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# EXTRINSIC EIGENVALUES ESTIMATES FOR HYPERSURFACES IN PRODUCT SPACES

JULIEN ROTH

ABSTRACT. We prove extrinsic upper bounds for the first eigenvalue of second order operator of divergence type as well as for Paneitz-like operators and two generalized Steklov problems on hypersurfaces of product spaces  $N \times \mathbb{R}$ . Examples of equality cases are given.

## 1. INTRODUCTION AND STATEMENTS OF THE RESULTS

In his seminal paper [18], Reilly proved the following well-known upper bound for the first non-zero eigenvalue of the Laplace operator on a closed  $n$ -dimensional submanifold  $M$  of a Euclidean space  $\mathbb{R}^m$

$$(1) \text{ ?Reilly1?} \quad \lambda_1(M) \leq \frac{n}{V(M)} \int_M \|H\|^2 dv_g,$$

where  $V(M)$  is the volume of  $(M, g)$ ,  $dv_g$  its volume element and  $H$  is the mean curvature vector of the isometric immersion of  $(M, g)$  into  $\mathbb{R}^m$ . This inequality has been extended by many authors in different contexts: for other ambient spaces [15, 12], in terms of higher order mean curvatures [18], other operators [1, 2, 4, 14, 20], in the anisotropic setting [19], for weighted ambient spaces [5, 11, 20] or for differential forms [21] and spinors [3, 13]. Recently, Xiong [23] obtained extrinsic estimates of Reilly type for closed hypersurfaces of product spaces  $(\mathbb{R} \times N, dt^2 \oplus h)$ , where  $(N^n, h)$  is a complete Riemannian manifold. In particular, he proved that the first eigenvalue  $\lambda_1$  of the Laplace operator and the first eigenvalue  $\sigma_1$  of the Steklov problem for mean-convex hypersurfaces (bounding a domain for the second one) satisfy respectively

$$\lambda_1 \leq n\kappa_+(M)\|H\|_\infty \quad \text{and} \quad \sigma_1 \leq \kappa_+(M) \frac{\|H\|_\infty}{\inf_M H}.$$

In the present note, we prove extrinsic eigenvalue estimates for four types of eigenvalues, namely for divergence-type operators  $L_T$  (Theorem 1.1), Paneitz-like operators (Theorem 1.3), Steklov-Wentzell problem (Theorem 1.5) and biharmonic Steklov problem (Theorem 1.6). These results are extensions for product manifolds  $\mathbb{R} \times N$  of the estimates obtained by the author in [20] for hypersurfaces of Euclidean spaces.

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**1.1.  $L_T$  operators.** Let  $(M^n, g)$  be a closed connected and oriented Riemannian manifold and consider  $T$  a symmetric, divergence-free and positive definite  $(1, 1)$ -tensor over  $M$ . We associate with  $T$  the following second order differential operator  $L_T$  defined by  $L_T f = -\operatorname{div}(\nabla f)$  for any  $\mathcal{C}^2$  function on  $M$ , where  $\operatorname{div}$  and  $\nabla$  are respectively the divergence and the gradient over  $(M^n, g)$ . Under the above assumptions on  $T$ , the operator  $L_T$  is self-adjoint, elliptic and positive. In particular, its spectrum is a increasing sequence of real numbers

$$0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty.$$

The eigenvalue 0 is simple and corresponds to constant eigenfunctions. In the sequel, we will consider the first positive eigenvalue  $\lambda_1$ . Now, assume that  $(M^n, g)$  is isometrically immersed into a Riemannian product  $\mathbb{R} \times N$ . We set

$$(2) \quad \boxed{\text{defHT}} \quad H_T = \operatorname{tr}(TS),$$

where  $S$  is the shape operator of the immersion.

For the well understanding of the statement of the result, we will introduce at this point the following notations: if  $A$  is a  $(1, 1)$ -tensor over  $M$ , then we denote

$$A_- = \min\{A_-(x) | x \in M\}$$

where  $A_-(x)$  is the smallest eigenvalue of  $A$  at the point  $x$  and

$$A_+ = \max\{A_+(x) | x \in M\}$$

where  $A_+(x)$  is the biggest eigenvalue of  $A$  at the point  $x$ .

Now, we can state the first result of this note which gives an extrinsic upper bound for the first eigenvalue of  $L_T$ . Namely, we have the following

(thm1)

**Theorem 1.1.** *Let  $(N^n, h)$  be a complete Riemannian manifold and  $(M^n, g)$  be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product  $(\mathbb{R} \times N, dt^2 \oplus h)$ . Moreover, let  $T$  be a symmetric, positive definite and divergence-free  $(1, 1)$ -tensor over  $M$  and assume that  $H_T$  is a positive function. Then, the first eigenvalue  $\lambda_1$  of the operator  $L_T$  on  $M$  satisfies*

$$\lambda_1 \leq \frac{(TS)_+}{T_-} \|H_T\|_\infty.$$

Moreover, if  $T$  and  $S$  commute, then we have

$$\lambda_1 \leq \kappa_+(M) \|H_T\|_\infty.$$

**Remarks 1.2.** (1) Note that since  $H_T = \operatorname{tr}(TS) > 0$ , then  $(TS)_+ > 0$  and the upper bound in the theorem is positive.

(2) We also want to point out that in the case where  $T$  and  $S$  commute, the hypotheses that  $T$  is positive definite and  $H_T$  is a positive function imply that  $M$  has necessarily at least one positive principal curvature and so the upper bound  $\kappa_+(M) \|H_T\|_\infty$  is positive.

(3) In particular  $T$  and  $S$  commute if  $T$  is one the tensors  $T_r$  associated with the higher order mean curvatures  $H_r$ . They will be considered in the first example of Section 3.

**1.2. Paneitz-like operators.** On a 4-dimensional Riemannian manifold  $(M^4, g)$ , the Paneitz operator, first introduced in [16] by Paneitz (see also [17]), is the fourth order differential operator  $P$  defined by

$$Pu = \Delta^2 u - \operatorname{div} \left( \frac{2}{3} \operatorname{Scal} \nabla u - 2 \operatorname{Ric}(\nabla u) \right),$$

for any  $C^4$  function  $u$ , where  $\operatorname{div}$  is the divergence,  $\Delta = -\operatorname{div} \nabla$  the Laplacian,  $\operatorname{Scal}$  the scalar curvature and  $\operatorname{Ric}$  the  $(1, 1)$ -Ricci tensor associated with the metric  $g$ . It has been generalized in any dimension by Branson [6]. Namely, we have for  $n \geq 5$ ,

$$Pu = \Delta^2 u - \operatorname{div} \left( \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \operatorname{Scal} \nabla u - \frac{4}{n-2} \operatorname{Ric}(\nabla u) \right) + \frac{n-4}{2} Qu,$$

where  $Q$  is the Branson  $Q$ -curvature associated with the metric  $g$ . The Paneitz operator is conformally covariant and plays a crucial role in the problem of prescribing  $Q$ -curvature. In the last two decades, the Paneitz operator (and its links with  $Q$ -curvature) has been intensively studied by many authors (see [10] and reference therein for instance).

Here, we are interesting in the spectrum of the Paneitz operator and more generally of Paneitz-like operators (for which the classical Paneitz operator in dimension 4 is a particular case). In [20] we obtain general Reilly-type upper bounds generalizing previous estimates proved by Chen and Li in [8]. The Paneitz-like operators are defined for some constants  $a$  and  $b$  with  $b \geq -\frac{n}{n-1}$  by

$$P_{a,b}u = \Delta^2 u - \operatorname{div}(a \operatorname{Scal} \nabla u + b \operatorname{Ric} \nabla u),$$

for any smooth function  $u$  on  $M$ . The fourth order operator  $P_{a,b}$  is elliptic and self-adjoint so that it has a discrete real spectrum. In the sequel, we will restrict to the case where  $P_{a,b}$  is positive. The positivity of  $P_{a,b}$  is ensured under some curvature lower bounds (see [24] for more details). Here, we give upper bounds for the first eigenvalue of  $P_{a,b}$  for hypersurfaces in products spaces  $\mathbb{R} \times N$ . Namely, we prove the following

(thm2)

**Theorem 1.3.** *Let  $(N^n, h)$  be a complete Riemannian manifold and  $(M^n, g)$  be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product  $(\mathbb{R} \times N, dt^2 \oplus h)$ . Let  $a$  and  $b$  two real constants with  $b \geq -\frac{n}{n-1}$  and  $na + b \geq 0$ . Moreover, assume that  $M$  has nonnegative scalar curvature and that the Paneitz-like operator  $P_{a,b}$  is positive. Then, the first eigenvalue  $\Lambda_1$  of the Paneitz-like operator  $P_{a,b}$  on  $M$  satisfies*

$$\Lambda_1 \leq n\kappa_+(M) \|H\|_\infty \left( n\kappa_+(M) \|H\|_\infty + (a \operatorname{Scal} \operatorname{Id} + b \operatorname{Ric})_+ \right).$$

**Remark 1.4.** *This result is of interest only if the operator  $P_{a,b}$  is positive. As mentioned, see [24] for details about the positivity of  $P_{a,b}$ .*

**1.3. Steklov-Wentzell problem.** Let  $\Omega$  be a smooth domain of the Riemannian product  $\mathbb{R} \times N$  with non-empty boundary  $M = \partial\Omega$  and  $b$  a nonnegative constant. We will denote by  $g$  the induced metric on  $M$  and denote by  $\bar{\Delta}$  and  $\Delta$  the Laplacian on  $\Omega$  and  $M$  respectively. We consider the following Steklov-type problem for the

Laplacian  $\bar{\Delta}$  with the so-called Wentzell boundary condition. Namely, we consider

$$(SW) \text{ ?Wentzell?} \quad \begin{cases} \bar{\Delta}f = 0 & \text{on } \Omega \\ -b\Delta f - \frac{\partial f}{\partial \nu} = \alpha f & \text{on } M. \end{cases}$$

where  $\frac{\partial f}{\partial \nu} = \langle \bar{\nabla}f, \nu \rangle$  is the derivative of the function  $f$  with the respect to the inner unit normal  $\nu$ . Here,  $\bar{\nabla}$  is the gradient over  $\Omega$ . Note that, if  $b = 0$ , then, we recover the classical Steklov problem. The spectrum of this problem is an increasing sequence (see [9, 22])

$$0 = \alpha_0 < \alpha_1 \leq \alpha_2 \cdots \leq \alpha_k \leq \cdots \longrightarrow +\infty.$$

The eigenvalue 0 is simple and the corresponding eigenfunctions are the constant ones. Here again, we prove a Reilly-type upper bound for the first positive eigenvalue of this problem. Namely, we have

(thm3)

**Theorem 1.5.** *Let  $(N^n, h)$  be a complete Riemannian manifold and  $(M^n, g)$  be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product  $(\mathbb{R} \times N, dt^2 \oplus h)$ . Moreover, assume that  $M$  is mean-convex and bounds a domain  $\Omega$  in  $\mathbb{R} \times N$ . Then, the first eigenvalue  $\alpha_1$  of the Steklov-Wentzell problem satisfies*

$$\alpha_1 \leq \kappa_+(M) \|H\|_\infty \left( \frac{1}{\inf_M H} + bn \right).$$

**1.4. Biharmonic Steklov problem.** Let  $\Omega$  be a smooth domain of the Riemannian product  $\mathbb{R} \times N$  with non-empty boundary  $M = \partial\Omega$  and  $\tau$  a positive constant. We consider the following biharmonic Steklov problem.

$$(BS) \text{ [Steklovbih]} \quad \begin{cases} \bar{\Delta}^2 f - \tau \bar{\Delta}f = 0 & \text{on } \Omega, \\ \frac{\partial^2 f}{\partial \nu^2} = 0 & \text{on } M, \\ \tau \frac{\partial f}{\partial \nu} - \operatorname{div}_{\partial M}(P_{\partial M}((\nabla^2 f)\nu)) - \frac{\partial \Delta f}{\partial \nu} = \beta f & \text{on } M. \end{cases}$$

where  $P_{\partial M}$  is the projection over the tangent space of  $\partial M$ . This problem has a discrete spectrum consisting in an increasing sequence (see [7])

$$0 = \beta_0 < \beta_1 \leq \beta_2 \cdots \leq \beta_k \leq \cdots \longrightarrow +\infty.$$

The eigenvalue 0 is simple and the corresponding eigenfunctions are the constant functions. In the next theorem, we prove an extrinsic upper for the first positive eigenvalue of this problem.

(thm4)

**Theorem 1.6.** *Let  $(N^n, h)$  be a complete Riemannian manifold and  $(M^n, g)$  be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product  $(\mathbb{R} \times N, dt^2 \oplus h)$ . Moreover, assume that  $M$  is mean-convex and bounds a domain  $\Omega$  in  $\mathbb{R} \times N$ . Then, the first eigenvalue of the biharmonic Steklov problem satisfies*

$$\beta_1 \leq \tau \kappa_+(M) \frac{\|H\|_\infty}{\inf_M H}.$$

After giving the proof of these four theorems in Section 2, we will give some examples of their equality cases in Section 3.

## 2. PROOFS OF THE RESULTS

(sec2)

**2.1. Proof of Theorem 1.1.** We recall that the variational characterization of  $\lambda_1$  given by

$$\lambda_1 = \inf \left\{ \frac{\int_M \langle T\nabla f, \nabla f \rangle dv_g}{\int_M f^2 dv_g} \mid u \neq 0, \int_M f dv_g = 0 \right\}.$$

Here,  $\nabla u$  stands for the gradient of the function  $u$  over  $M$  and  $dv_g$  is the Riemannian volume form of  $M$ . Note that in the sequel, we will also use, without confusion,  $\nabla$  for the Levi-Civita connection of  $(M, g)$ .

We will use as test function the function  $t$  which is the coordinate in the factor  $\mathbb{R}$  of the product  $\mathbb{R} \times N$ . First, obviously,  $M$  is invariant by translation in the direction of  $\mathbb{R}$ , so we can assume that  $\int_M t dv_g = 0$ . Second, since the function  $H_T$  is positive, we deduce that  $t$  does not vanish identically. Indeed, if  $t$  vanishes identically over  $M$ , then  $M$  is included in the slice  $\{0\} \times N$ . Since  $M$  is a closed manifold, this is possible if and only if  $M = N$  and so  $M$  is totally geodesic in the product  $N \times \mathbb{R}$ . This is a contradiction with the fact that  $H_T > 0$ . Hence,  $t$  does not vanish identically and can be used as a test function. Thus, we have

$$\lambda_1 \leq \frac{\int_M \langle T(\nabla t), \nabla t \rangle dv_g}{\int_M t^2 dv_g}.$$

Now, let us compute  $L_T t$ . For more convenience, let  $p \in M$  and consider  $\{e_1, \dots, e_n\}$  be a normal frame at  $p$ . We have

$$\begin{aligned} L_T t &= -\operatorname{div}(T\nabla t) \\ &= -\sum_{i=1}^n \langle \nabla_{e_i}(T\nabla t), e_i \rangle \\ &= -\sum_{i,j=1}^n \langle \nabla_{e_i}(\langle \nabla t, e_j \rangle T e_j), e_i \rangle \\ &= -\sum_{i,j=1}^n e_i(\langle \nabla t, e_j \rangle T e_j, e_i) + \sum_{i,j=1}^n \langle \nabla t, e_j \rangle \nabla_{e_i}(T e_j), e_i \rangle \\ &= -\sum_{i,j=1}^n e_i(\langle \nabla t, e_j \rangle T e_j, e_i), \end{aligned}$$

where the second part of the right hand side vanishes in the last line since  $T$  is divergence free. Hence, denoting by  $\bar{\nabla}$  the Levi-Civita connection of  $(\mathbb{R} \times N, dt^2 \oplus h)$ ,

and  $T_{i,j} = \langle Te_i, e_j \rangle$ , we have

$$\begin{aligned} L_T t &= - \sum_{i,j=1}^n \langle \bar{\nabla}_{e_i} \nabla t, e_j \rangle T_{i,j} \\ &= - \sum_{i,j=1}^n \langle \bar{\nabla}_{e_i} (\partial_t - \langle \partial_t, \nu \rangle \nu), e_j \rangle T_{i,j}, \end{aligned}$$

where  $\nu$  is a unit normal vector field. Moreover, since  $\partial_t$  is parallel for  $\bar{\nabla}$  and  $-\bar{\nabla}_{\langle \cdot, \nu \rangle} \nu$  is the shape operator  $S$ , we get

$$\begin{aligned} L_T t &= - \sum_{i,j=1}^n \langle \partial_t, \nu \rangle \langle Se_i, e_j \rangle T_{i,j} \\ &= - \sum_{i=1}^n \langle \partial_t, \nu \rangle \langle Se_i, Te_i \rangle \\ &= -H_T u, \end{aligned}$$

where we have set  $u = \langle \partial_t, \nu \rangle$ . Then, we have

$$\begin{aligned} \lambda_1 &\leq \frac{\int_M t L_T t dv_g}{\int_M t^2 dv_g} \\ &\leq \frac{\left( \int_M t L_T t dv_g \right)^2}{\left( \int_M t^2 dv_g \right) \left( \int_M \langle T \nabla t, \nabla t \rangle dv_g \right)} \end{aligned}$$

But, since  $L_T t = -H_T u$ , we have from the Cauchy-Schwarz inequality

$$\left( \int_M t L_T t dv_g \right)^2 \leq \left( \int_M H_T^2 u^2 dv_g \right) \left( \int_M t^2 dv_g \right)$$

and so

$$\lambda_1 \leq \frac{\int_M H_T^2 dv_g}{\int_M \langle T \nabla t, \nabla t \rangle dv_g}.$$

On the other hand, we have  $u L_T t = -H_T u^2$ , which after integration gives

$$(3) \quad \int_M H_T u^2 dv_g = - \int_M \langle T \nabla u, \nabla t \rangle dv_g = \int_M \langle T S \nabla t, \nabla t \rangle dv_g,$$

since  $\nabla u = \sum_{i=1}^n e_i(u) e_i = \sum_{i=1}^n e_i(\langle \nu, \partial_t \rangle) e_i = - \sum_{i=1}^n \langle Se_i, \partial_t \rangle e_i = -S(\nabla t)$ . Finally, we get

$$\lambda_1 \leq \|H_T\|_\infty \frac{\int_M \langle T S \nabla t, \nabla t \rangle dv_g}{\int_M \langle T \nabla t, \nabla t \rangle dv_g}$$

and so

$$\lambda_1 \leq \frac{(TS)_+}{T_-} \|H_T\|_\infty.$$

We recall that  $T_- = \min\{T_-(x)|x \in M\}$  where  $T_-(x)$  is the smallest eigenvalue of  $T$  at the point  $x$ . Note that  $T_-$  is a positive number since  $T$  is positive definite and  $M$  is compact. Also,  $(TS)_+ = \max\{(TS)_+(x)|x \in M\}$  where  $(TS)_+(x)$  is the largest eigenvalue of  $TS$  at the point  $x$ . Since  $H_T$  is a positive function,  $(TS)_+$  is also a positive number.

Now, assume that  $T$  and  $S$  commute. Since  $T$  is positive definite and symmetric, there exists a square root of  $T$ , denoted  $U$  which is also symmetric, positive definite and which also commutes with  $S$ . Hence, we have

$$\begin{aligned} \lambda_1 &\leq \|H_T\|_\infty \frac{\int_M \langle TS\nabla t, \nabla t \rangle dv_g}{\int_M \langle T\nabla t, \nabla t \rangle dv_g} \\ &\leq \|H_T\|_\infty \frac{\int_M \langle SU\nabla t, U\nabla t \rangle dv_g}{\int_M \langle U\nabla t, U\nabla t \rangle dv_g} \\ &\leq \|H_T\|_\infty \kappa_+(M). \end{aligned}$$

This concludes the proof.  $\square$

**2.2. Proof of Theorem 1.3.** From the variational characterization of  $\Lambda_1$ , we obtain, using  $t$  as test function

$$\begin{aligned} \Lambda_1 \int_M t^2 dv_g &\leq \int_M t P_{a,b} t dv_g \\ &\leq \int_M (t\Delta^2 t - t \operatorname{div}(a\operatorname{Scal} \nabla t + b\operatorname{Ric}(\nabla t))) dv_g \\ &\leq \int_M (|\Delta t|^2 + a\operatorname{Scal} |\nabla t|^2 + b\langle \operatorname{Ric}(\nabla t), \nabla t \rangle) dv_g \\ &\leq \int_M (|\Delta t|^2 + (a\operatorname{Scal} + b\operatorname{Ric})_+ \|\nabla t\|^2) dv_g. \end{aligned}$$

Note that from the assumption  $na + b \geq 0$  and  $\operatorname{Scal} \geq 0$ , then  $(a\operatorname{Scal} + b\operatorname{Ric})_+$  is nonnegative. Moreover, as we have seen in the proof of Theorem 1.1 (with  $T = \operatorname{Id}$ ), we have  $\Delta t = -nHu$ , with  $u = \langle \partial_t, \nu \rangle$ . Hence, we get

$$\begin{aligned} \Lambda_1 &\leq \frac{n^2 \int_M H^2 u^2 dv_g}{\int_M t^2 dv_g} + (a\operatorname{Scal} + b\operatorname{Ric})_+ \frac{\int_M \|\nabla t\|^2 dv_g}{\int_M t^2 dv_g} \\ (4) \quad \boxed{\Lambda_1} &\leq \left( \frac{n^2 \int_M H^2 u^2 dv_g}{\int_M \|\nabla t\|^2 dv_g} + (a\operatorname{Scal} + b\operatorname{Ric})_+ \right) \frac{\int_M \|\nabla t\|^2 dv_g}{\int_M t^2 dv_g}. \end{aligned}$$



First, since  $t\Delta t = -nHut$ , we have

$$(5) \quad \int_M \|\nabla t\|^2 dv_g = n \int_M Hut dv_g \\ \leq n \left( \int_M H^2 u^2 dv_g \right)^{\frac{1}{2}} \left( \int_M t^2 dv_g \right)^{\frac{1}{2}}$$

by the Cauchy-Schwarz inequality. Hence, we get

$$\frac{\int_M \|\nabla t\|^2 dv_g}{\int_M t^2 dv_g} \leq \frac{n^2 \int_M H^2 u^2 dv_g}{\int_M \|\nabla t\|^2 dv_g} \\ \leq n \|H\|_\infty \frac{n \int_M H u^2 dv_g}{\int_M \|\nabla t\|^2 dv_g} \\ \leq n \|H\|_\infty \frac{\int_M \langle S \nabla t, \nabla t \rangle dv_g}{\int_M \|\nabla t\|^2 dv_g} \\ \leq n \|H\|_\infty \kappa_+(M)$$

where we have used (3) with  $T = \text{Id}$ . Thus, reporting in (4), we obtain

$$\Lambda_1 \leq n \kappa_+(M) \|H\|_\infty \left( n \kappa_+(M) \|H\|_\infty + (aS\text{Id} + b\text{Ric})_+ \right),$$

which concludes the proof of Theorem 1.3.  $\square$

**2.3. Proof of Theorem 1.5.** First, we recall that the first eigenvalue  $\alpha_1$  of Steklov-Wentzell problem has the following variational characterization (see [9, 22])

$$(6) \quad \alpha_1 = \inf \left\{ \frac{\int_\Omega \|\bar{\nabla} f\|^2 dv_{\bar{g}} + b \int_M \|\nabla f\|^2 dv_g}{\int_M f^2 dv_g} \mid \int_{\partial M} f dv_g = 0 \right\}.$$

As in the proof of Theorem 1.1, we may assume that the function  $t$  satisfies

$\int_{\partial M} t dv_g = 0$  and thus use it as a test function. So, we get

$$\alpha_1 \leq \frac{\int_\Omega \|\bar{\nabla} t\|^2 dv_{\bar{g}}}{\int_M t^2 dv_g} + b \frac{\int_M \|\nabla t\|^2 dv_g}{\int_M t^2 dv_g}.$$

First, we have

$$\int_\Omega \|\bar{\nabla} t\|^2 dv_{\bar{g}} = - \int_\Omega t \bar{\Delta} t dv_{\bar{g}} + \int_\Omega \text{div}_{\bar{g}}(t \bar{\nabla} t) dv_{\bar{g}}$$

Since  $\bar{\Delta} t = 0$ , using the Stokes theorem, we get

$$\int_\Omega \|\bar{\nabla} t\|^2 dv_{\bar{g}} = \int_M \langle t \bar{\nabla} t, \nu \rangle dv_g = \int_M t u dv_g,$$

where  $u$  is defined as above by  $u = \langle \partial_t, \nu \rangle = \langle \bar{\nabla} t, \nu \rangle$ . Hence, by the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega} \|\bar{\nabla} t\|^2 dv_{\bar{g}} \leq \left( \int_M t^2 dv_g \right)^{\frac{1}{2}} \left( \int_M u^2 dv_g \right)^{\frac{1}{2}}$$

and thus

$$\frac{\int_{\Omega} \|\bar{\nabla} t\|^2 dv_{\bar{g}}}{\int_M t^2 dv_g} \leq \frac{\left( \int_M u^2 dv_g \right)^{\frac{1}{2}}}{\left( \int_M t^2 dv_g \right)^{\frac{1}{2}}}.$$

On the other hand, we have

$$\begin{aligned} n \inf_M(H) \int_M u^2 dv_g &\leq \int_M n H u^2 dv_g \\ &\leq \int_M \langle S \nabla t, \nabla t \rangle dv_g \\ &\leq \kappa_+(M) \int_M \|\nabla t\|^2 dv_g \\ &\leq \kappa_+(M) \int_M n H u t dv_g \\ &\leq n \kappa_+(M) \|H\|_{\infty} \int_M u t dv_g \\ &\leq n \kappa_+(M) \|H\|_{\infty} \left( \int_M t^2 dv_g \right)^{\frac{1}{2}} \left( \int_M u^2 dv_g \right)^{\frac{1}{2}} \end{aligned}$$

where we have used (3) with  $T = \text{Id}$  and (5) successively. Finally, we get

$$(7) \quad \frac{\int_{\Omega} \|\bar{\nabla} t\|^2 dv_{\bar{g}}}{\int_M t^2 dv_g} \leq \frac{\kappa_+(M) \|H\|_{\infty}}{\inf_M(H)}.$$

Moreover, proceeding as in the proof of Theorem 1.1 with  $T = \text{Id}$ , we obtain immediately that

$$\frac{\int_M \|\nabla t\|^2 dv_g}{\int_M t^2 dv_g} \leq n \kappa_+ \|H\|_{\infty}$$

which gives finally

$$\alpha_1 \leq \kappa_+(M) \|H\|_{\infty} \left( \frac{1}{\inf_M H} + bn \right).$$

□

**2.4. Proof of Theorem 1.6.** The boundary conditions in the biharmonic Steklov problem (BS) are the natural one so that the weak formulation of this problem is the following (see [7]):

$$\int_{\Omega} \left( \langle \bar{\nabla}^2 f, \bar{\nabla}^2 \phi \rangle + \tau \langle \bar{\nabla} f, \bar{\nabla} \phi \rangle \right) dv_{\bar{g}} = \beta \int_M f \phi dv_g,$$

Hence, the first positive eigenvalue  $\beta_1$  has the following variational characterization

$$(8) \text{ ?characterized?} = \inf \left\{ \frac{\int_{\Omega} \left( \|\bar{\nabla}^2 f\|^2 + \tau \|\bar{\nabla} f\|^2 \right) dv_{\bar{g}}}{\int_M f^2 dv_g} \mid \int_M f dv_g = 0 \right\}.$$

As previously, up to a possible translation, we use  $t$  as test function in the above variational characterization so that

$$\begin{aligned} \beta_1 \int_M t^2 dv_g &\leq \int_{\Omega} \left( \|\bar{\nabla}^2 t\|^2 + \tau \|\bar{\nabla} t\|^2 \right) dv_{\bar{g}} \\ &\leq \tau \int_{\Omega} \|\bar{\nabla} t\|^2 dv_{\bar{g}}, \end{aligned}$$

since  $\bar{\nabla}^2 t = 0$ . Thus, we have

$$\beta_1 \leq \tau \frac{\int_{\Omega} \|\bar{\nabla} t\|^2 dv_{\bar{g}}}{\int_M t^2 dv_g}$$

From the proof of Theorem 1.5, we have (7)

$$\frac{\int_{\Omega} \|\bar{\nabla} t\|^2 dv_{\bar{g}}}{\int_M t^2 dv_g} \leq \frac{\kappa_+(M) \|H\|_{\infty}}{\inf_M(H)},$$

which concludes the proof of Theorem 1.6.  $\square$

### 3. EXAMPLES OF EQUALITY CASE

(sec3)

We finish this note by giving examples of equality cases for each of the four theorems.

**Example 3.1.** We consider here the well-known operators  $L_r$  associated to the higher order mean curvatures. The higher order mean curvatures are extrinsic quantities defined from the second fundamental form and generalizing the notion of mean curvature. Up to a normalisation constant the mean curvature  $H$  is the trace of the second fundamental form  $B$ :

$$H = \frac{1}{n} \text{tr}(B).$$

In other words the mean curvature is

$$H = \frac{1}{n} S_1(\kappa_1, \dots, \kappa_n),$$

where  $S_1$  is the first elementary symmetric polynomial and  $\kappa_1, \dots, \kappa_n$  are the principal curvatures. Higher order mean curvatures are defined in a similar way for  $r \in \{1, \dots, n\}$  by

$$H_r = \frac{1}{\binom{n}{r}} S_r(\kappa_1, \dots, \kappa_n),$$

where  $S_r$  is the  $r$ -th elementary symmetric polynomial, that is for any  $n$ -tuple  $(x_1, \dots, x_n)$ ,

$$S_r(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdots x_{i_r}.$$

By convention we set  $H_0 = 1$  and  $H_{n+1} = 0$ . Finally, for convenience we also set  $H_{-1} = -\langle X, \nu \rangle$ .

To each  $H_r$  we associate a symmetric  $(2, 0)$ -tensor, which is in coordinates given by

$$T_r = (T_r^{ij}) = \left( \frac{\partial S_{r+1}}{\partial B_{ij}} \right),$$

where  $S_{r+1}$  is now understood to depend on the second fundamental form and the metric. The relation between these two notions can be found in [4] for example. These tensors  $T_r$  are divergence-free (see [14] for instance) and satisfy the following relations:

$$\operatorname{tr}(T_r) = c(r)H_r \quad \text{and} \quad H_{T_r} = -c(r)H_{r+1}\nu,$$

where  $c(r) = (n-r)\binom{n}{r}$  and  $H_{T_r}$  is given by the relation (2). The operator  $L_r$  is defined as the operator  $L_{T_r}$  associated with the tensor  $T_r$ . Note that in space forms, if  $H_{r+1} > 0$ , then  $L_r$  is a positive operator (see [4]).

Now, we consider the sphere  $\mathbb{S}^n(R)$  of radius  $R$  into  $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ . The first eigenvalue of  $L_r$  is  $\frac{c(r)}{R^{n+2}}$  (see [1]). On the other hand, since  $T_r$  and  $B$  commute, the bound of Theorem 1.1 is

$$\kappa_+(M) \|H_{T_r}\|_\infty^2 = \frac{c(r)}{R^{r+2}},$$

since all the principal curvature are  $\frac{1}{R}$  and  $\|H_{T_r}\| = c(r)H_{r+1} = \frac{c(r)}{R^{r+1}}$ . Hence equality occurs in Theorem 1.1.

**Example 3.2.** Now, we consider the sphere  $\mathbb{S}^4(R)$  into  $\mathbb{R}^5 = \mathbb{R} \times \mathbb{R}^4$  and  $P = P_{\frac{2}{3}, -2}$  the Paneitz operator on  $\mathbb{S}^4(R)$ . The upper bounds of Theorem 1.3 is

$$4\kappa_+(M) \|H\|_\infty \left( 4\kappa_+(M) \|H\|_\infty + \left( \frac{2}{3} \operatorname{Scal} \operatorname{Id} - 2 \operatorname{Ric} \right)_+ \right) = \frac{24}{R^4},$$

which is the first eigenvalue of the Paneitz operator on  $\mathbb{S}^4(R)$  (see [8]). Hence equality in Theorem 1.3 is attained.

**Example 3.3.** For the Steklov-Wentzell problem, the same example provide the sharpness of Theorem 1.5. Indeed, for the sphere  $\mathbb{S}^n(R)$  of radius  $R$  into  $\mathbb{R}^{n+1}$ , the upper bound of Theorem 1.5 is

$$\kappa_+(M) \|H\|_\infty \left( \frac{1}{\inf_M H} + bn \right) = \frac{R + bn}{R^2}.$$

On the other hand, in [20, Theorem 3.2], we prove that for the sphere

$$\alpha_1 V(\mathbb{S}^n(R)) = \frac{1}{R^2} \left( nV(B^{n+1}(R)) + bnV(\mathbb{S}^n(R)) \right).$$

An immediate computation using the fact that  $V(\mathbb{S}^n(R)) = \omega_n R^2$  and  $V(B^{n+1}(R)) = \frac{\omega_n R^{n+1}}{n+1}$  where  $\omega_n$  is the volume  $\mathbb{S}^n$ , we obtain that  $\alpha_1 = \frac{R+bn}{R^2}$  and equality occurs in Theorem 1.5.

**Example 3.4.** Finally, for the biharmonic Steklov problem, we consider again  $\mathbb{S}^n(R)$  into  $\mathbb{R}^{n+1}$ . In that case, the first eigenvalue is (see [20, Theorem 3.3])

$$\beta_1 = \frac{(n+1)\tau V(B^{n+1}(R))}{R^2 V(\mathbb{S}^n(R))} = \frac{\tau}{R}$$

which coincides with the upper bound of Theorem 1.6, that is,  $\tau\kappa_+(M)\frac{\|H\|_\infty}{\inf_M(H)}$ .

## REFERENCES

- [ACR] [1] H. Alencar, M.P. Do Carmo & H. Rosenberg, *On the first eigenvalue of Linearized operator of the  $r$ -th mean curvature of a hypersurface*, Ann. Glob. Anal. Geom., **11** (1993), 387-395.
- [AM] [2] L.J. Alias & J.M. Malacarne, *On the first eigenvalue of the linearized operator of the higher order mean curvature for closed hypersurfaces in space forms*, Illinois J. Math. **48** (1) (2004) 219-240.
- [Ba] [3] C. Bär, *Extrinsic Bounds for Eigenvalues of the Dirac Operator*, Ann. Glob. Anal. Geom. **16**(6) (1998), 573-596.
- [BC] [4] J.L.M. Barbosa & A.G. Colares, *Stability of hypersurfaces with constant  $r$ -mean curvature*, Ann. Global Anal. Geom. **15** (1997), 277-297.
- [BCP] [5] M. Batista, M.P. Cavalcante & J. Pyo, *Some isoperimetric inequalities and eigenvalue estimates in weighted manifolds*, J. Math. Anal. Appl., **419**(1) (2014), 617-626.
- [Br] [6] T. Branson, *Differential Operators Canonically Associated to a Conformal Structure*, Math. Scand. **57** (1985), 293-345.
- [BP] [7] D. Buoso & L. Provenzano, *On the eigenvalues of a biharmonic Steklov problem*, Integral Methods in Science and Engineering: Theoretical and Computational Advances, Birkhauser, 2015.
- [CL] [8] D. Chen & H. Li, *The sharp estimates for the first eigenvalue of Paneitz operator on 4-dimensional submanifolds*, arXiv:1010.3102.
- [DKL] [9] M. Dambrine, D. Kateb and J. Lamboley, *An extremal eigenvalue problem for the Wentzell-Laplace operator*, Ann. I. H. Poincaré, **33** (2016), no. 2, 409-450.
- [DHL] [10] Z. Djadli, E. Hebey & M. Ledoux, *Paneitz-type operators and applications*, Duke Math. J., **104**(1) (2000), 129-169.
- [CM] [11] M.C. Domingo-Juan & V. Miquel, *Reilly's type inequality for the Laplacian associated to a density related with shrinkers for MCF*, arXiv:1503.01332.
- [EI] [12] A. El Soufi & S. Ilias, *Une inégalité de type "Reilly" pour les sous-variétés de l'espace hyperbolique*, Comment. Math. Helv. **67**(2) (1992), 167-181.
- [Gi] [13] N. Ginoux, *Reilly-type spinorial inequalities*, Math. Z. **241**(3) (2002), 513-525.
- [Gr] [14] J.F. Grosjean, *Upper bounds for the first eigenvalue of the Laplacian on compact manifolds*, Pac. J. Math. **206**(1) (2002), 93-111.
- [He] [15] E. Heintze, *Extrinsic upper bounds for  $\lambda_1$* , Math. Ann. **280** (1988), 389-402.
- [Pan0] [16] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.
- [Pan] [17] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, SIGMA **4** (2008), article 036.
- [Re] [18] R.C. Reilly, *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52** (1977), 525-533.
- [Ro1] [19] J. Roth, *Upper bounds for the first eigenvalue of the Laplacian in terms of anisotropic mean curvatures*, Results Math. **64** (3-4) (2013), pp 383-403.
- [Ro] [20] J. Roth, *Reilly-type inequalities for Paneitz and Steklov eigenvalues*, Potential Anal. (in press), <https://doi.org/10.1007/s11111>.
- [Sa] [21] A. Savo, *On the first Hodge eigenvalue of isometric immersions*, Proc. Amer. Math. Soc. **133** (2005), 587-594.
- [XW] [22] C. Xia & Q. Wang, *Eigenvalues of the Wentzell-Laplace Operator and of the Fourth Order Steklov Problems*, J. Diff. Equations **264**(10) (2018), 6486-6506.
- [Xi] [23] C. Xiong, *Eigenvalue estimates of Reilly type in product manifolds and eigenvalue comparison for strip domains*, Diff. Geom. Appl. **60** (2018), 104-115.
- [YX] [24] P. Yang & X. Xu, *Positivity of Paneitz operators*, Discrete Cont. Dyn. Syst. **7**(2) (2001), 329-342.

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