Transfer of regularity for Markov semigroups

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Abstract

We study the regularity of a Markov semigroup $(P_t)_{t>0}$, that is, when $P_t(x,dy)=p_t(x,y)dy$ for a suitable smooth function $p_t(x,y)$. This is done by transferring the regularity from an approximating Markov semigroup sequence $(P^n_t)_{t>0}$, $n\in\mathbb{N}$, whose associated densities $p^n_t(x,y)$ are smooth and can blow up as $n\to\infty$. We use an interpolation type result and we show that if there exists a good equilibrium between the blow up and the speed of convergence, then $P_t(x,dy)=p_t(x,y)dy$ and p_t has some regularity properties.

Keywords: Markov semigroups; regularity of probability laws; interpolation spaces.

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Contents

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1 Introduction

In this paper we study Markov semigroups, that is, strongly continuous and positive semigroups P_t, $t \geq 0$, such $P_t 1 = 1$. We set the domain equal to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of the $C^\infty(\mathbb{R}^d)$ functions all of whose derivatives are rapidly decreasing.

The link with Markov processes gives the representation

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x, dy), \quad t \geq 0, \ f \in \mathcal{S}(\mathbb{R}^d).$$

We study here the regularity of a Markov semigroup, which is the property $P_t(x, dy) = p_t(x, y)dy$, $t > 0$, for a suitable smooth function $p_t(x, y)$, by transferring the regularity from an approximating Markov semigroup sequence P_t^n, $n \in \mathbb{N}$.

To be more precise, let P_t be a Markov semigroup on $\mathcal{S}(\mathbb{R}^d)$ with infinitesimal operator L and let $P_t^n, n \in \mathbb{N}$, be a sequence of Markov semigroups on $\mathcal{S}(\mathbb{R}^d)$ with infinitesimal operators L_n, $n \in \mathbb{N}$. Classical results (Trotter Kato theorem, see e.g. [13]) assert that, if $L_n \to L$ then $P_t^n \to P_t$. The problem that we address in this paper is the following. We suppose that P_t^n has the regularity (density) property $P_t^n(x, dy) = p_t^n(x, y)dy$ with $p_t^n \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and we ask under which hypotheses this property is inherited by the limit semigroup P_t. If we know that p_t^n converges to some p_t in a sufficiently strong sense, of course we obtain $P_t(x, dy) = p_t(x, y)dy$. But in our framework p_t^n does not converge: here, p_t^n can even "blow up" as $n \to \infty$. However, if we may find a good equilibrium between the blow up and the speed of convergence, then we are able to conclude that $P_t(x, dy) = p_t(x, y)dy$ and p_t has some regularity properties. This is an interpolation type result.

Roughly speaking our main result is as follows. We assume that the speed of convergence is controlled in the following sense: there exists some $a \in \mathbb{N}$ such that for every $q \in \mathbb{N}$

$$\|(L - L_n)f\|_{q,\infty} \leq \varepsilon_n \|f\|_{q+a,\infty} \tag{1.1}$$

Here $\|f\|_{q,\infty}$ is the norm in the standard Sobolev space $W^{q,\infty}$. In fact we will work with weighted Sobolev spaces, and this is an important point. And also, we will assume a similar hypothesis for the adjoint $(L - L_n)^*$ (see Assumption 2.1 for a precise statement).

Moreover we assume a "propagation of regularity" property: there exist $b \in \mathbb{N}$ and $\Lambda_n \geq 1$ such that for every $q \in \mathbb{N}$

$$\|P_t^n f\|_{q,\infty} \leq \Lambda_n \|f\|_{q+b,\infty} \tag{1.2}$$

Here also we will work with weighted Sobolev norms. And a similar hypothesis is supposed to hold for the adjoint $P_t^{*,n}$ (see Assumption 2.2 for a precise statement).

Finally we assume the following regularity property: for every $t \in (0, 1]$, $P_t^n(x, dy) = p_t^n(x, y)dy$ with $p_t^n \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and for every $\kappa \geq 0$, $t \in (0, 1]$,

$$\left| \partial_x^\alpha \partial_y^\beta p_t^n(x, y) \right| \leq \frac{C}{(\lambda_n t)^{\theta_0(|\alpha|+|\beta|+\theta_1)}} \times \frac{(1 + |x|^2)^{\pi(\kappa)}}{(1 + |x - y|^2)^\kappa}. \tag{1.3}$$

Here, α, β are multi-indexes and $\partial_x^\alpha, \partial_y^\beta$ are the corresponding differential operators. Moreover, $\pi(\kappa)$, θ_0 and θ_1 are suitable parameters and $\lambda_n \to 0$ as $n \to \infty$ (we refer to Assumption 2.9).

By (1.1)–(1.3), the rate of convergence is controlled by $\varepsilon_n \to 0$ and the blow up of p_t^n is controlled by $\lambda_n^{-\theta_0} \to \infty$. So the regularity property may be lost as $n \to \infty$. However, if there is a good equilibrium between $\varepsilon_n \to 0$ and $\lambda_n^{-\theta_0} \to \infty$ and $\Lambda_n \to \infty$ then the regularity is saved: we ask that for some $\delta > 0$

$$\overline{\lim_n} \frac{\varepsilon_n \Lambda_n}{\lambda_n^{\theta_0(a+b+\delta)}} < \infty, \tag{1.4}$$

the parameters a, b and θ_0 being given in (1.1), (1.2) and (1.3) respectively. Then $P_t(x,dy) = p_t(x,y)dy$ with $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the following upper bound holds: for every $\varepsilon > 0$ and $\kappa \in \mathbb{N}$ one may find some constants $C, \pi(\kappa) > 0$ such that for every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq \frac{C}{t^{\theta_0(1+\frac{a+b}{\delta})(|\alpha|+|\beta|+2d+\varepsilon)}} \times \frac{(1+|x|^2)^{\pi(\kappa)}}{(1+|x-y|^2)^\kappa}. \tag{1.5}$$

This is the "transfer of regularity" that we mention in the title and which is stated in Theorem 2.7. The proof is based on a criterion of regularity for probability measures given in [3], which is close to interpolation spaces techniques.

A second result concerns a perturbation of the semigroup P_t by adding a compound Poisson process: we prove that if P_t verifies (1.2) and (1.3) then the perturbed semigroup still verifies (1.3) – see Theorem 2.11. A similar perturbation problem is discussed in [19] (but the arguments there are quite different).

The regularity criterion presented in this paper is tailored in order to handle the following example (which will be treated in a forthcoming paper). We consider the integro-differential operator

$$Lf(x) = \langle b(x), \nabla f(x)\rangle + \int_E \big(f(x+c(z,x)) - f(x) - \langle c(z,x), \nabla f(x)\rangle\big) d\mu(z) \tag{1.6}$$

where μ is an infinite measure on the normed space $(E, |\circ|_E)$ such that $\int_E 1 \wedge |c(z,x)|^2 \, d\mu(z) < \infty$. Moreover, we consider a sequence $\varepsilon_n \downarrow \emptyset$, we denote

$$A_n^{i,j}(x) = \int_{\{|z|_E \leq \varepsilon_n\}} c^i(z,x) c^j(z,x) d\mu(z)$$

and we define

$$\begin{aligned} L_n f(x) = \quad & \langle b(x), \nabla f(x)\rangle + \int_{\{|z|_E \geq \varepsilon_n\}} (f(x+c(z,x)) - f(x) - \langle c(z,x), \nabla f(x)\rangle) d\mu(z) \\ & + \frac{1}{2}\mathrm{tr}(A_n(x)\nabla^2 f(x)). \end{aligned} \tag{1.7}$$

By Taylor's formula,

$$\|Lf - L_n f\|_\infty \leq \|f\|_{3,\infty} \varepsilon_n \quad \text{with} \quad \varepsilon_n = \sup_x \int_{\{|z|_E \leq \varepsilon_n\}} |c(z,x)|^3 \, d\mu(z)$$

Under the uniform ellipticity assumption $A_n(x) \geq \lambda_n$ for every $x \in \mathbb{R}^d$, the semigroup P_t^n associated to L_n has the regularity property (1.3) with θ_0 depending on the measure μ. The speed of convergence in (1.1), with $a = 3$, is controlled by $\varepsilon_n \downarrow 0$. So, if (1.4) holds, then we obtain the regularity of P_t and the short time estimates (1.5).

The semigroup P_t associated to L corresponds to stochastic equations driven by the Poisson point measure $N_\mu(dt,dz)$ with intensity measure μ, so the problem of the regularity of P_t has been extensively discussed in the probabilistic literature. A first approach initiated by Bismut [8], Leandre [16] and Bichteler, Gravereaux and Jacod [7] (see also the recent monograph of Bouleau and Denis [9] and the bibliography therein), is done under the hypothesis that $E = \mathbb{R}^m$ and $\mu(dz) = h(z)dz$ with $h \in C^\infty(\mathbb{R}^m)$. Then one constructs a Malliavin type calculus based on the amplitude of the jumps of the Poisson point measure N_μ and employs this calculus in order to study the regularity of P_t. A second approach initiated by Carlen and Pardoux [11] (see also Bally and Clément [5]) follows the ideas in Malliavin calculus based on the exponential density of the jump times in order to study the same problem. Finally a third approach, due to Picard [17, 18] (see also the recent monograph by Ishikawa [14] for many references and developments in this direction), constructs a Malliavin type calculus based on finite differences (instead of standard Malliavin derivatives) and obtains the regularity of P_t for a general class of intensity measures μ including purely atomic measures (in contrast with $\mu(dz) = h(z)dz$). We stress that all the above approaches work under different non degeneracy hypotheses, each of them corresponding to the specific noise that is used in the calculus. So in some sense we have not a single problem but three different classes of problems. The common feature is that the strategy in order to solve the problem follows the ideas from Malliavin calculus based on some noise contained in N_μ. Our approach is completely different because, as described above, we use the regularization effect of $\mathrm{tr}(A_n(x)\nabla^2)$. This regularization effect may be exploited either by using the standard Malliavin calculus based on the Brownian motion or using some analytical arguments. The approach that we propose in [4] is probabilistic, so employs the standard Malliavin calculus. But anyway, as mentioned above, the regularization effect vanishes as $n \to \infty$ and a supplementary argument based on the equilibrium given in (1.4) is used. We precise that the non degeneracy condition $A_n(x) \geq \lambda_n > 0$ is of the same nature as the one employed by J. Picard so the problem we solve is in the same class.

The idea of replacing "small jumps" (the ones in $\{|z|_E \leq \varepsilon_n\}$ here) by a Brownian part (that is $\mathrm{tr}(A_n(x)\nabla^2)$ in L_n) is not new - it has been introduced by Asmussen and Rosinski in [2] and has been extensively employed in papers concerned with simulation problems: since there is a huge amount of small jumps, they are difficult to simulate and then one approximates them by the Brownian part corresponding to $\mathrm{tr}(A_n(x)\nabla^2)$. See for example [1, 6, 12] and many others. However, at our knowledge, this idea has not been yet used in order to study the regularity of P_t.

The paper is organized as follows. In Section 2 we give the notation and the main results mentioned above and in Section 4 we give the proof of these results. Section 3 is devoted to some preliminary results about regularity. Namely, in Section 3.1 we recall and develop some results concerning regularity of probability measures, based on interpolation type arguments, coming from [3]. These are the main instruments used in the paper. In Section 3.2 we prove a regularity result which is a key point in our approach. In fact, it allows to handle the multiple integrals coming from the application of a Lindeberg method for the decomposition of $P_t - P_t^n$. The results stated in Section 2 are then proved in the subsections in which Section 4 is split. Finally, in Appendix A.1, A.2 and A.3 we prove some technical results used in the paper.

2 Notation and main results

2.1 Notation

For a multi-index $\alpha = (\alpha_1, ..., \alpha_m) \in \{1, ..., d\}^m$ we denote $|\alpha| = m$ (the length of the multi-index) and ∂^α is the derivative corresponding to α, that is $\partial^{\alpha_m} \cdots \partial^{\alpha_1}$, with $\partial^{\alpha_i} = \partial_{x_{\alpha_i}}$. For

$f \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and two multi-indexes α and β, we denote by ∂_x^α the derivative with respect to x and by ∂_y^α the derivative with respect to y.

Moreover, for $f \in C^\infty(\mathbb{R}^d)$ and $q \in \mathbb{N}$ we denote

$$\left|f\right|_q(x) = \sum_{0 \leq |\alpha| \leq q} \left|\partial^\alpha f(x)\right|. \tag{2.1}$$

If f is not a scalar function, that is, $f = (f^i)_{i=1,\dots,d}$ or $f = (f^{i,j})_{i,j=1,\dots,d}$, we denote $\left|f\right|_q = \sum_{i=1}^d \left|f^i\right|_q$ respectively $\left|f\right|_q = \sum_{i,j=1}^d \left|f^{i,j}\right|_q$.

We will work with the weights

$$\psi_\kappa(x) = (1 + |x|^2)^\kappa, \quad \kappa \in \mathbb{Z}. \tag{2.2}$$

The following properties hold:

- for every $\kappa \geq \kappa' \geq 0$,
$$\psi_\kappa(x) \leq \psi_{\kappa'}(x); \tag{2.3}$$
- for every $\kappa \geq 0$, there exists $C_\kappa > 0$ such that
$$\psi_\kappa(x) \leq C_\kappa \psi_\kappa(y) \psi_\kappa(x - y); \tag{2.4}$$
- for every $\kappa \geq 0$, there exists $C_\kappa > 0$ such that for every $\phi \in C_b^\infty(\mathbb{R}^d)$,
$$\psi_\kappa(\phi(x)) \leq C_\kappa \psi_\kappa(\phi(0))(1 + \|\nabla \phi\|_\infty^2)^\kappa \psi_\kappa(x); \tag{2.5}$$
- for every $q \in \mathbb{N}$ there exists $\overline{C}_q \geq \underline{C}_q > 0$ such that for every $\kappa \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^d)$,
$$\underline{C}_q \psi_\kappa \left|f\right|_q(x) \leq \left|\psi_\kappa f\right|_q(x) \leq \overline{C}_q \psi_\kappa \left|f\right|_q(x). \tag{2.6}$$

Note that (2.3)–(2.5) are immediate, whereas (2.6) is proved in Appendix A.1 (see Lemma A.1).

For $q \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $p \in (1, \infty]$ (we stress that we include the case $p = +\infty$), we set $\|\cdot\|_p$ the usual norm in $L^p(\mathbb{R}^d)$ and

$$\|f\|_{q,\kappa,p} = \left\|\left|\psi_\kappa f\right|_q\right\|_p. \tag{2.7}$$

We denote $W^{q,\kappa,p}$ to be the closure of $C^\infty(\mathbb{R}^d)$ with respect to the above norm. If $\kappa = 0$ we just denote $\|f\|_{q,p} = \|f\|_{q,0,p}$ and $W^{q,p} = W^{q,0,p}$ (which is the usual Sobolev space). So, we are working with weighted Sobolev spaces. The following properties hold:

- for every $q \in \mathbb{N}$ there exists $\overline{C}_q \geq \underline{C}_q > 0$ such that for every $\kappa \in \mathbb{R}$, $p > 1$ and $f \in W^{q,k,p}(\mathbb{R}^d)$,
$$\underline{C}_q \|\psi_\kappa |f|_q\|_p \leq \|f\|_{q,\kappa,p} \leq \overline{C}_q \|\psi_\kappa |f|_q\|_p; \tag{2.8}$$
- for every $q \in \mathbb{N}$ and $p > 1$ there exists $C_{q,p} > 0$ such that for every $\kappa \in \mathbb{R}$ and $f \in W^{q,k,p}(\mathbb{R}^d)$,
$$\|f\|_{q,\kappa,p} \leq C_{q,p} \|f\|_{q,\kappa+d,\infty} \tag{2.9}$$
and if $p > d$,
$$\|f\|_{q,\kappa,\infty} \leq C_{q,p} \|f\|_{q+1,\kappa,p}; \tag{2.10}$$

- for $\kappa, \kappa' \in \mathbb{R}$, $q, q' \in \mathbb{N}$, $p \in (1, \infty]$ and $U : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$, the following two assertions are equivalent: there exists a constant $C_* \geq 1$ such that for every f,

$$\|Uf\|_{q,\kappa,\infty} \leq C_* \|f\|_{q',\kappa',p} \tag{2.11}$$

and there exists a constant $C^* \geq 1$ such that for every f,

$$\Big\|\psi_\kappa U\Big(\frac{1}{\psi_{\kappa'}} f\Big)\Big\|_{q,\infty} \leq C^* \|f\|_{q',p} . \tag{2.12}$$

Notice that (2.8) is a consequence of (2.6). The inequality (2.9) is an immediate consequence of (2.6) and of the fact that $\psi_{-d} \in L^p(\mathbb{R}^d)$ for every $p \geq 1$. And the inequality (2.10) is a consequence of Morrey's inequality (Corollary IX.13 in [10]), whose use gives $\|f\|_{0,0,\infty} \leq \|f\|_{1,0,p}$, and of (2.6). In order to prove the equivalence between (2.11) and (2.12), one takes $g = \psi_{\kappa'} f$ (respectively $g = \frac{1}{\psi_{\kappa'}} f$) and uses(2.6) as well.

2.2 Main results

We consider a Markov semigroup P_t on $\mathcal{S}(\mathbb{R}^d)$ with infinitesimal operator L and a sequence $P_t^n, n \in \mathbb{N}$ of Markov semigroups on $\mathcal{S}(\mathbb{R}^d)$ with infinitesimal operator L_n. We suppose that $\mathcal{S}(\mathbb{R}^d)$ is included in the domain of L and of L_n and we suppose that for $f \in \mathcal{S}(\mathbb{R}^d)$ we have $Lf \in \mathcal{S}(\mathbb{R}^d)$ and $L_n f \in \mathcal{S}(\mathbb{R}^d)$. We denote $\Delta_n = L - L_n$. Moreover, we denote by $P_t^{*,n}$ the formal adjoint of P_t^n and by Δ_n^* the formal adjoint of Δ_n that is

$$\left\langle P_t^{*,n} f, g\right\rangle = \left\langle f, P_t^n g\right\rangle \quad \text{and} \quad \left\langle \Delta_n^* f, g\right\rangle = \left\langle f, \Delta_n g\right\rangle , \tag{2.13}$$

$\langle \cdot, \cdot \rangle$ being the scalar product in $L^2(\mathbb{R}^d, dx)$.

We present now our hypotheses. The first one concerns the speed of convergence of $L_n \to L$.

Assumption 2.1 *Let $a \in \mathbb{N}$, and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $\lim_n \varepsilon_n = 0$. We assume that for every $q \in \mathbb{N}, \kappa \geq 0$ and $p > 1$ there exists $C > 0$ such that for every n and f,*

$$(A_1) \qquad \|\Delta_n f\|_{q,-\kappa,\infty} \leq C\varepsilon_n \|f\|_{q+a,-\kappa,\infty} , \tag{2.14}$$

$$(A_1^*) \qquad \|\Delta_n^* f\|_{q,\kappa,p} \leq C\varepsilon_n \|f\|_{q+a,\kappa,p} . \tag{2.15}$$

Our second hypothesis concerns the "propagation of regularity" for the semigroups P_t^n.

Assumption 2.2 *Let $\Lambda_n \geq 1, n \in \mathbb{N}$ be an increasing sequence such that $\Lambda_{n+1} \leq \gamma\Lambda_n$ for some $\gamma \geq 1$. For every $q \in \mathbb{N}$ and $\kappa \geq 0, p > 1$, there exist $C > 0$ and $b \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ and f,*

$$(A_2) \qquad \sup_{s\leq t} \|P_s^n f\|_{q,-\kappa,\infty} \leq C\Lambda_n \|f\|_{q+b,-\kappa,\infty} , \tag{2.16}$$

$$(A_2^*) \qquad \sup_{s\leq t} \|P_s^{*,n} f\|_{q,\kappa,p} \leq C\Lambda_n \|f\|_{q+b,\kappa,p} . \tag{2.17}$$

The hypothesis (A_2^*) is rather difficult to verify so, in Appendix A.2, we give some sufficient conditions in order to check it (see Proposition A.7).

Our third hypothesis concerns the "regularization effect" of the semi-group P_t^n.

Assumption 2.3 *We assume that*

$$P_t^n f(x) = \int_{\mathbb{R}^d} p_t^n(x,y) f(y) dy \tag{2.18}$$

with $p_t^n \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. *Moreover, we assume there exist* $\theta_0 > 0$ *and a sequence* λ_n, $n \in \mathbb{N}$ *with*

$$\lambda_n \downarrow 0, \quad \lambda_n \leq \gamma \lambda_{n+1}, \tag{2.19}$$

foe some $\gamma \geq 1$, *such that the following property holds: for every* $\kappa \geq 0, q \in \mathbb{N}$ *there exist* $\pi(q,\kappa)$, *increasing in* q *and in* κ, *a constant* $\theta_1 \geq 0$, *and a constant* $C > 0$ *such that for every* $n \in \mathbb{N}$, $t \in (0,1]$, *for every multi-indexes* α *and* β *with* $|\alpha| + |\beta| \leq q$ *and* $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$(A_3) \qquad \left| \partial_x^\alpha \partial_y^\beta p_t^n(x,y) \right| \leq C \frac{1}{(\lambda_n t)^{\theta_0(q+\theta_1)}} \times \frac{\psi_{\pi(q,\kappa)}(x)}{\psi_\kappa(x-y)} \tag{2.20}$$

Note that in (2.20) we are quantifying the possible blow up of $|\partial_x^\alpha \partial_y^\beta p_t^n(x,y)|$ as $n \to \infty$.

We also assume the following statement will holds for the semigroup P_t.

Assumption 2.4 *For every* $\kappa \geq 0, q \in \mathbb{N}$ *there exists* $C \geq 1$ *such that*

$$(A_4) \qquad \|P_t f\|_{q,-\kappa,\infty} \leq C \|f\|_{q,-\kappa,\infty} . \tag{2.21}$$

For $\delta \geq 0$ we denote

$$\Phi_n(\delta) = \varepsilon_n \Lambda_n \times \lambda_n^{-\theta_0(a+b+\delta)}, \tag{2.22}$$

where a and b are the constants in Assumption 2.1 and 2.2 respectively. Notice that

$$\Phi_n(\delta) \leq \gamma^{1+\theta_0(a+b+\delta)} \Phi_{n+1}(\delta). \tag{2.23}$$

And, for $\kappa \geq 0, \eta \geq 0$ we set

$$\Psi_{\eta,\kappa}(x,y) := \frac{\psi_\kappa(y)}{\psi_\eta(x)}, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.24}$$

Our first result concerns the regularity of the semigroup P_t :

Theorem 2.5 *Suppose that Assumption 2.1, 2.2 , 2.3 and 2.4 hold. Moreover we suppose there exists* $\delta > 0$ *such that*

$$\limsup_n \Phi_n(\delta) < \infty, \tag{2.25}$$

$\Phi_n(\delta)$ *being given in (2.22). Then the following statements hold.*

A. $P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) dy$ *with* $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

B. *Let* $n \in \mathbb{N}$ *and* $\delta_* > 0$ *be such that*

$$\overline{\Phi}_n(\delta_*) := \sup_{n' \geq n} \Phi_{n'}(\delta_*) < \infty. \tag{2.26}$$

We fix $q \in \mathbb{N}$, $p > 1$, $\varepsilon_* > 0$, $\kappa \geq 0$ *and we put* $\mathfrak{m} = 1 + \frac{q+2d/p_*}{\delta_*}$ *with* p_* *the conjugate of* p. *There exist* $C \geq 1$ *and* $\eta_0 \geq 1$ *(depending on* $q, p, \varepsilon_*, \delta_*, \kappa$ *and* γ*) such that for every* $\eta > \eta_0$ *and* $t > 0$

$$\|\Psi_{\eta,\kappa} p_t\|_{q,p} \leq C \times Q_n(q,\mathfrak{m}) \times t^{-\theta_0((a+b)\mathfrak{m}+q+2d/p_*)(1+\varepsilon_*)} \quad \textit{with} \tag{2.27}$$

$$Q_n(q,\mathfrak{m}) = \Big(\frac{1}{\lambda_n^{\theta_0(a+b)\mathfrak{m}+q+2d/p_*}} + \overline{\Phi}_n^{\mathfrak{m}}(\delta_*) \Big)^{1+\varepsilon_*} \tag{2.28}$$

C. *Let $p > 2d$. Set $\bar{\mathfrak{m}} = 1 + \frac{q+1+2d/p_*}{\delta_*}$. There exist $C \geq 1, \eta \geq 0$ (depending on $q, p, \varepsilon_*, \delta_*, \kappa$) such that for every $t > 0$, $n \in \mathbb{N}$ and for every multi-indexes α, β such that $|\alpha| + |\beta| \leq q$,*

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq C \times Q_n(q+1, \bar{\mathfrak{m}}) \times t^{-\theta_0((a+b)\bar{\mathfrak{m}}+q+1+2d/p_*)(1+\varepsilon_*))} \times \frac{\psi_{\eta+\kappa}(x)}{\psi_\kappa(x-y)} \tag{2.29}$$

for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$.

Remark 2.6 *We stress that in hypothesis (2.26) the order of derivation q does not appear. However the conclusions (2.27) and (2.29) hold for every q. The motivation of this is given by the following heuristics. The hypothesis (2.20) says that the semi-group P_t^n has a regularization effect controlled by $1/(\lambda_n t)^{\theta_0}$. If we want to decouple this effect m_0 times we write $P_t^n = P_{t/m_0}^n....P_{t/m_0}^n$ and then each of the m_0 operators P_{t/m_0}^n acts with a regularization effect of order $(\lambda_n \times t/m_0)^{\theta_0}$. But this heuristics does not work directly: in order use it, we have to use a development in Taylor series coupled with the interpolation type criterion given in the following section.*

The proof of Theorem 2.5 is developed in Section 4.1. We give now a variant of the estimate (2.29), whose proof can be found in Section 4.2.

Theorem 2.7 *Suppose that Assumption 2.1, 2.2, 2.3 and 2.4 hold. Suppose also that (2.25) holds for some $\delta > 0$ and that for every $\kappa > 0$ there exist $\bar{\kappa}, \bar{C} > 0$ such that $P_t\psi_\kappa(x) \leq \bar{C}\psi_{\bar{\kappa}}(x)$, for all $x \in \mathbb{R}^d$ and $t > 0$. Then $P_t(x,y) = p_t(x,y)$ with $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and for every $\kappa \in \mathbb{N}$, $\varepsilon > 0$ and for every multi-indexes α and β there exists $C = C(\kappa, \varepsilon, \delta, \alpha, \beta)$ such that for every $t > 0$ and $x, y \in \mathbb{R}^d$*

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq C \times t^{-\theta_0(1+\frac{a+b}{\delta})(|\alpha|+|\beta|+2d+\varepsilon)} \times \frac{\psi_{\eta+\kappa}(x)}{\psi_\kappa(x-y)} \tag{2.30}$$

with θ_0 from (2.20).

We give now a result which goes in another direction (but the techniques used to prove it are the same): we assume that the semigroup $P_t : C_b^\infty(\mathbb{R}^d) \to C_b^\infty(\mathbb{R}^d)$ verifies hypothesis of type (A_2) (see (2.16) and (2.17)) and (A_3) (see (2.20)), we perturb it by a compound Poisson process, and we prove that the new semigroup still verifies a regularity property of type (A_3). This result will be used in [**?**] in order to cancel the "big jumps".

Let us give our hypotheses.

Assumption 2.8 *For every $q \in \mathbb{N}, \kappa \geq 0$ and $p \geq 1$ there exist $C_{q,\kappa,p}(P), C_{q,\kappa,\infty}(P) \geq 1$ such that*

$$(H_2) \qquad \|P_t f\|_{q,-\kappa,\infty} \leq C_{q,\kappa,\infty}(P)\, \|f\|_{q,-\kappa,\infty}\,, \tag{2.31}$$

$$(H_2^*) \qquad \|P_t^* f\|_{q,\kappa,p} \leq C_{q,\kappa,p}(P)\, \|f\|_{q,\kappa,p} \tag{2.32}$$

This means that the hypotheses (A_2) and(A_2^*) (see (2.16) and (2.17)) from Section 3.2 hold for P_t (instead of P_t^n) with Λ_n replaced by $C_{q,\kappa,\infty}(P) \vee C_{q,\kappa,p}(P)$ and with $b = 0$.

Assumption 2.9 *We assume that* $P_t(x,dy) = p_t(x,y)dy$ *with* $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *and the blow up of* $p_t \to \infty$ *as* $t \to 0$ *is controlled in the following way. For every fixed* $q \in \mathbb{N}, \kappa \geq 0$ *there exist some constants* $C \geq 1, 0 < \lambda \leq 1$ *and* $\eta > 0$ *such that for every two multi-indexes* α, β *with* $|\alpha| + |\beta| \leq q$

$$(H_3) \qquad \left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \times \frac{\psi_\eta(x)}{\psi_\kappa(x-y)} \tag{2.33}$$

Here $\theta_i \geq 0, i = 0,1$ *are some fixed parameters.*

We construct now the perturbed semigroup. We consider a Poisson process $N(t)$ of parameter $\rho > 0$ and we denote by T_k, $k \in \mathbb{N}$, its jump times. We also consider a sequence Z_k, $k \in \mathbb{N}$, of independent random variables of law ν, on a measurable space $(E, \mathcal{E})$, which are independent of $N(t)$ as well. Moreover we take a function $\phi : E \times \mathbb{R}^d \to \mathbb{R}^d$ and we denote $\phi_z(x) = \phi(z,x)$. We will precise in a moment the hypothesis on ϕ. We associate the operator

$$U_z f(x) = f(\phi_z(x)) \tag{2.34}$$

and we define $\overline{P}_t$ to be the perturbation of P_t in the following way. Conditionally to T_k and $Z_k, k \in \mathbb{N}$ we define

$$\begin{aligned} P_t^{N,Z} &= P_t && \text{for} \quad t < T_1, \\ P_{T_k}^{N,Z} &= U_{Z_k} P_{T_k-}^{N,Z}, && \\ P_t^{N,Z} &= P_{t-T_k} P_{T_k}^{N,Z} && \text{for} \quad T_k \leq t < T_{k+1} \end{aligned} \tag{2.35}$$

The second equality reads $P_{T_k}^{N,Z} f(x) = P_{T_k-}^{N,Z} f(\phi_{Z_k})$. Essentially (2.35) means that on $[T_{k-1}, T_k)$ we have a process which evolves according to the semigroup P_t and at time T_k it jumps according to ϕ_{Z_k}. Then we define

$$\overline{P}_t f(x) = \mathbb{E}(P_t^{N,Z} f(x)) = \sum_{m=0}^{\infty} I_m(f)(x)$$

with

$$I_0(f)(x) = \mathbb{E}(1_{\{N(t)=0\}} P_t f(x)) = e^{-\rho t} P_t f(x)$$

and for $m \geq 1$,

$$\begin{aligned} I_m(f)(x) &= \mathbb{E}\Big(1_{\{N(t)=m\}} \frac{m!}{t^m} \int_{0<t_1<\ldots<t_{m-1}<t_m\leq t} P_{t-t_m} \prod_{i=0}^{m-1} U_{Z_{m-i}} P_{t_{m-i}-t_{m-i-1}} f(x) dt_1 \ldots dt_m\Big) \\ &= \rho^m e^{-\rho t} \mathbb{E}\Big(\int_{0<t_1<\ldots<t_{m-1}<t_m\leq t} \Big(\prod_{i=0}^{m-1} P_{t_{m-i+1}-t_{m-i}} U_{Z_{m-i}} \Big) P_{t_1} f(x) dt_1 \ldots dt_m \Big), \end{aligned}$$

in which $t_0 = 0$ and $t_{m+1} = t$. We come now to the hypothesis on ϕ. We assume that for every $z \in E$, $\phi_z \in C^\infty(\mathbb{R}^d)$ and $\nabla \phi_z \in C_b^\infty(\mathbb{R}^d)$ and that for every $q \in \mathbb{N}$

$$\|\phi\|_{1,q,\infty} := \sup_{z\in E} \|\phi_z\|_{1,q,\infty} = \sum_{1\leq|\alpha|\leq q} \sup_{z\in E} \sup_{x\in\mathbb{R}^d} |\partial_x^\alpha \phi(z,x)| < \infty, \tag{2.36}$$

$$\widehat{\phi} := \sup_{z\in E} |\phi_z(0)| < \infty. \tag{2.37}$$

Moreover we define $\sigma(\phi_z) = \nabla \phi_z (\nabla \phi_z)^*$ and we assume that there exists a constant $\varepsilon(\phi) > 0$ such that for every $z \in E$ and $x \in \mathbb{R}^d$

$$\det \sigma(\phi_z)(x) \geq \varepsilon(\phi). \tag{2.38}$$

Remark 2.10 *We recall that in Appendix A.3 we have denoted $V_{\phi_z}f(x)=f(\phi_z(x))=U_zf(x)$. With this notation, under (2.36), (2.37), (2.38) we have proved in (A.21) and (A.23) that, for every $z\in E$,*

$$\|U_zf\|_{q,-\kappa,\infty}\leq C1\vee\widehat{\phi}^{2\kappa}\,\|\phi\|_{1,q,\infty}^{q+2\kappa}\,\|f\|_{q,-\kappa,\infty}\,, \tag{2.39}$$

$$\|U_z^*f\|_{q,\kappa,p}\leq C\,\frac{1\vee\widehat{\phi}^{2\kappa}}{\varepsilon(\phi)^{q(q+1)+1/p_*}}\times(1\vee\|\phi\|_{1,q+2,\infty}^{2dq+1+2\kappa})\times\|f\|_{q,\kappa,p}\,. \tag{2.40}$$

$$\tag{2.41}$$

This means that Assumption 3.5 from Section 3.2 hold uniformly in $z\in E$ and the constant given in (3.20) is upper bounded by

$$C_{q,\kappa,\infty,p}(U,P)\leq C\frac{1\vee\widehat{\phi}^{\kappa}}{\varepsilon(\phi)^{q(q+1)+1/p_*}}\times(1\vee\|\phi\|_{1,q+2,\infty}^{2dq+1+2\kappa})\times(C_{q,\kappa,\infty}(P)\vee C_{q,\kappa,p}(P)). \tag{2.42}$$

We are now able to give our result (the proof being postponed for Section 4.3):

Theorem 2.11 *Suppose that P_t satisfies assumptions 2.8 and 2.9. Suppose moreover that ϕ satisfies (2.36), (2.37), (2.38). Then $\overline{P}_t(x,dy)=\overline{p}_t(x,y)dy$ and we have the following estimates. Let $q\in\mathbb{N},\kappa\geq 0$ and $\delta>0$ be given. There exist some constants C,χ such that for every α,β with $|\alpha|+|\beta|\leq q$*

$$\left|\partial_x^\alpha\partial_y^\beta\overline{p}_t(x,y)\right|\leq\frac{C^\rho}{((\lambda t)^{\theta_0(q+2d)(1+\delta)}}\times\frac{\psi_\chi(x)}{\psi_\kappa(x-y)}. \tag{2.43}$$

We stress that the constant C depends on $C_{k,\kappa,\infty,p}(U,P)$ (see (2.42)) and on q,κ and δ ***but not on*** *t,ρ and λ.*

This gives the following consequence concerning the semigroup P_t itself:

Corollary 2.12 *Suppose that (2.31),(2.32),(2.33) hold. Then, does not matter the value of θ_1 in (2.33), the inequality (2.33) holds with $\theta_1'=2d+\varepsilon$ for every $\varepsilon>0$.*

Proof. Just take $\phi_z(x)=x$. □

3 Regularity results

This section is devoted to some preliminary results allowing us to prove the statements resumed in Section 2.2: in Section 3.1 we give an abstract regularity criterion, in Section 3.2 we prove a regularity result for iterated integrals.

3.1 A regularity criterion based on interpolation

Let us first recall some results obtained in [3] concerning the regularity of a measure μ on $\mathbb{R}^d$ (with the Borel σ-field). For two signed finite measures μ,ν and for $k\in\mathbb{N}$ we define the distance

$$d_k(\mu,\nu)=\sup\Big\{\Big|\int fd\mu-\int fd\nu\Big|:\|f\|_{k,\infty}\leq 1\Big\}. \tag{3.1}$$

If μ and ν are probability measures, d_0 is the total variation distance and d_1 is the Fortét Mourier distance. In this paper we will work with an arbitrary $k \in \mathbb{N}$. Notice also that $d_k(\mu,\nu) = \|\mu - \nu\|_{W_*^{k,\infty}}$ where $W_*^{k,\infty}$ is the dual of $W^{k,\infty}$.

We fix now $k, q, h \in \mathbb{N}$, with $h \geq 1$, and $p > 1$. Hereafter, we denote by $p_* = p/(p-1)$ the conjugate of p. Then, for a signed finite measure μ and for a sequence of absolutely continuous signed finite measures $\mu_n(dx) = f_n(x)dx$ with $f_n \in C^{2h+q}(\mathbb{R}^d)$, we define

$$\pi_{k,q,h,p}(\mu,(\mu_n)_n) = \sum_{n=0}^{\infty} 2^{n(k+q+d/p_*)} d_k(\mu,\mu_n) + \sum_{n=0}^{\infty} \frac{1}{2^{2nh}} \|f_n\|_{2h+q,2h,p} \,. \tag{3.2}$$

Remark 3.1 *Notice that $\pi_{k,q,h,p}$ is a particular case of $\pi_{k,q,h,\mathbf{e}}$ treated in [3]: just choose the Young function $\mathbf{e}(x) \equiv \mathbf{e}_p(x) = |x|^p$, giving $\beta_{\mathbf{e}_p}(t) = t^{1/p_*}$ (see Example 1 in [3]). Moreover, $\pi_{k,q,h,p}$ is strongly related to interpolation spaces. More precisely, let*

$$\overline{\pi}_{k,q,h,p}(\mu) = \inf\{\pi_{k,q,h,p}(\mu,(\mu_n)_n) : \mu_n(dx) = f_n(x)dx, \quad f_n \in C^{2h+q}(\mathbb{R}^d)\}.$$

Then $\overline{\pi}_{k,q,h,p}$ is equivalent with the interpolation norm of order $\rho = \frac{k+q+d/p_*}{2h}$ between the spaces $W_*^{k,\infty}$ (the dual of $W^{k,\infty}$) and $W^{2h+q,2h,p} = \{f : \|f_n\|_{2h+q,2h,p} < \infty\}$. This is proved in [3], see Section 2.4 and Appendix B. So the inequality (3.3) below says that the Sobolev space $W^{q,p}$ is included in the above interpolation space. However we prefer to remain in an elementary framework and to derive directly the consequences of (3.3) - see Lemma 3.4 below

The following result is the key point in our approach (this is Proposition 2.5 in [3]):

Lemma 3.2 *Let $k, q, h \in \mathbb{N}$ with $h \geq 1$, and $p > 1$ be given. There exists a constant C_* (depending on k, q, h and p only) such that the following holds. Let μ be a finite measure for which one may find a sequence $\mu_n(dx) = f_n(x)dx$, $n \in \mathbb{N}$ such that $\pi_{k,q,h,p}(\mu,(\mu_n)_n) < \infty$. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$ and moreover*

$$\|f\|_{q,p} \leq C_* \times \pi_{k,q,h,p}(\mu,(\mu_n)_n). \tag{3.3}$$

The proof of Lemma 3.2 is given in [3], being a particular case (take $\mathbf{e} = \mathbf{e}_p$) of Proposition A.1 in Appendix A.

We give a first simple consequence.

Lemma 3.3 *Let $p_t \in C^\infty(\mathbb{R}^d), t > 0$ be a family of non negative functions and let $\varphi = \varphi(x) \geq 0$ be such that $\int \varphi(x)p_t(x)dx \leq m < \infty$ for every $t < 1$. We assume that for some $\theta_0 > 0$ and $\theta_1 > 0$ the following holds: for every $q \in \mathbb{N}$ and $p > 1$ there exists a constant $C = C(q,p)$ such that*

$$\|\varphi p_t\|_{q,p} \leq C t^{-\theta_0(q+\theta_1)}, \quad t < 1. \tag{3.4}$$

Let $\delta > 0$. Then, there exists a constant $C_* = C_*(q,p,\delta)$ such that

$$\|\varphi p_t\|_{q,p} \leq C_* t^{-\theta_0(q+\frac{d}{p_*}+\delta)}, \quad t < 1, \tag{3.5}$$

where p_* is the conjugate of p. So, does not matter the value of θ_1, one may replace it by $\frac{d}{p_*}$.

Proof We take $n_* \in \mathbb{N}$ and we define $f_n = 0$ for $n \leq n_*$ and $f_n = \varphi p_t$ for $n > n_*$. Notice that $d_0(\varphi p_t, 0) \leq m$. Then (3.3) with $k = 0$ gives (C denoting a positive constant which may change from a line to another)

$$\begin{aligned}\left\|\varphi p_t\right\|_{q,p} &\leq C\Big(m\sum_{n=0}^{n_*} 2^{n(q+\frac{d}{p_*})} + \left\|\varphi p_t\right\|_{2h+q,2h,p} \sum_{n=n_*+1}^{\infty} \frac{1}{2^{2nh}}\Big) \\ &\leq C\Big(m2^{n_*(q+\frac{d}{p_*})} + \left\|\varphi p_t\right\|_{2h+q,2h,p} \frac{1}{2^{2n_*h}}\Big).\end{aligned}$$

We denote $\rho_h = (q + \frac{d}{p_*})/2h$. We optimize over n_* and we obtain

$$\begin{aligned}\left\|\varphi p_t\right\|_{q,p} &\leq 2C \times m^{\frac{1}{1+\rho_h}} \times \left\|\varphi p_t\right\|_{2h+q,2h,p}^{\frac{\rho_h}{1+\rho_h}} \\ &\leq 2Cm^{\frac{1}{1+\rho_h}} \times Ct^{-\theta_0(2h+q+\theta_1)\frac{\rho_h}{1+\rho_h}}.\end{aligned}$$

Since $\lim_{h\to\infty} \rho_h = 0$ and $\lim_{h\to\infty}(2h+q+\theta_1)\frac{\rho_h}{1+\rho_h} = q + \frac{d}{p_*}$ the proof is completed, we choose h large enough and we obtain (3.5). $\square$

We will also use the following consequence of Lemma 3.2.

Lemma 3.4 *Let $k, q, h \in \mathbb{N}$, with $h \geq 1$, and $p > 1$ be given and set*

$$\rho_h := \frac{k + q + d/p_*}{2h}. \tag{3.6}$$

We consider an increasing sequence $\theta(n) \geq 1, n \in \mathbb{N}$ such that $\lim_n \theta(n) = \infty$ and $\theta(n + 1) \leq \Theta \times \theta(n)$ for some constant $\Theta \geq 1$. Suppose that we may find a sequence of functions $f_n \in C^{2h+q}(\mathbb{R}^d), n \in \mathbb{N}$ such that

$$\left\|f_n\right\|_{2h+q,2h,p} \leq \theta(n) \tag{3.7}$$

and, with $\mu_n(dx) = f_n(x)dx$,

$$\limsup_n d_k(\mu, \mu_n) \times \theta^{\rho_h+\varepsilon}(n) < \infty \tag{3.8}$$

for some $\varepsilon > 0$. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$.

Moreover, for $\delta, \varepsilon > 0$ and $n_* \in \mathbb{N}$, let

$$A(\delta) = |\mu|(\mathbb{R}^d) \times 2^{l(\delta)(1+\delta)(q+k+d/p_*)} \quad \text{with } l(\delta) = \min\{l : 2^{l\times\frac{\delta}{1+\delta}} \geq l\}, \tag{3.9}$$

$$B(\varepsilon) = \sum_{l=1}^{\infty} \frac{l^{2(q+k+d/p_*+\varepsilon)}}{2^{2\varepsilon l}}, \tag{3.10}$$

$$C_{h,n_*}(\varepsilon) = \sup_{n\geq n_*} d_k(\mu, \mu_n) \times \theta^{\rho_h+\varepsilon}(n). \tag{3.11}$$

Then, for every $\delta > 0$

$$\|f\|_{q,p} \leq C_*(\Theta + A(\delta)\theta(n_*)^{\rho_h(1+\delta)} + B(\varepsilon)C_{h,n_*}(\varepsilon)), \tag{3.12}$$

C_* being the constant in (3.3) and ρ_h being given in (3.6).

Proof of Lemma 3.4. We will produce a sequence of measures $\nu_l(dx) = g_l(x)dx, l \in \mathbb{N}$ such that

$$\pi_{k,q,h,p}(\mu,(\nu_l)_l) \leq \Theta + A(\delta)\theta(n_*)^{\rho_h(1+\delta)} + B(\varepsilon)C_{h,n_*}(\varepsilon) < \infty.$$

Then by Lemma 3.2 one gets $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$ and (3.12) follows from (3.3). Let us stress that the ν_l's will be given by a suitable subsequence $\mu_{n(l)}$, $l \in \mathbb{N}$, from the μ_n's.

Step 1. We define

$$n(l) = \min\{n : \theta(n) \geq \frac{2^{2hl}}{l^2}\}$$

and we notice that

$$\frac{1}{\Theta}\theta(n(l)) \leq \theta(n(l)-1) < \frac{2^{2hl}}{l^2} \leq \theta(n(l)). \tag{3.13}$$

Moreover we define

$$l_* = \min\{l : \frac{2^{2hl}}{l^2} \geq \theta(n_*)\}.$$

Since

$$\theta(n(l_*)) \geq \frac{2^{2hl_*}}{l_*^2} \geq \theta(n_*)$$

it follows that $n(l_*) \geq n_*$.

We take now $\varepsilon(\delta) = \frac{h\delta}{1+\delta}$ which gives $\frac{2h}{2(h-\varepsilon(\delta))} = 1+\delta$. And we take $l(\delta) \geq 1$ such that $2^{l\delta/(1+\delta)} \geq l$ for $l \geq l(\delta)$. Since $h \geq 1$ it follows that $\varepsilon(\delta) \geq \frac{\delta}{1+\delta}$ so that, for $l \geq l(\delta)$ we also have $2^{l\varepsilon(\delta)} \geq l$. Now we check that

$$2^{2(h-\varepsilon(\delta))l_*} \leq 2^{2hl(\delta)}\theta(n_*). \tag{3.14}$$

If $l_* \leq l(\delta)$ then the inequality is evident (recall that $\theta(n) \geq 1$ for every n). And if $l_* > l(\delta)$ then $2^{l_*\varepsilon(\delta)} \geq l_*$. By the very definition of l_* we have

$$\frac{2^{2h(l_*-1)}}{(l_*-1)^2} < \theta(n_*)$$

so that

$$2^{2hl_*} \leq 2^{2h}(l_*-1)^2\theta(n_*) \leq 2^{2h} \times 2^{2l_*\varepsilon(\delta)}\theta(n_*)$$

and this gives (3.14).

Step 2. We define

$$\nu_l = 0 \text{ if } l < l_* \text{ and } \nu_l = \mu_{n(l)} \text{ if } l \geq l_*$$

and we estimate $\pi_{k,q,h,p}(\mu,(\nu_l)_l)$. First, by (3.7) and (3.13)

$$\sum_{l=l_*}^{\infty} \frac{1}{2^{2hl}} \left\| f_{n(l)} \right\|_{q+2h,2h,p} \leq \sum_{l=l_*}^{\infty} \frac{1}{2^{2hl}}\theta(n(l)) \leq \Theta \sum_{l=l_*}^{\infty} \frac{1}{l^2} \leq \Theta.$$

Then we write

$$\sum_{l=1}^{\infty} 2^{(q+k+d/p_*)l} d_k(\mu,\nu_l) = S_1 + S_2$$

with

$$S_1 = \sum_{l=1}^{l_*-1} 2^{(q+k+d/p_*)l} d_k(\mu,0), \quad S_2 = \sum_{l=l_*}^{\infty} 2^{(q+k+d/p_*)l} d_k(\mu,\mu_{n(l)}).$$

Since $d_k(\mu,0) \le d_0(\mu,0) \le |\mu|(\mathbb{R}^d)$ we use (3.14) and we obtain

$$\begin{aligned} S_1 &\le |\mu|(\mathbb{R}^d) \times 2^{(q+k+d/p_*)l_*} = |\mu|(\mathbb{R}^d) \times (2^{2(h-\varepsilon(\delta))l_*})^{(q+k+d/p_*)/2(h-\varepsilon(\delta))} \\ &\le |\mu|(\mathbb{R}^d) \times (2^{2hl(\delta)}\theta(n_*))^{\rho_h(1+\delta)} = A(\delta)\theta(n_*)^{\rho_h(1+\delta)}. \end{aligned}$$

If $l \ge l_*$ then $n(l) \ge n(l_*) \ge n_*$ so that, from (3.11),

$$d_k(\mu,\mu_{n(l)}) \le \frac{C_{h,n_*}(\varepsilon)}{\theta^{\rho_h+\varepsilon}(n(l))} \le C_{h,n_*}(\varepsilon)\Big(\frac{l^2}{2^{2hl}}\Big)^{\rho_h+\varepsilon} = \frac{C_{h,n_*}(\varepsilon)}{2^{(q+k+d/p_*)l}} \times \frac{l^{2(\rho_h+\varepsilon)}}{2^{2h\varepsilon l}}.$$

We conclude that

$$S_2 \le C_{h,n_*}(\varepsilon)\sum_{l=l_*}^{\infty}\frac{l^{2(\rho_h+\varepsilon)}}{2^{2h\varepsilon l}} \le C_{h,n_*}(\varepsilon) \times B(\varepsilon).$$

□

3.2 A regularity lemma

We give here a regularization result in the following abstract framework. We consider a sequence of operators $U_j : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, $j \in \mathbb{N}$, and we denote by U_j^* the formal adjoint defined by $\langle U_j^* f, g\rangle = \langle f, U_j g\rangle$ with the scalar product in $L^2(\mathbb{R}^d)$.

Assumption 3.5 *Let $a \in \mathbb{N}$ be fixed. We assume that for every $q \in \mathbb{N}, \kappa \ge 0$ and $p \in [1,\infty)$ there exist constants $C_{q,\kappa,p}(U)$ and $C_{q,\kappa,\infty}(U)$ such that for every j and f,*

$$(H_1) \qquad \|U_j f\|_{q,-\kappa,\infty} \le C_{q,\kappa,\infty}(U)\,\|f\|_{q+a,-\kappa,\infty}\,, \tag{3.15}$$

$$(H_1^*) \qquad \left\|U_j^* f\right\|_{q,\kappa,p} \le C_{q,\kappa,p}(U)\,\|f\|_{q+a,\kappa,p}\,. \tag{3.16}$$

We assume that $C_{q,\kappa,p}(U)$, $p \in [1,\infty]$, is non decreasing with respect to q and κ.

We also consider a semigroup S_t, $t \ge 0$, of the form

$$S_t(x,dy) = s_t(x,y)dy \quad \text{with} \quad s_t \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d).$$

We define the formal adjoint operator

$$S_t^* f(y) = \int_{\mathbb{R}^d} s_t(x,y)f(x)dx, \quad t > 0.$$

Assumption 3.6 *If $f \in \mathcal{S}(\mathbb{R}^d)$ then $S_t f \in \mathcal{S}(\mathbb{R}^d)$. Moreover, there exist $b \in \mathbb{N}$ such that for every $q \in \mathbb{N}, \kappa \ge 0$ and $p \in [1,\infty)$ there exist constants $C_{q,\kappa,p}(S)$ such that for every $t > 0$,*

$$(H_2) \qquad \|S_t f\|_{q,-\kappa,\infty} \le C_{q,\kappa,\infty}(S)\,\|f\|_{q+b,-\kappa,\infty}\,, \tag{3.17}$$

$$(H_2^*) \qquad \|S_t^* f\|_{q,\kappa,p} \le C_{q,\kappa,p}(S)\,\|f\|_{q+b,\kappa,p}\,. \tag{3.18}$$

We assume that $C_{q,\kappa,p}(S)$, $p \in [1,\infty]$, is non decreasing with respect to q and κ.

We denote

$$C_{q,\kappa,\infty}(U,S)=C_{q,\kappa,\infty}(U)C_{q,\kappa,\infty}(S),\quad C_{q,\kappa,p}(U,S)=C_{q,\kappa,p}(U)C_{q,\kappa,p}(S), \tag{3.19}$$
$$C_{q,\kappa,\infty,p}(U,S)=C_{q,\kappa,\infty}(U,S)\vee C_{q,\kappa,p}(U,S). \tag{3.20}$$

Under Assumption 3.5 and 3.6, one immediately obtains

$$\|(S_tU_j)f\|_{q,-\kappa,\infty}\leq C_{q,\kappa,\infty}(U,S)\,\|f\|_{q+a+b,-\kappa,\infty}\,, \tag{3.21}$$
$$\left\|(S_t^*U_j^*)f\right\|_{q,\kappa,p}\leq C_{q,\kappa,p}(U,S)\,\|f\|_{q+a+b,\kappa,p}\,. \tag{3.22}$$

In fact these are the inequalities that we will employ in the following. We stress that the above constants $C_{q,\kappa,\infty}(U,S)$ and $C_{q,\kappa,p}(U,S)$ may depend on a,b and are increasing w.r.t. q and κ.

Finally we assume that the (possible) blow up of $s_t\to\infty$ as $t\to 0$ is controlled in the following way.

Assumption 3.7 *Let $\theta_0,\lambda>0$ be fixed. We assume that for every $\kappa\geq 0$ and $q\in\mathbb{N}$ there exist $\pi(q,\kappa)$, $\theta_1\geq 0$ and $C_{q,\kappa}>0$ such that for every multi-indexes α and β with $|\alpha|+|\beta|\leq q$, $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and $t\in(0,1]$ one has*

$$(H_3)\qquad \left|\partial_x^\alpha\partial_y^\beta s_t(x,y)\right|\leq\frac{C_{q,\kappa}}{(\lambda t)^{\theta_0(q+\theta_1)}}\times\frac{\psi_{\pi(q,\kappa)}(x)}{\psi_\kappa(x-y)}. \tag{3.23}$$

We also assume that $\pi(q,k)$ and $C_{q,\kappa}$ are both increasing in q and κ.

This property will be used by means of the following lemma:

Lemma 3.8 *Suppose that Assumption 3.7 holds.*

A. *For every $\kappa\geq 0$, $q\in\mathbb{N}$ and $p>1$ there exists $C>0$ such that for every $t\in(0,1]$ and f one has*

$$\|S_t^*f\|_{q,\kappa,p}\leq\frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}}\,\|f\|_{0,\nu,1} \tag{3.24}$$

where $\nu=\pi(q,\kappa+d)+\kappa+d$

B. *For every $\kappa\geq 0$, $q_1,q_2\in\mathbb{N}$ there exists $C>0$ such that for every $t\in(0,1]$, for every multi-index α with $|\alpha|\leq q_2$ and f one has*

$$\left\|\frac{1}{\psi_\eta}S_t(\psi_\kappa\partial^\alpha f)\right\|_{q_1,\infty}\leq\frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}}\,\|f\|_\infty \tag{3.25}$$

where $\eta=\pi(q_1+q_2,\kappa+d+1)+\kappa$.

Proof. In the sequel, C will denote a positive constant which may vary from a line to another and which may depend only on κ and q for the proof of **A.** and only on κ,q_1 and q_2 for the proof of **B.**

A. Using (3.23) if $|\alpha|\leq q$,

$$|\partial^\alpha S_t^*f(x)|\leq\int|\partial_x^\alpha s_t(y,x)|\times|f(y)|\,dy\leq\frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}}\int\frac{\psi_{\pi(q,\kappa+d)}(y)}{\psi_{\kappa+d}(x-y)}\times|f(y)|\,dy.$$

By (2.4) $\psi_{\kappa+d}(x)/\psi_{\kappa+d}(x-y)\leq C\psi_{\kappa+d}(y)$ so that

$$\begin{aligned}\psi_{\kappa+d}(x)\left|\partial^{\alpha}S_t^{*}f(x)\right| &\leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}}\int\frac{\psi_{\kappa+d}(x)\psi_{\pi(q,\kappa+d)}(y)}{\psi_{\kappa+d}(x-y)}\times|f(y)|\,dy\\ &\leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}}\int\psi_{\pi(q,\kappa+d)+\kappa+d}(y)\times|f(y)|\,dy\\ &= \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}}\left\|f\right\|_{0,\nu,1}\end{aligned}$$

We conclude that

$$\left\|S_t^{*}f\right\|_{q,\kappa+d,\infty}\leq\frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}}\left\|f\right\|_{0,\nu,1}.$$

By (2.9) $\left\|S_t^{*}f\right\|_{q,\kappa,p}\leq C\left\|S_t^{*}f\right\|_{q,\kappa+d,\infty}$ so the proof of (3.24) is completed.

B. Let γ with $|\gamma|\leq q_1$. Using integration by parts

$$\begin{aligned}\partial^{\gamma}S_t(\psi_\kappa\partial^{\alpha}f)(x) &= \int_{\mathbb{R}^d}\partial_x^{\gamma}s_t(x,y)\psi_\kappa(y)\partial^{\alpha}f(y)dy\\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^d}\partial_y^{\alpha}(\partial_x^{\gamma}s_t(x,y)\psi_\kappa(y))\times f(y)dy.\end{aligned}$$

Using (2.6), (3.23) and (2.4), it follows that

$$\begin{aligned}\left|\partial^{\gamma}S_t(\psi_\kappa\partial^{\alpha}f)(x)\right| &\leq \int_{\mathbb{R}^d}\left|\partial_y^{\alpha}(\partial_x^{\gamma}s_t(x,y)\psi_\kappa(y))\right|\times|f(y)|\,dy\\ &\leq \int_{\mathbb{R}^d}\left|s_t(x,y)\psi_\kappa(y)\right|_{q_1+q_2}\times|f(y)|\,dy\\ &\leq C\int_{\mathbb{R}^d}\left|s_t(x,y)\right|_{q_1+q_2}\psi_\kappa(y)\times|f(y)|\,dy\\ &\leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}}\left\|f\right\|_\infty\int_{\mathbb{R}^d}\frac{\psi_{\pi(q_1+q_2,\kappa+d+1)}(x)}{\psi_{\kappa+d+1}(x-y)}\times\psi_\kappa(y)dy\\ &\leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}}\left\|f\right\|_\infty\int_{\mathbb{R}^d}\frac{\psi_{\pi(q_1+q_2,\kappa+d+1)+\kappa}(x)}{\psi_{d+1}(x-y)}dy\\ &\leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}}\left\|f\right\|_\infty\psi_{\pi(q_1+q_2,\kappa+d+1)+\kappa}(x).\end{aligned}$$

This implies (3.25). □

We are now able to give the "regularity lemma". This is the core of our approach.

Lemma 3.9 *Suppose that Assumption 3.5, 3.6 and 3.7 hold. We fix $t\in(0,1]$, $m\geq1$ and $\delta_i>0$, $i=1,\ldots,m$ such that $\sum_{i=1}^{m}\delta_i=t$.*

A. *There exists a function $\tilde{p}_{\delta_1,\ldots,\delta_m}\in C^{\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ such that*

$$\prod_{i=1}^{m-1}(S_{\delta_i}U_i)S_{\delta_m}f(x)=\int\tilde{p}_{\delta_1,\ldots,\delta_m}(x,y)f(y)dy. \tag{3.26}$$

B. *We fix* $q_1, q_2 \in \mathbb{N}, \kappa \geq 0, p > 1$ *and we denote* $q = q_1 + q_2 + (a+b)(m-1)$. *One may find universal constants* $C, \chi, \bar{p} \geq 1$ *(depending on* κ, p *and* $q_1 + q_2$*) such that for every multi-index* β *with* $|\beta| \leq q_2$ *and every* $x \in \mathbb{R}^d$

$$\left\| \partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,\cdot) \right\|_{q_1,\kappa,p} \leq C \Big(\frac{2m}{\lambda t} \Big)^{\theta_0(q_1+q_2+d+2\theta_1)} \Big(C_{q,\chi,\bar{p},\infty}(U,S) \Big(\frac{2m}{\lambda t} \Big)^{\theta_0(a+b)} \Big)^{m-1} \psi_\chi(x). \tag{3.27}$$

Proof. A. For $g = g(x,y)$, we denote $g^x(y) := g(x,y)$. By the very definition of U_i^* one has

$$S_t U_i f(x) = \int_{\mathbb{R}^d} U_i^* s_t^x(y) f(y) dy.$$

As a consequence, one gets the kernel in (3.26):

$$\tilde{p}_{\delta_1,\ldots,\delta_m}(x,y) = \int_{\mathbb{R}^{d\times(m-1)}} U_1^* s_{\delta_1}^x(y_1) \Big(\prod_{j=2}^{m-1} U_j^* s_{\delta_j}^{y_{j-1}}(y_j) \Big) s_{\delta_m}(y_{m-1}, y) dy_1 \cdots dy_{m-1},$$

and the regularity immediately follows.

B. We split the proof in several steps.

Step 1: decomposition. Since $\sum_{i=1}^m \delta_i = t$ we may find $j \in \{1,...,m\}$ such that $\delta_j \geq \frac{t}{m}$. We fix this j and we write

$$\prod_{i=1}^{m-1} (S_{\delta_i} U_i) S_{\delta_m} = Q_1 Q_2$$

with

$$Q_1 = \prod_{i=1}^{j-1} (S_{\delta_i} U_i) S_{\frac{1}{2}\delta_j} \quad \text{and} \quad Q_2 = S_{\frac{1}{2}\delta_j} U_j \prod_{i=j+1}^{m-1} (S_{\delta_i} U_i) S_{\delta_m} = S_{\frac{1}{2}\delta_j} \prod_{i=j}^{m-1} (U_i S_{\delta_{i+1}}).$$

Here we use the semi-group property $S_{\frac{1}{2}\delta_j} S_{\frac{1}{2}\delta_j} = S_{\delta_j}$.

We suppose that $j \leq m-1$. In the case $j = m$ the proof is analogous but simpler. We will use Lemma 3.8 in order to estimate the terms corresponding to each of these two operators. As already seen, both Q_1 and Q_2 are given by means of smooth kernels, that we call $p_1(x,y)$ and $p_2(x,y)$ respectively.

Step 2. We take β with $|\beta| \leq q_2$ and we denote $g^{\beta,x}(y) := \partial_x^\beta g(x,y)$. For $h \in L^1$ we write

$$\begin{aligned}
\int_{\mathbb{R}^d} h(z) \partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,z) dz &= \int_{\mathbb{R}^d} h(z) \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) p_2(y,z) dy dz \\
&= \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) \int h(z) p_2(y,z) dz dy = \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) Q_2 h(y) dy \\
&= \int_{\mathbb{R}^d} Q_2^* p_1^{\beta,x}(y) h(y) dy.
\end{aligned}$$

It follows that

$$\partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,z) = Q_2^* p_1^{\beta,x}(z) = \prod_{i=1}^{m-j} (S_{\delta_{m-i+1}}^* U_{m-i}^*) S_{\frac{1}{2}\delta_j}^* p_1^{\beta,x}(z).$$

We will use (3.22) $m-j$ times first and (3.24) then. We denote

$$q_1' = q_1 + (m-j)(a+b)$$

and we write

$$\begin{aligned}\|\partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,\cdot)\|_{q_1,\kappa,p} &\leq C_{q_1',\kappa,p}^{m-j}(U,S)\|S^*_{\frac{1}{2}\delta_j} p_1^{\beta,x}\|_{q_1',\kappa,p} \\ &\leq C_{q_1',\kappa,p}^{m-j}(U,S)\, C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_1'+\theta_1)} \|p_1^{\beta,x}\|_{0,\nu,1}\end{aligned} \tag{3.28}$$

with

$$\nu = \pi(q_1', \kappa+d) + \kappa + d.$$

Step 3. We denote $g_z(u) = \prod_{l=1}^d 1_{(0,\infty)}(u_l - z_l)$, so that $\delta_0(u-z) = \partial_u^\rho g_z(u)$ with $\rho = (1,2,\ldots,d)$. We take $\mu = \nu + d + 1$ and we formally write

$$p_1(x,z) = \frac{1}{\psi_\mu(z)} Q_1(\psi_\mu \partial^\rho g_z)(x).$$

This formal equality can be rigorously written by using the regularization by convolution of the Dirac function.

We denote

$$q_2' = q_2 + (j-1)(a+b), \quad \eta = \pi(d+q_2', \mu+d+1) + \mu$$

and we write

$$|p_1^{\beta,x}(z)| = |\partial_x^\beta p_1(x,z)| \leq \frac{\psi_\eta(x)}{\psi_\mu(z)} \Big\| \frac{1}{\psi_\eta} \partial^\beta Q_1(\psi_\mu \partial^\rho g_z) \Big\|_\infty.$$

Since $\mu = \nu + d + 1$, $\int \psi_\nu \times \frac{1}{\psi_\mu} < \infty$, so using (2.6), we obtain (recall that $|\beta| \leq q_2$)

$$\begin{aligned}\|p_1^{\beta,x}\|_{0,\nu,1} &\leq C\psi_\eta(x) \sup_{z\in\mathbb{R}^d} \Big\| \frac{1}{\psi_\eta} \partial^\beta Q_1(\psi_\mu \partial^\rho g_z) \Big\|_\infty \leq C\psi_\eta(x) \sup_{z\in\mathbb{R}^d} \Big\| \frac{1}{\psi_\eta} Q_1(\psi_\mu \partial^\rho g_z) \Big\|_{q_2,\infty} \\ &\leq C\psi_{\eta'}(x) \sup_{z\in\mathbb{R}^d} \big\| Q_1(\psi_\mu \partial^\rho g_z) \big\|_{q_2,-\eta,\infty}.\end{aligned}$$

Using (3.21) $j-1$ times and (3.25) (with $\kappa = \mu$) we get

$$\begin{aligned}\|Q_1(\psi_\mu \partial^\rho g_z)\|_{q_2,-\eta,\infty} &\leq C_{q_2',\eta,\infty}^{j-1}(U,S)\|S_{\frac{1}{2}\delta_j}(\psi_\mu \partial^\rho g_z)\|_{q_2',-\eta,\infty} \\ &\leq C_{q_2',\eta,\infty}^{j-1}(U,S)\, \|g_z\|_\infty \, C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_2'+d+\theta_1)}.\end{aligned}$$

Since $\|g_z\|_\infty = 1$ we obtain

$$\|p_1^{\beta,x}\|_{0,\nu,1} \leq \psi_\eta(x) C_{q_2',\eta,\infty}^{j-1}(U,S)\, C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_2'+d+\theta_1)}.$$

By inserting in (3.28) we obtain (3.27), so the proof is completed. □

4 Proofs of the main results

In the present section, we use the results in Section 3 in order to prove Theorem 2.5 (Section 4.1) and Theorem 2.11 (Section 4.3).

4.1 Proof of Theorem 2.5

Step 0: constants and parameters set-up. In this step we will choose some parameters which will be used in the following steps. To begin we stress that we work with measures on $\mathbb{R}^d \times \mathbb{R}^d$ so the dimension of the space is $2d$ (and not d). We recall that in our statement the quantities $q, d, p, \delta_*, \varepsilon_*, \kappa$ and n are given and fixed. In the following we will denote by C a constant depending on all these parameters and which may change from a line to another. We define

$$m_0 = 1 + \left\lfloor \frac{q + 2d/p_*}{\delta_*} \right\rfloor > 0 \tag{4.1}$$

and given $h \in \mathbb{N}$ we denote

$$\rho_h = \frac{(a+b)m_0 + q + 2d/p_*}{2h}. \tag{4.2}$$

Notice that this is equal to the constant ρ_h defined in (3.6) corresponding to $k = (a+b)m_0$ and q and to $2d$ (instead of d).

Step 1: a Lindeberg-type method to decompose $P_t - P_t^n$. We fix (once for all) $t \in (0,1]$ and we write

$$P_t f - P_t^n f = \int_0^t \partial_s (P_{t-s}^n P_s) f ds = \int_0^t P_{t-s}^n (L - L_n) P_s f ds = \int_0^t P_{t-s}^n \Delta_n P_s f ds$$

We iterate this formula m_0 times (with m_0 chosen in (4.1)) and we obtain

$$P_t f(x) - P_t^n f(x) = \sum_{m=1}^{m_0 - 1} I_n^m f(x) + R_n^{m_0} f(x) \tag{4.3}$$

with (we put $t_0 = t$)

$$I_n^m f(x) = \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{m-1}} dt_m \prod_{i=0}^{m-1} (P_{t_i - t_{i+1}}^n \Delta_n) P_{t_m}^n f(x), \quad 1 \leq m \leq m_0 - 1,$$

$$R_n^{m_0} f(x) = \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{m_0 - 1}} dt_{m_0} \prod_{i=0}^{m_0 - 1} (P_{t_i - t_{i+1}}^n \Delta_n) P_{t_{m_0}} f(x).$$

In order to analyze $I_n^m f$ we use Lemma 3.9 for the semigroup $S_t = P_t^n$ and for the operators $U_i = \Delta_n = L - L_n$ (the same for each i), with $\delta_i = t_i - t_{i+1}$, $i = 0, \ldots, m$ (with $t_{m+1} = 0$). So the hypotheses (3.15) and (3.16) in Assumption 3.5 coincide with the requests (2.14) and (2.15) in Assumption 2.1. And we have $C_{q,\kappa,\infty}(U) = C_{q,\kappa,p}(U) = C\varepsilon_n$. Moreover the hypotheses (3.17) and (3.18) in Assumption 3.6 coincide with the hypotheses (2.16) and (2.17) in Assumption 2.2. And we have $C_{q,\kappa,\infty}(P^n) = C_{q,\kappa,p}(P^n) = \Lambda_n$. Hence,

$$C_{q,\kappa,\infty,p}(\Delta_n, P^n) = C\, \varepsilon_n \times \Lambda_n, \tag{4.4}$$

Finally, the hypothesis (3.23) in Assumption 3.7 coincides with (2.20) in Assumption 2.3. So, we can apply Lemma 3.9: by using (3.26) we obtain

$$I_n^m f(x) = \int_0^t dt_1 ... \int_0^{t_{m-1}} dt_m \int p_{t-t_1, t_1 - t_2, \ldots, t_m}^{n,m}(x,y) f(y) dy.$$

We denote

$$\phi_t^{n,m_0}(x,y)=p_t^n(x,y)+\sum_{m=1}^{m_0-1}\int_0^t dt_1...\int_0^{t_{m-1}}dt_m p^{n,m}_{t-t_1,t_2-t_1,...,t_m}(x,y)$$

so that (4.3) reads

$$\int f(y)P_t(x,dy)=\int f(y)\phi_t^{n,m_0}(x,y)dy+R_n^{m_0}f(x).$$

We recall that $\Psi_{\eta,\kappa}$ is defined in (2.24) and we define the measures on $\mathbb{R}^d\times\mathbb{R}^d$ defined by

$$\mu^{\eta,\kappa}(dx,dy)=\Psi_{\eta,\kappa}(x,y)P_t(x,dy)dx \quad \text{and} \quad \mu_n^{\eta,\kappa,m_0}(dx,dy)=\Psi_{\eta,\kappa}(x,y)\phi_t^{n,m_0}(x,y)dxdy.$$

So, the proof consists in applying Lemma 3.4 to $\mu=\mu^{\eta,\kappa}$ and $\mu_n=\mu_n^{\eta,\kappa,m_0}$.

Step 2: analysis of the principal term. We study here the estimates for $f_n(x,y)=\Psi_{\eta,\kappa}\phi_t^{n,m_0}(x,y)$ which are required in (3.7).

We first use (3.27) in order to get estimates for $p^{n,m}_{t-t_1,t_2-t_1,...,t_m}(x,y)$. We fix $q_1,q_2\in\mathbb{N},\kappa\geq 0,p>1$ and we recall that in Lemma 3.9 we introduced $\overline{q}=q_1+q_2+(a+b)(m_0-1)$. Moreover in Lemma 3.9 one produces χ such that (3.27) holds true: for every multi-index β with $|\beta|\leq q_2$

$$\begin{aligned}&\left\|\psi_\kappa\partial_x^\beta p^{n,m}_{t-t_1,t_1-t_2,...,t_m}(x,\cdot)\right\|_{q_1,p}\\ &\quad\leq C\Big(\frac{1}{\lambda_n t}\Big)^{\theta_0(q_1+q_2+d+2\theta_1)}\times\left(\varepsilon_n\Lambda_n\Big(\frac{1}{\lambda_n t}\Big)^{\theta_0(a+b)}\right)^m\psi_\chi(x).\end{aligned}$$

We recall the constant defined in (2.22):

$$\Phi_n(\delta)=\varepsilon_n\Lambda_n\times\frac{1}{\lambda_n^{\theta_0(a+b+\delta)}}.$$

Denote

$$\xi_1(q)=q+d+2\theta_1+m_0(a+b),\qquad \omega_1(q)=q+d+2\theta_1.$$

With this notation, if $|\beta|\leq q_2$ we have

$$\begin{aligned}\left\|\psi_\kappa\partial_x^\beta\phi_t^{n,m_0}(x,\cdot)\right\|_{q_1,p} &\leq C\Big(\frac{1}{\lambda_n t}\Big)^{\theta_0(q_1+q_2+d+2\theta_1)}\times\left(\varepsilon_n\Lambda_n\Big(\frac{1}{\lambda_n t}\Big)^{\theta_0(a+b)}\right)^{m_0}\psi_\chi(x) &(4.5)\\ &= Ct^{-\theta_0\xi_1(q_1+q_2)}\lambda_n^{-\theta_0\omega_1(q_1+q_2)}\Phi_n^{m_0}(0)\psi_\chi(x). &(4.6)\end{aligned}$$

We take $l=2h+q,l'=2h$ and we take $q(l)=l+(a+b)m_0$. Moreover we fix q_1 and q_2 (so $q=q_1+q_2\leq l$) and we take χ to be the one in (4.5). Moreover we take η is sufficiently large in order to have $p\eta-2h-p\chi\geq d+1$. This guarantees that

$$\int_{\mathbb{R}^d}\frac{dx}{\psi_{p\eta-l'-p\chi}(x)}=C<\infty. \tag{4.7}$$

By (2.6) and (4.5)

$$\begin{aligned}\|\Psi_{\eta,\kappa}\phi_t^{n,m_0}\|_{l,l',p}^p &\le C\sum_{|\alpha|+|\beta|\le l}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Psi_{\eta,\kappa}^p(x,y)\left|\partial_x^\alpha\partial_y^\beta\phi_t^{n,m_0}(x,y)\right|^p\psi_{l'}(x)\psi_{l'}(y)dydx\\
&= C\sum_{|\alpha|+|\beta|\le l}\int_{\mathbb{R}^d}\frac{1}{\psi_{p\eta-l'}(x)}\int_{\mathbb{R}^d}\left|\psi_{\kappa+l'/p}(y)\partial_x^\alpha\partial_y^\beta\phi_t^{n,m_0}(x,y)\right|^p dydx\\
&\le C\sum_{|\alpha|+|\beta|\le l}\int_{\mathbb{R}^d}\frac{1}{\psi_{p\eta-l'}(x)}\left\|\psi_{\kappa+l'/p}\partial_x^\alpha\phi_t^{n,m_0}(x,\cdot)\right\|_{|\beta|,p}^p dx\\
&\le C(t^{-\theta_0\xi_1(l)}\lambda_n^{-\theta_0\omega_1(l)}\Phi_n(0))^{pm_0}\int_{\mathbb{R}^d}\frac{dx}{\psi_{p\eta-l'-p\chi}(x)}.\end{aligned}$$

We conclude that

$$\|\Psi_{\eta,\kappa}\phi_t^{n,m_0}\|_{2h+q,2h,p}\le Ct^{-\theta_0\xi_1(q+2h)}\times\lambda_n^{-\theta_0\omega_1(q+2h)}\Phi_n^{m_0}(0)=:\theta(n). \tag{4.8}$$

By (2.23) $\theta(n)\uparrow+\infty$ and $\Theta\theta(n)\ge\theta(n+1)$ with

$$\Theta=\gamma^{\theta_0((a+b)m_0+q+2h+d+2\theta_1)+m_0}\ge 1.$$

In the following we will choose h sufficiently large, depending on δ_*,m_0,q,d and p. So Θ is a constant depending on $\delta_*,m_0,q,d,a,b,\gamma$ and p, as the constants considered in the statement of our theorem.

Step 3: analysis of the remainder. We study here $d_{m_0}(n):=d_{(a+b)m_0}(\mu^{\eta,\kappa},\mu_n^{\eta,\kappa,m_0})$ as required in (3.8): we prove that, if $\eta\ge\kappa+d+1$, then

$$d_{m_0}(n)\le C(\Lambda_n\varepsilon_n)^{m_0}\le\lambda_n^{\theta_0(a+b+\delta_*)m_0}\Phi_n^{m_0}(\delta_*). \tag{4.9}$$

Using first (A_1) and (A_2) (see (2.14) and (2.16)) and then (A_4) (see (2.21)) we obtain

$$\left\|\prod_{i=0}^{m_0-1}(P_{t_i-t_{i+1}}^n\Delta_n)P_{t_{m_0}}f\right\|_{0,-\kappa,\infty}\le C\|f\|_{(a+b)m_0,-\kappa,\infty}(\Lambda_n\varepsilon_n)^{m_0}$$

which gives

$$\|R_n^{m_0}f\|_{0,-\kappa,\infty}\le C\|f\|_{(a+b)m_0,-\kappa,\infty}(\Lambda_n\varepsilon_n)^{m_0}.$$

Using now the equivalence between (2.11) and (2.12) we obtain

$$\left\|\frac{1}{\psi_\kappa}R_n^{m_0}(\psi_\kappa f)\right\|_\infty\le C\|f\|_{(a+b)m_0,\infty}(\Lambda_n\varepsilon_n)^{m_0}. \tag{4.10}$$

We take now $g\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, we denote $g_x(y)=g(x,y)$, and we write

$$\begin{aligned}&\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}g(x,y)(\mu^{\eta,\kappa}-\mu_n^{,\kappa,m_0})(dx,dy)\right|\\
&\le\int_{\mathbb{R}^d}\frac{dx}{\psi_\eta(x)}\left|\int_{\mathbb{R}^d}g_x(y)\psi_\kappa(y)(P_t(x,dy)-\phi_t^{n,m_0}(x,y))dy\right|\\
&\le\int_{\mathbb{R}^d}\frac{dx}{\psi_{\eta-\kappa}(x)}\left|\frac{1}{\psi_\kappa(x)}R_n^{m_0}(\psi_\kappa g_x)(x)\right|dx\\
&\le C\sup_{x\in\mathbb{R}^d}\|g_x\|_{(a+b)m_0,\infty}(\Lambda_n\varepsilon_n)^{m_0}\end{aligned}$$

the last inequality being a consequence of (4.10) and of $\eta-\kappa\geq d+1$. Now (4.9) is proved because $\sup_{x\in\mathbb{R}^d}\|g_x\|_{(a+b)m_0,\infty}\leq\|g\|_{(a+b)m_0,\infty}$.

Step 4: use of Lemma 3.4 and proof of A. and B. We recall that ρ_h is defined in (4.2) and we estimate

$$d_{m_0}(n)\times\theta(n)^{\rho_h}\leq Ct^{-\theta_0\xi_2(h)}\lambda_n^{\theta_0\omega_2(h)}\Phi_n^{m_0(1+\rho_h)}(\delta_*)$$

with

$$\xi_2(h)=\rho_h\xi_1(q+2h)=\rho_h(q+2h+d+2\theta_1+m_0(a+b))$$

and

$$\begin{aligned}\omega_2(h) &= (a+b+\delta_*)m_0-\rho_h(q+2h+d+2\theta_1)\\ &= \delta_*m_0-\frac{(a+b)m_0+q+2d/p_*}{2h}(q+d+2\theta_1)-(q+2d/p_*).\end{aligned}$$

By our choice of m_0 we have

$$\delta_*m_0>q+2d/p_*$$

so, taking h sufficiently large we get $\omega_2(h)>0$. And we also have $\xi_2(h)\leq\xi_3:=(a+b)m_0+q+\frac{2d}{p_*}+\varepsilon_*$ and $\rho_h\leq\varepsilon_*$. So we finally get

$$d_{m_0}(n)\times\theta(n)^{\rho_h}\leq Ct^{-\theta_0\xi_3}\Phi_n^{m_0(1+\varepsilon_*)}(\delta_*). \tag{4.11}$$

The above inequality guarantees that (3.8) holds so that we may use Lemma 3.4. We take $\eta>\kappa+d$ and, using (A_4) (see (2.21)) we obtain

$$|\mu^{\eta,\kappa}|=\int_{\mathbb{R}^2}\frac{\psi_\eta(x)}{\psi_\kappa(y)}P_t(x,dy)dx\leq C\int_{\mathbb{R}}\frac{dx}{\psi_{\kappa-\eta}(x)}<\infty.$$

Then, $A(\delta)<C$ (see (3.9)). One also has $B(\varepsilon)<\infty$ (see (3.10)) and finally (see (3.11))

$$C_{h,n_*}(\varepsilon)\leq Ct^{-\theta_0\xi_3}\Phi_n^{m_0(1+\varepsilon_*)}(\delta_*).$$

We have used here (4.11). For large h we also have

$$\theta(n)^{\rho_h}\leq C(\lambda_nt)^{-\theta_0((a+b)m_0+q+\frac{2d}{p_*})(1+\varepsilon_*)}\Phi_n^{\varepsilon_*}(0).$$

Now (3.12) gives (2.27). So **A** and **B** are proved.

Step 5: proof of C. We apply **B.** with q replaced by $\bar{q} = q+1$, so $\Psi_{\eta,\kappa} p_t \in W^{\bar{q},p}(\mathbb{R}^d \times \mathbb{R}^d) = W^{\bar{q},p}(\mathbb{R}^{2d})$. Since $\bar{q} > 2d/p$ (here the dimension is $2d$), we can use the Morrey's inequality: for every α, β with $|\alpha|+|\beta| \leq \lfloor \bar{q} - 2d/p \rfloor = q$, then $|\partial_x^\alpha \partial_y^\beta (\Psi_{\eta,\kappa} p_t)(x,y)| \leq C \|\Psi_{\eta,\kappa} p_t\|_{\bar{q},p}$. By (2.27), one has (with $\bar{\mathfrak{m}} = 1 + \frac{\bar{q}+2d/p_*}{\delta_*}$)

$$\Big|\partial_x^\alpha \partial_y^\beta (\Psi_{\eta,\kappa} p_t)(x,y)\Big| \leq C\Big(1 + \Big(\frac{1}{\lambda_n t}\Big)^{(a+b)\bar{\mathfrak{m}}+\bar{q}+2d/p_*} + \Phi_{t,n,r}^{\bar{\mathfrak{m}}}(\delta_*)\Big)^{(1+\varepsilon_*)}$$

i.e. (using (2.6)),

$$\Big|\partial_x^\alpha \partial_y^\beta p_t(x,y)\Big| \leq C\Big(1 + \Big(\frac{1}{\lambda_n t}\Big)^{(a+b)\bar{\mathfrak{m}}+\bar{q}+2d/p_*} + \Phi_{t,n,r}^{\bar{\mathfrak{m}}}(\delta_*)\Big)^{(1+\varepsilon_*)} \times \frac{1}{\Psi_{\eta,\kappa}(x,y)}$$

Now, by a standard calculus, $\Psi_{\eta,\kappa}(x,y) \geq C_\kappa \frac{\psi_\kappa(x-y)}{\psi_{\eta+\kappa}(x)}$ (use that $\psi_\kappa(x-y) \leq C_\kappa \psi_\kappa(x)\psi_\kappa(-y) = C_\kappa \psi_\kappa(x)\psi_\kappa(y)$), so (2.29) follows. □

4.2 Proof of Theorem 2.7

By applying Theorem 2.5, $P_t(x,dy) = p_t(x,y)dy$ and p_t satisfies (2.27), which we rewrite here as

$$\|\Psi_{\eta,\kappa} p_t\|_{q,p} \leq C t^{-\theta_*(q+\theta_1)},$$

where $\theta_* = \theta_0(1+\frac{a+b}{\delta})(1+\varepsilon)$ and θ_1 is computed from (2.27) (the precise value is not important here). The constant C in the above inequality depends on $\kappa, \eta, \varepsilon, \delta, q$. Moreover, by choosing $\eta > \kappa + d$,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \Psi_{\eta,\kappa}(x,y) p_t(x,y) dx dy = \int_{\mathbb{R}^d} \frac{1}{\psi_\eta(x)} \times P_t \psi_\kappa(x) dx \leq \int_{\mathbb{R}^d} \frac{1}{\psi_{\eta-\kappa}(x)} dx = m < \infty.$$

So, Lemma 3.3 (recall that we are working here with $\mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$) gives

$$\|\Psi_{\eta,\kappa} p_t\|_{q,p} \leq C_* t^{-\theta_*(q+2d/p_*)}.$$

We choose now $p > 2d$ and by Morrey's inequality,

$$\|\Psi_{\eta,\kappa} p_t\|_{q,\infty} \leq C \|\Psi_{\eta,\kappa} p_t\|_{q+1,p} \leq C t^{-\theta_*(q+1+2d/p_*)}.$$

By taking $p = 2d/(1-\varepsilon)$, we get

$$\|\Psi_{\eta,\kappa} p_t\|_{r,\infty} \leq C t^{-\theta_*(r+2d+\varepsilon)},$$

where C denotes here a constant depending on $\kappa, \eta, \varepsilon$. This gives that, for every $x, y \in \mathbb{R}^d$ and for every multi-index α and β,

$$\Big|\partial_x^\alpha \partial_y^\beta p_t(x,y)\Big| \leq C \times t^{-\theta_*(|\alpha|+|\beta|+2d+\varepsilon)} \frac{\psi_\eta(x)}{\psi_\kappa(y)}.$$

The statement now follows from (2.4). □

4.3 Proof of Theorem 2.11

In this section we give the proof of Theorem 2.11.

Step 1. Let

$$\omega_t(dt_1,...,dt_m) = \frac{m!}{t^m} 1_{\{0<t_1<...<t_m<t\}} dt_1....dt_m$$

and (with $t_{m+1} = t$)

$$I_m(f)(x) = \mathbb{E}\Big(1_{\{N(t)=m\}} \int_{\mathbb{R}_+^m} \Big(\prod_{i=0}^{m-1} P_{t_{m-i+1}-t_{m-i}} U_{Z_{m-i}}\Big) P_{t_1} f(x) \omega_t(dt_1,...,dt_m))\Big),$$

Since, conditionally to $N(t) = m$, the law of $(T_1,...,T_m)$ is given by $\omega_t(dt_1,...,dt_m)$, it follows that

$$\overline{P}_t f(x) = \sum_{m=0}^{\infty} I_m(f)(x) = \sum_{m=0}^{m_0} I_m(f)(x) + R_{m_0} f(x)$$

with

$$R_{m_0} f(x) = \sum_{m=m_0+1}^{\infty} I_m(f)(x).$$

Step 2. We analyze first the regularity of $I_m(f)$. We apply Lemma 3.9. Here $S_t = P_t$, so assumptions 3.6 and 3.7 hold due to assumptions 2.8 and 2.9 respectively. Moreover, here $U_i = U_{m-Z_i}$, so Assumption 3.5 is satisfied uniformly in ω as observed in Remark 2.10. Notice that $a = b = 0$ in our case. Then Lemma 3.9 gives

$$\Big(\prod_{i=0}^{m-1} P_{t_{m-i+1}-t_{m-i}} U_{Z_{m-i}}\Big) P_{t_1} f(x) = \int p_{t_1,t_2-t_1,...,t-t_m}(x,y) f(y) dy$$

and, for $q_1, q_2 \in \mathbb{N}, \kappa \geq 0, p > 1$ and $|\beta| \leq q_2$ we have

$$\left\| \partial_x^\beta p_{t_1,t_2-t_1,...,t-t_m}(x,\cdot) \right\|_{q_1,\kappa,p} \leq \theta_{q,t}(m) \times \psi_\chi(x). \tag{4.12}$$

with

$$\theta_{q,t}(m) = \frac{C \times m^{q+d+2\theta_1}}{(\lambda t)^{\theta_0(q+d+2\theta_1)}} \times C_{q,\chi,p,\infty}^{m-1}(P,U).$$

Here C and χ are constants which depend on q_1, q_2 and κ. We notice that

$$\theta_{q,t}(m+1) \leq C \times C_{q,\chi,p,\infty}(P,U) \times \theta_{q,t}(m).$$

We summarize: for each fixed q, χ, κ, p and each $\delta > 0$ there exists some constants $\Theta \geq 1$ and $Q \geq 1$ (depending on q, χ, κ, p and δ but not on m and on t) such that for every $m \in \mathbb{N}$

$$\theta_{q,t}(m+1) \leq \Theta \times \theta_{q,t}(m), \qquad \textit{and} \tag{4.13}$$

$$\theta_{q,t}(m) \leq \frac{Q^{mq}}{(\lambda t)^{\theta_0(q+d+2\theta_1)}}. \tag{4.14}$$

We define now

$$\phi_t^{m_0}(x,y) = \sum_{m=0}^{m_0} \int_{\mathbb{R}_+^m} p_{t_1,t_2-t_1,...,t-t_m}(x,y) \omega_t(dt_m...dt_1).$$

Using (4.12), standard computations give: for every $l, l' \in \mathbb{N}$, $p > 1$ and $\kappa \in \mathbb{N}$ there exists $\eta_0 \in \mathbb{N}$ such that for every $\eta > \eta_0$,

$$\|\Psi_{\eta,\kappa}\phi_t^{m_0}\|_{l,l',p} \leq \theta_{l,t}(m_0). \tag{4.15}$$

Step 3. We fix $\eta > \eta_0$ and $\eta > \kappa + d$ and we define the measures

$$\mu^{\eta,\kappa}(dx, dy) = \Psi_{\eta,\kappa}(x, y)\overline{P}_t(x, dy)dx \quad and \quad \mu^{\eta,\kappa,m_0}(dx, dy) = \Psi_{\eta,\kappa}(x, y)\phi_t^{m_0}(x, y)dxdy.$$

Let $g = g(x, y)$ be a bounded function and set $g_x(y) = g(x, y)$. We have,

$$\begin{aligned}\Big|\int g d\mu^{\eta,\kappa} - \int g d\mu^{\eta,\kappa,m_0}\Big| &\leq \int \frac{1}{\psi_{\eta-\kappa}(x)}\Big\|\frac{1}{\psi_\kappa}R_{m_0}(\psi_\kappa g_x)\Big\|_\infty dx \\ &\leq \sum_{m \geq m_0+1} \int \frac{1}{\psi_{\eta-\kappa}(x)}\Big\|\frac{1}{\psi_\kappa}I_m(\psi_\kappa g_x)\Big\|_\infty dx.\end{aligned}$$

We deal with the norm in the integral above. By iterating (2.31) and (2.39) (with $q = 0$) we get

$$\Big\|\Big(\prod_{i=0}^{m-1} P_{t_{m-i+1}-t_{m-i}}U_{Z_{m-i}}\Big)P_{t_1}(\psi_\kappa g_x)\Big\|_{0,-\kappa,\infty} \leq c_\kappa^m \|\psi_\kappa g_x\|_{0,-\kappa,\infty} \leq c_\kappa^m \|g\|_\infty,$$

where $c_\kappa > 0$ is a constant depending on the constants appearing in (2.31) and (2.39). Therefore, we obtain

$$\Big\|\frac{1}{\psi_\kappa}I_m(\psi_\kappa g_x)\Big\|_\infty \leq c_\kappa^m P(N(t) = m)\|g\|_\infty.$$

Since $\eta - \kappa > d$, it follows that for every $m_0 \geq 1$ (recall that $t < 1$),

$$d_0(\mu^{\eta,\kappa}, \mu^{\eta,\kappa,m_0}) \leq \sum_{m \geq m_0+1} \frac{(c_\kappa\rho)^m}{m!} \leq \frac{(c_\kappa\rho)^{m_0}}{m_0!}\, e^{c_\kappa\rho}.$$

So, for $m \geq 1$ we have, for every l

$$\begin{aligned}\sup_{m\geq1} d_0(\mu^{\eta,\kappa}, \mu^{\eta,\kappa,m}) \times \theta_{l,t}(m)^r &\leq \frac{1}{(\lambda t)^{\theta_0(l+d+2\theta_1)r}} \times \sup_{m\geq1} \frac{(c_\kappa\rho Q^r)^m}{m!}\, e^{c_\kappa\rho} \\ &\leq \frac{e^{c_\kappa\rho(1+Q^r)}}{(\lambda t)^{\theta_0(l+d+2\theta_1)r}}.\end{aligned} \tag{4.16}$$

We now use Lemma 3.4 and we get $\mu^{\eta,\kappa}(dx, dy) = p^{\eta,\kappa}(x, y)dxdy$ with $p^{\eta,\kappa} \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. And one concludes that $\overline{P}_t(x, dy) = \overline{p}_t(x, y)dxdy$ with $\overline{p}_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

We will now obtain estimates of $\overline{p}_t$. We fix $h \in \mathbb{N}$ (to be chosen sufficiently large, in a moment) and we recall that in (3.6) we have defined $\rho_h = (q + 2d/p_*)/2h$ (in our case $k = 0$ and we work on $R^d \times R^d \sim R^{2d}$). So, with the notation from (3.11) (with $n_* = 1$)

$$C_{h,1}(\varepsilon) = \frac{e^{c_\kappa\rho(1+Q^{\rho_h+\varepsilon})}}{(\lambda t)^{\theta_0(2h+q+d+2\theta_1)(\rho_h+\varepsilon)}}.$$

We have used here (4.16) with $l = 2h+q$ and $r = \rho_h+\varepsilon$. Then by (3.12) with $n_* = 1$, for every $\delta > 0$

$$\|p^{\eta,\kappa}\|_{q,p} \leq C(\Theta + \theta_{2h+q,t}^{\rho_h(1+\delta)}(1) + C_{h,1}(\varepsilon)).$$

Taking h sufficiently large we have

$$C_{h,1}(\varepsilon) \leq \frac{e^{2c_\kappa \rho}}{(\lambda t)^{\theta_0(q+2d/p_*)(1+\delta)}}.$$

and, for $\delta > 0$,

$$\theta_{2h+q,t}^{\rho_h(1+\delta)}(c(r_h)\rho) \leq \frac{Q^{\rho_h+\varepsilon}}{(\lambda t)^{(2h+q+d/p_*)\rho_h(1+\delta)}} \leq C \times \frac{e^{2c_\kappa \rho}}{(\lambda t)^{(q+2d/p_*)(1+\delta)}}.$$

Since $\rho \geq 1$ we conclude that

$$\|p^{\eta,\kappa}\|_{q,p} \leq C \times \frac{e^{2c_\kappa \rho}}{(\lambda t)^{(q+2d/p_*)(1+\delta)}},$$

C denoting a constant which is independent of ρ. We take now $p = 2d + \varepsilon$ and, using now Morrey's inequality

$$\|p^{\eta,\kappa}\|_{q,\infty} \leq \|p^{\eta,\kappa}\|_{q+1,p} \leq C \times \frac{e^{2c_\kappa \rho}}{(\lambda t)^{(q+2d)(1+\delta)}}.$$

This proves (2.43). $\square$

A Appendix

A.1 Weights

We denote

$$\psi_k(x) = (1 + |x|^2)^k. \tag{A.1}$$

Lemma A.1 *For every multi-index α there exists a constant C_α such that*

$$\Big|\partial^\alpha\Big(\frac{1}{\psi_k}\Big)\Big| \leq \frac{C_\alpha}{\psi_k}. \tag{A.2}$$

Moreover, for every q there is a constant $C_q \geq 1$ such that for every $f \in C_b^\infty(\mathbb{R}^d)$

$$\frac{1}{C_q}\sum_{0\leq|\alpha|\leq q}\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big| \leq \sum_{0\leq|\alpha|\leq q}\frac{1}{\psi_k}\,|\partial^\alpha f| \leq C_q \sum_{0\leq|\alpha|\leq q}\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big|. \tag{A.3}$$

Proof. One checks by recurrence that

$$\partial^\alpha\Big(\frac{1}{\psi_k}\Big) = \sum_{q=1}^{|\alpha|}\frac{P_{\alpha,q}}{\psi_{k+q}}$$

where $P_{\alpha,q}$ is a polynomial of order q. And since

$$\frac{(1+|x|)^q}{(1+|x|^2)^{q+k}} \leq \frac{C}{(1+|x|^2)^k}$$

the proof (A.2) is completed. In order to prove (A.3) we write

$$\partial^\alpha\Big(\frac{f}{\psi_k}\Big)=\frac{1}{\psi_k}\partial^\alpha f+\sum_{\substack{(\beta,\gamma)=\alpha\\ |\beta|\geq 1}}c(\beta,\gamma)\partial^\beta\Big(\frac{1}{\psi_k}\Big)\partial^\gamma f.$$

This, together with (A.2) implies

$$\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big|\leq C\sum_{0\leq|\gamma|\leq|\alpha|}\frac{1}{\psi_k}\left|\partial^\gamma f\right|$$

so the first inequality in (A.3) is proved. In order to prove the second inequality we proceed by recurrence on q. The inequality is true for $q=0$. Suppose that it is true for $q-1$. Then we write

$$\frac{1}{\psi_k}\partial^\alpha f=\partial^\alpha\Big(\frac{f}{\psi_k}\Big)-\sum_{\substack{(\beta,\gamma)=\alpha\\ |\beta|\geq 1}}c(\beta,\gamma)\partial^\beta\Big(\frac{1}{\psi_k}\Big)\partial^\gamma f$$

and we use again (A.2) in order to obtain

$$\frac{1}{\psi_k}\left|\partial^\alpha f\right|\leq\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big|+C\sum_{|\gamma|<|\alpha|}\frac{1}{\psi_k}\left|\partial^\gamma f\right|\leq C\sum_{0\leq|\beta|\leq q}\Big|\partial^\beta\Big(\frac{f}{\psi_k}\Big)\Big|$$

the second inequality being a consequence of the recurrence hypothesis. □

Remark A.2 *The assertion is false if we define* $\psi_k(x)=(1+|x|)^k$ *because* $\partial_i\partial_j\left|x\right|=\frac{\delta_{i,j}}{|x|}-\frac{x_ix_j}{|x|^2}$ *blows up in zero.*

We look now to ψ_k itself.

Lemma A.3 *For every multi-index* α *there exists a constant* C_α *such that*

$$\left|\partial^\alpha\psi_k\right|\leq C_\alpha\psi_k. \tag{A.4}$$

Moreover, for every q *there is a constant* $C_q\geq 1$ *such that for every* $f\in C_b^\infty(\mathbb{R}^d)$

$$\frac{1}{C_q}\sum_{0\leq|\alpha|\leq q}\left|\partial^\alpha(\psi_k f)\right|\leq\sum_{0\leq|\alpha|\leq q}\psi_k\left|\partial^\alpha f\right|\leq C_q\sum_{0\leq|\alpha|\leq q}\left|\partial^\alpha(\psi_k f)\right|. \tag{A.5}$$

Proof. One proves by recurrence that, if $|\alpha|\geq 1$ then $\partial^\alpha\psi_k=\sum_{q=1}^{|\alpha|}\psi_{k-q}P_q$ with P_q a polynomial of order q. Since $1+|x|\leq 2(1+|x|^2)$ it follows that $|P_q|\leq C\psi_q$ and (A.4) follows. Now we write

$$\psi_k\partial^\alpha f=\partial^\alpha(\psi_k f)-\sum_{\substack{(\beta,\gamma)=\alpha\\ |\beta|\geq 1}}c(\beta,\gamma)\partial^\beta\psi_k\partial^\gamma f$$

and the same arguments as in the proof of (A.3) give (A.5).

A.2 Semigroup estimates

We consider a semigroup P_t on $C^\infty(\mathbb{R}^d)$ such that $P_t f(x) = \int f(y) P_t(x,dy)$ where $P_t(x,dy)$ is a probability transition kernel and we denote by P_t^* its formal adjoint.

Assumption A.4 *There exists $Q \geq 1$ such that for every $t \leq T$ and every $f \in C^\infty(\mathbb{R}^d)$*

$$\left\| P_t f \right\|_1 \leq Q \left\| f \right\|_1 . \tag{A.6}$$

Moreover, for every $k \in \mathbb{N}$ there exists $K_k \geq 1$ such that for every $x \in \mathbb{R}^d$

$$\left| P_t(\psi_k)(x) \right| \leq K_k \psi_k(x). \tag{A.7}$$

Lemma A.5 *Under Assumption A.4, one has*

$$\left\| \psi_k P_t^*(f/\psi_k) \right\|_p \leq K_{kp}^{1/p} Q^{1/p_*} \left\| f \right\|_p . \tag{A.8}$$

Proof. Using Hölder's inequality, the identity $\psi_k^p = \psi_{kp}$, and (A.7)

$$\left| P_t(\psi_k g)(x) \right| \leq \left| P_t(\psi_k^p)(x) \right|^{1/p} \left| P_t(\left| g \right|^{p_*})(x) \right|^{1/p_*} \leq K_{kp}^{1/p} \psi_k(x) \left| P_t(\left| g \right|^{p_*})(x) \right|^{1/p_*} .$$

Then, using (A.6)

$$\begin{aligned} \left\| \frac{1}{\psi_k} P_t(\psi_k g) \right\|_{p_*} &\leq K_{kp}^{1/p} \left\| \left| P_t(\left| g \right|^{p_*}) \right|^{1/p_*} \right\|_{p_*} = K_{kp}^{1/p} (\left\| P_t(\left| g \right|^{p_*}) \right\|_1)^{1/p_*} \\ &\leq K_{kp}^{1/p} Q^{1/p_*} (\left\| \left| g \right|^{p_*} \right\|_1)^{1/p_*} = K_{kp}^{1/p} Q^{1/p_*} \left\| g \right\|_{p_*} . \end{aligned}$$

Using Hölder's inequality first and the above inequality we obtain

$$\begin{aligned} \left| \langle g, \psi_k P_t^*(f/\psi_k) \rangle \right| &= \left| \left\langle \frac{1}{\psi_k} P_t(g\psi_k), f \right\rangle \right| \leq \left\| f \right\|_p \left\| \frac{1}{\psi_k} P_t(g\psi_k) \right\|_{p_*} \\ &\leq K_{kp}^{1/p} Q^{1/p_*} \left\| g \right\|_{p_*} \left\| f \right\|_p . \end{aligned}$$

□

We consider also the following hypothesis.

Assumption A.6 *There exists $\rho > 1$ such that for every $q \in \mathbb{N}$ there exists $D_{(q)}^*(\rho) \geq 1$ such that for every $x \in \mathbb{R}^d$*

$$\sum_{|\alpha| \leq q} \left| \partial^\alpha P_t^* f(x) \right| \leq D_{(q)}^*(\rho) \sum_{|\alpha| \leq q} (P_t^*(\left| \partial^\alpha f \right|^\rho)(x))^{1/\rho}. \tag{A.9}$$

Proposition A.7 *Suppose that Assumption A.4 and A.6 hold. Then for every $k, q \in \mathbb{N}$ and $p > \rho$ there exists a universal constant C (depending on k and q only) such that*

$$\left\| \psi_k P_t^*(f/\psi_k) \right\|_{q,p} \leq C K_{k\rho p}^{1/p} Q^{(p-\rho)/\rho p} D_{(q)}^*(\rho) \left\| f \right\|_{q,p} . \tag{A.10}$$

Proof. We will prove (A.10) first. Let α with $|\alpha| \leq q$. By (A.9)

$$\begin{aligned}
|\partial^\alpha(\psi_k P_t^*(f/\psi_k)(x))| &\leq C\psi_k(x)\sum_{|\gamma|\leq q}|\partial^\gamma(P_t^*(f/\psi_k)(x))| \\
&\leq CD^*_{(q)}(\rho)\psi_k(x)\sum_{|\beta|\leq q}(P_t^*(\left|\partial^\beta(f/\psi_k)\right|^\rho)(x))^{1/\rho} \\
&= CD^*_{(q)}(\rho)\sum_{|\beta|\leq q}(\psi_{\rho k}(x)P_t^*(\left|\partial^\beta(f/\psi_k)\right|^\rho)(x))^{1/\rho} \\
&= CD^*_{(q)}(\rho)\sum_{|\beta|\leq q}(\psi_{\rho k}(x)P_t^*(g/\psi_{\rho k})(x))^{1/\rho}
\end{aligned}$$

with

$$g(x) = \psi_{\rho k}(x)\left|\partial^\beta(f/\psi_k)(x)\right|^\rho = \left|\psi_k(x)\partial^\beta(f/\psi_k)(x)\right|^\rho.$$

Using (A.8)

$$\left\|(\psi_{\rho k}P_t^*(g/\psi_{\rho k}))^{1/\rho}\right\|_p = \left\|\psi_{\rho k}P_t^*(g/\psi_{\rho k})\right\|_{p/\rho}^{1/\rho} \leq K_{k\rho p}^{1/p}Q^{(p-\rho)/\rho p}\left\|g\right\|_{p/\rho}^{1/\rho}.$$

And we have

$$\left\|g\right\|_{p/\rho}^{1/\rho} = (\int\left|\psi_k(x)\partial^\beta(f/\psi_k)(x)\right|^p dx)^{1/p} \leq C\sum_{|\gamma|\leq q}(\int|\partial^\gamma f(x)|^p dx)^{1/p} = C\left\|f\right\|_{q,p}.$$

We conclude that

$$\left\|\psi_k P_t^*(f/\psi_k)\right\|_{q,p} \leq CK_{k\rho p}^{1/p}Q^{(p-\rho)/\rho p}D^*_{(q)}(\rho)\left\|f\right\|_{q,p}.$$

□

A.3 Integration by parts

We consider a function $\phi \in C^\infty(\mathbb{R}^d,\ \mathbb{R}^d)$ such that $\partial_j\phi \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $j = 1, \ldots, d$. We denote $\nabla\phi$ the $d \times d$ matrix field whose (i,j) entry is $\partial_j\phi^i$ and $\sigma(\phi) = \nabla\phi(\nabla\phi)^*$.

Lemma A.8 *We suppose that $\sigma(\phi)$ is invertible and we denote $\gamma(\phi) = \sigma^{-1}(\phi)$. Then*

$$\int(\partial_i f)(\phi(x))g(x)dx = \int f(\phi(x))H_i(\phi,g)(x)dx \tag{A.11}$$

with

$$H_i(\phi,g) = -\sum_{k=1}^d\partial_k\left(g\sum_{j=1}^d\gamma^{i,j}(\phi)\partial_k\phi^j\right). \tag{A.12}$$

Moreover, for a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$ we define

$$H_\alpha(\phi,g) = H_{\alpha_m}(\phi, H_{(\alpha_1,\ldots,\alpha_{m-1})}(\phi,g)) \tag{A.13}$$

and we obtain

$$\int(\partial^\alpha f)(\phi(x))g(x)dx = \int f(\phi(x))H_\alpha(\phi,g)(x)dx \tag{A.14}$$

Proof. The proof is standard: we use the chain rule and we obtain $\nabla(f(\phi)) = (\nabla\phi)^*(\nabla f)(\phi)$. By multiplying with $\nabla\phi$ first and with $\gamma(\phi)$ then, we get $(\nabla f)(\phi) = \gamma(\phi)\nabla\phi\nabla(f(\phi))$. Using standard integration by parts, (A.11) and (A.12) hold. And (A.13) follows by iteration. □

Our aim now is to give estimates of $\left|H_a(\phi, g)(x)\right|_q$. We use the notation introduced in (2.1) and for $q \in \mathbb{N}$, we denote

$$C_q(\phi)(x) = \frac{1 \vee |\phi(x)|_{1,q+2}^{2d-1}}{1 \wedge (\det \sigma(\phi)(x))^{q+1}}. \tag{A.15}$$

Lemma A.9 *For every multi index α and every $q \in \mathbb{N}$ there exists a universal constant $C \geq 1$ such that*

$$\left|H_\alpha(\phi, g)(x)\right|_q \leq C \left|g(x)\right|_{q+|\alpha|} \times C_{q+|\alpha|}^{|\alpha|}(\phi)(x). \tag{A.16}$$

Proof. We begin with some simple computational rules:

$$\left|f(x)g(x)\right|_q \leq C \sum_{k_1+k_2=q} \left|f(x)\right|_{k_1} \left|g(x)\right|_{k_2}, \tag{A.17}$$

$$\left|\langle\nabla f(x), \nabla g(x)\rangle\right|_q \leq C \sum_{k_1+k_2=q} \left|f(x)\right|_{1,k_1+1} \left|g(x)\right|_{1,k_2+1}, \tag{A.18}$$

$$\left|\frac{1}{g(x)}\right|_q \leq \frac{C}{|g(x)|} \sum_{l=0}^{q} \frac{\left|g(x)\right|_q^l}{\left|g(x)\right|^l}. \tag{A.19}$$

We denote by $\widehat{\sigma}^{i,j}(\phi)$ the algebraic complement and write $\gamma^{i,j}(\phi) = \widehat{\sigma}^{i,j}(\phi)/\det\sigma(\phi)$. Then, using the above computational rules we obtain

$$\left|\gamma^{i,j}(\phi)(x)\right|_q \leq C \times \frac{|\phi(x)|_{1,q+1}^{2(d-1)}}{1 \wedge (\det\sigma(\phi)(x))^{q+1}}$$

and moreover

$$\left|H_i(\phi, g)(x)\right|_q \leq C\left|g(x)\right|_{q+1} \times |\phi(x)|_{1,q+2} \times \frac{|\phi(x)|_{1,q+2}^{2(d-1)}}{1 \wedge (\det\sigma(\phi)(x))^{q+1}} \leq C\left|g(x)\right|_{q+1} \times C_q(\phi)(x).$$

Let $\alpha = (\beta, i)$. Iterating the above estimate we obtain

$$\begin{aligned}\left|H_\alpha(\phi, g)(x)\right|_q &= \left|H_i(\phi, H_\beta(\phi, g))(x)\right|_q \leq C\left|H_\beta(\phi, g)(x)\right|_{q+1} \times C_q(\phi)(x)\\ &\leq C\left|g(x)\right|_{q+|\alpha|} \times C_{q+|\alpha|}^{|\alpha|}(\phi)(x).\end{aligned}$$

□

We define now the operator $V_\phi : C_b^\infty(\mathbb{R}^d) \to C_b^\infty(\mathbb{R}^d)$ by

$$V_\phi f(x) = f(\phi(x)). \tag{A.20}$$

Lemma A.10 ***A***. *One has*

$$\left\|\frac{1}{\psi_\kappa} V_\phi(\psi_\kappa f)\right\|_{q,\infty} \leq C\psi_\kappa(\phi(0)) \left\|\phi\right\|_{1,q,\infty}^{q+2\kappa} \left\|f\right\|_{q,\infty}. \tag{A.21}$$

B. *Suppose that*

$$\inf_{x\in\mathbb{R}^d} \det\sigma(\phi)(x) \geq \varepsilon(\phi) > 0 \tag{A.22}$$

Then, for $\kappa, q \in \mathbb{N}$ and $p > 1$,

$$\left\|\psi_\kappa V_\phi^*(\frac{1}{\psi_\kappa}f)\right\|_{q,p} \leq C\psi_\kappa(\phi(0)) \times \frac{1 \vee \|\phi\|_{1,q+2,\infty}^{2dq+1+2\kappa}}{\varepsilon(\phi)^{q(q+1)+1/p_*}} \times \|f\|_{q+1,p}\,. \tag{A.23}$$

Proof. We notice first that

$$|g(\phi(x))|_q \leq C(1 \vee |\phi(x)|_{1,q}^q) \sum_{|\alpha|\leq q} |(\partial^\alpha g)(\phi(x))|\,. \tag{A.24}$$

Using (2.6) and the above inequality we obtain

$$\begin{aligned}\left|\frac{1}{\psi_\kappa(x)}V_\phi(\psi_\kappa f)(x)\right|_q &\leq \frac{C}{\psi_\kappa(x)}\,|V_\phi(\psi_\kappa f)(x)|_q \leq \frac{C(1\vee|\phi(x)|_{1,q}^q)}{\psi_\kappa(x)} \sum_{|\alpha|\leq q} |(\partial^\alpha(\psi_\kappa f))(\phi(x))| \\ &\leq \frac{C(1\vee|\phi(x)|_{1,q}^q)}{\psi_\kappa(x)} \times \psi_\kappa(\phi(x)) \sum_{|\alpha|\leq q} |(\partial^\alpha f)(\phi(x))|\,.\end{aligned}$$

And using (2.5) this gives (A.21).

B. We take now α with $|\alpha| \leq q$ and we write

$$\begin{aligned}\left\langle \partial^\alpha(\psi_\kappa V_\phi^*(\frac{1}{\psi_\kappa}f)), g\right\rangle &= (-1)^{|\alpha|}\left\langle \frac{f}{\psi_\kappa}, V_\phi(\psi_\kappa\partial^\alpha g)\right\rangle \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^d} \frac{f}{\psi_\kappa}(x)(\psi_\kappa\partial^\alpha g)(\phi(x))dx \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^d} g(\phi(x))H_\alpha\Big(\phi, \frac{f}{\psi_\kappa}\times\psi_\kappa(\phi)\Big)(x)dx.\end{aligned}$$

It follows that

$$\left|\left\langle \partial^\alpha(\psi_\kappa V_\phi^*(\frac{1}{\psi_\kappa}f)), g\right\rangle\right| \leq \|g(\phi)\|_{p_*}\left\|H_\alpha\Big(\phi, \frac{f}{\psi_\kappa}\times\psi_\kappa(\phi)\Big)\right\|_p.$$

Using (A.16) and (A.22) we obtain (recall that $|\alpha| \leq q$)

$$\begin{aligned}\left|H_\alpha\Big(\phi, \frac{f}{\psi_\kappa}\times\psi_\kappa(\phi)\Big)(x)\right| &\leq C\left|\Big(\frac{f}{\psi_\kappa}\times\psi_\kappa(\phi)\Big)(x)\right|_{q+1} \times C_q^q(\phi)(x) \\ &\leq C\,|f(x)|_{q+1}\left|\frac{1}{\psi_\kappa}\times\psi_\kappa(\phi)(x)\right|_{q+1} \times \Big(\frac{1\vee|\phi(x)|_{1,q+2}^{2d-1}}{\varepsilon(\phi)^{q+1}}\Big)^q\end{aligned}$$

By (A.24) we have

$$\begin{aligned}|\psi_\kappa(\phi)(x)|_{q+1} &\leq C(1\vee|\phi(x)|_{1,q+1}^{q+1}) \sum_{|\alpha|\leq k+1} |(\partial^\alpha\psi_\kappa)(\phi(x))| \leq C(1\vee|\phi(x)|_{1,q+1}^{q+1}) \times \psi_\kappa(\phi(x)) \\ &\leq C(1\vee|\phi(x)|_{1,q+1}^{q+1}) \times \psi_\kappa(\phi(0))(1\vee\|\nabla\phi\|_\infty^{2\kappa})\psi_\kappa(x).\end{aligned}$$

Finally

$$\left| H_\alpha\Big(\phi, \frac{f}{\psi_\kappa} \times \psi_\kappa(\phi)\Big)(x) \right| \leq C \left|f(x)\right|_{q+1} \psi_\kappa(\phi(0))(1 \vee \|\nabla\phi\|_\infty^{2\kappa}) \times \frac{1 \vee |\phi(x)|_{1,q+2}^{2dq+1}}{\varepsilon(\phi)^{q(q+1)}}$$
$$\leq C \left|f(x)\right|_{q+1} \psi_\kappa(\phi(0)) \times \frac{1 \vee \|\phi\|_{1,q+2,\infty}^{2dq+1+2\kappa}}{\varepsilon(\phi)^{q(q+1)}}$$

and this gives

$$\left\| H_\alpha\Big(\phi, \frac{f}{\psi_\kappa} \times \psi_\kappa(\phi)\Big) \right\|_p \leq C\psi_\kappa(\phi(0)) \times \frac{1 \vee \|\phi\|_{1,q+2,\infty}^{2dq+1+2\kappa}}{\varepsilon(\phi)^{q(q+1)}} \times \|f\|_{q+1,p}$$

Using a change of variable and (A.22) $\|g(\phi)\|_{p_*} \leq \varepsilon(\phi)^{-1/p_*} \|g\|_{p_*}$. These two inequalities prove (A.23). □

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