\varphi)=\int_{S}\left\langle y, n_{S}(y)\right\rangle e^{-\varphi(y)} d y
$$

Notice that if $S$ is the boundary of a cone with apex at the origin then $Q_{S}(\varphi)=0$. The following proposition generalizes ideas from the proof of Theorem 10 in [FM2] and Proposition 1 in [FHMRZ].

Proposition 4. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be convex such that $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Let $A$ be a Borel subset of $\mathbb{R}^{n}$ such that $\mu_{\varphi}(A)>0$ and such that $\partial A$ is an hypersurface of $\mathbb{R}^{n}$ and the exterior normal $n_{A}$ is defined a.e. on $\partial A$. Then for any $x \in \mathbb{R}^{n}$ one has

$$
\begin{equation*}
\left\langle x,-\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}\right\rangle-\varphi(x) \leq n-\int_{A} \varphi(y) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}(A)}-\frac{Q_{\partial A}(\varphi)}{\mu_{\varphi}(A)}, \tag{1}
\end{equation*}
$$

i.e. $\mathcal{L} \varphi\left(-\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}\right) \leq n-\int_{A} \varphi(y) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}(A)}-\frac{Q_{\partial A}(\varphi)}{\mu_{\varphi}(A)}$. Moreover, if for some $x_{0} \in \mathbb{R}^{n}$ there is equality in (1) then $x_{0} \in \operatorname{dom}(\varphi)$ and $\varphi$ is affine on $\left[x_{0}, z\right]$ for every $z \in A \cap \operatorname{dom}(\varphi)$.

Proof. By convexity, the function $\varphi$ is differentiable almost everywhere on $\operatorname{dom}(\varphi)$ and one has for almost all $y \in \operatorname{dom}(\varphi)$ and for all $x \in \mathbb{R}^{n}$

$$
\langle x, \nabla \varphi(y)\rangle-\varphi(x) \leq\langle y, \nabla \varphi(y)\rangle-\varphi(y)
$$

We multiply by $e^{-\varphi(y)}$, integrate in $y$ on $A$ and divide by $\mu_{\varphi}(A)$ to get

$$
\begin{equation*}
\left\langle x, \int_{A} \nabla \varphi(y) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}(A)}\right\rangle-\varphi(x) \leq \int_{A}(\langle y, \nabla \varphi(y)\rangle-\varphi(y)) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}(A)} . \tag{2}
\end{equation*}
$$

Recall the following consequence of Stokes formula, known as Green's identities : for any sufficiently smooth $f, g: A \rightarrow \mathbb{R}$ one has

$$
\int_{A}(f \Delta g+\langle\nabla f, \nabla g\rangle)=\int_{\partial A} f\left\langle\nabla g, n_{A}\right\rangle
$$

where the integrals are taken with respect to the Hausdorff measure. Applying this formula to $f(y)=e^{-\varphi(y)}$ and $g(y)=\langle x, y\rangle$, where $x$ is a fixed vector gives

$$
\begin{equation*}
\int_{A} \nabla \varphi d \mu_{\varphi}=-\int_{\partial A} n_{A}(y) e^{-\varphi(y)} d y=-V_{\partial A}(\varphi) . \tag{3}
\end{equation*}
$$

Applying it to $f(y)=e^{-\varphi(y)}$ and $g(y)=\frac{|y|^{2}}{2}$ gives

$$
\begin{equation*}
\int_{A}\langle y, \nabla \varphi(y)\rangle d \mu_{\varphi}(y)=n \mu_{\varphi}(A)-Q_{\partial A}(\varphi) . \tag{4}
\end{equation*}
$$

Replacing these values in the inequality (2) we conclude that inequality (1) is verified. Moreover, if for some $x_{0} \in \mathbb{R}^{n}$ there is equality in (1) then for almost all $y \in \operatorname{dom}(\varphi)$ one has $\varphi(y)+\left\langle x_{0}-\right.$ $y, \nabla \varphi(y)\rangle=\varphi\left(x_{0}\right)$. We conclude using Lemma 3 in [FGMR].
Corollary 5. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be convex such that $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Let $A, B$ be Borel subsets of $\mathbb{R}^{n}$ such that $\mu_{\varphi}(A), \mu_{\mathcal{L} \varphi}(B)>0$ and such that $\partial A$ and $\partial B$ are hypersurfaces of $\mathbb{R}^{n}$ and the exterior normal $n_{A}$ is defined a.e. on $\partial A$ and the exterior normal $n_{B}$ is defined a.e. on $\partial B$. Then

$$
\begin{equation*}
\left\langle\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}, \frac{V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)}\right\rangle \leq 2 n-\int_{A} \varphi(y) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}(A)}-\int_{B} \mathcal{L} \varphi(y) \frac{d \mu_{\mathcal{L} \varphi}(y)}{\mu_{\mathcal{L} \varphi}(B)}-\frac{Q_{\partial A}(\varphi)}{\mu_{\varphi}(A)}-\frac{Q_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)} . \tag{5}
\end{equation*}
$$

Moreover, if there is equality in (5) then $\varphi$ is affine on $\left[-\frac{V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)}, a\right]$ for every $a \in A \cap \operatorname{dom}(\varphi)$ and $\mathcal{L} \varphi$ is affine on $\left[-\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}, b\right]$ for every $b \in B \cap \operatorname{dom}(\mathcal{L} \varphi)$.

Proof. Since we are working with integrals, we may assume that $\varphi$ is lower semi-continuous. We apply the inequality $\langle x, y\rangle \leq \varphi(x)+\mathcal{L} \varphi(y)$ to $x=-\frac{V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)}$ and $y=-\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}$, apply Proposition 4 twice to $\varphi$ and $\mathcal{L} \varphi$ and use that $\mathcal{L}(\mathcal{L} \varphi)=\varphi$ to deduce that

$$
\begin{aligned}
\left\langle\frac{-V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}, \frac{-V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)}\right\rangle & \leq \mathcal{L} \varphi\left(-\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}\right)+\varphi\left(-\frac{V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)}\right) \\
& \leq 2 n-\int_{A} \varphi(y) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}(A)}-\int_{B} \mathcal{L} \varphi(y) \frac{d \mu_{\mathcal{L} \varphi}(y)}{\mu_{\mathcal{L} \varphi}(B)}-\frac{Q_{\partial A}(\varphi)}{\mu_{\varphi}(A)}-\frac{Q_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)} .
\end{aligned}
$$

Moreover if there is equality in (5) then there is equality in (1) for $x=\frac{-V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}(B)}$ hence from the equality case of Proposition 4 we deduce that $\varphi$ is affine on $\left[-\frac{V_{\partial B}(\mathcal{L} \varphi)}{\mu_{\mathcal{L}(B)}}, a\right]$ for every $a \in A \cap \operatorname{dom}(\varphi)$. The same argument gives that $\mathcal{L} \varphi$ is affine on $\left[-\frac{V_{\partial A}(\varphi)}{\mu_{\varphi}(A)}, b\right]$ for every $b \in B \cap \operatorname{dom}(\mathcal{L} \varphi)$.

Notice that, changing variables, for $t>0$, one has

$$
\mu_{t \varphi}(t A)=\int_{t A} e^{-t \varphi(x)} d x=t^{n} \int_{A} e^{-t \varphi(t z)} d z .
$$

Using again that $\mathcal{L}(t \varphi)(y)=t \mathcal{L} \varphi\left(\frac{y}{t}\right)$ we have also

$$
\mu_{\mathcal{L}(t \varphi)}(t B)=\int_{t B} e^{-\mathcal{L}(t \varphi)(y)} d y=t^{n} \int_{B} e^{-t \mathcal{L} \varphi(z)} d z
$$

We define

$$
F_{A, B}(t)=\mu_{t \varphi}(t A) \mu_{\mathcal{L}(t \varphi)}(t B) .
$$

In the next lemma, we compute the derivatives of $\mu_{t \varphi}(t A), \mu_{\mathcal{L}(t \varphi)}(t B)$ and $F_{A, B}(t)$.
Lemma 6. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be convex such that $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Let $A, B$ be Borel subsets of $\mathbb{R}^{n}$ such that $\mu_{\varphi}(A), \mu_{\mathcal{L} \varphi}(B)>0$ and such that $\partial A$ and $\partial B$ are hypersurfaces of $\mathbb{R}^{n}$ and the exterior normal $n_{A}$ is defined a.e. on $\partial A$. Then

1. $\left(\mu_{t \varphi}(t A)\right)^{\prime}=-\int_{t A} \varphi d \mu_{t \varphi}+\frac{1}{t} Q_{t \partial A}(t \varphi)$.
2. $\left(\mu_{\mathcal{L}(t \varphi)}(t B)\right)^{\prime}=\frac{n}{t} \mu_{\mathcal{L}(t \varphi)}(t B)-\frac{1}{t} \int_{t B} \mathcal{L}(t \varphi) e^{-\mathcal{L}(t \varphi)}$.
3. $\frac{F_{A, B}^{\prime}(t)}{F_{A, B}(t)}=\frac{n}{t}-\int_{t A} \varphi \frac{d \mu_{t \varphi}}{\mu_{t \varphi}(t A)}-\frac{1}{t} \int_{t B} \mathcal{L}(t \varphi) \frac{d \mu_{\mathcal{L}(t \varphi)}}{\mu_{\mathcal{L}(t \varphi)}(t B)}+\frac{1}{t} \frac{Q_{t \partial A}(t \varphi)}{\mu_{t \varphi}(t A)}$.
4. In particular, if $A$ is a cone with apex at the origin then

$$
F_{A, B}^{\prime}(1)=n F_{A, B}(1)-\int_{A} \varphi e^{-\varphi} \int_{B} e^{-\mathcal{L} \varphi}-\int_{A} e^{-\varphi} \int_{B} \mathcal{L} \varphi e^{-\mathcal{L} \varphi} .
$$

Proof. 1. We compute the derivative of $\mu_{t \varphi}(t A)$, change variables and apply Green's identity (4) to $t \varphi$ and $t A$, this gives

$$
\begin{aligned}
\left(\mu_{t \varphi}(t A)\right)^{\prime} & =n t^{n-1} \int_{A} e^{-t \varphi(t z)} d z-t^{n} \int_{A}(\varphi(t z)+t\langle\nabla \varphi(t z), z\rangle) e^{-t \varphi(t z)} d z \\
& =\frac{n}{t} \mu_{t \varphi}(t A)-\int_{t A} \varphi d \mu_{t \varphi}-\int_{t A}\langle\nabla \varphi(x), x\rangle d \mu_{t \varphi}(x) \\
& =-\int_{t A} \varphi d \mu_{t \varphi}+\frac{1}{t} Q_{t \partial A}(t \varphi) .
\end{aligned}
$$

2., 3. and 4. The computation of the derivatives of $\mu_{\mathcal{L}(t \varphi)}(t B)$ and $F_{A, B}(t)$ are direct.

Corollary 7. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be convex such that $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Let $A, B$ be Borel subsets of $\mathbb{R}^{n}$ such that $\mu_{\varphi}(A), \mu_{\mathcal{L} \varphi}(B)>0$ and such that $\partial A$ and $\partial B$ are hypersurfaces of $\mathbb{R}^{n}$ and the exterior normals $n_{A}$ and $n_{B}$ are defined a.e. on $\partial A$ and $\partial B$. Then

$$
\begin{equation*}
\frac{d}{d t}\left(t^{n} F_{A, B}\right) \geq t^{n-1}\left(\left\langle V_{t \partial A}(t \varphi), V_{t \partial B}(\mathcal{L}(t \varphi))\right\rangle+2 Q_{t \partial A}(t \varphi) \mu_{\mathcal{L}(t \varphi)}(t B)+Q_{t \partial B}(\mathcal{L}(t \varphi)) \mu_{t \varphi}(t A)\right) \tag{6}
\end{equation*}
$$

Moreover, if there is equality in (6) then $\varphi$ is affine on $\left[-\frac{V_{\partial B}(t \mathcal{L} \varphi)}{\mu_{t} \mathcal{L}(B)}\right.$,a] for every $a \in(t A) \cap \operatorname{dom}(\varphi)$ and $\mathcal{L} \varphi$ is affine on $\left[-\frac{V_{\partial t A}(t \varphi)}{t \mu_{t \varphi}(t A)}, b\right]$ for every $b \in B \cap \operatorname{dom}(\mathcal{L} \varphi)$.

Proof. Applying inequality (5) of Corollary 5 to $t A, t B$ and $t \varphi$, and using 3) of Lemma 6 we get

$$
\left\langle\frac{V_{t \partial A}(t \varphi)}{\mu_{t \varphi}(t A)}, \frac{V_{t \partial B}(\mathcal{L}(t \varphi))}{\mu_{\mathcal{L}(t \varphi)}(t B)}\right\rangle \leq n+\frac{t F_{A, B}^{\prime}(t)}{F_{A, B}(t)}-2 \frac{Q_{t \partial A}(t \varphi)}{\mu_{t \varphi}(t A)}-\frac{Q_{t \partial B}(\mathcal{L}(t \varphi))}{\mu_{\mathcal{L}(t \varphi)}(t B)}
$$

We multiply by $t^{n-1} F_{A, B}(t)$ and get inequality (6).
Lemma 8. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be convex such that $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Let $\varepsilon \in\{-1,1\}^{n}$. Then

1) $V_{\partial \mathbb{R}_{\varepsilon}^{n}}(\varphi)=-\sum_{i=1}^{n} \varepsilon_{i} e_{i} \int_{\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{+}} e^{-\varphi_{i}}$.
2) If moreover $\varphi$ is differentiable, strictly convex and the normal of $\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)$ is chosen exterior to $\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)$ then

$$
\left\langle V_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)}(\mathcal{L} \varphi), e_{i}\right\rangle=-\varepsilon_{i} \int_{\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}} e^{-\mathcal{L}\left(\varphi_{i}\right)}
$$

Proof. 1) It follows directly from the definition.
2) We assume that $\mathbb{R}_{\varepsilon}^{n}=\mathbb{R}_{+}^{n}$, the general case being the same. From the definition of $V$ one has

$$
\left\langle V_{\nabla \varphi\left(\mathbb{R}_{+}^{n} \cap e_{i}^{\perp}\right)}(\mathcal{L} \varphi), e_{i}\right\rangle=\int_{\nabla \varphi\left(\mathbb{R}_{+}^{n} \cap e_{i}^{\perp}\right)}\left\langle n_{\nabla \varphi\left(\mathbb{R}_{+}^{n}\right)}(y), e_{i}\right\rangle e^{-\mathcal{L} \varphi(y)} d y
$$

Using the parametrization $S_{i}:=\nabla \varphi\left(\mathbb{R}_{+}^{n} \cap e_{i}^{\perp}\right)=\left\{y+t_{i}(y) e_{i} ; y \in \mathbb{R}_{+}^{n} \cap e_{i}^{\perp}\right\}$ obtained in Lemma 3 the surface $S_{i}$ is the graph of the smooth function $t_{i}: \mathbb{R}_{+}^{n} \cap e_{i}^{\perp} \rightarrow \mathbb{R}$. Hence the surface element of $S_{i}$ is $\sqrt{1+\left|\nabla t_{i}(y)\right|^{2}} d y$ and so for any smooth function $g: S_{i} \rightarrow \mathbb{R}$ one has

$$
\int_{S_{i}} g(x) d x=\int_{\mathbb{R}_{+}^{n} \cap e_{i}^{+}} g\left(y+t_{i}(y) e_{i}\right) \sqrt{1+\left|\nabla t_{i}(y)\right|^{2}} d y .
$$

We apply this equality to $g(y)=\left\langle n_{S_{i}}(y), e_{i}\right\rangle e^{-\mathcal{L} \varphi(y)}$ and use that $n_{S_{i}}(y)=\frac{\nabla t_{i}(y)-e_{i}}{\sqrt{1+\left|\nabla t_{i}(y)\right|^{2}}}$ to deduce that $\left\langle n_{S_{i}}(y), e_{i}\right\rangle=\frac{-1}{\sqrt{1+\left|\nabla t_{i}(y)\right|^{2}}}$ and thus

$$
\int_{S_{i}}\left\langle n_{\nabla \varphi\left(\mathbb{R}_{+}^{n}\right)}(y), e_{i}\right\rangle e^{-\mathcal{L} \varphi(y)} d y=-\int_{\mathbb{R}_{+}^{n} \cap e_{i}^{\perp}} e^{-\mathcal{L} \varphi\left(y+t_{i}(y) e_{i}\right)} d y .
$$

From Lemma 3 we have $\mathcal{L} \varphi\left(y+t_{i}(y) e_{i}\right)=\mathcal{L}\left(\varphi_{i}\right)(y)$ for all $y \in e_{i}^{\perp}$ and thus we conclude.

For every convex function $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ and every $m \in \mathbb{N}^{*}$ we define the function $\varphi_{m}$ by

$$
\varphi_{m}(x)=\frac{|x|^{2}}{2 m}+\inf _{z}\left(\varphi(z)+\frac{m}{2}|x-z|^{2}\right) .
$$

Notice that $\varphi_{m}=\frac{|\cdot|^{2}}{2 m}+\varphi \square \frac{m|\cdot|^{2}}{2}$ thus $\mathcal{L} \varphi_{m}(y)=\inf _{z}\left(\mathcal{L} \varphi(z)+\frac{|z|^{2}}{2 m}+\frac{m}{2}|z-y|^{2}\right)$. We shall need the following approximation lemma.

Lemma 9. Let $n, m \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be even convex such that $0<\int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x<+\infty$. Then

1. $\operatorname{dom}\left(\varphi_{m}\right)=\operatorname{dom}\left(\mathcal{L} \varphi_{m}\right)=\mathbb{R}^{n}, \varphi_{m}$ and $\mathcal{L} \varphi_{m}$ are differentiable and strictly convex on $\mathbb{R}^{n}$ and $\nabla \varphi_{m}$ is bijective on $\mathbb{R}^{n}$.
2. When $m \rightarrow+\infty$ one has $\varphi_{m}(x) \rightarrow \varphi(x)$ and $\mathcal{L} \varphi_{m}(x) \rightarrow \mathcal{L} \varphi(x)$ a.e.
3. When $m \rightarrow+\infty$, for every measurable set $A$ one has $\int_{A} e^{-\varphi_{m}} \rightarrow \int_{A} e^{-\varphi}, \int_{A} \varphi_{m} e^{-\varphi_{m}} \rightarrow$ $\int_{A} \varphi e^{-\varphi}$ and moreover for every $t>0$

$$
\begin{equation*}
P\left(t \varphi_{m}\right) \rightarrow P(t \varphi) \quad \text { and } \quad \frac{d}{d t}\left(P\left(t \varphi_{m}\right)\right) \rightarrow \frac{d}{d t}(P(t \varphi)) . \tag{7}
\end{equation*}
$$

Proof. 1. For every $x \in \mathbb{R}^{n}$ one has $\varphi_{m}(x) \leq \frac{\mid x x^{2}}{2 m}+\varphi(0)+\frac{m}{2}|x|^{2}<+\infty$, hence $\operatorname{dom}\left(\varphi_{m}\right)=\mathbb{R}^{n}$. In the same way $\mathcal{L} \varphi_{m}(y) \leq \mathcal{L} \varphi(0)+\frac{m}{2}|y|^{2}<+\infty$, hence $\operatorname{dom}\left(\mathcal{L} \varphi_{m}\right)=\mathbb{R}^{n}$. Moreover it is clear that $\varphi_{m}$ is strictly convex and, using $[\mathrm{R}]$ Theorem 26.3 , it is not difficult to see that $\varphi_{m}$ is differentiable. It follows that the same holds for $\mathcal{L} \varphi_{m}$. The fact that $\nabla \varphi_{m}$ is bijective on $\mathbb{R}^{n}$ deduces from Lemma 3. 2. These convergences are classical. Let us prove for example the first one. On one hand one has $\varphi_{m}\left(x_{0}\right) \leq \frac{\left|x_{0}\right|^{2}}{2 m}+\varphi\left(x_{0}\right)$. On the other hand if $x_{0} \in \operatorname{dom}(\varphi)$ then there exists a hyperplane touching the epigraph of $\varphi$ at $\left(x_{0}, \varphi\left(x_{0}\right)\right)$ thus there exists $y \in \mathbb{R}^{n}$ such that $\varphi(x) \geq \varphi\left(x_{0}\right)+\left\langle x-x_{0}, y\right\rangle$, for all $x \in \mathbb{R}^{n}$. This implies that

$$
\varphi_{m}\left(x_{0}\right) \geq \inf _{x}\left(\varphi(x)+\frac{m}{2}\left|x-x_{0}\right|^{2}\right) \geq \inf _{x}\left(\varphi\left(x_{0}\right)+\left\langle x-x_{0}, y\right\rangle+\frac{m}{2}\left|x-x_{0}\right|^{2}\right)=\varphi\left(x_{0}\right)-\frac{|y|^{2}}{2 m} .
$$

Letting $m \rightarrow+\infty$ gives the convergence. If $x_{0} \notin \overline{\operatorname{dom}(\varphi)}$ then, using that $\varphi \geq \min \varphi+I_{\operatorname{dom}(\varphi)}$, we deduce that

$$
\varphi_{m}\left(x_{0}\right) \geq \min \varphi+\inf _{x \in \operatorname{dom}(\varphi)} \frac{m}{2}\left|x-x_{0}\right|^{2}=\min \varphi+\frac{m}{2} d\left(x_{0}, \operatorname{dom}(\varphi)\right)^{2} .
$$

Therefore $\varphi_{m}\left(x_{0}\right) \rightarrow \varphi\left(x_{0}\right)$ when $m \rightarrow+\infty$ for every $x_{0} \notin \partial(\operatorname{dom}(\varphi))$, that is a.e.
3. First notice that we may assume that $\varphi(0)=0$. From Lemma 2, it follows that there exists $a, b>0$ such that for every $x \in \mathbb{R}^{n}$ one has $a|x|-1 \leq \varphi(x) \leq I_{b B_{2}^{n}}(x)+1$. Hence we get

$$
\begin{equation*}
\varphi_{m}(x) \geq \inf _{z}\left(\varphi(z)+\frac{1}{2}|x-z|^{2}\right) \geq \inf _{z}\left(a|z|-1+\frac{1}{2}(|x|-|z|)^{2}\right)=a|x|-\frac{a^{2}}{2}-1 . \tag{8}
\end{equation*}
$$

Thus $e^{-\varphi_{m}(x)} \leq e^{\frac{a^{2}}{2}+1-a|x|}$ for all $m$, then from the dominated convergence theorem one deduces that $\int_{A} e^{-\varphi_{m}} \rightarrow \int_{A} e^{-\varphi}$ when $m \rightarrow+\infty$. In the same way, one has $b|y|-1 \leq \mathcal{L} \varphi(y) \leq I_{a B_{2}^{n}}(y)+1$ thus

$$
\mathcal{L} \varphi_{m}(y) \geq \inf _{z}\left(\mathcal{L} \varphi(z)+\frac{1}{2}|y-z|^{2}\right) \geq b|y|-\frac{b^{2}}{2}-1 .
$$

Hence from the dominated convergence theorem one deduces that $\int_{A} e^{-\mathcal{L} \varphi_{m}} \rightarrow \int_{A} e^{-\mathcal{L} \varphi}$ when $m \rightarrow$ $+\infty$. We conclude that $P\left(\varphi_{m}\right) \rightarrow P(\varphi)$ when $m \rightarrow+\infty$. Similarly we prove that for every $t>0$ one has $P\left(t \varphi_{m}\right) \rightarrow P(t \varphi)$ when $m \rightarrow+\infty$. Using that $u e^{-u} \leq \frac{2}{e} e^{-\frac{u}{2}}$ for every $u \in \mathbb{R}$ we get that $\varphi_{m} e^{-\varphi_{m}} \leq \frac{2}{e} e^{-\frac{\varphi_{m}}{2}}$ and we conclude again by the dominated convergence theorem. The same method gives the result for $t \varphi_{m}$ and $\mathcal{L}\left(t \varphi_{m}\right)$.

## 4 Proof in dimension 2

Theorem 10. Let $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R} \cup\{+\infty\}$ be even convex such that $0<\int_{\mathbb{R}^{2}} e^{-\varphi(x)} d x<+\infty$, then

$$
P(\varphi)=\int_{\mathbb{R}^{2}} e^{-\varphi(x)} d x \int_{\mathbb{R}^{2}} e^{-\mathcal{L} \varphi(y)} d y \geq 4^{2}=16,
$$

with equality if and only if there exists $a \in \mathbb{R}$ such that either $\varphi=I_{P}+a$, or $\varphi=\|\cdot\|_{P}+a$ with $P$ being a parallelogram centered at the origin or there exists a basis $\left(u_{1}, u_{2}\right)$ of $\mathbb{R}^{2}$ and $b, c>0$ such that $\varphi\left(x_{1} u_{1}+x_{2} u_{2}\right)=c\left|x_{1}\right|+I_{[-b, b]}\left(x_{2}\right)+a$ for every $x_{1}, x_{2} \in \mathbb{R}$.

### 4.1 The inequality in dimension 2

Proof of Theorem 10. Let $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R} \cup\{+\infty\}$ be even convex such that $0<\int_{\mathbb{R}^{2}} e^{-\varphi(x)} d x<+\infty$. First let us reduce to the case where $\varphi$ is strongly equipartioned, in the sense that

$$
\varphi(0)=0, \quad \int_{\mathbb{R}_{+}} e^{-\varphi\left(t e_{1}\right)} d t=\int_{\mathbb{R}_{+}} e^{-\varphi\left(t e_{2}\right)} d t=1, \quad \int_{\mathbb{R}_{+}^{2}} e^{-\varphi}=\frac{1}{4} \int_{\mathbb{R}^{2}} e^{-\varphi}, \quad \int_{\mathbb{R}_{+}^{2}} \varphi e^{-\varphi}=\frac{1}{4} \int_{\mathbb{R}^{2}} \varphi e^{-\varphi} .
$$

Since $P(\varphi)=P(\varphi-\varphi(0))$ we may assume that $\varphi(0)=0$. For any $u \in S^{1}$, let $C(u) \subset S^{1}$ be the open half-circle delimited by $u$ and $-u$ containing the vectors $v$ which are after $u$ with respect to the counterclockwise orientation of $S^{1}$. For $v \in C(u)$ let $C_{u, v}=\mathbb{R}_{+} u+\mathbb{R}_{+} v$ be the cone generated by $u$ and $v$ and define $f_{u}(v)=\mu_{\varphi}\left(C_{u, v}\right)$. The map $f_{u}$ is continuous and increasing on $C(u)$, $f_{u}(u)=0$ and $f_{u}(v) \longrightarrow \mu_{\varphi}\left(\mathbb{R}^{2}\right) / 2$ when $v \longrightarrow-u$, thus there exists a unique $v(u) \in C(u)$ such that $f_{u}(v(u))=\mu_{\varphi}\left(C_{u, v(u)}\right)=\mu_{\varphi}\left(\mathbb{R}^{2}\right) / 4$. Notice that $v: S^{1} \rightarrow S^{1}$ is continuous and, since $\varphi$ is even, one has $v(v(u))=-u$ for any $u \in S^{1}$. For $u \in S^{1}$, let $g(u)=\int_{C_{u, v(u)}} \varphi e^{-\varphi}-\frac{1}{4} \int_{\mathbb{R}^{2}} \varphi e^{-\varphi}$. Then $g$ is continuous on $S^{1}$ and, since $\varphi$ is even,

$$
g(u)+g(v(u))=\int_{C_{u, v(u)}} \varphi e^{-\varphi}+\int_{C_{v(u),-u}} \varphi e^{-\varphi}-\frac{1}{2} \int_{\mathbb{R}^{2}} \varphi e^{-\varphi}=0 .
$$

Hence $g(u)=-g(v(u))$. By the intermediate value theorem there exists $u \in S^{1}$ such that $g(u)=0$, thus

$$
\int_{C_{u, v(u)}} \varphi e^{-\varphi}=\frac{1}{4} \int_{\mathbb{R}^{2}} \varphi e^{-\varphi} \quad \text { and } \quad \mu_{\varphi}\left(C_{u, v(u)}\right)=\frac{1}{4} \mu_{\varphi}\left(\mathbb{R}^{2}\right) .
$$

Let $S$ be the linear map defined by $S\left(e_{1}\right)=u$ and $S\left(e_{2}\right)=v(u)$, then $S\left(\mathbb{R}_{+}^{2}\right)=C_{u, v(u)}$. Moreover, changing variables, for any Borel set $A$ in $\mathbb{R}^{2}$ we have $\mu_{\varphi \circ S}(A)=\mu_{\varphi}(S(A)) / \operatorname{det}(S)$ thus

$$
\mu_{\varphi \circ S}\left(\mathbb{R}_{+}^{2}\right)=\frac{\mu_{\varphi}\left(S\left(\mathbb{R}_{+}^{2}\right)\right)}{\operatorname{det}(S)}=\frac{\mu_{\varphi}\left(C_{u, v(u)}\right)}{\operatorname{det}(S)}=\frac{\mu_{\varphi}\left(\mathbb{R}^{2}\right)}{4 \operatorname{det}(S)}=\frac{\mu_{\varphi \circ S}\left(\mathbb{R}^{2}\right)}{4} .
$$

In the same way, one has

$$
\int_{\mathbb{R}_{+}^{2}}(\varphi \circ S) e^{-\varphi \circ S}=\frac{1}{4} \int_{\mathbb{R}^{2}}(\varphi \circ S) e^{-\varphi \circ S}
$$

Let $\alpha_{i}=\int_{0}^{+\infty} e^{-\varphi\left(r e_{i}\right)} d r$ and $\Delta$ be the linear map defined by $\Delta\left(e_{i}\right)=\alpha_{i} e_{i}$ and $T=S \circ \Delta$ then a change of variables shows that $\varphi \circ T$ is strongly equipartioned. Since $P(\varphi)=P(\varphi \circ T)$ we may assume that $\varphi$ is strongly equipartioned.

From Lemma 9 we can assume that $\operatorname{dom}(\varphi)=\operatorname{dom}(\mathcal{L} \varphi)=\mathbb{R}^{2}, \varphi$ is differentiable and strictly convex on $\mathbb{R}^{2}$. Then, up to sets of Lebesgue measure zero, we have the partition $\mathbb{R}^{2}=\cup_{\varepsilon \in\{-1,1\}^{2}} \nabla \varphi\left(\mathbb{R}_{\varepsilon}^{2}\right)$. Using the equipartition, we get

$$
P(\varphi)=\sum_{\varepsilon \in\{-1,1\}^{2}} \int_{\mathbb{R}^{2}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{2}\right)} e^{-\mathcal{L} \varphi}=4 \sum_{\varepsilon \in\{-1,1\}^{2}} \int_{\mathbb{R}_{\varepsilon}^{2}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{2}\right)} e^{-\mathcal{L} \varphi}
$$

Using the fact that $\varphi$ is even we get

$$
P(\varphi)=8 \int_{\mathbb{R}_{+}^{2}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} e^{-\mathcal{L} \varphi}+8 \int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)} e^{-\mathcal{L} \varphi}=8\left(F_{1}(1)+F_{2}(1)\right)
$$

where $F_{1}(t)=F_{\mathbb{R}_{+}^{2}, \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(t)$ and $F_{2}(t)=F_{\mathbb{R}_{+} \times \mathbb{R}_{-}, \nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}(t)$. Using 4) of Lemma 6 , we have

$$
\begin{gathered}
F_{1}^{\prime}(1)=2 F_{1}(1)-\int_{\mathbb{R}_{+}^{2}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} e^{-\mathcal{L} \varphi}-\int_{R_{+}^{2}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi} . \\
F_{2}^{\prime}(1)=2 F_{2}(1)-\int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)} e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi} .
\end{gathered}
$$

Then we get

$$
\begin{aligned}
\frac{d}{d t}\left(t^{2}\left(F_{1}(t)+F_{2}(t)\right)\right)_{\mid t=1} & =4\left(F_{1}(1)+F_{2}(1)\right)-\left(\int_{\mathbb{R}_{+}^{2}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} e^{-\mathcal{L} \varphi}+\int_{\mathbb{R}_{+}^{2}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}\right. \\
& \left.+\int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)} e^{-\mathcal{L} \varphi}+\int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}\right)
\end{aligned}
$$

Thus using the fact that $\varphi$ is equipartioned, we get that

$$
\begin{aligned}
\frac{d}{d t}\left(t^{2}\left(F_{1}(t)+F_{2}(t)\right)\right)_{\mid t=1} & =4\left(F_{1}(1)+F_{2}(1)\right)-\int_{\mathbb{R}_{+}^{2}} \varphi e^{-\varphi} \times \frac{1}{2} \int_{\mathbb{R}^{2}} e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}_{+}^{2}} e^{-\varphi} \times \frac{1}{2} \int_{\mathbb{R}^{2}} \mathcal{L} \varphi e^{-\mathcal{L} \varphi} \\
& =\frac{1}{2} P(\varphi)-\frac{1}{8}\left(\int_{\mathbb{R}^{2}} \varphi e^{-\varphi} \int_{\mathbb{R}^{2}} e^{-\mathcal{L} \varphi}+\int_{\mathbb{R}^{2}} e^{-\varphi} \int_{\mathbb{R}^{2}} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}\right) .
\end{aligned}
$$

On the other hand applying Lemma 6 one has

$$
\begin{aligned}
\frac{d}{d t}\left(t^{2} P(t \varphi)\right)_{\mid t=1} & =4 P(\varphi)-\left(\int_{\mathbb{R}^{2}} \varphi e^{-\varphi} \int_{\mathbb{R}^{2}} e^{-\mathcal{L} \varphi}+\int_{\mathbb{R}^{2}} e^{-\varphi} \int_{\mathbb{R}^{2}} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}\right) \\
& =8 \frac{d}{d t}\left(t^{2}\left(F_{1}(t)+F_{2}(t)\right)\right)_{\mid t=1}
\end{aligned}
$$

We apply Corollary 7 for $A=\mathbb{R}_{+}^{2}$ and $B=\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)$ and use the equipartition to get

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} F_{1}(t)\right)_{\mid t=1} \geq\left\langle V_{\partial \mathbb{R}_{+}^{2}}(\varphi), V_{\partial \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(\mathcal{L} \varphi)\right\rangle+\frac{\mu_{\varphi}\left(\mathbb{R}^{2}\right)}{4} Q_{\partial \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(\mathcal{L} \varphi) \tag{9}
\end{equation*}
$$

From Lemma 8 one has

$$
\begin{equation*}
V_{\partial \mathbb{R}_{+}^{2}}(\varphi)=-e_{1} \int_{0}^{+\infty} e^{-\varphi_{1}}-e_{2} \int_{0}^{+\infty} e^{-\varphi_{2}}=-\left(e_{1}+e_{2}\right) . \tag{10}
\end{equation*}
$$

Similarly one has $V_{\partial \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(\mathcal{L} \varphi)=-W_{1}-W_{2}$, where

$$
W_{1}=-\int_{\nabla \varphi\left(\{0\} \times \mathbb{R}_{+}\right)} n_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} e^{-\mathcal{L} \varphi} \quad \text { and } \quad W_{2}=-\int_{\nabla \varphi\left(\mathbb{R}_{+} \times\{0\}\right)} n_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)} e^{-\mathcal{L} \varphi} .
$$

Thus the equation (9) becomes

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} F_{1}(t)\right)_{\mid t=1} \geq\left\langle e_{1}+e_{2}, W_{1}+W_{2}\right\rangle+\frac{\mu_{\varphi}\left(\mathbb{R}^{2}\right)}{4} Q_{\partial \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(\mathcal{L} \varphi) \tag{11}
\end{equation*}
$$

Moreover, since $\varphi$ is even, one has

$$
V_{\partial\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}(\varphi)=-e_{1}+e_{2}, \quad V_{\partial \nabla^{\prime}\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}(\mathcal{L} \varphi)=-W_{1}+W_{2}
$$

and

$$
Q_{\partial \nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}(\mathcal{L} \varphi)=\int_{\partial \nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}\left\langle y, n_{\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}(y)\right\rangle e^{-\mathcal{L} \varphi(y)} d y=-Q_{\partial \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(\mathcal{L} \varphi)
$$

Applying Corollary 7 for $A=\mathbb{R}_{+} \times \mathbb{R}_{-}$and $B=\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)$and using the equipartition we get

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} F_{2}(t)\right)_{\mid t=1} \geq\left\langle e_{1}-e_{2}, W_{1}-W_{2}\right\rangle-\frac{\mu_{\varphi}\left(\mathbb{R}^{2}\right)}{4} Q_{\partial \nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(\mathcal{L} \varphi) . \tag{12}
\end{equation*}
$$

Adding (11) and (12) we obtain

$$
\frac{d}{d t}\left(t^{2} P(t \varphi)\right)_{\mid t=1}=8 \frac{d}{d t}\left(t^{2}\left(F_{1}(t)+F_{2}(t)\right)\right)_{\mid t=1} \geq 16\left(\left\langle e_{1}, W_{1}\right\rangle+\left\langle e_{2}, W_{2}\right\rangle\right) .
$$

Moreover from Lemma 8 for $i=1,2$ one has

$$
\left\langle e_{i}, W_{i}\right\rangle=\int_{\nabla \varphi\left(\mathbb{R}_{+}^{2} \cap e_{i}^{+}\right)}\left\langle-e_{i}, n_{\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)}(x)\right\rangle e^{-\mathcal{L} \varphi(x)} d x=\int_{0}^{+\infty} e^{-\mathcal{L}\left(\varphi_{i}\right)(y)} d y \geq 1
$$

where the last inequality comes from the result in dimension 1 proved in [FM2, FM3]. Thus we get

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} P(t \varphi)\right)_{\mid t=1} \geq 16\left(\int_{0}^{+\infty} e^{-\mathcal{L}\left(\varphi_{1}\right)}+\int_{0}^{+\infty} e^{-\mathcal{L}\left(\varphi_{2}\right)}\right) \geq 32 \tag{13}
\end{equation*}
$$

Applying this relation for $\varphi$ replaced by $s \varphi$ and using the fact that

$$
\frac{d}{d t}\left(t^{2} P(t s \varphi)\right)_{\mid t=1}=\lim _{t \rightarrow 1} \frac{t^{2} P(t s \varphi)-P(s \varphi)}{t-1}=\lim _{u \rightarrow s} \frac{\frac{u^{2}}{s^{2}} P(u \varphi)-P(s \varphi)}{\frac{u}{s}-1}=\frac{1}{s} \frac{d}{d u}\left(u^{2} P(u \varphi)\right)_{\mid u=s}
$$

one gets, $\forall t>0, \frac{d}{d t}\left(t^{2} P(t \varphi)\right) \geq 32 t$. Integrating this inequality we conclude that for every $0<\varepsilon<t$

$$
\begin{equation*}
t^{2} P(t \varphi)=\varepsilon^{2} P(\varepsilon \varphi)+\int_{\varepsilon}^{t} \frac{d}{d s}\left(s^{2} P(s \varphi)\right) d s \geq 32 \int_{\varepsilon}^{t} s d s=16\left(t^{2}-\varepsilon^{2}\right) \tag{14}
\end{equation*}
$$

Letting $\varepsilon$ tends to 0 we conclude that $t^{2} P(t \varphi) \geq 16 t^{2}$ and thus $P(\varphi) \geq 16$.

### 4.2 The equality case in dimension 2

Now we establish the equality case. Notice first that the inequalities (13) were so far established only for $\varphi$ being differentiable, strictly convex on $\mathbb{R}^{2}$ and strongly equipartioned. Let us prove that (13) still hold without these regularity assumption. We adapt the arguments of [FHMRZ] to the functional case. This requires to develop new functional inequalities. We first prove that the set of convex functions which are equipartioned has some compactness property.
Lemma 11. Let $\varphi: \mathbb{R}_{+} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a convex, non-decreasing function such that $\varphi(0)=0$ and $\int_{0}^{+\infty} e^{-\varphi(x)} d x=1$. Then for every $x \in \mathbb{R}_{+}$

$$
x-1 \leq \varphi(x) \leq I_{[0,1]}(x)+x \leq I_{[0,1]}(x)+1 .
$$

Proof. For $t \in[0,1]$ we define $\psi(t)=\sup \{x \geq 0 ; \varphi(x) \leq t\}$. One has $\varphi(\psi(t)) \leq t$ for almost all $t \geq 0$. From Jensen inequality we get

$$
\varphi(1)=\varphi\left(\int_{0}^{+\infty} e^{-\varphi(x)} d x\right)=\varphi\left(\int_{0}^{+\infty} \psi(t) e^{-t} d t\right) \leq \int_{0}^{+\infty} \varphi(\psi(t)) e^{-t} d t \leq \int_{0}^{+\infty} t e^{-t} d t=1
$$

By convexity we deduce that $\varphi(x)=\varphi((1-x) \cdot 0+x \cdot 1) \leq x$ for $x \in[0,1]$ and thus $\varphi(x) \leq$ $I_{[0,1]}(x)+x \leq I_{[0,1]}(x)+1$, for every $x \geq 0$. For the proof of the lower bound, the idea is exactly the same as in the proof of Proposition 4. By convexity, the function $\varphi$ is differentiable almost everywhere on $\operatorname{dom}(\varphi)$ and one has for almost all $y \in \operatorname{dom}(\varphi)$ and for all $x \geq 0$

$$
\varphi(x) \geq \varphi(y)+(x-y) \varphi^{\prime}(y) .
$$

We multiply by $e^{-\varphi(y)}$ and integrate in $y$ on $[0,+\infty)$ to get that for every $x \geq 0$

$$
\varphi(x) \geq \int_{0}^{+\infty} \varphi(y) e^{-\varphi(y)} d y+\int_{0}^{+\infty}(x-y) \varphi^{\prime}(y) e^{-\varphi(y)} d y
$$

Using that $\varphi \geq 0$ and integrating by parts we deduce that for every $x \geq 0$

$$
\varphi(x) \geq \int_{0}^{+\infty}(x-y) \varphi^{\prime}(y) e^{-\varphi(y)} d y=x-1
$$

Now we prove the analogue lemma in dimension 2 .
Lemma 12. Let $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a convex, even and strongly equipartioned function such that $\varphi(0)=0$. Then for every $x \in \mathbb{R}^{2}$, one has

$$
\frac{\|x\|_{1}}{e+2}-2 \leq \varphi(x) \leq I_{B_{1}^{2}}(x)+1 \quad \text { thus } \quad \frac{2}{e} \leq \int_{\mathbb{R}^{2}} e^{-\varphi(x)} d x \leq(2 e(e+2))^{2}
$$

Proof. The bounds on the integral follows directly from the bounds on the function. Let us prove these bounds. From Lemma 11 applied to $t \mapsto \varphi\left(t e_{i}\right)$ we deduce that $\varphi\left(e_{i}\right) \leq 1$ for $1 \leq i \leq 2$. Since $\varphi$ is even and convex we deduce that $\varphi(x) \leq 1$ for every $x \in B_{1}^{2}=\operatorname{Conv}\left( \pm e_{1}, \pm e_{2}\right)$. This proves the upper bound.

To prove the lower bound define $c=(e+2)^{-1}<1$. Let assume by contradiction that there exists $a=\left(a_{1}, a_{2}\right) \in \mathbb{R}^{2}$ such that

$$
\varphi(a)<c\|a\|_{1}-2 .
$$

By symmetry we may assume that $a_{1} \geq a_{2} \geq 0$. Moreover if $a_{2}=0$, from Lemma 11 applied to $\varphi_{2}$ one has $\varphi(a)=\varphi_{2}\left(a_{1}\right) \geq\left|a_{1}\right|-1=\|a\|_{1}-1$, which is not possible since $c<1$. Thus one has $a_{2}>0$. For every $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}_{+} \times \mathbb{R}_{-}$by convexity, one has

$$
\varphi\left(\frac{a_{2}}{a_{2}-x_{2}} x+\frac{-x_{2}}{a_{2}-x_{2}} a\right) \leq \frac{a_{2}}{a_{2}-x_{2}} \varphi(x)+\frac{-x_{2}}{a_{2}-x_{2}} \varphi(a) .
$$

Since $a_{2} x-x_{2} a=\left(a_{2} x_{1}-x_{2} a_{1}\right) e_{1} \in \mathbb{R}_{+} e_{1}$, we may apply Lemma 11 and get

$$
\varphi\left(\frac{a_{2}}{a_{2}-x_{2}} x+\frac{-x_{2}}{a_{2}-x_{2}} a\right) \geq \frac{a_{2}}{a_{2}-x_{2}} x_{1}+\frac{-x_{2}}{a_{2}-x_{2}} a_{1}-1 .
$$

Thus we deduce that

$$
\varphi(x) \geq x_{1}+\frac{\left(-x_{2}\right)}{a_{2}}\left(-\varphi(a)-1+a_{1}\right)-1
$$

Since $a_{1} \geq a_{2}>0$, using the upper bound on $\varphi(a)$ we get

$$
-\varphi(a)-1+a_{1} \geq-c\left(a_{1}+a_{2}\right)+a_{1}+1 \geq(1-c) a_{1}-c a_{2} \geq(1-2 c) a_{2} .
$$

Therefore for every $x \in \mathbb{R}_{+} \times \mathbb{R}_{-}$we deduce that

$$
\varphi(x) \geq x_{1}+(1-2 c)\left(-x_{2}\right)-1
$$

Integrating on $\mathbb{R}_{+} \times \mathbb{R}_{-}$, and replacing $c$ by its value, we get

$$
\int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} e^{-\varphi(x)} d x \leq \frac{e}{1-2 c}=\frac{1}{c}
$$

Now we apply Proposition 4 to the cone $A=\mathbb{R}_{+}^{2}$. Recall that in this case the term $Q_{\partial A}(\varphi)$ vanishes. Thus we get

$$
\varphi(a) \geq-2+\left\langle a,-\frac{V_{\partial \mathbb{R}_{+}^{2}}(\varphi)}{\mu_{\varphi}\left(\mathbb{R}_{+}^{2}\right)}\right\rangle+\int_{\mathbb{R}_{+}^{2}} \varphi(y) \frac{d \mu_{\varphi}(y)}{\mu_{\varphi}\left(\mathbb{R}_{+}^{2}\right)}
$$

Using that $\varphi \geq 0$ and $V_{\partial \mathbb{R}_{+}^{2}}(\varphi)=-\left(e_{1}+e_{2}\right)$ that was established in equation (10) we get

$$
\varphi(a) \geq-2+\left\langle a, \frac{e_{1}+e_{2}}{\mu_{\varphi}\left(\mathbb{R}_{+}^{2}\right)}\right\rangle=-2+\frac{\|a\|_{1}}{\mu_{\varphi}\left(\mathbb{R}_{+}^{2}\right)} \geq-2+c\|a\|_{1}
$$

which is in contradiction with the definition of $c$.
Remark 13. The same argument applies inductively and shows that there exists a constant $c_{n}>0$ such that for any even convex function $\varphi$ on $\mathbb{R}^{n}$ such that all its restrictions to coordinate planes are equipartitioned and $\varphi(0)=0$ one has

$$
c_{n}\|x\|_{1}-n \leq \varphi(x) \leq I_{B_{1}^{n}}(x)+1 \quad \text { thus } \quad \frac{2^{n}}{e n!} \leq \int_{\mathbb{R}^{n}} e^{-\varphi(x)} d x \leq\left(\frac{2 e}{c_{n}}\right)^{n}
$$

Lemma 14. Let $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R} \cup\{+\infty\}$ be an even convex function such that $0<\int e^{-\varphi}<+\infty$ and $\varphi(0)=0$. Then there exists a sequence $\left(\psi_{k}\right)_{k}$ of differentiable strongly equipartioned even strictly convex functions with $\operatorname{dom}\left(\psi_{k}\right)=\mathbb{R}^{2}$ and there exists an invertible linear map $T$ such that
(i) for every $x \in \mathbb{R}^{2},\left(\psi_{k}(x)\right)_{k}$ converges to $\varphi \circ T(x)$ and $e^{-\psi_{k}(x)} \leq C e^{-d|x|}$, for some $C, d>0$
(ii) for every $x \in \mathbb{R}^{2},\left(\mathcal{L} \psi_{k}(x)\right)_{k}$ converges to $\mathcal{L}(\varphi \circ T)(x)$ and $e^{-\mathcal{L} \psi_{k}(x)} \leq C e^{-d|x|}$, for some $C, d>0$.

Proof. We define the set $K_{\varphi}=\left\{x \in \mathbb{R}^{2} ; \varphi(x) \leq 1\right\}$. From Lemma $2, K_{\varphi}$ is a symmetric convex body, and there exists $a, b>0$ such that $a|x|-1 \leq \varphi(x) \leq I_{b B_{2}^{2}}(x)+1$. We recall the definition of $\varphi_{m}$ used in Lemma 9: for every $x \in \mathbb{R}^{2}$

$$
\varphi_{m}(x)=\frac{|x|^{2}}{2 m}+\inf _{z}\left(\varphi(z)+\frac{m}{2}|x-z|^{2}\right)
$$

Using the lower bound obtained in (8) we have $\varphi_{m}(x) \geq a|x|-\frac{a^{2}}{2}-1$. Hence for every $m \in \mathbb{N}^{*}$,

$$
\left\{x ; \varphi_{m}(x) \leq 1\right\} \subset R B_{2}^{2}
$$

where $R=\frac{a}{2}+\frac{2}{a}$. There exists a sequence of invertible linear maps $T_{m}$ such that $\varphi_{m} \circ T_{m}$ is strongly equipartioned. From Lemma 12 one has $\varphi_{m}\left(T_{m}\left(e_{i}\right)\right) \leq 1$, for all $1 \leq i \leq 2$ thus

$$
T_{m}\left(B_{1}^{2}\right) \subset\left\{x ; \varphi_{m}(x) \leq 1\right\} \subset R B_{2}^{2}
$$

Thus the sequence $\left(T_{m}\right)_{m}$ is bounded in the normed spaces of linear maps and thus there exists a subsequence $\left(T_{m_{k}}\right)_{k}$ of linear maps that converges to some linear map $T$. Let us prove that $T$ is invertible. For every $m \in \mathbb{N}^{*}$, using Lemma 12 and denoting $c=(e+2)^{-1}$ one has $c\|x\|_{1}-2 \leq$ $\varphi_{m}\left(T_{m} x\right)$ for every $x$. Moreover since $\varphi_{m}(x) \leq \varphi(x)+\frac{|x|^{2}}{2}$ and $\varphi(x) \leq I_{b B_{2}^{2}}(x)+1$, it follows that

$$
\varphi_{m}(x) \leq I_{b B_{2}^{2}}(x)+1+\frac{|x|^{2}}{2} \leq I_{b B_{2}^{2}}(x)+1+\frac{b^{2}}{2}
$$

Thus for any $x \in \mathbb{R}^{2}$

$$
\frac{b c\|x\|_{1}}{\left|T_{m} x\right|} \leq \varphi_{m}\left(\frac{b T_{m} x}{\left|T_{m} x\right|}\right)+2 \leq 3+\frac{b^{2}}{2}
$$

This gives that for every $x \in \mathbb{R}^{2}$

$$
\left(\frac{3}{b}+\frac{b}{2}\right)\left|T_{m} x\right| \geq c\|x\|_{1}
$$

Hence $T$ satisfies the same bound and thus is invertible. Moreover since $\varphi_{m}\left(T_{m} x\right) \geq c\|x\|_{1}-2$, we conclude that the sequence $\psi_{k}=\varphi_{m_{k}} \circ T_{m_{k}}$ is a sequence of strongly equipartioned even differentiable strictly convex functions such that $\left(\psi_{k}(x)\right)_{k}$ converges to $\varphi \circ T(x)$ and $e^{-\psi_{k}(x)} \leq e^{2-c\|x\|_{1}}$. Thus $\varphi \circ T$ is strongly equipartioned. Moreover, from Lemma 12 one has $\varphi_{m}\left(T_{m} x\right) \leq I_{B_{1}^{2}}(x)+1$, hence

$$
\mathcal{L}\left(\varphi_{m} \circ T_{m}\right)(x) \geq \mathcal{L}\left(I_{B_{1}^{2}}+1\right)(x)=\|x\|_{\infty}-1
$$

Therefore $\mathcal{L} \psi_{k}=\mathcal{L}\left(\varphi_{m_{k}} \circ T_{m_{k}}\right)$ satisfies the same bound. From Lemma 9 one has $\mathcal{L} \varphi_{m}(x) \rightarrow \mathcal{L} \varphi(x)$, for every $x \in \mathbb{R}^{2}$, when $m \rightarrow+\infty$. Since $T_{m_{k}}$ converges to $T$ we conclude that for every $x \in \mathbb{R}^{2}$, $\mathcal{L} \psi_{k}(x)$ converges to $\mathcal{L}(\varphi \circ T)(x)$.

Proof of the equality case. Let $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R} \cup\{+\infty\}$ be an even convex function such that $0<\int_{\mathbb{R}^{2}} e^{-\varphi(x)} d x<+\infty$ and $P(\varphi)=16$. By Lemma 14 there exists a sequence $\left(\psi_{k}\right)_{k}$ of differentiable strongly equipartioned even strictly convex functions with $\operatorname{dom}\left(\psi_{k}\right)=\mathbb{R}^{2}$ and a bijective linear map $T$ such that $\varphi \circ T$ is strongly equipartioned. Since $P(\varphi \circ T)=P(\varphi)$ and our equality case is invariant by invertible linear maps, we replace $\varphi \circ T$ by $\varphi$ in the rest of the proof. We have thus established that $\varphi$ is the limit of a sequence $\left(\psi_{k}\right)_{k}$ of differentiable and strongly equipartioned strictly convex functions. Thus the inequalities (13) and (14) are valid for the functions $\psi_{k}$. By taking the limit
and using Lemma 9 we deduce that these inequalities are also valid for $\varphi$. Then applying the same reasoning as in inequality (14) for $t=1$ and using that $P(\varphi)=16$ and $P(\varepsilon \varphi) \geq 16$ we get

$$
16=P(\varphi)=\varepsilon^{2} P(\varepsilon \varphi)+\int_{\varepsilon}^{1} \frac{d}{d s}\left(s^{2} P(s \varphi)\right) d s \geq 16 \varepsilon^{2}+32 \int_{\varepsilon}^{1} s d s=16 \varepsilon^{2}+16\left(1-\varepsilon^{2}\right)=16
$$

Thus there is equality in the intermediate inequalities. Hence for every $0<\varepsilon \leq 1$ one has $P(\varepsilon \varphi)=$ 16. Thus for all $0<t \leq 1$ we have $\frac{d}{d t}\left(t^{2} P(t \varphi)\right)=\frac{d}{d t}\left(16 t^{2}\right)=32 t$. Hence there is equality in (13). This implies that

$$
\int_{0}^{+\infty} e^{-\mathcal{L}\left(\varphi_{1}\right)}=1 \quad \text { and } \quad \int_{0}^{+\infty} e^{-\mathcal{L}\left(\varphi_{2}\right)}=1 .
$$

From the equality case in dimension 1 and since $\varphi_{i}$ is equipartioned, we deduce that for $i=1,2$ either $\varphi_{i}(x)=I_{[-1,1]}(x)$ or $\varphi_{i}(x)=|x|$ for every $x \in \mathbb{R}$. Following the proof in [FGMR] we distinguish three cases.
A. If $\varphi_{2}(x)=\varphi(x, 0)=I_{[-1,1]}(x)$ and $\varphi_{1}(x)=\varphi(0, x)=|x|$ for every $x \in \mathbb{R}$. Then let us prove that for all $\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$ one has $\varphi\left(x_{1}, x_{2}\right)=I_{[-1,1]}\left(x_{1}\right)+\left|x_{2}\right|$. This deduces from the following more general lemma, which extends observations done in the unconditional case in [FGMR].
Lemma 15. Let $n \geq 1$ and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be an even convex function such that there exists two convex bodies $K_{1} \subset F_{1}$ and $K_{2} \subset F_{2}$, where $F_{1}$ and $F_{2}$ are two complementary linear subspaces in $\mathbb{R}^{n}$ such that $\varphi\left(x_{1}\right)=\left\|x_{1}\right\|_{K_{1}}$ for all $x_{1} \in F_{1}$ and $\varphi\left(x_{2}\right)=I_{K_{2}}\left(x_{2}\right)$ for all $x_{2} \in F_{2}$. Then $\varphi\left(x_{1}+x_{2}\right)=\left\|x_{1}\right\|_{K_{1}}+I_{K_{2}}\left(x_{2}\right)$, for all $x_{1} \in F_{1}$ and $x_{2} \in F_{2}$.

Proof. Let $x_{1} \in F_{1}$ and $x_{2} \in K_{2}$. From the convexity of $\varphi$ and using that

$$
x_{1}+x_{2}=\left(1-\left\|x_{2}\right\|_{K_{2}}\right) \frac{x_{1}}{1-\left\|x_{2}\right\|_{K_{2}}}+\left\|x_{2}\right\|_{K_{2}} \frac{x_{2}}{\left\|x_{2}\right\|_{K_{2}}},
$$

we deduce that

$$
\varphi\left(x_{1}+x_{2}\right) \leq\left(1-\left\|x_{2}\right\|_{K_{2}}\right) \varphi\left(\frac{x_{1}}{1-\left\|x_{2}\right\|_{K_{2}}}\right)+\left\|x_{2}\right\|_{K_{2}} \varphi\left(\frac{x_{2}}{\left\|x_{2}\right\|_{K_{2}}}\right)=\left\|x_{1}\right\|_{K_{1}} .
$$

On the other hand, using that $\frac{x_{1}}{2}=\frac{1}{2}\left(x_{1}+x_{2}\right)+\frac{1}{2}\left(-x_{2}\right)$ one gets

$$
\frac{\left\|x_{1}\right\|_{K_{1}}}{2}=\varphi\left(\frac{x_{1}}{2}\right) \leq \frac{1}{2} \varphi\left(x_{1}+x_{2}\right)+\frac{1}{2} \varphi\left(-x_{2}\right)=\frac{1}{2} \varphi\left(x_{1}+x_{2}\right) .
$$

We deduce that $\varphi\left(x_{1}+x_{2}\right)=\left\|x_{1}\right\|_{K_{1}}$.
Let $x_{1} \in F_{1}$ and $x_{2} \notin K_{2}$. Let $1<\mu<\left\|x_{2}\right\|_{K_{2}}$ and $\lambda=\mu /\left\|x_{2}\right\|_{K_{2}} \in(0,1)$. Then $\lambda x_{2} \notin K_{2}$ and

$$
\lambda x_{2}=\lambda\left(x_{1}+x_{2}\right)+(1-\lambda) \frac{-\lambda x_{1}}{1-\lambda} .
$$

Hence using the convexity of $\varphi$ we get

$$
+\infty=\varphi\left(\lambda x_{2}\right) \leq \lambda \varphi\left(x_{1}+x_{2}\right)+(1-\lambda) \varphi\left(\frac{-\lambda x_{1}}{1-\lambda}\right) .
$$

Since $(1-\lambda) \varphi\left(\frac{-\lambda x_{1}}{1-\lambda}\right)=\lambda\left\|x_{1}\right\|_{K_{1}}<+\infty$, we deduce that $\varphi\left(x_{1}+x_{2}\right)=+\infty=I_{K_{2}}\left(x_{2}\right)$. We conclude that $\varphi\left(x_{1}+x_{2}\right)=\left\|x_{1}\right\|_{K_{1}}+I_{K_{2}}\left(x_{2}\right)$, for all $x_{1} \in F_{1}$ and $x_{2} \in F_{2}$.
B. If $\varphi_{2}(s)=\varphi\left(s e_{1}\right)=I_{[-1,1]}(s)$ and $\varphi_{1}(s)=\varphi\left(s e_{2}\right)=I_{[-1,1]}(s)$ for every $s \in \mathbb{R}$. Let $U=$ $\{x ; \varphi(x)=0\}$ and $K=\operatorname{dom}(\varphi)$. From the hypothesis one has $\varphi\left( \pm e_{i}\right)=0$ thus $\pm e_{i} \in U$. Since $\min \varphi=0$, the convexity of $\varphi$ implies that $U$ is convex. Thus one has $B_{1}^{2} \subset U \subset K$. Since $\pm e_{i} \in \partial K$ for $i=1,2$ one deduces there exists $u_{i} \in \partial K^{*}$ such that $\left\langle e_{i}, u_{i}\right\rangle=1$ and one has $K \subset\left\{x ;\left|\left\langle x, u_{i}\right\rangle\right| \leq 1, i=1,2\right\}$. We distinguish two cases:

- if $u_{1}=u_{2}$ : since $\left\langle e_{i}, u_{i}\right\rangle=1$ one has $u_{1}=u_{2}=e_{1}+e_{2}$. Thus $K \subset\left\{x \in \mathbb{R}^{2} ;\left|x_{1}+x_{2}\right| \leq 1\right\}:=D$. Hence $\operatorname{Conv}\left(0, e_{1}, e_{2}\right) \subset U \cap \mathbb{R}_{+}^{2} \subset K \cap \mathbb{R}_{+}^{2} \subset D \cap \mathbb{R}_{+}^{2}=\operatorname{Conv}\left(0, e_{1}, e_{2}\right)$. Therefore $\varphi_{\mid \mathbb{R}_{+}^{2}}=I_{B_{1}^{2} \cap \mathbb{R}_{+}^{2}}$. Using the equipartition and the fact that $\varphi \leq I_{B_{1}^{2}}$ we conclude that $\varphi=I_{B_{1}^{2}}$.
- if $u_{1} \neq u_{2}$ : using that for every $x \in K$ one has $\left\langle u_{i}, x\right\rangle \leq 1$ and $\varphi(x) \geq 0$ then for every $s>0$ and for $i=1,2$ we get

$$
\mathcal{L} \varphi\left(s u_{i}\right)=\sup _{x}\left(\left\langle s u_{i}, x\right\rangle-\varphi(x)\right)=\sup _{x \in K}\left(s\left\langle u_{i}, x\right\rangle-\varphi(x)\right)=s
$$

with equality for $x=e_{i}$. Since $\varphi$ is even we deduce that for every $s \in \mathbb{R}$

$$
\mathcal{L} \varphi\left(s u_{i}\right)=|s| .
$$

Define $C_{+}=\mathbb{R}_{+} u_{1}+\mathbb{R}_{+} u_{2}$ and $C_{-}=\mathbb{R}_{+} u_{1}+\mathbb{R}_{-} u_{2}$. Since $e_{i} \in K$ and $u_{i} \in K^{*}$ one has $\left|\left\langle u_{1}, e_{2}\right\rangle\right| \leq 1$ and $\left|\left\langle u_{2}, e_{1}\right\rangle\right| \leq 1$. Denote by $v_{i}$ the unitary exterior normal of $C_{+}$to the line $\mathbb{R} u_{i}$. We have $V_{\partial C_{+}}(\mathcal{L} \varphi)=-V_{1}-V_{2}$ where

$$
V_{1}=-v_{2} \int_{\mathbb{R}_{+} u_{2}} e^{-\mathcal{L} \varphi}=-v_{2} \int_{0}^{+\infty} e^{-\mathcal{L} \varphi\left(s u_{2}\right)} d s\left|u_{2}\right|=-v_{2} \int_{0}^{+\infty} e^{-s} d s\left|u_{2}\right|=-v_{2}\left|u_{2}\right|
$$

and in the same way $V_{2}=-v_{1}\left|u_{1}\right|$. Hence $V_{\partial C_{+}}(\mathcal{L} \varphi)=v_{1}\left|u_{1}\right|+v_{2}\left|u_{2}\right|$. It is easy to see that $V_{\partial C_{+}}(\mathcal{L} \varphi) \in \mathbb{R}_{+}^{2}$. We also have $V_{\partial \mathbb{R}_{+}^{2}}(\varphi)=-\left(e_{1}+e_{2}\right)$. Using that $\left\langle e_{i}, u_{i}\right\rangle=1$ one has $\left\langle e_{1}, V_{1}\right\rangle=$ $-\left\langle e_{1}, v_{2}\right\rangle\left|u_{2}\right|=\left\langle e_{2}, u_{2}\right\rangle=1$. In the same way one also has $\left\langle e_{2}, V_{2}\right\rangle=1$. We reproduce the same argument as before with $\nabla \varphi\left(\mathbb{R}_{+}^{2}\right)$ replaced by $C_{+}$and $\nabla \varphi\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)$replaced by $C_{-}$. Some terms are simplified because $C_{+}$and $C_{-}$are cones. Using again that $\varphi$ is even we have

$$
P(\varphi)=8 \int_{\mathbb{R}_{+}^{2}} e^{-\varphi} \int_{C_{+}} e^{-\mathcal{L} \varphi}+8 \int_{\mathbb{R}_{+} \times \mathbb{R}_{-}} e^{-\varphi} \int_{C_{-}} e^{-\mathcal{L} \varphi}=8\left(F_{1}(1)+F_{2}(1)\right)
$$

where $F_{1}(t)=F_{\mathbb{R}_{+}^{2}, C_{+}}(t)$ and $F_{2}(t)=F_{\mathbb{R}_{+} \times \mathbb{R}_{-}, C_{-}}(t)$. We apply Corollary 7 for $A=\mathbb{R}_{+}^{2}$ and $B=C_{+}$ and use the equipartition to get

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} F_{1}(t)\right)_{\mid t=1} \geq\left\langle V_{\partial \mathbb{R}_{+}^{2}}(\varphi), V_{\partial C_{+}}(\mathcal{L} \varphi)\right\rangle=\left\langle e_{1}+e_{2}, V_{1}+V_{2}\right\rangle \tag{15}
\end{equation*}
$$

Applying Corollary 7 for $A=\mathbb{R}_{+} \times \mathbb{R}_{-}$and $B=C_{-}$we also get

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} F_{2}(t)\right)_{\mid t=1} \geq\left\langle V_{\partial\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right)}(\varphi), V_{\partial C_{-}}(\mathcal{L} \varphi)\right\rangle=\left\langle e_{1}-e_{2}, V_{1}-V_{2}\right\rangle \tag{16}
\end{equation*}
$$

Adding (15) and (16) we obtain

$$
32=\frac{d}{d t}\left(t^{2} P(t \varphi)\right)_{\mid t=1}=8 \frac{d}{d t}\left(t^{2} F_{1}(t)+t^{2} F_{2}(t)\right)_{\mid t=1} \geq 16\left(\left\langle e_{1}, V_{1}\right\rangle+\left\langle e_{2}, V_{2}\right\rangle\right)=32
$$

Hence we have equality in the inequalities (15) and (16). From the equality case of Corollary 7 with $A=\mathbb{R}_{+}^{2}$ and $B=C_{+}$we deduce that $\varphi$ is affine on $\left[-\frac{V_{\partial C_{+}}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)}, a\right]$ for every $a \in \mathbb{R}_{+}^{2} \cap K$. Moreover
from Proposition 4 one has $-\frac{V_{\partial C_{+}}(\mathcal{L} \varphi)}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)} \in K \cap \mathbb{R}_{+}^{2}$. Since $\varphi$ is affine on $\left[\frac{V_{1}+V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)}, 0\right]$ and vanishes on $B_{1}^{2}$ then $\frac{V_{1}+V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)} \in U$. In the same way we prove that $\frac{V_{1}-V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{-}\right)} \in U$. Hence

$$
\operatorname{Conv}\left(B_{1}^{2}, \pm \frac{V_{1}+V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)}, \pm \frac{V_{1}-V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{-}\right)}\right) \subset U .
$$

Then

$$
\begin{equation*}
\int_{\mathbb{R}_{+}^{2}} e^{-\varphi} \geq\left|U \cap \mathbb{R}_{+}^{2}\right| \geq\left|\operatorname{Conv}\left(0, e_{1}, e_{2}, \frac{V_{1}+V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)}\right)\right|=\frac{1}{2}\left\langle\frac{V_{1}+V_{2}}{\mu_{\mathcal{L} \varphi}\left(C_{+}\right)}, e_{1}+e_{2}\right\rangle . \tag{17}
\end{equation*}
$$

Thus we get $\mu_{\varphi}\left(\mathbb{R}_{+}^{2}\right) \mu_{\mathcal{L} \varphi}\left(C_{+}\right) \geq \frac{1}{2}\left\langle V_{1}+V_{2}, e_{1}+e_{2}\right\rangle$. Similarly we have $\mu_{\varphi}\left(\mathbb{R}_{+} \times \mathbb{R}_{-}\right) \mu_{\mathcal{L}_{\varphi}}\left(C_{-}\right) \geq$ $\frac{1}{2}\left\langle V_{1}-V_{2}, e_{1}-e_{2}\right\rangle$. Adding these two inequalities we obtain

$$
2=\mu_{\varphi}\left(\mathbb{R}_{+}^{2}\right)\left(\mu_{\mathcal{L} \varphi}\left(C_{+}\right)+\mu_{\mathcal{L} \varphi}\left(C_{-}\right)\right) \geq\left\langle V_{1}, e_{1}\right\rangle+\left\langle V_{2}, e_{2}\right\rangle=2,
$$

so we get equality in (17). Hence $\int_{R_{+}^{2}} e^{-\varphi}=\left|U \cap \mathbb{R}_{+}^{2}\right|$ and $\varphi=I_{U}=I_{K}$, then $P(\varphi)=16=2 P(K)$ therefore $P(K)=8$. Thus $K$ satisfies the equality case of Mahler inequality in dimension 2, which implies that $K$ is a symmetric parallelogram by $[\mathrm{Me}]$ and $[\mathrm{Re}]$.
C. If $\varphi_{2}(s)=\varphi\left(s e_{1}\right)=|s|$ and $\varphi_{1}(s)=\varphi\left(s e_{2}\right)=|s|$ for every $s \in \mathbb{R}$, then $\mathcal{L} \varphi_{1}=\mathcal{L} \varphi_{2}=I_{[-1,1]}$ and $\operatorname{dom}(\mathcal{L} \varphi)$ is bounded. Indeed, let's prove that $\varphi(x) \leq\|x\|_{1}$. For all $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}_{+}^{2}$

$$
\begin{aligned}
\varphi\left(x_{1}, x_{2}\right) & =\varphi\left(\frac{x_{1}}{x_{1}+x_{2}}\left(x_{1}+x_{2}\right) e_{1}+\frac{x_{2}}{x_{1}+x_{2}}\left(x_{1}+x_{2}\right) e_{2}\right) \\
& \leq \frac{x_{1}}{x_{1}+x_{2}} \varphi\left(\left(x_{1}+x_{2}\right) e_{1}\right)+\frac{x_{2}}{x_{1}+x_{2}} \varphi\left(\left(x_{1}+x_{2}\right) e_{2}\right)=x_{1}+x_{2}
\end{aligned}
$$

Applying this in the other quadrants, we get that $\varphi(x) \leq\|x\|_{1}$ for all $x \in \mathbb{R}^{2}$. Hence $\mathcal{L} \varphi(x) \geq$ $I_{B_{\infty}^{2}}(x)$ and $\operatorname{dom}(\mathcal{L} \varphi) \subset B_{\infty}^{2}$ is bounded. Thus there exists a linear invertible map $T$ such that $\psi=(\mathcal{L} \varphi) \circ T$ is strongly equipartioned and $P(\psi)=P(\mathcal{L} \varphi)=16$ then for all $i=1,2, P\left(\psi_{i}\right)=4$ and $\psi_{i}(x)=I_{[-1,1]}(x)$ or $|x|$. Since $\operatorname{dom}(\psi)$ is bounded, then $\psi_{i}=I_{[-1,1]}$. From case B one concludes that $\psi=I_{K}$ where $K$ is a symmetric parallelogram hence $\mathcal{L} \varphi=I_{L}$ where $L=T^{-1}(K)$ is a symmetric parallelogram.

## 5 The inequality in dimension $n$

Theorem 16. Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be even convex such that $\varphi$ and $\varphi_{i}$ are equipartioned for all $1 \leq i \leq n$ and $P\left(\varphi_{i}\right) \geq 4^{n-1}$ then $P(\varphi) \geq 4^{n}$.

Proof. We can assume that $\varphi(0)=0$. From Lemma 9 we reduce to the case where $\operatorname{dom}(\varphi)=$ $\operatorname{dom}(\mathcal{L} \varphi)=\mathbb{R}^{n}, \varphi$ is differentiable and strictly convex on $\mathbb{R}^{n}$ and $\mathbb{R}^{n}=\cup_{\varepsilon \in\{-1,1\}^{n}} \nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)$. Using the equipartition, we have

$$
P(\varphi)=\int_{\mathbb{R}^{n}} e^{-\varphi} \int_{\mathbb{R}^{n}} e^{-\mathcal{L} \varphi}=2^{n} \int_{\mathbb{R}_{\varepsilon}^{n}} e^{-\varphi} \int_{\mathbb{R}^{n}} e^{-\mathcal{L} \varphi}=2^{n} \sum_{\varepsilon \in\{-1,1\}^{n}} F_{\varepsilon}(1),
$$

where $F_{\varepsilon}(t)=F_{\mathbb{R}_{\varepsilon}^{n}, \nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)}(t)$ for every $\varepsilon \in\{-1,1\}^{n}$. Using 4) of Lemma 6 for $A=\mathbb{R}_{\varepsilon}^{n}$ and $B=\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)$ we get that for every $\varepsilon \in\{-1,1\}^{n}$ one has

$$
F_{\varepsilon}^{\prime}(1)=n F_{\varepsilon}(1)-\int_{\mathbb{R}_{\varepsilon}^{n}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}_{\varepsilon}^{n}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} e^{-\mathcal{L} \varphi}
$$

Thus, using the equipartition, we have

$$
\begin{aligned}
\frac{d}{d t}\left(t^{n} F_{\varepsilon}(t)\right)_{\mid t=1} & =n F_{\varepsilon}(1)+F_{\varepsilon}^{\prime}(1) \\
& =2 n F_{\varepsilon}(1)-\int_{\mathbb{R}_{\varepsilon}^{n}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}_{\varepsilon}^{n}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} e^{-\mathcal{L} \varphi} \\
& =2 n F_{\varepsilon}(1)-\frac{1}{2^{n}}\left(\int_{\mathbb{R}^{n}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}^{n}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} e^{-\mathcal{L} \varphi}\right) .
\end{aligned}
$$

Summing these terms and using again the equipartition, we get

$$
\begin{aligned}
\sum_{\varepsilon} \frac{d}{d t}\left(t^{n} F_{\varepsilon}(t)\right)_{\mid t=1} & =2 n \sum_{\varepsilon} F_{\varepsilon}(1)-\frac{1}{2^{n}} \sum_{\varepsilon}\left(\int_{\mathbb{R}^{n}} e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}^{n}} \varphi e^{-\varphi} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)} e^{-\mathcal{L} \varphi}\right) \\
& =2 n \frac{P(\varphi)}{2^{n}}-\frac{1}{2^{n}}\left(\int_{\mathbb{R}^{n}} e^{-\varphi} \int_{\mathbb{R}^{n}} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}^{n}} \varphi e^{-\varphi} \int_{\mathbb{R}^{n}} e^{-\mathcal{L} \varphi}\right) .
\end{aligned}
$$

Now applying Lemma 6 one has

$$
\begin{aligned}
\frac{d}{d t}\left(t^{n} P(t \varphi)\right)_{\mid t=1} & =n P(\varphi)+\left(n P(\varphi)-\int_{\mathbb{R}^{n}} e^{-\varphi} \int_{\mathbb{R}^{n}} \mathcal{L} \varphi e^{-\mathcal{L} \varphi}-\int_{\mathbb{R}^{n}} \varphi e^{-\varphi} \int_{\mathbb{R}^{n}} e^{-\mathcal{L} \varphi}\right) \\
& =2^{n} \sum_{\varepsilon} \frac{d}{d t}\left(t^{n} F_{\varepsilon}(t)\right)_{\mid t=1}
\end{aligned}
$$

We apply Corollary 7 for $A=\mathbb{R}_{\varepsilon}^{n}$ and $B=\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)$ and use the equipartition to get

$$
\frac{d}{d t}\left(t^{n} P(t \varphi)\right)_{\mid t=1} \geq 2^{n} \sum_{\varepsilon}\left(\left\langle V_{\partial \mathbb{R}_{\varepsilon}^{n}}(\varphi), V_{\partial \nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)}(\mathcal{L} \varphi)\right\rangle+\frac{\mu_{\varphi}\left(\mathbb{R}^{n}\right)}{2^{n}} Q_{\partial \nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)}(\mathcal{L} \varphi)\right)
$$

Notice that

$$
\sum_{\varepsilon} Q_{\partial \nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)}(\mathcal{L} \varphi)=\sum_{\varepsilon} \sum_{i=1}^{n} Q_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)}(\mathcal{L} \varphi)=\sum_{i=1}^{n} \sum_{\varepsilon} \int_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)}\left\langle y, n_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)}(y)\right\rangle e^{-\mathcal{L} \varphi(y)} d y,
$$

where in $\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)$, the normal is chosen exterior to $\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n}\right)$. Since in each hyperplane $e_{i}^{\perp}$, each cone $\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}$ appears twice with two opposite orientations thus the sum of these two terms vanishes. Hence the whole sum vanishes. Using that in each $e_{i}^{\perp}$ the function $\varphi_{i}$ is equipartitioned and Lemma 8 it follows that

$$
\begin{aligned}
\frac{d}{d t}\left(t^{n} P(t \varphi)\right)_{\mid t=1} & \geq 2^{n} \sum_{\varepsilon} \sum_{1 \leq i, j \leq n} \int_{\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{+}} e^{-\varphi_{i}}\left\langle-\varepsilon_{i} e_{i}, V_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{j}^{\perp}\right)}(\mathcal{L} \varphi)\right\rangle \\
& \geq 2 \sum_{1 \leq i, j \leq n} \sum_{\varepsilon} \int_{e_{i}^{+}} e^{-\varphi_{i}}\left\langle-\varepsilon_{i} e_{i}, V_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap \cap \frac{\perp}{j}\right)}(\mathcal{L} \varphi)\right\rangle .
\end{aligned}
$$

Noticing again that in each hyperplane $e_{j}^{\perp}$, each cone $\mathbb{R}_{\varepsilon}^{n} \cap e_{j}^{\perp}$ appears twice with two opposite orientations one has for every fixed $i \neq j$

$$
\sum_{\varepsilon} \varepsilon_{i} V_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{j}^{+}\right)}=0
$$

Thus

$$
\frac{d}{d t}\left(t^{n} P(t \varphi)\right)_{\mid t=1} \geq 2 \sum_{i=1}^{n} \int_{e_{i}^{\perp}} e^{-\varphi_{i}} \sum_{\varepsilon}\left\langle-\varepsilon_{i} e_{i}, V_{\nabla \varphi\left(\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}\right)}(\mathcal{L} \varphi)\right\rangle .
$$

Using Lemma 8 we get

$$
\frac{d}{d t}\left(t^{n} P(t \varphi)\right)_{\mid t=1} \geq 2 \sum_{i=1}^{n} \int_{e_{i}^{\perp}} e^{-\varphi_{i}} \sum_{\varepsilon} \int_{\mathbb{R}_{\varepsilon}^{n} \cap e_{i}^{\perp}} e^{-\mathcal{L}\left(\varphi_{i}\right)}=2 \sum_{i=1}^{n} \int_{e_{i}^{\perp}} e^{-\varphi_{i}}\left(2 \int_{e_{i}^{\perp}} e^{-\mathcal{L}\left(\varphi_{i}\right)}\right)=4 \sum_{i=1}^{n} P\left(\varphi_{i}\right) .
$$

Since $P\left(\varphi_{i}\right) \geq 4^{n-1}$ we get

$$
\frac{d}{d t}\left(t^{n} P(t \varphi)\right)_{\mid t=1} \geq 4^{n} n
$$

Applying this to $s \varphi$ we deduce that for all $t>0$

$$
\frac{d}{d t}\left(t^{n} P(t \varphi)\right) \geq 4^{n} n t^{n-1}
$$

Integrating this inequality we conclude that for every $0<\varepsilon<t$ one has

$$
t^{n} P(t \varphi)=\varepsilon^{n} P(\varepsilon \varphi)+\int_{\varepsilon}^{t} \frac{d}{d s}\left(s^{n} P(s \varphi)\right) d s \geq 4^{n} n \int_{\varepsilon}^{t} s^{n-1} d s=4^{n}\left(t^{n}-\varepsilon^{n}\right)
$$

Letting $\varepsilon$ tends to 0 we conclude that $t^{n} P(t \varphi) \geq 4^{n} t^{n}$ and thus $P(\varphi) \geq 4^{n}$.

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