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On the Rajchman property for self-similar measures

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Abstract

For classical Bernoulli convolutions, the convergence to zero at infinity of the Fourier transform was characterized by successive works of Erdös [2] and Salem [13]. We prove similar results for general self-similar measures associated to real affine contractions.

1 Introduction

In the present article we consider the extension of some well-known results concerning Bernoulli convolutions to a more general context of self-similar measures. For a Borel probability measure m on the real line, define its Fourier transform as :

$$\hat{m}(t) = \int_{\mathbb{R}} e^{itx}\ dm(x),\ t \in \mathbb{R}.$$

We say that m is Rajchman, if $\hat{m}(t) \to 0$, as $t \to +\infty$. This property is very important in Harmonic Analysis, for example for its key role in the analysis of sets of multiplicity for trigonometric series, cf Lyons [9]. Let us now recall standard notions on self-similar measures, from a probabilistic angle.

We write $\mathcal{L}(X)$ for the law of a real random variable X. Let $N \geq 0$ and affine contractions $\varphi_k(x) = r_k x + b_k$, with $0 < r_k < 1$, $x \in \mathbb{R}$, $0 \leq k \leq N$. For $n \geq 0$, compositions have the form :

$$\varphi_{j_{n-1}} \circ \cdots \circ \varphi_{j_0}(x) = r_{j_{n-1}} \cdots r_{j_0} x + \sum_{l=0}^{n-1} b_{j_l} r_{j_{n-1}} \cdots r_{j_{l+1}}.$$

Introduce the convex set $\mathcal{C}_N = \{p = (p_0, \cdots, p_N) \mid p_i \geq 0,\ \sum_i p_i = 1\}$ and fix a probability vector $p \in \mathcal{C}_N$. We now compose the contractions at random, independently, according to p. Precisely, let X_0 be any real random variable and $(\varepsilon_n)_{n\geq 0}$ be independent and identically distributed random variables (*i.i.d.*), independent from X_0, with $\mathbb{P}(\varepsilon_n = k) = p_k$, $0 \leq k \leq N$. We consider the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}$ defined by $X_n = \varphi_{\varepsilon_{n-1}} \circ \cdots \circ \varphi_{\varepsilon_0}(X_0)$, $n \geq 0$.

It is classical that $(X_n)_{n\geq 0}$ has a unique invariant measure ν. This can be seen for example from the fact that $\mathcal{L}(X_n) = \mathcal{L}(\tilde{X}_n)$, where :

$$\tilde{X}_n = \varphi_{\varepsilon_0} \circ \cdots \circ \varphi_{\varepsilon_{n-1}}(X_0) = r_{\varepsilon_0} \cdots r_{\varepsilon_{n-1}} X_0 + \sum_{l=0}^{n-1} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}.$$

Since $\tilde{X}_n$ converges almost-surely to $X := \sum_{l\geq 0} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}$, this implies that $\nu_n := \mathcal{L}(X_n)$ weakly converges to $\nu := \mathcal{L}(X)$. By construction :

$$\nu_{n+1} = \sum_{0\leq k\leq N} p_k \nu_n \circ \varphi_k^{-1},$$

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so, taking the limit as $n \to +\infty$, we obtain that ν is a solution of the equation :

$$\nu = \sum_{0 \leq k \leq N} p_k \nu \circ \varphi_k^{-1}. \tag{1}$$

The previous convergence implies that the solution of this equation is unique among Borel probability measures. Moreover ν has to be of pure type, i.e. either absolutely continuous with respect to Lebesgue measure or atomic or else singular continuous, since each term in its Radon-Nikodym decomposition with respect to Lebesgue measure verifies equation (1). Using the repartition function, it is not difficult to observe that ν is continuous if and only if the fixed points $b_k/(1-r_k)$ of the φ_k, $0 \leq k \leq N$, are not all equal (see for example Feng-Lau [5]). In case of equality, ν is the Dirac mass at the common fixed point. This trivial case excluded, a difficult problem is to characterize absolute continuity in terms of the parameters $r := (r_k)$ and (b_k).

An example with a long and well-known history is that of Bernoulli convolutions, corresponding to $N = 1$, the system of contractions $\varphi_0(x) = \lambda x - 1$, $\varphi_1(x) = \lambda x + 1$, $0 < \lambda < 1$, and $p = (1/2, 1/2)$. Notice that when the contraction rates are equal, the situation is a little simplified, as ν is an infinite convolution (this is not true in general). Although we discuss below some works in this context, we will not present here the vast subject of Bernoulli convolutions, addressing the reader to detailed surveys, Peres-Schlag-Solomyak [11] or more recently Solomyak [16].

For general self-similar measures, an important aspect of the problem, that we shall not enter, and an active line of research, concerns the Hausdorff dimension of the measure ν. In a large generality, cf for example Falconer [3], one has an "entropy/Lyapunov exponent" upper-bound :

$$\mathrm{Dim}_{\mathcal{H}}(\nu) \leq \min\{1, s(p,r)\}, \text{ where } s(p,r) := \frac{-\sum_{i=0}^N p_i \log p_i}{-\sum_{i=0}^N p_i \log r_i}.$$

The quantity $s(p,r)$ is called the singularity dimension of the measure. The equality $\mathrm{Dim}_{\mathcal{H}}(\nu) = 1$ does not mean that ν is absolutely continuous, but the inequality $s(p,r) < 1$ implies that ν is singular. The interesting domain of parameters therefore corresponds to $s(p,r) \geq 1$.

We focus here on another fundamental tool, the Fourier transform $\hat{\nu}$. If ν is not Rajchman, the Riemann-Lebesgue lemma implies that ν is singular. This property was used by Erdös [2] in the context of Bernoulli convolutions. Erdös proved that if $1/2 < \lambda < 1$ is such that $1/\lambda$ is a Pisot number, then ν is not Rajchman. The reciprocal statement was next shown by Salem [13]. As a result, for Bernoulli convolutions the Rajchman property holds except for a very particular countable set of parameters λ.

The aim of the present article is to study the Rajchman property and prove results in the same spirit for more general self-similar measures. The non-Rajchman character was recently shown to hold only for a very small set of parameters by Solomyak [17] : as soon as the (φ_k) do not have a common fixed point and p is not degenerated, then outside a zero-Hausdorff dimensional set for the (r_k), the Fourier transform even has a power decay at infinity. Our purpose is to focus on the exceptional set and to show that the parameters (r_k) and (b_k) have to be rather specific, as for Bernoulli convolutions. In the sequel, we write ν_p instead of ν for the invariant measure, in order to emphasize its dependence with respect to $p \in \mathcal{C}_N$. We shall first prove the following result.

Theorem 1.1
Let $0 < \lambda < 1$ be such that $1/\lambda$ is a Pisot number. Let $b \in \mathbb{R}$, $c \in \mathbb{R}$, $N \geq 0$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = \lambda^{n_k} x + b_k$, for integers $n_k \geq 1$ and $b_k = ba_k + c(1-\lambda^{n_k})$, with $a_k \in \mathbb{Q}[\lambda]$. Then for $p \in \mathcal{C}_N$ outside a finite set, the invariant measure ν_p is not Rajchman.

This is a way of producing continuous singular invariant measures. We in fact give very concrete examples with $1/\lambda$ the Plastic number in the last section. Concerning the existence of singular measures in the inhomogeneous case, we are essentially aware of the non-explicit examples, using algebraic curves, of Neunhäuserer [10].

Since the set of Pisot numbers contains the integers ≥ 2, one may observe that the previous theorem is in some sense optimal, as the finite set involved in the conclusion of the theorem can be

non-empty, as soon as $N \geq 1$. Indeed when $N \geq 1$, taking $\varphi_k(x) = (x+k)/(N+1)$, $0 \leq k \leq N$, with $p = (1/(N+1), \cdots, 1/(N+1))$, gives for ν_p Lebesgue measure on $[0,1]$, of course Rajchman.

The conditions of Theorem 1.1 seem rather restrictive, but somehow surprisingly we shall show a full reciprocal statement :

Theorem 1.2
Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, with no common fixed point, and $p \in \mathcal{C}_N$, with $p_j > 0$ for all $0 \leq j \leq N$. If the invariant measure ν_p is not Rajchman, there exist $0 < \lambda < 1$ with $1/\lambda$ Pisot, real numbers $b \neq 0$ and c, relatively prime positive integers $(n_k)_{0 \leq k \leq N}$ and $a_k \in \mathbb{Q}[\lambda]$, $0 \leq k \leq N$, such that for any $0 \leq j \leq N$: $r_j = \lambda^{n_j}$, $b_j = ba_j + c(1-\lambda^{n_j})$.

The first step in the proof of the theorem is to show that $\log r_i / \log r_j \in \mathbb{Q}$, for any $0 \leq i, j \leq N$. Credit for this is due to Li and Sahlsten [8], who showed that ν_p is Rajchman whenever $\log r_i / \log r_j \notin \mathbb{Q}$, for some i, j, with moreover some logarithmic decay at infinity of $\hat{\nu}$ under a Diophantine condition on $\log r_i / \log r_j$. Their work, involving renewal theory, was one of the motivation for the present paper. Coming after them, we simplify their proof and relate it to the standard renewal theorem.

As a general fact, Theorem 1.2 has some consequences in terms of sets of multiplicity for trigonometric series, cf for example Salem [14], chap. 5 for details. Recall that a subset E of the torus $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ is a set of uniqueness, if whenever a trigonometric series $\sum_{n \geq 0}(a_n \cos(2\pi x) + b_n \sin(2\pi x))$ converges to 0, for $x \notin E$, then $a_n = b_n = 0$ for all $n \geq 0$. Otherwise E is a set of multiplicity. If ever $E \neq \mathbb{T}$ supports a Rajchman probability measure, then it is a set of multiplicity. We thus deduce the following corollary for certain self-similar sets (cf Huchinson [6] for general properties of such sets) :

Corollary 1.3
Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$ from the interval $(0,1)$ to itself, with no common fixed point. Suppose that the (r_k) and (b_k) are not of the form stated in the conclusion of Theorem 1.2. Then the unique closed self-similar set $F \neq \emptyset$ verifying $F = \cup_{0 \leq k \leq N} \varphi_k(F)$ is a set of multiplicity for trigonometric series.

Indeed, taking any $p \in \mathcal{C}_N$, with $p_j > 0$ for all $0 \leq j \leq N$, gives a Rajchman invariant measure ν_p supported by $F \neq \mathbb{T}$. Reciprocally, when $N \geq 1$ and the (φ_k) are affine contractions preserving $(0,1)$ and of the form $\varphi_k(x) = \lambda^{n_k} x + ba_k + c(1-\lambda^{n_k})$, for some $0 < \lambda < 1$ with $1/\lambda$ Pisot, integers $n_k \geq 1$ and $a_k \in \mathbb{Q}[\lambda]$, an interesting question, that we leave open, is whether the self-similar set $F = \cup_{0 \leq k \leq N} \varphi_k(F)$ is a set of uniqueness for trigonometric series.

Plan of the article. In sections 2 and 3, we successively prove Theorems 1.1 and 1.2. In the last section, we give some complements, first surprising numerical simulations involving the Plastic number and next a partial analysis in a more general "contracting on average" context.

2 Proof of Theorem 1.1

Recall that a Pisot number is an algebraic integer > 1 with Galois conjugates (the other roots of its minimal unitary polynomial) of modulus strictly less than 1. We write $Q \in \mathbb{Z}[X]$ for the minimal polynomial of $1/\lambda$, of degree $s+1$, with roots $\alpha_0 = 1/\lambda$, $\alpha_1, \cdots, \alpha_s$, where $|\alpha_k| < 1$, for $1 \leq k \leq s$. The case $s = 0$ corresponds to $1/\lambda$ an integer ≥ 2 (using then usual conventions regarding sums or products). A classical fact, used in *Step 2* and *Step 3*, is that, for $n \geq 0$:

$$\lambda^{-n} + \sum_{1 \leq i \leq s} \alpha_i^n \in \mathbb{Z},$$

as a symmetric polynomial of the roots of Q. Introduce the torus $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ and write equality in $\mathbb{T}$ as $x \equiv y$, for $x, y \in \mathbb{T}$. Recall that $\varphi_k(x) = \lambda^{n_k} x + b_k$, with $b_k = ba_k + c(1-\lambda^{n_k})$. The $a_k \in \mathbb{Q}[\lambda]$ can be written as :

$$a_k = \frac{1}{q} \sum_{0 \le l \le s} p_{l,k} \lambda^l,$$

for integers $p_{l,k}$ and $q \geq 1$, with q independent on k. One can also freely assume $1 \leq n_0 \leq \cdots \leq n_N$. As in the introduction, the $(\varepsilon_n)_{n \geq 0}$ are *i.i.d.* random variables with law $p \in \mathcal{C}_N$. The probability $\mathbb{P}$ and the corresponding expectation $\mathbb{E}$ are related to these random variables.

Recall that ν_p is the law of $\sum_{l \geq 0} b_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}$. As a preliminary computation :

$$\begin{aligned} \sum_{l \geq 0} b_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}} &= b \sum_{l \geq 0} a_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}} + c \sum_{l \geq 0} (1 - \lambda^{n_{\varepsilon_l}}) \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}} \\ &= b \sum_{l \geq 0} a_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}} + c. \end{aligned}$$

We thus assume $b \neq 0$ and also $N \geq 1$, otherwise ν_p is a Dirac mass. In the sequel, $m \neq 0$ is an integer, fixed at the end of the proof independently on $p \in \mathcal{C}_N$.

Step 1. We introduce the following quantities, where we mark the dependence in p :

$$F_p(k) = \mathbb{E}\left(e^{2i\pi m q \lambda^k \sum_{l \geq 0} a_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}} \right), \ k \in \mathbb{Z}.$$

Notice that $F_p(k) = \hat{\nu}_p(2\pi m q \lambda^k / b) e^{-2i\pi m q \lambda^k c/b}$. In a nearly symmetric way, using that $\lambda^{-n} \to 0$ in $\mathbb{T}$ exponentially fast, as $n \to +\infty$, and that $q a_k \in \mathbb{Z}[\lambda]$, we define :

$$G_p(k,r) = \mathbb{E}\left(e^{2i\pi m \sum_{l \geq 0} (q a_{\varepsilon_l}) \lambda^{k - (n_{\varepsilon_0} + \cdots + n_{\varepsilon_l})}} 1_{n_{\varepsilon_0} > r} \right), \ k \in \mathbb{Z}, \ r \geq 0.$$

When $r = 0$, the indicator $1_{n_{\varepsilon_0} > r}$ can be removed. Conditioning now with respect to the value of ε_0, we get recursive relations, for $k \in \mathbb{Z}$, $r \geq 0$:

$$\begin{cases} F_p(k) &= \sum_{0 \leq j \leq N} p_j e^{2i\pi m q \lambda^k a_j} F_p(k + n_j), \\ \\ G_p(k,r) &= \sum_{0 \leq j \leq N, n_j > r} p_j e^{2i\pi m q \lambda^{k - n_j} a_j} G_p(k - n_j, 0). \end{cases}$$

The following lemma is central in this work. An extended version is given in the last section.

Lemma 2.1
For $k \in \mathbb{Z}$, define :

$$\Delta_p(k) = \sum_{0 \leq r < n_N} F_p(k + r) G_p(k + r, r).$$

Then $\Delta_p(k) = \Delta_p(k+1)$, $k \in \mathbb{Z}$.

Proof of the lemma :
Notice that $\Delta_p(k) = \sum_{0 \leq r \leq n_N} F_p(k + r) G_p(k + r, r)$, since $G_p(k + n_N, n_N) = 0$. Also :

$$\Delta_p(k+1) = \sum_{0 \leq r < n_N} F_p(k + 1 + r) G_p(k + 1 + r, r) = \sum_{1 \leq r \leq n_N} F_p(k + r) G_p(k + r, r - 1).$$

Therefore, using first the second recursive relation (for G_p) and next the first one (for F_p) :

$$\begin{aligned}
\Delta_p(k)-\Delta_p(k+1) &= F_p(k)G_p(k,0)+\sum_{1\le r\le n_N} F_p(k+r)\left(G_p(k+r,r)-G_p(k+r,r-1)\right)\\
&= F_p(k)G_p(k,0)-\sum_{1\le r\le n_N} F_p(k+r)\sum_{0\le j\le N, n_j=r} p_j e^{2i\pi mq\lambda^{k+r-n_j}a_j}G_p(k+r-n_j,0)\\
&= F_p(k)G_p(k,0)-\sum_{1\le r\le n_N} F_p(k+r)\sum_{0\le j\le N, n_j=r} p_j e^{2i\pi mq\lambda^{k}a_j}G_p(k,0)\\
&= F_p(k)G_p(k,0)-G_p(k,0)\sum_{0\le j\le N} p_j e^{2i\pi mq\lambda^{k}a_j}\sum_{1\le r\le n_N} 1_{r=n_j}F_p(k+r)\\
&= G_p(k,0)\left(F_p(k)-\sum_{0\le j\le N} p_j e^{2i\pi mq\lambda^{k}a_j}F_p(k+n_j)\right)=0.
\end{aligned}$$

This is the desired result. □

As a consequence of this lemma, we set $\Delta_p=\Delta_p(k)$. Now, the argument is simply the following. If ν_p were Rajchman, we would have $\lim_{k\to-\infty}F_p(k)=0$. The quantities appearing in the definition of $\Delta_p(k)$ being all bounded by 1, this would imply $\lim_{k\to-\infty}\Delta_p(k)=0$, hence $\Delta_p=0$. We next show that this can happen only for finitely many values of $p\in\mathcal{C}_N$ (for a well-chosen $m\neq 0$).

Step 2. Let us consider the regularity of $p\longmapsto\Delta_p$ on $\mathcal{C}_N$. We shall prove using standard methods that it is continuous and real-analytic on $\mathcal{C}_N$, in a sense precised below. This will result from the same properties for all $p\longmapsto F_p(k+r)$ and $p\longmapsto G_p(k+r,r)$, the case of the second functions needing to rewrite the λ^{-n}, $n\ge 0$, appearing in the definition of G_p as $-\sum_{1\le k\le s}\alpha_k^n$, equal to λ^{-n} in $\mathbb{T}$. We treat the case of $p\longmapsto F_p(0)$, the other ones being exactly similar. Continuity is immediate, as this function is the uniform limit on $\mathcal{C}_N$, as $L\to+\infty$, of the continuous maps :

$$p\longmapsto\mathbb{E}\left(e^{2i\pi mq\sum_{l=0}^{L}a_{\varepsilon_l}\lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}}\right).$$

Fix now $p\in\mathcal{C}_N$. Let $\mathbb{N}=\{0,1,\cdots\}$ and the symbolic space $S=\{0,\cdots,N\}^{\mathbb{N}}$, equipped with the left shift σ. For $x=(x_0,x_1,\cdots)\in S$, we define :

$$g(x)=e^{2i\pi mq\left(\sum_{l\ge 0}a_{x_l}\lambda^{n_{x_0}+\cdots+n_{x_{l-1}}}\right)}.$$

Introducing the product measure $\mu_p=(\sum_{0\le j\le N}p_j\delta_j)^{\otimes\mathbb{N}}$ on S, we can write :

$$F_p(0)=\int_S g\;d\mu_p.$$

Denote by $C(S)$ the space of continuous functions $f:S\to\mathbb{C}$ and introduce the operator $P_p:C(S)\to C(S)$ defined by :

$$P_p(f)(x)=\sum_{0\le j\le N}p_j f((j,x)),\ x\in S,$$

where $(j,x)\in S$ is the word obtained by the left concatenation of the symbol j to x. The operator P_p is Markovian, i.e. $f\ge 0\Rightarrow P_p(f)\ge 0$ and verifies $P_p\mathbf{1}=\mathbf{1}$, where $\mathbf{1}(x)=1$, $x\in S$. The measure μ_p has the invariance property $\int_S P_p(f)\;d\mu_p=\int_S f\;d\mu_p$, $f\in C(S)$. For $f\in C(S)$ and $k\ge 0$, introduce the variation :

$$\mathrm{Var}_k(f)=\sup\{|f(x)-f(y)|,\ x_i=y_i,\ 0\le i<k\}.$$

For any $0<\theta<1$, let $|f|_\theta=\sup\{\theta^{-k}\mathrm{Var}_k(f),\ k\ge 0\}$, as well as $\|f\|_\theta=|f|_\theta+\|f\|_\infty$. We denote by $\mathcal{F}_\theta$ the complex Banach space of fonctions f on S such that $\|f\|_\theta<\infty$. Any $\mathcal{F}_\theta$ is preserved by P_p. Observe now that $g\in\mathcal{F}_\theta$ for $\lambda\le\theta<1$. We take for example $\theta=\lambda$ and write $\mathcal{F}$ for $\mathcal{F}_\theta$.

As a classical fact from Spectral Theory, cf Ruelle [12] or Baladi [1], the operator $P_p : \mathcal{F} \to \mathcal{F}$ satisfies a Perron-Frobenius theorem : the eigenvalue 1 is simple and the rest of its spectrum is contained in a closed disk of radius $\rho < 1$. By standard functional holomorphic calculus, cf Kato [7], when taking for Γ the circle centered at 1 with radius $0 < r < 1 - \rho$, the following operator, involving the resolvent, is a continuous (Riesz) projector on Vect($\mathbf{1}$) :

$$\Pi_p = \int_\Gamma (zI - P_p)^{-1} dz.$$

Moreover, $\Pi_p(\mathcal{F})$ and $(I - \Pi_p)(\mathcal{F})$ are closed P_p-invariant subspaces with $\mathcal{F} = \Pi_p(\mathcal{F}) \oplus (I - \Pi_p)(\mathcal{F})$. In restriction to $(I - \Pi_p)(\mathcal{F})$, the spectral radius of P_p is less than ρ. In particular $\int_S f \, d\mu_p = 0$, for $f \in (I - \Pi_p)(\mathcal{F})$. This implies that for any $f \in \mathcal{F}$:

$$\Pi_p(f) = \left(\int_S f \, d\mu_p \right) \mathbf{1}.$$

Applying this to the function g of interest to us, we obtain that :

$$F_p(0)\mathbf{1} = \int_\Gamma (zI - P_p)^{-1}(g) dz.$$

Recall now that $N \geq 1$. Let $\eta' = (\eta_0, \cdots, \eta_{N-1})$ and $\eta = (\eta_0, \cdots, \eta_{N-1}, -(\eta_0 + \cdots + \eta_{N-1}))$. The condition on η' for $p + \eta \in \mathcal{C}_N$ is written as $\eta' \in D_N(p)$. Explicitly the condition is :

$$-p_i \leq \eta_i \leq 1 - p_i, \ 0 \leq i \leq N-1, \text{ and } p_N - 1 \leq \eta_0 + \cdots + \eta_{N-1} \leq p_N.$$

For the sequel, let $B_N(0, R)$ be the open Euclidean ball in $\mathbb{R}^N$ centered at 0, of radius R.

Definition 2.2
A function $h : \mathcal{C}_N \to \mathbb{C}$ admits a development in series around a point $p \in \mathcal{C}_N$, if there exists $\varepsilon > 0$ such that for $\eta' = (\eta_0, \cdots, \eta_{N-1}) \in D_N(p) \cap B_N(0, \varepsilon)$ and writing $\eta = (\eta_0, \cdots, \eta_{N-1}, -(\eta_0 + \cdots + \eta_{N-1}))$, then $h(p + \eta)$ is given by an absolutely converging series :

$$h(p + \eta) = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} A_{l_0, \cdots, l_{N-1}} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}}.$$

A function is real-analytic in $\mathcal{C}_N$ if it admits a development in series around every $p \in \mathcal{C}_N$.

For such a function, when non-constant, its zeroes are in finite number in $\mathcal{C}_N$, by the standard argument that the set of points where there is a null development in series is open and closed for the relative topology and thus equal to $\mathcal{C}_N$ by connexity if non-empty. In case of infinitely many zeros, any accumulation point (which exists, as $\mathcal{C}_N$ is compact) is such a point.

We now check below that $p \longmapsto F_p(0)$ is real-analytic in the previous sense. As already indicated, this property will be inheritated by $p \longmapsto \Delta_p$. In this direction, notice that :

$$P_{p+\eta} = P_p + \sum_{0 \leq j \leq N-1} \eta_j Q_j,$$

where $Q_j(f)(x) = f(j, x) - f(N, x)$. For $z \in \Gamma$ and η' small enough :

$$\begin{aligned}
(zI - P_{p+\eta})^{-1} &= \left(I - (zI - P_p)^{-1} \sum_{0 \leq j \leq N-1} \eta_j Q_j \right)^{-1} (zI - P_p)^{-1} \\
&= \sum_{n \geq 0} \sum_{0 \leq j_1, \cdots, j_n \leq N-1} \eta_{j_1} \cdots \eta_{j_n} (zI - P_p)^{-1} Q_{j_1} \cdots (zI - P_p)^{-1} Q_{j_n} (zI - P_p)^{-1}.
\end{aligned}$$

This is clearly absolutely convergent in the Banach operator algebra, for small enough η', uniformly in $z \in \Gamma$. We rewrite it as :

$$(zI - P_{p+\eta})^{-1} = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} B_{l_0, \cdots, l_{N-1}}(z) \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}},$$

converging for the operator norm, uniformly in $z \in \Gamma$. This leads to :

$$F_{p+\eta}(0)\mathbf{1} = \int_\Gamma (zI - P_{p+\eta})^{-1}(g)\ dz = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}} \int_\Gamma B_{l_0, \cdots, l_{N-1}}(z)(g)\ dz.$$

Applying this equality at some particular $x \in S$, we obtain the desired development in series around p. This completes this step.

Step 3. We finish the argument. If ever $\Delta_p = 0$ for infinitely many $p \in \mathcal{C}_N$, then by *Step 2*, $p \longmapsto \Delta_p$ has to be constant on $\mathcal{C}_N$ and equal to zero. We shall show that if $m \neq 0$ has been appropriately chosen at the beginning it is not possible. We start with a lemma.

Lemma 2.3
Let $d \geq 1$ and $u \in \mathbb{Z}$. The series $\sum_{l \in \mathbb{Z}} \lambda^{ld+u}$, well-defined as an element of $\mathbb{T}$, verifies, for integers $A_{d,u}$, with $A_{d,d+u} = A_{d,u}$, and $B_d \neq 0$:

$$\sum_{l \in \mathbb{Z}} \lambda^{ld+u} \equiv -\frac{A_{d,u}}{B_d}.$$

Proof of the lemma :
It is enough to take $0 \leq u < d$. Cutting the sum of the left-hand side in two and next using the conjugates $\alpha_1, \cdots, \alpha_s$ of $\alpha_0 = 1/\lambda$, we have the following equalities on the torus :

$$\begin{aligned}
\sum_{l \in \mathbb{Z}} \lambda^{ld+u} &\equiv \frac{\lambda^u}{1-\lambda^d} + \sum_{l \geq 1} \lambda^{u-ld} \equiv \frac{\lambda^u}{1-\lambda^d} - \sum_{1 \leq i \leq s} \sum_{l \geq 1} \alpha_i^{ld-u} \equiv \frac{\lambda^u}{1-\lambda^d} - \sum_{1 \leq i \leq s} \frac{\alpha_i^{d-u}}{1-\alpha_i^d} \\
&\equiv -\left(\frac{(1/\lambda)^{d-u}}{1-(1/\lambda)^d} + \sum_{1 \leq i \leq s} \frac{\alpha_i^{d-u}}{1-\alpha_i^d} \right) = -\sum_{0 \leq i \leq s} \frac{\alpha_i^{d-u}}{1-\alpha_i^d} \\
&\equiv -\frac{\sum_{0 \leq i \leq s} \alpha_i^{d-u} \prod_{0 \leq j \leq s, j \neq i} (1-\alpha_j^d)}{\prod_{0 \leq i \leq s} (1-\alpha_i^d)} \equiv -\frac{A_{d,u}}{B_d},
\end{aligned}$$

as the numerator and denominator of the last fraction are symmetric polynomials of the $(\alpha_i)_{0 \leq i \leq s}$, roots of a polynomial in $\mathbb{Z}[X]$.

□

We conclude the argument. Fixing $0 \leq j \leq N$ and $p^j = (0, \cdots, 0, 1, 0, \cdots, 0)$, where the 1 is at place j, we have for $k \in \mathbb{Z}$ and $r \geq 0$:

$$F_{p^j}(k) = e^{2i\pi m q \lambda^k a_j \sum_{l \geq 0} \lambda^{l n_j}} \text{ and } G_{p^j}(k, r) = e^{2i\pi m \sum_{l \geq 1} (q a_j) \lambda^{k - l n_j}} 1_{n_j > r}.$$

Summing on $0 \leq r < n_N$ the $F_{p^j}(k+r) G_{p^j}(k+r, r)$ and making use of the indicator function $1_{n_j > r}$, we obtain :

$$\Delta_{p^j}(k) = \sum_{0 \leq r < n_j} e^{2i\pi m \sum_{l \in \mathbb{Z}} (q a_j) \lambda^{k+r+l n_j}} = \sum_{0 \leq r < n_j} e^{2i\pi m \sum_{0 \leq u \leq s} p_{u,j} \sum_{l \in \mathbb{Z}} \lambda^{u+k+r+l n_j}}.$$

Notice in passing that the constant character with respect to k is now obvious, as we sum on r on a full period of length n_j. Taking $k = 0$ and using the previous lemma :

$$\Delta_{p^j} = \Delta_{p^j}(0) = \sum_{0 \leq r < n_j} e^{-2i\pi m \sum_{0 \leq u \leq s} p_{u,j} \frac{A_{n_j, u+r}}{B_{n_j}}}.$$

If for example m is a multiple of B_{n_j} for any $0 \leq j \leq N$, we get $\Delta_{p^j} = n_j \geq 1$, for all $0 \leq j \leq N$, which is more than enough. This ends the proof of the theorem.

□

Remark. — A word on the method. Trying to follow the proof of Erdös [2], probabilistic computations involving the renewal theorem lead to the convergence of some sequence $(\hat{\nu}_p(\alpha\lambda^{-n}))$, as $n \to +\infty$. The limit was the product of a positive constant (involving some Green function) with $\Delta_p(0)$. Replacing n by $n-k$, one gets a necessarily invariant function of k. It appeared more efficient (but perhaps more abrupt) to restart the analysis directly from $\Delta_p(k)$.

Remark. — In the context of Theorem 1.1, it would be important to have an interpretation of the quantity Δ_p, in terms of Hermitian product, or else (cf also the last section). Another question is whether the condition $\Delta_p = 0$ is equivalent to ν_p Rajchman ? When the Pisot number $1/\lambda$ is irrational, it would be interesting to determine classes of parameters where $\Delta_p \neq 0$, for all $p \in \mathcal{C}_N$. See the interesting pictures in the last section.

We detail in section 4 concrete examples falling in the interesting domain of parameters. For the moment we turn to the reverse side of the question.

3 Proof of Theorem 1.2

This time $N \geq 1$ and we assume that $p \in \mathcal{C}_N$ with $p_k > 0$ for all $0 \leq k \leq N$ and the $\varphi_k(x) = r_k x + b_k$ do not all have the same fixed point. The $(\varepsilon_n)_{n\geq 0}$ are still $i.i.d$ random variables with law p, to which $\mathbb{P}$ and $\mathbb{E}$ refer.

Step 1. We reprove in a simpler form the result of Li-Sahlsten [8], that if for some $0 \leq i \neq j \leq N$ one has $\log r_i / \log r_j \notin \mathbb{Q}$, then ν_p is Rajchman.

For $n \geq 1$, consider the random walk $S_n = -\log r_{\varepsilon_0} - \cdots - \log r_{\varepsilon_{n-1}}$, with $S_0 = 0$. For a real $s \geq 0$, introduce the finite stopping time $\tau_s = \min\{n \geq 0,\ S_n \geq s\}$ and write $\mathcal{T}_s$ for the corresponding sub-σ-algebra of the underlying σ-algebra. Taking $\alpha > 0$ and $s \geq 0$:

$$\begin{aligned}\hat{\nu}_p(\alpha e^s) &= \mathbb{E}\left(e^{i\alpha e^s \sum_{l\geq 0} b_{\varepsilon_l} e^{-S_l}}\right) \\ &= \mathbb{E}\left(e^{i\alpha e^{-S_{\tau_s}+s} \sum_{0\leq l<\tau_s} b_{\varepsilon_l} e^{-S_l+S_{\tau_s}}} e^{i\alpha e^{-S_{\tau_s}+s} \sum_{l\geq \tau_s} b_{\varepsilon_l} e^{-S_l+S_{\tau_s}}}\right).\end{aligned}$$

In the expectation, the first exponential term is $\mathcal{T}_s$-measurable. Also, the conditional expectation of the second exponential term with respect to $\mathcal{T}_s$ is just $\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s})$, as a consequence of the strong Markov property. It follows that :

$$\hat{\nu}_p(\alpha e^s) = \mathbb{E}\left(\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s}) e^{i\alpha e^{-S_{\tau_s}+s} \sum_{0\leq l<\tau_s} b_{\varepsilon_l} e^{-S_l+S_{\tau_s}}}\right).$$

This gives $|\hat{\nu}_p(\alpha e^s)| \leq \mathbb{E}\left(|\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s})|\right)$, so by the Cauchy-Schwarz inequality and a safe Fubini theorem consecutively :

$$\begin{aligned}|\hat{\nu}_p(\alpha e^s)|^2 \leq \mathbb{E}\left(|\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s})|^2\right) &= \mathbb{E}\left(\int_{\mathbb{R}^2} e^{i\alpha e^{-S_{\tau_s}+s}(x-y)}\ d\nu_p(x) d\nu_p(y)\right) \\ &= \int_{\mathbb{R}^2} \mathbb{E}\left(e^{i\alpha e^{-S_{\tau_s}+s}(x-y)}\right)\ d\nu_p(x) d\nu_p(y).\end{aligned}$$

Let $Y := -\log r_{\varepsilon_0}$ and $a = 1/\mathbb{E}(Y)$. As the law of Y is non-lattice and with finite moment, it is a well-known and easy consequence of the Blackwell theorem on the law of the overshoot, see Feller [4] (p. 369-370), that for any Riemann-integrable g, $\mathbb{E}(g(S_{\tau_s} - s)) \to a \int_0^{+\infty} g(x)\mathbb{P}(Y > x)\ dx$, as $s \to +\infty$. By dominated convergence, for any $\alpha > 0$:

$$\limsup_{t\to+\infty} |\hat{\nu}_p(t)|^2 \leq a \int_{\mathbb{R}^2} \left| \int_0^{+\infty} e^{i\alpha e^{-u}(x-y)} \mathbb{P}(Y > u) du \right| d\nu_p(x) d\nu_p(y).$$

The inside term is uniformly bounded with respect to $(x, y) \in \mathbb{R}^2$, as Y has finite support, so $\mathbb{P}(Y > u) = 0$ for large u. We shall use dominated convergence once more, this time with $\alpha \to +\infty$. It is sufficient to show that for $\nu_p^{\otimes 2}$-almost every (x, y), the inside term goes to zero. Since ν_p is non-atomic, $\nu_p^{\otimes 2}$-almost-surely, $x \neq y$. If for example $x > y$:

$$\int_0^{+\infty} e^{i\alpha e^{-u}(x-y)} \mathbb{P}(Y > u) du = \int_0^{x-y} e^{i\alpha t} \mathbb{P}(Y > \log((x-y)/t)\ \frac{dt}{t},$$

making the change of variable $t = e^{-u}(x - y)$. Remark that the integrated term is zero for small enough $t > 0$. The integral now converges to 0, as $\alpha \to +\infty$, by the Riemann-Lebesgue lemma. This shows that $\lim_{t\to+\infty} \hat{\nu}_p(t) = 0$ and completes the proof of this step.

Step 2. From *Step 1*, if ν_p is not Rajchman, then $\log r_i / \log r_j \in \mathbb{Q}$, for all (i, j), hence $r_j = r_0^{p_j/q_j}$, with integers $p_j \geq 1$, $q_j \geq 1$, for $1 \leq j \leq N$. Let :

$$n_0 = \prod_{1\leq l\leq N} q_l \text{ and } n_j = p_j \prod_{1\leq l\leq N, l\neq j} q_l,\ 1 \leq j \leq N.$$

Setting $\lambda = r_0^{1/n_0} \in (0, 1)$, one has $r_j = \lambda^{n_j}$, $0 \leq j \leq N$. Up to taking some positive power of λ, one can assume that $\gcd(n_0, \cdots, n_N) = 1$. Recall in passing that the set of Pisot numbers is stable under positive powers. Using now some sub-harmonicity, one can reinforce the assumption that $\hat{\nu}_p(t)$ is not converging to 0, as $t \to +\infty$.

Lemma 3.1
There exists $1 \leq \alpha \leq 1/\lambda$ and $c > 0$ such that $\hat{\nu}_p(2\pi\alpha\lambda^{-k}) = c_k e^{2i\pi\theta_k}$, written in polar form, verifies $c_k \to c$, as $k \to +\infty$.

Proof of the lemma :
Let us write this time $S_n = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{n-1}}$, for $n \geq 1$, with $S_0 = 0$. Using that $\gcd(n_0, \cdots, n_N) = 1$, we fix $r \geq 1$ and $M \geq 1$ such that the support of S_r is included in $\{1, \cdots, M\}$ and contains two consecutive integers $1 \leq u \leq u + 1 \leq M$. Since for all $t \in \mathbb{R}$:

$$\hat{\nu}_p(t) = \mathbb{E}\left(e^{it \sum_{l\geq 0} b_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}} \right),$$

we get the relation $\hat{\nu}(t) = \sum_{0\leq j\leq N} p_j e^{itb_j} \hat{\nu}(\lambda^{n_j} t)$, hence $|\hat{\nu}(t)| \leq \sum_{0\leq j\leq N} p_j |\hat{\nu}(\lambda^{n_j} t)|$. Iterating :

$$|\hat{\nu}(t)| \leq \mathbb{E}\left(|\hat{\nu}_p(\lambda^{S_r} t)| \right),\ t \in \mathbb{R}. \tag{2}$$

In particular, $|\hat{\nu}(t)| \leq \max_{1\leq l\leq M} |\hat{\nu}(\lambda^l t)|$. We now set, for $\alpha > 0$:

$$V_\alpha(k) := \max_{k\leq l<k+M} |\hat{\nu}(\alpha\lambda^l)|,\ k \in \mathbb{Z}.$$

The previous remarks imply that $V_\alpha(k) \leq V_\alpha(k + 1)$, $k \in \mathbb{Z}$, $\alpha > 0$.

We now have $|\hat{\nu}_p(2\pi t_l)| \geq c' > 0$, along some sequence $t_l \to +\infty$. Write $t_l = \alpha_l \lambda^{-k_l}$, with $1 \leq \alpha_l \leq 1/\lambda$ and $k_l \to +\infty$. Up to extracting a subsequence, $\alpha_l \to \alpha \in [1, 1/\lambda]$. Fixing $k \in \mathbb{Z}$:

$$c' \leq V_{2\pi\alpha_l}(-k_l) \leq V_{2\pi\alpha_l}(-k),$$

as soon as l is large enough. By continuity, taking $l \to +\infty$, we get $c' \leq V_{2\pi\alpha}(-k)$, $k \in \mathbb{Z}$. As $k \longmapsto V_{2\pi\alpha}(-k)$ is non-increasing, $V_{2\pi\alpha}(-k) \to c \geq c'$, as $k \to +\infty$. We now show that necessarily, $|\hat{\nu}_p(2\pi\alpha\lambda^{-k})| \to c$, as $k \to +\infty$.

If this were not true, there would exist $\varepsilon > 0$ and $(m_k) \to +\infty$, with $|\hat{\nu}_p(2\pi\alpha\lambda^{-m_k})| \leq c - \varepsilon$. Recalling that $V_{2\pi\alpha}(-k) \to c$ and $|\hat{\nu}_p(2\pi\alpha\lambda^{-m_k})| \leq c - \varepsilon$, as $k \to +\infty$, consider (2) with $t = 2\pi\alpha\lambda^{-m_k-u}$ and next $t = 2\pi\alpha\lambda^{-m_k-u-1}$. Since all p_j are > 0, we obtain the existence of some $c_1 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(2\pi\alpha\lambda^{-m_k-u})|, |\hat{\nu}(2\pi\alpha\lambda^{-m_k-u-1})|\} \leq c_1 < c.$$

Again via (2), with successively $t = 2\pi\alpha\lambda^{-m_k-2u}$, $t = 2\pi\alpha\lambda^{-m_k-2u-1}$ and $t = 2\pi\alpha\lambda^{-m_k-2u-2}$ and still using that the p_j are all > 0, we obtain some $c_2 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(2\pi\alpha\lambda^{-m_k-2u})|,\ |\hat{\nu}(2\pi\alpha\lambda^{-m_k-2u-1})|,\ |\hat{\nu}(2\pi\alpha\lambda^{-m_k-2u-2})|\} \leq c_2 < c.$$

Etc, for some $c_{M-1} < c$ and k large enough :

$$\max\{|\hat{\nu}(2\pi\alpha\lambda^{-m_k-(M-1)u})|,\ \cdots,\ |\hat{\nu}(2\pi\alpha\lambda^{-m_k-(M-1)u-(M-1)})|\} \leq c_{M-1} < c.$$

This contradicts the fact that $V_{2\pi\alpha}(-k) \to c$, as $k \to \infty$. This allows to conclude that $|\hat{\nu}_p(2\pi\alpha\lambda^{-k})| \to c$, as $k \to \infty$, and this ends the proof of the lemma.

□

Step 3. We complete the proof of Theorem 1.2. In this section, introduce the notation $\|x\| = \mathrm{dist}(x, \mathbb{Z})$, for $x \in \mathbb{R}$. As in the last lemma, let $1 \leq \alpha \leq 1/\lambda$ with $\hat{\nu}_p(2\pi\alpha\lambda^{-k}) = c_k e^{2i\pi\theta_k}$, verifying $c_k \to c > 0$, as $k \to +\infty$. We start from the relation :

$$\hat{\nu}_p(2\pi\alpha\lambda^{-k}) = \sum_{0\leq j\leq N} p_j e^{2i\pi\alpha\lambda^{-k}b_j}\hat{\nu}_p(2\pi\alpha\lambda^{-k+n_j}).$$

This furnishes for $k \geq 0$:

$$c_k = \sum_{0\leq j\leq N} p_j e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)}c_{k-n_j}.$$

We rewrite this as :

$$\sum_{0\leq j\leq N} p_j\left[e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)}-1\right]c_{k-n_j} = c_k - \sum_{0\leq j\leq N} p_j c_{k-n_j} = \sum_{0\leq j\leq N} p_j(c_k - c_{k-n_j}).$$

Let $K > 0$ be such that $c_{k-n_j} \geq c/2 > 0$ for $k \geq K$ and all $0 \leq j \leq N$. For $L > \max_{0\leq j\leq N} n_j$, we sum the previous equality on $K \leq k \leq K + L$:

$$\sum_{0\leq j\leq N} p_j \sum_{k=K}^{K+L} c_{k-n_j}\left[e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)}-1\right] = \sum_{0\leq j\leq N} p_j\left(\sum_{k=K+L-n_j+1}^{K+L} c_k - \sum_{k=K-n_j}^{K-1} c_k\right).$$

Observe that the right-hand side is bounded, uniformly in K and L. In the left hand-hand side, we take the real part and use that $1 - \cos(2\pi x) = 2(\sin\pi x)^2$, which, as is well-known, has the same order as $\|x\|^2$. We obtain, for some constant C, that for K and L large enough :

$$\frac{c}{2}\sum_{0\leq j\leq N} p_j \sum_{k=K}^{K+L} \|\alpha\lambda^{-k}b_j + \theta_{k-n_j} - \theta_k\|^2 \leq C.$$

Let $p_* = \min_{0\leq j\leq N} p_j$. Since $p_* > 0$, we get that for all $0 \leq j \leq N$ and K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k}b_j + \theta_{k-n_j} - \theta_k\|^2 \leq C',$$

with $C' = 2C/(cp_*)$. Hence, for any $r \geq 0$ and K, L large enough (depending on r) :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k+rn_j} b_j + \theta_{k-(r+1)n_j} - \theta_{k-rn_j}\|^2 \leq C'.$$

Fixing $l_j \geq 1$ and summing over $0 \leq r \leq l_j - 1$, making use of the triangular inequality and of $(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2)$, we obtain, for K, L large enough (depending on l_j) :

$$\sum_{k=K}^{K+L} \left\| \alpha\lambda^{-k} b_j \left(\frac{1-\lambda^{l_j n_j}}{1-\lambda^{n_j}} \right) + \theta_{k-l_j n_j} - \theta_k \right\|^2 \leq l_j C'.$$

From the hypothesis, there exist $0 \leq i \neq j \leq N$, such that $b_i/(1-\lambda^{n_i}) \neq b_j/(1-\lambda^{n_j})$, otherwise the contractions φ_k have a common fixed point. Taking $l_j = n_i$ in the previous relation, we deduce that for K, L large enough :

$$\sum_{k=K}^{K+L} \left\| \alpha\lambda^{-k} b_j \left(\frac{1-\lambda^{n_i n_j}}{1-\lambda^{n_j}} \right) + \theta_{k-n_i n_j} - \theta_k \right\|^2 \leq n_i C'.$$

Using the symmetric one when permuting i and j, the triangular inequality and $(a+b)^2 \leq 2(a^2+b^2)$, we obtain, for K, L large enough :

$$\sum_{k=K}^{K+L} \left\| \alpha(1-\lambda^{n_i n_j}) \left(\frac{b_j}{1-\lambda^{n_j}} - \frac{b_i}{1-\lambda^{n_i}} \right) \lambda^{-k} \right\|^2 \leq 2(n_i + n_j) C'.$$

This implies that $(\|\alpha'\lambda^{-k}\|)_{k\geq 0} \in l^2(\mathbb{N})$, for some $\alpha' \neq 0$. By a famous theorem of Pisot, cf Salem [14], Theorem A, chap. 1, we obtain that $1/\lambda$ is a Pisot number, with moreover :

$$\alpha(1-\lambda^{n_i n_j}) \left(\frac{b_j}{1-\lambda^{n_j}} - \frac{b_i}{1-\lambda^{n_i}} \right) \in \mathbb{Q}[\lambda], \text{ hence } \alpha \left(\frac{b_j}{1-\lambda^{n_j}} - \frac{b_i}{1-\lambda^{n_i}} \right) \in \mathbb{Q}[\lambda].$$

To conclude, let $b = 1/\alpha \neq 0$. We also fix $0 \leq i_0 \leq N$ and set $\beta = b_{i_0}/(1-\lambda^{n_{i_0}})$. Then for any $0 \leq j \leq N$, either $b_j/(1-\lambda^{n_j}) = \beta$, in which case $b_j = (1-\lambda^{n_j})\beta$, or $b_j/(1-\lambda^{n_j}) \neq \beta$ and the previous reasoning for (i_0, j) furnishes $b_j = (1-\lambda^{n_j})\beta + ba_j$, for some $a_j \in \mathbb{Q}[\lambda]$. This ends the proof of the theorem.

□

4 Complements

4.1 A numerical example

Considering an example as simple as possible which is not homogeneous, take $N = 1$ and the two contractions $\varphi_0(x) = \lambda x$, $\varphi_1(x) = 1 + \lambda^2 x$, where $1/\lambda > 1$ is a Pisot number, with probability vector $p = (p_0, p_1)$. Then $n_0 = 1$ and $n_1 = 2$ and ν_p is the law of $\sum_{l\geq 0} \varepsilon_l \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}$, with $(\varepsilon_n)_{n\geq 0}$ *i.i.d.*, with common law $\mathrm{Ber}(p_1)$.

Taking $m = 1$, we have for $k \in \mathbb{Z}$, $r \in \{0, 1\}$:

$$F_p(k) = \mathbb{E}\left(e^{2i\pi\lambda^k \sum_{l\geq 0} \varepsilon_l \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}} \right),\ G_p(k,r) = \mathbb{E}\left(e^{2i\pi \sum_{l\geq 0} \varepsilon_l \lambda^{k-(n_{\varepsilon_0} + \cdots + n_{\varepsilon_l})}} 1_{n_{\varepsilon_0} > r} \right),$$

leading to $\Delta_p = \Delta_p(k) = F_p(k)G_p(k,0) + F_p(k+1)G_p(k+1,1)$. We now discuss the choice of p and Pisot number $1/\lambda$.

A degenerated example (the invariant measure being automatically singular) is for instance given by $\lambda = (3-\sqrt{5})/2 < 1/2$. Nevertheless, it is interesting to notice that $\lambda^{-n} \equiv -\lambda^n$, $n \geq 0$. Taking $p = (1/2, 1/2)$, one can check that $\Delta_p = \Delta_p(1) = |F_p(1)|^2 + |F_p(2)|^2/2$. Necessarily $\Delta_p > 0$. Indeed, $k \longmapsto F_p(k)$ verifying a linear recurrence of order two, the equality $\Delta_p = 0$ would give $F_p(k) = 0$ for all k, but $F_p(k) \to 1$, as $k \to +\infty$. Notice that $(3-\sqrt{5})/2$ is the largest λ with this

property (it has to be a root of some $X^2 - aX + 1$, $a \geq 0$). Let us mention that in general Δ_p is not a real number; cf the pictures below.

To study an interesting example, we take into account the similarity dimension $s(p, r)$, rewritten here as $s(p, \lambda)$:

$$s(p, \lambda) = \frac{p_0 \ln p_0 + p_1 \ln p_1}{p_0 \ln \lambda + p_1 \ln(\lambda^2)}.$$

The condition $s(p, \lambda) \geq 1$ is equivalent to $p_0 \ln p_0 + (1 - p_0) \ln(1 - p_0) - (2 - p_0) \ln \lambda \leq 0$. As a function of p_0, the left-hand side has a minimum value $-\ln(\lambda + \lambda^2)$, attained at $p_0 = 1/(1 + \lambda)$. As a first attempt, taking for $1/\lambda$ the golden mean $(\sqrt{5}+1)/2 = 1,618...$ appears in fact not to be a good idea, as in this case $\lambda + \lambda^2 = 1$, giving $s(p, \lambda) \leq 1$.

We instead take for $1/\lambda$ the Plastic number, the smallest Pisot number (cf Siegel [15]). It is defined as the unique real root of $X^3 - X - 1$. Approximately, $1/\lambda = 1.324718....$ For this λ :

$$s(p, \lambda) > 1 \Longleftrightarrow 0,203... < p_0 < 0,907....$$

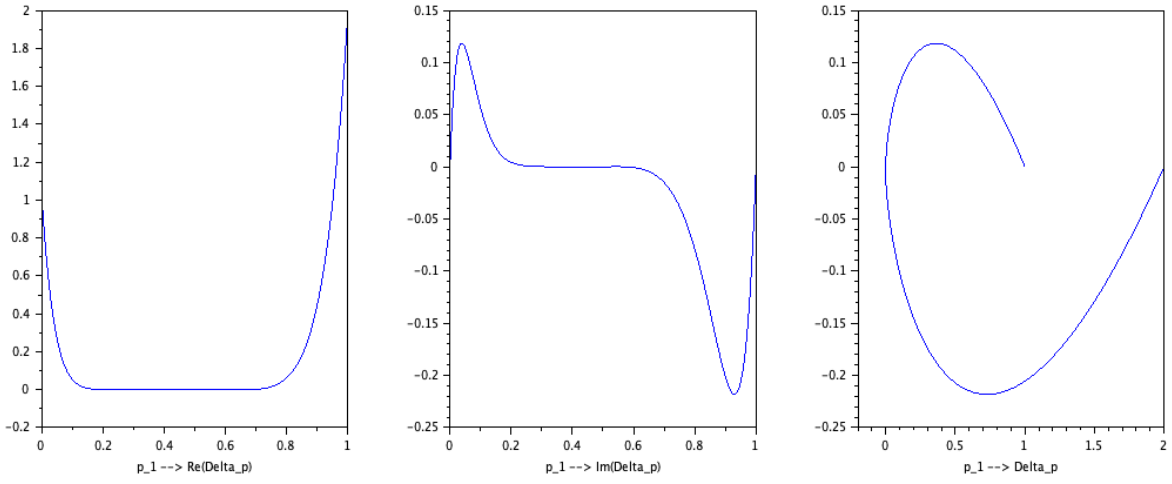
The other roots of $X^3 - X - 1 = 0$ are conjugate numbers $\rho e^{\pm i\theta}$. From the relations $1/\lambda + 2\rho \cos\theta = 0$ and $(1/\lambda)\rho^2 = 1$, we deduce $\rho = \sqrt{\lambda}$ and $\cos\theta = -1/(2\lambda^{3/2})$, thus $\theta = \pm 2.43...$ rad. For computations, the relations $\lambda^{-n} + \rho^n e^{in\theta} + \rho^n e^{-in\theta} \in \mathbb{Z}$, $n \geq 0$, furnish :

$$\lambda^{-n} \equiv -2(\sqrt{\lambda})^n \cos(n\theta).$$

Let us finally compute the extreme values of $p \longmapsto \Delta_p$. We have $\Delta_{(1,0)} = F_{(1,0)}(0)G_{(1,0)}(0, 0) = 1$. At the other extremity :

$$\begin{aligned}
\Delta_{(0,1)} &= F_{(0,1)}(0)G_{(0,1)}(0, 0) + F_{(0,1)}(1)G_{(0,1)}(1, 1) \\
&= e^{2i\pi \sum_{l\geq 0} \lambda^{2l}} e^{2i\pi \sum_{l\geq 0} \lambda^{-2(l+1)}} + e^{2i\pi\lambda \sum_{l\geq 0} \lambda^{2l}} e^{2i\pi \sum_{l\geq 0} \lambda^{1-2(l+1)}} \\
&= e^{2i\pi \left(\frac{1}{1-\lambda^2} - 2\sum_{l\geq 0} (\sqrt{\lambda})^{2l} \cos(2l\theta) \right)} + e^{2i\pi \left(\frac{\lambda}{1-\lambda^2} - 2\sum_{l\geq 0} (\sqrt{\lambda})^{2l+1} \cos((2l+1)\theta) \right)} \\
&= e^{2i\pi \left(\frac{1}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\lambda e^{2i\theta}}{1-\lambda e^{2i\theta}} \right) \right)} + e^{2i\pi \left(\frac{\lambda}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\sqrt{\lambda} e^{i\theta}}{1-\lambda e^{2i\theta}} \right) \right)}.
\end{aligned}$$

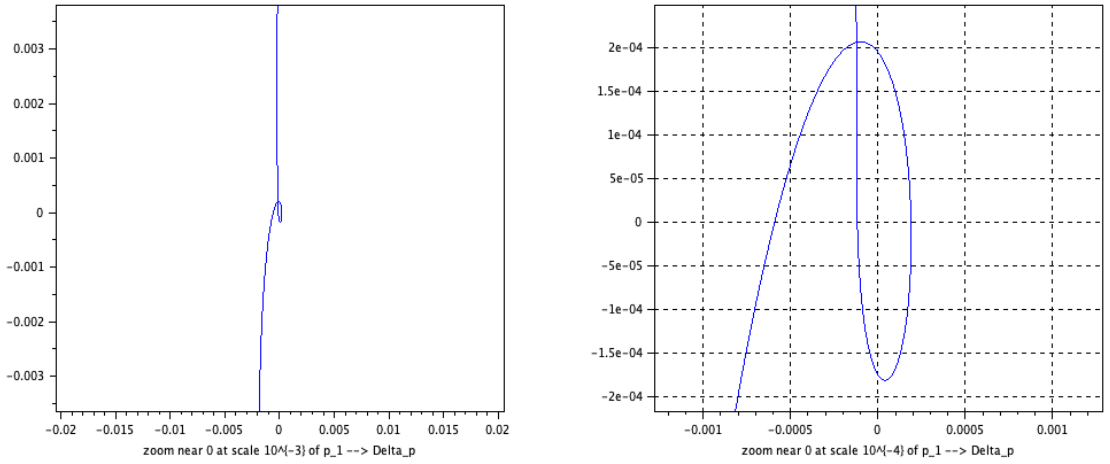
A not difficult computation, shortened by the observation that $(1 - \lambda e^{2i\theta})(1 - \lambda e^{-2i\theta}) = 1/\lambda$, shows that the arguments in the exponential terms (after the $2i\pi$) are respectively equal to 3 and 0, leading to $\Delta_{(0,1)} = 2$. Recalling that $p = (1 - p_1, p_1)$, below are respectively drawn the real-analytic maps $p_1 \longmapsto \mathrm{Re}(\Delta_p)$, $p_1 \longmapsto \mathrm{Im}(\Delta_p)$ and the parametric curve $p_1 \longmapsto \Delta_p$, $0 \leq p_1 \leq 1$.



The first two pictures indicate that $p_1 \longmapsto \Delta_p$ spends a rather long time near 0, with $\mathrm{Re}(\Delta_p)$ and $\mathrm{Im}(\Delta_p)$ both around 10^{-4}. Let us precise here that one can exploit the product form (given by the exponential) inside the expectation appearing in $F_p(k)$ and $G_p(k,r)$ and make a deterministic numerical computation of Δ_p, with nearly an arbitrary precision, based on a dynamical programming (using a binomial tree). For example, one can obtain the rather remarquable value :

$$\Delta_{(1/2,1/2)} = 0,000178... + i0,0000491...,$$

where all digits are exact. In this case, $s((1/2,1/2),\lambda) = 1,64... > 1$. The above pictures were drawn with 1000 points, each one determined with a sufficient precision. This allows to safely zoom on the neighbourhood of 0, the interesting region. We obtain the following surprising pictures, the one on the right-hand side containing around 500 points :



One might guess the existence of profound reasons behind these pictures, that would in particular clarify the condition $\Delta_p \neq 0$. Further investigations are necessary.

From the previous numerical analysis, we conclude that the curve $p_1 \longmapsto \Delta_p$ is rather convincingly not touching 0. It may certainly be possible to build a rigorous numerical proof of this fact, but this is not the purpose of the present paper. Being confident in this, we can state :

Numerical Theorem 4.1
Let $N = 1$ and the two contractions $\varphi_0(x) = \lambda x$ and $\varphi_1(x) = 1 + \lambda^2 x$, where $1/\lambda > 1$ is the Plastic number. Then for any probability vector $p \in \mathcal{C}_1$, the invariant measure ν_p is not Rajchman.

4.2 On the extension to a "contracting on average" context

In this last part, we investigate a more general situation. Let $N \geq 0$ and affine maps $\varphi_k(x) = r_k x + b_k$, with $r_k > 0$ but not necessarily $r_k < 1$, for $0 \leq k \leq N$. Let $p \in \mathcal{C}_N$ and suppose that the (r_k) and p verify the following "contraction on average" condition :

$$\sum_{0 \leq k \leq N} p_k \log r_k < 0.$$

When this holds, as in the introduction, existence, unicity and purity of ν_p are guaranteed and ν_p is the law of $\sum_{l \geq 0} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}$, where $(\varepsilon_l)_{l \in \mathbb{Z}}$ are *i.i.d.* random variables with law p.

Let us now place in a context similar to that of Theorem 1.1. Let $0 < \lambda < 1$ be such that $1/\lambda$ is a Pisot number and for all $0 \leq k \leq N$, $r_k = \lambda^{n_k}$, for an integer $n_k \in \mathbb{Z}$, assuming that $n_0 \leq \cdots \leq n_N$. The "contraction on average" condition reads as $\sum_{0 \leq k \leq N} p_k n_k > 0$. We also suppose that $b_k = b a_k + c(1 - \lambda^{n_k})$, with $a_k \in \mathbb{Q}[\lambda]$, for all $0 \leq k \leq N$, with common reals b and c.

Considering the Birkhoff sums associated to the (n_{ε_l}), let us introduce cocycle notations $(S_l)_{l\in\mathbb{Z}}$, where $S_l = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}$, for $l \geq 1$, $S_0 = 0$ and $S_l = -n_{\varepsilon_l} - \cdots - n_{\varepsilon_{-1}}$, for $l \leq -1$. Denote by θ the formal shift such that $\theta\varepsilon_l = \varepsilon_{l+1}$, $l \in \mathbb{Z}$. We have for all k and l in $\mathbb{Z}$:

$$S_{k+l} = S_k + \theta^k S_l.$$

We still write $a_k = (1/q)\sum_{0\leq l\leq s} p_{l,k}\lambda^l$, $0 \leq k \leq N$, and take some integer $m \neq 0$. Starting as in the proof of Theorem 1.1, we introduce :

$$F_p(k) = \mathbb{E}\left(e^{2i\pi m\sum_{l\geq 0}(qa_{\varepsilon_l})\lambda^{k+S_l}}\right),\ k \in \mathbb{Z}.$$

On the other side define in the same way :

$$G_p(k,r) = \mathbb{E}\left(e^{2i\pi m\sum_{l\leq -1}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{\max_{l\geq 1} S_{-l}<-r}\right),\ k \in \mathbb{Z},\ r \in \mathbb{Z}.$$

We have the following strict extension of Lemma 2.1 :

Lemma 4.1
Let $\Delta_p(k) = \sum_{0\leq r<n_N} F_p(k+r)G_p(k+r,r)$. Then $\Delta_p(k) = \Delta_p(k+1)$, $k \in \mathbb{Z}$.

Proof of the lemma :
This is a little more involved than the proof of Lemma 2.1. First of all, we notice that $\Delta_p(k) = \sum_{0\leq r\leq n_N} F_p(k+r)G_p(k+r,r)$, since again $G_p(k+n_N, n_N) = 0$. We also have as before $\Delta_p(k+1) = \sum_{1\leq r\leq n_N} F_p(k+r)G_p(k+r,r-1)$. Introducing :

$$H_p(k,r) = \mathbb{E}\left(e^{2i\pi m\sum_{l\leq -1}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{\max_{l\geq 1} S_{-l}=-r}\right),\ k \in \mathbb{Z},\ r \in \mathbb{Z},$$

we obtain :

$$\Delta_p(k) - \Delta_p(k+1) = F_p(k)G_p(k,0) - \sum_{1\leq r\leq n_N} F_p(k+r)H_p(k+r,r).$$

Using independence :

$$\Delta_p(k) - \Delta_p(k+1) = \mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{\max_{l\geq 1} S_{-l}<0}\right) - \sum_{1\leq r\leq n_N} L_p(k+r,r),$$

with $L_p(k+r,r) = \mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+r+S_l}}1_{\max_{l\geq 1} S_{-l}=-r}\right)$. For $r \geq 1$, introduce the set :

$$A_r = \left\{\max_{l\geq 1} S_{-l} < 0 \text{ and the first } l \geq 1 \text{ with } S_l > 0 \text{ checks } S_l = r\right\}.$$

We shall show that for $1 \leq r \leq n_N$:

$$L_p(k+r,r) = \mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{A_r}\right). \tag{3}$$

From this we deduce :

$$\Delta_p(k)-\Delta_p(k+1) = \mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{\max_{l\geq 1} S_{-l}<0}\right) - \mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{\cup_{1\leq r\leq n_N} A_r}\right).$$

Since $\cup_{1\leq r\leq n_N} A_r = \{\max_{l\geq 1} S_{-l} < 0\}$, we conclude that $\Delta_p(k) - \Delta_p(k+1) = 0$, as desired.

We now show (3). Going to $-\infty$ and looking at the last moment $p \geq 1$ when $S_{-p} = -r$, we can decompose $L_p(k+r,r)$ into :

$$
\begin{aligned}
L_p(k+r,r) &= \sum_{p\geq 1}\mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+r+S_{-p}+(S_l-S_{-p})}}1_{S_{-1}\leq -r,\cdots,S_{-p+1}\leq -r,S_{-p}=-r,S_{-p-u}<-r,u\geq 1}\right)\\
&= \sum_{p\geq 1}\mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+(S_l-S_{-p})}}1_{S_{-1}-S_{-p}\leq 0,\cdots,S_{-p+1}-S_{-p}\leq 0,S_{-p}=-r,S_{-p-u}-S_{-p}<0,u\geq 1}\right)\\
&= \sum_{p\geq 1}\mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+\theta^{-p}S_{l+p}}}1_{\theta^{-p}S_{-1+p}\leq 0,\cdots,\theta^{-p}S_{-p+1+p}\leq 0,\theta^{-p}S_p=r,\theta^{-p}S_{-p-u+p}<0,u\geq 1}\right).
\end{aligned}
$$

Using the invariance of the law of $(\varepsilon_l)_{l\in\mathbb{Z}}$ with respect to shift of coordinates :

$$
\begin{aligned}
L_p(k+r,r) &= \sum_{p\geq 1}\mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_{l+p}})\lambda^{k+S_{l+p}}}1_{S_{p-1}\leq 0,\cdots,S_1\leq 0,S_p=r,S_{-u}<0,u\geq 1}\right)\\
&= \sum_{p\geq 1}\mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{S_{p-1}\leq 0,\cdots,S_1\leq 0,S_p=r,S_{-u}<0,u\geq 1}\right)\\
&= \mathbb{E}\left(e^{2i\pi m\sum_{l\in\mathbb{Z}}(qa_{\varepsilon_l})\lambda^{k+S_l}}1_{A_r}\right).
\end{aligned}
$$

This completes the proof of (3) and finishes the proof of the lemma.

□

We thus set again $\Delta_p = \Delta_p(k)$. At this point and following the plan of the proof of Theorem 1.1, we try to analyze the regularity of $p \longmapsto \Delta_p$ and as a first step that of $p \longmapsto F_p(0)$ on the convex domain :

$$\mathcal{D} = \{p \in \mathcal{C}_N,\ \sum_{0\leq k\leq N} p_k n_k > 0\}.$$

Continuity is rather clear, but the real-analytic character a priori requires more work. Introducing again $S = \{0,\cdots,N\}^{\mathbb{N}}$ and $\mu_p = (\sum_{0\leq j\leq N} p_j\delta_j)^{\otimes\mathbb{N}}$ on S, we still have :

$$F_p(0) = \int_S g\ d\mu_p,$$

with $g(x) = e^{2i\pi mq\left(\sum_{l\geq 0} a_{x_l}\lambda^{n_{x_0}+\cdots+n_{x_{l-1}}}\right)}$, where $x = (x_0, x_1, \cdots) \in S$. However this function is not continuous on S and in fact only defined μ_p-almost-everywhere.

Mention finally that *Step 3.* in the proof of Theorem 1.1 directly goes through as there is always some $0 \leq j \leq N$, such that $p^j = (0,\cdots,0,1,0\cdots,0) \in \mathcal{D}$, with the 1 at place j. The extension of Theorem 1.2 a priori seems less delicate. Upgrading Theorems 1.1 and 1.2 to the more general "contracting on average" situation deserves a separate study.

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References

[1] V. Baladi, *Positive transfer operators and decay of correlations.* Advanced Series in Nonlinear Dynamics, 16. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[2] P. Erdös, *On a family of symmetric Bernoulli convolutions.* Amer. J. Math. 61 (1939), 974-975.

[3] K. Falconer, *Fractal geometry. Mathematical foundations and applications.* Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[4] W. Feller, *An introduction to probability theory and its applications.* Vol. II. Second edition John Wiley & Sons, Inc., New York-London-Sydney 1971.

[5] D.-J. Feng, K.-S. Lau, *Multifractal formalism for self-similar measures with weak separation condition*. J. Math. Pures Appl., 92 (2009), 407-428.

[6] J. Hutchinson, *Fractals and self-similarity*. Indiana U. J. of Math., 30, (1981), p. 713-747.

[7] T. Kato, *Perturbation theory for linear operators*. Second edition, Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1976.

[8] J. Li, T. Sahlsten, *Trigonometric series and self-similar sets*. 2019, arXiv:1902.00426.

[9] R. Lyons, *Seventy years of Rajchman measures*. In Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), number Special Issue, pages 363-377, 1995.

[10] J. Neunhäuserer, *A family of exceptional parameters for non-uniform self-similar measures*. Electron. Commun. Probab. 16 (2011), 192-199.

[11] Y. Peres, W. Schlag and B. Solomyak, *Sixty years of Bernoulli convolutions*. In Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), volume 46 of Progr. Probab., pages 39-65. Birkhäuser, Basel, 2000.

[12] D. Ruelle, *Statistical mechanics of one-dimensional lattice gas*, Comm. Math. Phys. 9 (1968), 267-278.

[13] R. Salem, *A remarkable class of algebraic integers, proof of a conjecture by Vijayaraghavan.* Duke Math. J. 11 (1944), 103-108.

[14] R. Salem, *Algebraic numbers and Fourier analysis*. D. C. Heath and Co., Boston, Mass. 1963.

[15] C. Siegel, *Algebraic numbers whose conjugates lie in the unit circle.* Duke Math. J. 11, 597-602, 1944.

[16] B. Solomyak, *Notes on Bernoulli convolutions.* Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1, 207-230, Proc. Sympos. Pure Math., 72, Part 1, Amer. Math. Soc., Providence, RI, 2004.

[17] B. Solomyak, *Fourier decay for self-similar measures.* 2019, arXiv:1906.12164.

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