Self-similar measures and the Rajchman property

Julien Brémont

Self-similar measures and the Rajchman property

Julien Brémont

Université Paris-Est-Créteil, novembre 2020

Abstract

For Bernoulli convolutions, the convergence to zero of the Fourier transform at infinity was characterized by successive works of Erdös [4] and Salem [17]. We provide a quasi-complete extension of these results to general self-similar measures on the real line.

1 Introduction

Rajchman measures. In the present article we consider the question of extending some classical results on Bernoulli convolutions to a more general context of self-similar measures. For a Borel probability measure μ on $\mathbb{R}$, define its Fourier transform as :

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{2i\pi tx} \, d\mu(x), \ t \in \mathbb{R}.$$

We say that μ is Rajchman, whenever $\hat{\mu}(t) \to 0$, as $t \to +\infty$. When μ is a Borel probability measure on the torus $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$, we introduce its Fourier coefficients, defined as :

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{2i\pi nx} \, d\mu(x), \ n \in \mathbb{Z}.$$

In this study, starting from a Borel probability measure μ on $\mathbb{R}$, Borel probability measures on $\mathbb{T}$ will naturally appear, quantifying the non-Rajchman character of μ.

For a Borel probability measure μ on $\mathbb{R}$, the Rajchman property holds for example if μ is absolutely continuous with respect to Lebesgue measure $\mathcal{L}_{\mathbb{R}}$, by the Riemann-Lebesgue lemma. The situation can be more subtle and for instance there exist Cantor sets of zero Lebesgue measure and even of zero-Hausdorff dimension which support a Rajchman measure; cf Menshov [13], Bluhm [2]. Questions on the Rajchman property of a measure naturally arise in Harmonic Analysis, for example when studying sets of multiplicity for trigonometric series; cf Lyons [12] or Zygmund [29]. We shall say a word on this topic at the end of the article. A classical counter-example is the uniform measure μ on the standard middle-third Cantor set, which is a continuous singular measure, not

AMS 2010 *subject classifications : 11K16, 37A45, 42A38, 42A61, 60K20.*
Key words and phrases : Rajchman measure, self-similar measure, Pisot number, Plastic number.

Rajchman (due to $\hat{\mu}(3n) = \hat{\mu}(n)$, $n \in \mathbb{Z}$). As in this last example, the obstructions for a measure to be Rajchman are often seen to be of arithmetical nature. The present work goes in this direction.

As it concerns $t \to +\infty$, the Rajchman character of a measure μ on $\mathbb{R}$ is an information of local regularity. As is well-known, it says for example that μ has no atom; if $\hat{\mu} \in L^2(\mathbb{R})$, then μ is absolutely continuous with respect to $\mathcal{L}_{\mathbb{R}}$ with an $L^2(\mathbb{R})$ density; if $\hat{\mu}$ has some polynomial decay at infinity, one gets a lower bound on the Hausdorff dimension of μ; etc. The Rajchman character can be reformulated as an equidistribution property modulo 1. Since $\hat{\mu}(t) \to 0$ is equivalent to $\hat{\mu}(mt) \to 0$ for any integer $m \neq 0$, if X is a real random variable with law μ, then μ is Rajchman if and only if the law of $tX \mod 1$ converges, as $t \to +\infty$, to Lebesgue measure $\mathcal{L}_{\mathbb{T}}$ on $\mathbb{T}$.

Self-similar measures. We now recall standard notions about self-similar measures on the real line $\mathbb{R}$, with a probabilistic point of view. We write $\mathcal{L}(X)$ for the law of a real random variable X. Let $N \geq 0$ and real affine maps $\varphi_k(x) = r_k x + b_k$, with $r_k > 0$, for $0 \leq k \leq N$, and at least one $r_k < 1$. We shall talk of "strict contractions" in the case when $0 < r_k < 1$, for all $0 \leq k \leq N$. This assumption will be considered principally in the second half of the article. For the sequel, we introduce the vectors $r = (r_k)_{0 \leq k \leq N}$ and $b = (b_k)_{0 \leq k \leq N}$.

Notice for what follows that for $n \geq 0$, a composition $\varphi_{k_{n-1}} \circ \cdots \circ \varphi_{k_0}$ has the explicit expression :

$$\varphi_{k_{n-1}} \circ \cdots \circ \varphi_{k_0}(x) = r_{k_{n-1}} \cdots r_{k_0} x + \sum_{l=0}^{n-1} b_{k_l} r_{k_{n-1}} \cdots r_{k_{l+1}}.$$

Consider the convex set $\mathcal{C}_N = \{p = (p_0, \cdots, p_N) \mid \forall j,\ p_j > 0,\ \sum_j p_j = 1\}$, open for the topology of the affine hyperplane $\{\sum_j p_j = 1\}$. We denote its closure by $\bar{\mathcal{C}}_N$. Define :

$$\mathcal{D}_N(r) = \{p \in \bar{\mathcal{C}}_N \mid \sum_{0 \leq j \leq N} p_j \log r_j < 0\}.$$

This is a non-empty open subset of $\bar{\mathcal{C}}_N$, for the relative topology. Notice that $\mathcal{D}_N(r) = \bar{\mathcal{C}}_N$, in the case when the $(\varphi_k)_{0 \leq k \leq N}$ are strict contractions.

Fixing a probability vector $p \in \mathcal{D}_N(r)$, we now compose the contractions at random, independently, according to p. Precisely, let X_0 be any real random variable and $(\varepsilon_n)_{n \geq 0}$ be independent and identically distributed (*i.i.d.*) random variables, independent from X_0, and with law p, in other words $\mathbb{P}(\varepsilon_0 = k) = p_k$, $0 \leq k \leq N$. We consider the Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{R}$ defined by :

$$X_n = \varphi_{\varepsilon_{n-1}} \circ \cdots \circ \varphi_{\varepsilon_0}(X_0),\ n \geq 0.$$

The condition $p \in \mathcal{D}_N(r)$, of contraction on average, can be rewritten as $\mathbb{E}(\log r_{\varepsilon_0}) < 0$. It implies that $(X_n)_{n \geq 0}$ has a unique stationary (time invariant) measure, written as ν. This follows for example from the fact that $\mathcal{L}(X_n) = \mathcal{L}(Y_n)$, where :

$$Y_n := \varphi_{\varepsilon_0} \circ \cdots \circ \varphi_{\varepsilon_{n-1}}(X_0) = r_{\varepsilon_0} \cdots r_{\varepsilon_{n-1}} X_0 + \sum_{l=0}^{n-1} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}.$$

As usual, (Y_n) is more stable than (X_n). Since $n^{-1} \log(r_{\varepsilon_0} \cdots r_{\varepsilon_{n-1}}) \to \mathbb{E}(\log r_{\varepsilon_0}) < 0$, a.-s., as $n \to +\infty$, by the Law of Large Numbers, we get that Y_n converges a.-s., as $n \to +\infty$, to :

$$X := \sum_{l\geq 0} b_{\varepsilon_l} r_{\varepsilon_0}\cdots r_{\varepsilon_{l-1}}.$$

Setting $\nu = \mathcal{L}(X)$, we obtain that $\mathcal{L}(X_n)$ weakly converges to ν. By construction, we have $\mathcal{L}(X_{n+1}) = \sum_{0\leq j\leq N} p_j \mathcal{L}(X_n)\circ\varphi_j^{-1}$. Taking the limit as $n\to+\infty$, the measure ν verifies :

$$\nu = \sum_{0\leq j\leq N} p_j \nu\circ\varphi_j^{-1}. \tag{1}$$

The previous convergence implies that the solution of this "stable fixed point equation" is unique among Borel probability measures. Also, ν has to be of pure type, i.e. either purely atomic or absolutely continuous with respect to $\mathcal{L}_{\mathbb{R}}$ or else singular continuous, since each term in its Radon-Nikodym decomposition with respect to $\mathcal{L}_{\mathbb{R}}$ verifies (1). A few remarks are in order :

i) If $p\in\mathcal{C}_N$, the measure ν is purely atomic if and only if the φ_j have a common fixed point c, in which case ν is the Dirac mass at c. Indeed, consider the necessity and suppose that ν has an atom. Let $a>0$ be the maximal mass of an atom and E the finite set of points having mass a. Fixing any $c\in E$, the relation $\nu(\{c\}) = \sum_j p_j\nu(\{\varphi_j^{-1}(c)\})$ furnishes $\varphi_j^{-1}(c)\in E$, $0\leq j\leq N$. Hence $\varphi_j^{-n}(c)\in E$, $n\geq 0$, for all j. If $\varphi_j\neq id$, then $\varphi_j^{-1}(c)=c$, the set $\{\varphi_j^{-n}(c),\ n\geq 0\}$ being infinite otherwise. If $\varphi_j = id$, it fixes all points.

ii) The equation for a hypothetical density f of ν with respect to $\mathcal{L}_{\mathbb{R}}$, coming from (1), is :

$$f = \sum_{0\leq j\leq N} p_j r_j^{-1} f\circ\varphi_j^{-1}.$$

This "unstable fixed point equation" is difficult to solve directly. It is equivalently reformulated into the fact that $((r_{\varepsilon_{n-1}}^{-1}\cdots r_{\varepsilon_0}^{-1}) f\circ\varphi_{\varepsilon_{n-1}}^{-1}\cdots\circ\varphi_{\varepsilon_0}^{-1}(x))_{n\geq 0}$ is a non-negative martingale (for its natural filtration), for Lebesgue a.-e. $x\in\mathbb{R}$. Notice that when f exists and is bounded, then $p_j\leq r_j$ for all j, because $p_j r_j^{-1}\|f\|_\infty = \|p_j r_j^{-1} f\circ\varphi_j^{-1}\|_\infty \leq \|f\|_\infty$ and $\|f\|_\infty\neq 0$.

iii) Let $f(x) = ax+b$ be an affine map, with $a\neq 0$. With the same $p\in\mathcal{D}_N(r)$, consider the conjugate system $(\psi_j)_{0\leq j\leq N}$, with $\psi_j(x) = f\circ\varphi_j\circ f^{-1}(x) = r_j x + b(1-r_j) + ab_j$. It has an invariant measure $w = \mathcal{L}(aX+b)$ verifying the relation $\hat{w}(t) = \hat{\nu}(at)e^{2i\pi tb}$, $t\in\mathbb{R}$. In particular, ν is Rajchman if and only if w is Rajchman.

iv) When supposing that the $(\varphi_k)_{0\leq k\leq N}$ are strict contractions, some self-similar set F can be introduced, where $F\subset\mathbb{R}$ is the unique non-empty compact set verifying the self-similarity relation $F = \cup_{0\leq k\leq N}\varphi_k(F)$. See for example Huchinson [7] for general properties of such sets. Introducing $\mathbb{N} = \{0,1,\cdots\}$ and the compact $S = \{0,\cdots,N\}^{\mathbb{N}}$, the hypothesis that the $(\varphi_k)_{0\leq k\leq N}$ are strict contractions implies that F is a continuous (and even hölderian) image of S, in other words we have the following description :

$$F \quad = \quad \left\{\sum_{l\geq 0} b_{x_l} r_{x_0}\cdots r_{x_{l-1}},\ (x_0,x_1,\cdots)\in S\right\}.$$

Whereas in the general case a self-similar invariant measure can have $\mathbb{R}$ as topological support, when the $(\varphi_k)_{0\leq k\leq N}$ are strict contractions the compact self-similar set F exists and supports any self-similar measure.

Background and content of the article. Coming back to the general case, we assume in the sequel that the $(\varphi_j)_{0\leq j\leq N}$ do not have a common fixed point (in particular $N \geq 1$), so that μ is a continuous measure. A difficult problem is to characterize the absolute continuity of ν with respect to $\mathcal{L}_{\mathbb{R}}$ in terms of the parameters r, b and p. An example with a long and well-known history is that of Bernoulli convolutions, corresponding to $N = 1$, the affine contractions $\varphi_0(x) = \lambda x - 1$, $\varphi_1(x) = \lambda x + 1$, $0 < \lambda < 1$, and the probability vector $p = (1/2, 1/2)$. Notice that when the r_i are all equal (to some real in $(0, 1)$), the situation is a little simplified, as ν is an infinite convolution (this is not true in general). Although we discuss below some works in this context, we will not present here the vast subject of Bernoulli convolutions, addressing the reader to detailed surveys, such as Peres-Schlag-Solomyak [15] or Solomyak [21].

For general self-similar measures, an important aspect of the problem, that we shall not enter, and an active line of research, concerns the Hausdorff dimension of the measure ν; cf the recent fundamental work of Hochman [6] for example. In a large generality, see Falconer [5] and more recently Jaroszewska and Rams [9], there is an "entropy/Lyapunov exponent" upper-bound :

$$\mathrm{Dim}_{\mathcal{H}}(\nu) \leq \min\{1, s(p,r)\}, \text{ where } s(p,r) := \frac{-\sum_{i=0}^{N} p_i \log p_i}{-\sum_{i=0}^{N} p_i \log r_i}.$$

The quantity $s(p, r)$ is called the singularity dimension of the measure and can be > 1. The equality $\mathrm{Dim}_{\mathcal{H}}(\nu) = 1$ does not mean that ν is absolutely continuous, but the inequality $s(p, r) < 1$ surely implies that ν is singular. The interesting domain of parameters for the question of the absolute continuity of the invariant measure therefore corresponds to $s(p, r) \geq 1$.

We focus in this work on another fundamental tool, the Fourier transform $\hat{\nu}$. If ν is not Rajchman, the Riemann-Lebesgue lemma implies that ν is singular. This property was used by Erdös [4] in the context of Bernoulli convolutions. He proved that if $1/2 < \lambda < 1$ is such that $1/\lambda$ is a Pisot number, then ν is not Rajchman. The reciprocal statement (for $1/2 < \lambda < 1$) was next shown by Salem [17]. As a result, for Bernoulli convolutions the Rajchman property always holds, except for a very particular set of parameters. Some works have next focused on the decay on average of the Fourier transform for general self-similar measures associated to strict contractions; cf Strichartz [24, 25], Tsuji [26]. In the same context, the non-Rajchman character was recently shown to hold for only a very small set of parameters by Li and Sahlsten [11], who showed that ν is Rajchman when $\log r_i / \log r_j$ is irrational for some (i, j), with moreover some logarithmic decay of $\hat{\nu}$ at infinity, under a Diophantine condition. Next, Solomyak [22] proved that outside a set of r of zero Hausdorff dimension, $\hat{\nu}$ even has a power decay at infinity.

The aim of the present article is to study for general self-similar measures the exceptional set of parameters where the Rajchman property is not true, trying to follow the line of [4] and [17]. We essentially show that r and b have to be closely related to some fixed Pisot number, as for Bernoulli convolutions. We first prove a general extension of the result of Salem [17], reducing to a small island the set of parameters where the Rajchman property may not hold. Focusing then on this island of parameters, we provide a general characterization of the Rajchman character, appearing in this particular case as equivalent to absolute continuity with respect to $\mathcal{L}_{\mathbb{R}}$. Next, supposing that the $(\varphi_k)_{0\leq k\leq N}$ are strict contractions, we prove a partial extension of the theorem of Erdös [4], showing that for most parameters in the small island the Rajchman property is not true, with in general a few exceptions. We finally give some complements, first rather surprising numerical simulations involving the Plastic number, then an application to sets of uniqueness for trigonometric series.

2 Statement of the results

Let us place in the general context considered in the Introduction. Pisot numbers will play a central role in the analysis. Let us introduce a few definitions concerning Algebraic Number Theory; cf for example Samuel [19] for more details.

Definition 2.1
A Pisot number is a real algebraic integer $\theta > 1$, with conjugates (the other roots of its minimal unitary polynomial) of modulus strictly less than 1. Fixing such a $\theta > 1$, denote its minimal polynomial as $Q = X^{s+1} + a_s X^s + \cdots + a_0 \in \mathbb{Z}[X]$, of degree $s+1$, with $s \geq 0$. If $s = 0$, then θ is an integer ≥ 2. The images of $\mu \in \mathbb{Q}[\theta]$ by the $s+1$ $\mathbb{Q}$-homomorphisms $\mathbb{Q}[\theta] \to \mathbb{C}$ are the conjugates of μ corresponding to the field $\mathbb{Q}[\theta]$, in general denoted by $\mu = \mu^{(0)}, \mu^{(1)}, \cdots, \mu^{(s)}$.

i) For $\alpha \in \mathbb{Q}[\theta]$, the trace $Tr_\theta(\alpha)$ is the trace of the linear operator $x \longmapsto \alpha x$ of multiplication by α, considered from $\mathbb{Q}[\theta]$ to itself. As a general fact, $Tr_\theta(\alpha) \in \mathbb{Q}$.

ii) Let $\mathbb{Z}[\theta] = \mathbb{Z}\theta^0 + \cdots + \mathbb{Z}\theta^s$ be the subring generated by θ of the ring of algebraic integers of $\mathbb{Q}[\theta]$. We write $\mathcal{D}(\theta)$ for its $\mathbb{Z}$-dual (as a $\mathbb{Z}$-lattice), i.e. :

$$\mathcal{D}(\theta) = \{\alpha \in \mathbb{Q}[\theta],\ Tr_\theta(\theta^n \alpha) \in \mathbb{Z},\ \text{for } 0 \leq n \leq s\}.$$

It can be shown that $\mathcal{D}(\theta) = (1/Q'(\theta))\mathbb{Z}[\theta]$. As a classical fact, $Tr_\theta(\theta^n\alpha) \in \mathbb{Z}$, for all $n \geq 0$, if this holds for $0 \leq n \leq s$. Define :

$$\mathcal{T}(\theta) = \{\alpha \in \mathbb{Q}[\theta],\ Tr_\theta(\theta^n \alpha) \in \mathbb{Z},\ \text{for large } n \geq 0\}.$$

Then $\mathcal{T}(\theta) = \cup_{n\geq 0} \theta^{-n}\mathcal{D}(\theta) = \dfrac{\mathbb{Z}[\theta, 1/\theta]}{Q'(\theta)}$, with $\mathbb{Z}[\theta, 1/\theta]$ the subring of $\mathbb{Q}[\theta]$ generated by $\{\theta, 1/\theta\}$.

Remark. — In the context of the previous definition, introduce the integer-valued $(s+1)\times(s+1)$-companion matrix M of Q :

$$M = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & 1 \\ -a_0 & \cdots & -a_{s-1} & -a_s \end{pmatrix}.$$

One may show that for any $\mu \in \mathbb{Q}[\theta]$, setting $V = (Tr_\theta(\theta^0\mu), \cdots, Tr_\theta(\theta^s\mu))$, then $\mu \in \mathcal{T}(\theta)$ if and only if there exists $n \geq 0$ such that VM^n has integral entries.

We introduce special families of affine maps, that will play the role of canonical models for the analysis of the Rajchman property.

Definition 2.2
Let $N \geq 1$. A family of real affine maps $(\varphi_k)_{0\leq k\leq N}$ is said in Pisot form, if there exist a Pisot number $1/\lambda > 1$, relatively prime integers $(n_k)_{0\leq k\leq N}$ and $\mu_k \in \mathcal{T}(1/\lambda)$, $0 \leq k \leq N$, such that $\varphi_j(x) = \lambda^{n_j} x + \mu_j$, for all $0 \leq j \leq N$.

Remark. — If $(\varphi_j)_{0\leq j\leq N}$ is in Pisot form, then the $(\lambda, (n_j), (\mu_j))$ are uniquely determined. Indeed, if the $(\lambda', (n'_j), (\mu'_j))$ also convene, we just need to show that $\lambda = \lambda'$. Taking some collection of integers (a_j) realizing a Bezout relation $1 = \sum_j a_j n_j$, we have :

$$\lambda = \lambda^{\sum_j a_j n_j} = \lambda'^{\sum_j a_j n'_j} = \lambda'^p,$$

for some $p \geq 1$. Idem, $\lambda' = \lambda^q$, for some $q \geq 1$. Hence $pq = 1$, giving $p = q = 1$ and $\lambda = \lambda'$.

As a first result, extending [17], the analysis of the non-Rajchman character of the invariant measure requires to consider families in Pisot form.

Theorem 2.3
Let $N \geq 1$, $p \in \mathcal{C}_N$ and affine maps $\varphi_k(x) = r_k x + b_k$, $r_k > 0$, for $0 \leq k \leq N$, with no common fixed point, and $\sum_{0\leq j\leq N} p_j \log r_j < 0$. The invariant measure ν is not Rajchman if and only if there exists $f(x) = ax + b$, $a \neq 0$, such that the conjugate system $(f \circ \varphi_j \circ f^{-1})_{0\leq j\leq N}$ is in Pisot form, for some Pisot number $1/\lambda > 1$, with invariant measure w verifying $\hat{w}(\lambda^{-k}) \not\to_{k+\infty} 0$.

In particular, one gets that $r_j = \lambda^{n_j}$, for all j, for some Pisot number $1/\lambda > 1$ et relatively prime integers $(n_k)_{0\leq k\leq N}$. Hence, up to an affine change of variables, the non-Rajchman character of the invariant measure ν can be read on the sequence $(\lambda^{-k})_{k\geq 0}$, as in [4]. In a second step, we provide a general analysis of families in Pisot form.

We now fix a Pisot number $1/\lambda > 1$, an integer $N \geq 1$, relatively prime integers $(n_k)_{0\leq k\leq N}$ and $(\mu_k)_{0\leq k\leq N} \in (\mathcal{T}(1/\lambda))^{N+1}$, such that $\varphi_k(x) = \lambda^{n_k} x + \mu_k$, for $0 \leq k \leq N$. Let $p \in \mathcal{C}_N$ be such that $\sum_{0\leq j\leq N} p_j n_j > 0$ and *i.i.d.* random variables $(\varepsilon_n)_{n\in\mathbb{Z}}$, with $\mathbb{P}(\varepsilon_0 = k) = p_k$, $0 \leq k \leq N$. We introduce cocycle notations $(S_l)_{l\in\mathbb{Z}}$, where $S_0 = 0$ and for $l \geq 1$:

$$S_l = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}, \quad S_{-l} = -n_{\varepsilon_{-l}} - \cdots - n_{\varepsilon_{-1}}.$$

An important preliminary remark is that when $\mu \in \mathcal{T}(1/\lambda)$ and $k \geq 0$ is large enough, we have :

$$\lambda^{-k}\mu + \sum_{1\leq j\leq s} \alpha_j^k \mu^{(j)} = Tr_{1/\lambda}(\lambda^{-k}\mu) \in \mathbb{Z},$$

where the $(\alpha_j)_{0\leq j\leq s}$ are the conjugates of $1/\lambda =: \alpha_0$ and the $(\mu^{(j)})_{0\leq j\leq s}$ that of $\mu = \mu^{(0)}$, corresponding to the field $\mathbb{Q}[\lambda]$. Since $|\alpha_j| < 1$, for $1 \leq j \leq s$, and (S_l) is a.-s. transient with a non-zero linear speed to $-\infty$, as $l \to -\infty$, by the Law of Large Numbers, this ensures that for any $k \in \mathbb{Z}$, the random variable $\sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l} \mod 1$ is well-defined as a $\mathbb{T}$-valued random variable.

In the sequel we use standard inner products and Euclidean norms on all spaces $\mathbb{R}^n$.

Theorem 2.4
Let $1/\lambda > 1$ be a Pisot number of degree $s + 1$. Let $N \geq 1$, relatively prime integers $(n_k)_{0\leq k\leq N}$ and $(\mu_k)_{0\leq k\leq N} \in (\mathcal{T}(1/\lambda))^{N+1}$, such that $\varphi_k(x) = \lambda^{n_k} x + \mu_k$, $0 \leq k \leq N$. Let $p \in \mathcal{C}_N$ be such

that $\sum_{0\leq j\leq N} p_j n_j > 0$ and i.i.d. random variables $(\varepsilon_n)_{n\in\mathbb{Z}}$, with law p. Let $(S_l)_{l\in\mathbb{Z}}$ be the cocycle notations associated to the $(n_{\varepsilon_i})_{i\in\mathbb{Z}}$. The real random variable $X = \sum_{l\geq 0} \mu_{\varepsilon_l} \lambda^{S_l}$ has law ν.

i) Let the $\mathbb{T}$-valued random variables $Z_k = \sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l}$, $k\in\mathbb{Z}$. Then $\lambda^{-n}X \mod 1$ converges, as $n\to+\infty$, to a probability measure m on $\mathbb{T}$, verifying, for all $f\in C(\mathbb{T},\mathbb{R})$ and all $k\in\mathbb{Z}$:

$$\int_{\mathbb{T}} f(x)\ dm(x) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\leq r<n^*} \mathbb{E}\left[f\left(Z_{k+r}\right) 1_{S_{-u}<-r, u\geq 1}\right],$$

where $n^* = \max_{0\leq k\leq N} n_k$. More generally, $\lambda^{-n}(X, \lambda^{-1}X, \cdots, \lambda^{-s}X) \mod \mathbb{Z}^{s+1}$ converges in law, as $n\to+\infty$, to a probability measure $\mathcal{M}$ on $\mathbb{T}^{s+1}$, with one-dimensional marginals m, verifying :

$$\int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\leq r<n^*} \mathbb{E}\left[f\left(Z_{k+r}, Z_{k+r-1}, \cdots, Z_{k+r-s}\right) 1_{S_{-u}<-r, u\geq 1}\right],$$

for all $f\in C(\mathbb{T}^{s+1},\mathbb{R})$ and all $k\in\mathbb{Z}$.

ii) If the $(\varphi_k)_{0\leq k\leq N}$ do not have a common fixed point (i.e. if ν is continuous), denoting by Z a $\mathbb{T}^{s+1}$-valued random variable with law $\mathcal{M}$, then for any $0\neq n = (n_0,\cdots,n_s)^t \in \mathbb{Z}^{s+1}$, $\langle Z, n\rangle$ has a continuous law; in particular, m and $\mathcal{M}$ are continuous measures. If the $(\varphi_k)_{0\leq k\leq N}$ have a common fixed point, there exists a rational number p/q such that $m = \delta_{p/q}$ and $\mathcal{M} = (\delta_{p/q})^{\otimes(s+1)}$.

iii) Either $\mathcal{M}\perp\mathcal{L}_{\mathbb{T}^{s+1}}$ or $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}}$. Also, $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}} \Leftrightarrow \nu$ is Rajchman $\Leftrightarrow \nu \ll \mathcal{L}_{\mathbb{R}}$.

In the context of the previous theorem, ν and $\mathcal{M}$ are always of the same nature, with respect to the uniform measure of the space they live on. In particular, $\mathcal{M}$ is also of pure type. We finally consider families in Pisot form, when supposing that the $(\varphi_k)_{0\leq k\leq N}$ are strict contractions.

Theorem 2.5
Let $N\geq 1$ and $\varphi_k(x) = \lambda^{n_k}x + \mu_k$, for $0\leq k\leq N$, with $1/\lambda > 1$ a Pisot number, relatively prime integers $(n_k)_{0\leq k\leq N}$, with $n_k\geq 1$ and $\mu_k\in\mathcal{T}(1/\lambda)$, for $0\leq k\leq N$. When $p\in\mathcal{C}_N$ is fixed, we denote by m the measure on $\mathbb{T}$ of Theorem 2.4, i).

i) For any $p\in\mathcal{C}_N$, if the invariant measure ν is Rajchman, then it is absolutely continuous with respect to $\mathcal{L}_{\mathbb{R}}$, with a density bounded and with compact support.

ii) There exists $0\neq a\in\mathbb{Z}$ such that for any $k\neq 0$, for any $p\in\mathcal{C}_N$ outside finitely many real-analytic graphs of dimension $\leq N-1$ (points if $N=1$), we have $\hat{m}(ak)\neq 0$. In this case, $m\neq\mathcal{L}_{\mathbb{T}}$ and ν is not Rajchman.

Remark. — In Theorem 2.5 *ii*), observe that when making k vary, we obtain that for all $p\in\mathcal{C}_N$ outside a countable number of real-analytic graphs of dimension less than or equal to $N-1$ (points if $N=1$), then $\hat{m}(ak)\neq 0$, for all $k\in\mathbb{Z}$. Part *ii*) of Theorem 2.5 relies on an indirect argument, based on the analysis of the regularity of $\hat{m}(n)$, for some fixed $n\in\mathbb{Z}$, as a function of $p\in\mathcal{C}_N$.

Remark. — On the existence of singular measures in the non-homogeneous case, we are essentially aware of the non-explicit examples, using algebraic curves, of Neunhäuserer [14]. As suggested by the referee, Theorem 2.5 allows to give in the non-homogeneous case an explicit example of a continuous singular and not Rajchman invariant measure ν with singularity dimension > 1. Indeed, take for $1/\lambda > 1$ the Plastic number, i.e. the real root of $X^3 - X - 1$. This is the smallest Pisot number; cf Siegel [20]. We have $1/\lambda = 1.3247....$ Let $N = 1$ and $\varphi_0(x) = \lambda x,\ \varphi_1(x) = \lambda^2 x + 1$. For $p = (p_0, p_1) \in \mathcal{C}_1$, if ν is absolutely continuous with respect to $\mathcal{L}_\mathbb{R}$, then, by Theorem 2.5 *i*), the density has to be bounded. By remark *ii*) in the Introduction, this implies that $p_0 \le \lambda = 0.7548...$ and $p_1 \le \lambda^2$. Now, as detailed in the last section, the similarity dimension in this case is > 1 if and only if $0,203... < p_0 < 0,907...$ For example we can conclude that for $p_0 \in [0.76, 0.90]$, the measure ν is continuous, singular with respect to $\mathcal{L}_\mathbb{R}$, not Rajchman and with similarity dimension > 1. Still for the system $\varphi_0(x) = \lambda x$, $\varphi_1(x) = \lambda^2 x + 1$, we will give in the last section a strong numerical support for the fact that ν is in fact continuous singular and not Rajchman for all $p \in \mathcal{C}_1$.

Remark. — In the context of Theorem 2.5, it would be important to determine all the exceptional parameters where ν is absolutely with respect to $\mathcal{L}_\mathbb{R}$. Let us give some examples where the exceptional set in Theorem 2.5 *ii*) is non-empty :

1) Let $1/\lambda = N \ge 1$ and $\varphi_k(x) = (x + k)/(N + 1)$, with $p_k = 1/(N + 1)$, for $0 \le k \le N$; then ν is Lebesgue measure on $[0, 1]$.

2) Take for $1/\lambda > 1$ the Plastic number, $N = 1$ and this time $\varphi_0(x) = \lambda^2 x,\ \varphi_1(x) = \lambda^3 x + 1$. One may verify that the similarity dimension is < 1 for all $p \in \mathcal{C}_1$, except for $p = (\lambda^2, \lambda^3)$, where it equals one. Thus the invariant measure ν is singular for $p \in \mathcal{C}_1$ with $p \ne (\lambda^2, \lambda^3)$. Another way, if ν is absolutely continuous with respect to $\mathcal{L}_\mathbb{R}$, then its density has to be bounded by Theorem 2.5. Therefore, $p_0 \le \lambda^2$ and $p_1 \le \lambda^3$, using remark *ii*) in the Introduction. Since $\lambda^2 + \lambda^3 = 1$, we have $p_0 = \lambda^2$ and $p_1 = \lambda^3$. As a result, when $p = (p_0, p_1) \ne (\lambda^2, \lambda^3)$ and $p_0 > 0$, $p_1 > 0$, then ν is continuous singular and not Rajchman. When $p = (\lambda^2, \lambda^3)$, set $I = [0, 1 + \lambda]$ and notice that $\varphi_0(I) = [0, 1]$, $\varphi_1(I) = [1, 1 + \lambda]$. Hence, Lebesgue a.-e. :

$$1_I = 1_{\varphi_0(I)} + 1_{\varphi_1(I)} = p_0 \lambda^{-2} 1_{\varphi_0(I)} + p_1 \lambda^{-3} 1_{\varphi_1(I)},$$

meaning that $\nu = \frac{1}{1+\lambda} \mathcal{L}_I$. Taking for $1/\lambda$ the supergolden ratio (the real root of $X^3 - X^2 - 1$; the fourth Pisot number), one gets the same situation with the system $(\lambda x + 1, \lambda^3 x)$, the exceptional parameters being then (λ, λ^3), giving for ν the uniform probability measure on $[0, \lambda^{-3}]$.

3) When $1/\lambda > 1$ is the Plastic number, $N = 2$, $\varphi_0(x) = \lambda^2 x$, $\varphi_1(x) = \lambda^3 x + 1$, $\varphi_2(x) = \lambda^3 x + 1$ and $p_0 = \lambda^2$, $p_1 = \lambda^3 \alpha$, $p_2 = \lambda^3(1 - \alpha)$, then $\nu = \frac{1}{1+\lambda} \mathcal{L}_{[0,1+\lambda]}$, for all $0 < \alpha < 1$. This is an example, a little degenerated, of a one-dimensional real-analytic graph where the corresponding invariant measure ν is absolutely continuous with respect to $\mathcal{L}_\mathbb{R}$.

It would be interesting to find more developed examples, where ν is absolutely continuous with respect to $\mathcal{L}_\mathbb{R}$. A difficulty is that a priori the probability vector p has to be chosen in accordance with the polynomial equations verified by λ.

3 Proof of Theorem 2.3

Let $N \ge 1$ and $(\varphi_k)_{0 \le k \le N}$, with $\varphi_k(x) = r_k x + b_k$, where $r_k > 0$, and having no common fixed point. Fixing $p \in \mathcal{C}_N$, introduce *i.i.d.* random variables $(\varepsilon_n)_{n \ge 0}$ with law p, to which $\mathbb{P}$ and $\mathbb{E}$ refer. By hypothesis, $\mathbb{E}(\log r_{\varepsilon_0}) < 0$. Recall that the invariant measure ν is the law of the random variable $\sum_{l \ge 0} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}$ and that ν is supposed to be non Rajchman. Without loss of generality, we assume that $0 < r_0 \le r_1 \le \cdots \le r_N$, with therefore $r_0 < 1$.

The proof has three parts. First we show that $\log r_i / \log r_j \in \mathbb{Q}$, for all $0 \leq i \neq j \leq N$. From this, we will get that $r_j = \lambda^{n_j}$, for some $0 < \lambda < 1$ and integers (n_j). We then show that the non Rajchman character of ν can be seen on a subsequence of the form $(\alpha\lambda^{-k})_{k \geq 0}$. We finally prove that $1/\lambda$ is a Pisot number and the family $(\varphi_k)_{0 \leq k \leq N}$ is affinely conjugated with one in Pisot form.

Step 1. Let us show that if ever $\log r_i / \log r_j \notin \mathbb{Q}$, for some $0 \leq i \neq j \leq N$, then ν is Rajchman. This is established in [11] for strict contractions. We simplify their proof.

For $n \geq 1$, consider the random walk $S_n = -\log r_{\varepsilon_0} - \cdots - \log r_{\varepsilon_{n-1}}$, with $S_0 = 0$. For a real $s \geq 0$, introduce the finite stopping time $\tau_s = \min\{n \geq 0,\ S_n > s\}$ and write $\mathcal{T}_s$ for the corresponding sub-σ-algebra of the underlying σ-algebra. Taking $\alpha > 0$ and $s \geq 0$:

$$\begin{aligned}
\hat{\nu}(\alpha e^s) &= \mathbb{E}\left(e^{2\pi i \alpha e^s \sum_{l \geq 0} b_{\varepsilon_l} e^{-S_l}}\right) \\
&= \mathbb{E}\left(e^{2\pi i \alpha e^s \sum_{0 \leq l < \tau_s} b_{\varepsilon_l} e^{-S_l}} e^{2\pi i \alpha e^{-S_{\tau_s}+s} \sum_{l \geq \tau_s} b_{\varepsilon_l} e^{-S_l + S_{\tau_s}}}\right).
\end{aligned}$$

In the expectation, the first exponential term is $\mathcal{T}_s$-measurable. Also, the conditional expectation of the second exponential term with respect to $\mathcal{T}_s$ is just $\hat{\nu}(\alpha e^{-S_{\tau_s}+s})$, as a consequence of the strong Markov property. It follows that :

$$\hat{\nu}(\alpha e^s) = \mathbb{E}\left(\hat{\nu}(\alpha e^{-S_{\tau_s}+s}) e^{2\pi i \alpha e^s \sum_{0 \leq l < \tau_s} b_{\varepsilon_l} e^{-S_l}}\right).$$

This gives $|\hat{\nu}(\alpha e^s)| \leq \mathbb{E}\left(|\hat{\nu}(\alpha e^{-S_{\tau_s}+s})|\right)$, so by the Cauchy-Schwarz inequality and the Fubini theorem, which directly applies, consecutively :

$$\begin{aligned}
|\hat{\nu}(\alpha e^s)|^2 \leq \mathbb{E}\left(|\hat{\nu}(\alpha e^{-S_{\tau_s}+s})|^2\right) &= \mathbb{E}\left(\int_{\mathbb{R}^2} e^{2\pi i \alpha e^{-S_{\tau_s}+s}(x-y)}\ d\nu(x) d\nu(y)\right) \\
&= \int_{\mathbb{R}^2} \mathbb{E}\left(e^{2\pi i \alpha e^{-S_{\tau_s}+s}(x-y)}\right)\ d\nu(x) d\nu(y) \\
&\leq \int_{\mathbb{R}^2} \left|\mathbb{E}\left(e^{2\pi i \alpha e^{-S_{\tau_s}+s}(x-y)}\right)\right|\ d\nu(x) d\nu(y).
\end{aligned}$$

Let $Y := -\log r_{\varepsilon_0}$. The law of Y is non-lattice, since some $\log r_i / \log r_j \notin \mathbb{Q}$ and $p_k > 0$ for all $0 \leq k \leq N$. As Y is integrable, with $0 < \mathbb{E}(Y) < \infty$, it is a well-known consequence of the Blackwell theorem on the law of the overshoot that (see for instance Woodroofe [28], chap. 2, thm 2.3), that :

$$\mathbb{E}(g(S_{\tau_s} - s)) \to \frac{1}{\mathbb{E}(S_{\tau_0})} \int_0^{+\infty} g(x) \mathbb{P}(S_{\tau_0} > x)\ dx, \text{ as } s \to +\infty,$$

for any Riemann-integrable g defined on $\mathbb{R}^+$. Here, all $S_{\tau_s} - s$, for $s \geq 0$, (and in particular S_{τ_0}) have support in some $[0, A]$. Therefore, $\mathbb{P}(S_{\tau_0} > x) = 0$ for large $x > 0$. For any $\alpha > 0$, by dominated convergence (letting $s \to +\infty$) :

$$\limsup_{t \to +\infty} |\hat{\nu}(t)|^2 \leq \frac{1}{\mathbb{E}(S_{\tau_0})} \int_{\mathbb{R}^2} \left| \int_0^{+\infty} e^{2\pi i \alpha e^{-u}(x-y)} \mathbb{P}(S_{\tau_0} > u) du \right| d\nu(x) d\nu(y).$$

The inside term (in the modulus) is uniformly bounded with respect to $(x,y) \in \mathbb{R}^2$. We shall use dominated convergence once more, this time with $\alpha \to +\infty$. It is sufficient to show that for $\nu^{\otimes 2}$-almost every (x,y), the inside term goes to zero. Since the measure ν is non-atomic, $\nu^{\otimes 2}$-almost-surely, $x \neq y$. If for example $x > y$:

$$\int_0^{+\infty} e^{2\pi i\alpha e^{-u}(x-y)}\mathbb{P}(S_{\tau_0} > u)du = \int_0^{x-y} e^{2\pi i\alpha t}\mathbb{P}(S_{\tau_0} > \log((x-y)/t)\ \frac{dt}{t},$$

making the change of variable $t = e^{-u}(x-y)$. The last integral now converges to 0, as $\alpha \to +\infty$, by the Riemann-Lebesgue lemma. Hence, $\lim_{t\to+\infty} \hat{\nu}(t) = 0$. This ends the proof of this step.

Step 2. As ν is not Rajchman, from *Step 1*, $\log r_i / \log r_j \in \mathbb{Q}$, for all (i,j). Hence $r_j = r_0^{p_j/q_j}$, with integers $p_j \in \mathbb{Z}$, $q_j \geq 1$, for $1 \leq j \leq N$. Let :

$$n_0 = \prod_{1\leq l\leq N} q_l \geq 1 \text{ and } n_j = p_j \prod_{1\leq l\leq N, l\neq j} q_l \in \mathbb{Z},\ 1 \leq j \leq N.$$

Recall that $0 < r_0 < 1$. Setting $\lambda = r_0^{1/n_0} \in (0,1)$, one has $r_j = \lambda^{n_j}$, $0 \leq j \leq N$. Up to taking some positive integral power of λ, one can assume that $\gcd(n_0, \cdots, n_N) = 1$. Recall in passing that the set of Pisot numbers is stable under positive integral powers. The condition $\mathbb{E}(\log r_{\varepsilon_0}) < 0$ rewrites into $\mathbb{E}(n_{\varepsilon_0}) > 0$ and we have $n_N \leq \cdots \leq n_0$, with $n_0 \geq 1$.

Using some sub-harmonicity, we shall now show that one can reinforce the assumption that $\hat{\nu}(t)$ is not converging to 0, as $t \to +\infty$.

Lemma 3.1
There exists $1 \leq \alpha \leq 1/\lambda$ and $c > 0$ such that $\hat{\nu}(\alpha\lambda^{-k}) = c_k e^{2i\pi\theta_k}$, written in polar form, verifies $c_k \to c$, as $k \to +\infty$.

Proof of the lemma :
Let us write this time $S_n = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{n-1}}$, for $n \geq 1$, with $S_0 = 0$. Since $\mathbb{E}(n_{\varepsilon_0}) > 0$, (S_n) is transient to $+\infty$. Introduce the random ladder epochs $0 = \sigma_0 < \sigma_1 < \cdots$, where inductively σ_{k+1} is the first time $n \geq 0$ with $S_n > S_{\sigma_k}$. Let $S'_k = S_{\sigma_k}$. The $(S'_k - S'_{k-1})_{k\geq 1}$ are *i.i.d.* random variables with law $\mathcal{L}(S_{\tau_0})$ and support in $\{1, \cdots, n_0\}$. Since $\gcd(n_0, \cdots, n_N) = 1$, the support of the law of S_{τ_0} generates $\mathbb{Z}$ as an additive group (cf for example Woodroofe [28], thm 2.3, second part). For an integer $u \geq 1$ large enough, we can fix integers $r \geq 1$ and $s \geq 1$ such that the support of the law of S'_r contains u and that of S'_s contains $u+1$, both supports being included in some $\{1, \cdots, M\}$, with therefore $1 \leq u \leq u+1 \leq M$. Proceeding as in *Step 1*, for any $t \in \mathbb{R}$:

$$\hat{\nu}(t) = \mathbb{E}\left(e^{2\pi it\sum_{l\geq 0} b_{\varepsilon_l}\lambda^{S_l}}\right) = \mathbb{E}\left(\hat{\nu}(t\lambda^{S'_r})e^{2\pi it\sum_{0\leq l<\sigma_r} b_{\varepsilon_l}\lambda^{S_l}}\right).$$

Doing the same thing with S'_s and taking modulus gives :

$$|\hat{\nu}(t)| \leq \mathbb{E}\left(|\hat{\nu}(t\lambda^{S'_r})|\right) \text{ and } |\hat{\nu}(t)| \leq \mathbb{E}\left(|\hat{\nu}(t\lambda^{S'_s})|\right). \tag{2}$$

In particular, $|\hat{\nu}(t)| \leq \max_{1\leq l\leq M} |\hat{\nu}(\lambda^l t)|$. We now set :

$$V_\alpha(k) := \max_{k \le l < k+M} |\hat{\nu}(\alpha\lambda^l)|, \ k \in \mathbb{Z}, \ \alpha > 0.$$

The previous remarks imply that $V_\alpha(k) \le V_\alpha(k+1)$, $k \in \mathbb{Z}$, $\alpha > 0$.

Since ν is not Rajchman, $|\hat{\nu}(t_l)| \ge c' > 0$, along some sequence $t_l \to +\infty$. Write $t_l = \alpha_l \lambda^{-k_l}$, with $1 \le \alpha_l \le 1/\lambda$ and $k_l \to +\infty$. Up to taking a subsequence, $\alpha_l \to \alpha \in [1, 1/\lambda]$. Fixing $k \in \mathbb{Z}$:

$$c' \le V_{\alpha_l}(-k_l) \le V_{\alpha_l}(-k),$$

as soon as l is large enough. By continuity, letting $l \to +\infty$, we get $c' \le V_\alpha(-k)$, $k \in \mathbb{Z}$. As $k \longmapsto V_\alpha(-k)$ is non-increasing, $V_\alpha(-k) \to c \ge c'$, as $k \to +\infty$. We now show that necessarily $|\hat{\nu}(\alpha\lambda^{-k})| \to c$, as $k \to +\infty$.

If this were not true, there would exist $\varepsilon > 0$ and $(m_k) \to +\infty$, with $|\hat{\nu}(\alpha\lambda^{-m_k})| \le c - \varepsilon$. Using $V_\alpha(-k) \to c$ and $|\hat{\nu}(\alpha\lambda^{-m_k})| \le c - \varepsilon$, as $k \to +\infty$, consider (2) with r and $t = \alpha\lambda^{-m_k-u}$ and next with s and $t = \alpha\lambda^{-m_k-u-1}$. Since u is in the support of the law of S'_r and $u+1$ is in the support of the law of S'_s, we obtain the existence of some $c_1 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-u})|, |\hat{\nu}(\alpha\lambda^{-m_k-u-1})|\} \le c_1 < c.$$

Again via (2), with successively r and $t = \alpha\lambda^{-m_k-2u}$, next r and $t = \alpha\lambda^{-m_k-2u-1}$ and finally s and $t = \alpha\lambda^{-m_k-2u-2}$, still using that u is in the support of the law of S'_r and $u+1$ in the support of the law of S'_s, we get some $c_2 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-2u})|, \ |\hat{\nu}(\alpha\lambda^{-m_k-2u-1})|, \ |\hat{\nu}(\alpha\lambda^{-m_k-2u-2})|\} \le c_2 < c.$$

Etc, for some $c_{M-1} < c$ and k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-(M-1)u})|, \ \cdots, \ |\hat{\nu}(\alpha\lambda^{-m_k-(M-1)u-(M-1)})|\} \le c_{M-1} < c.$$

This contradicts the fact that $V_\alpha(-k) \to c$, as $k \to \infty$. We conclude that $|\hat{\nu}(\alpha\lambda^{-k})| \to c$, as $k \to \infty$, and this ends the proof of the lemma.

□

Step 3. We complete the proof of Theorem 2.3. In this part, introduce the notation $\|x\| = \mathrm{dist}(x, \mathbb{Z})$, for $x \in \mathbb{R}$. Let us consider any $1 \le \alpha \le 1/\lambda$, with $\hat{\nu}(\alpha\lambda^{-k}) = c_k e^{2i\pi\theta_k}$, verifying $c_k \to c > 0$, as $k \to +\infty$. The existence of such a α was shown in *Step 2.* We start from the relation :

$$\hat{\nu}(\alpha\lambda^{-k}) = \sum_{0 \le j \le N} p_j e^{2i\pi\alpha\lambda^{-k} b_j} \hat{\nu}(\alpha\lambda^{-k+n_j}),$$

obtained when conditioning with respect to the value of ε_0. This furnishes for $k \ge 0$:

$$c_k = \sum_{0 \le j \le N} p_j e^{2i\pi(\alpha\lambda^{-k} b_j + \theta_{k-n_j} - \theta_k)} c_{k-n_j}.$$

We rewrite this as :

$$\sum_{0\le j\le N} p_j \left[e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)}-1\right] c_{k-n_j} = c_k - \sum_{0\le j\le N} p_j c_{k-n_j} = \sum_{0\le j\le N} p_j (c_k - c_{k-n_j}).$$

Let $K>0$ be such that $c_{k-n_j}\ge c/2>0$, for $k\ge K$ and all $0\le j\le N$. For $L>n^*$, where $n^*=\max_{0\le j\le N}|n_j|$, we sum the previous equality on $K\le k\le K+L$:

$$\sum_{0\le j\le N} p_j \sum_{k=K}^{K+L} c_{k-n_j}\left[e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)}-1\right] = \sum_{0\le j\le N} p_j \left(\sum_{k=K}^{K+L} c_k - \sum_{k=K-n_j}^{K+L-n_j} c_k\right).$$

Observe that the right-hand side involves a telescopic sum and is bounded by $2n^*$ (using that $|c_k|\le 1$), uniformly in K and L. In the left hand-hand side, we take the real part and use that $1-\cos(2\pi x)=2(\sin\pi x)^2$, which, as is well-known, has the same order as $\|x\|^2$. We obtain, for some constant C, that for K and L large enough :

$$\frac{c}{2}\sum_{0\le j\le N} p_j \sum_{k=K}^{K+L} \|\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k\|^2 \le C.$$

Introducing the constants $p_*=\min_{0\le j\le N} p_j>0$ and $C'=2C/(cp_*)$, we get that for all $0\le j\le N$ and K,L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k\|^2 \le C'. \tag{3}$$

In the sequel, we distinguish two cases : there exists a non-zero translation among the $(\varphi_k)_{0\le k\le N}$ (case 1) or not (case 2).

- *Case 1.* For any non-zero-translation $\varphi_j(x)=x+b_j$, we have $n_j=0$ and $b_j\ne 0$. Then (3) gives that for K,L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k}b_j\|^2 \le C'.$$

This implies that $(\|\alpha b_j\lambda^{-k}\|)_{k\ge 0}\in l^2(\mathbb{N})$. By a classical theorem of Pisot, cf Cassels [3], chap. 8, Theorems I and II, we obtain that $1/\lambda$ is a Pisot number and $b_j=(1/\alpha)\mu_j$, with $\mu_j\in\mathcal{T}(1/\lambda)$. Consider now the non-translations $\varphi_j(x)=\lambda^{n_j}x+b_j$, $n_j\ne 0$. By (3), for any $r\ge 0$ and K,L large enough (depending on r) :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k+rn_j}b_j+\theta_{k-(r+1)n_j}-\theta_{k-rn_j}\|^2 \le C'.$$

Fixing $l_j\ge 1$ and summing over $0\le r\le l_j-1$, making use of the triangular inequality and of $(x_1+\cdots+x_n)^2\le n(x_1^2+\cdots+x_n^2)$, we obtain, for K,L large enough (depending on l_j) :

$$\sum_{k=K}^{K+L} \left\|\alpha\lambda^{-k}b_j\left(\frac{1-\lambda^{l_jn_j}}{1-\lambda^{n_j}}\right)+\theta_{k-l_jn_j}-\theta_k\right\|^2 \le l_jC'. \tag{4}$$

Changing k into $k + l_j n_j$, we obtain, for K, L large enough (depending on l_j) :

$$\sum_{k=K}^{K+L} \left\| \alpha\lambda^{-k} b_j \left(\frac{1-\lambda^{-l_j n_j}}{1-\lambda^{n_j}} \right) + \theta_{k+l_j n_j} - \theta_k \right\|^2 \leq l_j C'. \tag{5}$$

Let $1 = \sum_{0 \leq j \leq N} l_j n_j$ be a Bezout relation and $J \subset \{0, \cdots, N\}$ be the subset of j where $l_j n_j \neq 0$, equipped with its natural order. Using successively for $j \in J$ either (4) or (5), according to the sign of l_j, we obtain with :

$$b := \sum_{j \in J} b_j \lambda^{\sum_{k \in J, k<j} l_k n_k} \left(\frac{1-\lambda^{l_j n_j}}{1-\lambda^{n_j}} \right), \tag{6}$$

the following relation, for a new constant C' and all K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k} b + \theta_{k-1} - \theta_k\|^2 \leq C'.$$

Now, for any $n_j \neq 0$, whatever the sign of n_j is, we arrive at, for some constant C' and all K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k} b \left(\frac{1-\lambda^{n_j}}{1-\lambda} \right) + \theta_{k-n_j} - \theta_k\|^2 \leq C'.$$

Set $b' = b/(1-\lambda)$. Hence, for any $0 \leq j \leq N$ with $n_j \neq 0$, for some new constant C' and all K, L large enough, using (3) :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k}(b_j - b'(1-\lambda^{n_j}))\|^2 \leq C'.$$

Let $0 \leq j \leq N$, with $n_j \neq 0$. If $b_j \neq b'(1-\lambda^{n_j})$, then we deduce again (still by Cassels [3], chap. 8, Theorems I and II) that $1/\lambda$ is a Pisot number and $b_j = b'(1-\lambda^{n_j}) + (1/\alpha)\mu_j$, with $\mu_j \in \mathcal{T}(1/\lambda)$. The other case is $b_j = b'(1-\lambda^{n_j})$. In any case, we obtain that for all $0 \leq j \leq N$:

$$\varphi_j(x) = b' + \lambda^{n_j}(x - b') + (1/\alpha)\mu_j, \tag{7}$$

for some $\mu_j \in \mathcal{T}(1/\lambda)$. Finally, remark that (7) says that the $(\varphi_j)_{0 \leq j \leq N}$ are conjugated with the $(\psi_j)_{0 \leq j \leq N}$, where $\psi_j(x) = \lambda^{n_j} x + \mu_j$. Precisely $\varphi_j = f \circ \psi_j \circ f^{-1}$, with $f(x) = x/\alpha + b'$.

- *Case 2.* Any φ_j with $n_j = 0$ is the identity. The conclusion is the same, because there now necessarily exists some $0 \leq j \leq N$ with $n_j \neq 0$ and $b_j \neq b'(1-\lambda^{n_j})$, otherwise b' is a common fixed point for all $(\varphi_j)_{0 \leq j \leq N}$.

This ends the proof of Theorem 2.3.

□

4 Proof of Theorem 2.4

Let $N \geq 1$ and affine maps $\varphi_k(x) = \lambda^{n_k} x + \mu_k$, for $0 \leq k \leq N$, with $1/\lambda > 1$ a Pisot number, relatively prime integers $(n_k)_{0 \leq k \leq N}$ and $\mu_k \in \mathcal{T}(1/\lambda)$, for $0 \leq k \leq N$. Let $p \in \mathcal{C}_N$ and denote by $(\varepsilon_n)_{n \in \mathbb{Z}}$ a two-sided family of *i.i.d.* random variables with law p, to which again the probability $\mathbb{P}$ and the expectation $\mathbb{E}$ refer. We suppose that $\mathbb{E}(n_{\varepsilon_0}) > 0$. Without loss of generality, $n_N \leq \cdots \leq n_0$ and in particular $n_0 \geq 1$. For general background on Markov chains, cf Spitzer [23].

Recall the cocycle notations for the $(n_{\varepsilon_i})_{i \in \mathbb{Z}}$ introduced before the statement of the theorem and denote by θ the formal shift such that $\theta \varepsilon_l = \varepsilon_{l+1}$, $l \in \mathbb{Z}$. We have for all k and l in $\mathbb{Z}$:

$$S_{k+l} = S_k + \theta^k S_l.$$

Then ν is the law of $X = \sum_{l \geq 0} \mu_{\varepsilon_l} \lambda^{S_l}$. We write $Q \in \mathbb{Z}[X]$ for the minimal polynomial of $1/\lambda$, of degree $s+1$, with roots $\alpha_0 = 1/\lambda$, $\alpha_1, \cdots, \alpha_s$, where $|\alpha_k| < 1$, for $1 \leq k \leq s$. The case $s = 0$ corresponds to $1/\lambda$ an integer ≥ 2 (using then usual conventions regarding sums or products). Recall that for any $k \in \mathbb{Z}$, $\sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l} \mod 1$ is a well-defined $\mathbb{T}$-valued random variable.

Step 1. In order to prove the convergence in law of $(\lambda^{-n}X, \lambda^{-n-1}X, \cdots, \lambda^{-n-s}X) \mod \mathbb{Z}^{s+1}$, as $n \to +\infty$, it is enough to prove, for any $(m_0, \cdots, m_s) \in \mathbb{Z}^{s+1}$, the convergence of :

$$\mathbb{E}\left(e^{2i\pi \sum_{0 \leq u \leq s} m_u \lambda^{-n-u} X}\right) = \mathbb{E}\left(e^{2i\pi \sum_{l \geq 0} (\alpha \mu_{\varepsilon_l}) \lambda^{-n+S_l}}\right),$$

with $\alpha = \sum_{0 \leq u \leq s} m_u \lambda^{-u}$. Notice that $\alpha \mu_j \in \mathcal{T}(1/\lambda)$, for $0 \leq j \leq N$. We make the proof when $\alpha = 1$, the one for α being obtained by changing (μ_j) into $(\alpha \mu_j)$.

Since $\sum_{l<0} \mu_{\varepsilon_l} \lambda^{-n+S_l} \mod 1$ converges a.-s. to 0 in $\mathbb{T}$, as $n \to +\infty$, it is enough to consider expectations with $\sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{-n+S_l} \mod 1$ in the exponential. Let $k \in \mathbb{Z}$ be a fixed integer. For $n \geq 0$, that will tend to $+\infty$, consider $(S_l)_{l \in \mathbb{Z}}$ and the first $q \in \mathbb{Z}$ such that $S_q \geq n$. We have :

$$\begin{aligned}
\mathbb{E}\left(e^{2i\pi \sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k-n+S_l}}\right) &= \sum_{0 \leq r < n_0} \sum_{q \in \mathbb{Z}} \mathbb{E}\left(e^{2i\pi \sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{(k-n+S_q)+(S_l-S_q)}} 1_{S_{q-u} < n, u \geq 1, S_q = n+r}\right) \\
&= \sum_{0 \leq r < n_0} \sum_{q \in \mathbb{Z}} \mathbb{E}\left(e^{2i\pi \sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+r+\theta^q S_{l-q}}} 1_{\theta^q S_{-u} < -r, u \geq 1, \theta^q S_{-q} = -n-r}\right) \\
&= \sum_{0 \leq r < n_0} \sum_{q \in \mathbb{Z}} \mathbb{E}\left(e^{2i\pi \sum_{l \in \mathbb{Z}} \mu_{\varepsilon_{l-q}} \lambda^{k+r+S_{l-q}}} 1_{S_{-u} < -r, u \geq 1, S_{-q} = -n-r}\right) \\
&= \sum_{0 \leq r < n_0} \sum_{q \in \mathbb{Z}} \mathbb{E}\left(e^{2i\pi \sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+r+S_l}} 1_{S_{-u} < -r, u \geq 1, S_{-q} = -n-r}\right).
\end{aligned}$$

For each $0 \leq r < n_0$, we now observe that we can move the sum $\sum_{q \in \mathbb{Z}}$ inside the expectation, using the theorem of Fubini, if we first show the finiteness of :

$$\sum_{q \in \mathbb{Z}} \mathbb{E}\left(1_{S_{-q} = -n-r}\right) = \mathbb{E}\left(\sum_{q \geq 0} 1_{S_{-q} = -n-r}\right) + \mathbb{E}\left(\sum_{q \geq 1} 1_{S_q = -n-r}\right).$$

This is true, since, as soon as n is larger than some constant (because of the missing term for $q=0$ in the second sum), this equals $G^-(0,-n-r)+G^+(0,-n-r)<+\infty$, where $G^-(x,y)$ and $G^+(x,y)$ are the Green functions, finite for every integers x and y, respectively associated to the *i.i.d.* transient random walks $(S_{-q})_{q\geq 0}$ and $(S_q)_{q\geq 0}$. Let σ_k^+, for $k\in\mathbb{Z}$, be the first time ≥ 0 when $(S_q)_{q\geq 0}$ touches k. We have $G^+(x,y)=\mathbb{P}_0(\sigma_{y-x}^+<\infty)G^+(0,0)$. With some symmetric quantities, one has $G^-(x,y)=\mathbb{P}_0(\sigma_{y-x}^-<\infty)G^-(0,0)$.

We therefore obtain :

$$\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k-n+S_l}}\right)=\sum_{0\leq r<n_0}\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,u\geq 1}\left(\sum_{q\in\mathbb{Z}}1_{S_{-q}=-n-r}\right)\right).$$

Let us now fix $0\leq r<n_0$ and consider the corresponding term of the right-hand side. First of all, for $n>0$ larger than some constant (so that $S_0\neq -n-r$) :

$$\mathbb{E}\left(\sum_{q<0}1_{S_{-q}=-n-r}\right)=\mathbb{P}_0(\sigma_{-n-r}^+<\infty)G^+(0,0)\to 0, \tag{8}$$

as $n\to+\infty$, since $(S_q)_{q\geq 0}$ is transient to the right. We thus only need to consider :

$$T(-n):=\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,u\geq 1}N(-n-r)\right),$$

where $N(-k-r):=\sum_{q\geq 0}1_{S_{-q}=-n-r}$. Consider an integer M_0, that will tend to $+\infty$ at the end. The difference of $T(-n)$ with the following expression :

$$\mathbb{E}\left(e^{2i\pi\sum_{l\geq -M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,1\leq u\leq M_0}N(-n-r)\right)$$

is bounded by $A+B$, where, first :

$$\begin{aligned}
A &= \mathbb{E}\left[\left|e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}-e^{2i\pi\sum_{l\geq -M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right|N(-n-r)\right]\\
&= \mathbb{E}\left[\left|1-e^{2i\pi\sum_{l<-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right|N(-n-r)\right]\\
&\leq \left(\mathbb{E}\left[\left|1-e^{2i\pi\sum_{l<-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right|^2\right]\right)^{1/2}\left(\mathbb{E}(N(-n-r)^2)\right)^{1/2}\\
&\leq \left(\mathbb{E}\left[\left|1-e^{2i\pi\sum_{l<-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right|^2\right]\right)^{1/2}\left(\mathbb{E}(N(0)^2)\right)^{1/2},
\end{aligned}$$

because $N(-n-r)$ is stochastically dominated by $N(0)$. Notice that $N(0)$ is square integrable, as it has exponential tail. The first term on the right-hand side also goes to 0, as $M_0\to+\infty$, by dominated convergence. The other term B is :

$$\begin{aligned} B &= \mathbb{E}\left(1_{S_{-u}<-r,1\le u\le M_0,\exists v>M_0,S_{-v}\ge -r} N(-n-r)\right) \\ &\le \mathbb{P}(\exists v>M_0, S_{-v}\ge -r)^{1/2}\left(\mathbb{E}(N(-n-r)^2)\right)^{1/2} \\ &\le \mathbb{P}(\exists v>M_0, S_{-v}\ge -r)^{1/2}\left(\mathbb{E}(N(0)^2)\right)^{1/2}, \end{aligned}$$

as before. The first term on the right-hand side goes to 0, as $M_0 \to +\infty$, since (S_{-v}) is transient to $-\infty$, as $v \to +\infty$. As a result :

$$T(-n) = \mathbb{E}\left(e^{2i\pi \sum_{l\ge -M_0} \mu_{\varepsilon_l} \lambda^{k+r+S_l}} 1_{S_{-u}<-r,1\le u\le M_0} N(-n-r)\right) + o_{M_0}(1),$$

where $o_{M_0}(1)$ goes to 0, as $M_0 \to +\infty$, uniformly in n. Now, when $n > 0$ is large enough, $N(-k-r) = \sum_{q\ge 0} 1_{S_{-q}=-n-r} = \sum_{q\ge M_0} 1_{S_{-q}=-n-r}$, for all ω. Taking inside the expectation the conditional expectation with respect to the σ-algebra generated by the $(\varepsilon_l)_{l\ge -M_0}$, we obtain :

$$T(-n) = \mathbb{E}\left(e^{2i\pi \sum_{l\ge -M_0} \mu_{\varepsilon_l} \lambda^{k+r+S_l}} 1_{S_{-u}<-r,1\le u\le M_0} G^-(S_{-M_0}, -n-r)\right) + o_{M_0}(1).$$

Now, things are simpler because $G^-(S_{-M_0}, -n-r)$ is bounded by the constant $G^-(0,0)$. Hence, for some new $o_{M_0}(1)$, with the same properties :

$$T(-n) = \mathbb{E}\left(e^{2i\pi \sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+r+S_l}} 1_{S_{-u}<-r,u\ge 1} G^-(S_{-M_0}, -n-r)\right) + o_{M_0}(1).$$

Since $G^-(S_{-M_0}, -n-r) \to 1/\mathbb{E}(n_{\varepsilon_0})$, as $n\to\infty$, by renewal theory (since the (n_j) are relatively prime and $p_j > 0$, for all $0 \le j \le N$; cf Woodroofe [28], chap. 2, thm 2.1), staying bounded by $G^-(0,0)$, we get by dominated convergence and next $M_0 \to +\infty$:

$$\lim_{n\to+\infty} T(-n) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})}\mathbb{E}\left(e^{2i\pi \sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+r+S_l}} 1_{S_{-u}<-r,u\ge 1}\right).$$

From the initial expression, the limit, if existing, had to be independent on the parameter k. So this gives the announced convergence and invariance, hence proving item *i)* in Theorem 2.4.

Step 2. We now consider the proof of Theorem 2.4 *ii)* and suppose that ν is continuous. We first show that m is a continuous measure. For a continuous $f : \mathbb{T} \to \mathbb{R}^+$ and any $k \in \mathbb{R}$, we have :

$$\int_{\mathbb{T}} f(x)\, dm(x) \le \frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\le r<n^*} \mathbb{E}\left[f(Z_{k+r})\right].$$

Letting $k \in \mathbb{Z}$, we have $Z_k = \sum_{l<0} \mu_{\varepsilon_l}\lambda^{k+S_l} + \lambda^k X \mod 1$. Since $\mathcal{L}(\lambda^k X)$ on $\mathbb{R}$ is continuous, $\mathcal{L}(\lambda^k X \mod 1)$ on $\mathbb{T}$ is continuous. Since $\sum_{l<0} \mu_{\varepsilon_l}\lambda^{k+S_l} \mod 1$ and $\lambda^k X \mod 1$ are independent random variables, the law of Z_k on $\mathbb{T}$ is continuous. Thus m is a continuous measure (hence $\mathcal{M}$).

More generally, if $0 \ne n = (n_0, \cdots, n_s)^t \in \mathbb{Z}^{s+1}$ and if Z is random variable with law $\mathcal{M}$, then the law of $\langle Z, n\rangle$ on $\mathbb{T}$ is m_α, measure corresponding to m when replacing the (μ_j) by $(\alpha\mu_j)$, thus the (φ_j) by the (ψ_j), with $\psi_j(x) = \lambda^{n_j}x + \alpha\mu_j$, where $\alpha = \sum_{0\le u\le s} n_u \lambda^{-u}$. Since $\alpha \ne 0$, because

$(\lambda^{-u})_{0\le u\le s}$ is a basis of $\mathbb{Q}[\lambda]$ over $\mathbb{Q}$, the (ψ_j) do not have a common fixed point and thus m_α is continuous, by the previous reasoning.

Suppose now that the (φ_j) have a common fixed point c. Hence $\mu_j = c(1-\lambda^{n_j})$, $0 \le j \le N$, and $\nu = \delta_c$. Necessarily $c \in \mathbb{Q}[\lambda]$, since the n_j are not all zero. We shall show that $\lambda^{-n}c \mod 1$ converges to a rational number in $\mathbb{T}$, as $n \to +\infty$. First of all, for n large enough, for all $0 \le j \le N$:

$$Tr_{1/\lambda}(c\lambda^{-n}) - Tr_{1/\lambda}(c\lambda^{-n+n_j}) = Tr_{1/\lambda}(\lambda^{-n}\mu_j) \in \mathbb{Z}.$$

Hence, for any fixed sequence $(k_j)_{0\le j\le N}$, for n large enough, for all $0 \le j \le N$:

$$Tr_{1/\lambda}(c\lambda^{-n}) - Tr_{1/\lambda}(c\lambda^{-n+k_j n_j}) \in \mathbb{Z}.$$

Supposing that $\sum_{0\le j\le N} k_j n_j = 1$, using the previous expression successively n replaced by $n, n - k_0 n_0, \cdots, n - \sum_{0\le j\le N-1} k_j n_j$, respectively with $j=0, j=1, \cdots, j=N$, and finally adding the results, we obtain that for some large $K>0$, for all $n > K$:

$$Tr_{1/\lambda}(c\lambda^{-n}) - Tr_{1/\lambda}(c\lambda^{-n+1}) \in \mathbb{Z}.$$

Let $Tr_{1/\lambda}(c\lambda^{-K}) = p/q$. For $n > K$, there exists an integer l_n such that $Tr_{1/\lambda}(c\lambda^{-n}) = p/q + l_n$. As a result, denoting by $c = c_0, c_1, \cdots, c_s$ the conjugates of c corresponding to $\mathbb{Q}[\lambda]$ (reminding that $(\alpha_j)_{0\le j\le s}$ are that of $1/\lambda = \alpha_0$), we get :

$$c\lambda^{-n} = p/q + l_n - \sum_{1\le j\le s} c_j \alpha_j^n.$$

Consequently $\lambda^{-n}c \mod 1$ converges to p/q in $\mathbb{T}$, as $n \to +\infty$, as announced.

Step 3. Consider the proof of Theorem 2.4 *iii*). We show that when ν is Rajchman, then $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}}$. Fix any $0 \neq (n_0, \cdots, n_s)^t \in \mathbb{Z}^{s+1}$ and set $\beta = \sum_{0\le u\le s} n_u \lambda^{-u}$. Again $\beta \neq 0$. We have :

$$\sum_{0\le u\le s} n_u(\lambda^{-n-u}X) = \beta\lambda^{-n}X.$$

Since ν is Rajchman, $\mathbb{E}(e^{2i\pi\beta\lambda^{-n}X}) \to 0$, as $n \to +\infty$. As a result, the Fourier coefficient of $\mathcal{M}$ corresponding to $(n_0, \cdots, n_s)$ is zero. Hence $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}}$. This implies that $m = \mathcal{L}_{\mathbb{T}}$.

To complete the proof of *iii*), we show that $\nu \perp \mathcal{L}_{\mathbb{R}}$ implies $\mathcal{M} \perp \mathcal{L}_{\mathbb{T}^{s+1}}$. Recall that $Z_k = \sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l} \mod 1$. For any $f \in C(\mathbb{T}^{s+1}, \mathbb{R})$ and $k \in \mathbb{Z}$:

$$\frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\le r<n^*} \mathbb{E}\left[f(Z_{-k+r}, Z_{-k+r-1}, \cdots, Z_{-k+r-s}) 1_{S_{-v}<-r, v\ge 1}\right] = \int_{\mathbb{T}^{s+1}} f(x)\, d\mathcal{M}(x),$$

with $n^* = \max_{0\le j\le N} n_j$. We now fix $k \ge n^*$ so that $Tr_{1/\lambda}(\lambda^{-l}\mu_j) \in \mathbb{Z}$, for $0 \le j \le N$, $l \ge k - n^*$.

For $0 \le j \le N$, denote by $(\mu_j^{(t)})_{0\le t\le s}$ the conjugates of $\mu_j = \mu_j^{(0)}$ corresponding to the field $\mathbb{Q}[\lambda]$. Let $0 \le r < n^*$. Taking any $0 \le u \le s$ and $l < 0$, we have :

$$\mu_{\varepsilon_l}\lambda^{-u-k+r+S_l} = Tr_{1/\lambda}(\mu_{\varepsilon_l}\lambda^{-u-k+r+S_l}) - \sum_{1\le t\le s}\mu_{\varepsilon_l}^{(t)}\alpha_t^{u+k-r-S_l}.$$

The role of the indicator function is now fundamental. On the event $\{S_{-v} < -r, v \ge 1\}$, we have $Tr_{1/\lambda}(\mu_{\varepsilon_l}\lambda^{-u-k+r+S_l}) \in \mathbb{Z}$, by our choice of k, since $l \le -1$. As a result, introducing the *real* random variables :

$$Y_u^{(r)} = \lambda^{-u}\sum_{l\ge 0}\mu_{\varepsilon_l}\lambda^{-k+r+S_l} - \sum_{1\le t\le s}\alpha_t^{u+k-r}\sum_{l<0}\mu_{\varepsilon_l}^{(t)}\alpha_t^{-S_l}, \tag{9}$$

together with $Y^{(r)} = (Y_0^{(r)}, \cdots, Y_s^{(r)})$, we obtain that for any $f \in C(\mathbb{T}^{s+1}, \mathbb{R})$:

$$\frac{1}{\mathbb{E}(n_{\varepsilon_0})}\sum_{0\le r<n^*}\mathbb{E}\left[f(Y^{(r)})1_{S_{-v}<-r, v\ge 1}\right] = \int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x). \tag{10}$$

Hence, for any $f \in C(\mathbb{T}^{s+1}, \mathbb{R}^+)$:

$$\int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x) \le \frac{1}{\mathbb{E}(n_{\varepsilon_0})}\sum_{0\le r<n^*}\mathbb{E}\left[f(Y^{(r)})\right]. \tag{11}$$

Fix any $0 \le r < n^*$ and let $X_0 = \sum_{l\ge 0}\mu_{\varepsilon_l}\lambda^{-k+r+S_l}$ and for $1 \le j \le s$, $X_j = -\sum_{l<0}\mu_{\varepsilon_l}^{(j)}\alpha_j^{k-r-S_l}$. By definition, $(Y^{(r)})^t = V(X_0, \cdots, X_s)^t$, where V is the Vandermonde matrix :

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda^{-1} & \alpha_1 & \cdots & \alpha_s \\ \vdots & \vdots & \vdots & \vdots \\ \lambda^{-s} & \alpha_1^s & \cdots & \alpha_s^s \end{pmatrix}.$$

The matrix V is invertible (since the roots of the minimal polynomial Q of $1/\lambda$ are simple). By Cramer's formula :

$$X_0 = \sum_{0\le i\le s}\gamma_i Y_i^{(r)},$$

with $\gamma_i = \det(V^{(i)})/\det(V)$, where $V^{(i)}$ is obtained from V by replacing the first column by e_i, denoting by $(e_i)_{0\le i\le s}$ the canonical basis of $\mathbb{R}^{s+1}$.

Notice now that each γ_i is real (first of all, $1/\lambda$ is a real root of Q; next, regrouping the other roots in conjugate pairs, when conjugating γ_i one gets permutations in the numerator $\det(V^{(i)})$ and the denominator $\det(V)$, the same ones, so $\bar{\gamma_i} = \gamma_i$). As V is invertible, $\gamma := (\gamma_i)_{0\le i\le s} \neq 0$.

We have $X_0 = \langle Y^{(r)}, \gamma\rangle$. Since ν is singular with respect to $\mathcal{L}_\mathbb{R}$, we also have $\mathcal{L}(X_0) \perp \mathcal{L}_\mathbb{R}$, as $X_0 = \lambda^{-k+r}X$. As $\gamma \neq 0$, we get that $\mathcal{L}(Y^{(r)}) \perp \mathcal{L}_{\mathbb{R}^{s+1}}$. As a result, $\mathcal{L}(Y^{(r)} \mod \mathbb{Z}^{s+1}) \perp \mathcal{L}_{\mathbb{T}^{s+1}}$, for all $0 \le r < n^*$. Finally, (11) implies that $\mathcal{M} \perp \mathcal{L}_{\mathbb{T}^{s+1}}$, as announced.

This ends the proof of Theorem 2.4.

□

5 Proof of Theorem 2.5

The context is the same as that of Theorem 2.4, but now the $(\varphi_k)_{0\le k\le N}$ are strict contractions. Precisely, let $N \ge 1$ and $\varphi_k(x) = \lambda^{n_k} x + \mu_k$, for $0 \le k \le N$, with $1/\lambda > 1$ a fixed Pisot number, relatively prime integers $(n_k)_{0\le k\le N}$, with now $n_0 \ge \cdots \ge n_N \ge 1$, without loss of generality, and $\mu_k \in \mathcal{T}(1/\lambda)$, for $0 \le k \le N$.

Step 1. We first show Theorem 2.5 *i*), using again the arguments appearing in the previous section. If ν is absolutely continuous with respect to $\mathcal{L}_{\mathbb{R}}$, then $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}}$. The event $\{S_{-v} < 0, v \ge 1\}$ has this time probability one. Looking at (10) with $r = 0$, we get that the law of $Y^{(0)} \mod \mathbb{Z}^{s+1}$ is absolutely continuous with respect to $\mathcal{L}_{\mathbb{T}^{s+1}}$, with a density bounded by $\mathbb{E}(n_{\varepsilon_0})$. Hence the law of $Y^{(0)}$ on $\mathbb{R}^{s+1}$ is absolutely continuous with respect to $\mathcal{L}_{\mathbb{R}^{s+1}}$, with a density also bounded by $\mathbb{E}(n_{\varepsilon_0})$. Since the $(\varphi_k)_{0\le k\le N}$ are strict contractions, the n_j are ≥ 1, so the random variable $Y^{(0)}$ is evidently bounded, cf (9). As a result the density of the law of $Y^{(0)}$ with respect to $\mathcal{L}_{\mathbb{R}^{s+1}}$ is bounded and with compact support in $\mathbb{R}^{s+1}$. Hence this is also the case of $X_0 = \langle Y^{(0)}, \gamma\rangle$, where $\gamma = (\gamma_i)_{0\le i\le s} \ne 0$ is the first line of the inverse of the Vandermonde matrix V. Therefore this is also verified for $X = \lambda^{k-r} X_0$. This ends the proof of Theorem 2.5 *i*).

We turn to the proof of Theorem 2.5 *ii*). We shall focus on some Fourier coefficient $\hat{m}(n)$, thus for some fixed $n \in \mathbb{Z}$, of the measure m appearing in Theorem 2.4 *i*). We study its regularity as a function of $p \in \mathcal{C}_N$, showing its real-analytic character. We then conclude the proof of Theorem 2.5 *ii*) using a theorem on the structure of the set of zeros of a non constant real-analytic function.

Step 2. Considering $p \in \mathcal{C}_N$, denote by $(\varepsilon_n)_{n\in\mathbb{Z}}$ a sequence of *i.i.d.* random variables with law p. Let us fix an integer $n \ne 0$, whose exact value will be precised at the end of the proof. We focus on the Fourier coefficient $\hat{m}(n)$ of the measure m introduced in Theorem 2.4 *i*). Let us write m_p in place of m to mark the dependence in $p \in \mathcal{C}_N$. As $n_j \ge 1$, for $0 \le j \le N$, we have the simplified expression for this Fourier coefficient :

$$\hat{m}_p(n) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})}\Delta_p, \text{ with } \Delta_p = \Delta_p(k) = \sum_{0\le r<n_0} \mathbb{E}\left(e^{2i\pi n\sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{n_{\varepsilon_{-1}}>r}\right),$$

where this last quantity is independent on $k \in \mathbb{Z}$, by Theorem 2.4 *i*). The expectation $\mathbb{E}(n_{\varepsilon_0})$ also depends on p, but to study the zeros of $p \longmapsto \hat{m}_p(n)$ we just need to focus on Δ_p. We now consider the regularity of $p \longmapsto \Delta_p$ on the domain $\mathcal{C}_N$.

For any $k \in \mathbb{Z}$, observe first that $\Delta_p(k)$ is well-defined, with the same formula as above, on the closure $\bar{\mathcal{C}}_N$. Fixing $k \in \mathbb{Z}$, the map $p \longmapsto \Delta_p(k)$ is continuous on $\bar{\mathcal{C}}_N$, as this function is the uniform limit on $\bar{\mathcal{C}}_N$, as $L \to +\infty$, of the continuous maps :

$$p \longmapsto \sum_{0\le r<n_0} \mathbb{E}\left(e^{2i\pi n\sum_{-L\le l\le L} \mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{n_{\varepsilon_{-1}}>r}\right).$$

It follows that $p \longmapsto \Delta_p(k) = \Delta_p$ is well-defined on $\bar{\mathcal{C}}_N$, continuous and independent on k. We shall now prove using standard methods that it is in fact real-analytic in a classical sense, precised below. Let us take $k = 0$ and fix $0 \le r < n_0$. Using independence, write :

$$\mathbb{E}\left(e^{2i\pi n\sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l}\lambda^{r+S_l}} 1_{n_{\varepsilon_{-1}}>r}\right) = \mathbb{E}\left(e^{2i\pi n\sum_{l\ge 0} \mu_{\varepsilon_l}\lambda^{r+S_l}}\right)\mathbb{E}\left(e^{2i\pi n\sum_{l\le -1} \mu_{\varepsilon_l}\lambda^{r+S_l}} 1_{n_{\varepsilon_{-1}}>r}\right).$$

Call $F(p)$ and $G(p)$ respectively the terms appearing in the right-hand side. We shall show that both functions are real-analytic functions of p. This property will be inheritated by $p \longmapsto \Delta_p$. We treat the case of $p \longmapsto F(p)$, the case of $G(p)$ needing only to rewrite first the $\mu_{\varepsilon_l}\lambda^{r+S_l}$, appearing in the definition of $G(p)$, as soon as $l < 0$ is large enough (depending only the $(\mu_j)_{0\leq j\leq N}$, since $n_k \geq 1$, for all k), as $-\sum_{1\leq j\leq s}\alpha_j^{-r-S_l}\mu_{\varepsilon_l}^{(j)}$, quantity equal to $\mu_{\varepsilon_l}\lambda^{r+S_l}$ in $\mathbb{T}$, where the $(\mu_k^{(j)})_{1\leq j\leq s}$ are the conjugates of μ_k corresponding to the field $\mathbb{Q}[\lambda]$.

Fix now $p \in \bar{\mathcal{C}}_N$. Let $\mathbb{N} = \{0, 1, \cdots\}$ and the symbolic space $S = \{0, \cdots, N\}^{\mathbb{N}}$, equipped with the left shift σ. For $x = (x_0, x_1, \cdots) \in S$, we define :

$$g(x) = e^{2i\pi n\left(\sum_{l\geq 0}\mu_{x_l}\lambda^{r+n_{x_0}+\cdots+n_{x_{l-1}}}\right)}.$$

Introducing the product measure $\mu_p = (\sum_{0\leq j\leq N} p_j\delta_j)^{\otimes\mathbb{N}}$ on S, we can write :

$$F(p) = \int_S g\ d\mu_p.$$

Denote by $C(S)$ the space of continuous functions $f : S \to \mathbb{C}$ and introduce the operator $P_p : C(S) \to C(S)$ defined by :

$$P_p(f)(x) = \sum_{0\leq j\leq N} p_j f((j, x)),\ x \in S, \tag{12}$$

where $(j, x) \in S$ is the word obtained by the left concatenation of the symbol j to x. The operator P_p is Markovian, i.e. $f \geq 0 \Rightarrow P_p(f) \geq 0$ and verifies $P_p\mathbf{1} = \mathbf{1}$, where $\mathbf{1}(x) = 1$, $x \in S$. The measure μ_p has the invariance property $\int_S P_p(f)\ d\mu_p = \int_S f\ d\mu_p$, $f \in C(S)$. For $f \in C(S)$ and $k \geq 0$, introduce the variation :

$$\mathrm{Var}_k(f) = \sup\{|f(x) - f(y)|,\ (x, y) \in S^2, x_i = y_i,\ 0 \leq i < k\}.$$

For any $0 < \alpha < 1$, let $|f|_\alpha = \sup\{\alpha^{-k}\mathrm{Var}_k(f),\ k \geq 0\}$, as well as $\|f\|_\alpha = |f|_\alpha + \|f\|_\infty$. We denote by $\mathcal{F}_\alpha$ the complex Banach space of fonctions f on S such that $\|f\|_\alpha < \infty$. Any $\mathcal{F}_\alpha$ is preserved by P_p. Observe now that $g \in \mathcal{F}_\alpha$ for $\lambda \leq \alpha < 1$. We fix $\alpha = \lambda$.

As a classical fact from Spectral Theory, cf for example Baladi [1], the operator $P_p : \mathcal{F}_\lambda \to \mathcal{F}_\lambda$ satisfies a Perron-Frobenius theorem. Let us show this elementarily. For $f \in \mathcal{F}_\lambda$, we have :

$$P_p^n f(x) = \sum_{0\leq j_1,\cdots,j_n\leq N} p_{j_1}\cdots p_{j_n} f((j_1, \cdots, j_n, x)).$$

This furnishes $\mathrm{Var}_k(P_p^n f - \mathbf{1}\int_S f\ d\mu_p) = \mathrm{Var}_k(P_p^n f) \leq \mathrm{Var}_{k+n}(f)$. Therefore :

$$\left|P_p^n(f) - \mathbf{1}\int_S f\ d\mu_p\right|_\lambda \leq \lambda^n|f|_\lambda.$$

In a similar way, we can write :

$$(P_p^n f - \mathbf{1}\int_S f\ d\mu_p)(x) = P_p^n(f)(x) - \mathbf{1}(x)\int_S P_p^n(f)\ d\mu_p$$
$$= \sum_{0\le j_1,\cdots,j_n\le N} p_{j_1}\cdots p_{j_n}\int_S (f((j_1,\cdots,j_n,x)) - f((j_1,\cdots,j_n,y)))\ d\mu_p(y).$$

Consequently, $\|P_p^n f - \mathbf{1}\int_S f\ d\mu_p\|_\infty \le \mathrm{Var}_n(f) \le \lambda^n |f|_\lambda$. Putting things together, finally :

$$\|P_p^n(f - \mathbf{1}\int_S f\ d\mu_p)\|_\lambda \le 2\lambda^n \|f\|_\lambda.$$

This shows that 1 is a simple eigenvalue and that the rest of the spectrum of P_p is contained in the closed disk of radius $\lambda < 1$. Remark that this holds uniformly on $p \in \bar{\mathcal{C}}_N$.

Fix some circle Γ centered at 1 and with radius $0 < r < 1-\lambda$. By standard functional holomorphic calculus, cf Kato [10], for any $p \in \bar{\mathcal{C}}_N$, the following operator, involving the resolvent, is a continuous (Riesz) projector on Vect($\mathbf{1}$) :

$$\Pi_p = \int_\Gamma (zI - P_p)^{-1} dz. \tag{13}$$

Moreover $\Pi_p(\mathcal{F}_\lambda)$ and $(I - \Pi_p)(\mathcal{F}_\lambda)$ are closed P_p-invariant subspaces, with :

$$\mathcal{F}_\lambda = \Pi_p(\mathcal{F}_\lambda) \oplus (I - \Pi_p)(\mathcal{F}_\lambda).$$

Also, in restriction to $(I - \Pi_p)(\mathcal{F}_\lambda)$, the spectral radius of P_p is less than λ.

Recall that $N \ge 1$. We view a function of $p \in \bar{\mathcal{C}}_N$ in terms of the first N variables $(p_0, \cdots, p_{N-1}) \in \mathbb{R}^N$. Let $\eta' = (\eta_0, \cdots, \eta_{N-1})$ and $\eta = (\eta_0, \cdots, \eta_{N-1}, -(\eta_0 + \cdots + \eta_{N-1}))$. For any $p \in \bar{\mathcal{C}}_N$ and any η' (even when $p + \eta \notin \bar{\mathcal{C}}_N$), we can define the continuous operator $P_{p+\eta} : \mathcal{F}_\lambda \to \mathcal{F}_\lambda$ by (12). It always verifies the relation :

$$P_{p+\eta} = P_p + \sum_{0\le j\le N-1} \eta_j Q_j,$$

where $Q_j(f)(x) = f(j,x) - f(N,x)$. Denote by $B_N(0,\delta)$ the open Euclidean ball in $\mathbb{R}^N$ of radius δ. Let $\lambda < \lambda' < 1 - r$. For any p in $\bar{\mathcal{C}}_N$, there exists $\delta > 0$ such that when $\eta' \in B_N(0,\delta)$, then $\mathbf{1}$ is still a simple eigenfunction of $P_{p+\eta}$, with $P_{p+\eta}\mathbf{1} = \mathbf{1}$, the rest of the spectrum of $P_{p+\eta}$ being contained in the disk of radius λ' and $\Pi_{p+\eta}$, also defined by (13), is a continuous projector on Vect($\mathbf{1}$); this follows from the implicit function theorem, cf Rosenbloom [16], Kato [10]. By compacity of $\bar{\mathcal{C}}_N$, we can choose $\delta > 0$ uniformly on $p \in \bar{\mathcal{C}}_N$. This defines some open δ-neighborhood $\mathcal{C}_N^\delta$ of $\bar{\mathcal{C}}_N$.

When $p \in \bar{\mathcal{C}}_N$, we have $\int_S f\ d\mu_p = 0$, for $f \in (I - \Pi_p)(\mathcal{F}_\lambda)$. Thus for any $f \in \mathcal{F}_\lambda$:

$$\Pi_p(f) = \left(\int_S f\ d\mu_p\right)\mathbf{1}.$$

Applying this to the function g of interest to us, we obtain that when $p \in \bar{\mathcal{C}}_N$:

$$F(p)\mathbf{1} = \int_\Gamma (zI - P_p)^{-1}(g)dz.$$

The function F is next extended to $\mathcal{C}_N^\delta$ by the previous formula. Recall the following definition :

Definition 5.1
A function $h : \mathcal{C}_N^\delta \to \mathbb{C}$, seen as a function of $(p_0, \cdots, p_{N-1})$, admits a development in series around $p \in \mathcal{C}_N^\delta$, if there exists $\varepsilon > 0$ such that for $\eta' = (\eta_0, \cdots, \eta_{N-1}) \in B_N(0, \varepsilon)$ and writing $\eta = (\eta', -(\eta_0 + \cdots + \eta_{N-1}))$, then $h(p + \eta)$ is given by an absolutely converging series :

$$h(p+\eta) = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} A_{l_0, \cdots, l_{N-1}} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}}.$$

A function is real-analytic in $\mathcal{C}_N^\delta$ if it admits a development in series around all $p \in \mathcal{C}_N^\delta$.

Let us now check that $p \longmapsto F(p)$ is real-analytic on $\mathcal{C}_N^\delta$ in the previous sense. Let $p \in \mathcal{C}_N^\delta$. For $z \in \Gamma$ and η' small enough (and the corresponding η), we can write :

$$\begin{aligned}
(zI - P_{p+\eta})^{-1} &= \left(I - (zI - P_p)^{-1} \sum_{0 \leq j \leq N-1} \eta_j Q_j \right)^{-1} (zI - P_p)^{-1} \\
&= \sum_{n \geq 0} \sum_{0 \leq j_1, \cdots, j_n \leq N-1} \eta_{j_1} \cdots \eta_{j_n} (zI - P_p)^{-1} Q_{j_1} \cdots (zI - P_p)^{-1} Q_{j_n} (zI - P_p)^{-1}.
\end{aligned}$$

For small enough η', uniformly in $z \in \Gamma$, this is absolutely convergent in the Banach operator algebra. We rewrite it as :

$$(zI - P_{p+\eta})^{-1} = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} B_{l_0, \cdots, l_{N-1}}(z) \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}},$$

converging for the operator norm, uniformly in $z \in \Gamma$. Hence, for small enough η' (and thus η) :

$$F(p+\eta)\mathbf{1} = \int_\Gamma (zI - P_{p+\eta})^{-1}(g)\ dz = \sum_{l_0 \geq 0, \cdots, l_{N-1} \geq 0} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}} \int_\Gamma B_{l_0, \cdots, l_{N-1}}(z)(g)\ dz.$$

Applying this equality at some particular $x \in S$, we obtain the desired development in series around p. This completes this step.

Step 3. Maybe restricting $\delta > 0$, taking into account the finite number of functions appearing in the expression of Δ_p, we obtain that $p \longmapsto \Delta_p$ is real-analytic on $\mathcal{C}_N^\delta$. We shall show that if $n \neq 0$ has been appropriately chosen at the beginning, then Δ_p is not zero at some extremal points of $\bar{\mathcal{C}}_N$. The point will be that if ever Δ_p has a zero on $\bar{\mathcal{C}}_N$, then this will imply that either $p \longmapsto \mathrm{Re}(\Delta_p)$ or $p \longmapsto \mathrm{Im}(\Delta_p)$ is non-constant on $\mathcal{C}_N^\delta$.

Now if $h : \mathcal{C}_N^\delta \to \mathbb{R}$ is real-analytic and non-constant, Lojasiewicz's stratification theorem (cf Krantz-Parks [8], theorem 5.2.3) says that the real-analytic set $\{p \in \mathcal{C}_N^\delta \mid h(p) = 0\}$ is locally a finite union of real-analytic graphs of dimension $\leq N-1$ (points if $N=1$). By compacity of $\bar{\mathcal{C}}_N$, the set $\{p \in \bar{\mathcal{C}}_N \mid h(p)=0\}$ is included in a finite union of real-analytic graphs of dimension $\leq N-1$.

For the sequel, let us write $x \equiv y$ for equality of x and y in $\mathbb{T}$.

Lemma 5.2
Let $d \geq 1$ and $\mu \in \mathcal{T}(1/\lambda)$. The series $\sum_{l\in\mathbb{Z}} \mu\lambda^{ld} \mod 1$, well-defined as an element of $\mathbb{T}$, equals a rational number modulo 1.

Proof of the lemma :
Let $l_0 \geq 1$ be such that $Tr_{1/\lambda}(\lambda^{-l}\mu) \in \mathbb{Z}$, for $l > l_0$. Denote by $(\mu^{(j)})_{0\leq j\leq s}$ the conjugates of μ, with $\mu^{(0)} = \mu$, and $\alpha_1, \cdots, \alpha_s$ that of $\alpha_0 = 1/\lambda$. We have the following equalities on the torus :

$$\begin{aligned}
\sum_{l\in\mathbb{Z}} \mu\lambda^{ld} &\equiv \frac{\mu\lambda^{-l_0 d}}{1-\lambda^d} + \sum_{l>l_0} \mu\lambda^{-ld} \equiv \frac{\mu\lambda^{-l_0 d}}{1-\lambda^d} - \sum_{1\leq i\leq s} \mu^{(i)} \sum_{l>l_0} \alpha_i^{ld} \equiv \frac{\mu\lambda^{-l_0 d}}{1-\lambda^d} - \sum_{1\leq i\leq s} \mu^{(i)} \frac{\alpha_i^{(l_0+1)d}}{1-\alpha_i^d} \\
&\equiv -\left(\frac{\mu\lambda^{-(l_0+1)d}}{1-\lambda^{-d}} + \sum_{1\leq i\leq s} \mu^{(i)} \frac{\alpha_i^{(l_0+1)d}}{1-\alpha_i^d} \right) = -Tr_{1/\lambda}\left(\frac{\mu\lambda^{-(l_0+1)d}}{1-\lambda^{-d}} \right) \in \mathbb{Q}.
\end{aligned}$$

□

We complete the argument. Fixing $0 \leq j \leq N$ and $p^j = (0, \cdots, 0, 1, 0, \cdots, 0)$, where the 1 is at place j, we have for $k \in \mathbb{Z}$, recalling that $1 \leq n_j \leq n_0$:

$$\Delta_{p^j} = \Delta_{p^j}(k) = \sum_{0\leq r<n_0} e^{2i\pi n \sum_{l\in\mathbb{Z}} \mu_j \lambda^{k+r+ln_j}} 1_{n_j>r} = \sum_{0\leq r<n_j} e^{2i\pi n \sum_{l\in\mathbb{Z}} \mu_j \lambda^{k+r+ln_j}}.$$

Notice in passing that the invariance with respect to k is now obvious, as we sum over r on a full period of length n_j. Now, taking $k=0$, we have :

$$\Delta_{p^j} = \sum_{0\leq r<n_j} e^{2i\pi n (A_{j,r}/B_{j,r})},$$

for rational numbers $A_{j,r}/B_{j,r}$, making use of the previous lemma, since $\lambda^r \mu_j \in \mathcal{T}(1/\lambda)$, for any r. If for example n is a multiple of $B_{j,r}$ for any $0 \leq r < n_j$, we get $\Delta_{p^j} = n_j \geq 1$, which gives what was desired. This ends the proof of the theorem.

□

Remark. — Lojasiewicz's stratification theorem, giving the local structure of $\{p \in \mathcal{C}_N^\delta \mid h(p) = 0\}$, is a difficult theorem. In an elementary way, using the implicit function theorem, one can show that the set of zeros of a real-valued real analytic non constant function is locally included in a countable union of connected real-analytic graphs of codimension one.

Remark. — In the general case, when the $(\varphi_k)_{0\leq k\leq N}$ are not all strict contractions, the method seems to reach some limit. Using the notation $\mathcal{D}_N(r)$ of the Introduction, with $r = (\lambda^{n_k})_{0\leq k\leq N}$, and considering as in *Step 2* the regularity of $p \longmapsto F(p)$ on $\mathcal{D}_N(r)$, it is not difficult to show continuity, using some standard coupling argument. The real-analytic character, if ever true, a priori requires more work. Still setting $S = \{0, \cdots, N\}^{\mathbb{N}}$ and $\mu_p = (\sum_{0 \le j \le N} p_j \delta_j)^{\otimes \mathbb{N}}$ on S, we again have :

$$F(p) = \int_S g \, d\mu_p,$$

with $g(x) = e^{2i\pi n \left(\sum_{l \ge 0} \mu_{x_l} \lambda^{r + n_{x_0} + \cdots + n_{x_{l-1}}} \right)}$, but this function is only defined μ_p-almost-everywhere.

6 Complements

6.1 A numerical example

Considering an example as simple as possible which is not homogeneous, take $N = 1$ and the two contractions $\varphi_0(x) = \lambda x$, $\varphi_1(x) = \lambda^2 x + 1$, where $1/\lambda > 1$ is a Pisot number, with probability vector $p = (p_0, p_1)$. Then $n_0 = 1$, $n_1 = 2$ and ν is the law of $\sum_{l \ge 0} \varepsilon_l \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}$, with $(\varepsilon_n)_{n \ge 0}$ *i.i.d.*, with common law $\mathrm{Ber}(p_1)$, i.e. $\mathbb{P}(\varepsilon_0 = 1) = p_1$ and $\mathbb{P}(\varepsilon_0 = 0) = 1 - p_1$. We shall take $0 \le p_1 \le 1$ as parameter for simulations. Notice that $\mathbb{E}(n_{\varepsilon_0}) = p_0 + 2p_1 = 1 + p_1$,

Taking $n = 1$, $k \in \mathbb{Z}$ and $r \in \{0, 1\}$, let us define :

$$F_p(k) = \mathbb{E}\left(e^{2i\pi \lambda^k \sum_{l \ge 0} \varepsilon_l \lambda^{n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}}} \right), \; G_p(k, r) = \mathbb{E}\left(e^{2i\pi \sum_{l \ge 0} \varepsilon_l \lambda^{k - (n_{\varepsilon_0} + \cdots + n_{\varepsilon_l})}} 1_{n_{\varepsilon_0} > r} \right),$$

leading to $\Delta_p = F_p(k) G_p(k, 0) + F_p(k+1) G_p(k+1, 1)$, for all $k \in \mathbb{Z}$. Writing m_p in place of m for the measure on $\mathbb{T}$ in Theorem 2.4 *i)* (defined when $0 < p_1 < 1$), we get $\hat{m}_p(1) = \Delta_p / (1 + p_1)$. Let us first discuss the choice of probability vector $p = (1 - p_1, p_1)$ and Pisot number $1/\lambda$.

A degenerated example (the invariant measure being automatically singular with respect to $\mathcal{L}_{\mathbb{R}}$) is for instance given by $\lambda = (3 - \sqrt{5})/2 < 1/2$. Nevertheless, it is interesting to notice that $\lambda^{-n} \equiv -\lambda^n$, $n \ge 0$. Taking $p_1 = 1/2$, one can check that $\Delta_p = |F_p(1)|^2 + |F_p(2)|^2/2$. Necessarily $\Delta_p > 0$. Indeed, $k \longmapsto F_p(k)$ verifying a linear recurrence of order two, the equality $\Delta_p = 0$ would give $F_p(k) = 0$ for all k, but $F_p(k) \to 1$, as $k \to +\infty$. Notice that $(3 - \sqrt{5})/2$ is the largest λ with this property (it has to be a root of some $X^2 - aX + 1$, for some integer $a \ge 0$). Mention that in general Δ_p is not real; cf the pictures below.

To study an interesting example, we take into account the similarity dimension $s(p, r)$, rewritten here as $s(p, \lambda)$:

$$s(p, \lambda) := \frac{(1 - p_1) \ln(1 - p_1) + p_1 \ln p_1}{(1 - p_1) \ln \lambda + p_1 \ln(\lambda^2)}.$$

The condition $s(p, \lambda) \ge 1$ is equivalent to $(1 - p_1) \ln(1 - p_1) + p_1 \ln p_1 - (1 + p_1) \ln \lambda \le 0$. As a function of p_1, the left-hand side has a minimum value $-\ln(\lambda + \lambda^2)$, attained at $p_1 = \lambda/(1 + \lambda)$. As a first attempt, taking for $1/\lambda$ the golden mean $(\sqrt{5} + 1)/2 = 1,618...$ is in fact not interesting, as in this case $\lambda + \lambda^2 = 1$, giving $s(p, \lambda) \le 1$.

We instead take (as considered in Section 2) for $1/\lambda$ the Plastic number, i.e. the unique real root of $X^3 - X - 1$. Approximately, $1/\lambda = 1.324718....$ For this λ :

$$s(p,\lambda) > 1 \Longleftrightarrow 0,203... < p_0 < 0,907....$$

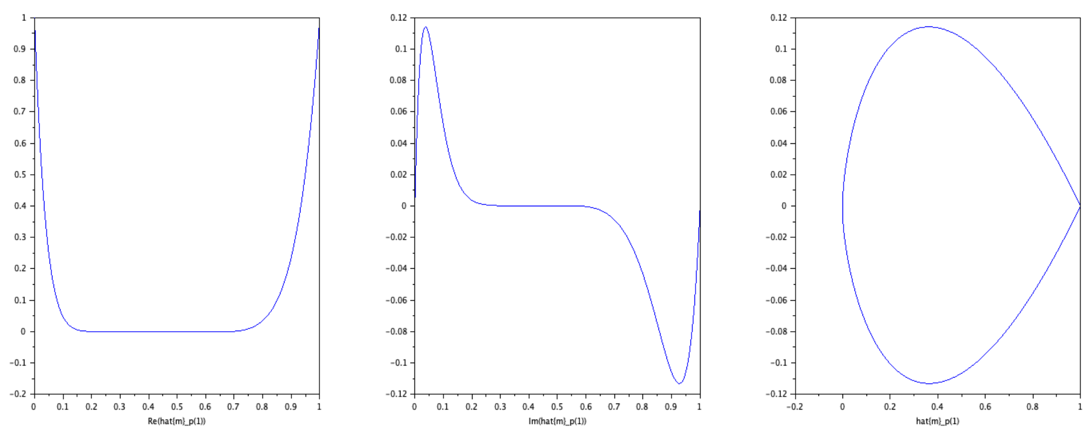
The other roots of $X^3 - X - 1 = 0$ are conjugate numbers $\rho e^{\pm i\theta}$. From the relations $1/\lambda + 2\rho\cos\theta = 0$ and $(1/\lambda)\rho^2 = 1$, we deduce $\rho = \sqrt{\lambda}$ and $\cos\theta = -1/(2\lambda^{3/2})$, thus $\theta = \pm 2.43...$ rad. For computations, the relations $\lambda^{-n} + \rho^n e^{in\theta} + \rho^n e^{-in\theta} \in \mathbb{Z}$, $n \geq 0$, furnish $\lambda^{-n} \equiv -2(\sqrt{\lambda})^n \cos(n\theta)$.

Let us finally compute the extreme values of $p_1 \longmapsto \hat{m}_p(1)$, abusively written as $\hat{m}_{(1,0)}(1)$ and $\hat{m}_{(0,1)}(1)$, since m_p has only been defined for $0 < p_1 < 1$. We first observe that $\hat{m}_{(1,0)}(1) = \Delta_{(1,0)} = F_{(1,0)}(0)G_{(1,0)}(0,0) = 1$. At the other extremity :

$$\begin{aligned}
\Delta_{(0,1)} &= F_{(0,1)}(0)G_{(0,1)}(0,0) + F_{(0,1)}(1)G_{(0,1)}(1,1) \\
&= e^{2i\pi\sum_{l\geq 0}\lambda^{2l}} e^{2i\pi\sum_{l\geq 0}\lambda^{-2(l+1)}} + e^{2i\pi\lambda\sum_{l\geq 0}\lambda^{2l}} e^{2i\pi\sum_{l\geq 0}\lambda^{1-2(l+1)}} \\
&= e^{2i\pi\left(\frac{1}{1-\lambda^2} - 2\sum_{l\geq 0}(\sqrt{\lambda})^{2l}\cos(2l\theta)\right)} + e^{2i\pi\left(\frac{\lambda}{1-\lambda^2} - 2\sum_{l\geq 0}(\sqrt{\lambda})^{2l+1}\cos((2l+1)\theta)\right)} \\
&= e^{2i\pi\left(\frac{1}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\lambda e^{2i\theta}}{1-\lambda e^{2i\theta}}\right)\right)} + e^{2i\pi\left(\frac{\lambda}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\sqrt{\lambda}e^{i\theta}}{1-\lambda e^{2i\theta}}\right)\right)}.
\end{aligned}$$

A not difficult computation, shortened by the observation that $(1-\lambda e^{2i\theta})(1-\lambda e^{-2i\theta}) = 1/\lambda$, shows that the arguments in the exponential terms (after the $2i\pi$) are respectively equal to 3 and 0, leading to $\Delta_{(0,1)} = 2$ and therefore $\hat{m}_{(0,1)}(1) = 1$.

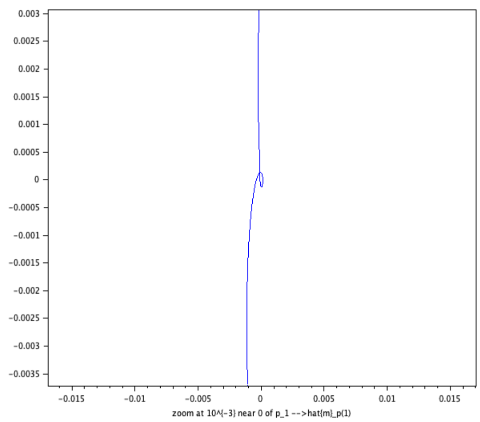
Recalling that $p = (1-p_1, p_1)$, below are respectively drawn the real-analytic maps $p_1 \longmapsto \mathrm{Re}(\hat{m}_p(1))$, $p_1 \longmapsto \mathrm{Im}(\hat{m}_p(1))$ and the parametric curve $p_1 \longmapsto \hat{m}_p(1)$, $0 \leq p_1 \leq 1$.

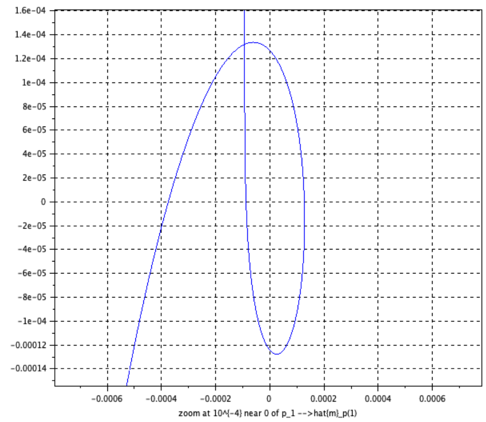


The first two pictures indicate that $p_1 \longmapsto \hat{m}_p(1)$ spends a rather long time near 0, with $\mathrm{Re}(\hat{m}_p(1))$ and $\mathrm{Im}(\hat{m}_p(1))$ both around 10^{-4}. Let us precise here that one can exploit the product form (given by the exponential) inside the expectation appearing in $F_p(k)$ and $G_p(k,r)$. Using a binomial tree, we make a deterministic numerical computation of $\hat{m}_p(1)$, with nearly an arbitrary precision. For example, one can obtain the rather remarquable value :

$$\hat{m}_{(1/2,1/2)}(1) = 0,0001186... + i0,0000327...,$$

where all digits are exact. In this case, $s((1/2,1/2),\lambda) = 1,64... > 1$. The above pictures were drawn with 1000 points, each one determined with a sufficient precision. This allows to safely zoom on the neighbourhood of 0 of $p_1 \longmapsto \hat{m}_p(1)$, the interesting region. We obtain the following surprising pictures, the one on the right-hand side containing around 500 points :



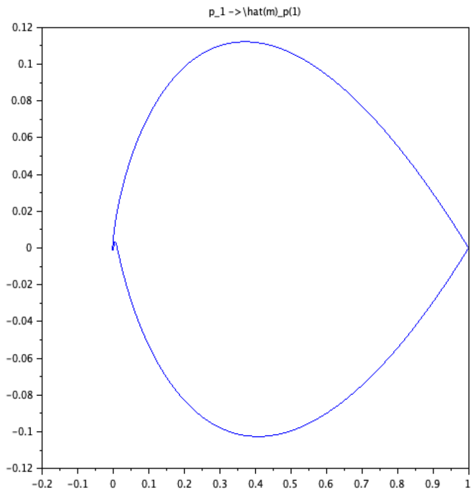


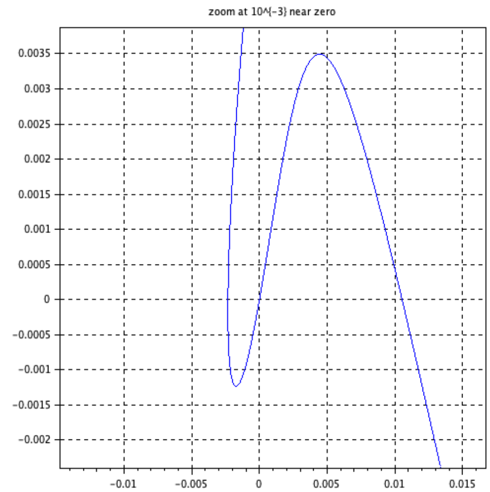
There are probably profound reasons behind these pictures, that would in particular clarify the condition of non-nullity of the Fourier coefficient $\hat{m}_p(1)$ and more generally of $\hat{m}_p(n)$, $n \in \mathbb{Z}$. Further investigations are necessary, but we can conclude that the curve $p_1 \longmapsto \hat{m}_p(1)$ is rather convincingly not touching 0. It may certainly be possible to build a rigorous numerical proof of this fact, but this is not the purpose of the present paper. We informally state :

Numerical Evidence 6.1
Let $N = 1$, $0 < \lambda < 1$, with $1/\lambda > 1$ the Plastic number, and $\varphi_0(x) = \lambda x$, $\varphi_1(x) = \lambda^2 x + 1$. Then for all $p \in \mathcal{C}_1$, the invariant measure ν is continuous singular and not Rajchman.

Remark. — For the same system, but taking for $1/\lambda$ the supergolden ratio, i.e. the fourth Pisot number (the real root of $X^3 - X^2 - 1$), one essentially gets the same pictures.

Still taking for $1/\lambda$ the Plastic number, but for the system $\varphi_0(x) = \lambda^2 x$ and $\varphi_1(x) = \lambda^3 x + 1$, already mentioned in Section 2, recall that the invariant measure ν is continuous singular and not Rajchman for all $p \in \mathcal{C}_1$, except when $p = (\lambda^2, \lambda^3)$, in which case $\nu = \frac{1}{1+\lambda}\mathcal{L}_{[0,1+\lambda]}$. We have drawn below the real analytic curve $p_1 \longmapsto \hat{m}_p(1)$, with next a zoom at 10^{-3} near the origin. This is also interesting, since this time the curve is not self-intersecting, being almost linear near zero and passing at zero exactly for the sole parameter $p_1 = \lambda^3$.





6.2 Applications to sets of uniqueness for trigonometric series

Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, with reals (r_k) and (b_k) such that $0 < r_k < 1$ for all k. As a general fact, Theorem 2.3 has some consequences in terms of sets of multiplicity for trigonometric series, cf for example Salem [18] or Zygmund [29] for details. As in the Introduction, let $F \subset \mathbb{R}$ be the unique non-empty compact set, verifying the self-similarity relation $F = \cup_{0 \leq k \leq N} \varphi_k(F)$. With $\mathbb{N} = \{0, 1, \cdots\}$ and $S = \{0, \cdots, N\}^{\mathbb{N}}$, one has :

$$F = \left\{ \sum_{l \geq 0} b_{x_l} r_{x_0} \cdots r_{x_{l-1}}, \ (x_0, x_1, \cdots) \in S \right\}.$$

Let us place on the torus $\mathbb{T}$ and consider trigonometric series. Recall that a subset E of $\mathbb{T}$ is a set of uniqueness (U-set), if whenever a trigonometric series $\sum_{n \geq 0}(a_n \cos(2\pi x) + b_n \sin(2\pi x))$, with complex numbers (a_n) and (b_n), converges to 0 for all $x \notin E$, then $a_n = b_n = 0$ for all $n \geq 0$. Otherwise E is said of multiplicity (M-set).

Theorem 6.1
Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, where $0 < r_k < 1$, with no common fixed point. Suppose that the system $(\varphi_k)_{0 \leq k \leq N}$ is not affinely conjugated to a family in Pisot form. Then $F \mod 1 \subset \mathbb{T}$ is a M-set.

Proof of the theorem :
Any $p \in \mathcal{C}_N$ gives a Rajchman invariant probability measure ν supported by $F \subset \mathbb{R}$. Hence $F \mod (1) \subset \mathbb{T}$ supports the probability $\tilde{\nu}$, image of ν under the projection $x \longmapsto x \mod 1$, from $\mathbb{R}$ to $\mathbb{T}$. Then $\tilde{\nu}$ is a Rajchman measure on $\mathbb{T}$, so, cf Salem [18] (chap. V), $F \mod 1$ is a M-set. $\square$

In the other direction, in general more delicate, we shall simply apply existing results. For the following statement, fixing $0 < \lambda < 1$ and integers $n_k \geq 1$, for $0 \leq k \leq N$, notice that for any $(x_0, x_1, \cdots) \in S$, we have $\sum_{l \geq 0} \lambda^{n_{x_0}+\cdots+n_{x_{l-1}}}(1-\lambda^{n_{x_l}}) = 1$.

Theorem 6.2
Let $N \geq 1$ and suppose that the (φ_k) are affine contractions of the form $\varphi_k(x) = \lambda^{n_k} x + b_k$, with $b_k = ba_k + c(1-\lambda^{n_k})$, for some $0 < \lambda < 1$ with $1/\lambda$ a Pisot number $> N+2$, relatively prime positive integers $n_k \geq 1$, $0 \leq a_k \in \mathbb{Q}[\lambda]$ and real numbers $b \geq 0$ and c. Then the non-empty compact self-similar set $F = \cup_{0 \leq k \leq N} \varphi_k(F) \subset \mathbb{R}$ can be written as $F = bG + c$, where G is the compact set :

$$G = \left\{ \sum_{l \geq 0} a_{x_l} \lambda^{n_{x_0}+\cdots+n_{x_{l-1}}},\ (x_0, x_1, \cdots) \in S \right\}.$$

Assume that $bG \subset [0,1)$, so that bG and F can be seen as subsets of $\mathbb{T}$. Then F is U-set.

Proof of the theorem :
Up to replacing b and the (a_k) respectively by br and (a_k/r), for some $r > 1$ in $\mathbb{Q}$, we may assume that $0 \leq a_k < 1/(1-\lambda)$, for all $0 \leq k \leq N$. Then :

$$G \subset H := \left\{ \sum_{l \geq 0} \eta_l \lambda^l,\ \eta_l \in \{0, a_0, \cdots, a_N\},\ l \geq 0 \right\} \subset [0,1).$$

Since $1/\lambda > N+2$ is a Pisot number and all $a_0, \cdots, a_N$ are in $\mathbb{Q}[\lambda]$, it follows from the Salem-Zygmund theorem, cf Salem [18], chap. VII, paragraph 3, on perfect homogeneous sets, that H is a perfect U-set. Mention that in this theorem, one also assumes that $\max_{0 \leq k \leq N} a_k = 1/(1-\lambda)$ and that successive $a_u < a_v$ in $[0,1)$ verify $a_v - a_u \geq \lambda$. These conditions serve to give a geometrical description of the perfect homogeneous set H in terms of dissection, without overlaps. They are in fact not used in the proof, where only the above description of H is important (one can indeed start reading Salem [18], chap. VII, paragraph 3, directly from line 9 of the proof).

As a subset of a U-set, G is also a U-set. This is also the case of bG, by hypothesis a subset of $[0,1)$, using Zygmund, Vol. I, chap. IX, Theorem 6.18 (the proof, not obvious, is in Vol. II, chap. XVI, 10.25, and relies on Fourier integrals). Hence, $F = bG + c$ is also a U-set, as any translate on $\mathbb{T}$ of a U-set is a U-set. This ends the proof of the theorem.

$\square$

Remark. — As a general fact, the hypothesis $1/\lambda > N+2$ ensures that H and F have zero Lebesgue measure, which is a necessary condition for a set to be a U-set.

Acknowledgments. We thank B. Kloeckner for initial discussions on this topic, J. Printems for his very precious help in the numerical part of the last section and K. Conrad for several enlightening discussions concerning Algebraic Number Theory questions. We also are grateful to P. Varju for pointing out an error in a first version of Theorem 2.5. His work with H. Yu [27], completes section 6.2 and fully characterizes sets of uniqueness among self-similar sets.

References

[1] V. Baladi, *Positive transfer operators and decay of correlations.* Advanced Series in Nonlinear Dynamics, 16. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[2] C. Bluhm, *Liouville numbers, Rajchman measures, and small Cantor sets.* Proceedings of the American Mathematical Society, Vol. 128, Num. 9, 2637-2640.

[3] J. Cassels, *An introduction to Diophantine approximation.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Cambridge University Press, New York, 1957.

[4] P. Erdös, *On a family of symmetric Bernoulli convolutions.* American Journal of Math. 61 (1939), 974-975.

[5] K. Falconer, *Fractal geometry. Mathematical foundations and applications.* Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[6] M. Hochman, *On self-similar sets with overlaps and inverse theorems for entropy.* Annals of Math. 180 (2014), no. 2, 773-822.

[7] J. Hutchinson, *Fractals and self-similarity.* Indiana U. J. of Math., 30, (1981), p. 713-747.

[8] S. Krantz and H. Parks, *A primer of real analytic functions.* Birkhäuser Verlag, Basel, Boston, Berlin. 1992.

[9] J. Jaroszewska and M. Rams, *On the Hausdorff dimension of invariant measures of weakly contracting on average measurable IFS.* J. Stat. Phys. 132 (2008), no. 5, 907-919.

[10] T. Kato, *Perturbation theory for linear operators.* Second edition, Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1976.

[11] J. Li and T. Sahlsten, *Trigonometric series and self-similar sets.* 2019, arXiv:1902.00426.

[12] R. Lyons, *Seventy years of Rajchman measures.* In Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), number Special Issue, pages 363-377, 1995.

[13] D. Menshov, *Sur l'unicité du développement trigonométrique.* C. R. Acad. Sc. Paris, Sér. A-B 163, 433-436. 1916.

[14] J. Neunhäuserer, *A family of exceptional parameters for non-uniform self-similar measures.* Electron. Commun. Probab. 16 (2011), 192-199.

[15] Y. Peres, W. Schlag and B. Solomyak, *Sixty years of Bernoulli convolutions.* In Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), volume 46 of Progr. Probab., pages 39-65. Birkhäuser, Basel, 2000.

[16] P. Rosenbloom, *Perturbation of linear operators in Banach space.* Arch. Math. (Basel) 6 (1955), 89-101.

[17] R. Salem, *A remarkable class of algebraic integers, proof of a conjecture by Vijayaraghavan.* Duke Math. J. 11 (1944), 103-108.

[18] R. Salem, *Algebraic numbers and Fourier analysis.* D. C. Heath and Co., Boston, Mass. 1963.

[19] P. Samuel, *Algebraic theory of numbers.* Translated from the French by Allan J. Silberger Houghton Mifflin Co., Boston, Mass. 1970.

[20] C. Siegel, *Algebraic numbers whose conjugates lie in the unit circle.* Duke Math. J. 11, 597-602, 1944.

[21] B. Solomyak, *Notes on Bernoulli convolutions.* Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1, 207-230, Proc. Sympos. Pure Math., 72, Part 1, Amer. Math. Soc., Providence, RI, 2004.

[22] B. Solomyak, *Fourier decay for self-similar measures.* 2019, arXiv:1906.12164.

[23] F. Spitzer, *Principles of random walks.* Second edition. Graduate Texts in Mathematics, vol. 34. Springer-Verlag, New-York Heidelberg, 1976.

[24] R. Strichartz, *Self-Similar Measures and Their Fourier Transforms I.* Indiana University Mathematics Journal. Vol. 39, No. 3 (Fall, 1990), pp. 797-817.

[25] R. Strichartz, *Self-Similar Measures and Their Fourier Transforms II.* Transactions of the American Mathematical Society. Volume 336, Number 1, 1993.

[26] M. Tsujii, *On the Fourier transforms of self-similar measures.* Dynamical Systems : an International Journal, Volume 30, Issue 4. Pages 468-484, 2015.

[27] P. Varju and H. Yu, *Fourier decay of self-similar measures and self-similar sets of uniqueness.* Preprint, April 2020.

[28] M. Woodroofe, *Nonlinear renewal theory in sequential analysis.* CBMS-NSF Regional Conference Series in Applied Mathematics, 39. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1982.

[29] A. Zygmund, *Trigonometric series.* Vol. I and II, second edition, Cambridge University Press, 1959.

Univ Paris Est Creteil, CNRS, LAMA, F-94010 Creteil, France
Univ Gustave Eiffel, LAMA, F-77447 Marne-la-Vallée, France
E-mail address : julien.bremont@u-pec.fr