On the Rajchman property for self-similar measures

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Abstract

For classical Bernoulli convolutions, the Rajchman property, i.e. the convergence to zero at infinity of the Fourier transform, was characterized by successive works of Erdös [2] and Salem [12]. We prove weak forms of their results for general self-similar measures associated to affine contractions of the real line.

1 Introduction

In the present work we consider the extension of some well-known results concerning Bernoulli convolutions to a more general context of self-similar measures. For a Borel probability measure m on the real line, define its Fourier transform as :

$$\hat{m}(t) = \int_{\mathbb{R}} e^{itx} \, dm(x), \;\; t \in \mathbb{R}.$$

We say that m is Rajchman, if $\hat{m}(t) \to 0$, as $t \to +\infty$. This property is very important in Harmonic Analysis, cf for example Lyons [8]. Let us now recall standard notions on self-similar measures, from a probabilistic angle.

We write $\mathcal{L}(X)$ for the law of a real random variable X. Let $N \geq 0$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, with $0 < r_k < 1$, $x \in \mathbb{R}$. For $n \geq 0$, compositions have the form :

$$\varphi_{j_{n-1}} \circ \cdots \circ \varphi_{j_0}(x) = r_{j_{n-1}} \cdots r_{j_0} x + \sum_{l=0}^{n-1} b_{j_l} r_{j_{n-1}} \cdots r_{j_{l+1}}.$$

Introduce the convex $\mathcal{C}_N = \{p = (p_0, \cdots, p_N) \mid p_i \geq 0, \; \sum_i p_i = 1\}$ and fix a probability vector $p \in \mathcal{C}_N$. We now compose the contractions at random, independently, according to p. Precisely, let X_0 be any random variable and $(\varepsilon_n)_{n \geq 0}$ be independent and identically distributed random variables (*i.i.d.*), independent from X_0, with $\mathbb{P}(\varepsilon_n = k) = p_k$, $0 \leq k \leq N$. We consider the Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{R}$ defined by $X_n = \varphi_{\varepsilon_{n-1}} \circ \cdots \circ \varphi_{\varepsilon_0}(X_0)$, $n \geq 0$.

It is classical that $(X_n)_{n \geq 0}$ has a unique invariant measure. This can be seen for example from the fact that $\mathcal{L}(X_n) = \mathcal{L}(\tilde{X}_n)$, where :

$$\tilde{X}_n = \varphi_{\varepsilon_0} \circ \cdots \circ \varphi_{\varepsilon_{n-1}}(X_0) = r_{\varepsilon_0} \cdots r_{\varepsilon_{n-1}} X_0 + \sum_{l=0}^{n-1} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}.$$

Since $\tilde{X}_n$ converges almost-surely to $X := \sum_{l \geq 0} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}$, this implies that $\nu_n := \mathcal{L}(X_n)$ weakly converges to $\nu := \mathcal{L}(X)$. By construction :

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$$\nu_{n+1} = \sum_{0 \le k \le N} p_k \nu_n \circ \varphi_k^{-1},$$

so, taking the limit as $n \to +\infty$, ν is a solution of the equation :

$$\nu = \sum_{0 \le k \le N} p_k \nu \circ \varphi_k^{-1}. \tag{1}$$

The previous convergence implies that the solution of this equation is unique in the class of Borel probability measures. Moreover ν has to be of pure type, i.e. either absolutely continuous with respect to Lebesgue measure or atomic or else singular continuous, since each term in its Radon-Nikodym decomposition verifies equation (1). Using the repartition function, it is not difficult to observe that ν is continuous if and only if the fixed points $b_k/(1-r_k)$ of the φ_k, $0 \le k \le N$, are not all equal (see for example Feng-Lau [5]). In case of equality, ν is the Dirac mass at the common fixed point. This trivial case excluded, a difficult problem is to characterize absolute continuity in terms of the parameters $r := (r_k)$ and (b_k).

An example with a long history is that of Bernoulli convolutions, corresponding to $N = 1$, the system of contractions $\varphi_0(x) = \lambda x - 1$, $\varphi_1(x) = \lambda x + 1$, $0 < \lambda < 1$, and $p = (1/2, 1/2)$. Notice that when the contraction rates are equal, ν is an infinite convolution (this is not true in general). Although we discuss below some works in this context, but we will not present here the vast subject of Bernoulli convolutions, addressing the reader to detailed surveys, Peres-Schlag-Solomyak [10] or more recently Solomyak [15].

For general self-similar measures, an important aspect of the problem, that we shall not enter, and an active line of research, concerns the Hausdorff dimension of the measure ν. In a large generality, cf for example Falconer [3], one has an “entropy/Lyapunov exponent” upper-bound :

$$\mathrm{Dim}_H(\nu) \le \min\{1, s(p,r)\}, \text{ where } s(p,r) := \frac{-\sum_{i=0}^N p_i \log p_i}{-\sum_{i=0}^N p_i \log r_i}.$$

The quantity $s(p,r)$ is called the singularity dimension of the measure. The equality $\mathrm{Dim}_H(\nu) = 1$ does not mean that ν has a density, but the inequality $s(p,r) < 1$ implies that ν is singular. The interesting domain of parameters therefore corresponds to $s(p,r) \ge 1$.

We focus here on another fundamental tool, the Fourier transform $\hat{\nu}$. If ν is not Rajchman, the Riemann-Lebesgue lemma implies that ν is singular. This property was used by Erdös [2] in the context of Bernoulli convolutions. Erdös proved that if $0 < \lambda < 1$ is such that $1/\lambda$ is a Pisot number, then ν is not Rajchman. The reverse implication was next shown by Salem [12], thus giving a complete characterization of the Rajchman property for Bernoulli convolutions.

The aim of the present article is to study the Rajchman property in the more general context of self-similar measures. The non-Rajchman character was shown to hold only for a very small set of parameters (it is countable for Bernoulli convolutions), by Solomyak [16] : as soon as the (φ_k) do not have a common fixed point and p is not degenerated, then outside a zero-Hausdorff dimensional set for the contractions rates, the Fourier transform even has a power decay at infinity. Our purpose is to focus on the exceptional set. In the sequel, we write ν_p instead of ν for the invariant measure, to state its dependence with respect to $p \in \mathcal{C}_N$. We shall first prove the following result.

Theorem 1.1
Let $0 < \lambda < 1$ be such that $1/\lambda$ is a Pisot number. Let $b \in \mathbb{R}$, $N \ge 0$ and for $0 \le k \le N$ affine contractions $\varphi_k(x) = \lambda^{n_k} x + b_k$, for integers $n_k \ge 1$ and $b_k = b a_k$, with $a_k \in \mathbb{Q}[\lambda]$. Then for $p \in \mathcal{C}_N$ outside a finite set, the invariant measure ν_p is not Rajchman.

This is a way of producing continuous singular invariant measures. We in fact give very concrete examples in the last section. Concerning the existence of singular measures in the inhomogeneous case, let us mention the non-explicit examples, using algebraic curves, of Neunhäuserer [9]. To present the result in the other direction, introduce the following set :

$$\mathcal{A} = \{x > 1 \mid \exists \alpha > 0,\ \mathrm{dist}(\alpha x^n, \mathbb{Z}) \to 0, \text{ as } n \to +\infty\}.$$

The set $\mathcal{A}$ is countable and contains the set of Pisot numbers. An old problem is whether there is equality or not, cf Salem [13], chap. I.4. We shall show :

Theorem 1.2
Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, with no common fixed point, and a probability vector p interior to $\mathcal{C}_N$. If the invariant measure ν_p is not Rajchman, there exist $0 < \lambda < 1$ such that $1/\lambda \in \mathcal{A}$ and relatively prime integers $n_k \geq 1$ such that $r_k = \lambda^{n_k}$, $0 \leq k \leq N$.

We have nothing to say on the (b_k). Credit for the existence of the (n_k) in the previous theorem is due to Li and Sahlsten [7], who showed that ν_p is Rajchman whenever $\log r_i / \log r_j \notin \mathbb{Q}$, for some i, j, with moreover some logarithmic decay at infinity of $\hat{\nu}$ under a Diophantine condition. Their work, involving renewal theory, was one of the motivation for the present paper. Coming after them, we simplify their proof and relate it to the standard renewal theorem.

In sections 2 and 3, we successively prove Theorems 1.1 and 1.2. In the last section, we discuss the results and show surprising numerical simulations involving the Plastic number. They suggest the possibility that, at the end of the story, in theorem 1.1 the finite set is actually empty and the Rajchman property in fact equivalent to the sufficient statement of this theorem. This would then characterize the Rajchman property in the context of self-similar measures.

2 Proof of Theorem 1.1

We can restrict to $N \geq 1$ and $b \neq 0$, otherwise ν_p is a Dirac mass. We next recall that a Pisot number is an algebraic integer > 1 with Galois conjugates (the other roots of its minimal unitary polynomial) of modulus strictly less than 1. We write $Q \in \mathbb{Z}[X]$ for the minimal polynomial of $1/\lambda$, of degree $s+1$, with roots $\alpha_0 = 1/\lambda, \alpha_1, \cdots, \alpha_s$, where $|\alpha_k| < 1$, for $1 \leq k \leq s$. A classical fact, used in *Step 3*, is that, for $n \geq 0$:

$$\lambda^{-n} + \sum_{1 \leq i \leq s} \alpha_i^n \in \mathbb{Z},$$

as a symmetric function of the roots of Q. We introduce the torus $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ and write equality in $\mathbb{T}$ as $x \equiv y$, for $x, y \in \mathbb{T}$. Recall that $\varphi_k(x) = \lambda^{n_k} x + b_k$, with $b_k = b a_k$. The a_k can be written as :

$$a_k = \frac{1}{q} \sum_{0 \leq l \leq s} p_{l,k} \lambda^l,$$

for integers $p_{l,k}$ and $q \geq 1$, with q independent on k. One can also freely assume $1 \leq n_0 \leq \cdots \leq n_N$. As in the introduction, the $(\varepsilon_n)_{n \geq 0}$ are *i.i.d.* random variables with law p. The probability $\mathbb{P}$ and the corresponding expectation $\mathbb{E}$ are related to these random variables. In the sequel, $m \neq 0$ is an integer, fixed at the end of the proof independently on $p \in \mathcal{C}_N$.

Step 1. We introduce the following quantities, where we mark the dependence in p :

$$F_p(k) = \mathbb{E}\left(e^{2i\pi m q \lambda^k \sum_{l \geq 0} a_{\varepsilon_l} \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}}\right),\ k \in \mathbb{Z}.$$

Notice that $F_p(k) = \hat{\nu}_p(2\pi m q \lambda^k / b)$. In a nearly symmetric way, using that $\lambda^{-n} \to 0$ in $\mathbb{T}$ exponentially fast and that $a_k \in \mathbb{Q}[\lambda]$, we define :

$$G_p(k, r) = \mathbb{E}\left(e^{2i\pi m q \sum_{l \geq 0} a_{\varepsilon_l} \lambda^{k - (n_{\varepsilon_0} + \cdots + n_{\varepsilon_l})}} 1_{n_{\varepsilon_0} > r}\right),\ k \in \mathbb{Z},\ r \geq 0.$$

When $r = 0$, the indicator $1_{n_{\varepsilon_0} > r}$ can be removed. Conditioning now with respect to the value of ε_0, we get recursive relations, for $k \in \mathbb{Z}$, $r \geq 0$:

$$\begin{cases} F_p(k) = \sum_{0\leq j\leq N} p_j e^{2i\pi mq\lambda^k a_j} F_p(k+n_j), \\ G_p(k,r) = \sum_{0\leq j\leq N, n_j>r} p_j e^{2i\pi mq\lambda^{k-n_j} a_j} G_p(k-n_j,0). \end{cases}$$

The following lemma is central in this work.

Lemma 2.1
For $k \in \mathbb{Z}$, define :

$$\Delta_p(k) = \sum_{0\leq r<n_N} F_p(k+r) G_p(k+r,r).$$

Then $\Delta_p(k) = \Delta_p(k+1)$, $k \in \mathbb{Z}$.

Proof of the lemma :
Notice that $\Delta_p(k) = \sum_{0\leq r\leq n_N} F_p(k+r)G_p(k+r,r)$, since $G_p(k+n_N,n_N)=0$. Also :

$$\Delta_p(k) = \sum_{0\leq r<n_N} F_p(k+1+r)G_p(k+1+r,r) = \sum_{1\leq r\leq n_N} F_p(k+r)G_p(k+r,r-1).$$

Therefore, using first the second recursive relation (for G_p) and next the first one (for F_p) :

$$\begin{aligned} \Delta_p(k)-\Delta_p(k+1) &= F_p(k)G_p(k,0) + \sum_{1\leq r\leq n_N} F_p(k+r)\left(G_p(k+r,r)-G_p(k+r,r-1)\right) \\ &= F_p(k)G_p(k,0) - \sum_{1\leq r\leq n_N} F_p(k+r) \sum_{0\leq j\leq N, n_j=r} p_j e^{2i\pi mq\lambda^{k+r-n_j} a_j} G_p(k+r-n_j,0) \\ &= F_p(k)G_p(k,0) - \sum_{1\leq r\leq n_N} F_p(k+r) \sum_{0\leq j\leq N, n_j=r} p_j e^{2i\pi mq\lambda^k a_j} G_p(k,0) \\ &= F_p(k)G_p(k,0) - G_p(k,0) \sum_{0\leq j\leq N} p_j e^{2i\pi mq\lambda^k a_j} \sum_{1\leq r\leq n_N} 1_{r=n_j} F_p(k+r) \\ &= G_p(k,0)\left(F_p(k) - \sum_{0\leq j\leq N} p_j e^{2i\pi mq\lambda^k a_j} \sum_{1\leq r\leq n_N} F_p(k+n_j)\right) = 0. \end{aligned}$$

This is the desired result. □

As a consequence, we can set $\Delta_p = \Delta_p(k)$. Now, the argument is simply the following. If ν_p were absolutely continuous with respect to Lebesgue measure, we would have $\lim_{k\to-\infty} F_p(k) = 0$. The quantities appearing in the definition of $\Delta_p(k)$ being all bounded by 1, this would imply $\lim_{k\to-\infty} \Delta_p(k) = 0$, hence $\Delta_p = 0$. We shall next show that this can only happen for finitely many values of $p \in \mathcal{C}_N$.

Step 2. Let us consider the regularity of $p \longmapsto \Delta_p$ on $\mathcal{C}_N$. We shall prove using standard methods that it is continuous and real-analytic on $\mathcal{C}_N$, in a sense precised below. This will result from the same properties for all $p \longmapsto F_p(k+r)$ and $p \longmapsto G_p(k+r,r)$. We treat the case of $p \longmapsto F_p(0)$, the other ones being exactly similar. Continuity is immediate, as this function is the uniform limit on $\mathcal{C}_N$, as $L \to +\infty$, of the continuous maps :

$$p \longmapsto \mathbb{E}\left(e^{2i\pi mq \sum_{l=0}^{L} a_{\varepsilon_l} \lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}}\right).$$

Fix now $p \in \mathcal{C}_N$. Let $\mathbb{N} = \{0,1,\cdots\}$ and the symbolic space $S = \{0,\cdots,N\}^{\mathbb{N}}$, equipped with the left shift σ. For $x = (x_0, x_1, \cdots) \in S$, we define :

$$g(x) = e^{2i\pi mq\left(\sum_{l\geq 0} a_{x_l}\lambda^{n_{x_0}+\cdots+n_{x_{l-1}}}\right)}.$$

Introducing the product measure $\mu_p = (\sum_{0\leq j\leq N} p_j\delta_j)^{\otimes\mathbb{N}}$ on S, we can write :

$$F_p(0) = \int_S g\ d\mu_p.$$

Denote by $C(S)$ the space of continuous functions $f : S \to \mathbb{C}$ and introduce the operator $P_p : C(S) \to C(S)$ defined by :

$$P_p(f)(x) = \sum_{0\leq j\leq N} p_j f((j,x)),\ x \in S,$$

where $(j,x) \in S$ is the word obtained by the left concatenation of the symbol j to x. The operator P_p is Markovian, i.e. $f \geq 0 \Rightarrow P_p(f) \geq 0$ and verifies $P_p\mathbf{1} = \mathbf{1}$, where $\mathbf{1}(x) = 1$, $x \in S$. The measure μ_p has the invariance property $\int_S P_p(f)\ d\mu_p = \int_S f\ d\mu_p$, $f \in C(S)$. For $f \in C(S)$ and $k \geq 0$, introduce the variation :

$$\mathrm{Var}_k(f) = \sup\{|f(x) - f(y)|,\ x_i = y_i,\ 0 \leq i < k\}.$$

For any $0 < \theta < 1$, let $|f|_\theta = \sup\{\theta^{-k}\mathrm{Var}_k(f),\ k \geq 0\}$, as well as $\|f\|_\theta = |f|_\theta + \|f\|_\infty$. We denote by $\mathcal{F}_\theta$ the complex Banach space of fonctions f on S such that $\|f\|_\theta < \infty$. Any $\mathcal{F}_\theta$ is preserved by P_p. Observe now that $g \in \mathcal{F}_\theta$ for $\lambda \leq \theta < 1$. We take for example $\theta = \lambda$ and write $\mathcal{F}$ for $\mathcal{F}_\theta$.

As a classical fact from Spectral Theory, cf Ruelle [11] or Baladi [1], the operator $P_p : \mathcal{F} \to \mathcal{F}$ satisfies a Perron-Frobenius theorem : the eigenvalue 1 is simple and the rest of its spectrum is contained in a closed disk of radius $\rho < 1$. By standard functional holomorphic calculus, cf Kato [6], when taking for Γ the circle centered at 1 with radius $0 < r < 1 - \rho$, the following operator, involving the resolvent, is a continuous (Riesz) projector on $\mathrm{Vect}(\mathbf{1})$:

$$\Pi_p = \int_\Gamma (zI - P_p)^{-1} dz.$$

Moreover, $\Pi_p(\mathcal{F})$ and $(I-\Pi_p)(\mathcal{F})$ are closed P_p-invariant subspaces with $\mathcal{F} = \Pi_p(\mathcal{F}) \oplus (I-\Pi_p)(\mathcal{F})$. In restriction to $(I - \Pi_p)(\mathcal{F})$, the spectral radius of P_p is less than ρ. In particular $\int_S f\ d\mu_p = 0$, for $f \in (I - \Pi_p)(\mathcal{F})$. This implies that for any $f \in \mathcal{F}$:

$$\Pi_p(f) = \left(\int_S f\ d\mu_p\right)\mathbf{1}.$$

Applying this to the function g of interest to us, we obtain that :

$$F_p(0)\mathbf{1} = \int_\Gamma (zI - P_p)^{-1}(g) dz.$$

Recall now that $N \geq 1$. Let $\eta' = (\eta_0, \cdots, \eta_{N-1})$ and $\eta = (\eta_0, \cdots, \eta_{N-1}, 1 - (\eta_0 + \cdots + \eta_{N-1}))$. The condition on η' for $p + \eta \in \mathcal{C}_N$ is written as $\eta' \in D_N(p)$. Explicitly the condition is :

$$-p_i \leq \eta_i \leq 1 - p_i,\ 0 \leq i \leq N-1, \text{ and } p_N \leq \eta_0 + \cdots + \eta_{N-1} \leq 1 + p_N.$$

For the sequel, let $B_N(0, R)$ be the open Euclidean ball in $\mathbb{R}^N$ centered at 0, of radius R.

Definition 2.2
A function $h : \mathcal{C}_N \to \mathbb{C}$ admits a development in series around a point $p \in \mathcal{C}_N$, if there exists $\varepsilon > 0$ such that for $\eta' = (\eta_0, \cdots, \eta_{N-1}) \in D_N(p) \cap B_N(0, \varepsilon)$ and writing $\eta = (\eta_0, \cdots, \eta_{N-1}, 1 - (\eta_0 + \cdots + \eta_{N-1}))$, then $h(p + \eta)$ is given by an absolutely converging series :

$$h(p + \eta) = \sum_{l_0\geq 0,\cdots,l_{N-1}\geq 0} A_{l_0,\cdots,l_{N-1}} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}}.$$

A function is real-analytic in $\mathcal{C}_N$ if it admits a development in series around every $p \in \mathcal{C}_N$.

For such a function, when non-constant, its zeroes are in finite number in $\mathcal{C}_N$, by the standard argument that the set of points where there is a null development in series is open and closed for the relative topology and thus equal to $\mathcal{C}_N$ by connexity if non-empty. In case of infinitely many zeros, any accumulation point (which exists, as $\mathcal{C}_N$ is compact) is such a point.

We now check below that $p \longmapsto F_p(0)$ is real-analytic in the previous sense. As already indicated, this property will be inheritated by $p \longmapsto \Delta_p$. In this direction, notice that :

$$P_{p+\eta} = P_p + \sum_{0 \leq j \leq N-1} \eta_j Q_j,$$

where $Q_j(f)(x) = f(j,x) - f(N,x)$. For $z \in \Gamma$ and η small enough :

$$\begin{aligned}(zI - P_{p+\eta})^{-1} &= \left(I - (zI - P_p)^{-1} \sum_{0 \leq j \leq N-1} \eta_j Q_j\right)^{-1} (zI - P_p)^{-1} \\ &= \sum_{n \geq 0} \sum_{0 \leq j_1, \cdots, j_n \leq N-1} \eta_{j_1} \cdots \eta_{j_n} (zI - P_p)^{-1} Q_{j_1} \cdots (zI - P_p)^{-1} Q_{j_n} (zI - P_p)^{-1}.\end{aligned}$$

This is clearly absolutely convergent in the Banach operator algebra, for small enough η, uniformly in $z \in \Gamma$. We rewrite it as :

$$(zI - P_{p+\eta})^{-1} = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} B_{l_0, \cdots, l_{N-1}}(z) \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}},$$

converging for the operator norm, uniformly in $z \in \Gamma$. This leads to :

$$F_{p+\eta}(0)\mathbf{1} = \int_\Gamma (zI - P_{p+\eta})^{-1}(g)\, dz = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}} \int_\Gamma B_{l_0, \cdots, l_{N-1}}(z)(g)\, dz.$$

Applying this equality at some particular $x \in S$, we obtain the desired development in series around p. This completes this step.

Step 3. We finish the argument. If ever $\Delta_p = 0$ for infinitely many $p \in \mathcal{C}_N$, by *Step 2*, $p \longmapsto \Delta_p$ has to be constant on $\mathcal{C}_N$ and equal to zero. We shall show that if $m \neq 0$ has been appropriately chosen at the beginning it is not possible. We start with a lemma.

Lemma 2.3
Let $d \geq 1$ and $u \in \mathbb{Z}$. The series $\sum_{l \in \mathbb{Z}} \lambda^{ld+u}$ is well-defined as an element of $\mathbb{T}$. There exists integers $A_{d,u}$, verifying $A_{d,d+u} = A_{d,u}$, and $B_d \neq 0$ such that :

$$\sum_{l \in \mathbb{Z}} \lambda^{ld+u} \equiv -\frac{A_{d,u}}{B_d}.$$

Proof of the lemma :
It is enough to take $0 \leq u < d$. Cutting the sum of the left-hand side in two and next using the conjugates $\alpha_1, \cdots, \alpha_s$ of $\alpha_0 = 1/\lambda$, we have the following equalities on the torus :

$$\begin{aligned}\sum_{l \in \mathbb{Z}} \lambda^{ld+u} &\equiv \frac{\lambda^u}{1 - \lambda^d} + \sum_{l \geq 1} \lambda^{u-ld} \equiv \frac{\lambda^u}{1 - \lambda^d} - \sum_{1 \leq i \leq s} \sum_{l \geq 1} \alpha_i^{ld-u} \equiv \frac{\lambda^u}{1 - \lambda^d} - \sum_{1 \leq i \leq s} \frac{\alpha_i^{d-u}}{1 - \alpha_i^d} \\ &\equiv -\left(\frac{(1/\lambda)^{d-u}}{1 - (1/\lambda)^d} + \sum_{1 \leq i \leq s} \frac{\alpha_i^{d-u}}{1 - \alpha_i^d}\right) = -\sum_{0 \leq i \leq s} \frac{\alpha_i^{d-u}}{1 - \alpha_i^d} \\ &\equiv -\frac{\sum_{0 \leq i \leq s} \alpha_i^{d-u} \prod_{0 \leq j \leq s, j \neq i}(1 - \alpha_j^d)}{\prod_{0 \leq i \leq s}(1 - \alpha_i^d)} \equiv -\frac{A_{d,u}}{B_d},\end{aligned}$$

as the numerator and denominator of the last fraction are symmetric functions of the $(\alpha_i)_{0\leq i\leq s}$, roots of a polynomial in $\mathbb{Z}[X]$.

$\square$

We conclude the argument. Fixing $0 \leq j \leq N$ and $p^j = (0, \cdots, 0, 1, 0, \cdots, 0)$, where the 1 is at place j, we have for $k \in \mathbb{Z}$ and $r \geq 0$:

$$F_{p^j}(k) = e^{2i\pi mq\lambda^k a_j \sum_{l\geq 0} \lambda^{ln_j}} \text{ and } G_{p^j}(k,r) = e^{2i\pi mq \sum_{l\geq 1} a_j \lambda^{k-ln_j}} 1_{n_j>r}.$$

Summing on $0 \leq r < n_N$ and making use of the indicator function $1_{n_j>r}$, we obtain :

$$\Delta_{p^j}(k) = \sum_{0\leq r<n_j} e^{2i\pi mq \sum_{l\in\mathbb{Z}} a_j \lambda^{k+r+ln_j}} = \sum_{0\leq r<n_j} e^{2i\pi m \sum_{0\leq u\leq s} p_{u,j} \sum_{l\in\mathbb{Z}} \lambda^{k+r+ln_j}}.$$

Notice in passing that the constant character with respect to k is now obvious, as we sum on r on a full period of length n_j. Taking $k = 0$ and using the previous lemma :

$$\Delta_{p^j} = \Delta_{p^j}(0) = \sum_{0\leq r<n_j} e^{-2i\pi m \sum_{0\leq u\leq s} p_{u,j} \frac{A_{n_j,r}}{B_{n_j}}}.$$

If for example m is a multiple of B_{n_j} for any $0 \leq j \leq N$, we get $\Delta_{p^j} = n_j \geq 1$, for all $0 \leq j \leq N$, which is more than enough. This ends the proof of the theorem.

$\square$

Remark. — A word on the method. Trying to follow the proof of Erdös [2], probabilistic computations involving the renewal theorem lead to the convergence of some sequence $(\hat{\nu}_p(\alpha\lambda^{-n}))$, as $n \to +\infty$. The limit was the product of a positive constant (involving some Green function) with $\Delta_p(0)$. Replacing n by $n-k$, one gets a necessarily invariant function of k. It appeared more efficient (but perhaps more abrupt) to restart the analysis directly from $\Delta_p(k)$.

We detail in the last section concrete examples falling in the interesting domain of parameters. For the moment we turn to the reverse side of the question.

3 Proof of Theorem 1.2

This time $N \geq 1$, p is interior to $\mathcal{C}_N$ and the $\varphi_k(x) = r_k + b_k$ do not all have the same fixed point. The $(\varepsilon_n)_{n\geq 0}$ are still *i.i.d* random variables with law p, to which $\mathbb{P}$ and $\mathbb{E}$ refer.

Step 1. We reprove in a simpler form the result of Li-Sahlsten [7], that if for some $0 \leq i \neq j \leq N$ one has $\log r_i / \log r_j \notin \mathbb{Q}$, then ν_p is Rajchman.

For $n \geq 1$, consider the random walk $S_n = -\log r_{\varepsilon_0} - \cdots - \log r_{\varepsilon_{n-1}}$, with $S_0 = 0$. For a real $s \geq 0$, introduce the finite stopping time $\tau_s = \min\{n \geq 0,\ S_n \geq s\}$ and write $\mathcal{T}_s$ for the corresponding sub-σ-algebra of the underlying σ-algebra. Taking $\alpha > 0$ and $s \geq 0$:

$$\begin{aligned}\hat{\nu}_p(\alpha e^s) &= \mathbb{E}\left(e^{i\alpha e^s \sum_{l\geq 0} b_{\varepsilon_l} e^{-S_l}}\right)\\ &= \mathbb{E}\left(e^{i\alpha e^{-S_{\tau_s}+s} \sum_{0\leq l<\tau_s} b_{\varepsilon_l} e^{-S_l+S_{\tau_s}}} e^{i\alpha e^{-S_{\tau_s}+s} \sum_{l\geq \tau_s} b_{\varepsilon_l} e^{-S_l+S_{\tau_s}}}\right).\end{aligned}$$

In the expectation, the first exponential term is $\mathcal{T}_s$-measurable. Also, the conditional expectation of the second exponential term with respect to $\mathcal{T}_s$ is just $\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s})$, as a consequence of the strong Markov property. It follows that :

$$\hat{\nu}_p(\alpha e^s) = \mathbb{E}\left(\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s}) e^{i\alpha e^{-S_{\tau_s}+s} \sum_{0\leq l<\tau_s} b_{\varepsilon_l}}\right).$$

This gives $|\hat{\nu}_p(\alpha e^s)| \leq \mathbb{E}\left(|\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s})|\right)$, so by the Cauchy-Schwarz inequality and a safe Fubini theorem consecutively :

$$
\begin{aligned}
|\hat{\nu}_p(\alpha e^s)|^2 \le \mathbb{E}\left(|\hat{\nu}_p(\alpha e^{-S_{\tau_s}+s})|^2\right) &= \mathbb{E}\left(\int_{\mathbb{R}^2} e^{i\alpha e^{-S_{\tau_s}+s}(x-y)}\, d\nu_p(x)d\nu_p(y)\right) \\
&\le \int_{\mathbb{R}^2} \mathbb{E}\left(e^{i\alpha e^{-S_{\tau_s}+s}(x-y)}\right)\, d\nu_p(x)d\nu_p(y).
\end{aligned}
$$

Let $Y := -\log r_{\varepsilon_0}$ and $a = 1/\mathbb{E}(Y)$. As the law of Y is non-lattice and with finite moment, it is a well-known consequence of the Blackwell theorem on the law of the overshoot, see Feller [4], that for any Riemann-integrable g, $\mathbb{E}(g(S_{\tau_s} - s)) \to a\int_0^{+\infty} g(x)\mathbb{P}(Y > x)\, dx$, as $s \to +\infty$. By dominated convergence, for any $\alpha > 0$:

$$
\limsup_{t\to+\infty} |\hat{\nu}_p(t)|^2 \le a \int_{\mathbb{R}^2} \left| \int_0^{+\infty} e^{i\alpha e^{-u}(x-y)}\mathbb{P}(Y > u)du \right| d\nu_p(x)d\nu_p(y).
$$

The inside term is uniformly bounded with respect to $(x, y) \in \mathbb{R}^2$, as Y has finite support, so $\mathbb{P}(Y > u) = 0$ for large u. We shall use dominated convergence once more, this time with $\alpha \to +\infty$. It is sufficient to show that for $\nu_p^{\otimes 2}$-almost every (x, y), the inside term goes to zero. Since ν_p is non-atomic, $\nu_p^{\otimes 2}$-almost-surely, $x \ne y$. If for example $x > y$:

$$
\int_0^{+\infty} e^{i\alpha e^{-u}(x-y)}\mathbb{P}(Y > u)du = \frac{1}{x-y}\int_0^{x-y} e^{i\alpha t}\mathbb{P}(Y > \log((x-y)/t)\; tdt,
$$

making the change of variable $t = e^{-u}(x - y)$. This now converges to 0, as $\alpha \to +\infty$, by the Riemann-Lebesgue lemma.

Step 2. From *Step 1*, if ν_p is not Rajchman, then $\log r_i/\log r_j \in \mathbb{Q}$, for all (i, j), hence $r_j = r_0^{p_j/q_j}$, with integers $p_j \ge 1$, $q_j \ge 1$, for $1 \le j \le N$. Let :

$$
n_0 = \prod_{1\le l\le N} q_l \text{ and } n_j = p_j \prod_{1\le l\le N, l\ne j} q_l,\ 1 \le j \le N.
$$

Setting $\lambda = r_0^{1/n_0} \in (0, 1)$, one has $r_j = \lambda^{n_j}$, $0 \le j \le N$. Up to taking some positive power of λ, one can assume that $\gcd(n_0, \cdots, n_N) = 1$. Recall in passing that $\mathcal{A}$ and the set of Pisot numbers are stable under positive powers. Using now some sub-harmonicity, one can reinforce the assumption that $\hat{\nu}_p(t)$ is not converging to 0, as $t \to +\infty$.

Lemma 3.1
There exists $1 \le \alpha \le 1/\lambda$ and $c > 0$ such that $\hat{\nu}_p(\alpha\lambda^{-k}) = c_k e^{i\theta_k}$, written in polar form, verifies $c_k \to c$, as $k \to +\infty$.

Proof of the lemma :
Let us write this time $S_n = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{n-1}}$, for $n \ge 1$, with $S_0 = 0$. Using that $\gcd(n_0, \cdots, n_N) = 1$, we fix $r \ge 1$ and $M \ge 1$ such that the support of S_r is included in $[1, M]$ and contains two consecutive points $1 \le u \le u + 1 \le M$. Since for all $t \in \mathbb{R}$:

$$
\hat{\nu}_p(t) = \mathbb{E}\left(e^{it\sum_{l\ge 0} b_{\varepsilon_l}\lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}}\right),
$$

we get the relation $\hat{\nu}(t) = \sum_{0\le j\le N} p_j e^{itb_j}\hat{\nu}(\lambda^{n_j}t)$, hence $|\hat{\nu}(t)| \le \sum_{0\le j\le N} p_j|\hat{\nu}(\lambda^{n_j}t)|$. Iterating :

$$
|\hat{\nu}(t)| \le \mathbb{E}\left(|\hat{\nu}_p(\lambda^{S_r}t)|\right),\ t \in \mathbb{R}. \tag{2}
$$

In particular, $|\hat{\nu}(t)| \le \max_{1\le l\le M} |\hat{\nu}(\lambda^l t)|$. We now set, for $\alpha > 0$:

$$
V_\alpha(k) := \max_{k\le l<k+M} |\hat{\nu}(\alpha\lambda^l)|,\ k \in \mathbb{Z}.
$$

The previous remarks imply that $V_\alpha(k) \leq V_\alpha(k+1)$, $k \in \mathbb{Z}$, $\alpha > 0$.

We now have $|\hat{\nu}_p(t_l)| \geq c' > 0$, along a sequence $t_l \to +\infty$. Write $t_l = \alpha_l \lambda^{-k_l}$, with $1 \leq \alpha_l \leq 1/\lambda$ and $k_l \to +\infty$. Up to taking a subsequence, $\alpha_l \to \alpha \in [1, 1/\lambda]$. Fixing $k \in \mathbb{Z}$, we have :

$$c' \leq V_{\alpha_l}(-k_l) \leq V_{\alpha_l}(-k),$$

as soon as l is large enough. By continuity, taking $l \to +\infty$, we get $c' \leq V_\alpha(-k)$, $k \in \mathbb{Z}$. As $k \longmapsto V_\alpha(-k)$ is non-increasing, $V_\alpha(-k) \to c \geq c'$, as $k \to +\infty$. We now show that necessarily, $|\hat{\nu}_p(\alpha\lambda^{-k})| \to c$, as $k \to +\infty$. If not, there exist $\varepsilon > 0$ and $(m_k) \to +\infty$, with $|\hat{\nu}_p(\alpha\lambda^{-m_k})| \leq c - \varepsilon$.

Recalling that $V_\alpha(-k) \to c$ and $|\hat{\nu}_p(\alpha\lambda^{-m_k})| \leq c - \varepsilon$, as $k \to +\infty$, consider (2) with $t = \alpha\lambda^{-m_k-u}$ and next $t = \alpha\lambda^{-m_k-u-1}$. Since the weights of p are all > 0, we obtain the existence of some $c_1 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-u})|, |\hat{\nu}(\alpha\lambda^{-m_k-u-1})|\} \leq c_1 < c.$$

Again via (2), with successively $t = \alpha\lambda^{-m_k-2u}$, $t = \alpha\lambda^{-m_k-2u-1}$ and $t = \alpha\lambda^{-m_k-2u-2}$ and still using that the weights of p are all > 0, we obtain some $c_2 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-2u})|,\ |\hat{\nu}(\alpha\lambda^{-m_k-2u-1})|,\ |\hat{\nu}(\alpha\lambda^{-m_k-2u-2})|\} \leq c_2 < c.$$

Etc, for some $c_{M-1} < c$ and k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-(M-1)u})|,\ \cdots,\ |\hat{\nu}(\alpha\lambda^{-m_k-(M-1)u-(M-1)})|\} \leq c_{M-1} < c.$$

This contradicts the fact that $V_\alpha(-k) \to c$, as $k \to \infty$. This allows to conclude that $|\hat{\nu}_p(\alpha\lambda^{-k})| \to c$, as $k \to \infty$.

□

Step 3. We complete the argument. As in the lemma, let $\hat{\nu}_p(\alpha\lambda^{-k}) = c_k e^{i\theta_k}$, $c_k \to c > 0$, as $k \to +\infty$. We start from the relation :

$$\hat{\nu}_p(\alpha\lambda^{-k}) = \sum_{0 \leq j \leq N} p_j e^{i\alpha\lambda^{-k} b_j} \hat{\nu}_p(\alpha\lambda^{-k+n_j}).$$

This furnishes for large k :

$$1 = \sum_{0 \leq j \leq N} p_j e^{i\alpha\lambda^{-k} b_j + i(\theta_{k-n_j} - \theta_k)} \frac{c_{k-n_j}}{c_k}.$$

For all j, we have $p_j > 0$ and $c_{k-n_j}/c_k \to 1$, as $k \to +\infty$. For obvious barycentric reasons, we obtain that for all $0 \leq j \leq N$:

$$\alpha\lambda^{-k} b_j + \theta_{k-n_j} - \theta_k \to_{k \to +\infty} 0, \text{ in } \mathbb{T}.$$

Hence, for all $r \geq 0$, $\alpha\lambda^{-k+rn_j} b_j + \theta_{k-(r+1)n_j} - \theta_{k-rn_j} \to_{k\to+\infty} 0$, in $\mathbb{T}$. Fixing $l_j \geq 1$ and summing over $0 \leq r \leq l_j - 1$, we obtain :

$$\alpha\lambda^{-k} b_j \left(\frac{1 - \lambda^{l_j n_j}}{1 - \lambda^{n_j}} \right) + \theta_{k-l_j n_j} - \theta_k \to_{k\to+\infty} 0, \text{ in } \mathbb{T}.$$

From the hypothesis, there exist $0 \leq i \neq j \leq N$, such that $b_i/(1 - \lambda^{n_i}) \neq b_j/(1 - \lambda^{n_j})$, otherwise the contractions φ_k have a common fixed point. Taking $l_j = n_i$ in the previous relation :

$$\alpha\lambda^{-k} b_j \left(\frac{1 - \lambda^{n_i n_j}}{1 - \lambda^{n_j}} \right) + \theta_{k-n_i n_j} - \theta_k \to_{k\to+\infty} 0, \text{ in } \mathbb{T}.$$

Writing the symmetric one when permuting i and j and subtracting, we obtain :

$$\alpha(1-\lambda^{n_i n_j})\left(\frac{b_j}{1-\lambda^{n_j}}-\frac{b_i}{1-\lambda^{n_i}}\right)\lambda^{-k}\to_{k\to+\infty}0,\text{ in }\mathbb{T}.$$

Thus $1/\lambda \in \mathcal{A}$, as announced, and this completes the proof of the theorem.

□

4 Discussion; examples

4.1 Some questions

In the context of Theorem 1.1, it would be important to have an interpretation of the quantity Δ_p, in terms of Hermitian product, or else. Also, is $\Delta_p = 0$ equivalent to ν_p Rajchman ? Is there an example where $\Delta_p = 0$ for some p ? See the interesting pictures at the end of this section.

Concerning the second part of Theorem 1.2, the method is essentially due to Salem [13]. For Bernoulli convolutions, the condition $|\hat{\nu}_p(t_l)| \geq c > 0$, with $t_l \to +\infty$, leads, using that $\hat{\nu}_p(t)$ is a product of cosinus, to $\sum \text{dist}(\alpha\lambda^{-n}, \mathbb{Z})^2 < \infty$, which furnishes $1/\lambda$ Pisot. Showing the convergence of this series would improve the statement of Theorem 1.2.

4.2 A numerical example

Considering an example as simple as possible which is not homogeneous, take $N = 1$ and the two contractions $\varphi_0(x) = \lambda x,\ \varphi_1(x) = 1 + \lambda^2 x$, where $1/\lambda > 1$ is a Pisot number, with probability vector $p = (p_0, p_1)$. Then $n_0 = 1$ and $n_1 = 2$ and ν_p is the law of $\sum_{l\geq 0} \varepsilon_l \lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}$, with $(\varepsilon_n)_{n\geq 0}$ *i.i.d.*, with common law Ber(p_1).

Taking $m = 1$, we have for $k \in \mathbb{Z}$, $r \in \{0, 1\}$:

$$F_p(k) = \mathbb{E}\left(e^{2i\pi\lambda^k \sum_{l\geq 0}\varepsilon_l \lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}}\right),\ G_p(k,r) = \mathbb{E}\left(e^{2i\pi\sum_{l\geq 0}\varepsilon_l\lambda^{k-(n_{\varepsilon_0}+\cdots+n_{\varepsilon_l})}}1_{n_{\varepsilon_0}>r}\right),$$

leading to $\Delta_p = \Delta_p(k) = F_p(k)G_p(k, 0) + F_p(k + 1)G_p(k + 1, 1)$. We now discuss the choice of p and Pisot number $1/\lambda$.

A degenerated example (the invariant measure being automatically singular) is for instance given by $\lambda = (3 - \sqrt{5})/2 < 1/2$. Nevertheless, it is interesting to notice that $\lambda^{-n} \equiv -\lambda^n$, $n \geq 0$. Taking $p = (1/2, 1/2)$, one can check that $\Delta_p = \Delta_p(1) = |F_p(1)|^2 + |F_p(2)|^2/2$. Necessarily $\Delta_p > 0$. Indeed, $k \longmapsto F_p(k)$ verifying a linear recurrence of order two, the equality $\Delta_p = 0$ would give $F_p(k) = 0$ for all k, but $F_p(k) \to 1$, as $k \to +\infty$. Notice that $(3 - \sqrt{5})/2$ is the largest λ with this property (it has to be a root of some $X^2 - aX + 1$, $a \geq 0$). Let us mention that in general Δ_p is not a real number; cf the pictures below.

To study an interesting example, we take into account the similarity dimension $s(p, r)$, rewritten here as $s(p, \lambda)$:

$$s(p,\lambda) = \frac{p_0 \ln p_0 + p_1 \ln p_1}{p_0 \ln \lambda + p_1 \ln(\lambda^2)}.$$

The condition $s(p, \lambda) \geq 1$ is equivalent to $p_0 \ln p_0 + (1 - p_0)\ln(1 - p_0) - (2 - p_0)\ln \lambda \leq 0$. As a function of p_0, the left-hand side has a minimum value $-\ln(\lambda + \lambda^2)$, attained at $p_0 = 1/(1 + \lambda)$. As a first attempt, taking for $1/\lambda$ the golden mean $(\sqrt{5}+1)/2 = 1,618...$ appears in fact not to be a good idea, as in this case $\lambda + \lambda^2 = 1$, giving $s(p, \lambda) \leq 1$.

We instead take for $1/\lambda$ the Plastic number, the smallest Pisot number (cf Siegel [14]). It is defined as the unique real root of $X^3 - X - 1$. Approximately, $1/\lambda = 1.324718....$ For this λ :

$$s(p,\lambda) > 1 \Longleftrightarrow 0,203... < p_0 < 0,907....$$

The other roots of $X^3 - X - 1 = 0$ are conjugate numbers $\rho e^{\pm i\theta}$. From the relations $1/\lambda + 2\rho\cos\theta = 0$ and $(1/\lambda)\rho^2 = 1$, we deduce $\rho = \sqrt{\lambda}$ and $\cos\theta = -1/(2\lambda^{3/2})$, thus $\theta = \pm 2.43...$ rad. For computations, the relations $\lambda^{-n} + \rho^n e^{in\theta} + \rho^n e^{-in\theta} \in \mathbb{Z}$, $n \geq 0$, furnish :

$$\lambda^{-n} \equiv -2(\sqrt{\lambda})^n \cos(n\theta).$$

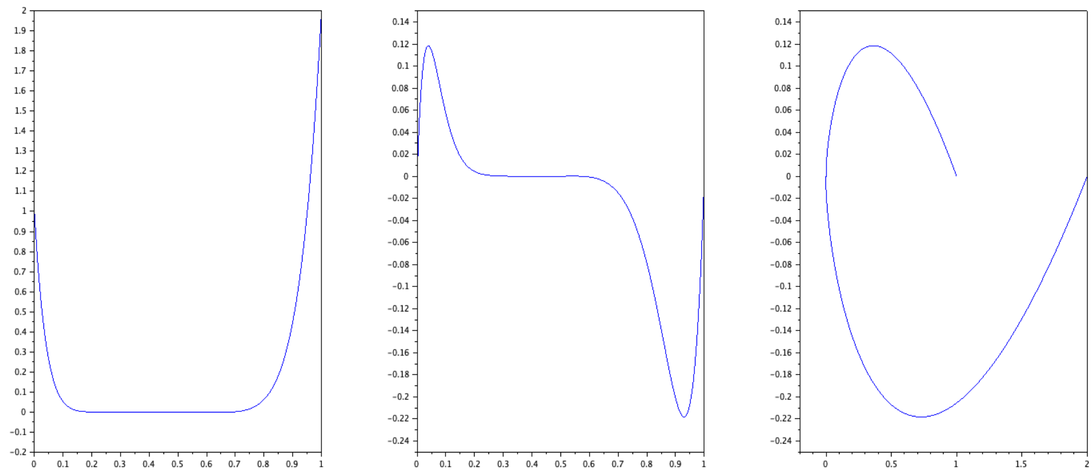
Let us finally compute the extreme values of $p \longmapsto \Delta_p$. We have $\Delta_{(1,0)} = F_{(1,0)}(0)G_{(1,0)}(0,0) = 1$. At the other extremity :

$$\begin{aligned}\Delta_{(0,1)} &= F_{(0,1)}(0)G_{(0,1)}(0,0) + F_{(0,1)}(1)G_{(0,1)}(1,1) \\ &= e^{2i\pi\sum_{l\geq 0}\lambda^{2l}} e^{2i\pi\sum_{l\geq 0}\lambda^{-2(l+1)}} + e^{2i\pi\lambda\sum_{l\geq 0}\lambda^{2l}} e^{2i\pi\sum_{l\geq 0}\lambda^{1-2(l+1)}}.\end{aligned}$$

As a result :

$$\begin{aligned}\Delta_{(0,1)} &= e^{2i\pi\left(\frac{1}{1-\lambda^2} - 2\sum_{l\geq 0}(\sqrt{\lambda})^{2l}\cos(2l\theta)\right)} + e^{2i\pi\left(\frac{\lambda}{1-\lambda^2} - 2\sum_{l\geq 0}(\sqrt{\lambda})^{2l+1}\cos((2l+1)\theta)\right)} \\ &= e^{2i\pi\left(\frac{1}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\lambda e^{2i\theta}}{1-\lambda e^{2i\theta}}\right)\right)} + e^{2i\pi\left(\frac{\lambda}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\sqrt{\lambda}e^{i\theta}}{1-\lambda e^{2i\theta}}\right)\right)}.\end{aligned}$$

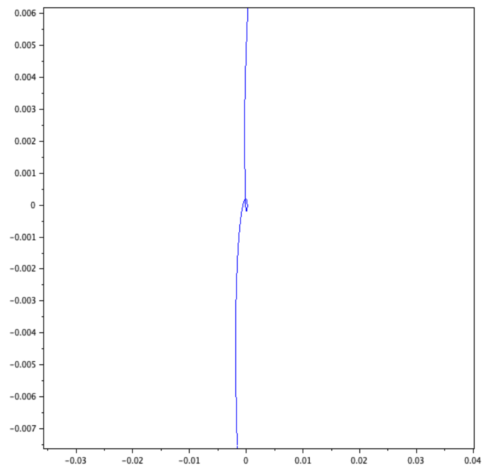
A not difficult computation, shortened by the observation that $(1-\lambda e^{2i\theta})(1-\lambda e^{-2i\theta}) = 1/\lambda$, shows that the arguments in the exponential terms (after the $2i\pi$) are respectively equal to 3 and 0, leading to $\Delta_{(0,1)} = 2$. Below are respectively drawn the real-analytic maps $p_1 \longmapsto \mathrm{Re}(\Delta_p)$, $p_1 \longmapsto \mathrm{Im}(\Delta_p)$ and the parametric curve $p_1 \longmapsto \Delta_p$, for $0 \leq p_1 \leq 1$, where we recall that $p = (1-p_1, p_1)$.

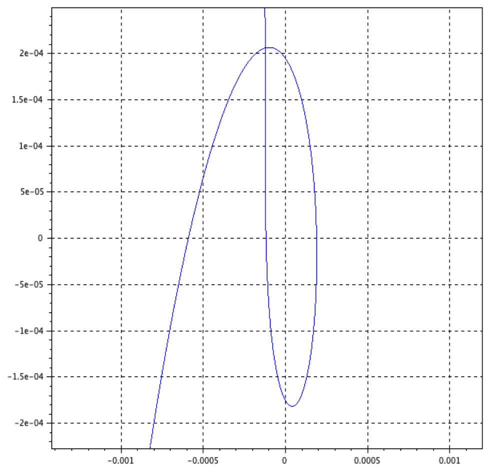


We observe that $p_1 \longmapsto \Delta_p$ spends a rather long time near 0, with $\mathrm{Re}(\Delta_p)$ and $\mathrm{Im}(\Delta_p)$ both around 10^{-4}. Mention here that one can exploit the product form (given by the exponential) inside the expectation appearing in $F_p(k)$ and $G_p(k,r)$ and make a deterministic numerical computation of Δ_p, with nearly an arbitrary precision, based on a dynamical programming (using a binomial tree). For example, one can obtain the value :

$$\Delta_{(1/2,1/2)} = 0,000178... + i0,0000491...,$$

where all digits are exact. In this case, $s((1/2,1/2),\lambda) = 1,64... > 1$. The pictures above were drawn with a sufficient precision, allowing to safely zoom on the neighbourhood of 0, the interesting region. We obtain the following pictures :





This came as a surprise and suggests the possible existence of profound reasons behind these pictures, but further investigations are necessary. A priori, there is no other singularity elsewhere. We conclude that the curve $p_1 \longmapsto \Delta_p$ is rather convincingly not touching 0; it may certainly be possible to build a rigorous numerical proof of this fact, but this is not the purpose of the present paper. Having faith in this, we can state :

Numerical Theorem 4.1
Let $N = 1$ and the two contractions $\varphi_0(x) = \lambda x$ and $\varphi_1(x) = 1 + \lambda^2 x$, where $1/\lambda > 1$ is the Plastic number. Then for any probability vector $p \in \mathcal{C}_1$, the invariant measure ν_p is not Rajchman.

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