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ABSTRACT. We prove Reilly-type upper bounds for the first non-zero eigenvalue of the Steklov problem associated with the p-Laplace operator on submanifolds with boundary of Euclidean spaces as well as for Riemannian products $\mathbb{R} \times M$ where M is a complete Riemannian manifold.

1. INTRODUCTION

Let (M^n, g) be a compact Riemannian manifold with a possibly non-empty boundary ∂M. For $p \in (1, +\infty)$, we consider the so-called p-Laplacian defined by

$$\Delta_p u = -\mathrm{div}(\|\nabla u\|^{p-2} \nabla u)$$

for any $\mathcal{C}^2$ function. For $p = 2$, Δ_2 is nothing else than the Laplace-Beltrami operator of (M^n, g).
Other the past years, this operator Δ_p, and especially its spectrum, has been intensively studied, mainly for Euclidean domains with Dirichlet or Neumann boundary conditions (see for instance [5] and references therein) but also on Riemannian manifolds [2, 6].
In the present paper, we will consider the Steklov problem associated with the p-Laplacian on submanifolds with boundary of the Euclidean space. Namely, we consider the following boundary value problem

$$\text{(S)} \qquad \begin{cases} \Delta_p u = 0 & \text{in } M, \\ \|\nabla u\|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial M, \end{cases}$$

where $\frac{\partial u}{\partial \nu}$ is the derivative of the function u with respect to the outward unit normal ν to the boundary ∂M. Note that for $p = 2$, (S) is the usual Steklov problem. Little is known about the spectrum of this problem. If M is a domain of $\mathbb{R}^N$, there exists a sequence of positive eigenvalues $\lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ consisting in the variational spectrum and obtained by the Ljusternik-Schnirelmann theory (see [5, 10] for instance). One can refer to [1] for details about the Ljusternik-Schnirelmann principle. Note that, as mentionned in [6, Remark 1.1], the arguments used in [5] can be extended to domains on Riemannian manifolds and we have that there exists a non-decreasing sequence of variational eigenvalues obtained by the Ljusternik–Schnirelman principle. Moreover, the eigenvalue 0 is simple with constant eigenfunctions and is isolated, that is there is no eigenvalue between 0 and λ_1. Then, the first positive eigenvalue of the Steklov problem λ_1 satisfies the following variational characterization

$$\lambda_1 = \inf\left\{ \frac{\displaystyle\int_M \|\nabla u\|^p dv_g}{\displaystyle\int_{\partial M} |u|^p dv_h} \,\middle|\, u \in W^{1,p}(M)\setminus\{0\},\ \int_{\partial M} |u|^{p-2} u dv_h = 0 \right\},$$

where dv_g and dv_h are the Riemannian volume forms respectively associated with the metric g on M and the induced metric h on ∂M.
Note that all the other eigenvalues λ_k of this sequence has also a variational characterization but we don't know if all the spectrum is contained in this sequence.

The aim of the present note is to obtain upper bounds for the first non-zero eigenvalue λ_1 of the p-Steklov problem, depending on the geometry of the boundary in the spirit of the classical Reilly upper bounds for the Laplacian on closed hypersurfaces. Reilly [7] showed that if (M^n, g) is a closed connected and oriented Riemannian manifold isometrically immersed into $\mathbb{R}^{n+1}$, then the first positive eigenvalue of the Laplacian on M satisfies

$$\lambda_1(\Delta) \leqslant \frac{n}{V(M)} \int_M H^2 dv_g,$$

where H is the mean curvature of the immersion. Note that M is not supposed to be embedded and so does not necessarily bounds a domain of $\mathbb{R}^{n+1}$. More generally, Reilly obtained the following inequalities for $r \in \{0, \cdots, n\}$

$$\lambda_1(\Delta)\left(\int_M H_r dv_g\right)^2 \leqslant V(M) \int_M H_{r+1}^2 dv_g,$$

where H_r and H_{r+1} stands for the higher order mean curvatures (that we will define in Section 3). For $r = 0$, we recover the first mentioned inequality. In addition, if equality holds in one these inequality, then M is immersed in a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1(\Delta)}}$. Note that, always in [7], Reilly also obtained similar estimates for higher codimension submanifolds. Namely, if (M^n, g) is isometrically immersed into $\mathbb{R}^N$, $N > n+1$, then

$$\lambda_1(\Delta)\left(\int_M H_r dv_g\right)^2 \leqslant V(M) \int_M \|H_{r+1}\|^2 dv_g,$$

for any even $r \in \{0, \cdots, n\}$ with equality if and only if M is minimally immersed in a geodesic sphere of $\mathbb{R}^N$. Note that, in the case of codimension greater that 1, H_r is a function and H_{r+1} is a normal vector field, contrary to the hypersurface case where both are functions (see again Section 3 for details).

Recently, Du and Mao [2] proved Reilly type upper bounds for the first eigenvalue $\lambda_1(\Delta_p)$ of the p-Laplace operator on closed submanifolds of $\mathbb{R}^N$. Namely, they proved that

$$\lambda_1(\Delta_p) \leqslant \frac{n^{p/2}}{V(M)^{p-1}} \left(\int_M \|H\|^{\frac{p}{p-1}} dv_g\right)^{p-1} \begin{cases} N^{\frac{p-2}{2}} & \text{if} \quad p \geqslant 2 \\ N^{\frac{2-p}{2}} & \text{if} \quad 1 < p \leqslant 2 \end{cases}$$

Moreover, equality occurs if and only if $p = 2$ and M is minimally immersed into a geodesic hypersphere. In particular, if $N = n+1$, M is a geodesic hypersphere. In addition, the authors proved analogous estimates with higher order mean curvatures.

On the other hand, Ilias and Makhoul [4] proved Reilly-type inequalities for the first eigenvalue σ_1 of the Steklov problem on submanifolds of $\mathbb{R}^N$. Namely, they proved the following estimate

$$\sigma_1 V(\partial M)^2 \leqslant nV(M) \int_{\partial M} \|H\|^2 dv_g,$$

where (M^n, g) is a compact submanifold of $\mathbb{R}^N$ with boundary ∂M and H denote the mean curvature of ∂M. We denote by X the isometric immersion.
The limitting case is also characterized¿ Namely, they proved that equality occurs if and only if M is minimally immersed into $B^N\left(\frac{1}{\lambda_1}\right)$ so that $X(\partial M) \subset \partial B^N\left(\frac{1}{\lambda_1}\right)$ minimally and orthogonally. In particular, if $n = N$, equality occurs if and only if $p = 2$ and $X(M) = B^N\left(\frac{1}{\lambda_1}\right)$. Here again, analogous estimates with higher order mean curvatures were proven.

The main result of this note is the following estimate for the first non-zero eigenvalue of the Steklov problem associated with the p-Laplacian. Namely, we prove

Theorem 1.1. *Let (M^n, g) be a compact connected and oriented Riemannian manifold with nonempty boundary ∂M and $p \in (1, +\infty)$. Assume that (M^n, g) is isometrically immersed into the Euclidean space $\mathbb{R}^N$ by X. Let λ_1 the first eigenvalue of the p-Steklov problem*

$$\begin{cases} \Delta_p u = 0 & \text{in } M, \\ \\ \|\nabla u\|^{p-2} \frac{\partial u}{\partial \nu} = \sigma |u|^{p-2} u & \text{on } \partial M \end{cases}$$

If $p \geqslant 2$, then λ_1 satisifes

$$\lambda_1 \leqslant N^{\frac{p-2}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} \|H\|^{\frac{p}{p-1}} dv_h \right)^{p-1} \frac{V(M)}{V(\partial M)^p}.$$

If $1 < p \leqslant 2$, then λ_1 satisifes

$$\lambda_1 \leqslant N^{\frac{2-p}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} \|H\|^{\frac{p}{p-1}} dv_h \right)^{p-1} \frac{V(M)}{V(\partial M)^p}.$$

Moreover, equality occurs in both inequality if and only if $p = 2$ and X is a minimal immersion of M into $B^N\left(\frac{1}{\lambda_1}\right)$ so that $X(\partial M) \subset \partial B^N\left(\frac{1}{\lambda_1}\right)$ minimally and orthogonally. In particular, if $n = N$, equality occurs if and only if $p = 2$ and $\phi(M) = B^N\left(\frac{1}{\lambda_1}\right)$.

After giving the proof of Theorem 1.1 in Section 2, we obtain more general inequalities involving higher order mean curvatures (Theorem 3.1 and Corollary 3.2) as well as an estimate for domains of products manifolds of the type $M \times \mathbb{R}$ (Theorem 4.1).

2. Proof of Theorem 1.1

First, we recall the variational characterization of λ_1:

$$\lambda_1=\inf\left\{\frac{\displaystyle\int_M|\nabla u|^p dv_g}{\displaystyle\int_{\partial M}|u|^p dv_h}\,\middle|\,u\in W^{1,p}(M)\setminus\{0\}\},\ \int_{\partial M}|u|^{p-2}u dv_h=0\right\}.$$

Replacing if needed, $|X^i|^{p-2}X^i$ by $|X^i|^{p-2}X^i-\dfrac{\int_{\partial M}|X^i|^{p-2}X^i dv_h}{V(\partial M)}$, we may assume without loss of generality that

$$\int_{\partial M}|X^i|^{p-2}X^i dv_h=0$$

for all $i\in\{1,\cdots,N\}$, so that we can use the coordinate functions as test functions. From this point, we will consider separately the cases $p\geqslant 2$ and $1<p\leqslant 2$.

The case $p\geqslant 2$.
Using the coordinates X^i, $1\leqslant i\leqslant N$, as test functions and summing for i from 1 to N, we get

$$\lambda_1\int_{\partial M}\sum_{i=1}^N|X^i|^p\leqslant\int_M\sum_{i=1}^N\|\nabla X^i\|^p.$$

First, since $p\geqslant 2$, we have

$$\sum_{i=1}^N\|\nabla X^i\|^p\leqslant\left(\sum_{i=1}^N\|\nabla X^i\|^2\right)^{\frac{p}{2}}=n^{\frac{p}{2}},\tag{1}$$

since we have $\sum_{i=1}^N\|\nabla X^i\|^2=n$ (see [8, Lemma 2.1] for instance).
Hence, we obtain

$$\lambda_1\int_{\partial M}\sum_{i=1}^N|X^i|^p dv_h\leqslant n^{\frac{p}{2}}V(M).$$

Moreover, by the Hölder inequality, we have

$$\|X\|^2\leqslant\left(\sum_{i=1}^N|X^i|^p\right)^{\frac{2}{p}}N^{\frac{p-2}{p}},\tag{2}$$

which gives

$$\sum_{i=1}^N|X^i|^p\geqslant\frac{1}{N^{\frac{p-2}{2}}}\|X\|^p$$

and so

$$\lambda_1\int_{\partial M}\|X\|^p dv_h\leqslant n^{\frac{p}{2}}N^{\frac{p-2}{2}}V(M).$$

We mulitply by $\left(\int_{\partial M}\|H\|^{\frac{p}{p-1}}dv_h\right)^{p-1}$ and use the integral Hölder inequality to get

$$\lambda_1\left|\int_{\partial M}\langle X,H\rangle dv_h\right|^p \leqslant n^{\frac{p}{2}}N^{\frac{p-2}{2}}\left(\int_{\partial M}\|H\|^{\frac{p}{p-1}}dv_h\right)^{p-1}V(M). \tag{3}$$

Thus, using the Hsiung-Minkoswki formula

$$\int_{\partial M}\Big(\langle H,X\rangle+1\Big)dv_h=0, \tag{4}$$

we get

$$\lambda_1 V(\partial M)^p \leqslant n^{\frac{p}{2}}N^{\frac{p-2}{2}}\left(\int_{\partial M}\|H\|^{\frac{p}{p-1}}dv_h\right)^{p-1}V(M).$$

which gives the desired upper bound for λ_1.
Moreover, if equality occurs, then equality holds in all the above inequalities and in particular in the inequality (1), which implies that $p=2$. Therefore, the end of the proof is similar to the proof of Ilias and Makhoul for the classical Steklov problem and we have that X is a minimal immersion of M into $B^N\left(\frac{1}{\lambda_1}\right)$ so that $X(\partial M)\subset\partial B^N\left(\frac{1}{\lambda_1}\right)$ minimally and orthogonally. In particular, if if $n=N$, then $X(M)=B^N\left(\frac{1}{\lambda_1}\right)$.

The case $1<p\leqslant 2$. First, since $p\leqslant 2$, we have

$$\|X\|^p=\left(\sum_{i=1}^N |X^i|^2\right)^{\frac{2}{p}} \leqslant \sum_{i=1}^N |X^i|^p. \tag{5}$$

On the other hand, by the Hölder inequality, we have

$$\sum_{i=1}^N\|\nabla X^i\|^p \geqslant N^{\frac{2-p}{p}}\left(\sum_{i=1}^N\|\nabla X^i\|^2\right)^{\frac{2}{p}} = N^{\frac{2-p}{p}}n^{\frac{2}{p}}.$$

Hence, using the last two inequalities in the variational characterization of λ_1, we obtain

$$\lambda_1\int_{\partial M}\|X\|^p dv_h \leqslant n^{\frac{2}{p}}N^{\frac{2-p}{p}}V(M).$$

The end of the proof is the same that in the case $p\leqslant 2$, we mulitply by $\left(\int_{\partial M}\|H\|^{\frac{p}{p-1}}dv_h\right)^{p-1}$, use the integral Hölder inequality and the Hsiung-Minkowski formula (4).
If equality holds, then equality occurs in (5). Thus, here again $p=2$ and we conclude as previously.

3. Inequalities with higher order mean curvatures

In this section, we extend Theorem 1.1 to estimates with higher order mean curvatures. It will appear as a particular space of a more general result. Before stating the result, we briefly give some recalls.

First of all, let T be a divergence-free symmetric $(1,1)$-tensor. We associate with T the second order differential operator L_T defined by $L_T u := -\mathrm{div}(T\nabla u)$, for any $\mathcal{C}^2$ function u on ∂M. We also associate with T the following normal vector field:

$$H_T = \sum_{i,j=1}^{n} \langle Te_i, e_j\rangle B(e_i, e_j), \tag{6}$$

where B is the second fundamental form of the immersion of M into $\mathbb{R}^N$ and $\{e_1, \cdots, e_n\}$ is a local orthonormal frame of $T\partial M$. Moreover, we recall the following generalized Hsiung-Minkowski formula (see [3, 8] for instance)

$$\int_{\partial M} \big(\langle X, H_T\rangle + \mathrm{tr}\,(T)\big) dv_h = 0. \tag{7}$$

Now, we can state the following

Theorem 3.1. *Let (M^n, g) be a compact connected and oriented Riemannian manifold with nonempty boundary ∂M and $p \in (1, +\infty)$. Assume that (M^n, g) is isometrically immersed into the Euclidean space $\mathbb{R}^N$ by X and let T be a symmetric and divergence-free $(2,0)$-tensor on ∂M. Let λ_1 the first eigenvalue of the p-Steklov problem*

$$\begin{cases} \Delta_p u = 0 & \text{in } M, \\ \|\nabla u\|^{p-2}\frac{\partial u}{\partial \nu} = \sigma |u|^{p-2}u & \text{on } \partial M \end{cases}$$

Then, the following holds

(1) *If $p \geqslant 2$, then λ_1 satisifes*
$$\lambda_1 \left| \int_{\partial M} \mathrm{tr}\,(T) \right|^p \leqslant N^{\frac{p-2}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} \|H_T\|^{\frac{p}{p-1}} \right)^{p-1} V(M).$$

(2) *If $1 < p \leqslant 2$, then λ_1 satisifes*
$$\lambda_1 \left| \int_{\partial M} \mathrm{tr}\,(T) \right|^p \leqslant N^{\frac{2-p}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} \|H_T\|^{\frac{p}{p-1}} \right)^{p-1} V(M).$$

Moreover, if H_T does not vanish identically, then equality occurs in one of both inequalities if and only if $p = 2$ and

(a) *if $N > n$, X is a minimal immersion of M into $B^N\left(\frac{1}{\lambda_1}\right)$ so that $X(\partial M) \subset \partial B^N\left(\frac{1}{\lambda_1}\right)$ minimally and orthogonally and H_T is proportional to $X_{|\partial M}$.*

(b) *if $N = n$, M is a ball and $\mathrm{tr}\,(T)$ is constant.*

Proof: The proof is similar to the proof of Theorem 1.1 with the difference that we use the generalized Hsiung-Minkowski formula (7) instead of the classical one. In case of equality, we also get that $p=2$ by equality in (1) or (5). Then, we conclude as in Theorem 1.1 by the argument of Ilias and Makhoul. $\square$

Now, let us consider higher order mean curvatures. For $r \in \{1, \cdots, n\}$, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r}} \epsilon \begin{pmatrix} i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r \end{pmatrix} \langle B_{i_1 j_1} B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}} B_{i_r j_r} \rangle e_i^* \otimes e_j^*,$$

if r is even and

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r}} \epsilon \begin{pmatrix} i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r \end{pmatrix} \langle B_{i_1 j_1} B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}} B_{i_r j_r} \rangle B_{i_r, j_r} \otimes e_i^* \otimes e_j^*,$$

where the B_{ij}'s are the coefficients of the second fundamental form B in a local orthonormal frame $\{e_1, \cdots, e_n\}$ and ϵ is the standard signature for permutations. Here, $\{e_1^*, \cdots, e_n^*\}$ is the dual coframe of $\{e_1, \cdots, e_n\}$. By definition, the r-th mean curvature is $H_r = \frac{1}{c(r)} \mathrm{tr}\,(T_r)$, where $c(r) = (n-r)\binom{r}{n}$. Note that H_r is a real function if r is even and a normal vector field if r is odd, in this case, we will denote it by $\mathbf{H}_r$. By convention, we set $H_0 = 1$. Moreover, always if r is even, we show easily that $H_{T_r} = c(r)\mathbf{H}_{r+1}$, where H_{T_r} is given by the relation (6).

In the case of hypersurfaces, we can consider the higher order mean curvatures as scalar functions also for odd indices by taking B as the real-valued second fundamental form.

By the symmetry of B, these tensors are clearly symmetric and it is also a classical fact that there are divergence-free (see [3] for instance). Hence, in this case, the Hsiung-Minkowski formula (7) becomes

$$\int_{\partial M} \big(\langle X, \mathbf{H}_{r+1} \rangle + H_r\big) dv_h = 0$$

for any even $r \in \{0, \cdots, n\}$ if $N > n+1$, and

$$\int_{\partial M} \big(\langle X, \nu \rangle H_{r+1} + H_r\big) dv_h = 0$$

for any $r \in \{0, \cdots, n\}$ if $N = n+1$, where ν is the normal unit vector field on ∂M choosen to define the shape operator.

We obtain directly from Theorem 3.1 the following corollary:

Corollary 3.2. *Let (M^n, g) be a compact connected and oriented Riemannian manifold with nonempty boundary ∂M and $p \in (1, +\infty)$. Assume that (M^n, g) is isometrically immersed into the Euclidean space $\mathbb{R}^N$ by X. Let λ_1 the first eigenvalue of the p-Steklov problem*

$$\begin{cases} \Delta_p u = 0 & \text{in } M, \\ \|\nabla u\|^{p-2} \frac{\partial u}{\partial \nu} = \sigma |u|^{p-2} u & \text{on } \partial M \end{cases}$$

(1) *If $N > n+1$, and $r \in \{0, \cdots, n-1\}$ is an even integer then we have*

(a) *If $p \geqslant 2$, then λ_1 satisifes*

$$\lambda_1 \left| \int_{\partial M} H_r \right|^p \leqslant N^{\frac{p-2}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} \|\mathbf{H}_{r+1}\|^{\frac{p}{p-1}} \right)^{p-1} V(M).$$

(b) *If $1 < p \leqslant 2$, then λ_1 satisifes*

$$\lambda_1 \left| \int_{\partial M} H_r \right|^p \leqslant N^{\frac{2-p}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} \|\mathbf{H}_{r+1}\|^{\frac{p}{p-1}} \right)^{p-1} V(M).$$

Moreover, if $\mathbf{H}_{r+1}$ does not vanish identically, then equality occurs in one of both inequalities if and only if $p = 2$ and X is a minimal immersion of M into $B^N\left(\frac{1}{\lambda_1}\right)$ so that $X(\partial M) \subset \partial B^N\left(\frac{1}{\lambda_1}\right)$ minimally and orthogonally.

(2) *If $N = n+1$ and $\in \{0, \cdots, n-1\}$ is any integer, then we have*

(a) *If $p \geqslant 2$, then λ_1 satisifes*

$$\lambda_1 \left| \int_{\partial M} H_r \right|^p \leqslant N^{\frac{p-2}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} |H_{r+1}|^{\frac{p}{p-1}} \right)^{p-1} V(M).$$

(b) *If $1 < p \leqslant 2$, then λ_1 satisifes*

$$\lambda_1 \left| \int_{\partial M} H_r \right|^p \leqslant N^{\frac{2-p}{2}} n^{\frac{p}{2}} \left(\int_{\partial M} |H_{r+1}|^{\frac{p}{p-1}} \right)^{p-1} V(M).$$

Moreover, if H_{r+1} does not vanish identically, then equality occurs in one of both inequalities if and only if $p = 2$ and $X(M) = B^N\left(\frac{1}{\lambda_1}\right)$.

4. An equality for product spaces

We finish the paper by a last result for hypersurfaces of product spaces for the p-Steklov problem, $p \geqslant 2$, in the spirit of those obtained in [11] and [9]. Namely, we prove the following

Theorem 4.1. *Let $p \geqslant 2$ and $(M^n, \overline{g})$ be a complete Riemannian manifold. Consider (Σ^n, g) a closed oriented Riemannian manifold isometrically immersed into the Riemannian product $(\mathbb{R} \times M, \widetilde{g} = dt^2 \oplus \overline{g})$. Moreover, assume that Σ is mean-convex and bounds a domain Ω in $\mathbb{R} \times M$. Let λ_1 be the first eigenvalue of the p-Steklov problem on Ω*

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ \\ \|\nabla u\|^{p-2} \frac{\partial u}{\partial \nu} = \sigma |u|^{p-2} u & \text{on } \partial\Omega \end{cases}$$

Then, λ_1 satisfies

$$\lambda_1 \leqslant \left(\frac{\kappa_+(\Sigma)\|H\|_\infty}{\inf\limits_\Sigma H} \right)^{p/2} \left(\frac{V(\Omega)}{V(\Sigma)} \right)^{1-\frac{p}{2}}.$$

Remark 4.2. *Note that for $p = 2$, we recover the result of Xiong* [11].

Proof: We will use as test function the function t which is the coordinate in the factor $\mathbb{R}$ of the product $\mathbb{R} \times M$. First, obviously, up to a possible translation in the direction of $\mathbb{R}$, we can assume that $\displaystyle\int_\Sigma t dv_g = 0$. Second, since Σ is mean-convex, we deduce that t does not vanish identically. Indeed, if t vanishes identically over Σ, then Σ is included in the slice $\{0\} \times M$ and thus is totally geodesic in the product $\mathbb{R} \times M$. This is a contradiction with the fact that Σ is mean-convex. Hence, t does not vanish identically and can be used as a test function. Thus, from the variational characterization of λ_1, we have

$$\lambda_1 \int_\Sigma |t|^p dv_g \leqslant \int_\Omega \|\widetilde{\nabla} t\|^p dv_{\widetilde{g}}. \tag{8}$$

First, since $\|\widetilde{\nabla} t\| = 1$, we have

$$\int_\Omega \|\widetilde{\nabla} t\|^p dv_{\widetilde{g}} = V(\Omega) = \left(\int_\Omega \|\widetilde{\nabla} t\|^2 dv_{\widetilde{g}}\right)^{\frac{p}{2}} V(\Omega)^{1-\frac{p}{2}}. \tag{9}$$

In addition, we have

$$\int_\Omega \|\widetilde{\nabla} t\|^2 dv_{\widetilde{g}} = -\int_\Omega t \widetilde{\Delta} t dv_{\widetilde{g}} + \int_\Omega \mathrm{div}_{\widetilde{g}}(t \widetilde{\nabla} t) dv_{\widetilde{g}}$$

Since $\widetilde{\Delta} t = 0$, using the Stokes theorem, we get

$$\int_\Omega \|\widetilde{\nabla} t\|^2 dv_{\widetilde{g}} = \int_\Sigma \langle t \widetilde{\nabla} t, \nu\rangle dv_g = \int_\Sigma t u dv_g,$$

where u is defined by $u = \langle \partial_t, \nu\rangle = \langle \widetilde{\nabla} t, \nu\rangle$. Hence, by the Hölder inequality, we obtain

$$\int_\Omega \|\widetilde{\nabla} t\|^2 dv_{\widetilde{g}} \leqslant \left(\int_\Sigma |t|^p dv_g\right)^{\frac{1}{p}} \left(\int_\Sigma |u|^{\frac{p}{p-1}} dv_g\right)^{\frac{p-1}{p}}. \tag{10}$$

Hence, using (9) and (10), (8) becomes

$$\lambda_1 \leqslant \frac{\left(\displaystyle\int_\Sigma |u|^{\frac{p}{p-1}} dv_{\widetilde{g}}\right)^{\frac{p-1}{2}}}{\left(\displaystyle\int_\Sigma |t|^p dv_g\right)^{\frac{1}{2}}} V(\Omega)^{1-\frac{p}{2}}. \tag{11}$$

On the other hand, we have

$$\begin{aligned}
\Delta t &= -\mathrm{div}_\Sigma(\nabla t) \\
&= -\sum_{i=1}^n \langle \nabla_{e_i}(\nabla t), e_i\rangle \\
&= -\sum_{i=1}^n \langle \widetilde{\nabla}_{e_i}(\partial t - \langle \partial t, \nu\rangle \nu), e_i\rangle
\end{aligned}$$

where ν is a unit normal vector field. Moreover, since ∂_t is parallel for $\widetilde{\nabla}$ and $-\widetilde{\nabla}_{(\cdot)}\nu$ is the shape operator S, we get

$$\begin{aligned}\Delta t &= -\sum_{i=1}^{n}\langle\partial_t,\nu\rangle\langle Se_i,e_i\rangle\\ &= -nHu.\end{aligned}$$

Hence, multilying respectively by t and u, we get immediately $t\Delta t = -nHut$ and $u\Delta t = -nHu^2$ which after integration over Σ gives

$$\int_\Sigma \|\nabla t\|^2 dv_g = \int_\Sigma nHut dv_g \tag{12}$$

$$\int_\Sigma \langle S\nabla t,\nabla t\rangle dv_g = \int_\Sigma nHu^2. \tag{13}$$

Note that for the second one, we have used the fact that $\nabla u = -S\nabla t$. Indeed, we have

$$\nabla u = \sum_{i=1}^{n} e_i(u)e_i = \sum_{i=1}^{n} e_i(\langle\nu,\partial t\rangle)e_i = -\sum_{i=1}^{n}\langle Se_i,\partial_t\rangle e_i = -S(\nabla t).$$

Moreover, we have, using (13),

$$\begin{aligned}n\inf_\Sigma(H)\int_M u^2 dv_g &\leqslant \int_M nHu^2 dv_g\\ &\leqslant \int_\Sigma\langle S\nabla t,\nabla t\rangle dv_g\\ &\leqslant \kappa_+(\Sigma)\int_\Sigma\|\nabla t\|^2 dv_g,\end{aligned}$$

where $\kappa_+(\Sigma) = \max\{\kappa_+(x)|x\in M\}$ with $\kappa_+(x)$ the biggest principal curvature of Σ at the point x. Now, we use (12) and the Hölder inequality to get

$$\begin{aligned}n\inf_\Sigma(H)\int_\Sigma u^2 dv_g &\leqslant \kappa_+(\Sigma)\int_\Sigma nHut dv_g\\ &\leqslant n\kappa_+(\Sigma)\|H\|_\infty\int_\Sigma ut dv_g\\ &\leqslant n\kappa_+(\Sigma)\|H\|_\infty\left(\int_\Sigma |t|^p dv_g\right)^{\frac{1}{p}}\left(\int_\Sigma |u|^{\frac{p}{p-1}}dv_g\right)^{\frac{p-1}{p}}.\end{aligned}$$

Finally, using the Hölder inequality a last time, we have

$$\inf_\Sigma(H)\left(\int_\Sigma |u|^{\frac{p}{p-1}}dv_g\right)^{\frac{2(p-1)}{p}}V(\Sigma)^{\frac{2-p}{p}} \leqslant \kappa_+(\Sigma)\|H\|_\infty\left(\int_M |t|^p dv_g\right)^{\frac{1}{p}}\left(\int_M |u|^{\frac{p}{p-1}}dv_g\right)^{\frac{p-1}{p}}.$$

and so

$$\frac{\left(\int_\Sigma |u|^{\frac{p}{p-1}}dv_g\right)^{\frac{p-1}{p}}}{\left(\int_M |t|^p dv_g\right)^{\frac{1}{p}}} \leqslant \frac{\kappa_+\Sigma\|H\|_\infty}{\inf_\Sigma(H)}V(\Sigma)^{\frac{p-2}{p}}.$$

Reporting this in (11), we get the desired inequality:

$$\lambda_1 \leqslant \left(\frac{\kappa_+(\Sigma)\|H\|_\infty}{\inf\limits_\Sigma H} \right)^{p/2} \left(\frac{V(\Omega)}{V(\Sigma)} \right)^{1-\frac{p}{2}}.$$

□

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