The functional form of Mahler conjecture for even log-concave functions in dimension 2

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Abstract

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an even convex function and $\mathcal{L}\varphi$ be its Legendre transform. We prove the functional form of Mahler conjecture concerning the functional volume product $P(\varphi) = \int e^{-\varphi} \int e^{-\mathcal{L}\varphi}$ in dimension 2: we give the sharp lower bound of this quantity and characterize the equality case. The proof uses the computation of the derivative in t of $P(t\varphi)$ and ideas due to Meyer [M] for unconditional convex bodies, adapted to the functional case by Fradelizi-Meyer [FM2] and extended for symmetric convex bodies in dimension 3 by Iriyeh-Shibata [IS] (see also [FHMRZ]).

1 Introduction

In the theory of convex bodies, many geometric inequalities can be generalized to functional inequalities. This is the case of Prékopa-Leindler, which is the functional form of Brunn-Minkowski inequality. Let us mention also Blaschke-Santaló inequality [San], which states in the symmetric case that if K is a symmetric convex body in $\mathbb{R}^n$ (in this paper, K symmetric means $K = -K$) and

$$P(K) = |K||K^*|,$$

where $K^* = \{y \in \mathbb{R}^n; \langle y, x\rangle \leq 1, \text{ for all } x \in K\}$ is the polar body of K then

$$P(K) \leq P(B_2^n),$$

with equality if and only if K is an ellipsoid, where B_2^n is the Euclidean ball associated to the standard scalar product in $\mathbb{R}^n$ and $|B|$ stands for the Lebesgue measure of a Borel subset B of $\mathbb{R}^n$ ([P], see [MP] or also [MR] for a simple proof of both the inequality and the case of equality).

Mahler conjectured an inverse form of the Blaschke-Santaló inequality for symmetric convex bodies in $\mathbb{R}^n$ in [Mah1]. He asked if for every symmetric convex body K,

$$P(K) \geq P([-1,1]^n) = \frac{4^n}{n!}.$$

It was later conjectured that the equality case occurs if and only if K is a Hanner polytope (see [RZ] for the definition). The inequality was proved by Mahler for $n = 2$ [Mah1] (see also [Me] and [S] Section 10.7, for other proofs and the characterization of the equality case). This conjecture has been proved also in a number of particular cases, for zonoids by Reisner [Re] (see also [GMR]) and for unconditional convex bodies by Saint Raymond [SR] (see also [M]), for hyperplane sections of $B_p^n = \{x \in \mathbb{R}^n; \sum |x_i|^p \leq 1\}$ and Hanner polytopes by Karasev [K]. The 3-dimensional case of the conjecture was proved by Iriyeh and Shibata [IS] (see [FHMRZ] for a shorter proof).

Some functional versions of the previous inequalities were proposed with convex bodies replaced with log-concave functions, and polarity with Legendre transform. Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an even convex function. Then the Legendre transform $\mathcal{L}\varphi$ of φ is defined for $y \in \mathbb{R}^n$ by

$$\mathcal{L}\varphi(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(x)).$$

We define the functional volume product of an even convex function to be

$$P(\varphi) = \int e^{-\varphi(x)} dx \int e^{-\mathcal{L}\varphi(y)} dy.$$

The functional version of the Blaschke-Santaló inequality for even convex functions states that

$$P(\varphi) \leq P\left(\frac{|\cdot|^2}{2}\right) = (2\pi)^n,$$

where $|.|$ stands here for the Euclidean norm in $\mathbb{R}^n$, with equality if and only if φ is a positive quadratic form. This statement was proved by Ball [B] (see also Artstein-Klartag-Milman [AKM], Fradelizi-Meyer [FM1] and Lehec [L1, L2] for more general results). For $\varphi(x) = \|x\|_K^2/2$, one has $\mathcal{L}\varphi(y) = \|y\|_{K^*}^2/2$, thus it is not difficult to see that

$$P(\varphi) = \frac{(2\pi)^n}{|B_2^n|^2} P(K).$$

This shows that the functional form indeed implies the geometric form of the inequality. We deal in this article with a functional version of Mahler conjecture for even convex functions. The following conjecture was stated in [FM2]. We prove it for $n = 2$.

Conjecture 1. *Let $n \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an even convex function such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)} dx < +\infty$. Then*

$$\int e^{-\varphi} \int e^{-\mathcal{L}\varphi} \geq 4^n,$$

with equality if and only if there exists $c \in \mathbb{R}$ and two Hanner polytopes $K_1 \subset F_1$ and $K_2 \subset F_2$, where F_1 and F_2 are two complementary subspaces in $\mathbb{R}^n$, such that for all $(x_1, x_2) \in F_1 \times F_2$

$$\varphi(x_1 + x_2) = c + \|x_1\|_{K_1} + I_{K_2}(x_2),$$

where I_K is the function defined by $I_K(x) = 0$ if $x \in K$ and $I_K(x) = +\infty$ if $x \notin K$.

For unconditional functions (and in particular if $n = 1$) the inequality in conjecture 1 was proved in [FM2, FM3] and the equality case was proved in [FGMR]. For a symmetric convex body K of $\mathbb{R}^n$ and for any $y \in \mathbb{R}^n$ one has

$$\mathcal{L}I_K(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - I_K(x)) = \sup_{x \in K} \langle x, y \rangle = h_K(y) = \|y\|_{K^*},$$

where h_K denotes the support function of K. In addition, using Fubini, we have

$$\int_{\mathbb{R}^n} e^{-\|y\|_{K^*}} dy = \int_{\mathbb{R}^n} \int_{\|y\|_{K^*}}^{+\infty} e^{-t} dt dy = \int_0^{+\infty} \int_{\{\|y\|_{K^*} \leq t\}} e^{-t} dy dt = \int_0^{+\infty} |tK^*| e^{-t} dt = n! |K^*|.$$

Thus if $\varphi = I_K$ we get

$$P(\varphi) = \int_{\mathbb{R}^n} e^{-I_K} \int_{\mathbb{R}^n} e^{-\|y\|_{K^*}} = |K| \int_{\mathbb{R}^n} e^{-\|y\|_{K^*}} = n! |K| |K^*| = n! P(K).$$

Hence Conjecture 1 implies Mahler conjecture for symmetric convex bodies. Notice that recently Gozlan [G] established precise relationships between the functional form of Mahler conjecture and the deficit in the Gaussian log-Sobolev inequality, thus our results implies better bounds in dimension 2 for these deficits.
In the proof, as in [IS, FHMRZ], we use the notion of equipartition. Denote by $(e_1, ..., e_n)$ the canonical basis of $\mathbb{R}^n$. A function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is equipartioned if

$$\int_{\mathbb{R}^n_\varepsilon} \varphi e^{-\varphi} = \frac{1}{2^n} \int_{\mathbb{R}^n} \varphi e^{-\varphi} \quad \text{and} \quad \int_{\mathbb{R}^n_\varepsilon} e^{-\varphi} = \frac{1}{2^n} \int_{\mathbb{R}^n} e^{-\varphi},$$

where $\forall \varepsilon \in \{-1, 1\}^n$, $\mathbb{R}^n_\varepsilon = \{x \in \mathbb{R}^n; \varepsilon_i x_i \geq 0, \forall i \in \{1, ..., n\}\}$. We use that in dimension $n \leq 2$, for any even convex function φ there exists a "position" of φ which is equipartioned. Moreover we also prove that if one has an even convex function φ on $\mathbb{R}^n$ such that φ and $\varphi_i = \varphi_{|e_i^\perp}$ are equipartioned for all $1 \leq i \leq n$ and φ_i satisfy the inequality of the conjecture in dimension $n-1$ then φ satisfies the conjectured inequality in dimension n.

This paper is organized in the following way. In section 2, we present some general results on the Legendre transform. In section 3, we establish some properties of the functional volume product. In section 4, we apply the results of sections 2 and 3 to prove the inequality and the case of equality of the functional volume product of φ in dimension 2. Finally, in section 5 we prove the inequality in dimension n for "strongly" equipartioned convex functions.

2 General results on convex functions and Legendre transform

Let us recall some useful facts about convex functions and the Legendre duality that can be found in the part I and V of the book of Rockafellar [R]. Recall that if K is convex, closed and contains 0 then $(K^*)^* = K$. Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. We denote the domain of φ by

$$\mathrm{dom}(\varphi) = \{x \in \mathbb{R}^n; \varphi(x) < +\infty\}.$$

It is a convex set. If φ is moreover lower semi-continuous and $\mathrm{dom}(\varphi) \neq \emptyset$ then $\mathcal{L}\mathcal{L}\varphi = \varphi$. The following lemma recalls some standard facts that can be found for example in Lemma 4 of [G].

Lemma 2. *Let $n \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that $\min \varphi = \varphi(0) = 0$. Then the following are equivalent.*

1. *One has $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)} dx < +\infty$.*

2. *The set $K_\varphi := \{x \in \mathbb{R}^n, \varphi(x) \leq 1\}$ is convex bounded and contains 0 in its interior.*

3. *There exists $a, b > 0$ such that for every $x \in \mathbb{R}^n$ one has $a|x| - 1 \leq \varphi(x) \leq I_{bB_2^n}(x) + 1$.*

Notice that for every $x \in \mathbb{R}^n$ one has $a|x| - 1 \leq \varphi(x) \leq I_{bB_2^n}(x) + 1$ if and only if for every $y \in \mathbb{R}^n$ one has $b|y| - 1 \leq \mathcal{L}\varphi(y) \leq I_{aB_2^n}(y) + 1$. Thus $0 < \int_{\mathbb{R}^n} e^{-\varphi} < +\infty$ is equivalent to $0 < \int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi} < +\infty$.

We define the analogue of sections and projections of convex sets for convex functions. The section of φ by an affine subspace F is simply the restriction of φ to this subspace and is denoted by $\varphi_{|F}$. The projection $P_F\varphi : F \to \mathbb{R} \cup \{+\infty\}$ of φ onto a linear subspace F is defined for $x \in F$ by

$$P_F\varphi(x) = \inf_{z \in F^\perp} \varphi(x + z).$$

The term projection comes from the fact that if $\tilde{P}_F : \mathbb{R}^n \times \mathbb{R} \to F \times \mathbb{R}$ denotes the orthogonal projection on $F \times \mathbb{R}$ parallel to $F^\perp$ then $\tilde{P}_F(\mathrm{Epi}(\varphi)) = \mathrm{Epi}(P_F\varphi)$ where $\mathrm{Epi}(\varphi) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}; \varphi(x) \leq t\}$. The infimal convolution of two convex functions $\varphi,\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$\varphi \Box \psi(x) = \inf_{z\in\mathbb{R}^n} (\varphi(x-z) + \psi(z)).$$

The infimal convolution is a convex function and it interacts with the Legendre transform in the following way: $\mathcal{L}(\varphi\Box\psi) = \mathcal{L}\varphi + \mathcal{L}\psi$. One can also define the projection using infimal convolution by noticing that for any $x \in \mathbb{R}^n$

$$\varphi \Box I_{F^\perp}(x) = \inf_{z\in\mathbb{R}^n} (\varphi(x-z) + I_{F^\perp}(z)) = \inf_{z\in F^\perp} \varphi(x-z).$$

Hence for $x \in F$ one has $\varphi\Box I_{F^\perp}(x) = P_F\varphi(x)$. Thus the same nice duality relationship between sections and projections that holds for convex sets holds also for convex functions, for any $y \in F$

$$\begin{aligned} P_F(\mathcal{L}\varphi)(y) &= \mathcal{L}\varphi\Box I_{F^\perp}(y) = \mathcal{L}\varphi\Box\mathcal{L}I_F(y) = \mathcal{L}(\varphi + I_F)(y) = \sup_{x\in\mathbb{R}^n} (\langle x,y\rangle - \varphi(x) - I_F(x)) \\ &= \sup_{x\in F} (\langle x,y\rangle - \varphi(x)) = \mathcal{L}(\varphi_F)(y), \end{aligned}$$

where, by an abuse of notation, we have denoted in the same way by $\mathcal{L}$ the Legendre transform applied to a function defined on $\mathbb{R}^n$ or on a subspace F. In each situation the supremum in the Legendre transform should be understood as taken in the subspace where the function is defined.

For $1 \leq i \leq n$ we denote the restriction of φ to $e_i^\perp$ by $\varphi_i = \varphi_{|e_i^\perp}$ and we define the analogue of the projection onto $e_i^\perp$ to be the function $P_i\varphi : e_i^\perp \to \mathbb{R}\cup\{+\infty\}$ defined for $x \in e_i^\perp$ by

$$P_i\varphi(x) = P_{e_i^\perp}\varphi(x) = \inf_{t\in\mathbb{R}} \varphi(x + te_i).$$

From the preceding, for every $y \in e_i^\perp$ one has

$$\mathcal{L}\varphi_i(y) = \sup_{x\in e_i^\perp} (\langle x,y\rangle - \varphi(x)) = P_i\mathcal{L}\varphi(y) = \inf_{t\in\mathbb{R}} \mathcal{L}\varphi(y+te_i).$$

Lemma 3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be differentiable and strictly convex such that $0 < \int e^{-\varphi} < +\infty$, and for $1 \leq i \leq n$ let $\varphi_i = \varphi_{|e_i^\perp}$ then*

1. *The function $\nabla\varphi$ is a bijection from $\mathbb{R}^n$ to $\mathrm{dom}(\mathcal{L}\varphi)$.*
2. *One has $\nabla\varphi(e_i^\perp) = \{y + t_i(y)e_i; y \in \mathrm{dom}(\mathcal{L}\varphi_i)\}$ where $t_i(y) = \langle\nabla\varphi\circ(\nabla\varphi_i)^{-1}(y), e_i\rangle$.*

Proof. Since φ is differentiable, we deduce from [R] Theorem 26.3 that $\mathcal{L}\varphi$ is strictly convex.
1. The fact that $\nabla\varphi$ is a bijection can be found in Corollary 26.3.1 in [R].
2. Since the supremum of $\mathcal{L}\varphi(y) = \sup(\langle x,y\rangle - \varphi(x))$ is reached at $x = (\nabla\varphi)^{-1}(y)$ one has

$$\mathcal{L}\varphi(\nabla\varphi(x)) = \langle x, \nabla\varphi(x)\rangle - \varphi(x)$$

and one can conclude from Corollary 23.5.1 of [R] that $(\nabla\varphi)^{-1} = \nabla(\mathcal{L}\varphi)$. Let now $y \in \mathrm{dom}(\mathcal{L}\varphi_i)$ be fixed and $g_y(t) = \mathcal{L}\varphi(y+te_i)$. The function g_y is strictly convex and tends to infinity at infinity so there exists a unique $t_i(y) \in \mathbb{R}$ at which the function g_y reaches its infimum and it satisfies $g_y'(t_i(y)) = 0$, *i.e.* $\langle\nabla\mathcal{L}\varphi(y+t_i(y)e_i), e_i\rangle = 0$ which means that $(\nabla\varphi)^{-1}(y+t_i(y)e_i) = \nabla\mathcal{L}\varphi(y+t_i(y)e_i) \in e_i^\perp$. This also means equivalently that $t_i(y)$ is the unique $t \in \mathbb{R}$ such that $y + te_i \in \nabla\varphi(e_i^\perp)$. Hence

$\nabla\varphi(e_i^\perp) = \{y + t_i(y)e_i; y \in \mathrm{dom}(\mathcal{L}\varphi_i)\}$ and the orthogonal projection P_i onto $e_i^\perp$ is a bijection from $\nabla\varphi(e_i^\perp)$ onto $e_i^\perp$. Moreover one has

$$P_i\mathcal{L}\varphi(y) = \inf_{t\in\mathbb{R}} \mathcal{L}\varphi(y + te_i) = \inf_{t\in\mathbb{R}} g_y(t) = g_y(t_i(y)) = \mathcal{L}\varphi(y + t_i(y)e_i).$$

Thus

$$P_i\mathcal{L}\varphi(y) = \sup_{x\in\mathbb{R}^n} (\langle x, y+t_i(y)e_i\rangle - \varphi(x)) \geq \sup_{x\in e_i^\perp} (\langle x, y+t_i(y)e_i\rangle - \varphi(x)) = \sup_{x\in e_i^\perp} (\langle x, y\rangle - \varphi(x)) = \mathcal{L}(\varphi_i)(y).$$

But in fact, we know that in the above, the left hand side supremum is reached at

$$x = (\nabla\varphi)^{-1}(y + t_i(y)e_i) = \nabla\mathcal{L}\varphi(y + t_i(y)e_i) \in e_i^\perp,$$

hence the above inequality is an equality. But the right hand side supremum is reached at $x = (\nabla\varphi_i)^{-1}(y)$. Since they are reached at the same point this implies that $(\nabla\varphi)^{-1}(y + t_i(y)e_i) = (\nabla\varphi_i)^{-1}(y)$, and $y + t_i(y)e_i = \nabla\varphi \circ (\nabla\varphi_i)^{-1}(y)$, thus $t_i(y) = \langle \nabla\varphi \circ (\nabla\varphi_i)^{-1}(y), e_i\rangle$, where $(\nabla\varphi_i)^{-1}$ is a bijection from $e_i^\perp$ to $\mathrm{dom}(\mathcal{L}\varphi_i) = \mathrm{dom}P_i(\mathcal{L}\varphi)$. □

3 General results on the functional volume product

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. Then putting $z = Tx$ we get for every $y \in \mathbb{R}^n$

$$\mathcal{L}(\varphi \circ T)(y) = \sup_x(\langle x, y\rangle - \varphi(Tx)) = \sup_z(\langle T^{-1}z, y\rangle - \varphi(z)) = (\mathcal{L}\varphi)\left((T^{-1})^*(y)\right).$$

Therefore $\mathcal{L}(\varphi \circ T) = (\mathcal{L}\varphi) \circ (T^{-1})^*$. Hence changing variables, we get $P(\varphi \circ T) = P(\varphi)$. The functional P admits another invariance: for any $c \in \mathbb{R}$ one has $\mathcal{L}(\varphi + c)(y) = \mathcal{L}\varphi(y) - c$. Thus

$$P(\varphi + c) = \int e^{-(\varphi+c)} \int e^{-\mathcal{L}\varphi+c} = P(\varphi).$$

Hence one may assume in the following that $\varphi(0) = 0$. Since we are dealing with even functions one has also $\varphi(0) = \min\varphi$ and $\mathcal{L}\varphi(0) = -\inf\varphi = -\varphi(0)$ thus if $\varphi(0) = 0$ then $\mathcal{L}\varphi(0) = 0$. On the opposite, when φ is replaced by $t\varphi$, for $t > 0$, the functional P is not invariant. We shall take advantage of this. For every $y \in \mathbb{R}^n$ and $t > 0$, one has

$$\mathcal{L}(t\varphi)(y) = \sup_x(\langle x, y\rangle - t\varphi(x)) = t\sup_x(\langle x, \frac{y}{t}\rangle - \varphi(x)) = t\mathcal{L}\varphi\left(\frac{y}{t}\right).$$

Hence, changing variables, we get

$$P(t\varphi) = \int_{\mathbb{R}^n} e^{-t\varphi(x)}dx \int_{\mathbb{R}^n} e^{-\mathcal{L}(t\varphi)(y)}dy = t^n \int_{\mathbb{R}^n} e^{-t\varphi(x)}dx \int_{\mathbb{R}^n} e^{-t\mathcal{L}\varphi(y)}dy.$$

In the following we denote by μ_φ the measure on $\mathbb{R}^n$ with density $e^{-\varphi}$ with respect to the Lebesgue measure and for an oriented hypersurface S of $\mathbb{R}^n$ whose normal is defined a.e., we denote

$$V_S(\varphi) = \int_S n_S(y)e^{-\varphi(y)}dy \quad \text{and} \quad Q_S(\varphi) = \int_S \langle y, n_S(y)\rangle e^{-\varphi(y)}dy.$$

Notice that if S is the boundary of a cone with apex at the origin then $Q_S(\varphi) = 0$. The following proposition generalizes ideas from the proof of Theorem 10 in [FM2] and Proposition 1 in [FHMRZ].

Proposition 4. *Let $n \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)} dx < +\infty$. Let A be a Borel subset of $\mathbb{R}^n$ such that $\mu_\varphi(A) > 0$ and such that ∂A is an hypersurface of $\mathbb{R}^n$ and the exterior normal n_A is defined a.e. on ∂A. Then for any $x \in \mathbb{R}^n$ one has*

$$\langle x, -\frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}\rangle - \varphi(x) \leq n - \int_A \varphi(y) \frac{d\mu_\varphi(y)}{\mu_\varphi(A)} - \frac{Q_{\partial A}(\varphi)}{\mu_\varphi(A)}, \tag{1}$$

i.e. $\mathcal{L}\varphi\left(-\frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}\right) \leq n - \int_A \varphi(y) \frac{d\mu_\varphi(y)}{\mu_\varphi(A)} - \frac{Q_{\partial A}(\varphi)}{\mu_\varphi(A)}$. *Moreover, if for some $x_0 \in \mathbb{R}^n$ there is equality in (1) then $x_0 \in \mathrm{dom}(\varphi)$ and φ is affine on $[x_0, z]$ for every $z \in A \cap \mathrm{dom}(\varphi)$.*

Proof. By convexity, the function φ is differentiable almost everywhere on $\mathrm{dom}(\varphi)$ and one has for almost all $y \in \mathrm{dom}(\varphi)$ and for all $x \in \mathbb{R}^n$

$$\langle x, \nabla\varphi(y)\rangle - \varphi(x) \leq \langle y, \nabla\varphi(y)\rangle - \varphi(y).$$

We multiply by $e^{-\varphi(y)}$, integrate in y on A and divide by $\mu_\varphi(A)$ to get

$$\langle x, \int_A \nabla\varphi(y) \frac{d\mu_\varphi(y)}{\mu_\varphi(A)}\rangle - \varphi(x) \leq \int_A \left(\langle y, \nabla\varphi(y)\rangle - \varphi(y)\right) \frac{d\mu_\varphi(y)}{\mu_\varphi(A)}. \tag{2}$$

Recall the following consequence of Stokes formula, known as Green's identities : for any sufficiently smooth $f, g : A \to \mathbb{R}$ one has

$$\int_A (f\Delta g + \langle \nabla f, \nabla g\rangle) = \int_{\partial A} f \langle \nabla g, n_A\rangle,$$

where the integrals are taken with respect to the Hausdorff measure. Applying this formula to $f(y) = e^{-\varphi(y)}$ and $g(y) = \langle x, y\rangle$, where x is a fixed vector gives

$$\int_A \nabla\varphi d\mu_\varphi = -\int_{\partial A} n_A(y) e^{-\varphi(y)} dy = -V_{\partial A}(\varphi). \tag{3}$$

Applying it to $f(y) = e^{-\varphi(y)}$ and $g(y) = \frac{|y|^2}{2}$ gives

$$\int_A \langle y, \nabla\varphi(y)\rangle d\mu_\varphi(y) = n\mu_\varphi(A) - Q_{\partial A}(\varphi). \tag{4}$$

Replacing these values in the inequality (2) we conclude that inequality (1) is verified. Moreover, if for some $x_0 \in \mathbb{R}^n$ there is equality in (1) then for almost all $y \in \mathrm{dom}(\varphi)$ one has $\varphi(y) + \langle x_0 - y, \nabla\varphi(y)\rangle = \varphi(x_0)$. We conclude using Lemma 3 in [FGMR]. □

Corollary 5. *Let $n \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)} dx < +\infty$. Let A, B be Borel subsets of $\mathbb{R}^n$ such that $\mu_\varphi(A), \mu_{\mathcal{L}\varphi}(B) > 0$ and such that ∂A and ∂B are hypersurfaces of $\mathbb{R}^n$ and the exterior normal n_A is defined a.e. on ∂A and the exterior normal n_B is defined a.e. on ∂B. Then*

$$\langle \frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}, \frac{V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}\rangle \leq 2n - \int_A \varphi(y) \frac{d\mu_\varphi(y)}{\mu_\varphi(A)} - \int_B \mathcal{L}\varphi(y) \frac{d\mu_{\mathcal{L}\varphi}(y)}{\mu_{\mathcal{L}\varphi}(B)} - \frac{Q_{\partial A}(\varphi)}{\mu_\varphi(A)} - \frac{Q_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}. \tag{5}$$

Moreover, if there is equality in (5) then φ is affine on $[-\frac{V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}, a]$ for every $a \in A \cap \mathrm{dom}(\varphi)$ and $\mathcal{L}\varphi$ is affine on $[-\frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}, b]$ for every $b \in B \cap \mathrm{dom}(\mathcal{L}\varphi)$.

Proof. Since we are working with integrals, we may assume that φ is lower semi-continuous. We apply the inequality $\langle x, y\rangle \le \varphi(x) + \mathcal{L}\varphi(y)$ to $x = -\frac{V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}$ and $y = -\frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}$, apply Proposition 4 twice to φ and $\mathcal{L}\varphi$ and use that $\mathcal{L}(\mathcal{L}\varphi) = \varphi$ to deduce that

$$\begin{aligned}\langle\frac{-V_{\partial A}(\varphi)}{\mu_\varphi(A)}, \frac{-V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}\rangle &\le \mathcal{L}\varphi\left(-\frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}\right) + \varphi\left(-\frac{V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}\right)\\ &\le 2n - \int_A \varphi(y)\frac{d\mu_\varphi(y)}{\mu_\varphi(A)} - \int_B \mathcal{L}\varphi(y)\frac{d\mu_{\mathcal{L}\varphi}(y)}{\mu_{\mathcal{L}\varphi}(B)} - \frac{Q_{\partial A}(\varphi)}{\mu_\varphi(A)} - \frac{Q_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}.\end{aligned}$$

Moreover if there is equality in (5) then there is equality in (1) for $x = \frac{-V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(B)}$ hence from the equality case of Proposition 4 we deduce that φ is affine on $[-\frac{V_{\partial B}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi(B)}}, a]$ for every $a \in A \cap \mathrm{dom}(\varphi)$. The same argument gives that $\mathcal{L}\varphi$ is affine on $[-\frac{V_{\partial A}(\varphi)}{\mu_\varphi(A)}, b]$ for every $b \in B \cap \mathrm{dom}(\mathcal{L}\varphi)$. □

Notice that, changing variables, for $t > 0$, one has

$$\mu_{t\varphi}(tA) = \int_{tA} e^{-t\varphi(x)}dx = t^n \int_A e^{-t\varphi(tz)}dz.$$

Using again that $\mathcal{L}(t\varphi)(y) = t\mathcal{L}\varphi(\frac{y}{t})$ we have also

$$\mu_{\mathcal{L}(t\varphi)}(tB) = \int_{tB} e^{-\mathcal{L}(t\varphi)(y)}dy = t^n \int_B e^{-t\mathcal{L}\varphi(z)}dz.$$

We define

$$F_{A,B}(t) = \mu_{t\varphi}(tA)\mu_{\mathcal{L}(t\varphi)}(tB).$$

In the next lemma, we compute the derivatives of $\mu_{t\varphi}(tA), \mu_{\mathcal{L}(t\varphi)}(tB)$ and $F_{A,B}(t)$.

Lemma 6. *Let $n \ge 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)}dx < +\infty$. Let A, B be Borel subsets of $\mathbb{R}^n$ such that $\mu_\varphi(A), \mu_{\mathcal{L}\varphi}(B) > 0$ and such that ∂A and ∂B are hypersurfaces of $\mathbb{R}^n$ and the exterior normal n_A is defined a.e. on ∂A. Then*

1. $(\mu_{t\varphi}(tA))' = -\int_{tA} \varphi d\mu_{t\varphi} + \frac{1}{t}Q_{t\partial A}(t\varphi).$
2. $(\mu_{\mathcal{L}(t\varphi)}(tB))' = \frac{n}{t}\mu_{\mathcal{L}(t\varphi)}(tB) - \frac{1}{t}\int_{tB} \mathcal{L}(t\varphi)e^{-\mathcal{L}(t\varphi)}.$
3. $\frac{F'_{A,B}(t)}{F_{A,B}(t)} = \frac{n}{t} - \int_{tA} \varphi\frac{d\mu_{t\varphi}}{\mu_{t\varphi}(tA)} - \frac{1}{t}\int_{tB} \mathcal{L}(t\varphi)\frac{d\mu_{\mathcal{L}(t\varphi)}}{\mu_{\mathcal{L}(t\varphi)}(tB)} + \frac{1}{t}\frac{Q_{t\partial A}(t\varphi)}{\mu_{t\varphi}(tA)}.$
4. *In particular, if A is a cone with apex at the origin then*
$$F'_{A,B}(1) = nF_{A,B}(1) - \int_A \varphi e^{-\varphi}\int_B e^{-\mathcal{L}\varphi} - \int_A e^{-\varphi}\int_B \mathcal{L}\varphi e^{-\mathcal{L}\varphi}.$$

Proof. 1. We compute the derivative of $\mu_{t\varphi}(tA)$, change variables and apply Green's identity (4) to $t\varphi$ and tA, this gives

$$\begin{aligned}(\mu_{t\varphi}(tA))' &= nt^{n-1}\int_A e^{-t\varphi(tz)}dz - t^n\int_A (\varphi(tz) + t\langle\nabla\varphi(tz), z\rangle)e^{-t\varphi(tz)}dz\\ &= \frac{n}{t}\mu_{t\varphi}(tA) - \int_{tA}\varphi d\mu_{t\varphi} - \int_{tA}\langle\nabla\varphi(x), x\rangle d\mu_{t\varphi}(x)\\ &= -\int_{tA}\varphi d\mu_{t\varphi} + \frac{1}{t}Q_{t\partial A}(t\varphi).\end{aligned}$$

2., 3. and 4. The computation of the derivatives of $\mu_{\mathcal{L}(t\varphi)}(tB)$ and $F_{A,B}(t)$ are direct. □

Corollary 7. *Let $n \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)}dx < +\infty$. Let A, B be Borel subsets of $\mathbb{R}^n$ such that $\mu_\varphi(A), \mu_{\mathcal{L}\varphi}(B) > 0$ and such that ∂A and ∂B are hypersurfaces of $\mathbb{R}^n$ and the exterior normals n_A and n_B are defined a.e. on ∂A and ∂B. Then*

$$\frac{d}{dt}\left(t^n F_{A,B}\right) \geq t^{n-1}\left(\langle V_{t\partial A}(t\varphi), V_{t\partial B}(\mathcal{L}(t\varphi))\rangle + 2Q_{t\partial A}(t\varphi)\mu_{\mathcal{L}(t\varphi)}(tB) + Q_{t\partial B}(\mathcal{L}(t\varphi))\mu_{t\varphi}(tA)\right). \quad (6)$$

Moreover, if there is equality in (6) then φ is affine on $[-\frac{V_{\partial B}(t\mathcal{L}\varphi)}{\mu_{t\mathcal{L}\varphi}(B)}, a]$ for every $a \in (tA) \cap \mathrm{dom}(\varphi)$ and $\mathcal{L}\varphi$ is affine on $[-\frac{V_{\partial tA}(t\varphi)}{t\mu_{t\varphi}(tA)}, b]$ for every $b \in B \cap \mathrm{dom}(\mathcal{L}\varphi)$.

Proof. Applying inequality (5) of Corollary 5 to tA, tB and $t\varphi$, and using 3) of Lemma 6 we get

$$\langle \frac{V_{t\partial A}(t\varphi)}{\mu_{t\varphi}(tA)}, \frac{V_{t\partial B}(\mathcal{L}(t\varphi))}{\mu_{\mathcal{L}(t\varphi)}(tB)}\rangle \leq n + \frac{tF'_{A,B}(t)}{F_{A,B}(t)} - 2\frac{Q_{t\partial A}(t\varphi)}{\mu_{t\varphi}(tA)} - \frac{Q_{t\partial B}(\mathcal{L}(t\varphi))}{\mu_{\mathcal{L}(t\varphi)}(tB)}.$$

We multiply by $t^{n-1}F_{A,B}(t)$ and get inequality (6). □

Lemma 8. *Let $n \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)}dx < +\infty$. Let $\varepsilon \in \{-1, 1\}^n$. Then*
1) $V_{\partial \mathbb{R}^n_\varepsilon}(\varphi) = -\sum_{i=1}^n \varepsilon_i e_i \int_{\mathbb{R}^n_\varepsilon \cap e_i^\perp} e^{-\varphi_i}.$
2) If moreover φ is differentiable, strictly convex and the normal of $\nabla\varphi(\mathbb{R}^n_\varepsilon \cap e_i^\perp)$ is chosen exterior to $\nabla\varphi(\mathbb{R}^n_\varepsilon)$ then

$$\langle V_{\nabla\varphi(\mathbb{R}^n_\varepsilon \cap e_i^\perp)}(\mathcal{L}\varphi), e_i\rangle = -\varepsilon_i \int_{\mathbb{R}^n_\varepsilon \cap e_i^\perp} e^{-\mathcal{L}(\varphi_i)}.$$

Proof. 1) It follows directly from the definition.
2) We assume that $\mathbb{R}^n_\varepsilon = \mathbb{R}^n_+$, the general case being the same. From the definition of V one has

$$\langle V_{\nabla\varphi(\mathbb{R}^n_+ \cap e_i^\perp)}(\mathcal{L}\varphi), e_i\rangle = \int_{\nabla\varphi(\mathbb{R}^n_+ \cap e_i^\perp)} \langle n_{\nabla\varphi(\mathbb{R}^n_+)}(y), e_i\rangle e^{-\mathcal{L}\varphi(y)}dy.$$

Using the parametrization $S_i := \nabla\varphi(\mathbb{R}^n_+ \cap e_i^\perp) = \{y + t_i(y)e_i; y \in \mathbb{R}^n_+ \cap e_i^\perp\}$ obtained in Lemma 3 the surface S_i is the graph of the smooth function $t_i : \mathbb{R}^n_+ \cap e_i^\perp \to \mathbb{R}$. Hence the surface element of S_i is $\sqrt{1 + |\nabla t_i(y)|^2}dy$ and so for any smooth function $g : S_i \to \mathbb{R}$ one has

$$\int_{S_i} g(x)dx = \int_{\mathbb{R}^n_+ \cap e_i^\perp} g(y + t_i(y)e_i)\sqrt{1 + |\nabla t_i(y)|^2}dy.$$

We apply this equality to $g(y) = \langle n_{S_i}(y), e_i\rangle e^{-\mathcal{L}\varphi(y)}$ and use that $n_{S_i}(y) = \frac{\nabla t_i(y) - e_i}{\sqrt{1+|\nabla t_i(y)|^2}}$ to deduce that $\langle n_{S_i}(y), e_i\rangle = \frac{-1}{\sqrt{1+|\nabla t_i(y)|^2}}$ and thus

$$\int_{S_i} \langle n_{\nabla\varphi(\mathbb{R}^n_+)}(y), e_i\rangle e^{-\mathcal{L}\varphi(y)}dy = -\int_{\mathbb{R}^n_+ \cap e_i^\perp} e^{-\mathcal{L}\varphi(y + t_i(y)e_i)}dy.$$

From Lemma 3 we have $\mathcal{L}\varphi(y + t_i(y)e_i) = \mathcal{L}(\varphi_i)(y)$ for all $y \in e_i^\perp$ and thus we conclude. □

For every convex function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and every $m \in \mathbb{N}^*$ we define the function φ_m by

$$\varphi_m(x) = \frac{|x|^2}{2m} + \inf_z \left(\varphi(z) + \frac{m}{2}|x-z|^2 \right).$$

Notice that $\varphi_m = \frac{|\cdot|^2}{2m} + \varphi \square \frac{m|\cdot|^2}{2}$ thus $\mathcal{L}\varphi_m(y) = \inf_z \left(\mathcal{L}\varphi(z) + \frac{|z|^2}{2m} + \frac{m}{2}|z-y|^2 \right)$. We shall need the following approximation lemma.

Lemma 9. *Let $n, m \geq 1$ and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be even convex such that $0 < \int_{\mathbb{R}^n} e^{-\varphi(x)} dx < +\infty$. Then*

1. $\mathrm{dom}(\varphi_m) = \mathrm{dom}(\mathcal{L}\varphi_m) = \mathbb{R}^n$*, φ_m and $\mathcal{L}\varphi_m$ are differentiable and strictly convex on $\mathbb{R}^n$ and $\nabla\varphi_m$ is bijective on $\mathbb{R}^n$.*
2. *When $m \to +\infty$ one has $\varphi_m(x) \to \varphi(x)$ and $\mathcal{L}\varphi_m(x) \to \mathcal{L}\varphi(x)$ a.e.*
3. *When $m \to +\infty$, for every measurable set A one has $\int_A e^{-\varphi_m} \to \int_A e^{-\varphi}$, $\int_A \varphi_m e^{-\varphi_m} \to \int_A \varphi e^{-\varphi}$ and moreover for every $t > 0$*

$$P(t\varphi_m) \to P(t\varphi) \quad \text{and} \quad \frac{d}{dt}(P(t\varphi_m)) \to \frac{d}{dt}(P(t\varphi)). \tag{7}$$

Proof. 1. For every $x \in \mathbb{R}^n$ one has $\varphi_m(x) \leq \frac{|x|^2}{2m} + \varphi(0) + \frac{m}{2}|x|^2 < +\infty$, hence $\mathrm{dom}(\varphi_m) = \mathbb{R}^n$. In the same way $\mathcal{L}\varphi_m(y) \leq \mathcal{L}\varphi(0) + \frac{m}{2}|y|^2 < +\infty$, hence $\mathrm{dom}(\mathcal{L}\varphi_m) = \mathbb{R}^n$. Moreover it is clear that φ_m is strictly convex and, using [R] Theorem 26.3, it is not difficult to see that φ_m is differentiable. It follows that the same holds for $\mathcal{L}\varphi_m$. The fact that $\nabla\varphi_m$ is bijective on $\mathbb{R}^n$ deduces from Lemma 3.
2. These convergences are classical. Let us prove for example the first one. On one hand one has $\varphi_m(x_0) \leq \frac{|x_0|^2}{2m} + \varphi(x_0)$. On the other hand if $x_0 \in \mathrm{dom}(\varphi)$ then there exists a hyperplane touching the epigraph of φ at $(x_0, \varphi(x_0))$ thus there exists $y \in \mathbb{R}^n$ such that $\varphi(x) \geq \varphi(x_0) + \langle x - x_0, y\rangle$, for all $x \in \mathbb{R}^n$. This implies that

$$\varphi_m(x_0) \geq \inf_x \left(\varphi(x) + \frac{m}{2}|x-x_0|^2 \right) \geq \inf_x \left(\varphi(x_0) + \langle x - x_0, y\rangle + \frac{m}{2}|x-x_0|^2 \right) = \varphi(x_0) - \frac{|y|^2}{2m}.$$

Letting $m \to +\infty$ gives the convergence. If $x_0 \notin \overline{\mathrm{dom}(\varphi)}$ then, using that $\varphi \geq \min\varphi + I_{\mathrm{dom}(\varphi)}$, we deduce that

$$\varphi_m(x_0) \geq \min\varphi + \inf_{x \in \mathrm{dom}(\varphi)} \frac{m}{2}|x-x_0|^2 = \min\varphi + \frac{m}{2} d(x_0, \mathrm{dom}(\varphi))^2.$$

Therefore $\varphi_m(x_0) \to \varphi(x_0)$ when $m \to +\infty$ for every $x_0 \notin \partial(\mathrm{dom}(\varphi))$, that is a.e.
3. First notice that we may assume that $\varphi(0) = 0$. From Lemma 2, it follows that there exists $a, b > 0$ such that for every $x \in \mathbb{R}^n$ one has $a|x| - 1 \leq \varphi(x) \leq I_{bB_2^n}(x) + 1$. Hence we get

$$\varphi_m(x) \geq \inf_z \left(\varphi(z) + \frac{1}{2}|x-z|^2 \right) \geq \inf_z \left(a|z| - 1 + \frac{1}{2}(|x| - |z|)^2 \right) = a|x| - \frac{a^2}{2} - 1. \tag{8}$$

Thus $e^{-\varphi_m(x)} \leq e^{\frac{a^2}{2}+1-a|x|}$ for all m, then from the dominated convergence theorem one deduces that $\int_A e^{-\varphi_m} \to \int_A e^{-\varphi}$ when $m \to +\infty$. In the same way, one has $b|y| - 1 \leq \mathcal{L}\varphi(y) \leq I_{aB_2^n}(y) + 1$ thus

$$\mathcal{L}\varphi_m(y) \geq \inf_z \left(\mathcal{L}\varphi(z) + \frac{1}{2}|y-z|^2 \right) \geq b|y| - \frac{b^2}{2} - 1.$$

Hence from the dominated convergence theorem one deduces that $\int_A e^{-\mathcal{L}\varphi_m} \to \int_A e^{-\mathcal{L}\varphi}$ when $m \to +\infty$. We conclude that $P(\varphi_m) \to P(\varphi)$ when $m \to +\infty$. Similarly we prove that for every $t > 0$ one has $P(t\varphi_m) \to P(t\varphi)$ when $m \to +\infty$. Using that $ue^{-u} \leq \frac{2}{e}e^{-\frac{u}{2}}$ for every $u \in \mathbb{R}$ we get that $\varphi_m e^{-\varphi_m} \leq \frac{2}{e}e^{-\frac{\varphi_m}{2}}$ and we conclude again by the dominated convergence theorem. The same method gives the result for $t\varphi_m$ and $\mathcal{L}(t\varphi_m)$. $\square$

4 Proof in dimension 2

Theorem 10. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be even convex such that $0 < \int_{\mathbb{R}^2} e^{-\varphi(x)}dx < +\infty$, then*

$$P(\varphi) = \int_{\mathbb{R}^2} e^{-\varphi(x)}dx \int_{\mathbb{R}^2} e^{-\mathcal{L}\varphi(y)}dy \geq 4^2 = 16,$$

with equality if and only if there exists $a \in \mathbb{R}$ such that either $\varphi = I_P + a$, or $\varphi = \|\cdot\|_P + a$ with P being a parallelogram centered at the origin or there exists a basis (u_1, u_2) of $\mathbb{R}^2$ and $b, c > 0$ such that $\varphi(x_1u_1 + x_2u_2) = c|x_1| + I_{[-b,b]}(x_2) + a$ for every $x_1, x_2 \in \mathbb{R}$.

4.1 The inequality in dimension 2

Proof of Theorem 10. Let $\varphi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be even convex such that $0 < \int_{\mathbb{R}^2} e^{-\varphi(x)}dx < +\infty$. First let us reduce to the case where φ is strongly equipartioned, in the sense that

$$\varphi(0) = 0, \quad \int_{\mathbb{R}_+} e^{-\varphi(te_1)}dt = \int_{\mathbb{R}_+} e^{-\varphi(te_2)}dt = 1, \quad \int_{\mathbb{R}^2_+} e^{-\varphi} = \frac{1}{4}\int_{\mathbb{R}^2} e^{-\varphi}, \quad \int_{\mathbb{R}^2_+} \varphi e^{-\varphi} = \frac{1}{4}\int_{\mathbb{R}^2} \varphi e^{-\varphi}.$$

Since $P(\varphi) = P(\varphi - \varphi(0))$ we may assume that $\varphi(0) = 0$. For any $u \in S^1$, let $C(u) \subset S^1$ be the open half-circle delimited by u and $-u$ containing the vectors v which are after u with respect to the counterclockwise orientation of S^1. For $v \in C(u)$ let $C_{u,v} = \mathbb{R}_+u + \mathbb{R}_+v$ be the cone generated by u and v and define $f_u(v) = \mu_\varphi(C_{u,v})$. The map f_u is continuous and increasing on $C(u)$, $f_u(u) = 0$ and $f_u(v) \longrightarrow \mu_\varphi(\mathbb{R}^2)/2$ when $v \longrightarrow -u$, thus there exists a unique $v(u) \in C(u)$ such that $f_u(v(u)) = \mu_\varphi(C_{u,v(u)}) = \mu_\varphi(\mathbb{R}^2)/4$. Notice that $v : S^1 \to S^1$ is continuous and, since φ is even, one has $v(v(u)) = -u$ for any $u \in S^1$. For $u \in S^1$, let $g(u) = \int_{C_{u,v(u)}} \varphi e^{-\varphi} - \frac{1}{4}\int_{\mathbb{R}^2} \varphi e^{-\varphi}$. Then g is continuous on S^1 and, since φ is even,

$$g(u) + g(v(u)) = \int_{C_{u,v(u)}} \varphi e^{-\varphi} + \int_{C_{v(u),-u}} \varphi e^{-\varphi} - \frac{1}{2}\int_{\mathbb{R}^2} \varphi e^{-\varphi} = 0.$$

Hence $g(u) = -g(v(u))$. By the intermediate value theorem there exists $u \in S^1$ such that $g(u) = 0$, thus

$$\int_{C_{u,v(u)}} \varphi e^{-\varphi} = \frac{1}{4}\int_{\mathbb{R}^2} \varphi e^{-\varphi} \quad \text{and} \quad \mu_\varphi\left(C_{u,v(u)}\right) = \frac{1}{4}\mu_\varphi\left(\mathbb{R}^2\right).$$

Let S be the linear map defined by $S(e_1) = u$ and $S(e_2) = v(u)$, then $S(\mathbb{R}^2_+) = C_{u,v(u)}$. Moreover, changing variables, for any Borel set A in $\mathbb{R}^2$ we have $\mu_{\varphi\circ S}(A) = \mu_\varphi(S(A))/\det(S)$ thus

$$\mu_{\varphi\circ S}(\mathbb{R}^2_+) = \frac{\mu_\varphi(S(\mathbb{R}^2_+))}{\det(S)} = \frac{\mu_\varphi(C_{u,v(u)})}{\det(S)} = \frac{\mu_\varphi(\mathbb{R}^2)}{4\det(S)} = \frac{\mu_{\varphi\circ S}(\mathbb{R}^2)}{4}.$$

In the same way, one has

$$\int_{\mathbb{R}^2_+} (\varphi\circ S)e^{-\varphi\circ S} = \frac{1}{4}\int_{\mathbb{R}^2} (\varphi\circ S)e^{-\varphi\circ S}.$$

Let $\alpha_i = \int_0^{+\infty} e^{-\varphi(re_i)}dr$ and Δ be the linear map defined by $\Delta(e_i) = \alpha_i e_i$ and $T = S \circ \Delta$ then a change of variables shows that $\varphi \circ T$ is strongly equipartioned. Since $P(\varphi) = P(\varphi \circ T)$ we may assume that φ is strongly equipartioned.

From Lemma 9 we can assume that $\mathrm{dom}(\varphi) = \mathrm{dom}(\mathcal{L}\varphi) = \mathbb{R}^2$, φ is differentiable and strictly convex on $\mathbb{R}^2$. Then, up to sets of Lebesgue measure zero, we have the partition $\mathbb{R}^2 = \cup_{\varepsilon\in\{-1,1\}^2}\nabla\varphi(\mathbb{R}^2_\varepsilon)$. Using the equipartition , we get

$$P(\varphi) = \sum_{\varepsilon\in\{-1,1\}^2}\int_{\mathbb{R}^2} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_\varepsilon)} e^{-\mathcal{L}\varphi} = 4\sum_{\varepsilon\in\{-1,1\}^2}\int_{\mathbb{R}^2_\varepsilon} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_\varepsilon)} e^{-\mathcal{L}\varphi}.$$

Using the fact that φ is even we get

$$P(\varphi) = 8\int_{\mathbb{R}^2_+} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_+)} e^{-\mathcal{L}\varphi} + 8\int_{\mathbb{R}_+\times\mathbb{R}_-} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)} e^{-\mathcal{L}\varphi} = 8(F_1(1)+F_2(1))$$

where $F_1(t) = F_{\mathbb{R}^2_+,\nabla\varphi(\mathbb{R}^2_+)}(t)$ and $F_2(t) = F_{\mathbb{R}_+\times\mathbb{R}_-,\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)}(t)$. Using 4) of Lemma 6, we have

$$F_1'(1) = 2F_1(1) - \int_{\mathbb{R}^2_+}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_+)} e^{-\mathcal{L}\varphi} - \int_{R^2_+} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_+)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}.$$

$$F_2'(1) = 2F_2(1) - \int_{\mathbb{R}_+\times\mathbb{R}_-}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)} e^{-\mathcal{L}\varphi} - \int_{\mathbb{R}_+\times\mathbb{R}_-} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}.$$

Then we get

$$\begin{aligned}\frac{d}{dt}(t^2(F_1(t)+F_2(t)))_{|t=1} = 4(F_1(1)+F_2(1)) - \Bigg(&\int_{\mathbb{R}^2_+}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_+)} e^{-\mathcal{L}\varphi} + \int_{\mathbb{R}^2_+} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^2_+)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}\\ &+ \int_{\mathbb{R}_+\times\mathbb{R}_-}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)} e^{-\mathcal{L}\varphi} + \int_{\mathbb{R}_+\times\mathbb{R}_-} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}\Bigg).\end{aligned}$$

Thus using the fact that φ is equipartioned, we get that

$$\begin{aligned}\frac{d}{dt}\left(t^2(F_1(t)+F_2(t))\right)_{|t=1} &= 4(F_1(1)+F_2(1)) - \int_{\mathbb{R}^2_+}\varphi e^{-\varphi}\times\frac{1}{2}\int_{\mathbb{R}^2} e^{-\mathcal{L}\varphi} - \int_{\mathbb{R}^2_+} e^{-\varphi}\times\frac{1}{2}\int_{\mathbb{R}^2}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}\\ &= \frac{1}{2}P(\varphi) - \frac{1}{8}\left(\int_{\mathbb{R}^2}\varphi e^{-\varphi}\int_{\mathbb{R}^2} e^{-\mathcal{L}\varphi} + \int_{\mathbb{R}^2} e^{-\varphi}\int_{\mathbb{R}^2}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}\right).\end{aligned}$$

On the other hand applying Lemma 6 one has

$$\begin{aligned}\frac{d}{dt}\left(t^2P(t\varphi)\right)_{|t=1} &= 4P(\varphi) - \left(\int_{\mathbb{R}^2}\varphi e^{-\varphi}\int_{\mathbb{R}^2} e^{-\mathcal{L}\varphi} + \int_{\mathbb{R}^2} e^{-\varphi}\int_{\mathbb{R}^2}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}\right)\\ &= 8\frac{d}{dt}(t^2(F_1(t)+F_2(t)))_{|t=1}.\end{aligned}$$

We apply Corollary 7 for $A = \mathbb{R}^2_+$ and $B = \nabla\varphi(\mathbb{R}^2_+)$ and use the equipartition to get

$$\frac{d}{dt}\left(t^2F_1(t)\right)_{|t=1} \geq \langle V_{\partial\mathbb{R}^2_+}(\varphi), V_{\partial\nabla\varphi(\mathbb{R}^2_+)}(\mathcal{L}\varphi)\rangle + \frac{\mu_\varphi(\mathbb{R}^2)}{4}Q_{\partial\nabla\varphi(\mathbb{R}^2_+)}(\mathcal{L}\varphi). \tag{9}$$

From Lemma 8 one has

$$V_{\partial\mathbb{R}^2_+}(\varphi) = -e_1 \int_0^{+\infty} e^{-\varphi_1} - e_2 \int_0^{+\infty} e^{-\varphi_2} = -(e_1+e_2). \tag{10}$$

Similarly one has $V_{\partial\nabla\varphi(\mathbb{R}^2_+)}(\mathcal{L}\varphi) = -W_1 - W_2$, where

$$W_1 = -\int_{\nabla\varphi(\{0\}\times\mathbb{R}_+)} n_{\nabla\varphi(\mathbb{R}^2_+)} e^{-\mathcal{L}\varphi} \quad \text{and} \quad W_2 = -\int_{\nabla\varphi(\mathbb{R}_+\times\{0\})} n_{\nabla\varphi(\mathbb{R}^2_+)} e^{-\mathcal{L}\varphi}.$$

Thus the equation (9) becomes

$$\frac{d}{dt}\left(t^2 F_1(t)\right)_{|t=1} \geq \langle e_1+e_2, W_1+W_2\rangle + \frac{\mu_\varphi(\mathbb{R}^2)}{4} Q_{\partial\nabla\varphi(\mathbb{R}^2_+)}(\mathcal{L}\varphi). \tag{11}$$

Moreover, since φ is even, one has

$$V_{\partial(\mathbb{R}_+\times\mathbb{R}_-)}(\varphi) = -e_1+e_2, \quad V_{\partial\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)}(\mathcal{L}\varphi) = -W_1+W_2$$

and

$$Q_{\partial\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)}(\mathcal{L}\varphi) = \int_{\partial\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)} \langle y, n_{\nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)}(y)\rangle e^{-\mathcal{L}\varphi(y)}dy = -Q_{\partial\nabla\varphi(\mathbb{R}^2_+)}(\mathcal{L}\varphi).$$

Applying Corollary 7 for $A = \mathbb{R}_+\times\mathbb{R}_-$ and $B = \nabla\varphi(\mathbb{R}_+\times\mathbb{R}_-)$ and using the equipartition we get

$$\frac{d}{dt}\left(t^2 F_2(t)\right)_{|t=1} \geq \langle e_1-e_2, W_1-W_2\rangle - \frac{\mu_\varphi(\mathbb{R}^2)}{4} Q_{\partial\nabla\varphi(\mathbb{R}^2_+)}(\mathcal{L}\varphi). \tag{12}$$

Adding (11) and (12) we obtain

$$\frac{d}{dt}\left(t^2 P(t\varphi)\right)_{|t=1} = 8\frac{d}{dt}(t^2(F_1(t)+F_2(t)))_{|t=1} \geq 16(\langle e_1, W_1\rangle + \langle e_2, W_2\rangle).$$

Moreover from Lemma 8 for $i=1,2$ one has

$$\langle e_i, W_i\rangle = \int_{\nabla\varphi(\mathbb{R}^2_+\cap e_i^\perp)} \langle -e_i, n_{\nabla\varphi(\mathbb{R}^2_+)}(x)\rangle e^{-\mathcal{L}\varphi(x)}dx = \int_0^{+\infty} e^{-\mathcal{L}(\varphi_i)(y)}dy \geq 1,$$

where the last inequality comes from the result in dimension 1 proved in [FM2, FM3]. Thus we get

$$\frac{d}{dt}\left(t^2 P(t\varphi)\right)_{|t=1} \geq 16\left(\int_0^{+\infty} e^{-\mathcal{L}(\varphi_1)} + \int_0^{+\infty} e^{-\mathcal{L}(\varphi_2)}\right) \geq 32. \tag{13}$$

Applying this relation for φ replaced by $s\varphi$ and using the fact that

$$\frac{d}{dt}\left(t^2 P(ts\varphi)\right)_{|t=1} = \lim_{t\to 1}\frac{t^2P(ts\varphi)-P(s\varphi)}{t-1} = \lim_{u\to s}\frac{\frac{u^2}{s^2}P(u\varphi)-P(s\varphi)}{\frac{u}{s}-1} = \frac{1}{s}\frac{d}{du}\left(u^2P(u\varphi)\right)_{|u=s}$$

one gets, $\forall t>0$, $\frac{d}{dt}(t^2P(t\varphi)) \geq 32t$. Integrating this inequality we conclude that for every $0<\varepsilon<t$

$$t^2P(t\varphi) = \varepsilon^2 P(\varepsilon\varphi) + \int_\varepsilon^t \frac{d}{ds}\left(s^2P(s\varphi)\right)ds \geq 32\int_\varepsilon^t sds = 16(t^2-\varepsilon^2). \tag{14}$$

Letting ε tends to 0 we conclude that $t^2P(t\varphi)\geq 16t^2$ and thus $P(\varphi)\geq 16$.

4.2 The equality case in dimension 2

Now we establish the equality case. Notice first that the inequalities (13) were so far established only for φ being differentiable, strictly convex on $\mathbb{R}^2$ and strongly equipartioned. Let us prove that (13) still hold without these regularity assumption. We adapt the arguments of [FHMRZ] to the functional case. This requires to develop new functional inequalities. We first prove that the set of convex functions which are equipartioned has some compactness property.

Lemma 11. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ be a convex, non-decreasing function such that $\varphi(0) = 0$ and $\int_0^{+\infty} e^{-\varphi(x)}dx = 1$. Then for every $x \in \mathbb{R}_+$*

$$x - 1 \leq \varphi(x) \leq I_{[0,1]}(x) + x \leq I_{[0,1]}(x) + 1.$$

Proof. For $t \in [0, 1]$ we define $\psi(t) = \sup\{x \geq 0; \varphi(x) \leq t\}$. One has $\varphi(\psi(t)) \leq t$ for almost all $t \geq 0$. From Jensen inequality we get

$$\varphi(1) = \varphi\left(\int_0^{+\infty} e^{-\varphi(x)}dx\right) = \varphi\left(\int_0^{+\infty} \psi(t)e^{-t}dt\right) \leq \int_0^{+\infty} \varphi(\psi(t))e^{-t}dt \leq \int_0^{+\infty} te^{-t}dt = 1.$$

By convexity we deduce that $\varphi(x) = \varphi((1 - x) \cdot 0 + x \cdot 1) \leq x$ for $x \in [0, 1]$ and thus $\varphi(x) \leq I_{[0,1]}(x) + x \leq I_{[0,1]}(x) + 1$, for every $x \geq 0$. For the proof of the lower bound, the idea is exactly the same as in the proof of Proposition 4. By convexity, the function φ is differentiable almost everywhere on $\mathrm{dom}(\varphi)$ and one has for almost all $y \in \mathrm{dom}(\varphi)$ and for all $x \geq 0$

$$\varphi(x) \geq \varphi(y) + (x - y)\varphi'(y).$$

We multiply by $e^{-\varphi(y)}$ and integrate in y on $[0, +\infty)$ to get that for every $x \geq 0$

$$\varphi(x) \geq \int_0^{+\infty} \varphi(y)e^{-\varphi(y)}dy + \int_0^{+\infty} (x - y)\varphi'(y)e^{-\varphi(y)}dy.$$

Using that $\varphi \geq 0$ and integrating by parts we deduce that for every $x \geq 0$

$$\varphi(x) \geq \int_0^{+\infty} (x - y)\varphi'(y)e^{-\varphi(y)}dy = x - 1.$$

□

Now we prove the analogue lemma in dimension 2.

Lemma 12. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be a convex, even and strongly equipartioned function such that $\varphi(0) = 0$. Then for every $x \in \mathbb{R}^2$, one has*

$$\frac{\|x\|_1}{e + 2} - 2 \leq \varphi(x) \leq I_{B_1^2}(x) + 1 \quad \textit{thus} \quad \frac{2}{e} \leq \int_{\mathbb{R}^2} e^{-\varphi(x)}dx \leq (2e(e + 2))^2.$$

Proof. The bounds on the integral follows directly from the bounds on the function. Let us prove these bounds. From Lemma 11 applied to $t \mapsto \varphi(te_i)$ we deduce that $\varphi(e_i) \leq 1$ for $1 \leq i \leq 2$. Since φ is even and convex we deduce that $\varphi(x) \leq 1$ for every $x \in B_1^2 = \mathrm{Conv}(\pm e_1, \pm e_2)$. This proves the upper bound.

To prove the lower bound define $c = (e + 2)^{-1} < 1$. Let assume by contradiction that there exists $a = (a_1, a_2) \in \mathbb{R}^2$ such that

$$\varphi(a) < c\|a\|_1 - 2.$$

By symmetry we may assume that $a_1 \geq a_2 \geq 0$. Moreover if $a_2 = 0$, from Lemma 11 applied to φ_2 one has $\varphi(a) = \varphi_2(a_1) \geq |a_1| - 1 = \|a\|_1 - 1$, which is not possible since $c < 1$. Thus one has $a_2 > 0$. For every $x = (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_-$ by convexity, one has

$$\varphi\left(\frac{a_2}{a_2 - x_2}x + \frac{-x_2}{a_2 - x_2}a\right) \leq \frac{a_2}{a_2 - x_2}\varphi(x) + \frac{-x_2}{a_2 - x_2}\varphi(a).$$

Since $a_2 x - x_2 a = (a_2 x_1 - x_2 a_1)e_1 \in \mathbb{R}_+ e_1$, we may apply Lemma 11 and get

$$\varphi\left(\frac{a_2}{a_2 - x_2}x + \frac{-x_2}{a_2 - x_2}a\right) \geq \frac{a_2}{a_2 - x_2}x_1 + \frac{-x_2}{a_2 - x_2}a_1 - 1.$$

Thus we deduce that

$$\varphi(x) \geq x_1 + \frac{(-x_2)}{a_2}\left(-\varphi(a) - 1 + a_1\right) - 1.$$

Since $a_1 \geq a_2 > 0$, using the upper bound on $\varphi(a)$ we get

$$-\varphi(a) - 1 + a_1 \geq -c(a_1 + a_2) + a_1 + 1 \geq (1 - c)a_1 - ca_2 \geq (1 - 2c)a_2.$$

Therefore for every $x \in \mathbb{R}_+ \times \mathbb{R}_-$ we deduce that

$$\varphi(x) \geq x_1 + (1 - 2c)(-x_2) - 1.$$

Integrating on $\mathbb{R}_+ \times \mathbb{R}_-$, and replacing c by its value, we get

$$\int_{\mathbb{R}_+ \times \mathbb{R}_-} e^{-\varphi(x)} dx \leq \frac{e}{1 - 2c} = \frac{1}{c}.$$

Now we apply Proposition 4 to the cone $A = \mathbb{R}^2_+$. Recall that in this case the term $Q_{\partial A}(\varphi)$ vanishes. Thus we get

$$\varphi(a) \geq -2 + \langle a, -\frac{V_{\partial \mathbb{R}^2_+}(\varphi)}{\mu_\varphi(\mathbb{R}^2_+)}\rangle + \int_{\mathbb{R}^2_+} \varphi(y) \frac{d\mu_\varphi(y)}{\mu_\varphi(\mathbb{R}^2_+)}.$$

Using that $\varphi \geq 0$ and $V_{\partial \mathbb{R}^2_+}(\varphi) = -(e_1 + e_2)$ that was established in equation (10) we get

$$\varphi(a) \geq -2 + \langle a, \frac{e_1 + e_2}{\mu_\varphi(\mathbb{R}^2_+)}\rangle = -2 + \frac{\|a\|_1}{\mu_\varphi(\mathbb{R}^2_+)} \geq -2 + c\|a\|_1,$$

which is in contradiction with the definition of c. □

Remark 13. *The same argument applies inductively and shows that there exists a constant $c_n > 0$ such that for any even convex function φ on $\mathbb{R}^n$ such that all its restrictions to coordinate planes are equipartitioned and $\varphi(0) = 0$ one has*

$$c_n\|x\|_1 - n \leq \varphi(x) \leq I_{B_1^n}(x) + 1 \quad \textit{thus} \quad \frac{2^n}{en!} \leq \int_{\mathbb{R}^n} e^{-\varphi(x)} dx \leq \left(\frac{2e}{c_n}\right)^n.$$

Lemma 14. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be an even convex function such that $0 < \int e^{-\varphi} < +\infty$ and $\varphi(0) = 0$. Then there exists a sequence $(\psi_k)_k$ of differentiable strongly equipartioned even strictly convex functions with $\mathrm{dom}(\psi_k) = \mathbb{R}^2$ and there exists an invertible linear map T such that*
(i) for every $x \in \mathbb{R}^2$, $(\psi_k(x))_k$ converges to $\varphi \circ T(x)$ and $e^{-\psi_k(x)} \leq Ce^{-d|x|}$, for some $C, d > 0$
(ii) for every $x \in \mathbb{R}^2$, $(\mathcal{L}\psi_k(x))_k$ converges to $\mathcal{L}(\varphi \circ T)(x)$ and $e^{-\mathcal{L}\psi_k(x)} \leq Ce^{-d|x|}$, for some $C, d > 0$.

Proof. We define the set $K_\varphi = \{x \in \mathbb{R}^2; \varphi(x) \le 1\}$. From Lemma 2, K_φ is a symmetric convex body, and there exists $a, b > 0$ such that $a|x| - 1 \le \varphi(x) \le I_{bB_2^2}(x) + 1$. We recall the definition of φ_m used in Lemma 9: for every $x \in \mathbb{R}^2$

$$\varphi_m(x) = \frac{|x|^2}{2m} + \inf_z \left(\varphi(z) + \frac{m}{2}|x - z|^2\right).$$

Using the lower bound obtained in (8) we have $\varphi_m(x) \ge a|x| - \frac{a^2}{2} - 1$. Hence for every $m \in \mathbb{N}^*$,

$$\{x; \varphi_m(x) \le 1\} \subset RB_2^2,$$

where $R = \frac{a}{2} + \frac{2}{a}$. There exists a sequence of invertible linear maps T_m such that $\varphi_m \circ T_m$ is strongly equipartioned. From Lemma 12 one has $\varphi_m(T_m(e_i)) \le 1$, for all $1 \le i \le 2$ thus

$$T_m(B_1^2) \subset \{x; \varphi_m(x) \le 1\} \subset RB_2^2.$$

Thus the sequence $(T_m)_m$ is bounded in the normed spaces of linear maps and thus there exists a subsequence $(T_{m_k})_k$ of linear maps that converges to some linear map T. Let us prove that T is invertible. For every $m \in \mathbb{N}^*$, using Lemma 12 and denoting $c = (e+2)^{-1}$ one has $c\|x\|_1 - 2 \le \varphi_m(T_m x)$ for every x. Moreover since $\varphi_m(x) \le \varphi(x) + \frac{|x|^2}{2}$ and $\varphi(x) \le I_{bB_2^2}(x) + 1$, it follows that

$$\varphi_m(x) \le I_{bB_2^2}(x) + 1 + \frac{|x|^2}{2} \le I_{bB_2^2}(x) + 1 + \frac{b^2}{2}.$$

Thus for any $x \in \mathbb{R}^2$

$$\frac{bc\|x\|_1}{|T_m x|} \le \varphi_m\left(\frac{bT_m x}{|T_m x|}\right) + 2 \le 3 + \frac{b^2}{2}.$$

This gives that for every $x \in \mathbb{R}^2$

$$\left(\frac{3}{b} + \frac{b}{2}\right)|T_m x| \ge c\|x\|_1.$$

Hence T satisfies the same bound and thus is invertible. Moreover since $\varphi_m(T_m x) \ge c\|x\|_1 - 2$, we conclude that the sequence $\psi_k = \varphi_{m_k} \circ T_{m_k}$ is a sequence of strongly equipartioned even differentiable strictly convex functions such that $(\psi_k(x))_k$ converges to $\varphi \circ T(x)$ and $e^{-\psi_k(x)} \le e^{2 - c\|x\|_1}$. Thus $\varphi \circ T$ is strongly equipartioned. Moreover, from Lemma 12 one has $\varphi_m(T_m x) \le I_{B_1^2}(x) + 1$, hence

$$\mathcal{L}(\varphi_m \circ T_m)(x) \ge \mathcal{L}\left(I_{B_1^2} + 1\right)(x) = \|x\|_\infty - 1.$$

Therefore $\mathcal{L}\psi_k = \mathcal{L}(\varphi_{m_k} \circ T_{m_k})$ satisfies the same bound. From Lemma 9 one has $\mathcal{L}\varphi_m(x) \to \mathcal{L}\varphi(x)$, for every $x \in \mathbb{R}^2$, when $m \to +\infty$. Since T_{m_k} converges to T we conclude that for every $x \in \mathbb{R}^2$, $\mathcal{L}\psi_k(x)$ converges to $\mathcal{L}(\varphi \circ T)(x)$. □

Proof of the equality case. Let $\varphi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be an even convex function such that $0 < \int_{\mathbb{R}^2} e^{-\varphi(x)} dx < +\infty$ and $P(\varphi) = 16$. By Lemma 14 there exists a sequence $(\psi_k)_k$ of differentiable strongly equipartioned even strictly convex functions with $\mathrm{dom}(\psi_k) = \mathbb{R}^2$ and a bijective linear map T such that $\varphi \circ T$ is strongly equipartioned. Since $P(\varphi \circ T) = P(\varphi)$ and our equality case is invariant by invertible linear maps, we replace $\varphi \circ T$ by φ in the rest of the proof. We have thus established that φ is the limit of a sequence $(\psi_k)_k$ of differentiable and strongly equipartioned strictly convex functions. Thus the inequalities (13) and (14) are valid for the functions ψ_k. By taking the limit and using Lemma 9 we deduce that these inequalities are also valid for φ. Then applying the same reasoning as in inequality (14) for $t=1$ and using that $P(\varphi)=16$ and $P(\varepsilon\varphi)\geq 16$ we get

$$16=P(\varphi)=\varepsilon^2 P(\varepsilon\varphi)+\int_\varepsilon^1 \frac{d}{ds}\left(s^2P(s\varphi)\right)ds\geq 16\varepsilon^2+32\int_\varepsilon^1 sds=16\varepsilon^2+16(1-\varepsilon^2)=16.$$

Thus there is equality in the intermediate inequalities. Hence for every $0<\varepsilon\leq 1$ one has $P(\varepsilon\varphi)=16$. Thus for all $0<t\leq 1$ we have $\frac{d}{dt}\left(t^2P(t\varphi)\right)=\frac{d}{dt}\left(16t^2\right)=32t$. Hence there is equality in (13). This implies that

$$\int_0^{+\infty} e^{-\mathcal{L}(\varphi_1)}=1 \quad\text{and}\quad \int_0^{+\infty} e^{-\mathcal{L}(\varphi_2)}=1.$$

From the equality case in dimension 1 and since φ_i is equipartioned, we deduce that for $i=1,2$ either $\varphi_i(x)=I_{[-1,1]}(x)$ or $\varphi_i(x)=|x|$ for every $x\in\mathbb{R}$. Following the proof in [FGMR] we distinguish three cases.

A. If $\varphi_2(x)=\varphi(x,0)=I_{[-1,1]}(x)$ and $\varphi_1(x)=\varphi(0,x)=|x|$ for every $x\in\mathbb{R}$. Then let us prove that for all $(x_1,x_2)\in\mathbb{R}^2$ one has $\varphi(x_1,x_2)=I_{[-1,1]}(x_1)+|x_2|$. This deduces from the following more general lemma, which extends observations done in the unconditional case in [FGMR].

Lemma 15. *Let $n\geq 1$ and $\varphi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be an even convex function such that there exists two convex bodies $K_1\subset F_1$ and $K_2\subset F_2$, where F_1 and F_2 are two complementary linear subspaces in $\mathbb{R}^n$ such that $\varphi(x_1)=\|x_1\|_{K_1}$ for all $x_1\in F_1$ and $\varphi(x_2)=I_{K_2}(x_2)$ for all $x_2\in F_2$. Then $\varphi(x_1+x_2)=\|x_1\|_{K_1}+I_{K_2}(x_2)$, for all $x_1\in F_1$ and $x_2\in F_2$.*

Proof. Let $x_1\in F_1$ and $x_2\in K_2$. From the convexity of φ and using that

$$x_1+x_2=(1-\|x_2\|_{K_2})\frac{x_1}{1-\|x_2\|_{K_2}}+\|x_2\|_{K_2}\frac{x_2}{\|x_2\|_{K_2}},$$

we deduce that

$$\varphi(x_1+x_2)\leq(1-\|x_2\|_{K_2})\varphi\left(\frac{x_1}{1-\|x_2\|_{K_2}}\right)+\|x_2\|_{K_2}\varphi\left(\frac{x_2}{\|x_2\|_{K_2}}\right)=\|x_1\|_{K_1}.$$

On the other hand, using that $\frac{x_1}{2}=\frac{1}{2}(x_1+x_2)+\frac{1}{2}(-x_2)$ one gets

$$\frac{\|x_1\|_{K_1}}{2}=\varphi\left(\frac{x_1}{2}\right)\leq\frac{1}{2}\varphi(x_1+x_2)+\frac{1}{2}\varphi(-x_2)=\frac{1}{2}\varphi(x_1+x_2).$$

We deduce that $\varphi(x_1+x_2)=\|x_1\|_{K_1}$.
Let $x_1\in F_1$ and $x_2\notin K_2$. Let $1<\mu<\|x_2\|_{K_2}$ and $\lambda=\mu/\|x_2\|_{K_2}\in(0,1)$. Then $\lambda x_2\notin K_2$ and

$$\lambda x_2=\lambda(x_1+x_2)+(1-\lambda)\frac{-\lambda x_1}{1-\lambda}.$$

Hence using the convexity of φ we get

$$+\infty=\varphi(\lambda x_2)\leq\lambda\varphi(x_1+x_2)+(1-\lambda)\varphi\left(\frac{-\lambda x_1}{1-\lambda}\right).$$

Since $(1-\lambda)\varphi\left(\frac{-\lambda x_1}{1-\lambda}\right)=\lambda\|x_1\|_{K_1}<+\infty$, we deduce that $\varphi(x_1+x_2)=+\infty=I_{K_2}(x_2)$. We conclude that $\varphi(x_1+x_2)=\|x_1\|_{K_1}+I_{K_2}(x_2)$, for all $x_1\in F_1$ and $x_2\in F_2$. □

B. If $\varphi_2(s) = \varphi(se_1) = I_{[-1,1]}(s)$ and $\varphi_1(s) = \varphi(se_2) = I_{[-1,1]}(s)$ for every $s \in \mathbb{R}$. Let $U = \{x; \varphi(x) = 0\}$ and $K = \mathrm{dom}(\varphi)$. From the hypothesis one has $\varphi(\pm e_i) = 0$ thus $\pm e_i \in U$. Since $\min \varphi = 0$, the convexity of φ implies that U is convex. Thus one has $B_1^2 \subset U \subset K$. Since $\pm e_i \in \partial K$ for $i = 1, 2$ one deduces there exists $u_i \in \partial K^*$ such that $\langle e_i, u_i\rangle = 1$ and one has $K \subset \{x; |\langle x, u_i\rangle| \leq 1, i = 1, 2\}$. We distinguish two cases:
- if $u_1 = u_2$: since $\langle e_i, u_i\rangle = 1$ one has $u_1 = u_2 = e_1 + e_2$. Thus $K \subset \{x \in \mathbb{R}^2; |x_1 + x_2| \leq 1\} := D$. Hence $\mathrm{Conv}(0, e_1, e_2) \subset U \cap \mathbb{R}^2_+ \subset K \cap \mathbb{R}^2_+ \subset D \cap \mathbb{R}^2_+ = \mathrm{Conv}(0, e_1, e_2)$. Therefore $\varphi_{|\mathbb{R}^2_+} = I_{B_1^2 \cap \mathbb{R}^2_+}$. Using the equipartition and the fact that $\varphi \leq I_{B_1^2}$ we conclude that $\varphi = I_{B_1^2}$.
- if $u_1 \neq u_2$: using that for every $x \in K$ one has $\langle u_i, x\rangle \leq 1$ and $\varphi(x) \geq 0$ then for every $s > 0$ and for $i = 1, 2$ we get

$$\mathcal{L}\varphi(su_i) = \sup_x(\langle su_i, x\rangle - \varphi(x)) = \sup_{x \in K}(s\langle u_i, x\rangle - \varphi(x)) = s,$$

with equality for $x = e_i$. Since φ is even we deduce that for every $s \in \mathbb{R}$

$$\mathcal{L}\varphi(su_i) = |s|.$$

Define $C_+ = \mathbb{R}_+u_1 + \mathbb{R}_+u_2$ and $C_- = \mathbb{R}_+u_1 + \mathbb{R}_-u_2$. Since $e_i \in K$ and $u_i \in K^*$ one has $|\langle u_1, e_2\rangle| \leq 1$ and $|\langle u_2, e_1\rangle| \leq 1$. Denote by v_i the unitary exterior normal of C_+ to the line $\mathbb{R}u_i$. We have $V_{\partial C_+}(\mathcal{L}\varphi) = -V_1 - V_2$ where

$$V_1 = -v_2 \int_{\mathbb{R}_+u_2} e^{-\mathcal{L}\varphi} = -v_2 \int_0^{+\infty} e^{-\mathcal{L}\varphi(su_2)} ds|u_2| = -v_2 \int_0^{+\infty} e^{-s} ds|u_2| = -v_2|u_2|$$

and in the same way $V_2 = -v_1|u_1|$. Hence $V_{\partial C_+}(\mathcal{L}\varphi) = v_1|u_1| + v_2|u_2|$. It is easy to see that $V_{\partial C_+}(\mathcal{L}\varphi) \in \mathbb{R}^2_+$. We also have $V_{\partial \mathbb{R}^2_+}(\varphi) = -(e_1 + e_2)$. Using that $\langle e_i, u_i\rangle = 1$ one has $\langle e_1, V_1\rangle = -\langle e_1, v_2\rangle|u_2| = \langle e_2, u_2\rangle = 1$. In the same way one also has $\langle e_2, V_2\rangle = 1$. We reproduce the same argument as before with $\nabla\varphi(\mathbb{R}^2_+)$ replaced by C_+ and $\nabla\varphi(\mathbb{R}_+ \times \mathbb{R}_-)$ replaced by C_-. Some terms are simplified because C_+ and C_- are cones. Using again that φ is even we have

$$P(\varphi) = 8 \int_{\mathbb{R}^2_+} e^{-\varphi} \int_{C_+} e^{-\mathcal{L}\varphi} + 8 \int_{\mathbb{R}_+ \times \mathbb{R}_-} e^{-\varphi} \int_{C_-} e^{-\mathcal{L}\varphi} = 8(F_1(1) + F_2(1)),$$

where $F_1(t) = F_{\mathbb{R}^2_+, C_+}(t)$ and $F_2(t) = F_{\mathbb{R}_+ \times \mathbb{R}_-, C_-}(t)$. We apply Corollary 7 for $A = \mathbb{R}^2_+$ and $B = C_+$ and use the equipartition to get

$$\frac{d}{dt}\left(t^2 F_1(t)\right)_{|t=1} \geq \langle V_{\partial \mathbb{R}^2_+}(\varphi), V_{\partial C_+}(\mathcal{L}\varphi)\rangle = \langle e_1 + e_2, V_1 + V_2\rangle. \tag{15}$$

Applying Corollary 7 for $A = \mathbb{R}_+ \times \mathbb{R}_-$ and $B = C_-$ we also get

$$\frac{d}{dt}\left(t^2 F_2(t)\right)_{|t=1} \geq \langle V_{\partial(\mathbb{R}_+ \times \mathbb{R}_-)}(\varphi), V_{\partial C_-}(\mathcal{L}\varphi)\rangle = \langle e_1 - e_2, V_1 - V_2\rangle. \tag{16}$$

Adding (15) and (16) we obtain

$$32 = \frac{d}{dt}\left(t^2 P(t\varphi)\right)_{|t=1} = 8\frac{d}{dt}\left(t^2 F_1(t) + t^2 F_2(t)\right)_{|t=1} \geq 16(\langle e_1, V_1\rangle + \langle e_2, V_2\rangle) = 32.$$

Hence we have equality in the inequalities (15) and (16). From the equality case of Corollary 7 with $A = \mathbb{R}^2_+$ and $B = C_+$ we deduce that φ is affine on $[-\frac{V_{\partial C_+}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(C_+)}, a]$ for every $a \in \mathbb{R}^2_+ \cap K$. Moreover from Proposition 4 one has $-\frac{V_{\partial C_+}(\mathcal{L}\varphi)}{\mu_{\mathcal{L}\varphi}(C_+)} \in K \cap \mathbb{R}^2_+$. Since φ is affine on $[\frac{V_1+V_2}{\mu_{\mathcal{L}\varphi}(C_+)}, 0]$ and vanishes on B_1^2 then $\frac{V_1+V_2}{\mu_{\mathcal{L}\varphi}(C_+)} \in U$. In the same way we prove that $\frac{V_1-V_2}{\mu_{\mathcal{L}\varphi}(C_-)} \in U$. Hence

$$\mathrm{Conv}\left(B_1^2, \pm\frac{V_1+V_2}{\mu_{\mathcal{L}\varphi}(C_+)}, \pm\frac{V_1-V_2}{\mu_{\mathcal{L}\varphi}(C_-)}\right) \subset U.$$

Then

$$\int_{\mathbb{R}^2_+} e^{-\varphi} \geq |U \cap \mathbb{R}^2_+| \geq \left|\mathrm{Conv}\left(0, e_1, e_2, \frac{V_1+V_2}{\mu_{\mathcal{L}\varphi}(C_+)}\right)\right| = \frac{1}{2}\langle \frac{V_1+V_2}{\mu_{\mathcal{L}\varphi}(C_+)}, e_1+e_2\rangle. \tag{17}$$

Thus we get $\mu_\varphi(\mathbb{R}^2_+)\mu_{\mathcal{L}\varphi}(C_+) \geq \frac{1}{2}\langle V_1+V_2, e_1+e_2\rangle$. Similarly we have $\mu_\varphi(\mathbb{R}_+\times\mathbb{R}_-)\mu_{\mathcal{L}\varphi}(C_-) \geq \frac{1}{2}\langle V_1-V_2, e_1-e_2\rangle$. Adding these two inequalities we obtain

$$2 = \mu_\varphi(\mathbb{R}^2_+)(\mu_{\mathcal{L}\varphi}(C_+) + \mu_{\mathcal{L}\varphi}(C_-)) \geq \langle V_1, e_1\rangle + \langle V_2, e_2\rangle = 2,$$

so we get equality in (17). Hence $\int_{R^2_+} e^{-\varphi} = |U \cap \mathbb{R}^2_+|$ and $\varphi = I_U = I_K$, then $P(\varphi) = 16 = 2P(K)$ therefore $P(K) = 8$. Thus K satisfies the equality case of Mahler inequality in dimension 2, which implies that K is a symmetric parallelogram by [Me] and [Re].

C. If $\varphi_2(s) = \varphi(se_1) = |s|$ and $\varphi_1(s) = \varphi(se_2) = |s|$ for every $s \in \mathbb{R}$, then $\mathcal{L}\varphi_1 = \mathcal{L}\varphi_2 = I_{[-1,1]}$ and $\mathrm{dom}(\mathcal{L}\varphi)$ is bounded. Indeed, let's prove that $\varphi(x) \leq \|x\|_1$. For all $x = (x_1, x_2) \in \mathbb{R}^2_+$

$$\begin{aligned}\varphi(x_1, x_2) &= \varphi\left(\frac{x_1}{x_1+x_2}(x_1+x_2)e_1 + \frac{x_2}{x_1+x_2}(x_1+x_2)e_2\right)\\ &\leq \frac{x_1}{x_1+x_2}\varphi((x_1+x_2)e_1) + \frac{x_2}{x_1+x_2}\varphi((x_1+x_2)e_2) = x_1+x_2.\end{aligned}$$

Applying this in the other quadrants, we get that $\varphi(x) \leq \|x\|_1$ for all $x \in \mathbb{R}^2$. Hence $\mathcal{L}\varphi(x) \geq I_{B^2_\infty}(x)$ and $\mathrm{dom}(\mathcal{L}\varphi) \subset B^2_\infty$ is bounded. Thus there exists a linear invertible map T such that $\psi = (\mathcal{L}\varphi)\circ T$ is strongly equipartioned and $P(\psi) = P(\mathcal{L}\varphi) = 16$ then for all $i = 1, 2$, $P(\psi_i) = 4$ and $\psi_i(x) = I_{[-1,1]}(x)$ or $|x|$. Since $\mathrm{dom}(\psi)$ is bounded, then $\psi_i = I_{[-1,1]}$. From case B one concludes that $\psi = I_K$ where K is a symmetric parallelogram hence $\mathcal{L}\varphi = I_L$ where $L = T^{-1}(K)$ is a symmetric parallelogram. □

5 The inequality in dimension n

Theorem 16. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be even convex such that φ and φ_i are equipartioned for all $1 \leq i \leq n$ and $P(\varphi_i) \geq 4^{n-1}$ then $P(\varphi) \geq 4^n$.*

Proof. We can assume that $\varphi(0) = 0$. From Lemma 9 we reduce to the case where $\mathrm{dom}(\varphi) = \mathrm{dom}(\mathcal{L}\varphi) = \mathbb{R}^n$, φ is differentiable and strictly convex on $\mathbb{R}^n$ and $\mathbb{R}^n = \cup_{\varepsilon\in\{-1,1\}^n}\nabla\varphi(\mathbb{R}^n_\varepsilon)$. Using the equipartition, we have

$$P(\varphi) = \int_{\mathbb{R}^n} e^{-\varphi}\int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi} = 2^n\int_{\mathbb{R}^n_\varepsilon} e^{-\varphi}\int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi} = 2^n\sum_{\varepsilon\in\{-1,1\}^n} F_\varepsilon(1),$$

where $F_\varepsilon(t) = F_{\mathbb{R}^n_\varepsilon,\nabla\varphi(\mathbb{R}^n_\varepsilon)}(t)$ for every $\varepsilon \in \{-1,1\}^n$. Using 4) of Lemma 6 for $A = \mathbb{R}^n_\varepsilon$ and $B = \nabla\varphi(\mathbb{R}^n_\varepsilon)$ we get that for every $\varepsilon \in \{-1,1\}^n$ one has

$$F'_\varepsilon(1) = nF_\varepsilon(1) - \int_{\mathbb{R}^n_\varepsilon} e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)} \mathcal{L}\varphi e^{-\mathcal{L}\varphi} - \int_{\mathbb{R}^n_\varepsilon}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)} e^{-\mathcal{L}\varphi}.$$

Thus, using the equipartition, we have

$$\begin{aligned}\frac{d}{dt}\left(t^nF_\varepsilon(t)\right)_{|t=1} &= nF_\varepsilon(1)+F'_\varepsilon(1)\\ &= 2nF_\varepsilon(1)-\int_{\mathbb{R}^n_\varepsilon}e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}-\int_{\mathbb{R}^n_\varepsilon}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)}e^{-\mathcal{L}\varphi}\\ &= 2nF_\varepsilon(1)-\frac{1}{2^n}\left(\int_{\mathbb{R}^n}e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}-\int_{\mathbb{R}^n}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)}e^{-\mathcal{L}\varphi}\right).\end{aligned}$$

Summing these terms and using again the equipartition, we get

$$\begin{aligned}\sum_\varepsilon\frac{d}{dt}\left(t^nF_\varepsilon(t)\right)_{|t=1} &= 2n\sum_\varepsilon F_\varepsilon(1)-\frac{1}{2^n}\sum_\varepsilon\left(\int_{\mathbb{R}^n}e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}-\int_{\mathbb{R}^n}\varphi e^{-\varphi}\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon)}e^{-\mathcal{L}\varphi}\right)\\ &= 2n\frac{P(\varphi)}{2^n}-\frac{1}{2^n}\left(\int_{\mathbb{R}^n}e^{-\varphi}\int_{\mathbb{R}^n}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}-\int_{\mathbb{R}^n}\varphi e^{-\varphi}\int_{\mathbb{R}^n}e^{-\mathcal{L}\varphi}\right).\end{aligned}$$

Now applying Lemma 6 one has

$$\begin{aligned}\frac{d}{dt}\left(t^nP(t\varphi)\right)_{|t=1} &= nP(\varphi)+\left(nP(\varphi)-\int_{\mathbb{R}^n}e^{-\varphi}\int_{\mathbb{R}^n}\mathcal{L}\varphi e^{-\mathcal{L}\varphi}-\int_{\mathbb{R}^n}\varphi e^{-\varphi}\int_{\mathbb{R}^n}e^{-\mathcal{L}\varphi}\right)\\ &= 2^n\sum_\varepsilon\frac{d}{dt}\left(t^nF_\varepsilon(t)\right)_{|t=1}.\end{aligned}$$

We apply Corollary 7 for $A=\mathbb{R}^n_\varepsilon$ and $B=\nabla\varphi(\mathbb{R}^n_\varepsilon)$ and use the equipartition to get

$$\frac{d}{dt}\left(t^nP(t\varphi)\right)_{|t=1}\geq 2^n\sum_\varepsilon\left(\left\langle V_{\partial\mathbb{R}^n_\varepsilon}(\varphi),V_{\partial\nabla\varphi(\mathbb{R}^n_\varepsilon)}(\mathcal{L}\varphi)\right\rangle+\frac{\mu_\varphi(\mathbb{R}^n)}{2^n}Q_{\partial\nabla\varphi(\mathbb{R}^n_\varepsilon)}(\mathcal{L}\varphi)\right).$$

Notice that

$$\sum_\varepsilon Q_{\partial\nabla\varphi(\mathbb{R}^n_\varepsilon)}(\mathcal{L}\varphi)=\sum_\varepsilon\sum_{i=1}^n Q_{\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_i^\perp)}(\mathcal{L}\varphi)=\sum_{i=1}^n\sum_\varepsilon\int_{\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_i^\perp)}\langle y,n_{\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_i^\perp)}(y)\rangle e^{-\mathcal{L}\varphi(y)}dy,$$

where in $\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_i^\perp)$, the normal is chosen exterior to $\nabla\varphi(\mathbb{R}^n_\varepsilon)$. Since in each hyperplane $e_i^\perp$, each cone $\mathbb{R}^n_\varepsilon\cap e_i^\perp$ appears twice with two opposite orientations thus the sum of these two terms vanishes. Hence the whole sum vanishes. Using that in each $e_i^\perp$ the function φ_i is equipartitioned and Lemma 8 it follows that

$$\begin{aligned}\frac{d}{dt}\left(t^nP(t\varphi)\right)_{|t=1} &\geq 2^n\sum_\varepsilon\sum_{1\leq i,j\leq n}\int_{\mathbb{R}^n_\varepsilon\cap e_i^\perp}e^{-\varphi_i}\langle-\varepsilon_ie_i,V_{\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_j^\perp)}(\mathcal{L}\varphi)\rangle\\ &\geq 2\sum_{1\leq i,j\leq n}\sum_\varepsilon\int_{e_i^\perp}e^{-\varphi_i}\langle-\varepsilon_ie_i,V_{\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_j^\perp)}(\mathcal{L}\varphi)\rangle.\end{aligned}$$

Noticing again that in each hyperplane $e_j^\perp$, each cone $\mathbb{R}^n_\varepsilon\cap e_j^\perp$ appears twice with two opposite orientations one has for every fixed $i\neq j$

$$\sum_\varepsilon\varepsilon_iV_{\nabla\varphi(\mathbb{R}^n_\varepsilon\cap e_j^\perp)}=0.$$

Thus

$$\frac{d}{dt}\left(t^n P(t\varphi)\right)_{|t=1} \geq 2\sum_{i=1}^n \int_{e_i^\perp} e^{-\varphi_i} \sum_{\varepsilon} \langle -\varepsilon_i e_i, V_{\nabla\varphi(\mathbb{R}^n_\varepsilon \cap e_i^\perp)}(\mathcal{L}\varphi)\rangle.$$

Using Lemma 8 we get

$$\frac{d}{dt}\left(t^n P(t\varphi)\right)_{|t=1} \geq 2\sum_{i=1}^n \int_{e_i^\perp} e^{-\varphi_i} \sum_{\varepsilon} \int_{\mathbb{R}^n_\varepsilon \cap e_i^\perp} e^{-\mathcal{L}(\varphi_i)} = 2\sum_{i=1}^n \int_{e_i^\perp} e^{-\varphi_i} \left(2\int_{e_i^\perp} e^{-\mathcal{L}(\varphi_i)}\right) = 4\sum_{i=1}^n P(\varphi_i).$$

Since $P(\varphi_i) \geq 4^{n-1}$ we get

$$\frac{d}{dt}\left(t^n P(t\varphi)\right)_{|t=1} \geq 4^n n.$$

Applying this to $s\varphi$ we deduce that for all $t > 0$

$$\frac{d}{dt}(t^n P(t\varphi)) \geq 4^n n t^{n-1}.$$

Integrating this inequality we conclude that for every $0 < \varepsilon < t$ one has

$$t^n P(t\varphi) = \varepsilon^n P(\varepsilon\varphi) + \int_\varepsilon^t \frac{d}{ds}\left(s^n P(s\varphi)\right) ds \geq 4^n n \int_\varepsilon^t s^{n-1} ds = 4^n(t^n - \varepsilon^n).$$

Letting ε tends to 0 we conclude that $t^n P(t\varphi) \geq 4^n t^n$ and thus $P(\varphi) \geq 4^n$.

□

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