# Planar random walk in a stratified quasi-periodic environment

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**Abstract**

Completing [3, 4, 5], we study the recurrence of inhomogeneous Markov chains in the plane, when the environment is horizontally stratified and the heterogeneity of quasi-periodic type.

## 1 Introduction

The present work investigates the question of the recurrence of a class of inhomogeneous Markov chains in the plane, assuming the environment invariant by horizontal translations. This type of random walks were first considered by Matheron and de Marsily [16] around 1980, motivated by hydrology and the modelization of pollutants diffusion in a porous and stratified ground. In 2003, a discrete version was introduced by Campanino and Petritis [6].

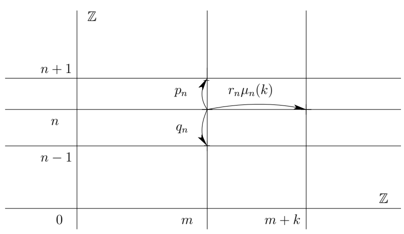
We consider an extension of the latter, studied in [3, 4, 5], restricting here to the plane and simplifying a little the hypotheses. We define a Markov chain $(S_k)_{k\geq 0}$ in $\mathbb{Z}^2$, starting at the origin, such that the transition laws are constant on each stratum $\mathbb{Z}\times\{n\}$, $n\in\mathbb{Z}$. Quantities relative to the first (resp. second) coordinate will be said "horizontal" (resp. "vertical"). For each vertical $n\in\mathbb{Z}$, let positive reals $p_n, q_n, r_n$ with $p_n+q_n+r_n=1$, and a probability measure $\mu_n$ so that :

**Hypothesis 1.1**
$\exists \eta>0$, $\forall n\in\mathbb{Z}$, $\min\{p_n,q_n,r_n\}\geq\eta$, $Supp(\mu_n)\subset]-1/\eta,1/\eta[\cap\mathbb{Z}$, $\mu_n(0)\leq 1-\eta$.

The transition laws are defined, for all $(m,n)\in\mathbb{Z}^2$ and $k\in\mathbb{Z}$, by :

$$(m,n)\overset{p_n}{\longrightarrow}(m,n+1),\ (m,n)\overset{q_n}{\longrightarrow}(m,n-1),\ (m,n)\overset{r_n\mu_n(k)}{\longrightarrow}(m+k,n).$$

Here is the corresponding picture :




The family of transition laws, here $((p_n,q_n,r_n,\mu_n))_{n\in\mathbb{Z}}$, is usually called the "environment". We introduce for the sequel the "local horizontal drift" at height $n\in\mathbb{Z}$, the expectation of $\mu_n$, i.e. $\varepsilon_n := \sum_{k\in\mathbb{Z}} k\mu_n(k)$. The special case when $p_n=q_n$, $n\in\mathbb{Z}$, is called the "vertically flat model".

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With respect to a completely inhomogeneous random walk in the plane, the horizontal stratification brings the notable simplification that the vertical component of $(S_n)$, restricted to vertical jumps, is a Markov chain on $\mathbb{Z}$. We call it the "vertical random walk". Its transition laws are :

$$n \xrightarrow{p'_n} n+1, \; n \xrightarrow{q'_n} n-1,$$

with $p'_n = p_n/(p_n + q_n)$ and $q'_n = q_n/(p_n + q_n)$. The model inherits from this some kind of "product structure". For instance, for $(S_n)$ to be recurrent, the vertical random walk has to be, the conditions for which being well-known. Placing in this case the study of the recurrence of $(S_n)$ essentially reduces to the analysis of the horizontal displacement.

We now discuss former results concerning recurrence/transience for this model. For the vertically flat case, the vertical random walk is just simple random walk on $\mathbb{Z}$ (recurrent). The main object of study has been the Campanino-Petritis model [6], which consists in taking $p_n = q_n = r_n = 1/3$ and $\mu_n = \delta_{\alpha_n}$, fixing some $(\alpha_n)_{n\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$, where $\delta_x$ is Dirac measure at $x$. Recurrence is shown in [6] when $\alpha_n = (-1)^n$ and transience when $\alpha_n = 1_{n\geq 1} - 1_{n\leq 0}$ or if $(\alpha_n)$ are typical realizations of *i.i.d* random variables $(\delta_1 + \delta_{-1})/2$-distributed. For random $(\alpha_n)$, transience results were shown by Guillotin-Plantard and Le Ny [12] when the $(\alpha_n)$ are independent with different marginals and by Pene [17], supposing some stationarity and decorrelation. Devulder and Pene [10] established transience for the model when $p_n = q_n$ and the $(r_n)$ are *i.i.d.* non-constant and $\mu_n = \delta_{\alpha_n}$, with an arbitrary $(\alpha_n) \in \{\pm 1\}^{\mathbb{Z}}$. In [7], Campanino and Petritis studied the case of a random perturbation of a periodic $(\alpha_n)$.

In [3], Theorem 1.2, for the general vertically flat case, a complete recurrence criterion was given. The asymptotics of the random walk is governed by the sums $(r_{-m}\varepsilon_{-m}/p_{-m} + \cdots + r_{n-1}\varepsilon_{n-1}/p_{n-1})$, to be seen as a horizontal flow, associated with the environment, transverse to the vertical layer $[-m, n)$. The central role is played by a two-variables function $\Phi(a, b)$, introduced below, measuring the "horizontal dispersion" of the previous flow between vertical levels $a < b$. The abstract form of the criterion in [3] comes from the computation of a Poisson kernel in a half-plane and the quantity deciding for the recurrence/transience of $(S_n)$ measures some "capacity of dispersion to infinity" of the environment. It involves the level lines of the function $\Phi(a, b)$, in fact some notion of curvature at infinity of these lines. Several examples were next presented in [3], showing in a broad sense that a growth condition like $(\log n)^{1+\delta}$ on $(r_0\varepsilon_0/p_0 + \cdots + r_{n-1}\varepsilon_{n-1}/p_{n-1})$ is sufficient for transience, confirming the natural prevalence of transience results in the litterature on this model.

For the general model, where $p_n$ need not equal $q_n$, a full recurrence criterion was shown in [4], Theorem 2.4, with the same form as in [3], involving some naturally generalized $\Phi(a, b)$; cf Definition 4.5 below. The criterion highlights the fact that the environment defines a new metrization of $\mathbb{Z}^2$, in fact notably more general than in the vertically flat case. Several examples were given in [4]. However, one has to point out that the methods employed for obtaining the structural results of [3, 4] are of very different nature than that used to treat examples. The analysis is in fact naturally divided in two parts, the second one never entering the mechanism of the random walk itself. The second aspect of the question involves the study of fine properties of certain ergodic sums. It is an open source of interesting and difficult problems, in fact closely related to temporal limit theorems and generalizations (cf Dolgopyat-Sarig [11]). In [5] for the general model, the particular case when the transition laws are independent was studied in detail, precisely quantifying the non-surprising fact that the transience regime largely prevails in the set of parameters.

The purpose of the present article is to complete [3, 4, 5], by extending the applications of [3, 4]. We study for both the vertically flat and the general model the case when the transition laws are described by functions defined above an irrational rotation on the one-dimensional torus.

## 2 Preliminaries

Let $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ be the one-dimensional torus. Unless otherwise stated, functions below are defined on $\mathbb{T}$, with arguments understood modulo one. We write $\|x\|$ for the distance of a real $x$ to $\mathbb{Z}$.

Let us recall classical facts about continued fractions. On this topic, cf Khinchin's book [14]. Any irrational $0 < \alpha < 1$ admits an infinite continued fraction expansion :

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [0, a_1, a_2, \cdots],$$

where the partial quotients $(a_i)$ are obtained by iterating the Gauss map $x \longmapsto \{1/x\}$, starting from $\alpha$. The truncations at level $n$ of this continued fraction are irreducible rationals $(p_n/q_n)_{n\geq 1}$, called convergents. Numerators $(p_n)$ and denominators $(q_n)$ check the same recurrence relation :

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, q_{n+1} = a_{n+1}q_n + q_{n-1}, n \geq 0,$$

with initial data $p_0 = 0, p_{-1} = 1$ and $q_0 = 1, q_{-1} = 0$. Taking $\alpha \notin \mathbb{Q}$, classical inequalities are :

$$\frac{1}{2q_{n+1}} \leq \frac{1}{q_n + q_{n+1}} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}}.$$

Fixing $\alpha \notin \mathbb{Q}$, we consider the rotation $Tx = x + \alpha \mod 1$ on $\mathbb{T}$. We often write $T^n f$ for $f \circ T^n$. For functions $f : \mathbb{T} \longmapsto \mathbb{R}$, we introduce cocycle notations :

$$f_n(x) = \begin{cases} f(x) + \cdots + f(T^{n-1}x), & n \geq 1, \\ 0, & n = 0, \\ -f(T^n x) - \cdots - f(T^{-1}x), & n \leq -1. \end{cases}$$

An important property is that $f_{n+p}(x) = f_n(x) + T^n f_p(x)$, for any $x \in \mathbb{T}$, $n, p \in \mathbb{Z}$.

A function $f : \mathbb{T} \longmapsto \mathbb{R}$ with bounded variation will be said BV, with total variation $V(f)$. When $f$ is BV with $\int_{\mathbb{T}} f(x)dx = 0$, the Denjoy-Koksma inequality says that :

$$|f_{q_n}(x)| \leq V(f), n \geq 1, x \in \mathbb{T}.$$

Still fixing a rotation $T$ of angle $\alpha$, with convergents $(q_n)$, let us recall known fact on Ostrowski's expansions. Every integer $q_m \leq n < q_{m+1}$ can be represented as :

$$n = \sum_{0\leq l\leq m} b_l q_l,$$

with $0 \leq b_0 < a_1$, $0 \leq b_j < a_{j+1}$, $1 \leq j < m$, and $1 \leq b_m \leq a_{m+1}$. Setting $A_0 = b_0$ and $A_l = \sum_{0\leq k\leq l} b_k q_k$, we have for a function $f$ :

$$f_n(x) = \sum_{l=0}^{m} f_{b_l q_l}(x + A_{l-1}\alpha).$$

When $f$ is BV and centered, using the Denjoy-Koksma inequality, one gets the upper-bound :

$$|f_n(x)| \leq \sum_{0\leq l\leq m} \|f_{q_l}\|_\infty b_l \leq V(f) \sum_{1\leq l\leq m} b_l, x \in \mathbb{T}.$$

Set $\mathbb{N} = \{0, 1, \cdots\}$. For $g : \mathbb{N} \to \mathbb{R}_+$ increasing to $+\infty$ and $x > 0$, let $g^{-1}(x)$ be the unique $n$ such that $g(n) \leq x < g(n + 1)$. Notice that $g(g^{-1}(x)) \leq x < g(g^{-1}(x) + 1)$ and $g^{-1}(g(n)) = n$, when $n \in \mathbb{N}$. Also, for $f, g : \mathbb{N} \to \mathbb{R}_+$, we write $g \preceq f$ if there exists $C > 0$ so that $g(n) \leq Cf(n)$ for large $n$. We write $f \asymp g$ if $g \preceq f$ and $f \preceq g$.

# 3 The vertically flat model in a quasi-periodic setting

In this section we study the vertically flat model, i.e. $p_n = q_n = (1 - r_n)/2$, $n \in \mathbb{Z}$. As a preliminary remark, consider the case when the sequence $(\varepsilon_n r_n/(1 - r_n))_{n\in\mathbb{Z}}$ is periodic of period $A \geq 1$. Then the random walk is either recurrent or transient, according to whether :

$$\sum_{0\leq n<A} \varepsilon_n \left(\frac{r_n}{1 - r_n}\right) = 0 \text{ or } \neq 0,$$

as follows from [3], respectively Prop 1.4, *i*) and Corollary 1.3, *i*). This extends the case of the Campanino-Petritis model in [6] when $\alpha_n = (-1)^n = \varepsilon_n$, $p_n = q_n = r_n = 1/3$.

Turning to quasi-periodic situations, our purpose is to generalize [3], Prop. 1.5, giving in particular a better understanding of the quasi-periodic Campanino-Petritis model.

**Theorem 3.1**
*Let* $\alpha \notin \mathbb{Q}$ *with expansion* $[0, a_1, a_2, \cdots]$ *and* $Tx = x + \alpha$ *on* $\mathbb{T}$. *Let* $f : \mathbb{T} \to \mathbb{R}$, *with finitely many discontinuities and* $K$*-Lipschitz on each contiguous open interval, having zero mean. Under hypothesis 1.1, let* $p_n = q_n$ *and* $\varepsilon_n r_n/(1 - r_n) = f(T^n x)$, $n \in \mathbb{Z}$, *for some* $x \in \mathbb{T}$. *Let us finally assume that the following condition holds :*

$$\sum_{n\geq 1} \frac{\log(1 + a_n)}{a_1 + \cdots + a_n} = +\infty.$$

*Then, for any* $x \in \mathbb{T}$, *the random walk is recurrent.*

*Remark.* — This was shown in [3], Prop. 1.5, for $x = 0$. The condition on $\alpha$ is generic in measure, since $\sum_{n\geq 1} 1/(a_1 + \cdots + a_n) = +\infty$ for a.-e. $\alpha$, by Khinchin [13] (cf [4], Prop. 7.1). As indicated in the previous section, the result applies in particular to piecewise constant $f$ with zero-mean and for example to the Campanino-Petritis model with $\alpha_n = 1_{[0,1/2)}(x + n\alpha) - 1_{[1/2,1)}(x + n\alpha)$, corresponding to $f(x) = (1_{[0,1/2)}(x) - 1_{[1/2,1)}(x))/2$.

A priori the condition on $\alpha$ should be sharp, but we cannot precise this. We just give an example in the other direction, more delicate, since requiring lower bounds on the ergodic sums.

**Proposition 3.2**
*Consider the context of Theorem 3.1, let* $f(x) = \gamma(1_{[0,1/2)}(x) - 1_{[1/2,1)}(x))$, $\gamma \neq 0$. *Let* $a_1$ *odd,* $a_n$ *even,* $n \geq 2$, *and suppose that for some* $\delta > 1$ *and large* $n$, $a_{n+1} \geq (a_n)^\delta$. *Then, for almost-every* $x \in \mathbb{T}$, *the random walk is transient.*

*Remark.* — In the statement of the proposition, one can for example take for angle $\alpha$ the irrational number in $(0, 1)$ with partial quotients $a_n = 2^{2^{n-1}-1}$, $n \geq 1$.

## 3.1 Proof of Theorem 3.1

Let us fix $\alpha \notin \mathbb{Q}$, $f$ and $x \in \mathbb{T}$, as in the statement of the theorem. Recalling cocycle notations for $f$, introduce for $n \geq 1$ the positive functions $\varphi(n)$ and $\varphi_+(n)$ such that :

$$\varphi^2(n) = n^2 + \sum_{-n\leq k<l\leq n} (f_l(x) - f_k(x))^2 \text{ and } \varphi_+^2(n) = n^2 + \sum_{-n\leq k<l\leq n, kl\geq 0} (f_l(x) - f_k(x))^2.$$

Obviously, $n \leq \varphi_+(n) \leq \varphi(n)$. The next lemma gives some control in the other direction.

**Lemma 3.3**
*There exists a constant* $C_0 > 0$, *independent of* $x \in \mathbb{T}$, *such that for all* $n \geq 1$ *and* $1 \leq m \leq 4a_{n+1}$, $\varphi^2(mq_n) \leq 2\varphi_+^2(mq_n) + C_0(m^2 q_n)^2$.

*Proof of the lemma :*
*Step 1.* For the sequel, we simplify $f_n(x)$ into $f_n$. Setting $A = \sum_{-n\leq k\leq -1<1\leq l\leq n}(f_l - f_k)^2$, we have $\varphi^2(n) = \varphi_+^2(n) + A$. Then :

$$A \;=\; n\sum_{1\leq l\leq n} f_l^2 + n\sum_{-n\leq k\leq -1} f_k^2 - 2\sum_{-n\leq k\leq -1} f_k \sum_{1\leq l\leq n} f_l.$$

Next we have :

$$-2\sum_{-n\leq k\leq -1} f_k \sum_{1\leq l\leq n} f_l = \left(\sum_{1\leq l\leq n}(f_l - f_{-l})\right)^2 - \left(\sum_{1\leq l\leq n} f_l\right)^2 - \left(\sum_{1\leq l\leq n} f_{-l}\right)^2.$$

Now, classically :

$$\begin{aligned}\sum_{1\leq k<l\leq n}(f_l - f_k)^2 &= \sum_{2\leq l\leq n}(l-1)f_l^2 + \sum_{1\leq l\leq n-1} f_l^2(n-l) - 2\sum_{1\leq k<l\leq n} f_k f_l \\ &= \sum_{1\leq l\leq n}(l-1)f_l^2 + \sum_{1\leq l\leq n} f_l^2(n-l) - 2\sum_{1\leq k<l\leq n} f_k f_l \\ &= n\sum_{1\leq l\leq n} f_l^2 - \left(\sum_{1\leq l\leq n} f_l\right)^2.\end{aligned}$$

Proceding symmetrically for the other part of $A$, we obtain :

$$A = \sum_{1\leq k<l\leq n}(f_l - f_k)^2 + \sum_{-n\leq k<l\leq -1}(f_l - f_k)^2 + \left(\sum_{1\leq l\leq n}(f_l - f_{-l})\right)^2.$$

Consequently :

$$\varphi^2(n) \leq 2\varphi_+^2(n) + \left(\sum_{1\leq l\leq n}(f_l - f_{-l})\right)^2.$$

*Step 2.* Let $n \geq 1$ and $1 \leq m \leq 4a_{n+1}$. Setting $B = \sum_{1\leq l\leq mq_n}(f_l - f_{-l})$, we have :

$$B \;=\; \sum_{0\leq u<m}\sum_{1\leq l\leq q_n}(f_{uq_n+l} - f_{-uq_n-l}) = \sum_{0\leq u<m}\sum_{1\leq l\leq q_n}(f_{uq_n} - f_{-uq_n} + T^{uq_n}f_l - T^{-uq_n}f_{-l}).$$

Using Denjoy-Koksma's inequality, we have $|f_{uq_n}(x)| \leq uV(f)$, idem for $f_{-uq_n}(x)$. As a result :

$$B \;=\; O(m^2q_n) + \sum_{0\leq u<m}\sum_{1\leq l\leq q_n}(T^{uq_n}f_l - T^{-uq_n}f_{-l}).$$

Fixing $0 \leq u < m$, we have :

$$\begin{aligned}\sum_{1\leq l\leq q_n}(T^{uq_n}f_l - T^{-uq_n}f_{-l}) &= \sum_{k=0}^{q_n-1}(q_n-k)f(x+uq_n\alpha+k\alpha) + \sum_{k=1}^{q_n}(q_n+1-k)f(x-uq_n\alpha-k\alpha) \\ &= \sum_{k=1}^{q_n} k(f(x+uq_n\alpha+(q_n-k)\alpha) - f(x-uq_n\alpha-k\alpha)) + O(q_n),\end{aligned}$$

using the Denjoy-Koksma inequality for removing $q_n+1$ in the second sum on the right-hand side of the first line. Set next $x_+^u = x+(u+1)q_n\alpha$ and $x_-^u = x - uq_n\alpha$. Then, on $\mathbb{T}$, the distance $d(x_+^u, x_-^u)$ is $\leq (2u+1)\|q_n\alpha\| \leq (8a_{n+1}+1)/q_{n+1} \leq 9/q_n$. Since there is exactly one $k\alpha$, $1 \leq k \leq q_n$, in each $[l/q_n, (l+1)/q_n)$, when denoting by $D$ the number of discontinuities of $f$, there is at most $10D$ terms in the above sum such that $[x_-^u, x_+^u] - k\alpha$ contains a discontinuity of $f$ (denoting by $[x_-^u, x_+^u]$ the short interval on $\mathbb{T}$ between $x_-^u$ and $x_+^u$). Hence an upper-bound for the sum is :

$$q_n^2 K \frac{9}{q_n} + 10Dq_n \times 2\|f\|_\infty = O(q_n),$$

where $K$ is a Lipschitz constant for $f$ on each open interval containing no discontinuity. As a result, $B = O(m^2 q_n) + O(mq_n) = O(m^2 q_n)$, which ends the proof of the lemma.

□

We turn to the proof of Theorem 3.1. In [3] Theorem 1.2, the recurrence of the random walk was shown to be equivalent to :

$$\sum_{n\geq 1} \frac{1}{n^2} \frac{(\Phi^{-1}(n))^2}{\Phi_+^{-1}(n)} = +\infty,$$

with the positive $\Phi$, $\Phi_+$ so that $\Phi^2(n) = n^2 + \sum_{-n\leq k\leq l\leq n} (R_k^l)^2$, $\Phi_+^2(n) = n^2 + \sum_{-n\leq k\leq l\leq n, kl>0} (R_k^l)^2$ and $R_k^l = \sum_{k\leq i\leq l} \varepsilon_i r_i/(1-r_i)$. The following property of dominated variation (cf [3], Lemma 6.1), depending only on $\eta$ of Hypothesis 1.1, was central :

$$\Phi^{-1}(2x) \leq C\Phi^{-1}(x) \text{ and } \Phi_+^{-1}(2x) \leq C\Phi_+^{-1}(x), x \geq 1.$$

It is a simple exercise to show that for some constant $C > 0$, depending only on $\eta$, for all $n \geq 1$, $\Phi(n)/C \leq \varphi(n) \leq C\Phi(n)$ and $\Phi_+(n)/C \leq \varphi_+(n) \leq C\Phi_+(n)$. As a result, $\varphi^{-1}$ and $\varphi_+^{-1}$ also check dominated variation and the recurrence criterion rewrites :

$$\sum_{n\geq 1} \frac{1}{n^2} \frac{(\varphi^{-1}(n))^2}{\varphi_+^{-1}(n)} = +\infty,$$

Let us reprove dominated variation in the following lemma, still removing the dependence in $x \in \mathbb{T}$. For $a < b$ in $\mathbb{Z}$, define $\psi(a,b) = \sum_{a\leq k<l\leq b} (f_l - f_k)^2$.

**Lemma 3.4**
*Let integers $a < b < c$. Then :*

$$\frac{\psi(a,c)}{c-a} \geq \frac{\psi(a,b-1)}{b-a} + \frac{\psi(b+1,c)}{c-b}.$$

*Proof of the lemma :*
We decompose $\psi(a,c) = \psi(a,b) + \psi(b,c) + \sum_{a\leq k<b<l\leq c} (f_l - f_k)^2$. Then :

$$\sum_{a\leq k<b<l\leq c} (f_l - f_k)^2 \quad = \quad (b-a)\sum_{b<l\leq c} f_l^2 + (c-b)\sum_{a\leq k<b} f_k^2 - 2\sum_{b<l\leq c} f_l \sum_{a\leq k<b} f_k.$$

As before, $\psi(b+1,c) = (c-b)\sum_{b<l\leq c} f_l^2 - (\sum_{b<l\leq c} f_l)^2$ and $\psi(a,b-1) = (b-a)\sum_{a\leq k<b} f_k^2 - (\sum_{a\leq k<b} f_k)^2$. We obtain :

$$\sum_{a\leq k<b<l\leq c} (f_l - f_k)^2 \quad = \quad \frac{b-a}{c-b}(\psi(b+1,c) + (\sum_{b<l\leq c} f_l)^2) + \frac{c-b}{b-a}(\psi(a,b-1) + (\sum_{a\leq k<b} f_k)^2).$$

As a consequence :

$$\psi(a,c) \quad \geq \quad \frac{c-a}{c-b}\psi(b+1,c) + \frac{c-a}{b-a}\psi(a,b-1) + \left(\sqrt{\frac{b-a}{c-b}}\sum_{b<l\leq c} f_l - \sqrt{\frac{c-b}{b-a}}\sum_{a\leq k<b} f_k\right)^2,$$

giving the result. □

*Remark.* — Observe that :

$$\frac{c-a}{c-b}\psi(b+1,c) + \frac{c-a}{b-a}\psi(a,b-1) \geq (\sqrt{\psi(a,b-1)} + \sqrt{\psi(b+1,c)})^2,$$

as this is equivalent to the true relation :

$$\frac{c-b}{b-a}\psi(a,b-1) + \frac{b-a}{c-b}\psi(b+1,c) \geq 2\sqrt{\psi(a,b-1)}\sqrt{\psi(b+1,c)}.$$

This thus implies some reverse triangular inequality :

$$\sqrt{\psi(a,c)} \geq \sqrt{\psi(a,b-1)} + \sqrt{\psi(b+1,c)}.$$

As a consequence, for integers $1 \leq a < b$, we have $\psi(-b,b) \geq \frac{2b}{a+b+1}\frac{a+b}{2a+1}\psi(-a,a)$, as well as $\psi(0,b) \geq b/(a+1)\psi(0,a)$ and $\psi(-b,0) \geq b/(a+1)\psi(-a,0)$. Hence :

$$\liminf_{b\geq a\to+\infty} \frac{\varphi^2(b)/b}{\varphi^2(a)/a} \geq 1 \text{ and } \liminf_{b\geq a\to+\infty} \frac{\varphi_+^2(b)/b}{\varphi_+^2(a)/a} \geq 1. \tag{1}$$

Concerning dominated variation, for any $K>0$ let $C_K>0$ be a constant so that :

$$\varphi^{-1}(Kx) \leq C_K\varphi^{-1}(x) \text{ and } \varphi_+^{-1}(Kx) \leq C_K\varphi_+^{-1}(x), x \geq 1.$$

We also rewrite the content of Lemma 3.3, as the fact that there exists a constant $C_0 > 0$, independent of $x \in \mathbb{T}$, such that for all $n \geq 1$ and $1 \leq m \leq 4a_{n+1}$ :

$$\varphi(mq_n) \leq C_0(\varphi_+(mq_n) + m^2q_n).$$

Let now $n \geq 1$ and $l \geq 0$ be such that $2^l \leq 4a_{n+1}$. We make the following discussion. If $\varphi_+(2^lq_n) \geq 2^{2l}q_n$, then :

$$\begin{aligned}\varphi^{-1}(\varphi_+(2^lq_n)) \geq \varphi^{-1}((\varphi_+(2^lq_n)+2^{2l}q_n)/2) \quad &\geq \quad \varphi^{-1}(\varphi(2^lq_n)/(2C_0)) \\ &\geq \quad \frac{\varphi^{-1}(\varphi(2^lq_n))}{C_{1/(2C_0)}} = \frac{2^lq_n}{C_{1/(2C_0)}}.\end{aligned}$$

Since by Ostrowski's expansion, $\varphi_+(2^lq_n) \leq C2^lq_n(a_1+\cdots+a_{n+1})$, we hence obtain that for some constant $C_1 > 0$ :

$$\frac{(\varphi^{-1}(\varphi_+(2^lq_n)))^2}{2^lq_n\varphi_+(2^lq_n)} \geq CC_1\frac{2^lq_n}{\varphi_+(2^lq_n)} \geq \frac{C_1}{a_1+\cdots+a_{n+1}}.$$

In the other case $\varphi_+(2^lq_n) < 2^{2l}q_n$, choose $0 \leq l' \leq l$ so that $2^{2l'}q_n \leq \varphi_+(2^lq_n) < 2^{2(l'+1)}q_n$. We get in this case :

$$\begin{aligned}\varphi^{-1}(\varphi_+(2^l q_n)) &\geq \varphi^{-1}(2^{2l'}q_n) \geq \varphi^{-1}((\varphi_+(2^{l'}q_n)+2^{2l'}q_n)/5) \geq \varphi^{-1}(\varphi(2^{l'}q_n)/(5C_0)) \\ &\geq \frac{\varphi^{-1}(\varphi(2^{l'}q_n))}{C_{1/(5C_0)}} = \frac{2^{l'}q_n}{C_{1/(5C_0)}}.\end{aligned}$$

As a result, also for some constant still written as $C_1 > 0$ :

$$\frac{(\varphi^{-1}(\varphi_+(2^l q_n)))^2}{2^l q_n \varphi_+(2^l q_n)} \geq 16C_1 \frac{2^{2l'}q_n^2}{2^l q_n \varphi_+(2^l q_n)} \geq 4\frac{C_1}{2^l} \geq \frac{C_1}{a_1+\cdots+a_{n+1}}.$$

We hence get the same inequality. As a conclusion :

$$\begin{aligned}\sum_{\varphi_+(q_n)\leq k<\varphi_+(4a_{n+1}q_n)} \frac{1}{k^2}\frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} &\geq \sum_{0\leq l\leq 1+\log_2 a_{n+1}} \sum_{\varphi_+(2^l q_n)\leq k<\varphi_+(2^{l+1}q_n)} \frac{1}{k^2}\frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \\ &\geq \sum_{0\leq l\leq 1+\log_2 a_{n+1}} \frac{(\varphi^{-1}(\varphi_+(2^l q_n)))^2}{\varphi_+^{-1}(\varphi_+(2^{l+1}q_n))} \sum_{\varphi_+(2^l q_n)\leq k<\varphi_+(2^{l+1}q_n)} \frac{1}{k^2} \\ &\geq \sum_{0\leq l\leq 1+\log_2 a_{n+1}} \frac{(\varphi^{-1}(\varphi_+(2^l q_n)))^2}{2^{l+1}q_n} \sum_{\varphi_+(2^l q_n)\leq k<\varphi_+(2^{l+1}q_n)} \frac{1}{k^2}.\end{aligned}$$

By the previous computations :

$$\sum_{\varphi_+(q_n)\leq k<\varphi_+(4a_{n+1}q_n)} \frac{1}{k^2}\frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \geq \frac{C_1/2}{a_1+\cdots+a_{n+1}} \sum_{0\leq l\leq 1+\log_2 a_{n+1}} \varphi_+(2^l q_n) \sum_{\varphi_+(2^l q_n)\leq k<\varphi_+(2^{l+1}q_n)} \frac{1}{k^2}.$$

Using $1/k^2 \geq 1/k - 1/(k+1)$, we obtain, using lower and upper integral parts :

$$\sum_{\varphi_+(2^l q_n)\leq k<\varphi_+(2^{l+1}q_n)} \frac{1}{k^2} \geq \frac{1}{\lceil\varphi_+(2^l q_n)\rceil} - \frac{1}{\lfloor\varphi_+(2^{l+1}q_n)\rfloor} \geq \frac{1-2^{-3/4}}{2\varphi_+(2^l q_n)},$$

for large $n$, applying (1). As a result, for some constant $C_2 > 0$ :

$$\sum_{\varphi_+(q_n)\leq k<\varphi_+(4a_{n+1}q_n)} \frac{1}{k^2}\frac{(\varphi^{-1}(k))^2}{\varphi_+^{-1}(k)} \geq C_2\frac{2+\log_2 a_{n+1}}{a_1+\cdots+a_{n+1}}.$$

To conclude, by hypothesis, the sum $\sum_{n\geq 1}$ of the generic term above diverges. Since $4a_{n+1} \leq q_{n+5}$, this sum is bounded by five times $\sum_{n\geq 1}\sum_{\varphi_+(q_n)\leq k<\varphi_+(q_{n+1})}$. The sum on all $k$ thus diverges. □

### 3.2 Proof of Proposition 3.2

Due to dominated variation of the inverses, we can take $\gamma = 1$ and thus $f(x) = 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x)$. From the relation $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $n \geq 0$, and $q_0 = 1$, $q_{-1} = 0$, we recursively obtain that $q_n$ is odd, $n \geq 1$. As a result, for any $n \geq 1$, $|f_{q_n}(x)| \geq 1$, $x \in \mathbb{T}$.

Let $2/3 < \beta < 1$ and $m_k = a_{k+1}^{\beta}$, $k \geq 1$. Introduce :

$$A_k = \{x \in \mathbb{T}, f_{mq_k}(x) = mf_{q_k}(x), 0 \leq m \leq m_k\}.$$

We have $\lambda(\mathbb{T}\backslash A_k) \leq Ca_{k+1}^{\beta-1}$, for some constant $C > 0$. Indeed, if $x \in \mathbb{T}\backslash A_k$, there exist $0 \leq m < m_k$ and $0 \leq l < q_k$ such that $[x + l\alpha + mq_k\alpha, x + l\alpha + (m+1)q_k\alpha]$ contains either 0 or 1/2. This is reformulated into the fact that $x$ belongs to the union of two intervals, each of length $\leq 1/q_{k+1}$. Since $m_k q_k/q_{k+1} \leq C m_k/a_{k+1}$, this gives the result.

Since $(a_k)$ grows at least geometrically, $\sum_k a_k^{-(1-\beta)} < +\infty$. By the first lemma of Borel-Cantelli, we deduce that for a.e $x$, $x \in A_k$ for large $k$.

Let $N_k = a_1 + \cdots + a_k$. By hypothesis, if $\beta < 1$ is close enough to 1, then $m_k \gg N_k$. Let $m_k \geq m \geq 100 N_k$. For $0 \leq l, l' \leq m q_k$, we make the Euclidean divisions of $l, l'$ by $q_k$ : $l = a q_k + b$, $l' = a' q_k + b'$, $0 \leq b, b' < q_k$. Almost-surely for large $k$, observe that :

$$f_l(x) - f_{l'}(x) = (a - a') f_{q_k}(x) + T^{a q_k} f_b(x) - T^{a' q_k} f_{b'}(x).$$

As a result, using that $V(f) = 2$, Denjoy-Koksma's inequality and Ostrowski's expansion, for a.-e. $x$, if $k$ is large enough :

$$|f_l(x) - f_{l'}(x)| \geq |a - a'||f_{q_k}(x)| - |T^{a q_k} f_b(x) - T^{a' q_k} f_{b'}(x)| \geq |a - a'| - 4N_k.$$

Consequently, for $m_k \geq m \geq 100 N_k$ :

$$\begin{aligned}
\varphi_+^2(m q_k) \geq \sum_{0 \leq l' \leq l \leq m q_k} (f_l - f_{l'})^2 \quad &\geq \sum_{0 \leq a' < m/4, m/2 < a < m, 0 \leq b', b < q_k} (f_{a q_k + b}(x) - f_{a' q_k + b'}(x))^2 \\
&\geq \sum_{0 \leq a' < m/4, m/2 < a < m, 0 \leq b', b < q_k} (m/4 - 4N_k)^2 \\
&\geq (m/5)^2 q_k^2 (m/4)^2 \geq m^4 q_k^2 / 400. \qquad (2)
\end{aligned}$$

Using [3], Corollary 1.3 *i*), the convergence of $\sum_{n \geq 1} 1/\varphi_+(n)$ is sufficient for the transience of the random walk. We have :

$$\begin{aligned}
\sum_{n \geq 1} \frac{1}{\varphi_+(n)} &\leq \sum_{k \geq 1} \sum_{1 \leq m \leq a_{k+1}} \sum_{m q_k \leq n < (m+1) q_k} \frac{1}{\varphi_+(n)} \leq \sum_{k \geq 1} \sum_{1 \leq m \leq a_{k+1}} \frac{q_k}{\varphi_+(m q_k)} \\
&\leq \sum_{k \geq 1} \left[ \sum_{1 \leq m \leq 100 N_k} + \sum_{100 N_k < m \leq m_k} + \sum_{m_k < m \leq a_{k+1}} \right] \frac{q_k}{\varphi_+(m q_k)} = \sum_{k \geq 1} [A_k + B_k + C_k].
\end{aligned}$$

1) When $1 \leq m \leq 100 N_k$, using Lemma 3.4 and next (2), for some (next generic) constant $c > 0$ :

$$\varphi_+(m q_k) \geq c \sqrt{m q_k / (m_{k-1} q_{k-1})} \varphi_+(m_{k-1} q_{k-1}) \geq c \sqrt{m q_k / (m_{k-1} q_{k-1})} m_{k-1}^2 q_{k-1}.$$

Hence $\varphi_+(m q_k) \geq c \sqrt{m} a_k^{1/2 + 3\beta/2} q_{k-1}$. Thus, using that $a_k$ grows at least geometrically and that $\beta > 2/3$, we obtain the following inequalities :

$$\sum_{k \geq 1} A_k \leq c \sum_{k \geq 1} \sum_{1 \leq m \leq 100 N_k} \frac{1}{\sqrt{m} a_k^{3\beta/2 - 1/2}} \leq c \sum_{k \geq 1} \frac{\sqrt{N_k}}{a_k^{3\beta/2 - 1/2}} \leq c \sum_{k \geq 1} \frac{\sqrt{a_k}}{a_k^{3\beta/2 - 1/2}} \leq c \sum_{k \geq 1} a_k^{1 - 3\beta/2} < +\infty.$$

2) When $100 N_k < m \leq m_k$, we have $\varphi_+(m q_k) \geq m^2 q_k / 20$, so :

$$\sum_{k \geq 1} B_k \leq c \sum_{k \geq 1} \sum_{100 N_k < m \leq m_k} \frac{1}{m^2} \leq c \sum_{k \geq 1} \frac{1}{N_k} < +\infty.$$

3) When $m_k < m \leq a_{k+1}$, by Lemma 3.4 and again (2), we have :

$$\varphi_+(m q_k) \geq \sqrt{m / m_k} \varphi_+(m_k q_k)/2 \geq \sqrt{m} m_k^{3/2} q_k / 2.$$

As a consequence :

$$\sum_{k\geq 1} C_k \leq c \sum_{k\geq 1} \sum_{m_k < m \leq a_{k+1}} \frac{1}{\sqrt{m} m_k^{3/2}} \leq \sum_{k\geq 1} \frac{\sqrt{a_{k+1}}}{m_k^{3/2}} \leq \sum_{k\geq 1} \frac{1}{a_{k+1}^{3\beta/2-1/2}} < +\infty,$$

as in case 1). This ends the proof of the proposition.

□

# 4 The general case in a quasi-periodic context

Let again $Tx = x + \alpha \mod (1)$ be an irrational rotation on $\mathbb{T}$. Our basic assumption will be that for some BV functions $f, g : \mathbb{T} \to \mathbb{R}$, with $f$ centered, some $x \in \mathbb{T}$ and all $n \in \mathbb{Z}$ :

$$q_n/p_n = e^{f(T^{n-1}x)}, r_n\varepsilon_n/p_n = g(T^n x).$$

Implicitly, Hypothesis 1.1 will be always realized, uniformly in $x \in \mathbb{T}$. Introduce some definitions.

**Definition 4.1**
*Let $\rho_n = e^{f_n(x)}$, $n \in \mathbb{Z}$. For $n \geq 0$, let :*

$$v_+(n) = \sum_{0\leq k\leq n} \rho_k \text{ and } v_-(n) = (q_0/p_0) \sum_{-n-1\leq k\leq -1} \rho_k.$$

*In the same way, let for $n \geq 0$ :*

$$w_+(n) = \sum_{0\leq k\leq n} 1/\rho_k \text{ and } w_-(n) = (p_0/q_0) \sum_{-n-1\leq k\leq -1} 1/\rho_k.$$

As already indicated in the Introduction, for the random walk to be recurrent, the vertical random walk has first to be. Classically, a necessary and sufficient condition for the latter (cf for instance [4], Lemma 3.2) is :

$$\lim_{n\to+\infty} v_+(n) = +\infty \;\&\; \lim_{n\to+\infty} v_-(n) = +\infty.$$

As $f$ is BV and centered, the Denjoy-Koksma inequality implies that $|f_{\pm q_n}(x)| \leq V(f)$, $x \in \mathbb{T}$, so $\rho_n$ does not go to zero, neither as $n \to +\infty$, nor as $n \to -\infty$. Hence the two conditions hold.

Some quasi-invariant measures on $\mathbb{T}$ will play a central role. Let us recall the following folklore result. Let $T\nu$ be the image by $T$ of a Borel probability measure $\nu$ on $\mathbb{T}$, i.e. $\int f dT\nu = \int Tf d\nu$.

**Theorem 4.2**
*Let $h : \mathbb{T} \to \mathbb{R}$ be BV and centered. There exists a unique Borel probability measure $\nu_h$ on $\mathbb{T}$ such that $dT\nu_h = e^{T^{-1}h} d\nu_h$. This measure has no atom.*

For a proof of unicity, cf [8], Proposition 5.8 or [1], Proposition 1.1. We reprove existence and atomicity below in a way suitable for us. Also, $\nu_h$ has a density with respect to Lebesgue measure $\mathcal{L}_{\mathbb{T}}$ if and only if $h = \log u - \log Tu$, for some $\mathcal{L}_{\mathbb{T}}$-integrable $u > 0$, otherwise it is singular. Notice the relation $\int_{\mathbb{T}} g d\nu_h = \int_{\mathbb{T}} e^{-h} Tg d\nu$, for any bounded measurable $g$. We shall show :

**Theorem 4.3**
*Let $Tx = x + \alpha \mod 1$ on $\mathbb{T}$, with $\alpha \notin \mathbb{Q}$. Let BV functions $f, g : \mathbb{T} \to \mathbb{R}$, with $f$ centered. Suppose that $q_n/p_n = e^{f(T^{n-1}x)}$ and $r_n\varepsilon_n/p_n = g(T^n x), n \in \mathbb{Z}$, $n \in \mathbb{Z}$, for some $x \in \mathbb{T}$.*

*i) Assume that $\int_{\mathbb{T}} g d\nu_f \neq 0$. Then for all $x \in \mathbb{T}$, the random walk is transient.*

*ii) Let $g = h - e^{-f} Th$, with $h$ bounded. If either $f(x_0 + x) = f(x_0 - x)$, for some $x_0$ and $\mathcal{L}_{\mathbb{T}}$-a.-e. $x \in \mathbb{T}$, or $f = u - Tu$ with $e^u \in L^1(\mathcal{L}_{\mathbb{T}})$, then for $\mathcal{L}_{\mathbb{T}}$-a.-e. $x$, the random walk is recurrent.*

Concerning the first item, the condition $\int_{\mathbb{T}} g d\nu_f \neq 0$ is a priori far from being necessary for transience. We shall construct an example below. As we shall see, when $g = 0$ for example, transience seems to require some frankly dissymmetric behaviour between $v_+(n)$ and $w_+(n)$ (or $v_-(n)$ and $w_-(n)$), such as $w_+(n) \gg v_+(n)$. The correct condition has not been identified.

**Proposition 4.4**
*In the context of Theorem 4.3, there exists $\alpha \notin \mathbb{Q}$ and some BV centered $f = u - Tu$, with $u \geq 0$ and $e^u \notin L^1(\mathcal{L}_{\mathbb{T}})$, such that for any bounded $g$, for $\mathcal{L}_{\mathbb{T}}$-a.-e. $x$, the random walk is transient.*

*Remark.* — One observes that, as soon as $f$ is not identically zero (i.e. $\nu_f \neq \mathcal{L}_{\mathbb{T}}$), it is possible to have $\int_{\mathbb{T}} g d\nu_f \neq 0$, while $\int_{\mathbb{T}} g(x)dx = 0$. Indeed, there are an interval $I$ and some $t$ such that $\nu_f(I) \neq \nu_f(I+t)$, so $g = 1_I - 1_{I+t}$ works.

*Remark.* — We shall discuss the improvement of the second item of the theorem after the proof. Notice that the symmetry condition for example applies to $f(x) = 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x)$, with $x_0 = 1/4$. The condition $g = h - e^{-f}Th$, with $h$ bounded, implies $\int_{\mathbb{T}} g d\nu_f = 0$. Reciprocally, if $\int_{\mathbb{T}} g d\nu_f = 0$, adding a regularity condition on both $g$ and $f$ together with a Diophantine condition on $\alpha$, then the equation $g = h - e^{-f}Th$ can be solved. For example, introducing the type of $\alpha$ :

$$\eta(\alpha) = \sup\{t > 0, \liminf q^t ||q\alpha|| = 0\} \geq 1,$$

suppose that $f$ is $C^{2m}$ and $g$ is $C^m$, for some integer $m > \eta(\alpha)$. By Arnold [2], cf also Conze-Marco [9] Thm 2.1, since $f^{(m)}$ is $C^m$ with $m > \eta(\alpha)$, one has $f^{(m)} = v - Tv$, for some continuous $v$. By integration, we have $f = u - Tu$, with $u$ of class $C^m$, with zero mean. Hence $e^{-f} = e^{Tu}/e^u$ and $\nu_f$ is the measure with density $e^u$ with respect to $\mathcal{L}_{\mathbb{T}}$. Since $\int (ge^u)(x)dx = 0$ and $ge^u$ is of class $C^m$, using one more time [2], we have $ge^u = H - TH$, for a continuous $H$. Finally, $h = e^{-u}H$ satisfies the desired equation and is bounded, as continuous on $\mathbb{T}$.

## 4.1 Preliminaries

Introduce, as in [4], the functions $\Phi_{str}(n)$, $\Phi(n)$ and $\Phi_+(n)$ describing the average horizontal macrodispersion of the environment, the last two corresponding to $\Phi_u(n)$ and $\Phi_{u,+}(n)$ in [4] Definition 2.3, with $d = 1$, $u = 1 \in \mathbb{R}_+$ and $\varepsilon_s = m_s$, with the notations of [4].

**Definition 4.5**
*i) The structure function, depending only on the vertical, is defined for $n \geq 0$ by :*

$$\Phi_{str}(n) = \left( n \sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} \frac{1}{\rho_k} \right)^{1/2}.$$

*2) For $m, n \geq 0$, introduce :*

$$\Phi(-m,n) = \left( \sum_{-v_-^{-1}(m) \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left[ \frac{1}{\rho_k^2} + \frac{1}{\rho_\ell^2} + \left( \sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right] \right)^{1/2}.$$

*For $n \geq 0$, set $\Phi(n) = \Phi(-n,n)$ and $\Phi_+(n) = \sqrt{\Phi^2(-n,0) + \Phi^2(0,n)}$.*

As in [5], we rectify a misleading point in [4] Definition 2.3 1), where the term "standard Lebesgue measure" on the half Euclidean ball $S_+^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1, \ x_1 \geq 0\}$ has to be understood as "uniform probability measure". The following result comes from [4], Theorem 2.4, Proposition 2.5 1) and Lemma 6.9.

**Theorem 4.6**
*i) The random walk is recurrent if and only if $\sum_{n \geq 1} \frac{1}{n^2} \frac{(\Phi^{-1}(n))^2}{\Phi_+^{-1}(n)} = +\infty$.*

*ii) The condition $\sum_{n \geq 1} 1/\Phi(n) < +\infty$ implies transience. It is necessary when $\Phi \preceq \Phi_+$.*

Rather clearly, direct functions verify $\Phi_{str} \preceq \Phi_+ \preceq \Phi$. Also, as a general fact, as detailed in [5], section 3.1 :

$$\Phi_+(n) \asymp \Phi_{str}(n) + \left( \sum_{-v_-^{-1}(n) \leq k \leq \ell \leq 0 \text{ or } 0 \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}, \quad (3)$$

as well as :

$$\Phi(n) \asymp \Phi_{str}(n) + \left( \sum_{-v_-^{-1}(n) \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}. \quad (4)$$

A crucial point, recalled in detail in [5], end of Section 3.1, is that the inverse functions $\Phi_{str}^{-1}$, $\Phi_+^{-1}$ and $\Phi^{-1}$ check dominated variation.

## 4.2 Proof of Theorem 4.3 $i)$

Let us start with a lemma, essentially inspired from [8], consequence of the unicity of the quasi-invariant measure. Introduce the notation :

$$A(n, g, x) = \frac{\sum_{k=0}^{n} g(T^k x)/\rho_k(x)}{\sum_{k=0}^{n} 1/\rho_k(x)}.$$

**Lemma 4.7**
*i) Let $(x_n)_{n\geq 1}$ in $\mathbb{T}$ and $(N_n)$ be an increasing sequence of integers. Then any cluster value $\mu$ of the following sequence of probability measures :*

$$\left( \frac{\sum_{k=0}^{N_n} \delta_{T^k x_n}/\rho_k(x_n)}{\sum_{k=0}^{N_n} 1/\rho_k(x_n)} \right)_{n\geq 1},$$

*for the weak-$*$ topology, is non-atomic and verifies $dT\mu = e^{T^{-1}f} d\mu$.*

*ii) Let $g : \mathbb{T} \to \mathbb{R}$ be BV. Then, as $n \to +\infty$, uniformly in $x$, $A(n, g, x) \to \int_{\mathbb{T}} g d\nu_f$.*

*Proof of the lemma :*
As a preliminary remark, for any $(x_n)$ and $(N_n)$, as $n \to +\infty$ :

$$\sum_{k=0}^{N_n} 1/\rho_k(x_n) \to +\infty \text{ and } (1/\rho_{N_n}(x_n))/(\sum_{k=0}^{N_n} 1/\rho_k(x_n)) \to 0. \quad (5)$$

The first point comes from the observation that $\rho_k(x_n)$ has order 1 at times $k = q_l$, independently of $x_n$, because of the Denjoy-Koksma inequality. The second one equivalently rewrites as :

$$\sum_{k=0}^{N_n} e^{-f_k(x_n)+f_{N_n}(x_n)} = \sum_{k=0}^{N_n} e^{T^k f_{N_n-k}(x_n)} = \sum_{k=0}^{N_n} e^{T^{N_n-k} f_k(x_n)} \to +\infty,$$

for the same reason.

$i)$ Consider a cluster value $\mu$ for the weak topology of this sequence, say $(\mu_n)$, of Borel probability measures on $\mathbb{T}$. Keeping the same notations $(x_n)$ and $(N_n)$, we can assume that $(\mu_n)$ weakly converges to $\mu$. Let $a \in \mathbb{T}$. Since $\mu((a-\delta, a+\delta)) \leq \liminf \mu_n((a-\delta, a+\delta))$, it is enough to show that $\mu_n((a-\delta, a+\delta))$ is small for large $n$, for a well-chosen $\delta > 0$.

Fixing an integer $K \geq 1$, take $\delta > 0$ so that the intervals $(a-\delta, a+\delta) - k\alpha$, $0 \leq k \leq 2q_K$, on $\mathbb{T}$ are disjoint. Then :

$$\mu_n((a-\delta,a+\delta)) = \frac{\sum_{k=0}^{N_n} 1_{(a-\delta,a+\delta)-k\alpha}(x_n)/\rho_k(x_n)}{\sum_{k=0}^{N_n} 1/\rho_k(x_n)}.$$

Let $L_n \geq 0$ and $0 \leq \tau_{1,n} < \cdots \tau_{L_n,n} \leq N_n$ be the subsequence of $0 \leq k \leq N_n$ such that $x_n \in (a-\delta,a+\delta)-k\alpha$. When $L_n \geq 2$, then $\tau_{k,n}+q_K < \tau_{k+1,n}$, for $1 \leq k < L_n$, and $\tau_{L_n-1,n}+2q_K < \tau_{L_n,n}$. If $L_n = 1$, then $\tau_{L_n,n} > q_K$ or $\tau_{L_n,n}+q_K < N_n$, if $n$ is large enough. As a result, using the Denjoy-Koksma inequality, giving $1/\rho_{k\pm q_l}(x_n) \geq e^{-V(f)}/\rho_k(x_n)$, we obtain when $L_n \geq 1$ :

$$\mu_n((a-\delta,a+\delta)) \leq \frac{\sum_{0\leq k\leq L_n} 1/\rho_{\tau_{k,n}}(x_n)}{\sum_{0\leq k\leq L_n}(1/\rho_{\tau_{k,n}}(x_n))\sum_{1\leq l\leq K} e^{-V(f)}} = \frac{e^{V(f)}}{K}.$$

This can be made arbitrary small if $K$ is large enough. If $L_n = 0$, we have $\mu_n((a-\delta,a+\delta)) = 0$. Hence $\mu$ is non-atomic.

For any continuous $h : \mathbb{T} \to \mathbb{R}$, $A(N_n,h,x_n) \to \int_{\mathbb{T}} h d\mu$. Since $\mu$ is non-atomic, this holds for any $h$ continuous except at countably many points, hence in particular if $h$ is BV. Since $f$ is BV, $e^{-f}$ is also BV. Thus for any continuous $h$, $A(N_n, e^{-f}Th, x_n) \to \int_{\mathbb{T}} e^{-f}Thd\mu$. It now follows from (5), that for any continuous $h$ :

$$\int_{\mathbb{T}} e^{-f}Thd\mu = \int_{\mathbb{T}} hd\mu,$$

so $dT\mu = e^{T^{-1}f}d\mu$. This completes the proof of this point.

*ii)* If the result is not true, by boundedness, there exists $a \in \mathbb{R}$, $(x_n)$ in $\mathbb{T}$ and $N_n \to +\infty$, such that $A(N_n,g,x_n) \to a \neq \int_{\mathbb{T}} gd\nu_f$. By compacity of the weak-$*$ topology, for some subsequence $(N_{n'})$ of $(N_n)$, the sequence of measures :

$$\frac{\sum_{k=0}^{N_{n'}} \delta_{T^k x_{n'}}/\rho_k(x_{n'})}{\sum_{k=0}^{N_{n'}} 1/\rho_k(x_{n'})}, n \geq 1,$$

weakly converges to some probability $\mu$ on $\mathbb{T}$. By point *i)*, $dT\mu = e^{T^{-1}f}d\mu$, so $\mu = \nu_f$, by unicity. Since $\nu_f$ is non-atomic, $A(N_{n'},g,x_{n'}) \to \int_{\mathbb{T}} gd\nu_f$, contradicting the hypothesis.

□

We now turn to the proof of the first item of the theorem. Observe first that :

$$\Phi_{str}(n) \preceq \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \left(\frac{\rho_k}{\rho_\ell}+\frac{\rho_\ell}{\rho_k}\right)\right)^{1/2} \preceq \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell\left(\sum_{s=k}^{\ell}\frac{1}{\rho_s}\right)^2\right)^{1/2},$$

considering only in the last inside sum the terms for $s = k$ and $s = l$. Introduce now the following function $\Psi$, essentially corresponding to $\Phi$ when $g$ is identically equal to 1 :

$$\Psi(n) = \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell\left(\sum_{s=k}^{\ell}\frac{1}{\rho_s}\right)^2\right)^{1/2}.$$

Notice now that :

$$\Phi(n) \asymp \Phi_{str}(n) + \left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}$$

$$\asymp \Phi_{str}(n) + \left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{g(T^s x)}{\rho_s} \right)^2 \right)^{1/2} \preceq \Psi(n),$$

as $g$ is bounded. In the other direction, using the second item of Lemma 4.7, let first $M \ge 1$ be such that for $n \ge M$ and all $x \in \mathbb{T}$ :

$$\left| \frac{\sum_{k=0}^{n} g(T^k x)/\rho_k(x)}{\sum_{k=0}^{n} 1/\rho_k(x)} - \int_{\mathbb{T}} g d\nu_f \right| \le \left| \int_{\mathbb{T}} g d\nu_f \right| /2.$$

Consequently :

$$\Phi(n) \ge \left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), \ell - k > M} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{g(T^s x)}{\rho_s} \right)^2 \right)^{1/2}$$

$$\ge \frac{1}{4} \left( \int_{\mathbb{T}} g d\nu_f \right)^2 \left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), \ell - k > M} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

For the remaining term, since $g$ is bounded by some $A$ :

$$\left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), l - k \le M} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{g(T^s x)}{\rho_s} \right)^2 \right)^{1/2} \le A \left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), l - k \le M} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

Since $\eta \le \rho_k(x)/\rho_{k+1}(x) \le 1/\eta$, we get that $\rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2$ has order one when $l - k \le M$. Therefore, for some constant $C$ :

$$\left( \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n), l - k \le M} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} \le C \left( \sum_{-v_-^{-1}(n) \le k \le v_+^{-1}(n)} 1 \right)^{1/2}$$

$$\preceq \sqrt{v_-^{-1}(n)} + \sqrt{v_+^{-1}(n)}.$$

We now show that $v_+^{-1}(n)/\Psi^2(n) \to 0$, $v_-^{-1}(n)/\Psi^2(n) \to 0$. Indeed, considering the first one :

$$\Psi^2(n) \ge \sum_{0 \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \succeq \sum_{0 \le k \le \ell \le v_+^{-1}(n)} \left( \frac{\rho_k}{\rho_l} + \frac{\rho_l}{\rho_k} \right)$$

$$\succeq \sum_{0 \le k \le v_+^{-1}(n)} \rho_k \sum_{0 \le l \le v_+^{-1}(n)} 1/\rho_l \succeq (v_+^{-1}(n))^2,$$

by Cauchy-Schwarz inequality. As $(v_+^{-1}(n))^2 \gg v_+^{-1}(n)$, since $v_+^{-1}(n) \to +\infty$, when $n \to +\infty$, we get the result. The same holds for $v_-^{-1}(n)/\Psi^2(n)$. As a consequence, $\Psi(n) \preceq \Phi(n)$ and thus $\Psi(n) \asymp \Phi(n)$. The same argumentation shows that one can also replace $\Phi_+(n)$ by some $\Psi_+(n)$, corresponding to $g$ identically equal to 1.

By Theorem 4.6, we are therefore left to proving :

$$\sum_{n\geq 1} \frac{1}{n^2} \frac{(\Psi^{-1}(n))^2}{\Psi_+^{-1}(n)} < +\infty.$$

We give two proofs. The first one is trivial, since we recognize the recurrence criterion when $g = 1$, which can be realized with $\mu_n = \delta_{+1}$, $p_n = r_n$, $q_n/p_n = e^{f(T^{n-1}x)}$, for example. In this case the random walk is obviously transient, so the above series is finite, which concludes the argument.

It is more satisfactory to give a direct proof of the convergence of the above series. In this direction, introduce the following functions :

$$\Psi_{++}(n) = \left( \sum_{0\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} \text{ and } \Psi_{+-}(n) = \left( \sum_{-v_-^{-1}(n)\leq k\leq \ell\leq 0} \rho_k \rho_\ell \left( \sum_{s=k}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

From what precedes, $\Psi_+(n) \asymp \Psi_{++}(n) + \Psi_{+-}(n)$, hence $\Psi_+^{-1}(n) \asymp \min\{\Psi_{++}^{-1}(n), \Psi_{+-}^{-1}(n)\}$, so :

$$\frac{1}{\Psi_+^{-1}(n)} \asymp \frac{1}{\Psi_{++}^{-1}(n)} + \frac{1}{\Psi_{+-}^{-1}(n)}.$$

We thus have to show the two convergences :

$$\sum_{n\geq 1} \frac{1}{n^2} \frac{(\Psi^{-1}(n))^2}{\Psi_{++}^{-1}(n)} < +\infty \text{ and } \sum_{n\geq 1} \frac{1}{n^2} \frac{(\Psi^{-1}(n))^2}{\Psi_{+-}^{-1}(n)} < +\infty.$$

Consider the first one, the other one being similar. As a first remark, using Hypothesis 1.1, for some $c > 1$, $\sum_{0\leq k\leq n+1} \rho_k \leq c \sum_{0\leq k\leq n} \rho_k$. Hence $\sum_{0\leq \ell\leq v_+^{-1}(n)} \rho_\ell \geq n/c$. Thus :

$$\sum_{v_+^{-1}(n/c^2)\leq \ell\leq v_+^{-1}(n)} \rho_\ell \asymp n,$$

We now have :

$$\begin{aligned} \Psi(n) \geq \left( \sum_{-v_-^{-1}(n)\leq k\leq 0\leq \ell\leq v_+^{-1}(n)} \rho_k \rho_\ell \left( \sum_{s=0}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} &\succeq \sqrt{n} \left( \sum_{0\leq \ell\leq v_+^{-1}(n)} \rho_\ell \left( \sum_{s=0}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2} \\ &\succeq n \sum_{s=0}^{v_+^{-1}(n/c^2)} \frac{1}{\rho_s} \geq (n/c^2) \sum_{s=0}^{v_+^{-1}(n/c^2)} \frac{1}{\rho_s}. \end{aligned}$$

Lowering $\Psi$, in a large sense, hence increasing $\Psi^{-1}$, we choose to redefine :

$$\Psi(n) := \sqrt{n} \left( \sum_{0\leq \ell\leq v_+^{-1}(n)} \rho_\ell \left( \sum_{s=0}^{\ell} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

Clearly $n \sum_{s=0}^{v_+^{-1}(n)} \frac{1}{\rho_s} \succeq \Psi(n)$, so setting $F(n) = \sum_{k=0}^{v_+^{-1}(n)} 1/\rho_k$ and $G(n) = nF(n)$, we have :

$$\Psi^{-1}(n) \asymp G^{-1}(n).$$

In order to treat $\Psi^2_{++}(n)$, fix $K > c^2$ and define $A_u = \sum_{v_+^{-1}(K^u)<k\leq v_+^{-1}(K^{u+1})} 1/\rho_k$. Then :

$$\left( \sum_{0\leq u<v-1<n-1} K^u K^v (A_u + \cdots + A_v)^2 \right)^{1/2} \preceq \Psi_{++}(K^n) \preceq \left( \sum_{0\leq u\leq v<n} K^u K^v (A_u + \cdots + A_v)^2 \right)^{1/2}.$$

On the one hand :

$$\begin{aligned} \Psi_{++}(K^n) &\preceq \sum_{0\leq u\leq v<n} \sqrt{K}^u \sqrt{K}^v (A_u + \cdots + A_v) \\ &\preceq \sum_{0\leq l<n} A_l \sum_{0\leq u\leq l} \sqrt{K}^u \sum_{l\leq v<n} \sqrt{K}^v \preceq \sqrt{K}^n \sum_{0\leq l<n} A_l \sqrt{K}^l. \end{aligned}$$

On the other hand :

$$\left( \sqrt{K}^n \sum_{0\leq l<n-2} A_l \sqrt{K}^l \right)^2 \preceq K^n \sum_{0\leq u\leq v<n-2} A_u A_v \sqrt{K}^u \sqrt{K}^v \preceq K^n \sum_{0\leq u<n-2} A_u^2 K^u \preceq \Psi^2_{++}(K^n).$$

Since $A^2_{n-1} K^{2n} \preceq \Psi^2_{++}(K^n)$, we get $\Psi_{++}(K^n) \asymp \sqrt{K}^n \sum_{0\leq l<n} A_l \sqrt{K}^l$. As a result, for $N \geq 1$ :

$$\sum_{n=0}^{N} \frac{\Psi_{++}(K^n)}{K^n} \asymp \sum_{0\leq l<N} A_l \sqrt{K}^l \sum_{l<n\leq N} \frac{1}{\sqrt{K}^n} \asymp \sum_{0\leq l<N} A_l \asymp F(K^n).$$

As a consequence, using dominated variation, maybe (also later) increasing $K$ so that $\Psi_{++}(K^n) \leq \Psi_{++}(K^{n+1})/2$, we obtain :

$$\sum_{0\leq k<\Psi_{++}(K^N)} \frac{1}{\Psi^{-1}_{++}(k)} = \sum_{0\leq n<N} \sum_{\Psi_{++}(K^n)\leq k<\Psi_{++}(K^{n+1})} \frac{1}{\Psi^{-1}_{++}(k)} \preceq \sum_{0\leq n<N} \frac{\Psi_{++}(K^{n+1})}{K^n} \preceq F(K^N).$$

In the same way, $F(K^{N-1}) \preceq \sum_{0\leq k<\Psi_{++}(K^N)} 1/\Psi^{-1}_{++}(k)$. From this :

$$F(n/K) \preceq \sum_{0\leq k<\Psi_{++}(n)} 1/\Psi^{-1}_{++}(k) \preceq F(Kn). \tag{6}$$

Up to increasing $K$, using dominated variation, we obtain :

$$F(\Psi^{-1}_{++}(n/K)) \preceq \sum_{0\leq k<n} 1/\Psi^{-1}_{++}(k) \preceq F(\Psi^{-1}_{++}(Kn)).$$

Let $Z(n) = \sum_{0\leq k<n} 1/\Psi^{-1}_{++}(k)$. We therefore deduce :

$$\begin{aligned} \sum_{0\leq k<\Psi(\Psi^{-1}_{++}(n))} 1/\Psi^{-1}_{++}(k) &= Z(n) + \sum_{n\leq k<\Psi(\Psi^{-1}_{++}(n))} 1/\Psi^{-1}_{++}(k) \\ &\leq Z(n) + \Psi(\Psi^{-1}_{++}(n))/\Psi^{-1}_{++}(n) \preceq Z(n) + F(\Psi^{-1}_{++}(n)) \preceq Z(Kn). \end{aligned}$$

From (6), we next have, using the previous computation and the fact that $\Psi\circ\Psi_{++}^{-1}\circ\Psi_{++}\circ\Psi^{-1}(n) = \Psi\circ\Psi^{-1}(n)\asymp n$ :

$$\frac{\Psi^{-1}(n)}{n} \preceq \frac{1}{\sum_{0\leq k<\Psi_{++}(\Psi^{-1}(n))} 1/\Psi_{++}^{-1}(k)} = \frac{1}{Z(\Psi_{++}(\Psi^{-1}(n)))} \preceq \frac{1}{Z(n/K)}.$$

From Theorem 3.1, we can now show the convergence of the desired series, using once more dominated variation :

$$\begin{aligned}\sum_{n\geq 1}\left(\frac{\Psi^{-1}(n)}{n}\right)^2\frac{1}{\Psi_{++}^{-1}(n)} \asymp \sum_{n\geq 1}\left(\frac{\Psi^{-1}(Kn)}{Kn}\right)^2\frac{1}{\Psi_{++}^{-1}(n)} &\preceq \sum_{n\geq 1}\left(\frac{1}{Z(n)}\right)^2\frac{1}{\Psi_{++}^{-1}(n)} \\ &\preceq \sum_{n\geq 1}\left(\frac{1}{Z(n-1)}-\frac{1}{Z(n)}\right) < +\infty.\end{aligned}$$

This concludes the second proof and ends the proof of this item of the theorem.

□

### 4.3 Proof of Theorem 4.3 *ii)*

We now assume that $g = h - e^{-f}Th$, for a bounded $h$. First of all :

$$\begin{aligned}\left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell\left(\sum_{s=k}^{\ell}\frac{r_s\varepsilon_s}{p_s\rho_s}\right)^2\right)^{1/2} &= \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell\left(\sum_{s=k}^{\ell}\frac{T^s g}{\rho_s}\right)^2\right)^{1/2} \\ &\preceq \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell\left(\sum_{s=k}^{\ell}\frac{T^s h}{\rho_s}-\frac{T^{s+1}h}{\rho_{s+1}}\right)^2\right)^{1/2} \\ &\preceq \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell\left(1/\rho_k^2+1\rho_l^2\right)\right)^{1/2} \asymp \Phi_{str}(n).\end{aligned}$$

Therefore $\Phi(n)\asymp\Phi_+(n)\asymp\Phi_{str}(n)$. As a corollary of Theorem 4.6, the recurrence of the random walk follows from the divergence of :

$$\sum_{n\geq 1}\frac{1}{\Phi_{str}(n)} = \sum_{n\geq 1}\frac{1}{\sqrt{n(w_+\circ v_+^{-1}(n)+w_-\circ v_-^{-1}(n))}}.$$

Because of monotonicity, for any $K>1$, this is equivalent to showing the divergence of :

$$\sum_{n\geq 1}\frac{\sqrt{K^n}}{\sqrt{w_+\circ v_+^{-1}(K^n)+w_-\circ v_-^{-1}(K^n)}}. \tag{7}$$

Suppose first that $f = u - Tu$, with $e^u$ Lebesgue-integrable. Then, by the Law of Large Numbers, a.-e., as $n\to+\infty$ :

$$w_+(n)(x)\sim ne^{-u(x)}\int_{\mathbb{T}} e^{u(y)}dy.$$

In the same way, a.-e., $w_-(n)(x)$ is linear, whereas $v_+(n)(x)/n\to+\infty$, $w_+(n)(x)/n\to+\infty$, as $n\to+\infty$. As a consequence, for large $n$, $w_+\circ v_+^{-1}(n)\leq w_-(n)\preceq n$ and $w_-\circ v_-^{-1}(n)\leq w_+(n)\preceq n$, making the series in (7) diverge.

Suppose next instead that $f(x+x_0)=f(x_0-x)$, for a.-e. $x\in\mathbb{T}$, for some $x_0$. Using the convergents of $\alpha$, at time $q_n$ one first observes that $v_+(q_n)\asymp v_-(q_n)$, since :

$$\sum_{0\le k\le q_n} e^{f_k(x)} = \sum_{0\le k\le q_n} e^{f_{q_n-k}(x)} = \sum_{0\le k\le q_n} e^{f_{-k}(x)+T^{-k}f_{q_n}(x)} \asymp \sum_{0\le k\le q_n} e^{f_{-k}(x)}, \qquad (8)$$

using Denjoy-Koksma's inequality. For the same reason, $w_+(q_n)\asymp w_-(q_n)$. Fix some large $K>1$ and $p_0$ such that for any $n\ge 1$, there exists $p$ so that $K^p\le v_+(q_n), v_-(q_n)\le K^{p+p_0}$. This gives $v_+^{-1}(K^p), v_-^{-1}(K^p)\le q_n$, so the corresponding term in (7) verifies :

$$\frac{\sqrt{K^p}}{\sqrt{w_+\circ v_+^{-1}(K^p)+w_-\circ v_-^{-1}(K^p)}} \ge \frac{\sqrt{K^{-p_0}v_+(q_n)}}{\sqrt{w_+(q_n)+w_-(q_n)}} \asymp \frac{\sqrt{v_+(q_n)}}{\sqrt{w_+(q_n)}}.$$

Immediately from the definition of the model, the set $\{x\in\mathbb{T},\text{ the random walk is transient}\}$ is measurable and $T$-invariant, hence has Lebesgue measure zero or one, by ergodicity of $(\mathbb{T},T,\mathcal{L}_{\mathbb{T}})$. As before, using at the end the symmetry assumption, we can write :

$$\frac{v_+(q_n)}{w_+(q_n)}(x_0+x) \asymp \frac{v_+(q_n)}{w_-(q_n)}(x_0+x) = \frac{\sum_{0\le k\le q_n} e^{f_k(x_0+x)}}{\sum_{0\le k\le q_n} e^{-f_{-k}(x_0+x)}} \asymp \frac{\sum_{0\le k\le q_n} e^{f_k(x_0+x)}}{\sum_{0\le k\le q_n} e^{f_k(x_0-x)}}.$$

If the random walk were transient for almost-every $x$, then a.-e. $x$, $(v_+(q_n)/w_+(q_n))(x_0+x)\to 0$, as $n\to+\infty$. Changing $x$ into $-x$, the symmetry $x\longmapsto -x$ preserving Lebesgue measure on $\mathbb{T}$, the inverse of the previous fraction would a.-e. go to zero as well. This contradiction completes the proof of the second item of the theorem.

□

*Remark.* — In a similar way, without the symmetry assumption, if $f(x)=u(x)-u(x+y)$, with $u$ BV, then for a.e. $(x,y)\in\mathbb{T}^2$ the random walk is recurrent. Indeed in the previous proof, if transience holds for some $(x,y)$, then :

$$\frac{v_+(n)}{w_+(n)} = \frac{\sum_{k=0}^{q_n} e^{u_k(x)-u_k(x+y)}}{\sum_{k=0}^{q_n} e^{-u_k(x)+u_k(x+y)}} \to 0.$$

The set of $(x,y)\in\mathbb{T}^2$ verifying this is invariant by the joint action of $T\times Id$ and $Id\times T$ on $\mathbb{T}^2$, which is ergodic. Hence it has measure 0 or 1. If this is 1, one has for a.e. $(x,y)$, as $n\to+\infty$ :

$$\frac{\sum_{k=0}^{q_n} e^{h_k(x)-h_k(y)}}{\sum_{k=0}^{q_n} e^{-h_k(x)+h_k(y)}} \to 0,$$

which is impossible when reversing the roles of $x$ and $y$. Thus we have recurrence. Rather generally, in the second item of the theorem, it would be interesting if the symmetry assumption on $f$ could be removed. This raises the question, for $f:\mathbb{T}\to\mathbb{R}$, BV and centered, of understanding the a.-e. behaviour of ratios of the form $\sum_{k=0}^n e^{f_k(x)}/\sum_{k=0}^n e^{-f_k(x)}$, as $n\to+\infty$. Similarly, the improvement of the condition on $g$ in the second item of the theorem, when $\int_{\mathbb{T}} g d\nu_f=0$, requires to find upper-bounds on sums of the form $\sum_{k=0}^n e^{-f_k(x)}T^k g(x)$, less evident than those given by the assumption $g=h-e^{-f}Th$, with $h$ bounded.

## 4.4 Proof of Proposition 4.4

By Theorem 4.6, it is enough to show the convergence of :

$$\sum_{n\ge 1} 1/\Phi_{str}(n) \le \sum_{n\ge 1} 1/\sqrt{n w_+\circ v_+^{-1}(n)}.$$

If $f = u - Tu$, with $u \geq 0$, then $e^{-u} \leq 1$ is integrable, so $v_+(n)$ is a.-e. linear. Setting $U = e^u$, we have $w_+(n) \sim e^{-u(x)} U_n(x)$, so it is enough to show that for a.-e. $x$ :

$$\sum_{n \geq 1} 1/\sqrt{nU_n} < +\infty.$$

Let us build $u$ and $f = u - Tu$. Let $\alpha \notin \mathbb{Q}$ be defined by the partial quotients $a_m = m^6$, $m \geq 1$. Introduce $h_{B,\Delta}(x) = B(1 - |x|/\Delta)_+$, for $\Delta > 0$, $B > 0$. For $m \geq 1$, let $h^m = h_{B_m,\Delta_m}$, with :

$$\Delta_m = 1/(m^2 q_m), B_m = m^2/q_m.$$

We now define $f = \sum_{m \geq 1} f^m$, where :

$$f^m = \sum_{k=0}^{q_m - 1} T^{-k}(h^m - T^{q_m} h^m).$$

For large $m$, the $k\alpha$, $0 \leq k < q_m$, are approximately equally spaced and in the above formula the sum involves functions with disjoint supports. For $m \geq 1$, we have $f^m$ centered and :

$$V(f^m) \leq q_m V(h^m - T^{q_m} h^m) \leq C q_m (B_m/\Delta_m) \|q_m \alpha\| \leq C/m^2.$$

As a result, $f$ is BV and centered. Now observe that $f = u - Tu$, with :

$$u = \sum_{m \geq 1} \sum_{k=0}^{q_m - 1} T^{-k} \sum_{l=0}^{q_m - 1} T^l h^m = \sum_{m \geq 1} \sum_{|l| < q_m} (q_m - |l|) T^l h^m.$$

Above, in the last sum, the measure of the support of the function corresponding to $m \geq 1$ is $\leq 2q_m \Delta_m$. As $\sum_{m \geq 1} q_m \Delta_m < +\infty$, by the first lemma of Borel-Cantelli, a.-s. a point $x$ belongs to only finitely many supports, so $u$ is well-defined a.-e..

Observe that the type $\eta(\alpha)$ of $\alpha$, also equal to $\limsup \log q_{n+1}/\log q_n$, has value 1. For $x \in \mathbb{T}$ and $r > 0$, let $\tau_r(x) = \min\{n \geq 1, \|T^n x\| < r\}$. By Kim and Marmi [15], for a.-e. $x$ :

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = 1.$$

Let $U = e^u$ and $s_m = \tau_{\Delta_m/2}(x)$, $m \geq 1$. Decompose now :

$$\sum_{n > s_1} \frac{1}{\sqrt{nU_n}} = \sum_{m \geq 1} \sum_{s_m < n \leq s_{m+1}} \frac{1}{\sqrt{nU_n}} \preceq \sum_{m \geq 1} \sqrt{s_{m+1}}/\sqrt{U_{s_m+1}}. \qquad (9)$$

We next have :

$$U_{s_m+1} \geq e^{u(T^{s_m} x)} \geq e^{q_m h^m (T^{s_m} x)} \geq e^{m^2/2}.$$

Also, for large $m$, $s_m \leq (2/\Delta_m)^2 \preceq (m^2 q_m)^2$. Next, $q_m = m^6 q_{m-1} + q_{m-2} \leq (2m)^6 q_{m-1}$, so brutally $q_m \leq (2m)^m \leq e^{cm \log m}$. Hence :

$$s_{m+1} = O(e^{3cm \log m}),$$

so the series in (9) is finite. This ends the proof of the proposition.

□

Let us conclude this work with the observation that in a rather large generality, when $f$ is not a coboundary, then the ratio $v_+(n)/w_+(n)$ has a somewhat typical behaviour.

**Lemma 4.8**
*Let $f \in \mathcal{L}^1(\mathbb{T} \to \mathbb{R})$, centered, not a.-e. equal to $u - Tu$, for some measurable $u$. Then either $v_+(n)/w_+(n) \to +\infty$ a.s., or $v_+(n)/w_+(n) \to 0$ a.s., or $\limsup v_+(n)/w_+(n) \to +\infty$ a.s. and $\liminf v_+(n)/w_+(n) \to 0$ a.s..*

*Proof of the lemma :*
A.-s., $v_+(n) \to 0$, since $(f_k)$ is recurrent to zero. We have $v_+(n)(x) \sim e^{f(x)} v_+(n-1)(Tx)$ and $w_+(n)(x) \sim e^{-f(x)} w_+(n-1)(Tx)$, so the set $\{\limsup v_+(n)(x)/w_+(n)(x) < +\infty\}$ is $T$-invariant, hence has Lebesgue measure 0 or 1 by ergodicity. If this is 1, let $\psi(x) = \limsup_n \frac{v_+(n)(x)}{w_+(n)(x)}$, a.-e..

From the opening remark on equivalents, we get $\psi(x) = e^{2f(x)}\psi(Tx)$. The set $\{\psi(x) > 0\}$ is $T$-invariant and thus again has measure 0 or 1. If this is 1, one has $f = (\log \psi)/2 - (\log T\psi)/2$, contrary to the hypothesis. Hence the set has measure 0.

As a result, $\limsup v_+(n)/w_+(n) = +\infty$ a.s., or $v_+(n)/w_+(n) \to 0$ a.s.. Symmetrically, $\limsup w_+(n)/v_+(n) = +\infty$ a.s., or $w_+(n)/v_+(n) \to 0$ a.s.. Intersecting the possibilities, we obtain the 3 cases given in the statement of the lemma.

□

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