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Do Generalized Draw-down Times Lead to Better Dividends? A Pontryaghin Principle-Based Answer

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In the context of maximizing cumulative dividends under barrier policies, generalized Azéma-Yor (draw-down) stopping times receive increasing attention during these last years. Based on Pontryaghin's maximality principle, we illustrate the necessity of such generalizations under the framework of spectrally negative Markov processes. Roughly speaking, starting from the explicit expression of the optimal value of discounted dividends in terms of the scale functions, we write down the optimality conditions (via Pontryaghin's principle). The use of generalized draw-downs is then quantified through a structure term (linked to the existence of non bang-bang optimal controls). We thoroughly study several classes of Lévy processes ([8, 15]) constituting the usual models of insurance claims and a particular piece-wise deterministic Markov model (extending the premium rate to reserve-dependent settings). In all these models we disprove the consistency of the aforementioned structure equation, thus denying the necessity of such generalizations. We end the paper with some heuristics on possible non-trivial cases for general Markov models.

Keywords:
Stochastic control; Stochastic jump process; Draw-down; Maximum principle; Dividends.

1. Introduction

The problem of dividend maximization for an insurance company is both very important and extensively studied under different assumptions on the structure of the claims. To give an historical overview, to a given process (roughly speaking associated to the reserve of the aforementioned insurance company) $(X_t)_{t\geqslant 0}$, we associate the running maximum and the draw-down

$$\bar{X}_t := \sup_{0\leqslant s\leqslant t} X_s,\ Y_t := \bar{X}_t - X_t,\ \forall t \geqslant 0.$$

In the first-passage theory for processes $(X_t)_{t\geqslant 0}$, one usually deals with first passage times above or bellow a specified level $a \in \mathbb{R}$

$$T^X_{a+(-)} := \inf\{t \geqslant 0 :\ X_t > (<) a\}, \tag{1.1}$$

where the infimum over empty sets is, as usual, set to be ∞. Whenever no confusion is at risk, we will drop the upper-script X.

The usual draw-down times are defined, for real values $d>0$, by setting

$$\tau_d := \inf\{t \geqslant 0 : Y_t > d\} = \inf\{t \geqslant 0 : X_t < \bar{X}_t - d\}. \tag{1.2}$$

These times intervene in the optimization of discounted dividends as follows. If one considers the classical DeFinetti problem of maximizing up to the ruin time T_{0-} the q-discounted dividends in connection to the wealth process X, then determining the optimal strategy starts by maximizing over levels $b \geqslant 0$ and using as policy the Skorohod regulators constraining the wealth X to remain smaller than b i.e. to consider

$$X_t^{b]} := X_t - U_t,\ \text{where } U_t \left(= U_t^b\right) := \left(\bar{X}_t - b\right)^+ = \max\left(\bar{X}_t - b, 0\right), \tag{1.3}$$

then to maximize the value functions

$$V^b(x) := \mathbb{E}_x\left[\int_0^{T_0^{b]}} e^{-qt} dU_t^b\right],\ \text{where } T_0^{b]} := T_{0-}\mathbf{1}_{T_{0-}<T_{b+}} + \tau_b \mathbf{1}_{T_{b+}<T_{0-}}. \tag{1.4}$$

It is well-known cf. [19] that under some further conditions on the Lévy (jump) measure, which include the case of completely monotone Lévy measures, the value function obtained by maximizing the discounted dividends is given by

$$V(x) = \sup_{b\in\mathbb{R}} V^b(x),\ \forall x \in \mathbb{R}_+.$$

1.1 *Scale Functions and Their Logarithmic Derivatives*

In the context of *spectrally-negative Lévy processes*, computations of the value function V^b given in (1.4) (and, hence, of V) rely on the scale function W_q. Alternatively, after restricting to processes having scale function $W_q \in C^1(0,\infty)$ [12, 13], one can use the logarithmic derivative

$$\nu_q(x) := \frac{W_q'(x)}{W_q(x)}, x \geqslant 0 \tag{1.5}$$

and the explicit formulae for the wining/survival probability, respectively the value function $V^b(x)$ defined in (1.4) are:

$$\begin{aligned} \mathbb{E}_x\left[e^{-qT_{b+}}\mathbf{1}_{T_{b+}<T_{a-}}\right] &= \frac{W_q(x-a)}{W_q(b-a)} = e^{-\int_x^b \nu_q(t-a)dt}, \\ V^b(x) &= \begin{cases} \dfrac{e^{-\int_x^b \nu_q(t)dt}}{\nu_q(b)},\ \text{if } x\in[0,b], \\ x-b+V^b(b),\ \text{if } x>b. \end{cases} \end{aligned} \tag{1.6}$$

Generalizing to a *spectrally-negative-Markov setting* relies on the (killed) survival probability:

$$\bar{\Psi}_q^b(x,a) := \mathbb{E}_x\left[e^{-qT_{b+}}\mathbf{1}_{T_{b+}<T_{a-}}\right], \tag{1.7}$$

for all $q \geqslant 0,\ a \leqslant x \leqslant b$.

Remark 1 *In the Lévy setting, the explicit formula connecting $\bar{\Psi}_q^b$ to the scale function is given in (1.6).*

Few theoretical results exist for spectrally-negative-Markov processes (in particular for the optimality of previously-treated barrier-type policies and/or their generalizations); in particular, the existence and explicit nature of W_q seem to be lost.

Under a mild regularity assumption for the functions $\bar{\Psi}_q$, one can infer a different definition of the function ν_q (which does not directly involve the scale function W_q and is, therefore, more appropriate in the general Markov setting) cf. [17].

Assumption 1 *For every discount parameter $q \geqslant 0$ and every $a \leqslant x$, the application $b \mapsto \bar{\Psi}_q^b(x,a)$ is differentiable at $b = x$. We are then able to define*

$$\nu_q(x,a) := \lim_{\varepsilon \downarrow 0} \frac{1-\bar{\Psi}_q^{x+\varepsilon}(x,a)}{\varepsilon}. \tag{1.8}$$

1.2 *Generalized Draw-down Times*

In analogy to (1.2), for a given positive real function d such that

- $s \mapsto \hat{d}(s) := Id(s) - d(s) = s - d(s)$ is non-decreasing;
- and $\hat{d}(s) \leqslant s,\ \forall s$ (or, equivalently, $d \geqslant 0$),

the generalized draw-time is defined as

$$\tau_d := \inf\{t \geqslant 0 : Y_t > d(\bar{X}_t)\} = \inf\left\{t \geqslant 0 : X_t < \hat{d}(\bar{X}_t)\right\}. \tag{1.9}$$

This kind of generalizations have received increasing attention in the last years [7, 18, 25, 4]. By considering $\tau_d^{b]}$ to be the generalized draw-down time for the reflected process $X^{b]}$, one defines, for the *spectrally-negative Markov setting*, in analogy with (1.4), $V_{b,d}^+(x) = \mathbb{E}_x\left[\int_0^{\tau_d^{b]}} e^{-qt} dU_t^b\right]$ and, owing to [5, Theorem 2.1],

$$V_{b,d}^+(x) := \frac{e^{-\int_x^b \nu_q\left(t,\hat{d}(t)\right)dt}}{\nu_q\left(b,\hat{d}(b)\right)}. \tag{1.10}$$

It is, therefore, natural to consider the following problem

Problem 1

$$V^+(x) := \sup_{b \geqslant x;\ d \in \mathscr{L}^{+,\uparrow}(x)} V_{b,d}^+(x), \text{ where } \mathscr{L}^{+,\uparrow} = \left\{ d : \mathbb{R} \to \mathbb{R}_+,\ [d]_1 := \sup_{t,s \in \mathbb{R}, t \neq s} \frac{|d(t)-d(s)|}{|t-s|} \leqslant 1 \right\},$$
$$V_{b,d}^+(x) = e^{-\int_x^b \nu_q\left(t,\hat{d}(t)\right)dt} \nu_q\left(b,\hat{d}(b)\right).$$

and ask whether V^+ gives better solutions than the use of couples $\left(b, d^b(x)\right) = (b, x-b)$ corresponding to classical (non-generalized) draw-down. In other words, we strive to give (at least for partial frameworks and/or examples) an answer to the question

Question 1 ***Are generalized draw-downs really necessary?***

Remark 2 *The reader is invited to note that the formulation of Problem 1 is stronger than the one encountered in [3]. Indeed, [3] replaces $\mathscr{L}^{+,\uparrow}(x)$ with the more optimization-friendly*

$$\mathscr{L}_1^{+,\uparrow} = \left\{ d : \mathbb{R} \to \mathbb{R}_+ \text{ non-decreasing, } [d]_1 := \sup_{t,s\in\mathbb{R}, t\neq s} \frac{|d(t)-d(s)|}{|t-s|} \in [0,1] \right\}.$$

The main features are

- *compactness of the control parameter $u := d'$ (linked to the non-decreasing assumption);*
- *no state constraint on d (other than $d(x) \geqslant 0$ for the starting datum x, condition incorporated into the initial data and which, together with the non-decreasing property, yields the non-negativeness of d on $[x,b]$).*

In particular, this leads to the application of standard Pontryaghin principle (for bounded controls and without state constraints) in [3, Theorem 5.1]. As a consequence of the compactness of the control parameter $u := d'$, a bang-bang-type result is obtained in the cited paper. Our present case needs (more recent) Pontryaghin results for systems exhibiting running state constraints (e.g. [10] or [11]).

Let us briefly explain the arguments developed throughout the remaining of the paper.

1. In section 2, we present the Pontryagin optimality approach to the study of the problem described. In particular, we focus on a structure equation (2.1) guaranteeing non-triviality of the draw-down function. To make it simple, since we look at the control parameter (derivative of d), it can happen that this belongs to the boundary of the admissible set (i.e. is null) whenever the structure condition is not satisfied. The structure condition will take into account the state-constraint ($d \geqslant 0$) and a bounded variation (increasing) multiplier. As it is usual, this multiplier will only act when the constraint is saturated i.e. $d(t) = 0$. Otherwise, we get a structure equation of PDE-type. Disproving solvability of such structure equations leads to trivial d and the effort of considering generalized Azèma-Yor times is not justified. In the same section, we focus on transformations of the structure equation for functions of Markov processes.

2. In section 3, we focus on the spectrally-negative Lévy processes. In this case, the structure equation leads to a particular (exponential) requirement (see Proposition 6) on the scale function in order for the structure equation (2.1) to hold true. We proceed with showing, for several classical (and important) classes of models, that the structure equation cannot hold true. For all these examples, the conclusion is that generalized draw-downs **do not** improve dividend policies.

3. Section 4 focuses on examples escaping the previous setting. We show that the use of the scale function for Segerdhal-Tichy processes (if optimal) needs not be combined with generalization of the draw-down functions. In this setting, the results are close to those in section 3. However, we design a (fictitious) example [1] showing that the structure equation is a key (non-trivial) feature for more complex systems.

[1] i.e. we use an optimization point of view rather than starting with the generator of a real Markov proceess

2. Pontryaghin Approach to Optimality

2.1 *The Structure Theorem*

We turn to the main theoretical result of the paper strengthening [3, Theorem 5.1] to a larger class of draw-down functions d. We will only focus on the structure assertions (derived from the Hamiltonian and costate component via a Pontryaghin principle for state-constrained trajectories) but, since our aim here is to disprove utility of the generalized draw-downs, we will not emphasize the transversality conditions and their implications on optimal barriers and initial positions.

Theorem 3 *Let us assume that the optimal draw-down function d^{opt} exists. Then it has the following form*

$$d^{opt}(t) = 0 \times \mathbf{1}_{[x,a)}(t) + (t-a)\,\mathbf{1}_{[a,b^{opt}]}(t)$$

(where $a \geqslant x$) unless the "structural equation"

$$\partial_y \nu_q\left(t, t - d^{opt}(t)\right) = 0, \tag{2.1}$$

admits solutions.
In this latter case, the domain $[x, b^{opt}]$ consists of three types of region:

1. *regions on which $d^{opt} = 0$ (such region are included in $\left\{t \geqslant x : \partial_y \nu_q(t,t) < 0\right\}$);*
2. *regions $[a,b]$ on which $u^{opt} = 1$ and, thus, $d^{opt}(t) = d^{opt}(a) + t - a$;*
3. *transition regions (a,b) on which $d^{opt} > 0$ and (2.1) is satisfied.*

Proof. By taking logarithm in (1.10), we get a minimization problem over $b \geqslant x$; $d \in \mathscr{L}^{+,\uparrow}$ for the cost

$$\begin{aligned}&-\log\left(V^{+}_{b,d}(x)\right)\\ =&\log\left(\nu_q\left(x,\hat{d}(x)\right)\right) + \int_x^b \nu_q\left(t,\hat{d}(t)\right)dt + \int_x^b \frac{\partial_x \nu_q\left(t,\hat{d}(t)\right) + \partial_y \nu_q\left(t,\hat{d}(t)\right)d'(t)}{\nu_q\left(t,\hat{d}(t)\right)}dt.\end{aligned}$$

By introducing the supplementary equation $d'(t) = u(t)$, we have a problem with

- freedom on the starting datum $d(x) \geqslant 0$,
- free end point b and datum $d(b)$,
- a state-restriction written in a standard form $g(d) := -d \leqslant 0$,
- the measurable control $u \in (-\infty, 1]$ to take care of upper restriction on the Lipschitz constant of d.

The associated Hamiltonian is

$$H(t,d,u,p) := \nu_q(t,t-d) + \frac{\partial_x \nu_q + \partial_y \nu_q}{\nu_q}(t,t-d) + \left(p - \frac{\partial_y \nu_q(t,t-d)}{\nu_q(t,t-d)}\right)u, \tag{2.2}$$

while the costate $p(t)$ obeys

$$\partial_t p(t) = -\partial_d H(t,d^{opt},u^{opt},p) - g'(d)d\eta_t = -\partial_d H(t,d^{opt},u^{opt},p) + d\eta_t, \tag{2.3}$$

where η is non-decreasing, of bounded variation on $[x, b^{opt}]$ and satisfies the **complementarity** condition

$$\int_x^{b^{opt}} d^{opt}(t)d\eta_t = 0. \tag{2.4}$$

The reader is invited to note that the multiplier η only acts on $d^{opt} = 0$. Due to the linearity (in the control variable u), $u^{opt} = 1$ (extremal control) unless the associated coefficient satisfies

$$p - \frac{\partial_y \nu_q(t, t-d)}{\nu_q(t, t-d)} = 0. \tag{2.5}$$

By taking ∂_t in (2.5) and using (2.3), one deduces

$$\partial_y \nu_q\left(t, t - d^{opt}(t)\right) + d\eta_t = 0. \tag{2.6}$$

We can now make explicit the three types of regions:

1. regions on which $u^{opt} < 1$, $d^{opt} = 0$ and on which η is strictly increasing (for sub-domains on which $d\eta = 0$, see the transition regions) for which (2.6) leads to
$$\partial_y \nu_q\left(t, t - d^{opt}(t)\right) < 0;$$
2. regions $[a, b]$ on which $u^{opt} = 1$ (and the structure equation might not be satisfied) leading to, $d^{opt}(t) = d^{opt}(a) + t - a$; (Please note that such region cannot directly lead back to $d^{opt} = 0$.)
3. transition regions (a, b) on which $u^{opt} < 1$ and $d^{opt} > 0$ (hence $d\eta_t = 0$ a.s.). In this case, (2.6) implies (2.1).

■

Remark 4
1. *For precise references for the Pontryaghin principle with running state constraints, the reader is invited to consult [10] or [11].*
2. *If the initial datum $d(x) > 0$ is not free, in absence of solutions to* (2.1)*, the optimal draw-down picked in this (a priori larger!) class $\mathscr{L}^{+,\uparrow}$ leads to the same optimal draw-down as [3, Lemma 6.1 (1)].*

2.2 *Functions of Markov Processes*

At this point we consider (a slight generalization of) spectrally-negative Markov processes. To be more precise, given a spectrally-negative Markov processes X_t and a C^∞-regular, strictly increasing function $F : \mathbb{R} \longrightarrow \mathbb{R}$, we define X^F by setting

$$F\left(X_t^F\right) = F(x_0) + X_t, \ \forall t \geqslant 0.$$

Corollary 5 *If, for every $x \geqslant 0$, the second partial application associated to the logarithmic derivative ν_q for the initial process X (i.e. $y \mapsto \nu_q(x, y)$) is strictly monotone, then the optimal draw-down function d for the discounted dividends where the wealth is given by X^F is piece-wise affine with slope* 0 *(thus constant functions) or* 1*. Generalizing the draw-down beyond this set is pointless.*

Proof. We proceed by contradiction. Let us assume, on the contrary, that, on some non-trivial time interval (α,β), the optimal draw-down d^{opt} has a derivative in $(-\infty,0)\cup(0,1)$.
Since we deal with two processes X and X^F, we make the convention that absence of superscript refers to the initial process X, while the superscript F indicates that we are talking about characteristics of X^F (e.g. ν_q vs. ν_q^F).
Step 1. Computing the function ν_q^F.
Due to the monotonicity of F, it is clear that T_{b+}^F and T_{a-}^F are simply $T_{(F(b)-F(x_0))+}$, respectively $T_{(F(a)-F(x_0))-}$. Hence, by definition of $\bar{\psi}$, one gets

$$\bar{\psi}_q^{b,F}(x,y)=\bar{\psi}_q^{F(b)-F(x_0)}\left(F(x)-F(x_0),F(y)-F(x_0)\right).$$

Due to the regularity of F and using chain derivation, we have

$$\nu_q^F(x,y)=F'(x)\nu_q\left(F(x)-F(x_0),F(y)-F(x_0)\right), \tag{2.7}$$

for $x,y\in\mathbb{R}_+, y\leqslant x$.
Step 2. The structure equation.
Using Theorem 3 (particularly the structure equation (2.1)), it follows that, due to our initial assumption, on (α,β), d^{opt} satisfies the equation

$$\begin{aligned} 0=&\partial_y\nu_q^F\left(t,t-d^{opt}(t)\right)\\ =&F'(t)F'\left(t-d^{opt}(t)\right)\left(1-\left(d^{opt}\right)'(t)\right)\\ &\times\partial_y\nu_q\left(F(t)-F(x_0),F(t-d^{opt}(t))-F(x_0)\right). \end{aligned} \tag{2.8}$$

Since F is assumed to be strictly monotone and $(d^{opt})'\neq 1$ on (α,β), it follows that

$$\partial_y\nu_q\left(F(t)-F(x_0),F(t-d^{opt}(t))-F(x_0)\right)=0.$$

This provides a contradiction with our assumption that $\partial_y\nu_q(x,y)\neq 0$, $\forall x,y\in\mathbb{R}_+,x\geqslant y$. ■

3. The Spectrally-Negative Lévy Framework

Let us consider the case when X is a *spectrally-negative-Lévy process*. In this case, the process is described by the Laplace transform and the Lévy-Khintchine triplet (μ,σ,Π)

$$\begin{aligned} &\mathbb{E}\left[e^{sX_t}\right]=e^{\psi(s)t},\text{ where}\\ &\psi(s)=-\mu s+\frac{\sigma^2}{2}s^2+\int_{\mathbb{R}_-}\left(e^{sx}-1-sx\mathbf{1}_{|x|<1}\right)\Pi(dx). \end{aligned} \tag{3.1}$$

Moreover, the scale function W_q is known to be the unique right-continuous function such that

$$\int_0^\infty e^{-\beta x}W_q(x)dx=\frac{1}{\psi(\beta)-q}. \tag{3.2}$$

Finally, let us point that, in this setting, the logarithmic derivative function is given by

$$\nu_q(x,y)=\frac{W_q'(x-y)}{W_q(x-y)},\ 0\leqslant y\leqslant x.$$

3.1 *Specialization of the Structure Theorem*

We deduce the following

Proposition 6 *For a spectrally-negative Lévy process whose scale function W_q is of class C^2, the following dichotomy holds true:*

1. *the optimal draw-down function d^{opt} is of bang-bang type between constant functions and 1-affine functions $d^{opt}(x) = x - c$, or*
2. *the function W_q is, locally, of exponential type (i.e. there exists a non-empty set $\left(\hat{\alpha},\hat{\beta}\right) \subset \mathbb{R}_+$ and the constants $c,d \in \mathbb{R}_+$ such that $W_q(x) = ce^{dx}$, $\forall x \in \left(\hat{\alpha},\hat{\beta}\right)$.*

Proof. In this framework, due to the previous considerations, the structure equation (2.1) is equivalent to

$$\left(W_q'' W_q - \left(W_q'\right)^2\right)(d(t)) = 0.$$

Let us assume that the first assertion fails to hold true on some interval (α,β), then, using the fact that d is not constant on (α,β), it follows that, on the non-empty interval $\left(\hat{\alpha},\hat{\beta}\right) := (d(\alpha),d(\beta))$ (recall that d is strictly increasing in this case), W_q satisfies

$$W_q'' W_q - \left(W_q'\right)^2(s) = 0,$$

or, again, by a simple calculus argument, that $W_q(x) = ce^{dx}$, $\forall x \in \left(\hat{\alpha},\hat{\beta}\right)$, for some (non-negative) real constants c,d. ■

Remark 7 *If the time interval $\left(\hat{\alpha},\hat{\beta}\right)$ is the whole real axis, then, by the definition of W_q in (3.2), it follows that X is a deterministic process (i.e. $\sigma = 0, \Pi(dx) = 0$).*

3.2 *First Examples*

We begin with the simplest non-diffusive example.

Example 8 (Cramér-Lundberg) *Let us now consider the usual Cramer-Lundberg model*

$$X_t = x + \mu t - \sum_{n=1}^{N_t} \xi_n,\ \forall t \geqslant 0, \mu > 0$$

where $(\xi_n)_{n\geqslant 1}$ are independent real-valued random variables whose common distribution is of $\rho > 0$-exponential-type and N_t is an independent Poisson process with intensity $\lambda > 0$. In this case, it is easy to check that the Laplace exponent is

$$\psi(s) = \mu s - \frac{\lambda s}{\rho + s}.$$

Computing the scale function starts by solving the equation $\psi(s) = q$. When $\mu \neq 0$, the roots are

$$\begin{aligned}
\zeta_1 &= \frac{1}{2\mu}\left(\sqrt{(\lambda + q - \mu\rho)^2 + 4\mu q\rho} - (\lambda + q - \mu\rho)\right),\\
\zeta_2 &= \frac{1}{2\mu}\left(-\sqrt{(\lambda + q - \mu\rho)^2 + 4\mu q\rho} - (\lambda + q - \mu\rho)\right).
\end{aligned} \tag{3.3}$$

Let us exclude the degenerate trivial case $\zeta_1 = \zeta_2$ which may happen if and only if

- $q = 0$ *and the safety coefficient* $\mu - \lambda \rho^{-1}$ *is null.*
- *no jumps (*$\rho = \infty$ *or* $\lambda = 0$*) when* X *is deterministic.*

Then,

$$W_q(x) = \frac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)} + \frac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)}.$$

In view of Proposition 6, non-trivial draw-downs can only happen if, on some interval $\left(\hat{\alpha}, \hat{\beta}\right)$ *the function* W_q *is of (pure) exponential type; however, this can not happen once outside the historically excluded cases above.*[2]

Example 9 (The tempered stable process) *For this class of processes, the Laplace exponent is given by*

$$\psi(s) = (s+a)^{\alpha} - a^{\alpha},$$

where $s \geqslant 0,\ a \geqslant 0,\ \alpha \in (0,1)$*. In this case, it is known (cf. [16], see also [9]) that*

$$W_q(x) = e^{-ax} x^{\alpha-1} \sum_{n \geqslant 0} \frac{y^n}{\Gamma\left(\alpha(n+1)\right)} \,|_{y=(q+a^{\alpha})x^{\alpha}}.$$

Let us now assume the existence of some $(\hat{\alpha}, \hat{\beta})$ *as in Proposition 6.*
1) On the set $\left(\hat{\alpha}, \hat{\beta}\right)$*, one should have* $x \mapsto \log W^q(x) = \tilde{c} + dx$ *is affine. Hence, by defining*

$$\mathscr{E}_{\alpha,\alpha}(y) := \sum_{n \geqslant 0} \frac{y^n}{\Gamma\left(\alpha(n+1)\right)},$$

one looks for solutions of the equation

$$\log \mathscr{E}_{\alpha,\alpha}\left((q+a^{\alpha})x^{\alpha}\right) + (\alpha-1)\log x = \tilde{c} + (d+a)x,$$

so for affine portions of $x \mapsto \mathscr{E}_{\alpha,\alpha}\left((q+a^{\alpha})x^{\alpha}\right) + (\alpha-1)\log x$*.*
In other words, one looks for the level lines of the modified log-derivative

$$\frac{d}{dx} \log \mathscr{E}_{\alpha,\alpha}\left((q+a^{\alpha})x^{\alpha}\right) + \frac{\alpha-1}{x} = (d+a) > 0.$$

Numerically checking that such level lines cannot yield non-trivial solution is then facilitated by the widely-spread implementation of Mittag-Leffler functions (or, at least, of the Gamma function). To illustrate this, we consider, in Fig. 1, the oscillating case ($a = 0$*) corresponding to* α*-stable processes close to exponential* $\alpha = 1.01$*, respectively for a larger* $\alpha = 1.5$*. We equally zoom on different regions for the initial fortune (where the global picture might appear controversial) to show that, indeed, these*

[2]Indeed, if one has the equality $\frac{e^{a_1 x}}{a_2} + \frac{e^{b_1 x}}{b_2} = e^{cx}$ on some interval, then the function $\frac{e^{(a_1-c)x}}{a_2} + \frac{e^{(b_1-c)x}}{b_2} = 1$ on that interval or, again, the derivative $(a_1-c)\frac{e^{(a_1-c)x}}{a_2} + (b_1-c)\frac{e^{(b_1-c)x}}{b_2} = 0$. Should $a_1 - c \neq 0$, it follows that $\frac{(b_1-c)a_2}{(a_1-c)b_2} e^{(b_1-a_1)x} = -1$ has an infinity of solutions. Then, necessarily, $a_1 = b_1$.

regions also fail to contain non-trivial level-solutions.
2) If $\hat{\beta} = \infty$, then, necessarily,

$$d = (q + a^{\alpha})^{\frac{1}{\alpha}} - a.$$

This is a consequence of the fact that the Mittag-Leffler function defining W_q is, asymptotically, of $\frac{1}{\alpha} y^{\frac{1-\alpha}{\alpha}} exp\left(y^{\frac{1}{\alpha}}\right)$-type (recall that $y = (q + a^{\alpha})x^{\alpha}$). It follows that

$$\lim_{x\to\infty} \frac{W_q(x)}{exp\left(\left((q + a^{\alpha})^{\frac{1}{\alpha}} - a\right)x\right)} \in (0,\infty)$$

which leads to our assertion.

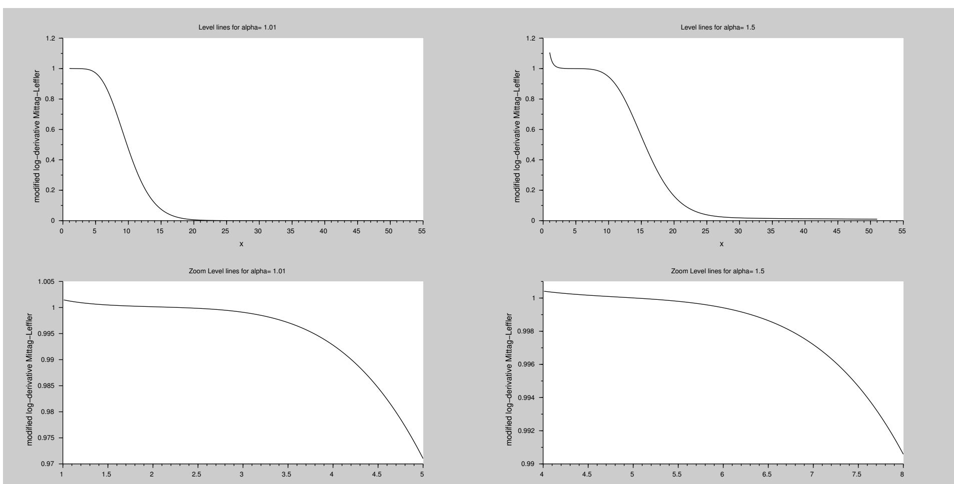

FIG. 1. Modified log-derivative of Mittag-Leffler function $x \mapsto \frac{d}{dx} \log \mathscr{E}_{\alpha,\alpha}((q + a^{\alpha})x^{\alpha}) + \frac{\alpha-1}{x}$.

3.3 *Erlang Mixtures with Finite Number of Components*

Let us now turn to the case in which

Assumption 2 *X has paths of bounded variation and furthermore the Lévy measure Π has rational transform, i.e.*

$$\Pi(dx) = \mathbf{1}_{x<0} \sum_{j=1}^{m} a_j |x|^{m_j - 1} e^{\rho_j x} dx,$$

(for some $m \in \mathbb{N}^*$, $\rho_j > 0$ and $m_j \in \mathbb{N}$).

Recall now that for any Lévy process without positive jumps there exists a function $\hat{u}_q$ with Radon-Nikodym derivative $\frac{\int_{\mathbb{R}_+} \mathbb{P}(-X \in dx)dt}{dx}$ such that

$$W_q(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} - \hat{u}_q(x),$$

where $\Phi(q) := \sup\{s \geqslant 0 : \ \psi(s) = q\}, \forall q \geqslant 0$. (For a proof of this result and the computability advantages of $\hat{u}_q$, the reader is referred to [14, Section 5] and [2, End of Section 3.1], and for numeric experiments with its Laguerre series, to [26].

Under the Assumption 2 one explicitly computes the function $\hat{u}_q$ as

$$\hat{u}_q(x) = -\sum_{j=1}^{n} e^{-\zeta_j x} \sum_{k=1}^{n_j} \frac{c_{j,k}}{(k-1)!} x^{k-1},$$

$\forall x > 0$. Here, the elements ζ_j are the distinct solutions with positive real part to $\psi(\zeta) = q$ and n_j are their respective multiplicities. The constants $c_{j,k}$ are obtained from decomposing $\frac{1}{\psi(x)-q}$ in simple fractions.

In this context, Proposition 6 implies:

Proposition 10 *Let X be a spectrally-negative Lévy process whose Lévy measure Π satisfies Assumption 2 and such that, for every $0 \leqslant s < t \leqslant \infty$ and every constant $k \in \mathbb{R}$,*

$$\hat{u}_q \notin Span_{C([s,t];\mathbb{R})} \left\{ e^{\Phi(q)\cdot}, e^{k\cdot} \right\}.$$

Then the optimal generalized draw-down associated to X is a combination of piecewise-constant functions and 1-affine functions $(x - c)$.

We may conclude:

1. If either $n > 2$ or the multiplicity $n_j > 1$ for some $1 \leqslant j \leqslant n$, then, by applying the previous proposition, d^{opt} is trivial (piece-wisely constant or 1-affine).
2. If $n = 2$ and $n_j = 1$ for $j \in \{1, 2\}$, then

$$\hat{u}_q(x) = -\frac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)} - \frac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)}.$$

   In this case, even if, say, $-\zeta_1 = \Phi(q)$, one gets W_q a linear combination (with non-null coefficients) of two exponentials. Due to Proposition 6, it follows, again, that d^{opt} is trivial.
3. The remaining case leads to $W_q(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))}$ or, again (see Remark 7), to deterministic processes $X_t = x + ct$ (where, of course the obvious strategy is to take dividends as $U_t^0 := x + ct,\ t \geqslant 0$).

3.4 *Completely Monotone Case*

Let us now turn our attention to another case of spectrally negative Lévy processes where results can be made more precise.

Assumption 3 *The measure Π admits a density π such that $\mathbb{R}_+^* \ni x \mapsto \pi(-x)$ is completely monotone*[3] *with respect to the Lebesgue measure on $\mathbb{R}_+$.*

[3] A C^∞-regular function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be completely monotone if, for every $n \geqslant 1$, the n-th order derivative satisfies $(-1)^n f^{(n)} \geqslant 0$ on $\mathbb{R}_+$.

In this case, due to Berstein's theorem (see e.g. [23] or [14, Proof of Theorem 3.4], there exist $a \geqslant 0$, $b \geqslant 0$ some Borel measure μ on $\mathbb{R}_+^*$ such that $t \mapsto 1 \wedge t$ is μ-integrable and

$$\begin{aligned} W_q(x) =& e^{\Phi(q)x}\left(a + bx + \int_{(0,\infty)} \left(1 - e^{-xt}\right)\mu(dt)\right) \\ =& e^{\Phi(q)x} x\left(b + \int_0^\infty e^{-xt}\bar{\mu}(t)dt,\right) \end{aligned} \tag{3.4}$$

where $\mu(\infty) := a,\ \bar{\mu}(t) := \mu\left((t,\infty]\right)$. In this case, the result of Proposition 6 specializes to

Proposition 11 *For a completely monotone spectrally-negative Lévy process (i.e. satisfying Assumption 3), either*

1. *the optimal draw-down function d^{opt} is of bang-bang type between constant functions and 1-affine functions*
   or
2. *there exists a non-empty set $\left(\hat{\alpha}, \hat{\beta}\right) \subset \mathbb{R}_+$ and the constants $b \geqslant 0$, $c, d \in \mathbb{R}$ such that the tail $\bar{\mu}$ satisfies*
$$\int_0^\infty e^{-xt}\bar{\mu}(t)dt = -b + \frac{ce^{dx}}{x}, \forall x \in \left(\hat{\alpha}, \hat{\beta}\right) \tag{3.5}$$

The proof is a consequence of the identity (3.4) and Proposition 6.

Remark 12

1. *The constant c in the second assertion must be non-negative (otherwise the equality cannot hold true).*
2. *The exponent constant d has to satisfy $d \leqslant \frac{1}{\hat{\beta}}$ (otherwise the right-hand member is non-decreasing and the equality in (3.5) cannot hold true).*
3. *If $\hat{\beta} = \infty$, then $b = 0$ (i.e. the diffusion coefficient is null).*

4. Beyond Lévy

Example 13 *An intreaguing, famous non- Lévy process is the Segerdahl-Tichy process studied in [22, 24] (see also [1], [20], [20]) is the Cramér-Lundberg-like example in which the premium function is state-dependent i.e.*

$$X_t = x + \int_0^t p\left(X_s\right)ds - \sum_{n=1}^{N_t} \xi_n.$$

Here, ξ_n are independent real-valued variables exponentially distributed (with parameter ρ) and N is an independent Poisson process of intensity $\lambda > 0$. The optimal dividends problem is far from being resolved in this case [6]. However, fixing a lower barrier $y = 0$ yields still one variable functions $W_q(x)$, $\nu_q(x) := \frac{W_q'(x)}{W_q(x)}$.

When further considering linear premiums

$$p(x) := \mu + \varepsilon x,\ \forall x \in \mathbb{R},$$

formulas become explicit

$$\begin{aligned} W_q(x) = & \varphi\left(\frac{q}{\varepsilon}+1, \frac{\lambda+q}{\varepsilon}+1, \rho\left(x+\frac{\mu}{\varepsilon}\right)\right) \\ & \times(\varepsilon x+\mu)^{\frac{\lambda+q}{\varepsilon}} \times e^{-\rho x} \end{aligned}, \tag{4.1}$$

where the function $\varphi(a,b,x)$ is a solution to the Kummer equation

$$x\varphi''(x)+(b-x)\varphi'(x)-a\varphi(x)=0 \tag{4.2}$$

(see [20, Section 5.1] and the original papers [22, 21] for the related ruin problem).

Let us show now that here as well, assuming the existence of some non-empty set $\left(\hat{\alpha},\hat{\beta}\right)$ on which $W_q(x)=ce^{dx}$ for some constants $c>0, d>0$ is impossible when we exclude the trivial purely deterministic case. Indeed, on this set, $W_q'(x)=dW_q(x)$. By computing derivative in (4.1), we get

$$dW_q(x)=\left(\rho\frac{\varphi'}{\varphi}\left(\rho\left(x+\frac{\mu}{\varepsilon}\right)\right)+\frac{\lambda+q}{\varepsilon x+\mu}-\rho\right)W_q(x),$$

or, again, for some constant $\tilde{c}\geqslant 0$,

$$\varphi(y)=\tilde{c}e^{\left(\frac{d}{\rho}+1\right)y}y^{-\frac{\lambda+q}{\varepsilon}}.$$

Then, by computing first and second order derivatives (if $\tilde{c}\neq 0$ which would lead to $W_q=0$) and substituting them into (4.2), one gets

$$\begin{aligned} & y\left[\frac{\lambda+q}{\varepsilon y^2}+\left(\frac{d}{\rho}+1-\frac{\lambda+q}{\varepsilon y}\right)^2-\left(\frac{d}{\rho}+1-\frac{\lambda+q}{\varepsilon y}\right)\right] \\ & +\left(\frac{\lambda+q}{\varepsilon}+1\right)\left(\frac{d}{\rho}+1-\frac{\lambda+q}{\varepsilon y}\right)-\left(\frac{q}{\varepsilon}+1\right) \\ & =0. \end{aligned} \tag{4.3}$$

By recalling that this should hold true for an infinity of solutions $y\in\left(\rho\left(\hat{\alpha}+\frac{\mu}{\varepsilon}\right),\rho\left(\hat{\beta}+\frac{\mu}{\varepsilon}\right)\right)$ and by multiplying the equality with y^2, it follows that the resulting polynomial should be always 0. This implies, in particular, that the coefficient of y^3 i.e. $\left(\frac{d}{\rho}+1\right)^2-\left(\frac{d}{\rho}+1\right)$ should be 0 which can only happen if $\rho=\infty$ (no actual jumps).

4.1 *A Hint on Possible Non-trivial Draw-downs*

Even though all the examples and frameworks studied so far in this paper turn out to discard the use of generalized draw-downs and even if we conjecture that this should be the case for every one-parameter scale function W_q, let us end the paper with a non-trivial draw-down strategy in a fictitious framework (where the scale function is given *a priori* without constructing it from a true Markov process).

Example 14 *We consider a continuous positive quadratic form $Q:\mathbb{R}^2\to\mathbb{R}_+$ and the two-parameter scale-like function*

$$W(x,y):=e^{\int_0^x Q(z,y)dz}.$$

One easily notes that

$$\frac{\partial_x W(x,y)}{W(x,y)} > 0$$

and sets

$$\nu(x,y) := \frac{\partial_x W(x,y)}{W(x,y)} = Q(x,y).$$

To simplify arguments, one starts with $x = 0$ and $d(x) = 0$ fixed. The aim is to minimize

$$\int_0^b \nu(t, t - d(t))dt + \log\left(\nu\left(b, b - d(b)\right)\right)$$

The best strategy is to pick d such that it minimizes the quadratic terms $Q(t, t - d(t))$ thus, necessarily,

$$\partial_y Q\left(t, t - d^{opt}(t)\right) = 0,$$

(of course, by taking care to insure $b^{opt} > 0$). A plethora of examples can be inferred (e.g. by taking $Q(x,y) = (y - \varphi(x))^2 + \varepsilon + \tilde{Q}(x)$ for some nonlinear φ, a non-negative function $\tilde{Q}$ and $\varepsilon > 0$). The reader is invited to take a look at [3, Section 8] where

$$\varepsilon := \frac{1}{4},\ \varphi(x) := x^2,\ \tilde{Q}(x) = 2\left(x - \frac{1}{2}\right)^2.$$

5. Conclusion

In this short paper, we raise the question of the utility of generalized draw-downs for the problem of maximization of dividends. For all the examples treated in either general classes of spectrally negative Lévy processes or on particularly relevant examples (e.g. particular piecewise-deterministic Markov processes), as soon as the scale function of the fortune process is of one variable, the answer is, invariably negative. The use of non-affine draw-down functions does not improve dividends. We conjecture that, for such processes, as shown in our cases, the draw-down is always trivial (in the sense that, piecewise, it is either constant or of form $x - c$ for some constant $c \geqslant 0$). As shown in the last example (Subsection 4.1), it is imperative to search for the utility of such generalizations in the general Markov case.

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