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Abstract. For a compact set $A \subset \mathbb{R}^d$ and an integer $k \geq 1$, let us denote by

$$A[k] = \{a_1 + \cdots + a_k : a_1, \ldots, a_k \in A\} = \sum_{i=1}^{k} A$$

the Minkowski sum of k copies of A. A theorem of Shapley, Folkmann and Starr (1969) states that $\frac{1}{k}A[k]$ converges to the convex hull of A in Hausdorff distance as k tends to infinity. Bobkov, Madiman and Wang (2011) conjectured that the volume of $\frac{1}{k}A[k]$ is non-decreasing in k , or in other words, in terms of the volume deficit between the convex hull of A and $\frac{1}{k}A[k]$, this convergence is monotone. It was proved by Fradelizi, Madiman, Marsiglietti and Zvavitch (2016) that this conjecture holds true if $d = 1$ but fails for any $d \geq 12$. In this paper we show that the conjecture is true for any star-shaped set $A \subset \mathbb{R}^d$ for arbitrary dimensions $d \geq 1$ under the condition $k \geq d - 1$. In addition, we investigate the conjecture for connected sets and present a counterexample to a generalization of the conjecture to the Minkowski sum of possibly distinct sets in $\mathbb{R}^d$, for any $d \geq 7$.

1. Introduction

The Minkowski sum of two sets $K, L \subset \mathbb{R}^d$ is defined as $K + L = \{x + y : x \in K, y \in L\}$, where, for brevity, we set $A[k] = \sum_{i=1}^{k} A$, for any $k \in \mathbb{N}$ and any compact set $A \subset \mathbb{R}^d$. Since Minkowski sum preserves the convexity of the summands and the set $\frac{1}{k}A[k]$ consists in some particular convex combinations of elements of A, the containment $\frac{1}{k}A[k] \subseteq \operatorname{conv} A$, and, for the special case of convex sets, the equality $\frac{1}{k}A[k] = \operatorname{conv} A$ trivially holds; here $\operatorname{conv} A$ denotes the convex hull of A. These observations suggest that for any compact set A, the set $\frac{1}{k}A[k]$ looks "more convex" for larger values of k. This intuition was formalized by Starr [St1, St2], crediting also Shapley and Folkman, and independently by Emerson and Greenleaf [EG], by proving that the set $\frac{1}{k}A[k]$ approaches $\operatorname{conv} A$ in Hausdorff distance as k approaches infinity and by giving bounds on the speed of this convergence (we refer to [FMMZ2] for more discussion of this fact).

A further step in the investigation of the sequence $\left\{\frac{1}{k}A[k]\right\}$ is to examine the monotonicity of this convergence. Whereas this sequence is clearly not monotonous

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in terms of containment, the main object of this paper is the following conjecture of Bobkov, Madiman, Wang [BMW], relating the volumes of the elements of the sequence, and in which $\mathrm{vol}(K)$ denotes the Lebesgue measure (volume) of the measurable set $K \subset \mathbb{R}^d$.

Conjecture 1 (Bobkov-Madiman-Wang)**.** Let A be a compact set in $\mathbb{R}^d$ for some $d \in \mathbb{N}$. Then the sequence

$$\left\{ \mathrm{vol}\left(\frac{1}{k} A[k]\right) \right\}_{k \geq 1}$$

is non-decreasing in k.

Equivalently, this conjecture asks whether for any integer $k \geq 1$ and compact set $A \subset \mathbb{R}^d$, the following inequality holds

$$\mathrm{vol}\left(\frac{1}{k} A[k]\right) \leq \mathrm{vol}\left(\frac{1}{k+1} A[k+1]\right). \tag{1}$$

This inequality trivially holds for any compact set A if $k = 1$ since $A \subset \frac{1}{2}A[2]$. In the same way, it is easy to find monotone subsequences of the sequence $\{\mathrm{vol}(\frac{1}{k}A[k])\}_{k\geq 1}$ by the same argument; one such example is $\{\mathrm{vol}(\frac{1}{2^m}A[2^m])\}_{m\geq 0}$. On the other hand, even the first nontrivial case; that is, the inequality $\mathrm{vol}\left(\frac{1}{2}A[2]\right) \leq \mathrm{vol}\left(\frac{1}{3}A[3]\right)$ seems to require new methods to approach. Conjecture 1 was partially resolved in [FMMZ1, FMMZ2], where, following the approach of [GMR], the authors proved it for any 1-dimensional compact set A, but constructed counterexamples in $\mathbb{R}^d$ for any $d \geq 12$. More precisely, they showed that for every $k \geq 2$, there is $d_k \in \mathbb{N}$ such that for every $d \geq d_k$ there is a compact set $A \subset \mathbb{R}^d$ such that $\mathrm{vol}\left(\frac{1}{k}A[k]\right) > \mathrm{vol}\left(\frac{1}{k+1}A[k+1]\right)$. In particular, one has $d_2 = 12$, whence Conjecture 1 fails for $\mathbb{R}^d$ if $d \geq 12$.

Our goal is to find additional conditions on A and k under which the statement in Conjecture 1, or more precisely when the inequality (1) is satisfied.

In the paper, for any set $A \subset \mathbb{R}^d$ we denote by $\dim A$ the dimension of the smallest affine subspace containing A, and for any $p, q \in \mathbb{R}^d$, we denote the closed segment with endpoints p, q by $[p, q]$. To state our main result, let us recall the following well-known concept.

Definition 1. A nonempty set $S \subset \mathbb{R}^d$ is called *star-shaped* with respect to a point p if for any $q \in S$, we have $[p, q] \subseteq S$.

Our main result is the following.

Theorem 1. *Let $d \geq 2$ and $k \geq d-1$ be positive integers. Then for any compact, star-shaped set $S \subset \mathbb{R}^d$ we have*

$$\mathrm{vol}\left(\frac{1}{k+1} S[k+1]\right) \geq \mathrm{vol}\left(\frac{1}{k} S[k]\right),$$

with equality if only if $\frac{1}{k}S[k] = \mathrm{conv}(S)$.

We feel it is worth noting that the compact sets A constructed in [FMMZ2] as counterexamples to Conjecture 1 are star-shaped, which makes Theorem 1 fairly unexpected.

We prove Theorem 1 in Section 2. In Section 3 we adapt our techniques to investigate connected sets. Our main result in this section is summarized in Theorem 2. Finally, in Section 4 we collect some additional remarks and questions, and, in particular, we construct low dimensional counterexamples to a generalization of Conjecture 1, which also appeared in [BMW].

2. Conjecture 1 for star-shaped sets: the proof of Theorem 1

We start this section with a couple of Lemmata which are needed for the proof. Throughout this section, we denote $X_d(t) = \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1 + \dots + x_d = t\}$ and $N_d(t) = \operatorname{card} X_d(t)$ to be the number of elements of $X_d(t)$.

Lemma 1. *For any integer $t \geq 1$, and $d \geq 2$, we have $N_d(t) = \binom{t+d-1}{d-1}$.*

Proof. If $d = 2$, then, clearly, $N_2(t) = t + 1 = \binom{t+2-1}{1}$. On the other hand, by induction, we have

$$N_d(t) = \sum_{s=0}^{t} N_{d-1}(s) = \sum_{s=0}^{t} \binom{s+d-2}{d-2} = \binom{t+d-1}{d-1}.$$

□

Lemma 2. *Let o be the origin of $\mathbb{R}^d$, $(p_1, \dots, p_d)$ be a basis of $\mathbb{R}^d$, and, let $B = \bigcup_{i=1}^{d} [o, p_i]$. Consider $M \subset \mathbb{R}^d$ such that $B[k] \subseteq M \subseteq k \operatorname{conv}(B)$, then*

$$\operatorname{vol}\left(\frac{1}{k+1}(M+B)\right) \geq \operatorname{vol}\left(\frac{1}{k}M\right), \tag{2}$$

where, equality holds if and only if $M = k \operatorname{conv}(B)$. Furthermore, if $\operatorname{vol}\left(\frac{1}{k}M\right) \geq \operatorname{vol}\left(\frac{1}{k+1}(M+B)\right) - \delta$ for some $\delta \geq 0$, then $\operatorname{vol}(M) \geq \operatorname{vol}(k \operatorname{conv}(B)) - C(d,k)\delta$ for some constant $C(d,k)$ depending only on d and k.

Proof. Since the inequality (2) is independent of a non-degenerate linear transformation applied to B and M simultaneously, we may assume that $(p_1, \dots, p_d)$ is the canonical basis of $\mathbb{R}^d$. Let

$$V(t) = \operatorname{vol}\{(x_1, \dots, x_d) \in [0,1]^d : x_1 + \dots + x_d \leq t\}.$$

Let $C_i = i + [0,1]^d$, $i \in \mathbb{Z}^d$ be the unit cube cells of the lattice $\mathbb{Z}^d$, and set $\mu_i = \operatorname{vol}(C_i \cap M)$, and $\lambda_i = \operatorname{vol}(C_i \cap (M+B))$.

Note that for any $i \in X_d(t)$, $\operatorname{vol}(C_i \cap k \operatorname{conv}(B))$ is independent of i, namely it is equal to 1, if $t \leq k-d$, and to $V(k-t)$ if $t = k-d+1, \dots, k-1$. A similar statement holds for $\operatorname{vol}(C_i \cap (k+1) \operatorname{conv}(B))$. The number of unit cells contained in $k \operatorname{conv}(B)$ is equal to the number of the solutions of the inequality $x_1 + x_2 \dots + x_d \leq k$, where each variable is a positive integer, and thus, it is $\binom{k}{d}$. This yields that the volume of these cells is $k^{\underline{d}} V$, where $k^{\underline{d}} = k(k-1)\dots(k-d+1)$, and $V = \operatorname{vol}(\operatorname{conv} B) = \frac{1}{d!}$.

Thus,

$$\operatorname{vol}(M) = k^{\underline{d}} V + \sum_{t=k-d+1}^{k-1} \sum_{i \in X_t} \mu_i, \tag{3}$$

and

$$\operatorname{vol}(M+B) = (k+1)^{\underline{d}} V + \sum_{t=k-d+2}^{k} \sum_{i \in X_t} \lambda_i.$$

In the following step, we give a lower bound on the λ_i's depending on the values of the μ_i's. We say that $i \in X_d(t)$ and $i' \in X_d(t+1)$ are *adjacent* if the corresponding cells C_i and $C_{i'}$ have a common facet, or in other words, if $i'-i$ coincides with one of the standard basis vectors p_j. In this case we write $ii' \in I$. Let $i \in X_d(t)$, and let $S = M \cap C_i$. Then, for every $j = 1, 2, \ldots, d$, $S + p_j \subset (M+B) \cap C_{i'}$ with $i' = i + p_j$. Thus, for any $i \in X_d(t+1)$,

$$\lambda_i \geq \max\{\mu_{i'} : i' \in X_d(t) \text{ is adjacent to } i\}. \tag{4}$$

Note that the right-hand side of this inequality is not less than any convex combination of the corresponding $\mu_{i'}$s. We show that, using a suitable convex combination for each $i \in X_d(t+1)$ this inequality implies that

$$\sum_{i \in X_d(t+1)} \lambda_i \geq \frac{t+d}{t+1} \sum_{i \in X_t} \mu_i. \tag{5}$$

Consider some $i = (i_1, i_2, \ldots, i_d) \in X_d(t+1)$. Then the indices in $X_d(t)$ adjacent to i are all of the form $i - p_j$ for some $i = 1, 2, \ldots, d$. Furthermore, $i - p_j$ is adjacent to i iff $i_j \geq 1$, or in other words, iff $i_j \neq 0$. Now, for any $i' \in X_d(t)$ adjacent to i we set $\alpha_{ii'} = \frac{i_j}{t+1}$, where $i - i' = p_j$ (cf. Figure 1). Then, since $i \in X_d(t+1)$, we clearly have $1 = \sum_{j=1}^{d} \frac{i_j}{t+1} = \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'}$. Thus, by (4), we have

$$\lambda_i \geq \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{i'} \tag{6}$$

for all $i \in X_d(t+1)$. Now, let $i' \in X_d(t)$, and $i' = (i'_1, i'_2, \ldots, i'_d)$. Then the indices in $X_d(t+1)$ adjacent to i' are exactly those of the form $i' + p_j$ for some $i = 1, 2, \ldots, d$. Hence,

$$\sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} = \sum_{j=1}^{d} \frac{i'_j + 1}{t+1} = \frac{t+d}{t+1}. \tag{7}$$

Finally, by (6) and (7)

$$\sum_{i \in X_d(t+1)} \lambda_i \geq \sum_{i \in X_d(t+1)} \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{i'} =$$

$$= \sum_{i' \in X_d(t)} \left(\sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} \right) \mu_{i'} = \frac{t+d}{t+1} \sum_{i' \in X_d(t)} \mu_{i'}.$$

Note that the sequence $\left\{ \frac{t+d}{t+1} \right\}$, where $t = 0, 1, 2, \ldots$, is strictly decreasing. Hence, using the fact that if $i \in X_d(t)$, then $\mu_i \leq V(k-t)$, it follows that

$$\operatorname{vol}(M+B) \geq (k+1)^{\underline{d}} V + \frac{k+1}{k-d+2} \sum_{t=k-d+1}^{k-1} \sum_{i \in X_d(t)} \mu_i \tag{8}$$

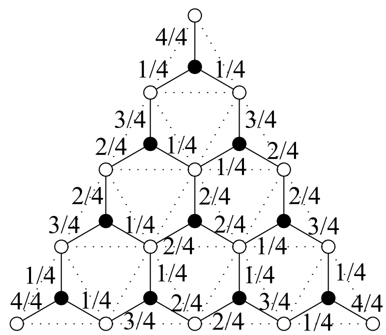

FIGURE 1. Illustration on choosing the weights if $d = 3$ and $t = 3$. The black and empty dots represent the elements of the set $X_3(3)$ and $X_3(4)$, respectively. Dots illustrating adjacent indices are connected by a segment. The weight assigned to the segment connecting the dots representing i and i' is equal to $\alpha_{ii'}$.

$$-\frac{k+1}{k-d+2}\sum_{t=k-d+1}^{k-1} V(k-t)N_d(t) + \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t)N_d(t).$$

Observe that $\sum_{t=k-d+1}^{k-1} V(k-t)N_d(t)$ is the volume of the part of $k\operatorname{conv}(B)$ belonging to cells that are not contained in it, and thus, it is equal to $(k^d - k^{\underline{d}})V$. Similarly,

$$\sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t)N_d(t) = \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t)\binom{t+d-1}{d-1} =$$

$$\sum_{t=k-d+1}^{k-1} \sum_{i\in X_d(t)} V(k-t)\binom{t+d}{d-1} = \sum_{t'=k-d+2}^{k} \sum_{i\in X_d(t')} V(k+1-t')N_d(t')$$

is the volume of the part of $(k+1)\operatorname{conv}(B)$ belonging to cells that are not contained in it. Thus, it is equal to $((k+1)^d - (k+1)^{\underline{d}})V$. Substituting these into (8) and by (3), we obtain

$$\begin{aligned}\operatorname{vol}(M+B) \geq (k+1)^{\underline{d}}V &+ \frac{k+1}{k-d+2}\left(\operatorname{vol}(M) - k^{\underline{d}}V\right) - \frac{k+1}{k-d+2}(k^d - k^{\underline{d}})V \\ &+ ((k+1)^d - (k+1)^{\underline{d}})V \\ = \frac{k+1}{k-d+2}&\operatorname{vol}(M) + \left((k+1)^d - \frac{k+1}{k-d+2}k^d\right)V.\end{aligned}$$

Thus,

$$\operatorname{vol}\left(\frac{1}{k+1}(M+B)\right) \geq \frac{k^d}{(k-d+2)(k+1)^{d-1}}\operatorname{vol}\left(\frac{1}{k}M\right) + \left(1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}}\right)V. \tag{9}$$

Since $\operatorname{vol}\left(\frac{1}{k}M\right) \leq V$, the first inequality of the lemma readily follows.

Now we prove the equality case. By (9), equality in the lemma implies that $\mathrm{vol}\left(\frac{1}{k}M\right) = V$, or equivalently, $\mathrm{vol}(k\,\mathrm{conv}(B)\backslash M) = 0$. Note that since $\mathrm{vol}(k\,\mathrm{conv}(B)) > 0$, its interior is not empty. Thus, $k\,\mathrm{conv}(B)$ is equal to the closure of its interior. On the other hand, $\mathrm{vol}(k\,\mathrm{conv}(B) \setminus M) = 0$ implies that $\mathrm{int}(k\,\mathrm{conv}\,B) \subset M$, but as M is compact, $M = k\,\mathrm{conv}\,B$ follows.

Finally, if $\mathrm{vol}\left(\frac{1}{k+1}(M+B)\right) - \delta \le \mathrm{vol}\left(\frac{1}{k}M\right)$, then in the same way (9) yields the inequality $\mathrm{vol}(M) \ge \mathrm{vol}(k\,\mathrm{conv}(B)) - C(d,k)\delta$, with $C(d,k) = \frac{k^d}{1-\frac{k^d}{(k-d+2)(k+1)^{d-1}}}$. □

Proof of Theorem 1. Without loss of generality, we may assume that S is star-shaped with respect to the origin. Let $\varepsilon > 0$ be an arbitrary positive number. By Carathéodory's theorem, we may choose a finite point set $A_0 \subset S$ such that $\mathrm{vol}(\mathrm{conv}(S)) - \varepsilon \le \mathrm{vol}(\mathrm{conv}(A_0))$, and without loss of generality, we may assume that the points of A_0 are in convex position. Clearly, the star-shaped set $A = \bigcup_{a \in A_0} [o,a]$ is a subset of S, satisfying $\mathrm{vol}(\mathrm{conv}(S)) - \varepsilon \le \mathrm{vol}(\mathrm{conv}(A))$. Consider a simplicial decomposition $\mathcal{F}$ of the boundary of $\mathrm{conv}(A)$ such that all vertices of $\mathcal{F}$ are vertices of $\mathrm{conv}(A)$. Let the $(d-1)$-dimensional faces of $\mathcal{F}$ be $F_1, F_2, \ldots, F_m$, and for $j = 1,2,\ldots,m$, let $B_j = \bigcup_{t=1}^{d} [o, p_t^j]$, where $p_1^j, p_2^j, \ldots, p_d^j$ are the vertices of F_j. Then $B_j \subseteq S$ for all values of j, the sets $\mathrm{conv}(B_j)$ are mutually non-overlapping, and $\mathrm{conv}(A) = \bigcup_{j=1}^{m} \mathrm{conv}(B_j)$. Finally, let $M_j = S[k] \cap (k\,\mathrm{conv}(B_j))$. Then, since $B_j \subseteq S$, we have $B_j[k] \subseteq M_j \subseteq (k\,\mathrm{conv}(B_j))$. Thus, Lemma 2 implies that $\mathrm{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) \ge \mathrm{vol}\left(\frac{1}{k}M_j\right)$, or in other words, $\mathrm{vol}(M_j + B_j) \ge \frac{(k+1)^d}{k^d}\mathrm{vol}(M_j)$. Thus, we have

$$\frac{(k+1)^d}{k^d}\mathrm{vol}(S[k] \cap \mathrm{conv}(kA)) = \sum_{j=1}^{m} \frac{(k+1)^d}{k^d}\mathrm{vol}(M_j) \le \sum_{j=1}^{m} \mathrm{vol}(M_j + B_j) \le \mathrm{vol}(S[k+1]).$$

On the other hand, since $0 \le \mathrm{vol}(\mathrm{conv}(S)) - \mathrm{vol}(\mathrm{conv}(A)) \le \varepsilon$, we have $0 < \mathrm{vol}((S[k] \setminus \mathrm{conv}(kA)) \le k^d\varepsilon$, implying that

$$\mathrm{vol}\left(\frac{1}{k}S[k]\right) - \varepsilon \le \mathrm{vol}\left(\frac{1}{k+1}S[k+1]\right).$$

This inequality is satisfied for all positive ε, and thus, the inequality part of Theorem 1 holds.

Now, assume that

$$\mathrm{vol}\left(\frac{1}{k}S[k]\right) = \mathrm{vol}\left(\frac{1}{k+1}S[k+1]\right).$$

Then, since $\mathrm{vol}((S[k] \setminus \mathrm{conv}(kA)) \le k^d\varepsilon$, it follows that $\mathrm{vol}(S[k+1]) - k^d\varepsilon \le \frac{(k+1)^d}{k^d}\mathrm{vol}(S[k] \cap \mathrm{conv}(kA))$, and thus,

$$\sum_{j=1}^{m}\left(\mathrm{vol}\left(\frac{1}{k+1}(M_j+B_j)\right) - \mathrm{vol}\left(\frac{1}{k}M_j\right)\right) \le \frac{\varepsilon}{(k+1)^d}.$$

For $j = 1, 2, \ldots, m$, set $\delta_j = \mathrm{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) - \mathrm{vol}\left(\frac{1}{k}M_j\right)$. Then, clearly $\sum \delta_j \leq \frac{\varepsilon}{(k+1)^d}$. On the other hand, by Lemma 2, for every $j = 1, 2, \ldots, m$, we have $\mathrm{vol}(k \,\mathrm{conv}\, B_j) - \mathrm{vol}(M_j) \leq C(k,d)\delta_j$ for some constant depending only on k and d. Thus, it follows that

$$\frac{\varepsilon C(k,d)}{(k+1)^d} \geq \mathrm{vol}(\mathrm{conv}(kA)) - \mathrm{vol}(S[k] \cap \mathrm{conv}(kA)),$$

implying that $\varepsilon\left(k^d + \frac{C(k,d)}{(k+1)^d}\right) \geq \mathrm{vol}(\mathrm{conv}(kS)) - \mathrm{vol}(S[k])$. This inequality holds for any value $\varepsilon > 0$, and hence, $\mathrm{vol}(\mathrm{conv}(S)) = \mathrm{vol}\left(\frac{1}{k}S[k]\right)$, or equivalently, $\mathrm{vol}\left(\mathrm{conv}(S) \setminus \frac{1}{k}S[k]\right) = 0$. Since $\mathrm{conv}(S)$ is a compact, convex set with nonempty interior, and $\frac{1}{k}S[k]$ is compact, to show the equality $\mathrm{conv}(S) = \frac{1}{k}S[k]$, we may apply the argument at the end of the proof of Lemma 2. □

3. Conjecture 1 for connected sets

In the first few lemmata we collect some elementary properties of the Minkowski sum of connected sets. Throughout this section, e_1, e_2 denotes the elements of the standard orthonormal basis of $\mathbb{R}^2$.

Lemma 3. *Let $A \subset \mathbb{R}^d$ be a compact set with a connected boundary and let $\partial A \subseteq B \subseteq A$. Then $B + B = A + A$.*

Proof. We have $\partial A + \partial A \subseteq B + B \subseteq A + A$. Thus it is sufficient to prove that $\partial A + \partial A = A + A$. Clearly, $A + A \supseteq \partial A + \partial A$. We show that $\frac{A+A}{2} \subseteq \frac{\partial A + \partial A}{2}$, which then yields the assertion. Consider a point $p \in \frac{A+A}{2}$. Then p is the midpoint of a segment whose endpoints are points of A. Let $\chi_p : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection about p. To prove that $p \in \frac{\partial A + \partial A}{2}$ we need to show that for some $q \in \partial A$, we have $\chi_p(q) \in \partial A$. To do this, let us define $f_p(x)$ ($x \in \mathbb{R}^d$) as the signed distance of $\chi_p(x)$ from the boundary of A, where the sign is positive if $\chi_p(x) \notin A$, and not positive if $\chi_p(x) \in A$. Here we remark that since A is compact, ∂A is compact as well. Let x_1 be a point of ∂A farthest from p. If $\chi_p(x_1) \in A$ then $\chi_p(x_1) \in \partial A$, and we are done. Thus, assume that $\chi_p(x_1) \notin A$, implying that $f_p(x_1) > 0$. Now, since $p \in \frac{A+A}{2}$, we have some $y \in A$ such that $\chi_p(y) \in A$. Let L be the line through y, p and $\chi_p(y)$. Let y' and y'' be points of $L \cap \partial A$ closest to y and $\chi_p(y)$, respectively. If $0 < |y' - y| \leq |y'' - \chi_p(y)|$, then $y' \in \partial A$ and $\chi_p(y') \in A$. If $0 < |y'' - \chi_p(y)| \leq |y' - y|$, then the same holds for y'' in place of y'. Thus, it follows that for some point $x_2 \in \partial A$, $\chi_p(x_2) \in A$. If $\chi_p(x_2) \in \partial A$, then we are done, and so we may assume that $\chi_p(x_2) \in \mathrm{int}\, A$, which yields that $f_p(x_2) < 0$.

We have shown that $f_p : \partial A \to \mathbb{R}$ attains both a positive and a negative value on its domain. On the other hand, since f is continuous and ∂A is connected, $f_p(q) = 0$ for some $q \in \partial A$, from which the assertion readily follows. □

Remark 1. Lemma 3 holds also for the boundary of the external connected component of $\mathbb{R}^d \setminus A$ in place of ∂A.

Remark 2. We note that the equality $A_1 + A_2 = \partial A_1 + \partial A_2$ does not hold in general for different compact sets A_1, A_2 with connected boundaries. To show it, one may consider the sets $A_1 = B_2^2$ and $A_2 = \varepsilon B_2^2$ for some sufficiently small value of ε, where B_2^d be the Euclidean unit ball of dimension d centered at the origin.

Remark 3. Lemma 3 does not hold if we omit the condition that ∂A is connected. To show it, we may choose A as the union of B_2^2 and a singleton $\{p\}$ with $|p|$ being sufficiently large.

Corollary 1. *If A is a compact set with a connected boundary then $A + A = A + \partial A = \partial A + \partial A$. Thus, for any positive integer $k \geq 2$, we have $\sum_{i=1}^k A = \sum_{i=1}^k \partial A$.*

Corollary 2. *For any $k \geq d-1$ and compact set A such that $\partial S \subseteq A \subseteq S$ for some compact, star-shaped set $S \subset \mathbb{R}^d$, we have*

$$\operatorname{vol}\left(\frac{1}{k}A[k]\right) \leq \operatorname{vol}\left(\frac{1}{k+1}A[k+1]\right)$$

Proof. Without loss of generality, we may assume that S is star-shaped with respect to the origin. Set $S' = S + \varepsilon B_2^d$ for some small value $\varepsilon > 0$.

First, we show that $\partial S'$ is path-connected. Let L be a ray starting at o. Since $o \in \operatorname{int} S'$, $L \cap \partial S' \neq \emptyset$. Let $p \in L \cap \partial S'$. Then there is a point $q \in S$ such that $|q-p| = \varepsilon$. Now, if x is any relative interior point of $[o,q]$, then the line through x and parallel to $[p,q]$ intersects $[o,q]$ at a point at distance less than ε from x. Since $[o,q] \subseteq S$, from this it follows that $x \in S + \varepsilon \operatorname{int} B_2^d \subseteq \operatorname{int} S'$. In other words, for any $p \in \partial S'$, all points of $[o,p]$ but p lie in $\operatorname{int} S'$. Thus, $L \cap \partial S'$ is a singleton for any ray L starting at o.

Let $0 < r < R$ such that $\partial S' \subset H = RB_2^d \setminus (r \operatorname{int} B_2^d)$. Let $P : H \to \mathbb{S}^{d-1}$ be the central projection to $\mathbb{S}^{d-1}$. Note that P is Lipschitz, and thus continuous on H, and its restriction $P|_{\partial S'}$ to $\partial S'$ is bijective. On the other hand, since $\partial S'$ (as also S') are compact, this implies that the inverse of $P|_{\partial S'}$ is continuous, that is, $\partial S'$ and $\mathbb{S}^{d-1}$ are homeomorphic. Thus, $\partial S'$ is path-connected.

On the other hand, $\partial S \subseteq A \subseteq S$ implies that $A' = A + \varepsilon B_2^d \subseteq S'$, and $\partial S' \subseteq \partial S + \varepsilon \mathbb{S}^{d-1} \subseteq \partial S + \varepsilon B_2^d \subseteq A'$. Now, we may apply Lemma 3 and Corollary 1, and obtain that for any value of $k \geq d-1$, $A'[k] = S'[k]$. Thus, by Theorem 1 it follows that $\operatorname{vol}\left(\frac{1}{k}A'[k]\right) \leq \operatorname{vol}\left(\frac{1}{k+1}A'[k+1]\right)$. On the other hand, since volume is continuous with respect to Hausdorff distance, we have $\lim_{\varepsilon \to 0^+} \operatorname{vol}\left(\frac{1}{m}A'[m]\right) = \lim_{\varepsilon \to 0^+} \operatorname{vol}\left(\frac{1}{m}A[m] + \varepsilon B_2^d\right) = \operatorname{vol}\left(\frac{1}{k}A[k]\right)$, which implies the corollary. □

Let us denote the closure of a set $A \subset \mathbb{R}^d$ by $\operatorname{cl}(A)$.

Proposition 1. *Let $\gamma \subset \mathbb{R}^2$ be a simple continuous curve connecting o and e_1 such that its intersection with the x-axis is $\{o, e_1\}$. Let D be the interior of the closed Jordan curve $\gamma \cup [o, e_1]$. For $i = 0, 1$, let $\gamma_i = \frac{i}{2}e_1 + \frac{1}{2}\gamma$, and $D_i = \frac{i}{2}e_1 + \frac{1}{2}D$. Then $\operatorname{cl}(D \setminus (D_0 \Delta D_1)) \subseteq \frac{1}{2}\gamma[2]$, where Δ denotes symmetric difference.*

Proof. For convenience, we assume that γ lies in the half plane $\{y \leq 0\}$. As in the proof of Lemma 3, let $\chi_p : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection about $p \in \mathbb{R}^2$, and note that $p \in \frac{1}{2}\gamma[2]$ if and only if there is some point $q \in \gamma$ such that $\chi_p(q) \in \gamma$, or in other words, if $\gamma \cap \chi_p(\gamma) \neq \emptyset$. Let L denote the x-axis, $L_p = \chi_p(L)$, and let S_p be the infinite strip between L and L_p (cf. Figure 2).

First, observe that $o, e_1 \in \gamma$ yields that $\gamma_0 \cup \gamma_1 \subset \frac{1}{2}\gamma[2]$, and $\gamma \subset \frac{1}{2}\gamma[2]$ trivially holds. Thus, we need to show that if for some point p we have $p \in D \setminus \operatorname{cl}(D_0 \cup D_1)$

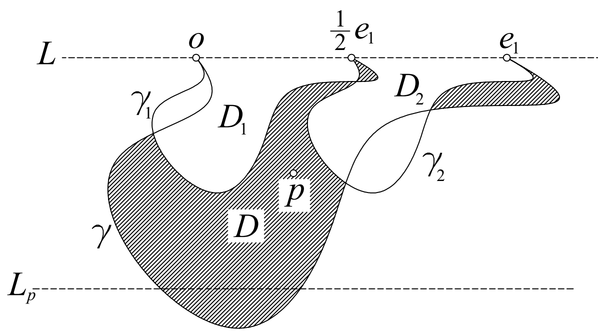

FIGURE 2. An illustration for Proposition 1. The dashed region belongs to $\frac{1}{2}\gamma[2]$.

or $p \in D_0 \cap D_1 \cap D$, then $p \in \frac{1}{2}\gamma[2]$. We do it only for the case $p \in D \setminus \mathrm{cl}(D_0 \cup D_1)$ since for the second case a similar argument can be applied.

Consider some point $p \in D \setminus (D_0 \cup D_1)$. Then $p \notin \mathrm{cl}(D_0 \cup D_1)$ yields that $\chi_p(o) = 2p \notin \mathrm{cl}\, D$, and the relation $\chi_p(e_1) \notin \mathrm{cl}\, D$ follows similarly.

Case 1: $\gamma \subset S$. Note that in this case $\chi_p(\gamma) \subset S$. Since $p \in D$ and $\chi_p(o) \notin \mathrm{cl}\, D$, $\partial D = \gamma \cup [o, e_1]$ and $[\chi_p(o), p] \cap [o, e_1] = \emptyset$, it follows by the continuity of γ that $\gamma \cap [\chi_p(o), p] \neq \emptyset$. Hence, by the compactness of γ, there is a point $x \in \gamma \cap [\chi_p(o), p]$ closest to p. By its choice, $\chi_p(x) \in D \cup \gamma$. If $\chi_p(x) \in \gamma$, we are done, and thus, we assume that $\chi_p(x) \in D$. This implies that $\chi_p(\gamma)$ contains both interior and exterior points of D. On the other hand, since $\chi_p(\gamma) \subset S$, this implies that $\chi_p(\gamma) \cap \gamma \neq \emptyset$.

Case 2: $\gamma \not\subset S$. Let $\gamma_p = \gamma \cap S_p$, and let γ_1 and γ_2 denote the connected components of γ_p containing o and e_1, respectively. For $i = 0, 1$, we denote the endpoint of γ_i on L_p by x_i. Clearly, since γ is simple and continuous, x_1 is on the left-hand side of x_2, and the curve $\gamma_1 \cup [x_1, x_2] \cup \gamma_2 \cup [o, e_1]$ is a Jordan curve. We denote the interior of this curve by D_p.

Consider the case where $p \notin D_p$. Then p is an exterior point of D_p, and there is a connected component γ^* of γ_p, with endpoints on L_p, that separates p from L. Since the reflections of the endpoints of γ^* about p lie on L, we may apply the argument in Case 1, and obtain that $\emptyset \neq \gamma^* \cap \chi_p(\gamma^*) \subseteq \gamma \cap \chi_p(\gamma)$. Thus, we may assume that $p \in D_p$.

If $\chi_p(x_1) \in [o, e_1]$, then the continuity of γ_1 and $\chi_p(o) \notin \mathrm{cl}\, D$ implies that $\emptyset \neq \gamma_1 \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$. If $\chi_p(x_2) \in [o, e_1]$, then we may apply a similar argument, and thus we may assume that $\chi_p(x_1), \chi_p(x_2) \notin [o, e_1]$. This implies that either $[\chi_p(x_1), \chi_p(x_2)]$ and $[o, p_1]$ are disjoint, or $[o, p_1] \subset [\chi_p(x_1), \chi_p(x_2)]$.

Consider the case where $[\chi_p(x_1), \chi_p(x_2)]$ and $[o, p_1]$ are disjoint; without loss of generality we may assume that $\chi_p(x_1)$, $\chi_p(x_2)$, o and e_1 are in this consecutive order on L. Let U be the closure of the connected component of $S \setminus \gamma_1$ containing γ_2. Then $\chi_p(p) = p \in \mathrm{int}\, U \cap \chi_p(U)$, implying that $\emptyset \neq \gamma_1 \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$. Thus, we may assume that $[o, p_1] \subset [\chi_p(x_1), \chi_p(x_2)]$. Since from this it follows that $[\chi_p(o), \chi_p(e_1)] \subset [x_1, x_2]$, $\chi_p(o) \notin \mathrm{cl}\, D$ yields that there is a connected component

γ', with endpoints on L_p, that separates $\chi_p(o)$ from L. Thus, γ' separates $\chi_p(o)$ also from $\chi_p(x_1) \in L$, which yields that $\emptyset \neq \gamma' \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$. □

The proof of Lemma 4 is based on the idea of the proof of Proposition 1, with some necessary modifications.

Lemma 4. *Let $k \geq 2$, and let $\gamma \subset \mathbb{R}^2$ be a convex, continuous curve connecting o and e_1 such that its intersection with the x-axis is $\{o, e_1\}$. Let D be the interior of the closed Jordan curve $\gamma \cup [o, e_1]$. For $i = 0, 1, \ldots, k-1$, let $\gamma_i = \frac{i}{k} e_1 + \frac{1}{k}\gamma$, and $D_i = \frac{i}{k} e_1 + \frac{i}{k} D$. Then $\operatorname{cl}\left(D \setminus \left(\bigcup_{i=1}^k D_i\right)\right) \subseteq \frac{1}{k}\gamma[k]$, and for any $i \neq j$, $D_i \cap D_j \subseteq \frac{1}{k}\gamma[k]$.*

Proof. First observe that D is convex, hence D_i is contained in D for all values of i. In the proof, we denote the x-axis by L, for any $p \in \mathbb{R}^2$ the homothety with center p and ratio $-\frac{1}{k-1}$ by $\chi_p^k : \mathbb{R}^2 \to \mathbb{R}^2$. Furthermore, we set $L_p^k = \chi_p^k(L)$, and denote the infinite strip between L and L_p^k by S. The assertion for $k = 2$ is a special case of Proposition 1. To prove it for $k \geq 3$, we apply induction on k, and assume that the lemma holds for $\gamma[k-1]$.

Let $p \in \operatorname{cl}(D) \setminus (\bigcup_{i=1}^k D_i)$. Clearly, since $\partial D = \gamma \subseteq \gamma[k]$, we may assume that $p \in D$. By the induction hypothesis for $\frac{k-1}{k}\gamma$, if $p \in X_1 = \frac{k-1}{k} \operatorname{cl} D$, then $p \in \frac{k-1}{k} \cdot \frac{1}{k-1}\gamma[k-1] = \frac{1}{k}\gamma[k-1] \subseteq \frac{1}{k}\gamma[k]$. Similarly, if $p \in X_2 = \frac{1}{k}e_1 + \frac{k-1}{k} \operatorname{cl} D$, then $p \in \frac{1}{k}e_1 + \frac{1}{k}\gamma[k-1] \subseteq \frac{1}{k}\gamma[k]$. Thus, assume that $p \notin X_1 \cup X_2$, which yields that $\chi_p^k(o)$ and $\chi_p^k(e_1)$ are in the exterior of D. Let the (unique) intersection point of $[p, \chi_p^k(o)]$ and γ be q_1 and the (unique) intersection point of $[p, \chi_p^k(e_1)]$ and γ be q_2. As $\chi_p^k(q_1) \in [o, p]$, the convexity of D implies that $\chi_p^k(q_1) \in D$, and the containment $\chi_p^k(q_2) \in D$ follows similarly.

Similarly like in Proposition 1, if $\gamma \subset S$, then by continuity, $\gamma \cap \chi_p^k(\gamma) \neq \emptyset$, which implies the containment $p \in \frac{1}{k}\gamma[k]$. Assume that $\gamma \not\subset S$. Then $S \cap \gamma$ has two connected components γ_1, γ_2, where we choose the indices such that $o \in \gamma_1$, and $e_1 \in \gamma_2$. Clearly, we have either $q_1 \in \gamma_2$, $q_2 \in \gamma_1$, or both. If $q_1 \in \gamma_2$, then the containment relations $\chi(q_1) \in D$, $\chi(e_1) \notin \operatorname{cl} D$, and $\chi_p^k(\gamma_2) \subset S$ yield that $\emptyset \neq \gamma_1 \cap \chi_p^k(\gamma_2) \subset \gamma \cap \chi_p^k(\gamma)$. If $q_2 \in \gamma_1$, then the assertion follows by a similar argument.

Finally, we consider the case that $p \in D_i \cap D_j$ for some $i < j$. In this case the convexity of D implies that $p \in D_s$ for any $i \leq s \leq j$. This yields that there are some distinct values $i, j \leq k-1$ or $i, j \geq 2$ such that $p \in D_i \cap D_j$. Thus, the assertion readily follows from the induction hypothesis. □

Lemma 5 is a variant of Lemma 2 for some path-connected sets in $\mathbb{R}^2$.

Lemma 5. *Let $k \geq 2$ and γ be a bounded convex curve in $\mathbb{R}^2$, and let $\gamma[k] \subseteq M \subseteq k \operatorname{conv} \gamma$. Then*

$$\operatorname{area}\left(\frac{1}{k} M\right) \leq \operatorname{area}\left(\frac{1}{k+1}(M + \gamma)\right).$$

Proof. If γ is closed, then Lemma 3 yields that $\frac{1}{k}\gamma[k] = \operatorname{conv}\gamma$ for all $k \geq 2$, which clearly implies the statement. Assume that γ is not closed. Since the inqualities in Lemma 5 do not change under affine transformations, we may assume that the endpoints of γ are o and e_1, and the x-axis is a supporting line of $\operatorname{conv}\gamma$.

Let us define

$$D = \operatorname{conv}\gamma, \alpha = \operatorname{area}(D \cap (e_1 + D)), \text{ and } \beta = \operatorname{area}(D \cap ((e_1 + D) \cup (-e_1 + D))).$$

Note that $0 \le \alpha \le \beta \le 2\alpha$. Let $D_i = ie_1 + D$ for $i = 0, 1, \dots, k$. For $0 \le i \le k-1$, let μ_i be the area of the region of M in D_i that do not belong to any D_j, $j \ne i$, where we note that since $k \ge 2$, by Lemma 4 we have that all other points of D_i belong to M. Similarly, for $0 \le i \le k$, let λ_i be the area of the region of $M + \gamma$ in D_i that do not belong to any D_j, $j \ne i$. An elementary computation shows that

$$\begin{aligned}\operatorname{area}(M) =& k^2 \operatorname{area}(D) - 2(\operatorname{area}(D) - \alpha) - (k-2)(\operatorname{area}(D) - \beta) + \sum_{i=0}^{k-1} \mu_i \\ =& (k^2 - k)\operatorname{area}(D) + 2\alpha + (k-2)\beta + \sum_{i=0}^{k-1} \mu_i,\end{aligned} \tag{10}$$

and similarly,

$$\operatorname{area}(M + \gamma) = (k^2 + k)\operatorname{area}(D) + 2\alpha + (k-1)\beta + \sum_{i=0}^{k} \lambda_i. \tag{11}$$

Since $o, e_1 \in \gamma$, we have $M, e_1 + M \subseteq M + \gamma$. Thus, $\lambda_0 \ge \mu_0$, $\lambda_k \ge \mu_{k-1}$, $\lambda_1 \ge \max\{\mu_0 - (\beta - \alpha), \mu_1\}$, $\lambda_{k-1} \ge \max\{\mu_{k-2}, \mu_{k-1} - (\beta - \alpha)\}$, and for $2 \le i \le k-2$, $\lambda_i \ge \max\{\mu_{i-1}, \mu_i\}$. Since $\lambda_i \ge \frac{i}{k}\mu_{i-1} + \frac{k-i}{k}\mu_i$ if $2 \le i \le k-2$, and $\lambda_i \ge \frac{i}{k}\mu_{i-1} + \frac{k-i}{k}\mu_i - \frac{1}{k}(\beta - \alpha)$ if $i = 1$ or $i = k-1$, it follows that $\sum_{i=0}^{k} \lambda_i \frac{k+1}{k} \ge \sum_{i=1}^{k-1} \mu_i - \frac{2}{k}(\beta - \alpha)$. Thus, by (10),

$$\sum_{i=0}^{k} \lambda_i \ge \frac{k+1}{k}\Big(\operatorname{area}(M) - (k^2 - k)\operatorname{area}(D) - 2\alpha - (k-2)\beta\Big) - \frac{2}{k}(\beta - \alpha).$$

After substituting this into (11) and simplifying, we obtain

$$\operatorname{area}(M + \gamma) \ge \frac{k+1}{k}\operatorname{area}(M) + (k+1)\operatorname{area}(D),$$

which yields

$$\operatorname{area}\left(\frac{1}{k+1}(M + \gamma)\right) \ge \frac{k}{k+1}\operatorname{area}\left(\frac{1}{k}M\right) + \frac{1}{k+1}\operatorname{area}(D).$$

Thus, the inequality $\operatorname{area}\left(\frac{1}{k}M\right) \le \operatorname{area}(D)$ yields the assertion. □

In Theorem 2, by an open topological disc we mean the bounded connected component defined by a Jordan curve.

Theorem 2. *Let $k \ge 2$. Let K be a plane convex body, and let $\mathcal{F} = \{F_i : i \in I\}$ be a family of pairwise disjoint topological discs open in K such that if $F_i \cap \partial K \ne \emptyset$ then $F_i \cap \partial K$ is a connected curve and F_i is convex. Let $X = K \setminus \left(\bigcup_{i \in I} F_i\right)$. Then*

$$\operatorname{area}\left(\frac{1}{k}X[k]\right) \le \operatorname{area}\left(\frac{1}{k+1}X[k+1]\right).$$

Proof. First, note that since each member of $\mathcal{F}$ has positive area, it has countably many elements; indeed, for any $\delta > 0$ there are only finitely many elements F_i of $\mathcal{F}$ for which $\operatorname{area}(F_i \cap K) \geq \delta$, and thus, we may list the elements according to area. Furthermore, since X is compact, $\operatorname{area}(X)$ exists.

By Lemma 3, we may assume that every member of $\mathcal{F}$ intersects ∂K. For any $i \in I$, let γ_i denote the part of ∂F_i in K. Clearly, γ_i is a convex curve, and the line through two of its endpoints supports $K \setminus F_i$. Choose some finite subfamily $I_\varepsilon \subseteq I$ such that $\operatorname{area}(X_\varepsilon \setminus X) \leq \varepsilon$, where $X_\varepsilon = K \setminus \left(\bigcup_{i \in I_\varepsilon F_i}\right)$. This is possible, since for any ordering of the elements, $\sum_{i \in I} \operatorname{area}(K \cap F_i)$ is a bounded series with positive elements, and hence, it is absolute convergent, and convex sets with small area and bounded diameter are contained in a small neighborhood of their boundary.

For any $i \in I_\varepsilon$, we set $D_i = F_i \cap K$, and observe that D_i is a convex set separated from X_ε by the convex curve γ_i. Clearly, for $k \geq 2$, only points in the D_i's may not belong to $\frac{1}{k} X_\varepsilon[k]$, and by Lemma 4, only points contained in exactly one homothetic copy $\frac{1}{k} D_{ij}$, $j = 1, 2, \ldots, k$ of $\frac{1}{k} D_i$ in D_i. Let $M_i = (X[k] \cap (kD_i))$. Then $M_i \subseteq \operatorname{conv}(kD_i)$, and thus, Lemma 5 yields that

$$\operatorname{area}\left(\frac{1}{k} M_i\right) \leq \operatorname{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right).$$

On the other hand, with the notation $D_\varepsilon = \bigcup_{i \in I_\varepsilon} D_i$, we have

$$\operatorname{area}\left(\frac{1}{k} X_[k] \cap D\right) = \sum_{i \in I_\varepsilon} \operatorname{area}\left(\frac{1}{k} M_i\right),$$

and

$$\operatorname{area}\left(\frac{1}{k} X_[k+1] \cap D\right) \geq \sum_{i \in I_\varepsilon} \operatorname{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right),$$

and thus, we have $\operatorname{area}\left(\frac{1}{k} X_[k] \cap D\right) \leq \operatorname{area}\left(\frac{1}{k+1} X[k+1] \cap D\right)$. On the other hand, since $\operatorname{area}(X_\varepsilon \setminus X) < \varepsilon$, $X_\varepsilon \cup D = \operatorname{conv} X$, and $X \subseteq X_\varepsilon$, we have that $\operatorname{area}(\frac{1}{m} X_[m] \setminus D) \leq \varepsilon$ for all $m \geq 1$. This implies that

$$\operatorname{area}\left(\frac{1}{k} X_[k]\right) \leq \operatorname{area}\left(\frac{1}{k+1} X[k+1]\right) - \varepsilon.$$

This holds for all $\varepsilon > 0$, which yields the assertion. □

4. Additional remarks and questions

Remark 4. One can ask if the statement of Theorem 1 holds for arbitrary measure instead of volume. The answer to this question is negative. Indeed, consider the measure $\mu(K) = \operatorname{vol}(K \cap C)$, where $C = \left[-\frac{1}{d}, \frac{1}{d}\right]^d$ and $S = \bigcup_{i=1}^d [o, e_i]$, where $e_1, e_2, \ldots, e_d$ are the vectors of the standard orthonormal basis. Then, clearly, we have

$$\mu\left(\frac{1}{2k} S[2k]\right) = \frac{1}{2^d} \operatorname{vol}(C) > \mu\left(\frac{1}{2k+1} S[2k+1]\right).$$

Remark 5. The statement of Theorem 1 does not hold for arbitrary measures even for rotationally invariant measures in the plane: for any value of k there is a compact, star-shaped set $S \subset \mathbb{R}^2$ such that $\operatorname{vol}\left(\frac{1}{k} S[k] \cap B_2^2\right) > \operatorname{vol}\left(\frac{1}{k+1} S[k+1] \cap B_2^2\right)$.

To prove this, set $S = [o, e_1] \cup [o, e_2]$, and let E denote the ellipse centered at o and containing the points $(1-1/k, 0)$ and $(1-2/k, 1/k)$. It is an elementary computation to check that in this case $\mathrm{vol}\left(\frac{1}{k}S[k] \cap E\right) = \frac{1}{4}\mathrm{vol}(E)$. On the other hand, the boundary point $(1-2/(k+1), 1/(k+1))$ of $\frac{1}{k+1}S[k+1]$ lies in $\mathrm{int}(E)$, which implies that $\mathrm{vol}\left(\frac{1}{k+1}S[k+1] \cap E\right) < \frac{1}{4}\mathrm{vol}(E)$. Now, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the linear transformation mapping E into B_2^2, then $f(S)$ satisfies the required conditions.

One can use star-shaped sets together with ideas from [FMMZ2] to give a negative answer to a more general version of Conjecture 1, also from [BMW].

Conjecture 2 (Bobkov-Madiman-Wang)**.** For any $k \geq 2$, star-shaped compact sets $A_1, A_2, \dots, A_{k+1}$ in $\mathbb{R}^d$, we have

$$\mathrm{vol}\left(\sum_{i=1}^{k+1} A_i\right)^{1/d} \geq \frac{1}{k}\sum_{i=1}^{k+1} \mathrm{vol}\left(\sum_{j\neq i} A_j\right)^{1/d}.$$

in particular, for $k = 2$,

$$\begin{aligned}(12)\qquad &\mathrm{vol}(A_1 + A_2 + A_3)^{1/d}\\ &\geq \frac{1}{2}\left(\mathrm{vol}_n\left(A_1 + A_2\right)^{1/d} + \mathrm{vol}\left(A_1 + A_3\right)^{1/d} + \mathrm{vol}\left(A_2 + A_3\right)^{1/d}\right).\end{aligned}$$

The above conjecture is trivial for convex sets. Moreover, (12) is true when $A_1 = A_2$ and A_1 is convex. Indeed, in this case (12) is equivalent to

$$\mathrm{vol}\left(A_1 + A_1 + A_3\right)^{1/d} \geq \frac{1}{2}\left(\mathrm{vol}\left(2A_1\right)^{1/d} + 2\,\mathrm{vol}_n\left(A_1 + A_3\right)^{1/d}\right).$$

$$\mathrm{vol}\left(A_1 + A_1 + A_3\right)^{1/d} \geq \mathrm{vol}\left(A_1\right)^{1/d} + \mathrm{vol}\left(A_1 + A_3\right)^{1/d},$$

where the last inequality follows from the Brunn-Minkowski inequality [Sch].

It was proved in [FMMZ2] that Conjecture 2 is true in $\mathbb{R}$. Since an affirmative answer to Conjecture 2 implies also Conjecture 1, the former is also false for $d \geq 12$ by [FMMZ1, FMMZ2]. Here we show that Conjecture 2 is false in $\mathbb{R}^d$ even for $d \geq 7$.

Proposition 2. *For any $d \geq 7$, there are compact, star-shaped sets $A_1, A_2, A_3 \subset \mathbb{R}^d$ satisfying*

$$\mathrm{vol}\left(A_1 + A_2 + A_3\right)^{1/d} < \frac{1}{2}\left(\mathrm{vol}\left(A_1 + A_2\right)^{1/d} + \mathrm{vol}\left(A_1 + A_3\right)^{1/d} + \mathrm{vol}\left(A_2 + A_3\right)^{1/d}\right).$$

Proof. We give the proof for $d = 7$ and the result follows for $d > 7$ by taking direct products with a cube. consider the sets

$$A_1 = [0,1]^4 \times \{0\}^3; A_2 = \{0\}^4 \times [0,1]^3 \text{ and } A_3 = ([0,a]^4 \times \{0\}^3) \cup (\{0\}^4 \times [0,b]^3),$$

where we select $a, b > 0$ later. An elementary consideration shows that

$$\mathrm{vol}(A_1 + A_3) = b^3, \mathrm{vol}(A_2 + A_3) = a^4 \text{ and } \mathrm{vol}(A_1 + A_2) = 1,$$

and

$$\mathrm{vol}(A_1 + A_2 + A_3) = (a+1)^4 + (b+1)^3 - 1.$$

The last step is to show that, with $a = 3$ and $b = 6$, the quantity

$$((a+1)^4 + (b+1)^3 - 1)^{1/7} - \frac{1}{2}\left(a^{4/7} + b^{3/7} + 1\right)$$

is negative, which gives a counterexample to (12). □

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