LORENTZ SPACES IN ACTION ON PRESSURELESS SYSTEMS ARISING FROM MODELS OF COLLECTIVE BEHAVIOR RAPHAËL DANCHIN AND PIOTR BOGUS LAW MUCHA

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Abstract. We are concerned with global-in-time and uniqueness results for models of pressureless gases that come up in the description of phenomena in astrophysics or collective behavior (like traffic, crowds or birds). The initial data are rough: in particular, the density is only bounded. Our results are based on interpolation and parabolic maximal regularity, where Lorentz spaces play a key role.

1. Introduction

We are concerned with some models coming from a special type of hydrodynamical systems, that do not include the effects of the internal pressure. The mathematical properties of these models are not quite the same as those that arise from the physical description of common fluids. The simplest example, giving the nature of the phenomenon, is the motion of dust, that is, of free particles evolving in the space. Here one can mention examples in astrophysics [13], or in multi-fluid systems [1, 3]. But leaving the world of inanimate matter, we can find models that describe collective behavior, where particles or rather agents exhibit some intelligence, and for which having a force like internal pressure is not so natural. A well known example in this area is given by equations of traffic flow [2, 19], where particles are just cars.

In order to specify and understand this class, let us go back to the kinetic description of a collective behaviour. Consider equations of the following form

$$f_t + v\cdot\nabla_x f + \mathrm{div}_v K(f)f = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^d_x\times\mathbb{R}^d_v, \tag{1.1}$$

where $f=f(t,x,v)$ is a distribution function of gas in the phase-space. Classically, if the operator K is given by the Poisson potential, then we obtain the Vlasov system. If taking a less singular K, then one may obtain for example the Cucker-Smale system that models collective behavior like flocking of birds [4].

Assuming a very special form of f, the so-called *mono-kinetic ansatz*, one can pass formally from the kinetic model to the hydrodynamical system, putting just

$$f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}. \tag{1.2}$$

This amounts to saying that the distribution of the gas under consideration is located on the curve $v=u(t,x)$. Although one cannot expect this simplification to be a correct description of a gas, it may be relevant for modelling collective behavior phenomena. Typically, one can expect a crowd of individuals to have the same speed (or tendency) at one point [26, 28].

Now, plugging (1.2) in (1.1) leads to the following general form of hydrodynamical system:

$$\begin{aligned} &\rho_t + \mathrm{div}\,(\rho u) = 0,\\ &\rho u_t + \rho u\cdot\nabla u = A(\rho,u). \end{aligned} \tag{1.3}$$

The first equation is the mass continuity law, and the second one is the momentum balance. Examples of A can be found in [30, 31] and in the survey [25]. In the case $A\equiv 0$, one just recovers inviscid compressible transport, namely the pressureless compressible Euler system, and there is no interaction whatsoever between the individuals.

In our note, we would like to put our attention on the following two cases:

$$A(\rho,u)=\mu\Delta u \quad \text{and} \quad A(\rho,u)=\mu\Delta u+\mu'\nabla\mathrm{div}\,u, \qquad \mu>0, \quad \mu+\mu'>0. \tag{1.4}$$

The first case is a viscous regularization of (1.4) that can be viewed as a simplification of the Euler alignment system. It corresponds to the hydrodynamical version of the Cucker-Smale model, namely

$$\begin{aligned}&\rho_t + \mathrm{div}\,(\rho u) = 0,\\ &\rho u_t + \rho u\cdot\nabla u = \int_{\mathbb{R}^d} \frac{u(t,y)-u(t,x)}{|x-y|^{d+\alpha}}\rho(t,y)\rho(t,x)dy, \qquad \alpha\in(0,2].\end{aligned} \tag{1.5}$$

The right-hand side of (1.5) involves the fractional Laplacian $(-\Delta)^{\alpha/2}$ (see details in [12, 15]) and, at least formally, the first case in (1.4) thus meets $\alpha = 2$. The second case of (1.4) is the Lamé operator that can be obtained from the Vlasov-Boltzmann equation (for more explanation, one may refer to the introduction of [29]).

The form of (1.4) does not take into account the effects of internal pressure. From the mathematical viewpoint, the lack of the pressure term P causes serious problems. In particular, all techniques for the compressible viscous systems based on the properties of the so-called effective viscous flux, namely $F := \mathrm{div}\, u - P$, which has better regularity than $\mathrm{div}\, u$ and P taken separately, are bound to fail. Recall that using F is one of the keys to the theory of weak solutions of the compressible Navier-Stokes equations [20, 21, 22, 16, 17], as it allows to exhibit compactness properties of the set of weak solutions. In the theory of regular solutions [24, 5, 27], the effective viscous flux provides the decay properties for the density.

In the case of pressureless systems, there is no such a possibility, so that we need to resort to more sophisticated techniques to control the density. This may partially explain the reason why the mathematical theory of pressureless models is poorer than the classical one.

The aim of this note is to present a novel technique coming from the maximal regularity theory for analytic semi-groups, in order to prove global-in-time properties of solutions to (1.3), (1.4). It will enable us to show existence and uniqueness results under rough assumptions on the density (only bounded), *even though one cannot take advantage of the effective viscous flux.* More precisely, by combining interpolation arguments, subtle embeddings, suitable time weighted norms and the magic properties of Lorentz spaces, we succeed in obtaining the $L_1(\mathbb{R}_+; L_\infty)$ regularity for the gradient of the solution to the linearized momentum equation in (1.3) and, eventually, to produce global-in-time strong solutions. Our main results concern global in time solvability for the two dimensional case for large velocity, and the three dimensional case in the small data regime.

2. Results

The idea of this note is to present an interesting application of Lorentz spaces for parabolic type systems. Lorentz spaces can be defined on any measure space (X,μ) via real interpolation between the classical Lebesgue spaces, as follows:

$$L_{p,r}(X,\mu) := (L_\infty(X,\mu), L_1(X,\mu))_{1/p,r} \ \text{ for } \ p\in(1,\infty) \quad\text{and}\quad r\in[1,\infty]. \tag{2.1}$$

Lorentz spaces may be endowed with the following (quasi)-norm (see e.g. [18, Prop. 1.4.9]):

$$\|f\|_{L_{p,r}} := \begin{cases} p^{\frac1r}\left(\displaystyle\int_0^\infty \left(s\,|\{|f|>s\}|^{\frac1p}\right)^r \frac{ds}{s}\right)^{\frac1r} & \text{if } r<\infty,\\ \sup_{s>0} s\,|\{|f|>s\}|^{\frac1p} & \text{if } r=\infty.\end{cases} \tag{2.2}$$

The reason for the pre-factor $p^{\frac1r}$ is to have $\|f\|_{L_{p,p}} = \|f\|_{L_p}$, according to Cavalieri's principle.

Let us first describe what we want to prove in the case where $A(\rho,u) = \mu\Delta u + \mu'\nabla\mathrm{div}\, u$, if the gas domain is the whole plane $\mathbb{R}^2$. So we consider:

$$\begin{aligned}&\rho_t + \mathrm{div}\,(\rho u) = 0 && \text{in } \mathbb{R}_+\times\mathbb{R}^2,\\ &\rho u_t + \rho u\cdot\nabla u = \mu\Delta u + \mu'\nabla\mathrm{div}\, u && \text{in } \mathbb{R}_+\times\mathbb{R}^2,\\ &\rho|_{t=0} = \rho_0, \qquad u|_{t=0} = u_0 && \text{at } \mathbb{R}^2.\end{aligned} \tag{2.3}$$

Following recent results of ours in [8, 10, 11] (in different contexts, though), we strive for global results for general initial velocities provided the volume (bulk) viscosity μ' is large enough. Owing to our approach based on a perturbative method, we need moreover the density to be close to a constant.

Our key solution space will be the set $\dot W^{2,1}_{4/3,1}$ of functions $z:\mathbb{R}_+\times\mathbb{R}^2\to\mathbb{R}$ such that

$$\partial_t z,\ \nabla_x^2 z\in L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2).$$

The corresponding trace space on constant times is the subspace $\dot W^{1/2}_{4/3,1}$ of $\dot W^{1/2}_{4/3}$ that is obtained if replacing the Lebesgue space $L_{4/3}$ by the Lorentz space $L_{4/3,1}$. That space may be characterized in terms of the heat semi-group $(e^{t\Delta})_{t>0}$ as follows:

$$u\in \dot W^{1/2}_{4/3,1}(\mathbb{R}^2)\iff \nabla^2 e^{t\Delta}u\in L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2).$$

Being a subspace of $\dot W^{1/2}_{4/3}(\mathbb{R}^2)$, the space $\dot W^{1/2}_{4/3,1}(\mathbb{R}^2)$ is embedded in $L_2(\mathbb{R}^2)$.

More generally, we define for any $1<p<\infty$ and $1\leq r\leq\infty$,

$$\dot W^{2-2/p}_{p,r}(\mathbb{R}^d)=\Big\{u\in L_{1,loc}(\mathbb{R}^d):\nabla^2e^{t\Delta}u\in L_{p,r}(\mathbb{R}_+\times\mathbb{R}^d)\Big\},\tag{2.4}$$

that coincides with the trace space of the set

$$\dot W^{2,1}_{p,r}:=\big\{z\in L_{1,loc}(\mathbb{R}_+\times\mathbb{R}^d):\partial_t z,\nabla_x^2 z\in L_{p,r}(\mathbb{R}_+\times\mathbb{R}^d)\big\}\cdot$$

Our first main result is a global existence and uniqueness statement for (2.3).

Theorem 2.1. *Let us fix some $M>0$ and consider any vector field u_0 on $\mathbb{R}^2$ with components in $\dot W^{1/2}_{4/3,1}$ such that, denoting by $\mathcal{P}$ and $\mathcal{Q}$ the Helmholtz projectors on divergence-free and potential vector fields, we have*

$$\mu^{-1}\|u_0\|^2_{L^2}+\|\mathcal{P}u_0\|^{1/2}_{\dot W^{1/2}_{4/3,1}}+(\nu/\mu)^{1/4}\|\mathcal{Q}u_0\|^{1/2}_{\dot W^{1/2}_{4/3,1}}\leq M\mu\quad\textit{with}\quad \nu:=\mu+\mu'.\tag{2.5}$$

There exist two universal constants c and C such if

$$\|\rho_0-1\|_{L_\infty}\leq c\quad\textit{and}\quad \nu\geq M^{1/2}e^{CM^4},\tag{2.6}$$

then System (2.3) admits a global finite energy solution (ρ,u) with $\rho\in\mathcal{C}^*_w(\mathbb{R}_+;L_\infty(\mathbb{R}^2))$ and $u\in\dot W^{2,1}_{4/3,1}$, satisfying

$$\|\rho-1\|_{L_\infty(\mathbb{R}_+\times\mathbb{R}^2)}\leq 2c.\tag{2.7}$$

Furthermore $\nabla u\in L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^2))$ and the following decay property holds:

$$\|t\mathcal{P}u\|_{L_\infty(\mathbb{R}_+;\dot W^{3/2}_{4,1}(\mathbb{R}^2))}+\Big(\frac{\nu}{\mu}\Big)^{3/4}\|t\mathcal{Q}u\|_{L_\infty(\mathbb{R}_+;\dot W^{3/2}_{4,1}(\mathbb{R}^2))}+\|(tu)_t,\nabla^2tu\|_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\frac{\nu}{\mu}\|\nabla\mathrm{div}\,tu\|_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}\leq e^{CM^4}.\tag{2.8}$$

Finally, the solution (ρ,u) is unique in the above regularity class.

Some comments are in order:

- Condition (2.5) means that global existence holds true for large ν, provided the divergence part of the velocity is $\mathcal{O}(\nu^{-1/4})$. A similar restriction (with other exponents, though), was found in our prior works dedicated to the global existence of strong solutions for the compressible Navier-Stokes equations with increasing pressure law [8, 10, 11].
- The above statement involves only quantities that are scaling invariant for System (2.3).
- Having Lorentz spaces with second index equal to 1 is crucial as it allows us to capture the limit case of the Sobolev embedding – see (A.2) in the Appendix. Other choices than $L_{4/3,1}$ and $L_{4,1}$ might be possible.

In the three-dimensional case, the energy space $L_2(\mathbb{R}^3)$ is super-critical by half a derivative, and there is no chance to prove a general result for large data, assuming only that one of the viscosity coefficients is large. Therefore, we decided for simplicity to focus on the second case of (1.4) and the system we want to consider is thus:

$$
\begin{aligned}
&\rho_t + \mathrm{div}\,(\rho u) = 0 && \text{in } \mathbb{R}_+ \times \mathbb{R}^3,\\
&\rho u_t + \rho u \cdot \nabla u = \mu \Delta u && \text{in } \mathbb{R}_+ \times \mathbb{R}^3,\\
&u|_{t=0} = u_0, \qquad \rho|_{t=0} = \rho_0 && \text{at } \mathbb{R}^3.
\end{aligned}
\tag{2.9}
$$

To simplify our analysis, we chose a functional framework for the velocity that is well beyond critical regularity. Here is our main result for (2.9):

Theorem 2.2. *Take initial data $\rho_0 \in L_\infty(\mathbb{R}^3)$ and $u_0 \in \dot W^{2/5}_{5/4}(\mathbb{R}^3) \cap \dot W^{6/5}_{5/2,1}(\mathbb{R}^3)$. There exists a constant $c > 0$ such that, if*

$$
\|\rho_0 - 1\|_{L_\infty} < c \quad \textit{and} \quad \|u_0\|^{1/2}_{\dot W^{6/5}_{5/2,1}} \|u_0\|^{1/2}_{\dot W^{2/5}_{5/4}} < c\mu,
\tag{2.10}
$$

then (2.9) has a global in time unique solution (ρ, u) with finite energy, and such that $\rho \in \mathcal{C}_w(\mathbb{R}_+; L_\infty(\mathbb{R}^3))$ and $u \in \dot W^{2,1}_{5/2,1} \cap \dot W^{2,1}_{5/4}$. Furthermore,

$$
\|1 - \rho\|_{L_\infty} < 2c,
\tag{2.11}
$$

the function tu belongs to $\dot W^{2,1}_{10/3,1}(\mathbb{R}^3 \times \mathbb{R}_+)$, and we have

$$
\int_0^\infty \|\nabla u\|_{L_\infty(\mathbb{R}^3)}\, dx \le C\mu^{-1} \|u_0\|^{1/2}_{\dot W^{6/5}_{5/2,1}} \|u_0\|^{1/2}_{\dot W^{2/5}_{5/4}}.
$$

Remark 2.1. *Let us emphasize that the initial velocity is actually is L^2, a consequence of the following interpolation inequality:*

$$
\|u_0\|_{L_2(\mathbb{R}^3)} \lesssim \|u_0\|^{3/4}_{\dot W^{2/5}_{5/4}(\mathbb{R}^3)} \|u_0\|^{1/4}_{\dot W^{6/5}_{5/2}(\mathbb{R}^3)}.
\tag{2.12}
$$

We also want to stress the fact that the smallness condition (2.10) is scaling invariant.

The rest of the paper is structured as follows. In the next section, we prove our two-dimensional global result for (2.3). Section 4 is devoted to the proof of Theorem 2.2. In Appendix, we recall some useful results on interpolation and Lorentz spaces.

We shall use standard notations and conventions. In particular, C will always designate harmless constants that do not depend on 'important' quantities, and we shall sometimes note $A \lesssim B$ instead of $A \le CB$.

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3. The two dimensional case

This part is dedicated to the proof of Theorem 2.1. The key observation is that the energy balance associated to (2.3) combined with some interpolation argument enables to bound the norm of u in $L_{4,2}(\mathbb{R}_+ \times \mathbb{R}^2)$, in terms of $\|u_0\|_{L_2(\mathbb{R}^2)}$. This will enable us to get a priori estimates for u in $\dot W^{2,1}_{4/3,1}$, then for tu in $\dot W^{2,1}_{4,1}$ provided the density is close to 1. Interpolating again will give a control on $\mathrm{div}\, u$ in $L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^2))$ and thus, going to the mass conservation equation, on $\rho - 1$. At this stage, we will use a bootstrap argument so as to justify a posteriori that, indeed, if $\nu := \mu + \mu'$ is large enough, then all the above global-in-time estimates hold true. Then, we observe that the very same arguments leading to the control of $\mathrm{div}\, u$ also allow to bound ∇u in $L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^2))$. From that point, we follow the energy method of [7, Sec. 4] going to Lagrangian coordinates in order to prove uniqueness, and the rigorous proof of existence is obtained by compactness arguments, after constructing a sequence of smoother solutions.

Let us now go to the details of the proof. We assume throughout that $\mu' \geq 0$ and, to simplify the computations, we take $\mu = 1$. That latter assumption is not restrictive, since (ρ, u) satisfies (2.3) with coefficients (μ, μ') if and only if

$$(\widetilde{\rho}, \widetilde{u})(t,x) := (\rho, \mu^{-1}u)(\mu^{-1}t, x) \tag{3.1}$$

satisfies (2.3) with coefficients 1 and μ'/μ.

Step 1. The energy balance and the control of the norm in $L_{4,2}(\mathbb{R}_+ \times \mathbb{R}^2)$. As already pointed out, the space for u_0 is continuously embedded in $L_2(\mathbb{R}^2)$. Hence the initial data have finite energy. Now, the energy balance for (2.3) reads

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \rho|u|^2 dx + \int_{\mathbb{R}^2} \big(|\nabla u|^2 + \mu'(\operatorname{div} u)^2\big) dx = 0. \tag{3.2}$$

Provided (2.7) is fulfilled with small enough c, we thus have, denoting by $\mathcal{P}$ the L_2 orthogonal projector on solenoidal vector-fields, and $\nu := 1 + \mu'$,

$$\|u\|^2_{L_\infty(\mathbb{R}_+;L_2(\mathbb{R}^2))} + 2\|\nabla\mathcal{P}u\|^2_{L_2(\mathbb{R}_+;L_2(\mathbb{R}^2))} + 2\nu\|\operatorname{div} u\|^2_{L_2(\mathbb{R}_+;L_2(\mathbb{R}^2))} \leq 2\|u_0\|^2_{L_2}. \tag{3.3}$$

We claim that the above inequality implies that

$$\|u\|_{L_{4,2}(\mathbb{R}_+\times\mathbb{R}^2)} \lesssim \|u_0\|_{L_2}. \tag{3.4}$$

To prove our claim, we shall use (3.3) and the fact that $\dot{H}^1(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}(\mathbb{R}^2)$. Hence it suffices to check that

$$L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^2)) \cap L_2(\mathbb{R}_+; \mathrm{BMO}(\mathbb{R}^2)) \hookrightarrow L_{4,2}(\mathbb{R}_+ \times \mathbb{R}^2). \tag{3.5}$$

Let us omit $\mathbb{R}_+$ and $\mathbb{R}^2$ in the following lines for better readability. As a start, we observe that, as a consequence of e.g. [32, Section 1.17],

$$L_\infty(L_2) = \big(L_\infty(L_{3/2}); L_\infty(L_3)\big)_{1/2,2} \quad \text{and} \quad L_2(\mathrm{BMO}) = \big(L_{3/2}(\mathrm{BMO}); \mathrm{L}_3(\mathrm{BMO})\big)_{1/2,2}. \tag{3.6}$$

Thus, one can write

$$\begin{aligned} L_\infty(L_2) \cap L_2(\mathrm{BMO}) &= L_\infty(L_2) \cap \big(L_{3/2}(\mathrm{BMO}); L_3(\mathrm{BMO})\big)_{1/2,2} \\ &= \big(L_\infty(L_2) \cap L_{3/2}(\mathrm{BMO}); L_\infty(L_2) \cap L_3(\mathrm{BMO})\big)_{1/2,2}. \end{aligned} \tag{3.7}$$

However, in light of the first relation of (3.6) and of obvious embedding,

$$\begin{aligned} L_\infty(L_2) \cap L_{3/2}(\mathrm{BMO}) &= \big(L_\infty(L_{3/2}) \cap L_{3/2}(\mathrm{BMO}); L_\infty(L_3) \cap L_{3/2}(\mathrm{BMO})\big)_{1/2,2} \\ &\hookrightarrow \big(L_\infty(L_{3/2}); L_{3/2}(\mathrm{BMO})\big)_{1/2,2} \\ &\hookrightarrow L_3(\mathbb{R}_+ \times \mathbb{R}^2). \end{aligned}$$

Analogously,

$$\begin{aligned} L_\infty(L_2) \cap L_3(\mathrm{BMO}) &= \big(L_\infty(L_{3/2}) \cap L_3(\mathrm{BMO}); L_\infty(L_3) \cap L_3(\mathrm{BMO})\big)_{1/2,2} \\ &\hookrightarrow \big(L_3(\mathrm{BMO}); L_\infty(L_3)\big)_{1/2,2} \\ &\hookrightarrow L_6(\mathbb{R}_+ \times \mathbb{R}^2). \end{aligned}$$

Plugging this in (3.7), we get

$$L_\infty(L_2) \cap L_2(\mathrm{BMO}) \hookrightarrow \big(L_3(\mathbb{R}_+ \times \mathbb{R}^2); L_6(\mathbb{R}_+ \times \mathbb{R}^2)\big)_{1/2,2} = L_{4,2}(\mathbb{R}_+ \times \mathbb{R}^2),$$

which completes the proof of (3.4).

Step 2. Control of the norm of the solution in $\dot W^{2,1}_{4/3,1}$. Rewrite the velocity equation as:

$$u_t-\Delta u-\mu'\nabla\mathrm{div}\,u=f:=(1-\rho)u_t-\rho u\cdot\nabla u.$$

Using the Helmholtz projectors $\mathcal{P}$ and $\mathcal{Q}$ on solenoidal and potential vector fields, respectively, yields

$$(\mathcal{P}u)_t-\Delta\mathcal{P}u=\mathcal{P}f\quad\text{and}\quad(\mathcal{Q}u)_t-\nu\Delta\mathcal{Q}u=\mathcal{Q}f.$$

Hence, thanks to parabolic maximal regularity (see Proposition A.1), we have

$$\|\mathcal{P}u\|_{L_\infty(\mathbb{R}_+;\dot W^{1/2}_{4/3,1})}+\|\nabla^2\mathcal{P}u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\|\mathcal{P}u_t\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}\lesssim\|\mathcal{P}u_0\|_{\dot W^{1/2}_{4/3,1}}+\|\mathcal{P}f\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}$$

$$\text{and}\quad\nu^{1/4}\|\mathcal{Q}u\|_{L_\infty(\mathbb{R}_+;\dot W^{1/2}_{4/3,1})}+\nu\|\nabla\mathrm{div}\,u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\|\mathcal{Q}u_t\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}\lesssim\nu^{1/4}\|\mathcal{Q}u_0\|_{\dot W^{1/2}_{4/3,1}}+\|\mathcal{Q}f\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}.$$

To bound the terms $\mathcal{P}f$ and $\mathcal{P}Q$, we observe that $\mathcal{P}$ and $\mathcal{Q}$ are continuous on $L_{4/3,1}$, so that it is thus enough to estimate f in $L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)$. We find that

$$\|f\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}\lesssim\|1-\rho\|_{L_\infty(\mathbb{R}_+\times\mathbb{R}^2)}\|u_t\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\|\rho\|_{L_\infty(\mathbb{R}_+\times\mathbb{R}^2)}\|u\cdot\nabla u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}.$$

Assuming (2.7), it will be eventually possible to absorb the first term in the right-hand side. Furthermore, by Hölder inequality for Lorentz spaces (see the Appendix),

$$\|u\cdot\nabla u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}\le\|u\|_{L_{4,2}(\mathbb{R}_+\times\mathbb{R}^2)}\|\nabla u\|_{L_2(\mathbb{R}_+\times\mathbb{R}^2)},$$

and thus, thanks to the first step,

$$\|u\cdot\nabla u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}\lesssim\|u_0\|^2_{L_2(\mathbb{R}^2)}.\tag{3.8}$$

Therefore, one eventually gets, as $\nu\ge1$,

$$\|\mathcal{P}u\|_{L_\infty(\mathbb{R}_+;\dot W^{1/2}_{4/3,1})}+\nu^{1/4}\|\mathcal{Q}u\|_{L_\infty(\mathbb{R}_+;\dot W^{1/2}_{4/3,1})}+\|u_t\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\|\nabla^2u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\nu\|\nabla\mathrm{div}\,u\|_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}\lesssim\|\mathcal{P}u_0\|_{\dot W^{1/2}_{4/3,1}}+\nu^{1/4}\|\mathcal{Q}u_0\|_{\dot W^{1/2}_{4/3,1}}+\|u_0\|^2_{L_2}.\tag{3.9}$$

Now, a real interpolation argument that is carried out at the end of the appendix implies that we also have the following control that will be the key to the next step:

$$\|\mathcal{P}u\|_{L_{4,1}(\mathbb{R}^2\times\mathbb{R}_+)}+\nu^{1/2}\|\mathcal{Q}u\|_{L_{4,1}(\mathbb{R}^2\times\mathbb{R}_+)}\lesssim\|\mathcal{P}u_0\|_{\dot W^{1/2}_{4/3,1}}+\nu^{1/4}\|\mathcal{Q}u_0\|_{\dot W^{1/2}_{4/3,1}}+\|u_0\|^2_{L_2}.\tag{3.10}$$

Step 3. A time weighted estimate. We now look at the momentum equation in the form

$$(t\,u)_t-\Delta(tu)-\mu'\nabla\mathrm{div}\,(tu)=(1-\rho)(t\,u)_t+\rho u-t\rho u\cdot\nabla u.\tag{3.11}$$

By definition, the initial data for tu is zero, and we know from (3.10) that the term ρu is in $L_{4,1}(\mathbb{R}^2\times\mathbb{R}_+)$. This gives us some hint on the regularity of the whole right-hand side. Now, projecting (3.11) by means of $\mathcal{P}$ and $\mathcal{Q}$, using the maximal regularity estimates of the appendix, and still assuming (2.7), one gets for all $0\le T\le T'\le\infty$,

$$\sup_{T\le t\le T'}\|t\mathcal{P}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\nu^{3/4}\sup_{T\le t\le T'}\|t\mathcal{Q}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\|\nabla^2tu\|_{L_{4,1}([T,T]'\times\mathbb{R}^2)}+\|(tu)_t\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}+\nu\|\nabla\mathrm{div}\,tu\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\le C\Big(\|T\mathcal{P}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\nu^{3/4}\|T\mathcal{Q}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\|u\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}+\|tu\cdot\nabla u\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\Big).\tag{3.12}$$

Since we have $\dot W^{1/2}_{4,1}(\mathbb{R}^2)\hookrightarrow L_\infty(\mathbb{R}^2)$ (see the Appendix), the term with $u\cdot\nabla u$ may be bounded as follows:

$$\|tu\cdot\nabla u\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\le C\|u\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\|tu\|_{L_\infty([T,T'];\dot W^{3/2}_{4,1}(\mathbb{R}^2))}.\tag{3.13}$$

Consequently, there exists a constant $c>0$ such that, if

$$\|u\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\le c,\tag{3.14}$$

then Inequality (3.12) reduces to

$$\sup_{T\le t\le T'}\|t\mathcal{P}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\nu^{3/4}\sup_{T\le t\le T'}\|t\mathcal{Q}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\|\nabla^2 tu\|_{L_{4,1}([T,T]'\times\mathbb{R}^2)}$$
$$+\|(tu)_t\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}+\nu\|\nabla\mathrm{div}\,tu\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\le C\Big(\|T\mathcal{P}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}$$
$$+\nu^{3/4}\|T\mathcal{Q}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\|u\|_{L_{4,1}([T,T']\times\mathbb{R}^2)}\Big)\cdot \tag{3.15}$$

Of course, if one can take $T=0$ and $T'=\infty$ in (3.14), then we control the left-hand side of (3.12) on $\mathbb{R}_+$, so assume from now on that $\|u\|_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}>c$. We claim that there exists a finite sequence $0=T_0<T_1<\cdots<T_{K-1}<T_K=\infty$ such that (3.14) if fulfilled on $[T_k,T_{k+1}]$ for each $k\in\{0,\cdots,K-1\}$. In order to prove our claim, we introduce

$$U(t):=\|u(t,\cdot)\|_{L_{4,1}(\mathbb{R}^2)}$$

and recall that (up to some constant)

$$\|U\|_{L_{4,1}(\mathbb{R}_+)}=\int_0^\infty|\{t\in\mathbb{R}_+:|U|>s\}|^{1/4}\,ds. \tag{3.16}$$

From Lebesgue dominated convergence theorem, we have

$$\|U\|_{L_{4,1}(T',T'')}=\int_0^\infty|\{t\in(T',T''):|U|>s\}|^{1/4}ds\to 0\quad\text{as}\quad T''-T'\to 0. \tag{3.17}$$

Hence one can construct inductively a family $0=T_0<T_1<\cdots<T_k<\cdots$ such that

$$\|U\|_{L_{4,1}(T_k,T_{k+1})}=c. \tag{3.18}$$

By simple Hölder inequality on series, we have

$$\sum_{k=1}^K|\{t\in(T_{k-1},T_k):|U|>s\}|^{1/4}\le K^{3/4}|\{t\in\mathbb{R}_+:|U|>s\}|^{1/4}. \tag{3.19}$$

Hence we find that for all $k=1,...,K$,

$$c=\|U\|_{L_{4,1}(T_{k-1},T_k)}\le K^{-1/4}\|U\|_{L_{4,1}(\mathbb{R}_+)}.$$

Consequently, we mus have

$$K\le(c^{-1}\|U\|_{L_{4,1}(\mathbb{R}_+)})^4. \tag{3.20}$$

Let $X_k:=\sup_{t\in(T_k,T_{k+1})}\big(\|t\mathcal{P}u(t)\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\nu^{3/4}\|t\mathcal{Q}u(t)\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}\big)\cdot$ Then, (3.15) and (3.19) ensure that

$$X_k\le C(c+X_{k-1}). \tag{3.21}$$

So, arguing by induction, we eventually get for all $m\in\{0,\cdots,K-1\}$,

$$X_m\le c\sum_{\ell=0}^m C^\ell\le c\sum_{l=1}^K(C)^\ell\le\frac{C^{K+1}}{C-1}\,c. \tag{3.22}$$

Reverting to (3.15), then using (3.20) and (3.10), we conclude that

$$\sup_{t\ge0}\|t\mathcal{P}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\nu^{1/4}\sup_{t\ge0}\|t\mathcal{Q}u\|_{\dot W^{3/2}_{4,1}(\mathbb{R}^2)}+\|\nabla^2tu\|_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}+\|(tu)_t\|_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}$$
$$+\nu\|\nabla\mathrm{div}\,tu\|_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}\le Cc\exp\Big\{\|\mathcal{P}u_0\|^4_{\dot W^{1/2}_{4/3,1}}+\nu\|\mathcal{Q}u_0\|^4_{\dot W^{1/2}_{4/3,1}}+\|u_0\|^8_{L_2}\Big\}\cdot \tag{3.23}$$

Step 4. Bounding $\operatorname{div} u$. In order to keep the density close to 1, we need to bound $\operatorname{div} u$ in $L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^2))$. To get it, the key observation is that

$$\operatorname{div}(tu)\in L_{4,1}(\mathbb{R}_+;\dot W^1_{4,1}(\mathbb{R}^2))\quad\text{and}\quad \operatorname{div} u\in L_{4/3,1}(\mathbb{R}_+;\dot W^1_{4/3,1}(\mathbb{R}^2)). \tag{3.24}$$

Now, from Gagliardo-Nirenberg inequality, we see that

$$\|z\|_{L_\infty(\mathbb{R}^2)}\leq C\|\nabla z\|^{1/2}_{L_4(\mathbb{R}^2)}\|\nabla z\|^{1/2}_{L_{4/3}(\mathbb{R}^2)}. \tag{3.25}$$

So we have, thanks to Hölder inequality in Lorentz spaces,

$$\begin{aligned}\int_0^\infty \|\operatorname{div} u\|_{L_\infty}\,dt &\leq C\int_0^\infty t^{-1/2}\|t\nabla \operatorname{div} u\|^{1/2}_{L_4}\|\nabla\operatorname{div} u\|^{1/2}_{L_{4/3}}\,dt\\ &\leq C\|t^{-1/2}\|_{L_{2,\infty}(\mathbb{R}_+)}\Big\|\|t\nabla\operatorname{div} u\|^{1/2}_{L_{4,1}(\mathbb{R}^2)}\Big\|_{L_{8,2}(\mathbb{R}_+)}\Big\|\|\nabla\operatorname{div} u\|^{1/2}_{L_{4/3,1}(\mathbb{R}^2)}\Big\|_{L_{8/3,2}(\mathbb{R}_+)}\\ &\leq C\|t\nabla\operatorname{div} u\|^{1/2}_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}\|\nabla\operatorname{div} u\|^{1/2}_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}.\end{aligned}$$

Hence, thanks to (3.9) and (3.23),

$$\nu\int_0^\infty\|\operatorname{div} u\|_{L_\infty}\,dt\leq Cc\Big(\|u_0\|_{L_2}+\|\mathcal{P}u_0\|^{1/2}_{\dot W^{1/2}_{4/3,1}}+\nu^{1/8}\|\mathcal{Q}u_0\|^{1/2}_{\dot W^{1/2}_{4/3,1}}\Big)e^{C\left(\|u_0\|^8_{L_2}+\|\mathcal{P}u_0\|^4_{\dot W^{1/2}_{4/3,1}}+\nu\|\mathcal{Q}u_0\|^4_{\dot W^{1/2}_{4/3,1}}\right)}. \tag{3.26}$$

Step 5. Bounding the density. The discrepancy of the density to 1 (that is $a:=\rho-1$) may be controlled by means of the mass equation:

$$\partial_t a+u\cdot\nabla a+(1+a)\operatorname{div} u=0,$$

which gives

$$\|a(t)\|_{L_\infty}\leq\|a_0\|_{L_\infty}+\int_0^t(1+\|a\|_{L_\infty})\|\operatorname{div} u\|_{L_\infty}\,d\tau,$$

and thus

$$\|a(t)\|_{L_\infty}\leq\|a_0\|_{L_\infty}\exp\Big(\int_0^t\|\operatorname{div} u\|_{L_\infty}\,d\tau\Big)+\exp\Big(\int_0^t\|\operatorname{div} u\|_{L_\infty}\,d\tau\Big)-1.$$

Hence, provided we have

$$\int_0^T\|\operatorname{div} u\|_{L_\infty}\,d\tau\leq\log(1+c/2), \tag{3.27}$$

the smallness property (2.7) is satisfied on $[0,T]$. Bearing in mind Inequality (3.26), assuming with no loss of generality that $Cc\leq 1$, and using a bootstrap argument, one can conclude that (2.7) is satisfied if

$$\nu\geq\Big(\|u_0\|_{L_2}+\|\mathcal{P}u_0\|^{1/2}_{\dot W^{1/2}_{4/3,1}}+\nu^{1/8}\|\mathcal{Q}u_0\|^{1/2}_{\dot W^{1/2}_{4/3,1}}\Big)e^{C\left(\|u_0\|^8_{L_2}+\|\mathcal{P}u_0\|^4_{\dot W^{1/2}_{4/3,1}}+\nu\|\mathcal{Q}u_0\|^4_{\dot W^{1/2}_{4/3,1}}\right)}. \tag{3.28}$$

Step 6: Uniqueness. The key to uniqueness is that ∇u is in $L_1(\mathbb{R}_+;L_\infty)$. To get that property, one may proceed exactly as for bounding $\operatorname{div} u$, writing that

$$\int_0^\infty\|\nabla u\|_{L_\infty}\,dt\leq C\|t^{-1/2}\|_{L_{2,\infty}(\mathbb{R}_+)}\|t\nabla^2 u\|^{1/2}_{L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)}\|\nabla^2 u\|^{1/2}_{L_{4/3,1}(\mathbb{R}_+\times\mathbb{R}^2)}, \tag{3.29}$$

and using that the r.h.s. is bounded in terms of u_0 according to (3.9) and (3.23). There is no need of bootstrap argument here.

Because of the hyperbolic nature of the continuity equation, the uniqueness issue is not straightforward in our case, as the regularity of the density is very low. However, having a control on ∇u in $L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^2))$ enables us to rewrite our system in Lagrangian coordinates. More precisely, for all $y\in\mathbb{R}^2$, consider the following ODE:

$$\frac{dX}{dt}(t,y)=u(t,X(t,y)),\qquad X|_{t=0}=y. \tag{3.30}$$

Having (3.29) at hand guarantees that (3.30) defines a C^1 flow X on $\mathbb{R}_+\times\mathbb{R}^2$.

Let us express the density and velocity in the new coordinates:

$$\eta(t,y)=\rho(t,X(t,y)),\qquad v(t,y)=u(t,X(t,y)). \tag{3.31}$$

Then, the system for (η,v) reads (see details in e.g. [6]):

$$\begin{aligned}&(J_v\eta)_t=0,\\ &\rho_0v_t-\mathrm{div}_v\left(\nabla_v v+\mu'(\mathrm{div}_v\,v)\,\mathrm{Id}\right)=0,\end{aligned} \tag{3.32}$$

where $\nabla_v:=A_v^\top\nabla_y$, $\mathrm{div}_v:=\mathrm{div}\,(J_v^{-1}A_v\cdot)=A_v^\top:\nabla_y$ with $A_v=(DX_v)^{-1}$ and $J_v=\det(DX_v)$. One points out that $J_v^{-1}A_v=\mathrm{adj}(DX_v)$ (the adjugate matrix of DX_v).

Since, in our framework the Lagrangian and Eulerian formulations are equivalent (see e.g. [6, 27]), it suffices to prove uniqueness at the level of Lagrangian coordinates. Therefore, consider two solutions (η,v) and $(\bar\eta,\bar v)$ of (3.32) emanating from the data (ρ_0,u_0). Then, the difference of velocities $\delta v:=\bar v-v$ satisfies

$$\begin{aligned}\rho_0\delta v_t-\mathrm{div}_v\left(\nabla_v\delta v+\mu'(\mathrm{div}_v\,\delta v)\,\mathrm{Id}\right)\\ =\big(\mathrm{div}_{\bar v}\nabla_{\bar v}-\mathrm{div}_{\,v}\nabla_v\big)\bar v+\mu'\big(\mathrm{div}_{\bar v}\,\mathrm{Id}\,\mathrm{div}_{\bar v}-\mathrm{div}_v\,\mathrm{Id}\,\mathrm{div}_v\big)\bar v.\end{aligned} \tag{3.33}$$

Note that

$$\begin{aligned}\big(\mathrm{div}_{\bar v}\nabla_{\bar v}-\mathrm{div}_{\,v}\nabla_v\big)\bar v&=\mathrm{div}\,\big((\mathrm{adj}(DX_{\bar v})A_{\bar v}^\top-\mathrm{adj}(DX_v)A_v^\top)\cdot\nabla\bar v\big),\\ \big(\mathrm{div}\bar v\,\mathrm{Id}\,\mathrm{div}_{\bar v}-\mathrm{div}_v\,\mathrm{Id}\,\mathrm{div}_v\big)\bar v&=\mathrm{div}\,\big((\mathrm{adj}(DX_{\bar v})A_{\bar v}^\top-\mathrm{adj}(DX_v)A_v^\top):\nabla\bar v\big)\cdot\end{aligned}$$

Now, taking the L^2 scalar product of (3.33) with δv and integrating by parts delivers

$$\begin{aligned}\frac12\frac{d}{dt}\|\sqrt{\rho_0}\,\delta v\|_{L_2}^2+\|\nabla_v\delta v\|_{L_2}^2+\mu'\|\mathrm{div}_v\,\delta v\|_{L_2}^2\\ \leq\nu\big\|\big(\mathrm{adj}(DX_{\bar v})A_{\bar v}^\top-\mathrm{adj}(DX_v)A_v^\top\big)\cdot\nabla\bar v\big\|_{L^2}\|\nabla\delta v\|_{L_2}.\end{aligned}$$

Let us take an interval $[0,T]$ for which

$$\max\bigg(\int_0^T\|\nabla v\|_{L_\infty}\,dt,\int_0^T\|\nabla\bar v\|_{L_\infty}\,dt\bigg)\quad\text{is small.} \tag{3.34}$$

Then, all terms like $\mathrm{Id}-A_w$, $1-J_w$ or $\mathrm{Id}-\mathrm{adj}(DX_w)$ (with $w=v,\bar v$) may be computed by using Neumann series expansions, and we end up with pointwise estimates of the following type:

$$|\,\mathrm{Id}-A_w|\lesssim\int_0^t|\nabla w|\,dt',\qquad|\,\mathrm{Id}-\mathrm{adj}(DX_w)|\lesssim\int_0^t|\nabla w|\,dt',\qquad|1-J_w|\lesssim\int_0^t|\nabla w|\,dt'.$$

From this, we deduce that

$$\frac{d}{dt}\|\sqrt{\rho_0}\,\delta v\|_{L_2}^2+\|\nabla\delta v\|_{L_2}^2\lesssim(\|\nabla v\|_{L_\infty}+\|\nabla\bar v\|_{L_\infty})\|\nabla\delta v\|_{L_2}\bigg\|\int_0^t\nabla\delta v\,d\tau\bigg\|_{L_2}.$$

Because we have, by Cauchy-Schwarz inequality,

$$t^{-1/2}\bigg\|\int_0^t\nabla\delta v\,d\tau\bigg\|_{L_2(\mathbb{R}^2)}\leq\|\nabla\delta v\|_{L_2(0,t\times\mathbb{R}^2)},$$

integrating the above inequality (and using again Cauchy-Schwarz inequality) yields

$$\|\sqrt{\rho_0}\,\delta v(t)\|_{L_2}^2+\int_0^t\|\nabla\delta v\|_{L_2}^2\,d\tau\leq C\biggl(\int_0^t\tau\|(\nabla v,\nabla\bar v)(\tau)\|_{L_\infty}^2\,d\tau\biggr)^{1/2}\int_0^t\|\nabla\delta v\|_{L_2}^2\,d\tau.$$

Hence, there exists $c>0$ such that if, in addition to (3.34), we have

$$\|t^{1/2}\nabla w\|_{L_2(0,T;L_\infty(\mathbb{R}^2))}\leq c\quad\text{for}\quad w=v,\bar v,\tag{3.35}$$

then we have $\delta v\equiv 0$ on $[0,T]$, that is to say $\bar v=v$. Since $\delta\eta=(J_{\bar v}^{-1}-J_v)\rho_0$, we get $\bar\eta=\eta$, too.

In light of the above arguments, in order to get uniqueness on the whole $\mathbb{R}_+$, it suffices to show that our solutions satisfy not only $\nabla u\in L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^2))$, but also

$$\int_0^\infty t\|\nabla u\|_{L_\infty(\mathbb{R}^2)}^2\,dt<\infty.\tag{3.36}$$

This is a consequence of (3.25), as it gives

$$\begin{aligned}\int_0^\infty t\|\nabla u\|_{L_\infty(\mathbb{R}^2)}^2\,dt&\lesssim\int_0^\infty\|t\nabla^2u\|_{L_4(\mathbb{R}^2)}\|\nabla^2u\|_{L_{4/3}(\mathbb{R}^2)}\\&\lesssim\|t\nabla^2u\|_{L_4(\mathbb{R}_+\times\mathbb{R}^2)}\|\nabla^2u\|_{L_{4/3}(\mathbb{R}_+\times\mathbb{R}^2)}.\end{aligned}$$

Step 7: Proof of existence. The idea is to smooth out the data, and to solve (2.3) supplemented with those data, according to the local-in-time existence result of [7] (that just requires the initial velocity to be smooth enough, and the initial density to be close to 1 in L_∞). Then, the previous steps provide uniform bounds that allow to show that those smoother solutions are actually global, and one can eventually pass to the limit. The reader may refer to the end of the next part where more details are given both for Theorems 2.1 and 2.2.

4. The three dimensional case

Our aim here is to prove a global existence result in the small data regime case for System (2.9). We want to stress the dependency of the smallness condition in terms of the viscosity coefficient, and to get the optimal one, it is convenient to use again the rescaling (3.1). So we assume from now on that $\mu=1$.

The bulk of the proof consists in exhibiting global-in-time bounds in terms of the data for

$$\Xi:=\sup_{t\geq0}\|u(t)\|_{\dot W^{6/5}_{5/2,1}(\mathbb{R}^3)}+\|\nabla^2u,u_t\|_{L_{5/2,1}(\mathbb{R}_+\times\mathbb{R}^3)}\tag{4.1}$$

$$\text{and}\quad\Psi:=\sup_{t\geq0}\|u(t)\|_{\dot W^{2/5}_{5/4}(\mathbb{R}^3)}+\|\nabla^2u,u_t\|_{L_{5/4}(\mathbb{R}_+\times\mathbb{R}^3)}.\tag{4.2}$$

From it, we will deduce a bound of u in the Lorentz space $L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)$ (that will play the same role as $L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2)$ in the previous section), and get a control on tu in $\dot W^{2,1}_{10/3,1}(\mathbb{R}^3\times\mathbb{R}_+)$, which will be the key to eventually bound ∇u in $L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^3))$. From that stage, the proof of uniqueness follows the same lines as for (2.3).

Step 1. Control by the energy. Remembering that our assumptions imply that u_0 is in $L_2(\mathbb{R}^3)$, we start with the basic energy balance:

$$\frac{d}{dt}\int_{\mathbb{R}^3}\rho|u|^2\,dx+\int_{\mathbb{R}^3}|\nabla u|^2\,dx=0.\tag{4.3}$$

By Sobolev embedding and provided that

$$\|1-\rho\|_{L_\infty(\mathbb{R}_+\times\mathbb{R}^3)}<2c\ll1,\tag{4.4}$$

this implies the following bound on u:

$$\|u\|_{L_\infty(\mathbb{R}_+;L_2(\mathbb{R}^3))}+\|u\|_{L_2(\mathbb{R}_+;L_6(\mathbb{R}^3))}\lesssim\|u_0\|_{L_2(\mathbb{R}^3)}.\tag{4.5}$$

That relation will enable us to control higher norms of the solution, globally in time, provided some scaling invariant quantity involving u_0 is small enough.

Step 2. Control of the high norm. This step is somehow standard: we want to construct smooth solutions like for the classical Navier-Stokes system. Now, assuming (4.4) and taking advantage of the maximal regularity estimate for the heat equation in $L_{5/2,1}(\mathbb{R}_+\times\mathbb{R}^3)$ stated in Proposition A.1 yields (recall the definition of Ξ in (4.1)):

$$\Xi\leq C\bigl(\|u\cdot\nabla u\|_{L_{5/2,1}(\mathbb{R}_+\times\mathbb{R}^3)}+\|u_0\|_{\dot W^{6/5}_{5/2,1}(\mathbb{R}^3)}\bigr).$$

We see by Hölder inequality and Sobolev embedding $\dot W^1_{5/2}(\mathbb{R}^3)\hookrightarrow L_{15}(\mathbb{R}^3)$ that

$$\begin{aligned}\|u\cdot\nabla u\|_{L_{5/2,1}(\mathbb{R}_+\times\mathbb{R}^3)}&\leq C\|u\|_{L_\infty(\mathbb{R}_+;L_{3,1}(\mathbb{R}^3))}\|\nabla u\|_{L_{5/2,1}(\mathbb{R}_+;L_{15}(\mathbb{R}^3))}\\&\leq C\|u\|_{L_\infty(\mathbb{R}_+;L_{3,1}(\mathbb{R}^3))}\|\nabla^2 u\|_{L_{5/2,1}(\mathbb{R}_+;L_{5/2}(\mathbb{R}^3))}.\end{aligned}$$

Moreover, we note that by interpolation, embedding (A.2) and (4.5),

$$\|u\|_{L_{3,1}(\mathbb{R}^3)}\leq C\|u\|^{2/3}_{L_2(\mathbb{R}^3)}\|u\|^{1/3}_{L_\infty(\mathbb{R}^3)}\leq C\|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\|u\|^{1/3}_{\dot W^{6/5}_{5/2,1}(\mathbb{R}^3)}.\tag{4.6}$$

Hence, altogether, this gives

$$\Xi\leq C\bigl(\|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\Xi^{1+1/3}+\Xi_0\bigr).\tag{4.7}$$

From this and a bootstrap argument, we deduce that

$$(2C)^{4/3}\Xi_0^{1/3}\|u_0\|^{2/3}_{L^2}\leq 1\quad\text{implies}\quad\Xi\leq 2C\Xi_0.\tag{4.8}$$

Step 3. Control of the low norm. It is now a matter of bounding the functional Ψ defined in (4.2). Using maximal regularity, we now get

$$\Psi\leq C\bigl(\|u\cdot\nabla u\|_{L_{5/4}(\mathbb{R}_+\times\mathbb{R}^3)}+\|u_0\|_{\dot W^{2/5}_{5/4}(\mathbb{R}^3)}\bigr).\tag{4.9}$$

By Hölder inequality and Sobolev embedding $\dot W^1_{5/4}(\mathbb{R}^3)\hookrightarrow L_{15/7}(\mathbb{R}^3)$, we discover that

$$\begin{aligned}\|u\cdot\nabla u\|_{L_{5/4}(\mathbb{R}_+\times\mathbb{R}^3)}&\leq C\|u\|_{L_\infty(\mathbb{R}_+;L_3(\mathbb{R}^3))}\|\nabla u\|_{L_{5/4}(\mathbb{R}_+;L_{15/7}(\mathbb{R}^3))}\\&\leq C\|u\|_{L_\infty(\mathbb{R}_+;L_3(\mathbb{R}^3))}\|\nabla^2 u\|_{L_{5/4}(\mathbb{R}_+\times\mathbb{R}^3))}.\end{aligned}$$

Hence, thanks to (4.6),

$$\|u\cdot\nabla u\|_{L_{5/4}(\mathbb{R}_+\times\mathbb{R}^3)}\leq C\|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\Xi^{1/3}\Psi.$$

Now, using (4.8) and reverting to (4.9) yields

$$\Psi\leq C\bigl(\Psi_0+\|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\Xi_0^{1/3}\Psi\bigr),$$

whence, thanks to the smallness condition in (4.8) (changing C if need be),

$$\Psi\leq 2C\Psi_0.\tag{4.10}$$

Step 4. Time weight. In order to get eventually the desired control on ∇u in $L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^3))$ that is needed to ensure (4.4) provided we have (2.10) for ρ_0, and, later on, uniqueness, we mimic the sharp approach of the two dimensional case, considering the momentum equation in the following form

$$(tu)_t-\Delta(tu)=(1-\rho)(tu)_t-t\rho u\cdot\nabla u+\rho u\quad\text{in }\ \mathbb{R}_+\times\mathbb{R}^3.\tag{4.11}$$

Now, we observe that

$$\dot W^{2,1}_{5/2,1}(\mathbb{R}^3\times\mathbb{R}_+)\hookrightarrow L_\infty(\mathbb{R}_+\times\mathbb{R}^3)\quad\text{and}\quad\dot W^{2,1}_{5/4}(\mathbb{R}^3\times\mathbb{R}_+)\hookrightarrow L_{5/2}(\mathbb{R}_+\times\mathbb{R}^3)$$

Since, furthermore, $(L_\infty(\mathbb{R}_+\times\mathbb{R}^3);L_{5/2}(\mathbb{R}_+\times\mathbb{R}^3))_{3/4,1}=L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)$, we obtain

$$\|u\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)}\lesssim\Xi_0^{1/4}\Psi_0^{3/4},\tag{4.12}$$

and it is thus natural (look at the last term of (4.11)) to expect that

$$tu \in \dot W^{2,1}_{10/3,1}(\mathbb{R}^3\times\mathbb{R}_+). \tag{4.13}$$

More precisely, Proposition A.1 and (4.4) lead to

$$\Pi := \sup_{t\geq0}\|tu\|_{\dot W^{7/5}_{10/3,1}} + \|\nabla^2(tu),(tu)_t\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|tu\cdot\nabla u\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)} + \|u\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)}.$$

In order to estimate the nonlinear term, we start from Hölder inequality:

$$\|u\cdot\nabla tu\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)} \leq C\|u\|_{L_{5,1}(\mathbb{R}_+\times\mathbb{R}^3)}\|t\nabla u\|_{L_{10}(\mathbb{R}_+\times\mathbb{R}^3)}. \tag{4.14}$$

Using suitable embedding, one may prove that

$$\Big(L_\infty(\mathbb{R}_+;\dot W^{2/5}_{10/3}); L_{10/3}(\mathbb{R}_+;\dot W^1_{10/3})\Big)_{1/3,1} \hookrightarrow L_{10}(\mathbb{R}_+\times\mathbb{R}^3). \tag{4.15}$$

Hence, owing to the definition of Π, we have

$$\|t\nabla u\|_{L_{10}(\mathbb{R}_+\times\mathbb{R}^3)} \leq C\Pi. \tag{4.16}$$

In order to bound u in $L_{5,1}(\mathbb{R}_+\times\mathbb{R}^3)$, we start with the observation that

$$(L_\infty(\mathbb{R}_+\times\mathbb{R}^3); L_{10/3}(\mathbb{R}_+\times\mathbb{R}^3))_{2/3,1} = L_{5,1}(\mathbb{R}_+\times\mathbb{R}^3),$$

and that, by Hölder inequality and (4.5),

$$\|u\|_{L_{10/3}(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|u_0\|_{L_2(\mathbb{R}^3)}, \qquad \text{hence} \qquad \|u\|_{L_{5,1}(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\Xi_0^{1/3}.$$

Putting together with (4.16) and reverting to (4.14), we end up with

$$\|u\cdot\nabla tu\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\,\Xi_0^{1/3}\,\Pi.$$

Therefore, using also (4.12), we get the following inequality for Π:

$$\Pi \leq C\big(\Xi_0^{1/4}\Psi_0^{3/4} + \|u_0\|^{2/3}_{L_2(\mathbb{R}^3)}\,\Xi_0^{1/3}\,\Pi\big)\cdot$$

Consequently, under the very same smallness condition as in (4.8) (up to a change of the constant maybe), we obtain

$$\Pi \leq 2C\Xi_0^{1/4}\Psi_0^{3/4}. \tag{4.17}$$

Step 5. Bounding ∇u. It is now easy to get the desired control on ∇u: we start from the following Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L_\infty(\mathbb{R}^3)} \leq C\|\nabla u\|^{2/3}_{\dot W^1_{10/3}(\mathbb{R}^3)}\|\nabla u\|^{1/3}_{\dot W^1_{5/2}(\mathbb{R}^3)}, \tag{4.18}$$

which implies

$$\int_0^\infty \|\nabla u\|_{L_\infty(\mathbb{R}^3)}dt \leq C\int_0^t t^{-2/3}\|t\nabla u\|^{2/3}_{\dot W^1_{10/3}(\mathbb{R}^3)}\|\nabla u\|^{1/3}_{\dot W^1_{5/2}(\mathbb{R}^3)}\,dt. \tag{4.19}$$

Using Hölder inequality (A.1) with respect to time in Lorentz spaces, we find that[1]

$$\int_0^\infty \|\nabla u\|_{L_\infty}dt \leq C\|t^{-2/3}\|_{L_{3/2,\infty}(\mathbb{R}_+)}\|t\nabla u\|^{2/3}_{L_{10/3,1}(\mathbb{R}_+;\dot W^1_{10/3}(\mathbb{R}^3))}\|\nabla u\|^{1/3}_{L_{5/2,1}(\mathbb{R}_+;\dot W^1_{5/2}(\mathbb{R}^3))}. \tag{4.20}$$

As the right-hand side is bounded, owing to (4.8) and (4.17), one may conclude that ∇u is in $L_1(\mathbb{R}_+;L_\infty(\mathbb{R}^3))$. More importantly, we have the inequality

$$\int_0^\infty \|\nabla u\|_{L_\infty}dt \leq C(\Xi_0^{1/4}\Psi_0^{3/4})^{2/3}\,\Xi_0^{1/3} = C\,\Xi_0^{1/2}\Psi_0^{1/2} \ll 1. \tag{4.21}$$

Therefore, arguing on the mass equation exactly as in the 2D case, one can justify (4.4), and thus all the previous steps provided (2.10) is satisfied. Indeed, in light of (2.12), the above smallness condition is stronger than (4.8).

[1]Here it is important to have the solution space built on a Lorentz space *with last index* 1.

Step 6. Uniqueness. Arguing as in the previous section and knowing (4.21) (so as to put our system in Lagrangian coordinates), it suffices to establish the additional property that $t^{1/2}\nabla u$ is in $L_2(0,T;L_\infty(\mathbb{R}^3))$. Now, one may write, owing to (4.18), that

$$\begin{aligned}\int_0^\infty t\|\nabla u\|_{L_\infty(\mathbb{R}^3)}^2\,dt &\lesssim \int_0^\infty t^{-1/3}\|t\nabla u\|_{\dot W^1_{10/3}(\mathbb{R}^3)}^{4/3}\|\nabla u\|_{\dot W^1_{5/2}}^{2/3}\,dt\\ &\lesssim \|t^{-1/3}\|_{L_{3,\infty}(\mathbb{R}_+)}\Big\|\|t\nabla u\|_{\dot W^1_{10/3}(\mathbb{R}^3)}^{4/3}\Big\|_{L_{5/2,1}(\mathbb{R}_+)}\Big\|\|\nabla u\|_{\dot W^1_{5/2}}^{2/3}\Big\|_{L_{15/4}(\mathbb{R}_+)}\\ &\lesssim \|t\nabla^2 u\|_{L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)}^{4/3}\|\nabla^2 u\|_{L_{5/2}(\mathbb{R}_+\times\mathbb{R}^3)}^{2/3}.\end{aligned}$$

Because $t\nabla^2 u$ is in $L_{10/3,1}(\mathbb{R}_+\times\mathbb{R}^3)$ and $\nabla^2 u$ is in $L_{5/2}(\mathbb{R}_+\times\mathbb{R}^3)$, the right-hand side is indeed bounded. This completes the proof of uniqueness.

Step 7. Existence. Here we sketch the proof of the existence of a global solution under our assumptions on the data. The overall strategy is essentially the same in dimensions $d=2$ and $d=3$.

As a first, we truncate ρ_0 and smooth out u_0 to meet the conditions of the local-in-time existence theorem of [7]. We get a sequence $(\rho_0^n,u_0^n)_{n\in\mathbb{N}}$ of data that generates a sequence of local solutions $(\rho^n,u^n)_{n\in\mathbb{N}}$ on $[0,T^n]$, in the classical maximal regularity space

$$E_{p,r}(T):=\bigl\{z\in L_{1,loc}(0,T^n\times\mathbb{R}^d):\partial_t z,\nabla_x^2 z\in L_r(0,T^n;L_p(\mathbb{R}^d))\bigr\},$$

with e.g. $p=2d$ and $r=7/6$.

It is shown in [7] that those solutions satisfy the energy balance and (4.4), and are such that

$$\nabla u^n\in L_{r_1}(0,T^n;L_p(\mathbb{R}^d))\quad\text{with}\quad \frac{1}{r_1}=\frac{1}{r}-\frac12\cdot$$

Since the computations of the previous step just follow from the properties of the heat flow and of basic functional analysis, each (ρ^n,u^n) satisfy the estimates therein. In particular, $\|\nabla u^n\|_{L_1(0,T^n;L_\infty)}$ is uniformly bounded like in (4.21), which provides control of (4.4). Now, applying the standard maximal regularity estimates to[2]

$$\partial_t u^n-\Delta u^n-\mu'\Delta u^n=(1-\rho^n)\partial_t u^n+\rho^n u^n\cdot\nabla u^n,$$

one gets for all $T<T^n$,

$$\|u^n\|_{E_{p,r}(T)}\lesssim\|u_0^n\|_{E^0_{p,r}}+\|u^n\cdot\nabla u^n\|_{L_r(0,T;L_p)}.$$

We have

$$\|u^n\cdot\nabla u^n\|_{L_r(0,T;L_p)}\leq\|u^n\|_{L_2(0,T;L_\infty)}\|\nabla u^n\|_{L_{r_1}(0,T^n;L_p)}.$$

The first term is small (controlled) by our method, in terms of the data. Indeed, in the framework of Theorem 2.1, one can use the fact that $\|u^n\|_{L_\infty(\mathbb{R}^2)}\lesssim\|u^n\|_{L_2(\mathbb{R}^2)}^{1/3}\|\nabla^2u^n\|_{L_{4/3}(\mathbb{R}^2)}^{3/2}$, and thus, by Hölder inequality,

$$\|u^n\|_{L_2(0,T^n;L_\infty(\mathbb{R}^2))}\lesssim\|u^n\|_{L_\infty(0,T^n;L_2(\mathbb{R}^2))}^{1/3}\|\nabla^2u^n\|_{L_{4/3}(0,T^n\times\mathbb{R}^2)}^{2/3}\lesssim\|u_0\|_{\dot W^{1/2}_{4/3,1}(\mathbb{R}^2)}.$$

In the framework of Theorem 2.2, we first use the Gagliardo-Nirenberg inequality

$$\|z\|_{L_{3/2}(\mathbb{R}_+;L_\infty(\mathbb{R}^3))}\lesssim\|z\|_{L_{5/2}(\mathbb{R}_+;\dot W^2_{5/2}(\mathbb{R}^3))}^{1/3}\|z\|_{L_{5/4}(\mathbb{R}_+;\dot W^2_{5/4}(\mathbb{R}^3))}^{2/3},$$

then the fact that $L_2(\mathbb{R}_+;L_\infty(\mathbb{R}^3))\subset L_{3/2}(\mathbb{R}_+;L_\infty(\mathbb{R}^3))\cap L_\infty(\mathbb{R}_+;L_\infty(\mathbb{R}^3))$ and that $\dot W^{6/5}_{5/2,1}(\mathbb{R}^3)\hookrightarrow L_\infty(\mathbb{R}^3)$, to eventually get the desired control of $\|u^n\|_{L_2(0,T;L_\infty)}$.

[2]Of course μ' is put to 0 if one wants to prove the existence part of Theorem 2.2.

Therefore, one can eventually bound u^n in $E_{p,r}(T^n)$ independently of T^n. Then, applying standard continuation arguments allow to prove that (ρ^n, u^n) is actually global, and may be bounded in terms of the original data (ρ_0, u_0) in the spaces of our main theorems, independently of n.

From this stage, passing to the limit *in the slightly larger space* $\dot W^{2,1}_{5/2}(\mathbb{R}^3\times\mathbb{R}_+)\cap\dot W^{2,1}_{5/4}(\mathbb{R}^3\times\mathbb{R}_+)$ (or in $\dot W^{2,1}_{4/3}(\mathbb{R}^2\times\mathbb{R}_+)$) for the velocity can be done as in [7] (passing to the limit directly in the nonreflexive space $\dot W^{2,1}_{5/2,1}$ would require more care). The mass conservation equation may be handled according to Di Perna and Lions' theory [14] (see details in [7]) and the momentum equation does not present any difficulty compared to works on global weak solutions, since a lot of regularity is available on the velocity and there is no pressure term.

Next, once we know that (ρ, u) is a solution, there is no difficulty to recover all the additional regularity, that are just based on 'linear' properties like interpolation or parabolic maximal regularity. □

Appendix A

For the reader's convenience, we here list (and even sometimes prove) some results concerning Lorentz spaces, embedding and parabolic maximal regularity that have been used repeatedly in the paper.

A.1. **Lorentz spaces.** The following classical properties of the Lorentz spaces may be found in e.g. [18]:

- Embedding : $L_{p,r_1}\hookrightarrow L_{p,r_2}$ if $r_1\le r_2$, and $L_{p,p}=L_p$.
- Hölder inequality : for $1<p,p_1,p_2<\infty$ and $1\le r,r_1,r_2\le\infty$, we have

$$\|fg\|_{L_{p,r}}\lesssim\|f\|_{L_{p_1,r_1}}\|g\|_{L_{p_2,r_2}}\quad\text{with}\quad\frac1p=\frac1{p_1}+\frac1{p_2}\quad\text{and}\quad\frac1r=\frac1{r_1}+\frac1{r_2},\tag{A.1}$$

  This still holds for couples $(1,1)$ and (∞,∞) with the convention $L_{1,1}=L_1$ and $L_{\infty,\infty}=L_\infty$.
- For any $\alpha>0$ and nonnegative measurable function f, we have

$$\|f^\alpha\|_{L_{p,r}}=\|f\|^\alpha_{L_{p\alpha,r\alpha}}.$$

The following continuous embedding holds true:

$$\dot W^{d/p}_{p,1}(\mathbb{R}^d)\hookrightarrow\mathcal{C}_b(\mathbb{R}^d),\quad 1<p<\infty.\tag{A.2}$$

Proof. Since all the embedding that will be used below are 'critical', a standard rescaling argument reduces the proof to nonhomogeneous spaces (the advantage being that the latter ones are Banach spaces, regardless of the regularity exponent).

Let us consider $1<p_1<p<p_2<\infty$ such that $2/p=1/p_1+1/p_2$. Then our definition in terms of trace space and interpolation gives us:

$$W^{d/p}_{p,1}(\mathbb{R}^d)=\big(W^{d/p}_{p_1}(\mathbb{R}^d),W^{d/p}_{p_2}(\mathbb{R}^d)\big)_{1/2,1}.$$

Now, we have the following classical embeddings in Besov spaces built on $L_\infty(\mathbb{R}^d)$:

$$W^{d/p}_{p_1}(\mathbb{R}^d)\hookrightarrow B^{d/p-d/p_1}_{\infty,\infty}(\mathbb{R}^d)\quad\text{and}\quad W^{d/p}_{p_2}(\mathbb{R}^d)\hookrightarrow B^{d/p-d/p_2}_{\infty,\infty}(\mathbb{R}^d).$$

Of course, $d/p-d/p_1=-(d/p-d/p_2)<0$. Consequently:

$$W^{d/p}_{p,1}(\mathbb{R}^d)\hookrightarrow\big(B^{d/p-d/p_1}_{\infty,\infty}(\mathbb{R}^d),B^{d/p-d/p_2}_{\infty,\infty}(\mathbb{R}^d)\big)_{1/2,1}=B^0_{\infty,1}(\mathbb{R}^d),$$

and it is obvious that $B^0_{\infty,1}(\mathbb{R}^d)$ is embedded in the set of continuous and bounded functions on $\mathbb{R}^d$. □

A.2. **Maximal regularity.** Consider the heat semi-group $(e^{t\Delta})_{t>0}$ on $\mathbb{R}^d$ and denote

$$M : f \mapsto \int_0^t e^{(t-\tau)\Delta} f(\tau)\, d\tau.$$

The fact that $\nabla^2 M : L_q(\mathbb{R}_+; L_p(\mathbb{R}^d)) \to L_q(\mathbb{R}_+; L_p(\mathbb{R}^d))$ whenever $1 < p, q < \infty$ belongs to the mathematical folklore (see the pioneering work in [23] for the particular case $p = q$). Now, observing that, for all $1 \le r \le \infty$ and $\theta \in (0,1)$, we have

$$\big(L_{p_1}(\mathbb{R}_+ \times \mathbb{R}^d); L_{p_2}(\mathbb{R}_+ \times \mathbb{R}^d)\big)_{\theta,r} = L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d), \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2},$$

one may conclude that

$$\nabla^2 M : L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d) \to L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d). \tag{A.3}$$

Next, observe that, for all $T \ge 0$, we have, by definition of Mf and of the space $\dot W^{2-2/p}_{p,r}(\mathbb{R}^d)$ in (2.4),

$$\begin{aligned}
\|Mf(T)\|_{\dot W^{2-2/p}_{p,r}(\mathbb{R}^d)} &= \left\| \int_0^T \nabla^2 e^{(t+T-\tau)\Delta} f(\tau)\, d\tau \right\|_{L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d)} \\
&= \left\| \int_0^{t'} \nabla^2 e^{(t'-\tau)\Delta} (1_{[0,T]} f)(\tau)\, d\tau \right\|_{L_{p,r}(T,\infty \times \mathbb{R}^d)} \\
&\le \left\| \int_0^{t'} \nabla^2 e^{(t'-\tau)\Delta} (1_{[0,T]} f)(\tau)\, d\tau \right\|_{L_{p,r}(0,\infty \times \mathbb{R}^d)} \\
&\lesssim \|f\|_{L_{p,r}(0,T \times \mathbb{R}^d)}.
\end{aligned}$$

Hence, combining with a density argument, one can deduce that M maps $L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d)$ to $\mathcal{C}_b(\mathbb{R}_+; \dot W^{2-2/p}_{p,r}(\mathbb{R}^d))$ (only weak continuity if $r = \infty$).

We see directly from the definition in (2.4) that $\nabla^2 e^{t\Delta} : \dot W^{2-2/p}_{p,r}(\mathbb{R}^d) \to L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d)$. Finally, the fact that $(e^{t\Delta})_{t>0}$ is a continuous semi-group on $\dot W^{2-2/p}_{p,r}(\mathbb{R}^d)$ just follows by interpolation, from the result of continuity in the more classical spaces $\dot W^{2-2/p}_{p}(\mathbb{R}^d)$. To conclude, combining, if need be, with a suitable rescaling argument, we have proved:

Proposition A.1. *Let $1 < p < \infty$ and $1 \le r < \infty$. Then, for all $u_0 \in \dot W^{2-2/p}_{p,r}(\mathbb{R}^d)$ and $f \in L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d)$, the following heat equation:*

$$\begin{aligned}
u_t - \mu\Delta u &= f \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\
u|_{t=0} &= u_0 \qquad \text{in } \mathbb{R}^d
\end{aligned}$$

has a unique solution in the space

$$\dot W^{2,1}_{p,r}(\mathbb{R}^d \times \mathbb{R}_+) := \Big\{ u \in \mathcal{C}_b\big(\mathbb{R}_+; \dot W^{2-2/p}_{p,r}(\mathbb{R}^d)\big) : u_t, \nabla^2 u \in L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d) \Big\},$$

and the following inequality holds true for all $t \ge 0$:

$$\mu^{1-1/p} \|u(t)\|_{\dot W^{2-2/p}_{p,r}(\mathbb{R}^d)} + \|u_t, \mu\nabla^2 u\|_{L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d)} \lesssim \mu^{1-1/p} \|u_0\|_{\dot W^{2-2/p}_{p,r}(\mathbb{R}^d)} + \|f\|_{L_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d)}.$$

Finally, we have to prove the following inequality that was one of the keys to Theorem 2.1:

$$\|u\|_{L_{4,1}(\mathbb{R}_+ \times \mathbb{R}^2)} \le C \|u\|_{\dot W^{2,1}_{4/3,1}(\mathbb{R}^2 \times \mathbb{R}_+)}. \tag{A.4}$$

Let us fix $1 < p_1 < 4/3 < p_2 < 2$ such that $1/p_1 + 1/p_2 = 3/2$. Then we have

$$L_{4/3,1}(\mathbb{R}_+ \times \mathbb{R}^2) = \big(L_{p_1}(\mathbb{R}_+ \times \mathbb{R}^2); L_{p_2}(\mathbb{R}_+ \times \mathbb{R}^2)\big)_{1/2,1}.$$

Now, from classical Sobolev embedding and Hölder inequality, it is not difficult to prove that

$$\|u\|_{L_{p_i^*}(\mathbb{R}_+ \times \mathbb{R}^2)} \lesssim \|u\|_{\dot W^{2,1}_{p_i}(\mathbb{R}^2 \times \mathbb{R}_+)} \quad \text{with} \quad \frac{1}{p_i^*} := \frac{1}{p_i} - \frac{1}{2}.$$

Since we also have $L_{4,1}(\mathbb{R}_+\times\mathbb{R}^2) = \big(L_{p_1^*}(\mathbb{R}_+\times\mathbb{R}^2); L_{p_2^*}(\mathbb{R}_+\times\mathbb{R}^2)\big)_{1/2,1}$, the desired inequality follows from real interpolation. □

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