Convection and total variation flow-erratum and improvement

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Abstract

This paper includes an erratum to "Convection and total variation flow" [3], that deals with a nonlinear hyperbolic scalar conservation law, regularised by the total variation flow operator (or 1-Laplacian), and in which a mistake has been written in the convergence proof of the numerical scheme to the continuous entropy solution. For correcting the proof, it is necessary to introduce an additional vanishing viscous term in the scheme. This modification imposes that the whole paper be cast in the framework of discrete and continuous solutions with unbounded support. This new version shows nevertheless a better result than the previous one, since the BV regularity and the compactness of the support of the initial data are no longer assumed.

Keywords: Hyperbolic scalar conservation law, total variation flow, 1-Laplacian, entropy formulation, finite volumes, finite elements

1 Introduction

Let us first briefly recall the problem studied in [3]. We consider a simplified model of unsteady Bingham flow with convection. This simplified model is scalar and consists in seeking $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$ and $\lambda : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ such that

$$\partial_t u + \operatorname{div} F(x,t,u) - \operatorname{div} \lambda = 0 \text{ and } \lambda \in \operatorname{Sgn}(\nabla u), \qquad \text{on} \quad Q_T := \mathbb{R}^d \times (0,T), \tag{1}$$

$$u(x,0) = u_{\text{ini}}(x), \qquad \text{on} \quad \mathbb{R}^d, \tag{2}$$

where $d \in \mathbb{N}^\star$, $T > 0$ is given, $F : \mathbb{R}^d \times (0,T) \times \mathbb{R} \to \mathbb{R}^d$ is divergence-free with respect to the space variables and u_{ini} is a given function from $\mathbb{R}^d$ to $\mathbb{R}$. We denote by Sgn the vector sign function, which is the set-valued map from $\mathbb{R}^d$ to $\mathcal{P}(\mathbb{R}^d)$ defined by

$$\lambda \in \operatorname{Sgn}(\mu) \Leftrightarrow \begin{cases} |\lambda| \le 1 & \text{if } \mu = 0 \\ \lambda = \frac{\mu}{|\mu|} & \text{if } \mu \neq 0, \end{cases}$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$.
Problem (1)-(2) is considered in this work under the following hypotheses, denoted by Hypotheses (HC) in this paper.

(HC1) The initial datum u_{ini} is assumed to belong to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The essential infimum and supremum of u_{ini} are denoted by a_0 and b_0, respectively.

(HC2) The flux function $F \in C^1(\overline{Q_T} \times \mathbb{R}, \mathbb{R}^d)$ is assumed to be divergence-free with respect to the space variables, that is

$$\sum_{i=1}^{d} \frac{\partial F_i}{\partial x_i}(x,t,u) = 0, \qquad \forall x = (x_i)_{i=1,\dots,d} \in \mathbb{R}^d,\ \forall t \in [0,T],\ \forall u \in \mathbb{R}. \tag{3}$$

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Furthermore, $\frac{\partial F}{\partial u}$ is assumed to be locally Lipschitz continuous and such that, for all compact set $K \subset \mathbb{R}$, $\left|\frac{\partial F}{\partial u}\right| \le C_K$ a.e. on $\overline{Q_T} \times K$, where C_K is a constant depending on K.

An entropy formulation for Problem (1)-(2) allows for the uniqueness of the entropy solution. A numerical approximation, based on a splitting scheme, is then proposed in the case where the initial data has a compact support. The hyperbolic flow is treated with finite volumes and the total variation flow is treated using P_1 finite elements. The finite volume mesh is built as a dual mesh of the finite element mesh, which makes simple the interpolation step between the two meshes. For the hyperbolic step (or finite volume step), we choose an explicit time discretisation. For the total variation flow step (or finite element step), we define an implicit scheme accounting for the non-regularity of the total variation flow operator. To guarantee the maximum principle, which is essential for the stability of the scheme, we use a non-obtuse finite element mesh.

Then the point is to prove the convergence of this numerical scheme to the unique entropy solution of Problem (1)-(2). Unfortunately, we made a mistake in the course of this proof. In [3, Proposition 4.2], the point was to derive an entropy inequality for the total variation approximation by the finite element step, holding for any regular convex entropy. This inequality, which holds at the continuous level, is also immediate at the discrete level if we restrict ourselves to the 1D case or to quadratic entropies. But we need here all the entropies in order to prove that the limit of the numerical approximation is solution to the entropy formulation. So we intended to prove such an inequality, using equation [3, (4.9)]. Unfortunately in this equation, the time and the space terms are not in agreement, the time test function is fixed and the space test function is let to be selected further, whereas it should in fact be the same function.

Turning to the correction of this wrong proof, we understood that the inequality that we tried to prove is in fact not true in the general multidimensional case (as written above, it holds in the 1D case or using quadratic entropies, but a two-dimensional counter-example is provided in Appendix A). We have therefore been led to modify the numerical scheme, by introducing a vanishing viscous term in the scheme (thanks to this term, the discrete solution is regular enough for passing to the limit in the above mentioned inequality). The compact support property, that was holding without this viscous term, is then no longer available. We have been therefore led to rewrite the whole numerical part of [3], accounting for solutions with non-compact support. Fortunately, this change does not imply to rewrite the uniqueness proof for the entropy solution of [3], since it does not account for the fact that the support of the solution is compact. Let us summarise in the next table the main differences between the original paper and this erratum.

	[3]	Erratum
support of the solution	bounded	bounded or unbounded
initial data	in L^∞ with bounded variations and bounded support	in $L^\infty \cap L^1$
finite element mass lumping	different from finite volume step	identical to finite volume step
finite element step	without additional viscosity	with additional viscosity
estimates on discrete solution	L^∞ and bounded support	$L^\infty \cap L^1$
inverted CFL	needed for keeping bounded support for discrete solution	no longer needed
convergence proof	uncorrect	use of the viscous term for passing to the limit

This erratum is organised as follows. In section 2, the concept of entropy solution for Problem (1)-(2) is redefined with respect to unbounded support for the solution. Section 3 describes the numerical approximation and the well-posedness and the maximum principle are proved. A priori estimates on the discrete solutions are provided in Section 4 and a discrete entropy formulation is established in Section 5. The convergence of the numerical approximation (and thus the existence of an entropy solution in the new framework) is finally proved in Section 6 using the results of the two previous sections. Appendix A provides a counter-example, justifying that we had to modify the scheme by addition of a vanishing viscous term.

2 Entropy formulation for nonlinear hyperbolic equation with total variation flow

In the usual entropy formulations of scalar conservation laws, the admissible entropies are the C^1 convex functions or the so-called Kruzhkov entropies. Let us recall that the Kruzhkov entropies are the functions $|\cdot - \kappa|$

with $\kappa \in \mathbb{R}$, the corresponding entropy fluxes being the functions $F(\cdot \vee \kappa) - F(\cdot \wedge \kappa)$, where $a \vee b$ denotes the maximum of a and b and $a \wedge b$ denotes the minimum of a and b.
The entropy formulation of the problem (1)-(2), owing to the term $\operatorname{div} \operatorname{Sgn}(\nabla u)$, requires more regular entropies.

Definition 2.1: Under Hypotheses (HC), an admissible entropy is a convex Lipschitz continuous function $\eta \in C^\infty(\mathbb{R})$. The corresponding entropy flux is the locally Lipschitz continuous function $\Phi \in C^0(\overline{Q_T} \times \mathbb{R}, \mathbb{R}^d)$ such that

$$\Phi(x,t,u) \;=\; \frac{1}{2}\int_{a_0}^{u} \eta'(s)\frac{\partial F}{\partial u}(x,t,s)\,ds + \frac{1}{2}\int_{b_0}^{u} \eta'(s)\frac{\partial F}{\partial u}(x,t,s)\,ds + \frac{1}{2}\left(\eta'(a_0)F(x,t,a_0) + \eta'(b_0)F(x,t,b_0)\right).$$

We then have $\frac{\partial \Phi}{\partial u}(x,t,u) = \eta'(u)\frac{\partial F}{\partial u}(x,t,u)$, and we remark that, since the flux function F is divergence-free with respect to the space variables, the entropy flux is divergence-free in the same sense as well.

Remark 2.2 (Consistency with Kruzhkov entropy pairs): Definition 2.1 is such that, for $\eta = |\cdot - \kappa|$ with $a_0 \le \kappa \le b_0$, then $\Phi(x,t,\cdot) = F(x,t,\cdot \vee \kappa) - F(x,t,\cdot \wedge \kappa)$. This consistency property is used in the course of the proof of the uniqueness theorem. Moreover, for any convex function $\eta \in C^\infty(\mathbb{R})$, letting Φ be given by Definition 2.1, integrate by parts in $[a_0,u]$ and $[u,b_0]$ shows that the following relations hold for a.e. $(x,t) \in \overline{Q_T}$:

$$\begin{aligned}
\eta(u) = &\quad \frac{1}{2}\int_{a_0}^{b_0} \eta''(s)|u-s|\,ds + \frac{1}{2}((\eta'(a_0)+\eta'(b_0))u + \eta(a_0) - \eta'(a_0)a_0 + \eta(b_0) - \eta'(b_0)b_0),\\
\Phi(x,t,u) = &\quad \frac{1}{2}\int_{a_0}^{b_0} \eta''(s)\left(F(x,t,u\vee s) - F(x,t,u\wedge s)\right)\,ds + \frac{\eta'(a_0)+\eta'(b_0)}{2}F(x,t,u),\ \forall u \in [a_0,b_0]. \quad (4)
\end{aligned}$$

Regular entropy-entropy flux pairs in the sense of Definition 2.1 can now be used in the following definition of an entropy solution. This definition is classically resulting from the vanishing viscosity method (the computations are detailed in [3, Section 1.3]): assuming that, for all $\epsilon > 0$, there exists $(u_\epsilon, \lambda_\epsilon)$ with

$$\partial_t u_\epsilon + \operatorname{div} F(x,t,u_\epsilon) - \operatorname{div} \lambda_\epsilon - \epsilon \Delta u_\epsilon = 0 \text{ and } \lambda_\epsilon \in \operatorname{Sgn}(\nabla u_\epsilon),$$

we multiply the preceding equation by $\eta'(u_\epsilon)\varphi$ for a given admissible entropy-entropy flux pairs (η,Φ) and for a nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$, we integrate over Q_T and we let $\epsilon \to 0$. Inequality (5) is then obtained, owing to $\eta'' \ge 0$ and to the semi-continuity of the total variation.

Definition 2.3 (Entropy solution): Under Hypotheses (HC), a function $u \in L^\infty(Q_T) \cap L^1(0,T;BV(\mathbb{R}^d))$ is said to be an entropy solution of (1)-(2) if there exists $\lambda \in L^\infty(Q_T)^d$, with $|\lambda| \le 1$ almost everywhere on Q_T, such that, for all admissible entropy-entropy flux pairs (η,Φ) in the sense of Definition 2.1 and all nonnegative test functions $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$,

$$\int_{Q_T} \Big(\eta(u)\partial_t\varphi + \big(\Phi(x,t,u) - \lambda\eta'(u)\big)\cdot\nabla\varphi\Big)\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} \varphi\,\big|D[\eta'(u)]\big|\mathrm{d}t + \int_{\mathbb{R}^d} \eta(u_{\rm ini}(x))\varphi(x,0)\ \mathrm{d}x \ge 0. \qquad (5)$$

Since η' is in $C^1(\mathbb{R})$, the function $\eta'(u)$ is in $L^1(0,T;BV(\mathbb{R}^d))$. Therefore, the term $\int_{Q_T}\varphi\,\big|D[\eta'(u)]\big|\mathrm{d}t$ is meaningful. The function λ, which is not necessarily unique, is called a multiplier by analogy with a Lagrange multiplier. More details on functions with bounded variation are provided in [3, Section 1.1].

Theorem 2.4: Under Hypotheses (HC), there exists one and only one entropy solution of (1)-(2) in the sense of Definition 2.3.

The proof of the uniqueness part of the preceding theorem is not modified with respect to that of [3, Theorem 1.3]. The proof of the existence part is done by the proof that the numerical scheme used below is converging.

3 Numerical approximation

3.1 Notation and hypotheses

The finite element mesh, denoted by $\mathcal{T}_h$, is a conforming simplicial mesh of $\mathbb{R}^d$ of size h: $\mathcal{T}_h$ is a (necessarily countable) set of disjoint open simplices such that $\bigcup_{K\in\mathcal{T}_h}\overline{K} = \mathbb{R}^d$, and h is the maximum value of the diameter of all $K \in \mathcal{T}_h$. The mesh is conforming in the sense that, for two distinct elements K, L of $\mathcal{T}_h$, $\overline{K}\cap\overline{L}$ is either empty or a simplex included in an affine subset of $\mathbb{R}^d$ with dimension strictly lower than d, whose vertices are simultaneously vertices of K and L. Therefore the set of the vertices of the mesh is countable as well, and denoted by $\{x_p,\ p \in \mathbb{N}\}$. In order to ensure the maximum principle, each element of $\mathcal{T}_h$ is assumed to be nonobtuse; we recall that a simplex is said to be nonobtuse if the angles between any two facets are less than or equal to $\pi/2$. For any $K \in \mathcal{T}_h$, we denote by $\mathcal{V}_K \subset \mathbb{N}$ the set of the $d+1$ indices of the vertices of K, and by $\mathcal{E}_K$ the set of the $d+1$ faces of K.

The finite volume mesh, denoted by $\mathcal{D}_h$, is a polyhedral mesh of $\mathbb{R}^d$ such that the interface between two cells is a finite union of faces. The mesh $\mathcal{D}_h$ is a dual mesh of $\mathcal{T}_h$ in the sense that each cell of $\mathcal{D}_h$ contains one and only one node of $\mathcal{T}_h$. For any $p \in \mathbb{N}$, the cell of $\mathcal{D}_h$ containing the node x_p is denoted by Q_p. We assume that

$$\forall p \in \mathbb{N},\ Q_p \subset \bigcup_{K \in \mathcal{T}_h \text{ s.t. } p \in \mathcal{V}_K} \overline{K}.$$

Let us introduce some additional notation about $\mathcal{D}_h$: $\mathcal{N}_p$ is the set containing the indices of the neighbouring cells of Q_p, $\mathcal{E}_h$ is the set of couples (p,q) such that Q_p and Q_q are neighbours and $p < q$, $\sigma_{p,q}$ is the interface between two neighbour cells Q_p and Q_q, $\nu_{p,q}$ is the unit normal vector to $\sigma_{p,q}$ pointing toward Q_q, m_p is the measure of Q_p, $m_{p,q}$ is the measure of $\sigma_{p,q}$.

In our scheme, the unknown function is simultaneously reconstructed from the values $v_h = (v_p)_{p\in\mathbb{N}}$ at the vertices using a continuous piecewise affine reconstruction denoted by $\widehat{v}_h$, and using a piecewise constant reconstruction denoted by $\overline{v}_h$:

$$\begin{aligned} \forall v_h \in \mathbb{R}^{\mathbb{N}},\ \widehat{v}_h \in C^0(\mathbb{R}^d)\ ;\ \widehat{v}_{h|K} \text{ is affine for each } K \in \mathcal{T}_h,\ \widehat{v}_h(x_p) = v_p,\ \forall p \in \mathbb{N}, \\ \overline{v}_h \in L^1_{\text{loc}}(\mathbb{R}^d)\ ;\ \overline{v}_{h|Q_p} = v_p,\ \forall p \in \mathbb{N}. \end{aligned}$$

Letting $K \in \mathcal{T}_h$, if we denote by $\{x_p\}_{p\in\mathcal{V}_K}$ the vertices of K and by $\{\phi_p\}_{p\in\mathcal{V}_K}$ the corresponding Lagrange basis function, we can write, for any $u_h, v_h \in \mathbb{R}^{\mathbb{N}}$,

$$\nabla\widehat{v}_{h|K} \cdot \nabla\widehat{u}_{h|K} = \sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} v_p u_q \nabla\phi_{p|K} \cdot \nabla\phi_{q|K}.$$

Using the fact that $\sum_{p\in\mathcal{V}_K} \phi_{p|K} = 1$, and thus $\sum_{p\in\mathcal{V}_K} \nabla\phi_{p|K} = 0$, the preceding equation can be rewritten as

$$\nabla\widehat{v}_{h|K} \cdot \nabla\widehat{u}_{h|K} = \sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} u_q(v_p - v_q) \nabla\phi_{p|K} \cdot \nabla\phi_{q|K}.$$

Exchanging the roles of p and q, we get

$$\nabla\widehat{v}_{h|K} \cdot \nabla\widehat{u}_{h|K} = \sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} u_p(v_q - v_p) \nabla\phi_{p|K} \cdot \nabla\phi_{q|K}.$$

Adding the two preceding relations provides

$$2\nabla\widehat{v}_{h|K} \cdot \nabla\widehat{u}_{h|K} = -\sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} (u_p - u_q)(v_p - v_q) \nabla\phi_{p|K} \cdot \nabla\phi_{q|K},$$

and therefore

$$\int_K \nabla\widehat{v}_h(x) \cdot \nabla\widehat{u}_h(x)\mathrm{d}x = \sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} T^K_{pq}(u_p - u_q)(v_p - v_q) \text{ with } T^K_{pq} = -\frac{1}{2}\int_K \nabla\phi_p(x) \cdot \nabla\phi_q(x)\mathrm{d}x. \tag{6}$$

Since the simplex K is nonobtuse, we have the standard inequality (see for example [4])

$$\nabla\phi_{p|K} \cdot \nabla\phi_{q|K} \le 0 \text{ and } T^K_{pq} \ge 0, \qquad \forall p, q \in \mathcal{V}_K, p \neq q. \tag{7}$$

Let us observe that, for any $u_h \in \mathbb{R}^{\mathbb{N}}$, we have, denoting for a.e. $x \in \mathbb{R}^d$ by $p(x) \in \mathbb{N}$ such that $x \in Q_p$,

$$\text{for a.e. } x \in \mathbb{R}^d,\ |\widehat{u}_h(x) - \overline{u}_h(x)| = |(x - x_{p(x)}) \cdot \nabla\widehat{u}_h(x)| \le h|\nabla\widehat{u}_h(x)|, \tag{8}$$

which implies, if $\nabla\widehat{u}_h \in L^1(\mathbb{R}^d)^d$,

$$\|\overline{u}_h - \widehat{u}_h\|_{L^1(\mathbb{R}^d)} \le h\|\nabla\widehat{u}_h\|_{L^1(\mathbb{R}^d)^d}. \tag{9}$$

For the finite volume step, we need numerical fluxes $F^n_{p,q}(u^n_p, u^n_q)$ between two neighbouring cells p and q at time t^n, function of u^n_p and u^n_q, the respective approximations of the unknown function in Q_p and Q_q at time t^n. We require that the family of numerical fluxes $(F^n_{p,q})_{p,q,n\in\mathbb{N}}$ is admissible and consistent with the flux F in the sense of the two definitions below.

Definition 3.1: A family of numerical fluxes $(F^n_{p,q})_{p,q,n\in\mathbb{N}}$ is said to be admissible if

- $F^n_{p,q} \in C^0(\mathbb{R}^2)$ and there exists $L > 0$ such that, for all $p \in \mathbb{N}$ and $q \in \mathcal{N}_p$ and for all $n \in \{0, ..., N-1\}$, the function $F^n_{p,q}$ is Lipschitz continuous with the constant $m_{p,q}L$ with respect to each of its variables.
- $F^n_{p,q}$ is monotone, in the sense that it is non-decreasing with respect to its first argument and non-increasing with respect to its second argument,
- $F^n_{p,q}$ is conservative, i.e. $F^n_{p,q}(u,v) = -F^n_{q,p}(v,u)$ for all $(u,v) \in \mathbb{R}^2$.

Definition 3.2: Let F be a flux function. A family of numerical fluxes $(F^n_{p,q})_{p,q,n\in\mathbb{N}}$ is said to be consistent with F if

$$F^n_{p,q}(u,u) = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}} F(x,t,u)\cdot \nu_{p,q}\ \mathrm{d}\gamma(x)\mathrm{d}t, \qquad \forall u \in \mathbb{R}^2. \tag{10}$$

Let us give two examples of consistent and admissible families of numerical fluxes:

- the Godunov numerical flux, defined by:

$$F^n_{p,q}(u,v) = \begin{cases} \displaystyle\min_{u\le s\le v} \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}} F(x,t,s)\cdot\nu_{p,q}\ \mathrm{d}\gamma(x)\mathrm{d}t & \text{if } u \le v \\ \displaystyle\max_{v\le s\le u} \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}} F(x,t,s)\cdot\nu_{p,q}\ \mathrm{d}\gamma(x)\mathrm{d}t & \text{if } v \le u, \end{cases}$$

- and the Rusanov numerical flux, defined, denoting by L_F a Lipschitz constant of F, by:

$$F^n_{p,q}(u,v) = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}} F(x,t,\frac{u+v}{2})\cdot\nu_{p,q}\ \mathrm{d}\gamma(x)\mathrm{d}t + \frac{m_{p,q}L_F}{2}(u-v).$$

The following lemma, which is a discrete version of the divergence theorem, will be used below.

Lemma 3.3: Let $(F^n_{p,q})_{p,q,n\in\mathbb{N}}$ be a family of numerical fluxes consistent with a flux function F in the sense of Definition 3.2. If F is divergence-free, then

$$\forall u \in \mathbb{R},\ \sum_{q\in\mathcal{N}_p} F_{p,q}(u,u) = 0. \tag{11}$$

Proof Owing to (10) and to the divergence theorem, we have, for a given $u \in \mathbb{R}$,

$$\sum_{q\in\mathcal{N}_p} F_{p,q}(u,u) = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\sum_{q\in\mathcal{N}_p}\int_{\sigma_{p,q}} F(x,t,u)\cdot\nu_{p,q}\ \mathrm{d}\gamma(x)\mathrm{d}t = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_p \sum_{i=1}^d \frac{\partial F_i}{\partial x_i}(x,t,u)\ \mathrm{d}x\mathrm{d}t,$$

which vanishes owing to the assumption that F is divergence-free. □

3.2 Description of the numerical scheme and well-posedness

We consider a family of discretisations $(\mathcal{F}_{h,\delta t})_{h,\delta t>0}$ – by discretisation, we mean a finite element mesh $\mathcal{T}_h$, a finite volume mesh $\mathcal{D}_h$, a time step δt, a family of numerical fluxes $(F^n_{p,q})_{p,q,n\in\mathbb{N}}$. We assume that the following hypotheses, denoted in the following by Hypotheses (HD), are satisfied uniformly by any element $\mathcal{F}_{h,\delta t}$ of the family.

(HD1) There exists $\alpha > 0$ such that, for all $p \in \mathbb{N}$, $m_p \ge \alpha\ h^d$, $\alpha\ |\partial Q_p| \le h^{d-1}$, and $|K| \ge \alpha\ h^d$, for all $K \in \mathcal{T}_h$.

(HD2) There exists an admissible family of numerical fluxes $(F^n_{p,q})_{p,q,n\in\mathbb{N}}$ in the sense of Definition 3.1, which is consistent with F in the sense of Definition 3.2. The constant L in Definition 3.1 is assumed to be independent of the discretisation.

(HD3) The time interval $[0,T]$ is divided into N equal intervals of length δt, such that the following CFL condition holds

$$\delta t \le \frac{\alpha^2 h}{L}. \tag{12}$$

Note that, thanks to Hypothesis (HD1), the condition (12) implies that

$$\delta t \leq \frac{1}{L}\frac{m_p}{\sum_{q\in\mathcal{N}_p} m_{p,q}}, \ \forall p \in \mathbb{N}. \tag{13}$$

The scheme for approximating (1)-(2) is given by:

- Initialisation of $(u_p^0)_{n\in\mathbb{N}}$ such that $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$:

$$u_p^0 = \frac{1}{m_p}\int_{Q_p} u_{\rm ini}(x)\mathrm{d}x, \qquad \forall p \in \mathbb{N}. \tag{14}$$

- Finite volume step. Letting $(u_p^n)_{p\in\mathbb{N}}$ such that $\overline{u}_h^n \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$, seek $(u_p^{n+\frac{1}{2}})_{p\in\mathbb{N}}$ such that $\overline{u}_h^{n+\frac{1}{2}} \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and

$$m_p\frac{u_p^{n+\frac{1}{2}} - u_p^n}{\delta t} + \sum_{q\in\mathcal{N}_p} F_{p,q}^n(u_p^n, u_q^n) = 0, \qquad \forall p \in \mathbb{N}. \tag{15}$$

- Finite element step. Let $\theta \in C^0((0,+\infty))$ be a positive function such that

$$\lim_{h\to 0}\theta(h) = \lim_{h\to 0}\frac{h}{\theta(h)} = 0, \tag{16}$$

  and let us define

$$X_h = \{v_h \in \mathbb{R}^\mathbb{N};\ \nabla\widehat{v}_h \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d),\ \overline{v}_h \in L^2(\mathbb{R}^d)\}$$
$$\Lambda_h := \{\mu_h \in L^\infty(\mathbb{R}^d)^d; \mu_{h|K} \text{ is constant for each } K\in\mathcal{T}_h\}.$$

  Then the finite element step consists in seeking $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h\times\Lambda_h$ such that

$$\int_{\mathbb{R}^d}\frac{\overline{u}_h^{n+1} - \overline{u}_h^{n+\frac{1}{2}}}{\delta t}\overline{v}_h\mathrm{d}x + \int_{\mathbb{R}^d}(\lambda_h^{n+1} + \theta(h)\nabla\widehat{u}_h^{n+1})\cdot\nabla\widehat{v}_h\,\mathrm{d}x = 0,\ \forall v_h\in X_h, \tag{17}$$

$$\lambda_h^{n+1} \in \mathrm{Sgn}(\nabla\widehat{u}_h^{n+1}). \tag{18}$$

An example of such a function θ is $\theta(h) = h^\gamma$ with $0<\gamma<1$. Note that, if $d=1$, it is possible to let $\theta(h)=0$. For each discretisation $\mathcal{F}_{h,\delta t}$, we define the approximate solutions $\widehat{u}_{h,\delta t} : Q_T \to \mathbb{R}$, $\overline{u}_{h,\delta t} : Q_T\to\mathbb{R}$, and $\lambda_{h,\delta t} : Q_T \to \mathbb{R}$ such that

$$\widehat{u}_{h,\delta t}(\cdot,t) := \widehat{u}_h^{n+1} \qquad \text{if } t\in(t^n,t^{n+1}],$$
$$\overline{u}_{h,\delta t}(\cdot,t) := \overline{u}_h^{n+1} \qquad \text{if } t\in(t^n,t^{n+1}],$$
$$\lambda_{h,\delta t}(\cdot,t) := \lambda_h^{n+1} \qquad \text{if } t\in(t^n,t^{n+1}].$$

The proposition below proves that the scheme has at least one solution, which is unique with respect to the unknown u_h.

Proposition 3.4: Let us assume Hypotheses (HC) and Hypotheses (HD). Then there exists at least one solution to Scheme (14)-(18) such that $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and, for all $n\in\mathbb{N}$, $\overline{u}_h^{n+\frac{1}{2}}, \overline{u}_h^{n+1} \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and $(u_h^{n+1},\lambda_h^{n+1}) \in X_h\times\Lambda_h$. Moreover, $\overline{u}_h^0$ and, for all $n\in\mathbb{N}$, $\overline{u}_h^{n+\frac{1}{2}}$ and u_h^{n+1} are unique and, for all $p\in\mathbb{N}$, $a_0 \le u_p^{n+\frac{1}{2}} \le b_0$ and $a_0\le u_p^n\le b_0$.

Proof Thanks to (14), we get from Hypothesis (HC1) that

$$\|\overline{u}_h^0\|_{L^1(\mathbb{R}^d)} \le \|u_{\rm ini}\|_{L^1(\mathbb{R}^d)},$$

and $a_0\le u_p^0\le b_0$ for all $p\in\mathbb{N}$, which completes the proof that $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$.

For any $n \in \mathbb{N}$, we get from Propositions 3.5 and 3.7, assuming $\overline{u}_h^n \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $a_0 \le u_p^n \le b_0$ for all $p \in \mathbb{N}$, that $\overline{u}_h^{n+\frac12} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $a_0 \le u_p^{n+\frac12} \le b_0$ for all $p \in \mathbb{N}$. Using Proposition 3.8, we get the existence of $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ such that (18) holds, and we get that u_h^{n+1} is unique. It now suffices to apply Proposition 3.9, for obtaining that $\overline{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $a_0 \le u_p^{n+1} \le b_0$ for all $p \in \mathbb{N}$. □

Let us now state and prove the propositions used in the proof of the preceding result. We have first the following result.

Proposition 3.5: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $n \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and a family $(u_p^n)_{p\in\mathbb{N}}$ be given such that $a_0 \le u_p^n \le b_0$ for all $p \in \mathbb{N}$. Let $(u_p^{n+\frac12})_{p\in\mathbb{N}}$ be given by (15). Then there holds

$$u_p^{n+\frac12} \vee \kappa \le u_p^n \vee \kappa - \frac{\delta t}{m_p} \sum_{q\in\mathcal{N}_p} F_{p,q}^n(u_p^n \vee \kappa, u_q^n \vee \kappa),\ \forall p \in \mathbb{N}, \tag{19}$$

$$u_p^{n+\frac12} \wedge \kappa \ge u_p^n \wedge \kappa - \frac{\delta t}{m_p} \sum_{q\in\mathcal{N}_p} F_{p,q}^n(u_p^n \wedge \kappa, u_q^n \wedge \kappa),\ \forall p \in \mathbb{N}, \tag{20}$$

$$m_p \frac{|u_p^{n+\frac12} - \kappa| - |u_p^n - \kappa|}{\delta t} + \sum_{q\in\mathcal{N}_p} \left(F_{p,q}^n(u_p^n \vee \kappa, u_q^n \vee \kappa) - F_{p,q}^n(u_p^n \wedge \kappa, u_q^n \wedge \kappa)\right) \le 0,\ \forall p \in \mathbb{N}. \tag{21}$$

Consequently $a_0 \le u_p^{n+\frac12} \le b_0$ for all $p \in \mathbb{N}$.

Proof The proof of this proposition is done in [5, Lemma 3] or [7, Lemma 27.1]. We recall it since it is very brief. We consider the function $H_p^n\ :\ \mathbb{R}^{1+\#\mathcal{N}_p} \to \mathbb{R}$, defined by

$$H_p^n(a, (b_q)_{q\in\mathcal{N}_p}) = a - \frac{\delta t}{m_p} \sum_{q\in\mathcal{N}_p} F_{p,q}^n(a, b_q).$$

We observe that, for any $a' > a$, there holds

$$H_p^n(a', (b_q)_{q\in\mathcal{N}_p}) - H_p^n(a, (b_q)_{q\in\mathcal{N}_p}) \ge (a' - a)(1 - \frac{\delta t L \sum_{q\in\mathcal{N}_p} m_{p,q}}{m_p}) \ge 0,$$

thanks to Definition 3.1 of admissible fluxes and to condition (13) implied by (12). Therefore the function H_p^n is non-decreasing with respect to all its arguments. Noticing that $\kappa = H_p^n(\kappa, (\kappa)_{q\in\mathcal{N}_p})$ and $u_p^{n+\frac12} = H_p^n(u_p^n, (u_q^n)_{q\in\mathcal{N}_p})$, we get that $\kappa \le H_p^n(u_p^n \vee \kappa, (u_q^n \vee \kappa)_{q\in\mathcal{N}_p})$ and $u_p^{n+\frac12} \le H_p^n(u_p^n \vee \kappa, (u_q^n \vee \kappa)_{q\in\mathcal{N}_p})$, which implies (19). The proof of (20) is similar, and (21) is obtained by the difference between (19) and (20).
Letting $\kappa = b_0$ in (19) and using (11) on one hand, letting $\kappa = a_0$ in (20) on the other hand complete the proof that $a_0 \le u_p^{n+\frac12} \le b_0$ for all $p \in \mathbb{N}$. □
We then deduce the following result.

Proposition 3.6: Let us assume Hypotheses (HC) and Hypotheses (HD). Let (η, Φ) be an entropy-entropy flux pair in the sense of Definition 2.1, and let $n \in \mathbb{N}$ be given. Then, the family $(\Phi_{p,q}^n)_{p,q,n\in\mathbb{N}}$ of admissible numerical fluxes defined by

$$\Phi_{p,q}^n(x,y) := \frac12 \int_{a_0}^{b_0} \eta''(\kappa) \left(F_{p,q}^n(x \vee \kappa, y \vee \kappa) - F_{p,q}^n(x \wedge \kappa, y \wedge \kappa)\right) d\kappa + \frac{\eta'(a_0) + \eta'(b_0)}{2} F_{p,q}^n(x,y), \tag{22}$$

is consistent with Φ in the sense of Definition 3.2, and is such that, if $u_h^{n+\frac12} \in \mathbb{R}^{\mathbb{N}}$ is obtained from $u_h^n \in \mathbb{R}^{\mathbb{N}}$ by (15) with $a_0 \le u_p^n \le b_0$, for all $p \in \mathbb{N}$, then

$$m_p \frac{\eta(u_p^{n+\frac12}) - \eta(u_p^n)}{\delta t} + \sum_{q\in\mathcal{N}_p} \Phi_{p,q}^n(u_p^n, u_q^n) \le 0, \qquad \forall n \in \{0, ..., N-1\}, \quad \forall p \in \mathbb{N}. \tag{23}$$

Furthermore, there is a constant L', depending only on L, η, a_0 and b_0, such that, for all $(p,q) \in \mathcal{E}_h$ and for all $n \in \{0, ..., N-1\}$, the function $\Phi_{p,q}^n$ is Lipschitz continuous with respect of each of its variables with the constant $m_{p,q} L'$.

Proof Thanks to Proposition 3.5, we have that all values u_p^n and $u_p^{n+\frac{1}{2}}$ belong to $[a_0, b_0]$, which enables to use relations (4). The consistency of $\Phi_{p,q}^n$ with Φ is a consequence of (4) and of the consistency of $F_{p,q}^n$ with F. Multiplying (21) by $\eta''(\kappa)$ and integrating on $\kappa \in [a_0, b_0]$ implies (23), owing to (4) and (22). □
From the above result, we get the following one.

Proposition 3.7: Let us assume Hypotheses (HC) and Hypotheses (HD). Assume that, for a given $n \in \mathbb{N}$, $a_0 \le u_p^n \le b_0$ for all $p \in \mathbb{N}$ and $\overline{u}_h^n \in L^1(\mathbb{R}^d)$. Then for all entropy η in the sense of Definition 2.1 such that $\eta(0) = 0$ and $\eta'(0) = 0$,

$$\int_{\mathbb{R}^d} \eta(\overline{u}_h^{n+\frac{1}{2}}(x))\mathrm{d}x \le \int_{\mathbb{R}^d} \eta(\overline{u}_h^n(x))\mathrm{d}x, \tag{24}$$

and consequently

$$\|\overline{u}_h^{n+\frac{1}{2}}\|_{L^1(\mathbb{R}^d)} \le \|\overline{u}_h^n\|_{L^1(\mathbb{R}^d)} \text{ and } \|\overline{u}_h^{n+\frac{1}{2}}\|_{L^2(\mathbb{R}^d)} \le \|\overline{u}_h^n\|_{L^2(\mathbb{R}^d)}. \tag{25}$$

Proof We get, from (23) and applying Lemma 3.3,

$$m_p \frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \sum_{q\in\mathcal{N}_p} (\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0,0)) \le 0, \qquad \forall n \in \{0, ..., N-1\}, \quad \forall p \in \mathbb{N},$$

For $\varepsilon > 0$, we multiply the preceding inequality by $\delta t \exp(-\varepsilon|x_p|)$ and we sum the result on $p \in \mathbb{N}$. We get

$$\sum_{p\in\mathbb{N}} m_p(\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)) \exp(-\varepsilon|x_p|) + \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \left(\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0,0)\right) \exp(-\varepsilon|x_p|) \le 0.$$

Defining, for any $(p,q) \in \mathcal{E}_h$, the point $x_{pq} = \frac{1}{2}(x_p + x_q)$ and using the property $\Phi_{p,q}^n(u,v) = -\Phi_{q,p}^n(v,u)$ for all $(u,v) \in [a_0, b_0]^2$ (which therefore implies $\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0,0) + \Phi_{q,p}^n(u_q^n, u_p^n) - \Phi_{q,p}^n(0,0) = 0$), we get

$$\left(\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0,0)\right) \exp(-\varepsilon|x_p|) + \left(\Phi_{q,p}^n(u_q^n, u_p^n) - \Phi_{q,p}^n(0,0)\right) \exp(-\varepsilon|x_q|) = A_{pq}^n + A_{qp}^n,$$

with

$$A_{pq}^n = \left(\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0,0)\right) (\exp(-\varepsilon|x_p|) - \exp(-\varepsilon|x_{pq}|)).$$

Hence the two preceding relations imply

$$\sum_{p\in\mathbb{N}} m_p(\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)) \exp(-\varepsilon|x_p|) + \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} A_{pq}^n \le 0,$$

Observing that Proposition 3.6 implies

$$|\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(0,0)| \le L' m_{pq}(|u_p^n| + |u_q^n|),$$

we get, using $|\exp(-\varepsilon|x_p|) - \exp(-\varepsilon|x_{pq}|)| \le \varepsilon||x_p| - |x_{pq}|| \le \varepsilon\frac{1}{2}|x_p - x_q|$, that

$$A_{pq}^n \ge -\frac{L'}{2} m_{pq}(|u_p^n| + |u_q^n|)\varepsilon h.$$

This leads to

$$\sum_{p\in\mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}) \exp(-\varepsilon|x_p|) \le \sum_{p\in\mathbb{N}} m_p \eta(u_p^n) \exp(-\varepsilon|x_p|) + \varepsilon\frac{L'}{2} \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} m_{pq} h(|u_p^n| + |u_q^n|).$$

Since we have

$$\frac{1}{2} \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} m_{pq} h(|u_p^n| + |u_q^n|) = \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} m_{pq} h|u_p^n| = \sum_{p\in\mathbb{N}} |u_p^n|\ |\partial Q_p| h,$$

we can write, using Hypothesis (HD1),

$$\sum_{p\in\mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}) \exp(-\varepsilon|x_p|) \le \sum_{p\in\mathbb{N}} m_p \eta(u_p^n) + \varepsilon\frac{L'}{\alpha^2} \sum_{p\in\mathbb{N}} m_p |u_p^n|.$$

Letting $\varepsilon \to 0$, we get by monotonous convergence (recall that thanks to the hypothesis $\overline{u}_h^n \in L^1(\mathbb{R}^d)$, we also have $\eta(\overline{u}_h^n) \in L^1(\mathbb{R}^d)$ and $\eta(s) \geq \eta(0) = 0$),

$$\sum_{p\in\mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}) \leq \sum_{p\in\mathbb{N}} m_p \eta(u_p^n),$$

which concludes the proof of (24).
Letting η tend to $\eta(s) = |s|$ and letting $\eta(s) = s^2$ allow for concluding (25).
□
The proposition below proves that the finite element step is well-posed, provided that $\overline{u}_h^{n+\frac{1}{2}} \in L^2(\mathbb{R}^d)$, which holds if $\overline{u}_h^{n+\frac{1}{2}} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, and gives a variational characterisation of u_h^{n+1}.

Proposition 3.8: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $n \in \mathbb{N}$ be given and let us assume that $\overline{u}_h^{n+\frac{1}{2}} \in L^2(\mathbb{R}^d)$. Then, equations (17)-(18) admit a solution $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$. Furthermore, u_h^{n+1} is unique and is the minimiser of the functional $J_h^{n+1} : X_h \to \mathbb{R}$ defined by

$$J_h^{n+1}(v_h) := \frac{1}{2\delta t}\int_{\mathbb{R}^d}\left(\overline{v}_h - \overline{u}_h^{n+\frac{1}{2}}\right)^2 \mathrm{d}x + \int_{\mathbb{R}^d}(|\nabla\widehat{v}_h| + \frac{1}{2}\theta(h)|\nabla\widehat{v}_h|^2)\,\mathrm{d}x. \tag{26}$$

Proof We remark that, defining the following norm on the Banach space X_h,

$$\|v_h\|_{X_h} = \|\overline{v}_h\|_{L^2(\mathbb{R}^d)} + \|\nabla\widehat{v}_h\|_{L^1(\mathbb{R}^d)} + \|\nabla\widehat{v}_h\|_{L^2(\mathbb{R}^d)},$$

we obtain that the stricly convex function J_h^{n+1} is such that

$$\lim_{\|v_h\|_{X_h}\to\infty} J_h^{n+1}(v_h) = +\infty.$$

Hence J_h^{n+1} reaches its unique minimum value at some point $u_h^{n+1} \in X_h$. Let us prove that this point is characterised by (17)-(18).

Let us first assume that there exists $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$, solution to (17)-(18). Writing, for any $w_h \in X_h$, $J_h^{n+1}(w_h) = J_h^{n+1}(u_h^{n+1} + v_h)$ with $v_h = w_h - u_h^{n+1}$, we have using (17)

$$J_h^{n+1}(w_h) - J_h^{n+1}(u_h^{n+1}) = \frac{1}{2\delta t}\int_{\mathbb{R}^d}\overline{v}_h(x)^2\mathrm{d}x + A + \int_{\mathbb{R}^d}\frac{1}{2}\theta(h)|\nabla\widehat{v}_h(x)|^2\mathrm{d}x,$$

with

$$\begin{aligned}A = \int_{\mathbb{R}^d}(|\nabla(\widehat{u}_h^{n+1} + \widehat{v}_h)(x)| - |\nabla\widehat{u}_h(x)| - \lambda_h^{n+1}(x)\cdot\nabla\widehat{v}_h(x))\mathrm{d}x \\ = \sum_{K\in\mathcal{T}_h}|K|(|\nabla(\widehat{u}_h^{n+1} + \widehat{v}_h)_{|K}| - |(\nabla\widehat{u}_h^{n+1})_{|K}| - \lambda_K^{n+1}\cdot(\nabla\widehat{v}_h)_{|K}).\end{aligned}$$

Recall that, if $(\nabla\widehat{u}_h^{n+1})_{|K} \neq 0$, then $\lambda_K^{n+1} = \frac{(\nabla\widehat{u}_h^{n+1})_{|K}}{|(\nabla\widehat{u}_h^{n+1})_{|K}|}$ and otherwise that $|\lambda_K^{n+1}| \leq 1$.
Using that for any $a, b \in \mathbb{R}^d$ with $a \neq 0$, we have $|a+b| - |a| - \frac{a\cdot b}{|a|} = \frac{|a||a+b| - a\cdot(a+b)}{|a|} \geq 0$, and $|b| - \lambda_K^{n+1}\cdot b \geq 0$, we get $A \geq 0$, which proves that $J_h^{n+1}(w_h) - J_h^{n+1}(u_h^{n+1}) \geq 0$, and therefore that J_h^{n+1} reaches its minimum value at u_h^{n+1}.

Reciprocally, let us assume that J_h^{n+1} reaches its minimum value at some point $u_h^{n+1} \in X_h$. Let us prove that there exists $\lambda_h^{n+1} \in \Lambda_h$ such that (17)-(18) holds. We denote by $\mathcal{T}_{h,0}^{n+1} = \{K \in \mathcal{T}_h,\ (\nabla\widehat{u}_h^{n+1})_{|K} = 0\}$, and we define the linear form $\mathcal{L}_h^{n+1}\ :\ X_h \to \mathbb{R}$ by

$$\mathcal{L}_h^{n+1}(v_h) := \frac{1}{\delta t}\int_{\mathbb{R}^d}\left(\overline{u}_h^{n+1} - \overline{u}_h^{n+\frac{1}{2}}\right)\overline{v}_h\mathrm{d}x + \int_{\{\nabla\widehat{u}_h^{n+1}\neq 0\}}\frac{\nabla\widehat{u}_h^{n+1}}{|\nabla\widehat{u}_h^{n+1}|}\cdot\nabla\widehat{v}_h\mathrm{d}x + \theta(h)\int_{\mathbb{R}^d}\nabla\widehat{u}_h^{n+1}\cdot\nabla\widehat{v}_h\,\mathrm{d}x.$$

For a given $\varepsilon \in \mathbb{R}$ with $\varepsilon \neq 0$ and for any $v_h \in X_h$, we write that

$$J_h^{n+1}(u_h^{n+1} + \varepsilon v_h) - J_h^{n+1}(u_h^{n+1}) \geq 0.$$

Assuming $\varepsilon > 0$, dividing the above equation by ε and letting $\varepsilon \to 0$, we get

$$\mathcal{L}_h^{n+1}(v_h) + \int_{\{\nabla \widehat{u}_h^{n+1}=0\}} |\nabla \widehat{v}_h| \mathrm{d}x \geq 0.$$

Assuming $\varepsilon < 0$, dividing the above equation by ε and letting $\varepsilon \to 0$, we get

$$\mathcal{L}_h^{n+1}(v_h) - \int_{\{\nabla \widehat{u}_h^{n+1}=0\}} |\nabla \widehat{v}_h| \mathrm{d}x \leq 0.$$

Hence we get that

$$\forall v_h \in X_h,\ |\mathcal{L}_h^{n+1}(v_h)| \leq \int_{\{\nabla \widehat{u}_h^{n+1}=0\}} |\nabla \widehat{v}_h| \mathrm{d}x = \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K|\ |\nabla \widehat{v}_{h|K}|. \tag{27}$$

We consider the set E of all functions f from $\mathcal{T}_{h,0}^{n+1} \to \mathbb{R}^d$ which are bounded for the norm

$$\|f\|_E = \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K|\ |f_K|.$$

We observe that, defining $F = \{f \in E,\ \exists v_h \in X_h,\ f = \nabla \widehat{v}_h\}$, the linear form $B : F \to \mathbb{R}$, such that $B(f) = \mathcal{L}_h^{n+1}(v_h)$ is well defined (since if $\nabla \widehat{v}_h = \nabla \widehat{w}_h$ on all $K \in \mathcal{T}_{h,0}^{n+1}$, then we get from (27) that $\mathcal{L}_h^{n+1}(v_h) = \mathcal{L}_h^{n+1}(w_h)$) and continuous (again from (27), which proves that $\|B\|_{F'} := \sup_{f \in F \setminus \{0\}} \frac{B(f)}{\|f\|_E} \leq 1$). Applying the Hahn-Banach theorem, this linear form can be extended on E by a linear form, again denoted by B, with the same norm (hence lower or equal to 1). Hence there exists $(\lambda_K^{n+1})_{K \in \mathcal{T}_{h,0}^{n+1}}$ with

$$\forall f \in E,\ B(f) = -\sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K|\ \lambda_K^{n+1} \cdot f_K.$$

and $\|B\|_{E'} = \sup_{K \in \mathcal{T}_{h,0}^{n+1}} |\lambda_K^{n+1}| \leq 1$. Therefore we have

$$\forall v_h \in X_h,\ \mathcal{L}_h^{n+1}(v_h) + \sum_{K \in \mathcal{T}_{h,0}^{n+1}} |K|\ \lambda_K^{n+1} \cdot (\nabla \widehat{v}_h)_{|K} = 0,$$

which concludes, denoting $\lambda_K^{n+1} = \frac{(\nabla \widehat{u}_h^{n+1})_{|K}}{|(\nabla \widehat{u}_h^{n+1})_{|K}|}$ if $(\nabla \widehat{u}_h^{n+1})_{|K} \neq 0$, the proof that $\lambda_h^{n+1} \in \Lambda_h$ is such that (17)-(18) holds. □

Proposition 3.9: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $n \in \mathbb{N}$. Let $\eta \in C^2(\mathbb{R})$ such that $\eta(0) = 0$, $\eta'(0) = 0$ and there exists $M \in \mathbb{R}^+$ with $\eta''(s) \in [0, M]$ for all $s \in \mathbb{R}$. Assume that $u_h^{n+\frac{1}{2}} \in \mathbb{R}^{\mathbb{N}}$ is such that

$$\sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}) < \infty.$$

Then

$$\sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+1}) \leq \sum_{p \in \mathbb{N}} m_p \eta(u_p^{n+\frac{1}{2}}). \tag{28}$$

As a consequence, if $a_0 \leq u_p^{n+\frac{1}{2}} \leq b_0$ for all $p \in \mathbb{N}$, then $a_0 \leq u_p^{n+1} \leq b_0$ for all $p \in \mathbb{N}$, and if $\overline{u}_h^{n+\frac{1}{2}} \in L^1(\mathbb{R}^d)$ then $\overline{u}_h^{n+1} \in L^1(\mathbb{R}^d)$.

Proof We remark that v_h, defined by $v_p = \eta'(u_p^{n+1})$, is such that $v_h \in X_h$. Indeed, there holds $|v_p| \leq M|u_p^{n+1}|$, and, using (6)-(7), we have

$$\|\nabla \widehat{v}_h\|_{L^2(\mathbb{R})^d}^2 = \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (\eta'(u_p^{n+1}) - \eta'(u_q^{n+1}))^2$$
$$\leq M^2 \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})^2 = M^2 \|\nabla \widehat{u}_h^{n+1}\|_{L^2(\mathbb{R})^d}^2,$$

which implies $v_h \in X_h$. We now remark that, for any $K \in \mathcal{T}_h$, if $\nabla \widehat{u}_{h|K}^{n+1} = 0$, then $u_p^{n+1} = u_q^{n+1}$ for any $p, q \in \mathcal{V}_K$, and therefore $\nabla \widehat{v}_{h|K} = 0$. So one can write

$$\lambda_{h|K}^{n+1} \cdot \nabla \widehat{v}_{h|K} = 0 = \nabla \widehat{u}_{h|K}^{n+1} \cdot \nabla \widehat{v}_{h|K}.$$

If $\nabla \widehat{u}_{h|K}^{n+1} \neq 0$, we then have

$$\lambda_{h|K}^{n+1} \cdot \nabla \widehat{v}_{h|K} = \frac{1}{|\nabla \widehat{u}_{h|K}^{n+1}|} \nabla \widehat{u}_{h|K}^{n+1} \cdot \nabla \widehat{v}_{h|K}.$$

Hence, defining α_K^{n+1} by $\alpha_K^{n+1} = 1$ if $\nabla \widehat{u}_{h|K}^{n+1} = 0$, and $\alpha_K^{n+1} = \frac{1}{|\nabla \widehat{u}_{h|K}^{n+1}|}$ otherwise, we get

$$\int_{\mathbb{R}^d} (\lambda_h^{n+1} + \theta(h) \nabla \widehat{u}_h^{n+1}) \cdot \nabla \widehat{v}_h \, \mathrm{d}x = \sum_{K \in \mathcal{T}_h} (\alpha_K^{n+1} + \theta(h)) \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})(\eta'(u_p^{n+1}) - \eta'(u_q^{n+1})) \geq 0.$$

Hence we can write from (17)

$$\sum_{p \in \mathbb{N}} m_p (u_p^{n+1} - u_p^{n+\frac{1}{2}}) \eta'(u_p^{n+1}) \leq 0.$$

Applying $\eta(b) - \eta(a) = \eta'(a)(b-a) + \eta''(c)\frac{(b-a)^2}{2}$ for c between a and b, we get

$$\sum_{p \in \mathbb{N}} m_p (\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})) \leq \sum_{p \in \mathbb{N}} m_p (u_p^{n+1} - u_p^{n+\frac{1}{2}}) \eta'(u_p^{n+1}) \leq 0,$$

which concludes the proof of (28).

Then, assuming $u_p^{n+\frac{1}{2}} \leq b_0$ for all $p \in \mathbb{N}$ and letting $\eta(s)$ tend to $s \vee b_0 - b_0$ (this is possible since $u_{\mathrm{ini}} \in L^1(\mathbb{R}^d)$ implies $a_0 \leq 0 \leq b_0$), we get that $\eta(u_p^{n+1}) = 0$, which shows that $u_p^{n+1} \leq b_0$. The same reasoning holds, letting $\eta(s)$ tend to $a_0 - s \wedge a_0$, which shows that $u_p^{n+1} \geq a_0$. Finally, letting $\eta(s)$ tend to $|s|$, we conclude that $\overline{u}_h^{n+1} \in L^1(\mathbb{R}^d)$. □

4 A priori estimates on the approximate solutions

A $L^\infty(Q_T)$ estimate on the family of approximate velocities $(\overline{u}_{h,\delta t})_{h,\delta t>0}$ has already been proved in Proposition 3.4. The aim of this section is to establish additional estimates on $(\widehat{u}_{h,\delta t})_{h,\delta t>0}$, namely a $L^1(0,T;BV(\mathbb{R}^d))$ estimate, and $L^1_{\mathrm{loc}}(Q_T)$ estimates on the space and time translates. The estimates on the space and time translates are deduced from the $L^1(0,T;BV(\mathbb{R}^d))$ estimate.

Remark 4.1: Hypothesis (HD1) implies that each cell of $\mathcal{D}_h$ has a finite number of neighbours (and this number is bounded independently of $\mathcal{F}_{h,\delta t}$).

Remark 4.2: Throughout this section and the next one, C denotes a generic constant independent of the discretisation $\mathcal{F}_{h,\delta t}$.

4.1 $L^1(0,T;BV(\mathbb{R}^d))$ estimate

Proposition 4.3: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and, for all $n \in \mathbb{N}$, $\overline{u}_h^{n+\frac{1}{2}}, \overline{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ be a solution to Scheme (14)-(18). Then there holds

$$\frac{1}{2} \|\overline{u}_{h,\delta t}\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))} + \sum_{n=1}^{N} \delta t \int_{\mathbb{R}^d} |\nabla \widehat{u}_h^n| \, \mathrm{d}x \leq \frac{1}{2} \|u_{\mathrm{ini}}\|^2_{L^2(\mathbb{R}^d)}, \tag{29}$$

and

$$\sum_{n=0}^{N-1} \delta t \int_{\mathbb{R}^d} \theta(h) |\nabla \widehat{u}_h^{n+1}|^2 \, \mathrm{d}x = \theta(h) \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{T}_h} \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})^2 \leq \frac{1}{2} \|u_{\mathrm{ini}}\|^2_{L^2(\mathbb{R}^d)}. \tag{30}$$

Proof Owing to (25) in Proposition 3.7, we have

$$\int_{\mathbb{R}^d} \overline{u}_h^{n+\frac{1}{2}}(x)^2 \mathrm{d}x \le \int_{\mathbb{R}^d} \overline{u}_h^n(x)^2 \mathrm{d}x. \tag{31}$$

We test (17) with $v_h = \delta t\, u_h^{n+1}$ and, since $\lambda_h^{n+1} \in \mathrm{Sgn}(\nabla \widehat{u}_h^{n+1})$, we obtain

$$\sum_{p\in\mathbb{N}} m_p u_p^{n+1} \left(u_p^{n+1} - u_p^{n+\frac{1}{2}}\right) + \delta t \int_{\mathbb{R}^d} (|\nabla \widehat{u}_h^{n+1}| + \theta(h)|\nabla \widehat{u}_h^{n+1}|^2)\ \mathrm{d}x = 0.$$

This leads to

$$\sum_{p\in\mathbb{N}} m_p \left(\frac{1}{2}(u_p^{n+1})^2 + \frac{1}{2}(u_p^{n+1} - u_p^{n+\frac{1}{2}})^2 - \frac{1}{2}(u_p^{n+\frac{1}{2}})^2\right) + \delta t \int_{\mathbb{R}^d} (|\nabla \widehat{u}_h^{n+1}| + \theta(h)|\nabla \widehat{u}_h^{n+1}|^2)\ \mathrm{d}x = 0,$$

hence giving, thanks to (31)

$$\sum_{p\in\mathbb{N}} m_p \left(\frac{1}{2}(u_p^{n+1})^2 - \frac{1}{2}(u_p^{n})^2\right) + \delta t \int_{\mathbb{R}^d} (|\nabla \widehat{u}_h^{n+1}| + \theta(h)|\nabla \widehat{u}_h^{n+1}|^2)\ \mathrm{d}x \le 0. \tag{32}$$

Summing (32) over $n \in \{0, ..., m\}$ for any $m = 1, \dots, N-1$ we obtain the inequality

$$\frac{1}{2}\sum_{p\in\mathbb{N}} m_p (u_p^{m+1})^2 + \sum_{n=0}^{m} \delta t \int_{\mathbb{R}^d} (|\nabla \widehat{u}_h^{n+1}| + \theta(h)|\nabla \widehat{u}_h^{n+1}|^2)\ \mathrm{d}x \le \sum_{p\in\mathbb{N}} m_p \frac{1}{2}(u_p^0)^2. \tag{33}$$

We conclude the proof of the lemma since the above inequality holds for any m. □

4.2 Time translate estimate

Proposition 4.4: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and, for all $n \in \mathbb{N}$, $\overline{u}_h^{n+\frac{1}{2}}, \overline{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ be a solution to Scheme (14)-(18). For all $R > 0$, there exists a constant C, independent of the family $(\mathcal{F}_{h,\delta t})_{h,\delta t>0}$, such that,

$$\int_0^{T-s} \int_{B(0,R)} |\overline{u}_{h,\delta t}(x, t+s) - \overline{u}_{h,\delta t}(x,t)|\ \mathrm{d}x\mathrm{d}t \le C\sqrt{s}, \qquad \forall s \in [0,T]. \tag{34}$$

Proof 1. Let $s \in [0,T]$. Using the Cauchy-Schwarz inequality, we have

$$\int_0^{T-s} \int_{B(0,R)} |\overline{u}_{h,\delta t}(x, t+s) - \overline{u}_{h,\delta t}(x,t)|\ \mathrm{d}x\mathrm{d}t \le \left(CR^dT \int_0^{T-s} \int_{\mathbb{R}^d} \left(\overline{u}_{h,\delta t}(x, t+s) - \overline{u}_{h,\delta t}(x,t)\right)^2\ \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}. \tag{35}$$

We define the function $\nu : \mathbb{R} \to \mathbb{Z}$ such that $\nu(t) = n+1$ if $t \in (n\delta t, (n+1)\delta t]$ and the function $\chi_n(t,s) : \mathbb{R}\times\mathbb{R} \to \{0,1\}$ such that $\chi_n(t,s) = 1$ if $\nu(t) \le n < \nu(t+s)$ and $\chi_n(t,s) = 0$ otherwise; so that

$$\int_0^{T-s} \int_{\mathbb{R}^d} \left(\overline{u}_{h,\delta t}(x, t+s) - \overline{u}_{h,\delta t}(x,t)\right)^2\ \mathrm{d}x\mathrm{d}t = \int_0^{T-s} \int_{\mathbb{R}^d} \left(\overline{u}_h^{\nu(t+s)} - \overline{u}_h^{\nu(t)}\right)^2\ \mathrm{d}x\mathrm{d}t$$

$$= \int_0^{T-s} \int_{\mathbb{R}^d} \left(\overline{u}_h^{\nu(t+s)} - \overline{u}_h^{\nu(t)}\right) \sum_{n=1}^{N-1} \chi_n(t,s) \left(\overline{u}_h^{n+1} - \overline{u}_h^{n}\right)\ \mathrm{d}x\mathrm{d}t. \tag{36}$$

2. We denote $\tau = 0$ or $\tau = s$, and we multiply (15), for $n \ge 1$, by $\delta t\, u_p^{\nu(t+\tau)}$ and sum over all $p \in \mathbb{N}$:

$$\sum_{p\in\mathbb{N}} m_p u_p^{\nu(t+\tau)} \left(u_p^{n+\frac{1}{2}} - u_p^n\right) + \delta t \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} u_p^{\nu(t+\tau)} \left(F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)\right) = 0. \tag{37}$$

Testing (17) with $v_h = \delta t u_h^{\nu(t+\tau)}$, we get:

$$\sum_{p\in\mathbb{N}} m_p u_p^{\nu(t+\tau)} \left(u_p^{n+1} - u_p^{n+\frac{1}{2}}\right) + \delta t \int_{\mathbb{R}^d} (\lambda_h^{n+1}\cdot\nabla\widehat{u}_h^{\nu(t+\tau)} + \theta(h)\nabla\widehat{u}_h^{n+1}\cdot\nabla\widehat{u}_h^{\nu(t+\tau)})\,\mathrm{d}x = 0. \tag{38}$$

Adding the above equalities (37) and (38), we find

$$\sum_{p\in\mathbb{N}} m_p u_p^{\nu(t+\tau)} \left(u_p^{n+1} - u_p^{n}\right) + \delta t \sum_{p\in\mathbb{N}} u_p^{\nu(t+\tau)} \sum_{q\in\mathcal{N}_p} \left(F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)\right)$$
$$+ \delta t \int_{\mathbb{R}^d} (\lambda_h^{n+1}\cdot\nabla\widehat{u}_h^{\nu(t+\tau)} + \theta(h)\nabla\widehat{u}_h^{n+1}\cdot\nabla\widehat{u}_h^{\nu(t+\tau)})\,\mathrm{d}x = 0. \tag{39}$$

3. Let $p\in\mathbb{N}$, $q\in\mathcal{N}_p$ and K an element of $\mathcal{T}_h$ for which x_p and x_q are vertices. In view of (HD1) and (HD2), for $n\geq 1$,

$$\left|F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)\right| \leq 2m_{p,q}L|u_p^n - u_q^n| \leq 2m_{p,q}Lh|\nabla\widehat{u}_{h|K}^n| \leq Ch^d|\nabla\widehat{u}_{h|K}^n|. \tag{40}$$

Therefore, owing to (HD1), to the finite number of neighbours in $\mathcal{D}_h$ and to the L^∞ estimate stated in Proposition 3.4,

$$\left|\sum_{p\in\mathbb{N}} u_p^{\nu(t+\tau)} \sum_{q\in\mathcal{N}_p} \left(F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)\right)\right| \leq C\int_{\mathbb{R}^d} |\nabla\widehat{u}_h^n|\,\mathrm{d}x. \tag{41}$$

The property $|\lambda_h^{n+1}|\leq 1$ gives immediately

$$\left|\int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\widehat{u}_h^{\nu(t+\tau)}\,\mathrm{d}x\right| \leq \int_{\mathbb{R}^d} |\nabla\widehat{u}_h^{\nu(t+\tau)}|\,\mathrm{d}x. \tag{42}$$

From (39), (41) and (42), it follows

$$\left|\int_{\mathbb{R}^d} \overline{u}_h^{\nu(t+\tau)} \left(\overline{u}_h^{n+1} - \overline{u}_h^n\right)\,\mathrm{d}x\right| \;\leq\; C\delta t\left(\int_{\mathbb{R}^d} (|\nabla\widehat{u}_h^n| + |\nabla\widehat{u}_h^{\nu(t+\tau)}| + \frac{\theta(h)}{2}(|\nabla\widehat{u}_h^{n+1}|^2 + |\nabla\widehat{u}_h^{\nu(t+\tau)}|^2))\,\mathrm{d}x\right). \tag{43}$$

4. Using the above estimate (43), we have

$$\left|\int_0^{T-s}\int_{\mathbb{R}^d} \overline{u}_h^{\nu(t+\tau)} \sum_{n=1}^{N-1} \chi_n(t,s)\left(\overline{u}_h^{n+1} - \overline{u}_h^n\right)\,\mathrm{d}x\mathrm{d}t\right| \leq C\sum_{n=1}^{N}\int_0^{T-s} \chi_n(t,s)\delta t(A^n + B^{\nu(t+\tau)})\,\mathrm{d}t,$$

with, for all $n = 1,\ldots,N$,

$$A^n = \int_{\mathbb{R}^d} (|\nabla\widehat{u}_h^n| + \frac{\theta(h)}{2}|\nabla\widehat{u}_h^{n+1}|^2)\,\mathrm{d}x$$

and

$$B^n = \int_{\mathbb{R}^d} (|\nabla\widehat{u}_h^n| + \frac{\theta(h)}{2}|\nabla\widehat{u}_h^{n}|^2)\,\mathrm{d}x.$$

We then apply Lemma 4.5 below, which leads to

$$\left|\int_0^{T-s}\int_{\mathbb{R}^d} \overline{u}_h^{\nu(t+\tau)} \sum_{n=1}^{N-1} \chi_n(t,s)\left(\overline{u}_h^{n+1} - \overline{u}_h^n\right)\,\mathrm{d}x\mathrm{d}t\right| \leq Cs\sum_{n=1}^{N}\delta t(A^n + B^n). \tag{44}$$

5. Recalling (29) and (30) which provide bounds on $\sum_{n=1}^N \delta t(A^n + B^n)$, and collecting (36) and (44), we obtain the desired estimate (34). □

Let us now state a lemma proved in [8, Lemma 4.6] and used in the preceding proof.

Lemma 4.5: Let $T>0$, $\delta t\in(0,T)$ and $(a^n)_{n\in\mathcal{N}}$ be a family of non negative real values. Then, defining the function $\nu:\mathbb{R}\to\mathbb{Z}$ such that $\nu(t) = n+1$ if $t\in(n\delta t,(n+1)\delta t]$

$$\int_0^{T-\delta t}\sum_{n=\nu(t)+1}^{\nu(t+\delta t)} a^n dt \leq \delta t\sum_{n=1}^{\nu(T)} a^n,$$

and for any $\sigma\in[0,\delta t]$

$$\int_0^{T-\delta t}\sum_{n=\nu(t)+1}^{\nu(t+\delta t)} a^{\nu(t+\sigma)} dt \leq \delta t\sum_{n=1}^{\nu(T)} a^n.$$

4.3 Space translate estimate

Proposition 4.6: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and, for all $n \in \mathbb{N}$, $\overline{u}_h^{n+\frac{1}{2}}, \overline{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ be a solution to Scheme (14)-(18). There exists a constant C, independent of the family $(\mathcal{F}_{h,\delta t})_{h,\delta t>0}$, such that

$$\int_0^T \int_{\mathbb{R}^d} |\widehat{u}_{h,\delta t}(x+y,t) - \widehat{u}_{h,\delta t}(x,t)| \,\mathrm{d}x\mathrm{d}t \le C|y|, \qquad \forall y \in \mathbb{R}^d. \tag{45}$$

and, for any $R > 0$, there exists C', independent of the family $(\mathcal{F}_{h,\delta t})_{h,\delta t>0}$,

$$\int_0^T \int_{B(0,R)} |\overline{u}_{h,\delta t}(x+y,t) - \overline{u}_{h,\delta t}(x,t)| \,\mathrm{d}x\mathrm{d}t \le C(|y| + 2CTR^d h), \qquad \forall y \in \mathbb{R}^d. \tag{46}$$

Proof For a given element $K \in \mathcal{T}_h$ and a given couple of points $(a,b) \in \mathbb{R}^d \times \mathbb{R}^d$, we denote by $\chi_K(a,b)$ the length of the segment $[a,b] \cap \overline{K}$. In particular, if $[a,b]$ does not intersect $\overline{K}$, then $\chi_K(a,b) = 0$. Let $n \in \{1, ..., N\}$ and $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. By applying the mean value theorem on each element intersected by the segment $[x, x+y]$, we find the inequality

$$|\widehat{u}_h^n(x+y) - \widehat{u}_h^n(x)| \le \sum_{K \in \mathcal{T}_h} \chi_K(x, x+y) |\nabla \widehat{u}_{h|K}^n|. \tag{47}$$

Next, an integration with respect to x yields

$$\int_{\mathbb{R}^d} |\widehat{u}_h^n(x+y) - \widehat{u}_h^n(x)| \,\mathrm{d}x \le \sum_{K \in \mathcal{T}_h} |\nabla \widehat{u}_{h|K}^n| \int_{\mathbb{R}^d} \chi_K(x, x+y) \,\mathrm{d}x. \tag{48}$$

For any $K \in \mathcal{T}_h$, the function $x \mapsto \chi_K(x, x+y)$ is bounded by $\min(h, |y|)$ and is zero outside a domain of measure lower than $h^{d-1}(h + |y|)$. Therefore,

$$\int_{\mathbb{R}^d} \chi_K(x, x+y) \,\mathrm{d}x \le h^{d-1}(h + |y|) \min(h, |y|) \le 2h^d |y|, \tag{49}$$

and

$$\int_{\mathbb{R}^d} |\widehat{u}_h^n(x+y) - \widehat{u}_h^n(x)| \,\mathrm{d}x \le C|y| \sum_{K \in \mathcal{T}_h} h^d |\nabla \widehat{u}_{h|K}^n|. \tag{50}$$

Using the hypothesis (HD1), we obtain

$$\sum_{K \in \mathcal{T}_h} h^d |\nabla \widehat{u}_{h|K}^n| \le C \int_{\mathbb{R}^d} |\nabla \widehat{u}_h^n| \,\mathrm{d}x. \tag{51}$$

Summing the above inequality (51) over $\{1, ..., N\}$, and using (29), we find the desired estimate (45). We then deduce (46) using (8). □

5 Entropy formulation for the approximate solutions

The aim of this section is to establish an entropy formulation, similar to (5), for the approximate solutions. We first prove a discrete entropy inequality for the finite volume step (Proposition 3.6). Then, we take into account the finite element step to obtain the complete discrete entropy formulation (Proposition 5.1). Error terms occur in this formulation. Proposition 5.1 ensures that they tend to zero when the meshsize and the time step tend to zero.

Proposition 5.1: Let us assume Hypotheses (HC) and Hypotheses (HD). Let $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and, for all $n \in \mathbb{N}$, $\overline{u}_h^{n+\frac{1}{2}}, \overline{u}_h^{n+1} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ be a solution to Scheme (14)-(18). Let (η, Φ) be an entropy-entropy flux pair. Then, for all nonnegative test functions $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$, there holds

$$\int_{Q_T} \eta(\overline{u}_{h,\delta t}) \partial_t \varphi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \Phi(x,t,\overline{u}_{h,\delta t}) \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t - \int_{Q_T} \eta'(\widehat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t$$
$$- \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h,\delta t})| \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} \eta(\overline{u}_h^0) \varphi(x,0) \,\mathrm{d}x + e_{h,\delta t} \ge 0, \tag{52}$$

where $e_{h,\delta t}$ satisfies

$$\lim_{h\to 0,\delta t\to 0} e_{h,\delta t} = 0. \tag{53}$$

Proof First step: proof of (52)

1. Let $\varphi^{n+\frac{1}{2}} \in C_c^\infty(\mathbb{R}^d,\mathbb{R}_+)$ be defined by $\varphi^{n+\frac{1}{2}}(x) := \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \varphi(x,s)\,\mathrm{d}s$. We test (17) with the function $v_h^{n+1} \in X_h$ such that $v_p^{n+1} = \eta'(u_p^{n+1})\varphi^{n+\frac{1}{2}}(x_p)$, for all $p \in \mathbb{N}$. We get:

$$\sum_{p\in\mathbb{N}} m_p \frac{u_p^{n+1} - u_p^{n+\frac{1}{2}}}{\delta t}\eta'(u_p^{n+1})\varphi^{n+\frac{1}{2}}(x_p) + \int_{\mathbb{R}^d} (\lambda_h^{n+1}\cdot\nabla\widehat{v}_h^{n+1} + \theta(h)\nabla\widehat{u}_h^{n+1}\cdot\nabla\widehat{v}_h^{n+1})\,\mathrm{d}x = 0. \tag{54}$$

Since η is convex,

$$\left(u_p^{n+1} - u_p^{n+\frac{1}{2}}\right)\eta'(u_p^{n+1}) \geq \eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}}) \qquad \forall p\in\mathbb{N},$$

and thus (54) leads to

$$\sum_{p\in\mathbb{N}} m_p \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t}\varphi^{n+\frac{1}{2}}(x_p) + \int_{\mathbb{R}^d} (\lambda_h^{n+1}\cdot\nabla\widehat{v}_h^{n+1} + \theta(h)\nabla\widehat{u}_h^{n+1}\cdot\nabla\widehat{v}_h^{n+1})\,\mathrm{d}x \leq 0. \tag{55}$$

Defining $\widetilde{v}^{n+1} := \eta'(\widehat{u}_h^{n+1})\varphi^{n+\frac{1}{2}}$, the second term of (55) can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\widehat{v}_h^{n+1}\,\mathrm{d}x &= \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\widetilde{v}^{n+1}\,\mathrm{d}x + \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\left(\widehat{v}_h^{n+1} - \widetilde{v}^{n+1}\right)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\left(\eta'(\widehat{u}_h^{n+1})\right)\varphi^{n+\frac{1}{2}}\,\mathrm{d}x + \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\varphi^{n+\frac{1}{2}}\,\eta'(\widehat{u}_h^{n+1})\,\mathrm{d}x + \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\left(\widehat{v}_h^{n+1} - \widetilde{v}^{n+1}\right)\,\mathrm{d}x.\end{aligned} \tag{56}$$

We have

$$\int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\left(\eta'(\widehat{u}_h^{n+1})\right)\varphi^{n+\frac{1}{2}}\,\mathrm{d}x = \int_{\mathbb{R}^d} \eta''(\widehat{u}_h^{n+1})\lambda_h^{n+1}\cdot\nabla(\widehat{u}_h^{n+1})\varphi^{n+\frac{1}{2}}\,\mathrm{d}x = \int_{\mathbb{R}^d} |\nabla\eta'(\widehat{u}_h^{n+1})|\varphi^{n+\frac{1}{2}}\,\mathrm{d}x. \tag{57}$$

The third term of (55) can be rewritten as

$$\int_{\mathbb{R}^d} \theta(h)\nabla\widehat{u}_h^{n+1}\cdot\nabla\widehat{v}_h^{n+1}\,\mathrm{d}x = \theta(h)\sum_{K\in\mathcal{T}_h}\sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} T_{pq}^K(u_p^{n+1} - u_q^{n+1})(\eta'(u_p^{n+1})\varphi^{n+\frac{1}{2}}(x_p) - \eta'(u_q^{n+1})\varphi^{n+\frac{1}{2}}(x_q)) = T_1 + T_2,$$

with

$$\begin{aligned}T_1 &= \theta(h)\sum_{K\in\mathcal{T}_h}\sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} T_{pq}^K(u_p^{n+1} - u_q^{n+1})(\eta'(u_p^{n+1}) - \eta'(u_q^{n+1}))\frac{\varphi^{n+\frac{1}{2}}(x_p) + \varphi^{n+\frac{1}{2}}(x_q)}{2} \geq 0 \\ T_2 &= \theta(h)\sum_{K\in\mathcal{T}_h}\sum_{p\in\mathcal{V}_K}\sum_{q\in\mathcal{V}_K} T_{pq}^K(u_p^{n+1} - u_q^{n+1})(\varphi^{n+\frac{1}{2}}(x_p) - \varphi^{n+\frac{1}{2}}(x_q))\frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2}.\end{aligned} \tag{58}$$

Collecting (55), (56), (58) and (57), we obtain

$$\begin{aligned}&\sum_{p\in\mathbb{N}} m_p \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t}\varphi^{n+\frac{1}{2}}(x_p) + \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\varphi^{n+\frac{1}{2}}\eta'(\widehat{u}_h^{n+1})\,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla\eta'(\widehat{u}_h^{n+1})|\varphi^{n+\frac{1}{2}}\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla\left(\widetilde{v}^{n+1} - \widehat{v}_h^{n+1}\right)\,\mathrm{d}x + \theta(h)\sum_{p\in\mathbb{N}}\sum_{q\in\mathbb{N}} T_{pq}|u_p^{n+1} - u_q^{n+1}|\;|\varphi^{n+\frac{1}{2}}(x_p) - \varphi^{n+\frac{1}{2}}(x_q)|\frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2}.\end{aligned} \tag{59}$$

Multiplying by δt and summing over $n \in \{0,...,N-1\}$, we eventually find

$$\begin{aligned}&\sum_{n=0}^{N-1}\sum_{p\in\mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^{n+\frac{1}{2}})}{\delta t}\int_{t^n}^{t^{n+1}} m_p\varphi(x_p,t)\,\mathrm{d}t + \int_{Q_T} \eta'(\widehat{u}_{h,\delta t})\lambda_{h,\delta t}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{Q_T} \varphi|\nabla\eta'(\widehat{u}_{h,\delta t})|\,\mathrm{d}x\mathrm{d}t \leq e_{h,\delta t}^{(1)} + e_{h,\delta t}^{(2)},\end{aligned} \tag{60}$$

where

$$e^{(1)}_{h,\delta t} := \sum_{n=0}^{N-1} \delta t \int_{\mathbb{R}^d} \lambda_h^{n+1}(x) \cdot \nabla(\widetilde{v}^{n+1}(x) - \widehat{v}_h^{n+1}(x))\, \mathrm{d}x, \tag{61}$$

and

$$e^{(2)}_{h,\delta t} := \sum_{n=0}^{N-1} \delta t \theta(h) \sum_{K\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_K} \sum_{q\in\mathcal{V}_K} T^K_{pq} |u_p^{n+1} - u_q^{n+1}| |\varphi^{n+\frac{1}{2}}(x_p) - \varphi^{n+\frac{1}{2}}(x_q)| \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2}. \tag{62}$$

2. Multiplying (23) by $\varphi_p^n := \int_{t^n}^{t^{n+1}} \varphi(x_p, s)\, \mathrm{d}s$, then adding to (60), we obtain

$$\sum_{n=0}^{N-1} \sum_{p\in\mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} m_p \varphi_p^n + \sum_{n=0}^{N-1} \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \Phi^n_{p,q}(u_p^n, u_q^n)\varphi_p^n$$
$$+ \int_{Q_T} \eta'(\widehat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla\varphi\, \mathrm{d}x\mathrm{d}t + \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h,\delta t})|\, \mathrm{d}x\mathrm{d}t - e^{(1)}_{h,\delta t} - e^{(2)}_{h,\delta t} \leq 0. \tag{63}$$

Now, observing that, since $\varphi(\cdot, t) = 0$ for $t \geq T$, we have $\varphi_p^N = 0$ and therefore

$$\sum_{n=0}^{N-1} \sum_{p\in\mathbb{N}} \frac{\eta(u_p^{n+1}) - \eta(u_p^n)}{\delta t} m_p \varphi_p^n = -\sum_{n=0}^{N-1} \sum_{p\in\mathbb{N}} \eta(u_p^{n+1}) m_p \frac{\varphi_p^{n+1} - \varphi_p^n}{\delta t} - \sum_{p\in\mathbb{N}} \eta(u_p^0) m_p \frac{1}{\delta t} \varphi_p^0$$

we can rewrite (63) as

$$\int_{Q_T} \eta(\overline{u}_{h,\delta t}) \partial_t \varphi\, \mathrm{d}x\mathrm{d}t + \int_{Q_T} \Phi(x, t, \overline{u}_{h,\delta t}) \cdot \nabla\varphi\, \mathrm{d}x\mathrm{d}t - \int_{Q_T} \eta'(\widehat{u}_{h,\delta t}) \lambda_{h,\delta t} \cdot \nabla\varphi\, \mathrm{d}x\mathrm{d}t$$
$$- \int_{Q_T} \varphi |\nabla \eta'(\widehat{u}_{h,\delta t})|\, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} \eta(\overline{u}_h^0) \varphi(x, 0)\, \mathrm{d}x + e^{(0)}_{h,\delta t} + e^{(1)}_{h,\delta t} + e^{(2)}_{h,\delta t} + e^{(3)}_{h,\delta t} + e^{(4)}_{h,\delta t} \geq 0, \tag{64}$$

where

$$e^{(0)}_{h,\delta t} := \sum_{p\in\mathbb{N}} \frac{1}{\delta t} \int_0^{\delta t} \int_{Q_p} \eta(u_p^0)(\varphi(x_p, t) - \varphi(x, 0))\mathrm{d}x\mathrm{d}t, \tag{65}$$

$$e^{(3)}_{h,\delta t} := \sum_{n=0}^{N-1} \sum_{p\in\mathbb{N}} \eta(u_p^{n+1}) \int_{t^n}^{t^{n+1}} \int_{Q_p} \left(\frac{\varphi(x_p, t + \delta t) - \varphi(x_p, t)}{\delta t} - \partial_t \varphi(x, t) \right) \mathrm{d}x\mathrm{d}t, \tag{66}$$

$$e^{(4)}_{h,\delta t} := -\int_{Q_T} \Phi(x, t, \overline{u}_{h,\delta t}) \cdot \nabla\varphi\, \mathrm{d}x\mathrm{d}t - \sum_{n=0}^{N-1} \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \Phi^n_{p,q}(u_p^n, u_q^n)\varphi_p^n. \tag{67}$$

Hence, setting

$$e_{h,\delta t} = e^{(0)}_{h,\delta t} + e^{(1)}_{h,\delta t} + e^{(2)}_{h,\delta t} + e^{(3)}_{h,\delta t} + e^{(4)}_{h,\delta t}, \tag{68}$$

we obtain (52).

Second step: proof of (53)

Study of $e^{(1)}_{h,\delta t}$: use of the vanishing viscous term

We define $\chi_K = 0$ if $\varphi(x, t) = 0$ on $K \times [0, T)$ and 1 otherwise. We observe that, for a given $n = 0, \ldots, N-1$,

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \left(\widetilde{v}^{n+1} - \widehat{v}_h^{n+1} \right) \mathrm{d}x = \sum_{K\in\mathcal{T}_h} \chi_K \lambda_h^{n+1} \cdot \int_K \nabla \left(\widetilde{v}^{n+1} - \widehat{v}_h^{n+1} \right) \mathrm{d}x,$$

and we have

$$\int_K \nabla \left(\widetilde{v}^{n+1} - \widehat{v}_h^{n+1} \right) \mathrm{d}x = \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \left(\widetilde{v}^{n+1}(x) - \widehat{v}_h^{n+1}(x) \right) \mathrm{d}\gamma(x).$$

We use the fact that, for all $f \in C^2(\overline{\sigma})$ and for all $x \in \sigma$, we have $|f(x) - \widehat{f}_h(x)| \le \max_{x\in\sigma} |D^2 f(x)| h^2$, denoting by $\widehat{f}_h(x)$ the affine function equal to f at the vertices of the simplex σ. We apply this inequality to the function $f(x) = \widetilde{v}^{n+1}(x) = \eta'(\widehat{u}_h^{n+1}(x))\varphi^{n+\frac{1}{2}}(x)$. Using that $\widehat{u}_h^{n+1}$ is affine on σ with tangential gradient bounded by $|\nabla \widehat{u}_{h|K}^{n+1}|$, we get, for all $x \in \sigma$, letting C_3^η be a bound of $|\eta'|$, $|\eta''|$ and $|\eta'''|$ and C_2^φ be a bound of $\varphi^{n+\frac{1}{2}}$, $|\nabla\varphi^{n+\frac{1}{2}}|$ and $|D^2\varphi^{n+\frac{1}{2}}|$,

$$|\widetilde{v}^{n+1}(x) - \widehat{v}_h^{n+1}(x)| \le h^2 C_3^\eta C_2^\varphi (|\nabla \widehat{u}_{h|K}^{n+1}|^2 + 2|\nabla \widehat{u}_{h|K}^{n+1}| + 1).$$

Since the above expression is integrated over σ,we get

$$\int_{\mathbb{R}^d} \lambda_h^{n+1} \cdot \nabla \left(\widetilde{v}^{n+1} - \widehat{v}_h^{n+1}\right) \mathrm{d}x \le \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathcal{E}_K} \chi_K |\sigma| h^2 C_3^\eta C_2^\varphi (|\nabla \widehat{u}_{h|K}^{n+1}|^2 + 2|\nabla \widehat{u}_{h|K}^{n+1}| + 1).$$

Multiplying by δt and summing on $n = 0, \ldots, N-1$, we obtain

$$e_{h,\delta t}^{(1)} \le \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathcal{E}_K} \chi_K |\sigma| h^2 C_3^\eta C_2^\varphi (|\nabla \widehat{u}_{h|K}^{n+1}|^2 + 2|\nabla \widehat{u}_{h|K}^{n+1}| + 1).$$

Owing to the geometrical hypotheses and to (30), we get

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathcal{E}_K} \chi_K |\sigma| h^2 |\nabla \widehat{u}_{h|K}^{n+1}|^2 \le Ch \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} |K| |\nabla \widehat{u}_{h|K}^{n+1}|^2 \le \frac{h}{\theta(h)} C. \tag{69}$$

Besides, we have, thanks to (29),

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathcal{E}_K} \chi_K |\sigma| h^2 |\nabla \widehat{u}_{h|K}^{n+1}| \le Ch \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} |K| |\nabla \widehat{u}_{h|K}^{n+1}| \le hC.$$

Finally

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathcal{E}_K} \chi_K |\sigma| h^2 \le ChT \sum_{K\in\mathcal{T}_h} \chi_K |K| \le ChT(|\mathrm{supp}(\varphi)| + h),$$

where $\mathrm{supp}(\varphi) = \{x \in \mathbb{R}^d,\ \exists t \in [0, T),\ \varphi(x, t) \ne 0\}$. Hence each of the above terms tends to 0 with h thanks to Hypotheses (HD), thus completing the proof that $\lim_{h,\delta t\to 0} e_{h,\delta t}^{(1)} = 0$.

Study of $e_{h,\delta t}^{(2)}$: proof that the viscous term is vanishing

Letting C_φ be a Lipschitz constant for $\varphi^{n+\frac{1}{2}}$, and $\overline{C}$ a bound for $\eta'(u_p^{n+1})$ (this term remains bounded since η' is continuous and $a_0 \le u_p^{n+1} \le b_0$, see Proposition 3.4), we get, thanks to the Cauchy-Schwarz inequality

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_K} \sum_{q\in\mathcal{V}_K} T_{pq}^K |u_p^{n+1} - u_q^{n+1}|\ |\varphi^{n+\frac{1}{2}}(x_p) - \varphi^{n+\frac{1}{2}}(x_q)| \frac{\eta'(u_p^{n+1}) + \eta'(u_q^{n+1})}{2}$$
$$\le \overline{C} C_\varphi \left(\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_K} \sum_{q\in\mathcal{V}_K} T_{pq}^K (u_p^{n+1} - u_q^{n+1})^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_K} \sum_{q\in\mathcal{V}_K} T_{pq}^K \chi_K |x_p - x_q|^2 \right)^{1/2}.$$

Thanks to the geometrical hypotheses and to (30), we get, multiplying by $\theta(h)$,

$$e_{h,\delta t}^{(2)} \le \overline{C} C_\varphi \theta(h) \left(\frac{C}{\theta(h)} C\ T\ (|\mathrm{supp}(\varphi)| + h) \right)^{1/2}.$$

We get $\lim_{h,\delta t\to 0} e_{h,\delta t}^{(2)} = 0$ thanks to the hypotheses (HD).

Study of $e_{h,\delta t}^{(0)}$

We observe that, for all $x \in Q_p$ and $t \in [0, \delta t]$,

$$|\varphi(x_p, t) - \varphi(x, 0)| \le h \max_{(x,t)\in Q_T} |\nabla_x \varphi(x,t)| + \delta t \max_{(x,t)\in Q_T} |\partial_t \varphi(x,t)|,$$

hence

$$\lim_{h,\delta t\to 0} e^{(0)}_{h,\delta t} = 0.$$

Study of $e^{(3)}_{h,\delta t}$

We have, for all $x \in Q_p$ and $t \in [t^n, t^{n+1}]$,

$$\left| \frac{\varphi(x_p, t+\delta t) - \varphi(x_p, t)}{\delta t} - \partial_t \varphi(x,t) \right| \le h \max_{(x,t)\in Q_T} |\nabla_x \partial_t \varphi(x,t)| + \delta t \max_{(x,t)\in Q_T} |\partial^2_{tt} \varphi(x,t)|,$$

which proves that

$$\lim_{h,\delta t\to 0} e^{(3)}_{h,\delta t} = 0.$$

Study of $e^{(4)}_{h,\delta t}$

Since we are using below the BV estimate (29), which only involves values $n \ge 1$, we define

$$I_0 := \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \Phi^0_{p,q}(u^0_p, u^0_q)\varphi^0_p, \quad I := \sum_{n=1}^{N-1} \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \Phi^n_{p,q}(u^n_p, u^n_q)\varphi^n_p,$$

and

$$I^*_0 := -\int_0^{\delta t} \int_{\mathbb{R}^d} \Phi(x, t, \overline{u}_{h,\delta t}) \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t, \quad I^* := -\int_{\delta t}^{T} \int_{\mathbb{R}^d} \Phi(x, t, \overline{u}_{h,\delta t}) \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t.$$

We then have $e^{(4)}_{h,\delta t} = -I_0 - I + I^*_0 + I^*$. We write, as in the proof of Proposition 3.7, again using the property $\Phi^n_{p,q}(u,v) = -\Phi^n_{q,p}(v,u)$ for all $(u,v) \in [a_0, b_0]^2$,

$$I_0 = \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} (\Phi^0_{p,q}(u^0_p, u^0_q) - \Phi^0_{p,q}(0,0))\varphi^0_p = \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} (\Phi^0_{p,q}(u^0_p, u^0_q) - \Phi^0_{p,q}(0,0))(\varphi^0_p - \frac{1}{2}(\varphi^0_p + \varphi^0_q)).$$

We observe that Proposition 3.6 implies

$$|\Phi^0_{p,q}(u^0_p, u^0_q) - \Phi^0_{p,q}(0,0)| \le L' m_{pq}(|u^0_p| + |u^0_q|) \le 2(b_0 - a_0)L' m_{pq},$$

and that

$$|\varphi^0_p - \varphi^0_q| \le \delta t \; h \max_{(x,t)\in Q_T} |\nabla\varphi(x,t)| \chi_{pq},$$

where we denote by $\chi_{pq} = 0$ if $\varphi(x,t) = 0$ on $(Q_p \cup Q_q) \times [0,T)$ and 1 otherwise. This leads to

$$|I_0| \le (b_0 - a_0)L' \delta t \max_{(x,t)\in Q_T} |\nabla\varphi(x,t)| \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \chi_{pq} m_{pq} h,$$

which leads, thanks to geometrical hypotheses (HD), to

$$|I_0| \le C\delta t.$$

We find as well that

$$|I^*_0| \le C\delta t.$$

Let us now turn to the study of $I - I^*$. From (3.3), it follows $\sum_{q\in\mathcal{N}_p} \Phi^n_{p,q}(u,u) = 0$ and thus

$$I = \sum_{n=1}^{N-1} \sum_{p\in\mathbb{N}} \sum_{q\in\mathcal{N}_p} \left(\Phi^n_{p,q}(u^n_p, u^n_q) - \Phi^n_{p,q}(u^n_p, u^n_p)\right) \varphi^n_p = I_1 - I_2,$$

where

$$I_1 := \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \left(\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_p^n, u_p^n)\right) \varphi_p^n, \quad I_2 := \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \left(\Phi_{p,q}^n(u_p^n, u_q^n) - \Phi_{p,q}^n(u_q^n, u_q^n)\right) \varphi_q^n.$$

Applying the divergence theorem on each cell in the expression of I^*, we find

$$\begin{aligned} I^* &= -\sum_{n=1}^{N-1} \sum_{p\in\mathbb{N}} \int_{t^n}^{t^{n+1}} \int_{\partial Q_p} \Phi(x,t,u_p^n)\cdot\nu_{p,q}\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t \\ &= -\sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left(\Phi(x,t,u_p^n)\cdot\nu_{p,q} - \Phi(x,t,u_q^n)\cdot\nu_{p,q}\right)\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t \\ &= I_1^* - I_2^*, \end{aligned}$$

where

$$\begin{aligned} I_1^* &:= \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left(\frac{1}{m_{p,q}}\Phi_{p,q}^n(u_p^n,u_q^n) - \Phi(x,t,u_p^n)\cdot\nu_{p,q}\right)\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t, \\ I_2^* &:= \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left(\frac{1}{m_{p,q}}\Phi_{p,q}^n(u_p^n,u_q^n) - \Phi(x,t,u_q^n)\cdot\nu_{p,q}\right)\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t. \end{aligned}$$

We first rewrite I_1^* as

$$\begin{aligned} I_1^* = \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} &\left(\Phi_{p,q}^n(u_p^n,u_q^n) - \Phi_{p,q}^n(u_p^n,u_p^n)\right)\frac{1}{m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t \\ &+ \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \left(\frac{1}{m_{p,q}}\Phi_{p,q}^n(u_p^n,u_p^n) - \Phi(x,t,u_p^n)\cdot\nu_{p,q}\right)\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t. \end{aligned}$$

Then, the consistency of the family $\{\Phi_{p,q}^n\}$ with Φ allows us to turn the above equation into

$$\begin{aligned} I_1^* = \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} &\left(\Phi_{p,q}^n(u_p^n,u_q^n) - \Phi_{p,q}^n(u_p^n,u_p^n)\right)\frac{1}{m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\varphi(x,t)\,\mathrm{d}\gamma(x)\mathrm{d}t \\ &+ \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \Phi(x,t,u_p^n)\cdot\nu_{p,q}\frac{1}{\delta t\, m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\left(\varphi(y,s)-\varphi(x,t)\right)\mathrm{d}\gamma(y)\mathrm{d}s\mathrm{d}\gamma(x)\mathrm{d}t. \end{aligned}$$

Hence we can now write $I_1 - I_1^* = A_1 + B_1$ with

$$\begin{aligned} A_1 &:= \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \left(\Phi_{p,q}^n(u_p^n,u_q^n) - \Phi_{p,q}^n(u_p^n,u_p^n)\right)\frac{1}{m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\left(\varphi(x_p,t)-\varphi(x,t)\right)\mathrm{d}\gamma(x)\mathrm{d}t, \\ B_1 &:= \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \Phi(x,t,u_p^n)\cdot\nu_{p,q}\frac{1}{\delta t\, m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\left(\varphi(x,t)-\varphi(y,s)\right)\mathrm{d}\gamma(y)\mathrm{d}s\mathrm{d}\gamma(x)\mathrm{d}t, \end{aligned}$$

and similarly $I_2 - I_2^* = A_2 + B_2$ with

$$\begin{aligned} A_2 &:= \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \left(\Phi_{p,q}^n(u_p^n,u_q^n) - \Phi_{p,q}^n(u_q^n,u_q^n)\right)\frac{1}{m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\left(\varphi(x_q,t)-\varphi(x,t)\right)\mathrm{d}\gamma(x)\mathrm{d}t, \\ B_2 &:= \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \int_{t^n}^{t^{n+1}} \int_{\sigma_{p,q}} \Phi(x,t,u_q^n)\cdot\nu_{p,q}\frac{1}{\delta t\, m_{p,q}}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}}\left(\varphi(x,t)-\varphi(y,s)\right)\mathrm{d}\gamma(y)\mathrm{d}s\mathrm{d}\gamma(x)\mathrm{d}t. \end{aligned}$$

From the mean value theorem and the uniform Lipschitz continuity of $\{\Phi^n_{p,q}\}$, we thus derive the estimates $|A_1| \le D$, $|A_2| \le D$ and $|B_1 - B_2| \le D$, with

$$D = C(h + \delta t) \sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \delta t m_{p,q} |u^n_p - u^n_q|.$$

Using the finite number of neighbours in $\mathcal{D}_h$, hypothesis (HD3), and estimate (29), we obtain the bound

$$\sum_{n=1}^{N-1} \sum_{(p,q)\in\mathcal{E}_h} \delta t m_{p,q} |u^n_p - u^n_q| \le C \sum_{n=1}^{N-1} \sum_{K\in\mathcal{T}_h} \delta t h^d |\nabla \widehat{u}^n_{h|K}| \le C \sum_{n=1}^{N-1} \delta t \int_{\mathbb{R}^d} |\nabla \widehat{u}^n_h| \,\mathrm{d}x \le C. \tag{70}$$

Finally we deduce the estimate $|I - I^*| \le C(h + \delta t)$, which yields $|e^{(4)}_{h,\delta t}| \le C(h + \delta t)$.

□

6 Convergence of the approximate solutions

The following lemma is used in the course of the proof of the convergence theorem.

Lemma 6.1: For all $n \in \mathbb{N}$, let $u_n \in L^1(0, T; BV(\mathbb{R}^d))$ be such that:

1. there exists $C \ge 0$ such that $\int_0^T TV(u_n(\cdot, t))\mathrm{d}t \le C$ for all $n \in \mathbb{N}$,
2. there exists $u \in L^1_{\mathrm{loc}}(Q_T)$ such that $u_n \to u$ in $L^1_{\mathrm{loc}}(Q_T)$.

Then $u \in L^1(0, T; BV(\mathbb{R}^d))$ and

$$\forall \varphi \in C_c(\mathbb{R}^d \times [0, T), \mathbb{R}^+), \liminf_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} \varphi(x, t)|Du_n(\cdot, t)|(\mathrm{d}x)\mathrm{d}t \ge \int_0^T \int_{\mathbb{R}^d} \varphi(x, t)|Du(\cdot, t)|(\mathrm{d}x)\mathrm{d}t. \tag{71}$$

Proof Since $u_n \to u$ in $L^1_{\mathrm{loc}}(Q_T)$, up to a subsequence, for a.e. $t \in (0, T)$, we have that $u_n(\cdot, t) \to u(\cdot, t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Therefore,

$$\text{for a.e. } t \in (0, T),\ TV(u(\cdot, t)) \le \liminf_{n\to\infty} TV(u_n(\cdot, t)),$$

where the quantities in the above inequality may be equal to $+\infty$. Integrating the above inequality and applying Fatou's lemma, we get

$$\int_0^T TV(u(\cdot, t))\mathrm{d}t \le \int_0^T \liminf_{n\to\infty} TV(u_n(\cdot, t))\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T TV(u_n(\cdot, t))\mathrm{d}t \le C.$$

Therefore $u \in L^1(0, T; BV(\mathbb{R}^d))$. Including the weight φ in the definition of the total variation, the proof works the same for getting (71).
□
We may now state the convergence theorem.

Theorem 6.2: Let us assume Hypotheses (HC). Let $(\mathcal{F}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ be a sequence of discretisations which uniformly satisfies the conditions (HD1)-(HD3) and whose meshsize and time step tend to zero. For any $k \in \mathbb{N}$, let $(\overline{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$, $(\widehat{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ and $(\lambda_{h_k,\delta t_k})_{k\in\mathbb{N}}$ be a solution to Scheme (14)-(18). Then there exists $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$, and $\lambda \in L^\infty(Q_T)^d$, with $|\lambda| \le 1$ almost everywhere on Q_T, such that, up to a subsequence,

$$\widehat{u}_{h_k,\delta t_k} \to u \quad \text{in } L^1_{\mathrm{loc}}(Q_T), \qquad \overline{u}_{h_k,\delta t_k} \to u \quad \text{in } L^1_{\mathrm{loc}}(Q_T), \qquad \lambda_{h_k,\delta t_k} \rightharpoonup \lambda \quad \text{weakly-}* \text{ in } L^\infty(Q_T). \tag{72}$$

Moreover, u is the unique entropy solution of (1)-(2), and the whole sequences $(\overline{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$, $(\widehat{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ converge to u in the above sense.

Proof 1. The estimates (34) and (46) allow us to apply Kolmogorov theorem to the sequence $(\overline{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$. Thus, there exist $u \in L^1_{\mathrm{loc}}(Q_T)$ and a subsequence of $(\mathcal{F}_{h_k,\delta t_k})_{k\in\mathbb{N}}$, again denoted by $(\mathcal{F}_{h_k,\delta t_k})_{k\in\mathbb{N}}$, such that $(\overline{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ converges to u in $L^1_{\mathrm{loc}}(Q_T)$. Applying (9) and (29), we get that $\widehat{u}_{h_k,\delta t_k} \to u$ in $L^1_{\mathrm{loc}}(Q_T)$ as well. Applying Lemma 6.1, we get that $u \in L^\infty(Q_T) \cap L^1(0, T; BV(\mathbb{R}^d))$, since the sequence $(\widehat{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ is uniformly

bounded in $L^\infty(Q_T)$ and $L^1(0,T;BV(\mathbb{R}^d))$.
2. As the sequence $(\lambda_{h_k,\delta t_k})_{k\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$, there exists $\lambda \in L^\infty(Q_T)^d$ such that, up to a subsequence, $(\lambda_{h_k,\delta t_k})_{k\in\mathbb{N}}$ converges weakly-$*$ to λ in $L^\infty(Q_T)$. Furthermore, $|\lambda| \le 1$ almost everywhere on Q_T, since $|\lambda_{h_k,\delta t_k}| < 1$ on Q_T.
3. Let us now consider an entropy η and a test function φ. Since the sequence $(\overline{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ is bounded and converges to u in $L^1_{\mathrm{loc}}(Q_T)$, we have

$$\int_{Q_T} \eta(\overline{u}_{h_k,\delta t_k})\partial_t\varphi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \Phi(x,t,\overline{u}_{h_k,\delta t_k})\cdot\nabla\varphi \,\mathrm{d}x\mathrm{d}t \to \int_{Q_T} \Big(\eta(u)\partial_t\varphi + \Phi(x,t,u)\cdot\nabla\varphi\Big)\,\mathrm{d}x\mathrm{d}t.$$

Since $\widehat{u}_{h_k,\delta t_k} \to u$ in $L^1_{\mathrm{loc}}(Q_T)$ and $\lambda_{h_k,\delta t_k} \rightharpoonup \lambda$ weakly-$*$ in $L^\infty(Q_T)$,

$$\int_{Q_T} \eta'(\widehat{u}_{h_k,\delta t_k})\lambda_{h_k,\delta t_k}\cdot\nabla\varphi\,\mathrm{d}t\mathrm{d}x \to \int_{Q_T} \eta'(u)\lambda\cdot\nabla\varphi\,\mathrm{d}t\mathrm{d}x.$$

Again applying Lemma 6.1 to the family $\eta'(\widehat{u}_{h_k,\delta t_k})$, we obtain

$$\liminf_{k\to+\infty}\int_{Q_T}\varphi|\nabla\eta'(\widehat{u}_{h_k,\delta t_k})|\,\mathrm{d}t\mathrm{d}x \ge \int_{Q_T}\varphi\,\big|D[\eta'(u)]\big|\mathrm{d}t. \tag{73}$$

From (14), we get that $\overline{u}^0_{h_k} \to u_{\mathrm{ini}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and thus

$$\int_{\mathbb{R}^d}\eta(\overline{u}^0_h)\varphi(x,0)\,\mathrm{d}x \to \int_{\mathbb{R}^d}\eta(u_{\mathrm{ini}})\varphi(x,0)\,\mathrm{d}x.$$

Finally, using the above limits and Proposition 5.1, we can pass to the limit in (52) and find

$$\int_{Q_T}\Big(\eta(u)\partial_t\varphi + \big(\Phi(x,t,u) - \lambda\eta'(u)\big)\cdot\nabla\varphi\Big)\,\mathrm{d}x\mathrm{d}t - \int_{Q_T}\varphi\,\big|D[\eta'(u)]\big|\mathrm{d}t + \int_{\mathbb{R}^d}\eta(u_{\mathrm{ini}})\varphi(x,0)\ \mathrm{d}x \ge 0,$$

which proves that u is the entropy solution. Owing to the uniqueness of the entropy solution (proved in Section 1.4), we conclude that, in fact, the whole sequence $(\widehat{u}_{h_k,\delta t_k})_{k\in\mathbb{N}}$ converges to u. □

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A A counter-example for an entropy inequality

We detail here why one argument in the proof of [3, Proposition 4.2] is wrong. The difficulties related to the discrete approximation of functions in BV are wellknown, see for example [1, 2]. Here, the difficulty is the use of affine approximation for nonlinear functions of the unknown (note that there exists nevertheless a density result for continuous piecewise functions in the set of BV functions [9]). For simplicity, dropping the function ψ and considering a finite domain $\Omega \subset \mathbb{R}^2$, the point is to show, only using that

1. $\widehat{u}_h$ converges to u in $L^1(\Omega)$ as $h \to 0$,
2. $\lambda_h \in \mathrm{Sgn}(\nabla \widehat{u}_h)$,
3. the mesh includes no obtuse angle,
4. $\beta := \eta'$ is a regular non-decreasing function,

that

$$\liminf_{h\to 0} \int_\Omega \lambda_h(x) \cdot \nabla \widehat{\beta(u_h)}(x) \mathrm{d}x \geq \int_\Omega |D\beta(u)(x)|, \tag{74}$$

where we denote by $\widehat{\beta(u_h)}$ the continuous piecewise affine reconstruction from the values $(\beta(u_p))_{p\in\mathcal{V}_h}$, and $D\beta(u)$ is the vector Radon measure such that

$$\int_\Omega \beta(u(x)) \,\mathrm{div}\phi(x)\,\mathrm{d}x = -\int_\Omega \phi\, D\beta(u), \qquad \forall \phi \in C^1_c(\Omega, \mathbb{R}^d).$$

Note that we always have

$$\liminf_{h\to 0} \int_\Omega \lambda_h(x) \cdot \nabla \beta(\widehat{u}_h)(x) \mathrm{d}x = \liminf_{h\to 0} \int_\Omega |\nabla \beta(\widehat{u}_h)(x)| \mathrm{d}x \geq \int_\Omega |D\beta(u)(x)|,$$

as it is used in (73) when passing to the limit on the approximate entropy inequality. But indeed, (74) is not identical to the preceding one: the affine reconstruction and the nonlinearity are applied in a different order. Let us also observe that (74) holds true in the case where β is affine, which means that η is quadratic (in this case, $\lambda_h(x) \cdot \nabla \widehat{\beta(u_h)}(x) = |\nabla \widehat{\beta(u_h)}(x)|$, see [6, p. 102] for similar comments in the case of nonlinear conservation equations using discontinuous Galerkin methods). It also holds in the one-dimensional case for any non-decreasing function β, since in this case we have $\lambda_h(x) \widehat{\beta(u_h)}'(x) = |\widehat{\beta(u_h)}'(x)|$. We show in this section that there exist a non-affine non-decreasing regular function β and a family $(\widehat{u}_h)_{h>0}$ which is converging to a function u, such that

$$\lim_{h\to 0} \int_\Omega \lambda_h(x) \cdot \nabla \widehat{\beta(u_h)}(x) \mathrm{d}x < \int_\Omega |D\beta(u)| \leq \liminf_{h\to 0} \int_\Omega |\nabla \widehat{\beta(u)}_h(x)| \mathrm{d}x, \tag{75}$$

for any $\lambda_h \in \mathrm{Sgn}(\nabla \widehat{u}_h)$, which is in contradiction with (74). The right inequality in (75) always holds true owing to the lower semi-continuity of the norm.

Lemma A.1: There exist a non-decreasing function $\beta \in C^\infty(\mathbb{R})$, and a family $(\mathcal{T}_h, u_h)_{h>0}$ such that:

1. $\mathcal{T}_h$ is a triangular mesh of $\Omega := (0,1)^2$ such that all triangles are rectangle triangles and h is the greatest diameter of the triangles; we denote by $(x_p)_{p\in\mathcal{V}_h}$ the vertices of the mesh;
2. $u_h = (u_p)_{p\in\mathcal{V}_h}$ is a family of real values at the vertices; we denote by $\widehat{u}_h$ the continuous piecewise affine reconstruction from the values $(u_p)_{p\in\mathcal{V}_h}$;
3. as $h \to 0$, $\widehat{u}_h$ tends in $L^p(\Omega)$ for all $p \in [1, +\infty)$ to the function u defined by $u(x) = 1$ for a.e. $x \in (0, \frac{1}{2}) \times (0,1)$, $u(x) = -1$ for a.e. $x \in (\frac{1}{2}, 1) \times (0,1)$;
4. the inequality (75) holds for any choice of $\lambda_h \in \mathrm{Sgn}(\nabla \widehat{u}_h)$.

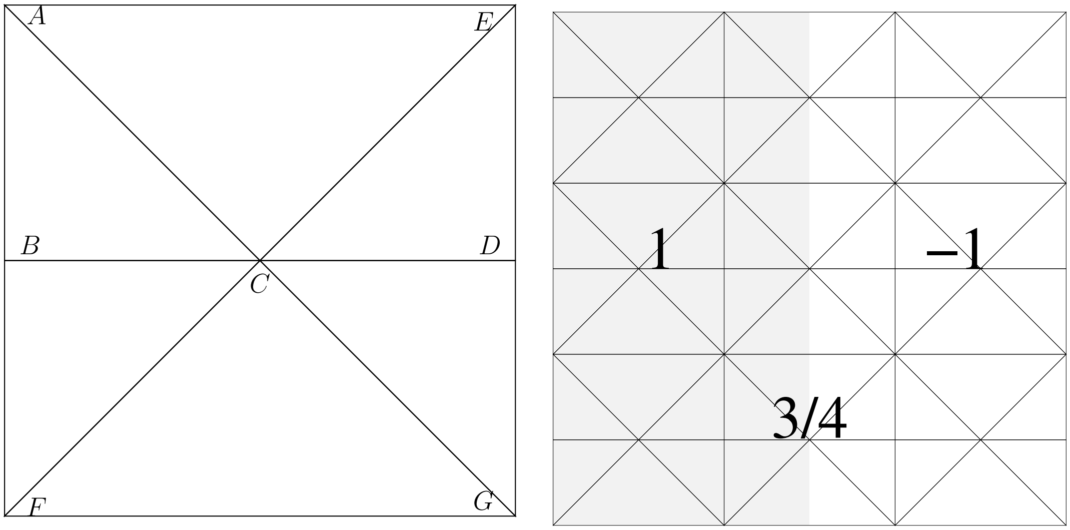

Fig. 1: The elementary pattern (left), the mesh for $N = 3$ (right)

Proof We consider any non-decreasing function $\beta \in C^\infty(\mathbb{R})$ such that $\beta(-1) = \beta(3/4) = 0$ and $\beta(1) = 1$. We build a family of meshes of Ω by assembling the pattern at the left part of Figure 1. So, for a given $n \in \mathbb{N}$, we let $N = 2n+1$, $h = \frac{1}{N}$, and we introduce all the points $(x_p)_{p\in\mathcal{V}_h}$ whose coordinates are (ih, jh) for $i = 0, \ldots, N$ and $j = 0, \ldots, N$ (points type A, E, F or G) or $(ih/2, (j - 1/2)h)$ for $i = 0, \ldots, 2N$ and $j = 1, \ldots, N$ (points type B, C, D). The mesh, represented in the right part of Figure 1, corresponds to the value $N = 3$. We then define $(u_p)_{p\in\mathcal{V}_h}$ by $u_p = 1$ if $x_p \in [0, \frac{1}{2}) \times [0, 1]$, $u_p = -1$ if $x_p \in (\frac{1}{2}, 1] \times [0, 1]$, and $u_p = 3/4$ if $x_p \in \{\frac{1}{2}\} \times [0, 1]$. Let us compute on this grid the expression $A = \int_\Omega \lambda_h(x) \cdot \nabla\widehat{\beta(u_h)}(x)\mathrm{d}x = \sum_{K\in\mathcal{T}_h} A_K$ with $A_K = |K|\,\lambda_K \cdot \nabla\widehat{\beta(u_h)}_K$. We can have $A_K \neq 0$ only for the triangles K whose one of the vertices of the type C is located on the line $\{\frac{1}{2}\} \times [0, 1]$ (for all the other triangles, $\nabla\widehat{\beta(u_h)}_K = 0$).

1. For a triangle type ACE, we have $|K| = h^2/4$, $\nabla\widehat{u}_{h|K} = \frac{1}{h}(-2, -3/2)^t$, hence $\lambda_K = (-4/5, -3/5)^t$ and $\nabla\widehat{\beta(u_h)}_K = \frac{1}{h}(-\beta(1) + \beta(-1), \beta(1) - 2\beta(3/4) + \beta(-1))^t = \frac{1}{h}(-1, 1)^t$, which leads to $A_K = h/20$.
2. For a triangle type CDE or CGD, we get that $\nabla\widehat{\beta(u_h)}_K = 0$ since $\beta(-1) = \beta(3/4)$, and therefore $A_K = 0$.
3. For a triangle type ABC or BCF, we have $|K| = h^2/8$, $\nabla\widehat{u}_{h|K} = \frac{2}{h}(-1/4, 0)^t$, hence $\lambda_K = (-1, 0)^t$ and $\nabla\widehat{\beta(u_h)}_K = \frac{2}{h}(\beta(3/4) - \beta(1), 0)^t = \frac{2}{h}(-1, 0)^t$, which leads to $A_K = h/4$.
4. For a triangle type FGC, we find the same result as that of ACE, leading to $A_K = h/20$.

The sum of all A_K for the pattern is then equal to $\frac{3h}{5}$. Since the number of patterns whose vertex type C is located on the line $\{\frac{1}{2}\} \times [0, 1]$ is equal to $N = 1/h$, we get $A = \int_\Omega \lambda_h(x) \cdot \nabla\widehat{\beta(u_h)}(x)\mathrm{d}x = \frac{3}{5}$, whereas $\int_\Omega |D\beta(u)| = \beta(1) - \beta(-1) = 1$. Hence (75) holds.
□