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ABSTRACT. We prove extrinsic upper bounds for the first eigenvalue of second order operator of divergence type as well as for Paneitz-like operators and two generalized Steklov problems on hypersurfaces of product spaces $N \times \mathbb{R}$. Examples of equality cases are given.

1. Introduction and statements of the results

In his seminal paper [18], Reilly proved the following well-known upper bound for the first non-zero eigenvalue of the Laplace operator on a closed n-dimensional submanifold M of a Euclidean space $\mathbb{R}^m$

(1) ?Reilly1?
$$\lambda_1(M) \leqslant \frac{n}{V(M)} \int_M \|H\|^2 dv_g,$$

where $V(M)$ is the volume of (M, g), dv_g its volume element and H is the mean curvature vector of the isometric immersion of (M, g) into $\mathbb{R}^m$. This inequality has been extended by many authors in different contexts: for other ambient spaces [15, 12], in terms of higher order mean curvatures [18], other operators [1, 2, 4, 14, 20], in the anisotropic setting [19], for weighted ambient spaces [5, 11, 20] or for differential forms [21] and spinors [3, 13]. Recently, Xiong [23] obtained extrinsic estimates of Reilly type for closed hypersurfaces of product spaces $(\mathbb{R} \times N, dt^2 \oplus h)$, where (N^n, h) is a complete Riemannian manifold. In particular, he proved that the first eigenvalue λ_1 of the Laplace operator and the first eigenvalue σ_1 of the Steklov problem for mean-convex hypersurfaces (bounding a domain for the second one) satisfy respectively

$$\lambda_1 \leqslant n\kappa_+(M)\|H\|_\infty \quad \text{and} \quad \sigma_1 \leqslant \kappa_+(M)\frac{\|H\|_\infty}{\inf\limits_M H}.$$

In the present note, we prove extrinsic eigenvalue estimates for four types of eigenvalues, namely for divergence-type operators L_T (Theorem 1.1), Paneitz-like operators (Theorem 1.3), Steklov-Wentzell problem (Theorem 1.5) and biharmonic Steklov problem (Theorem 1.6). These results are extensions for product manifolds $\mathbb{R} \times N$ of the estimates obtained by the author in [20] for hypersurfaces of Euclidean spaces.

Date: July 2, 2019.

2010 *Mathematics Subject Classification.* 35P15, 58C40.

Key words and phrases. Reilly-type inequality, eigenvalue estimates, hypersurfaces, product manifolds.

1.1. L_T **operators.** Let (M^n, g) be a closed connected and oriented Riemannian manifold and consider T a symmetric, divergence-free and positive definite $(1,1)$-tensor over M. We associate with T the following second order differential operator L_T defined by $L_T f = -\mathrm{div}(\nabla f)$ for any $\mathcal{C}^2$ function on M, where div and ∇ are respectively the divergence and the gradient over (M^n, g). Under the above assumptions on T, the operator L_T is self-adjoint, elliptic and positive. In particular, its spectrum is a increasing sequence of real numbers

$$0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k \leqslant\longrightarrow +\infty.$$

The eigenvalue 0 is simple and corresponds to constant eigenfunctions. In the sequel, we will consider the first positive eigenvalue λ_1. Now, assume that (M^n, g) is isometrically immersed into a Riemannian product $\mathbb{R} \times N$. We set

(2) defHT
$$H_T = \mathrm{tr}\,(TS),$$

where S is the shape operator of the immersion.
For the well understanding of the statement of the result, we will introducre at this point the following notations: if A is a $(1,1)$-tensor over M, then we denote

$$A_- = \min\{A_-(x) | x \in M\}$$

where $A_-(x)$ is the smallest eigenvalue of A at the point x and

$$A_+ = \max\{A_+(x) | x \in M\}$$

where $A_+(x)$ is the biggest eigenvalue of A at the point x.
Now, we can state the first result of this note which gives an extrinsic upper bound for the first eigenvalue of L_T. Namely, we have the following

⟨thm1⟩

Theorem 1.1. *Let (N^n, h) be a complete Riemannian manifold and (M^n, g) be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product $(\mathbb{R} \times N, dt^2 \oplus h)$. Moreover, let T be a symmetric, positive definite and divergence-free $(1,1)$-tensor over M and assume that H_T is a positive function. Then, the first eigenvalue λ_1 of the operator L_T on M satisfies*

$$\lambda_1 \leqslant \frac{(TS)_+}{T_-} \|H_T\|_\infty.$$

Moreover, if T and S commute, then we have

$$\lambda_1 \leqslant \kappa_+(M) \|H_T\|_\infty.$$

Remarks 1.2.
(1) *Note that since $H_T = \mathrm{tr}\,(TS) > 0$, then $(TS)_+ > 0$ and the upper bound in the theorem is positive.*
(2) *We also want to point out that in the case where T and S commute, the hypotheses that T is positive definite and H_T is a positive function imply that M has necessarily at least one positive principal curvature and so the upper bound $\kappa_+(M)\|H_T\|_\infty$ is positive.*
(3) *In particular T and S commute if T is one the tensors T_r associated with the higher order mean curvatures H_r. They will be considered in the first example of Section 3.*

1.2. **Paneitz-like operators.** On a 4-dimensional Riemannian manifold (M^4, g), the Paneitz operator, first introduced in [16] by Paneitz (see also [17]), is the fourth order differential operator P defined by

$$Pu = \Delta^2 u - \operatorname{div}\left(\frac{2}{3}\mathrm{Scal}\,\nabla u - 2\mathrm{Ric}(\nabla u)\right),$$

for any $\mathcal{C}^4$ function u, where div is the divergence, $\Delta = -\mathrm{div}\nabla$ the Laplacian, Scal the scalar curvature and Ric the $(1,1)$-Ricci tensor associated with the metric g. It has been generalized in any dimension by Branson [6]. Namely, we have for $n \geqslant 5$,

$$Pu = \Delta^2 u - -\operatorname{div}\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}\mathrm{Scal}\,\nabla u - \frac{4}{n-2}\mathrm{Ric}(\nabla u)\right) + \frac{n-4}{2}Qu,$$

where Q is the Branson Q-curvature associated with the metric g. The Paneitz operator is conformally covariant and plays a crucial role in the problem of prescribing Q-curvature. In the last two decades, the Paneitz operator (and its links with Q-curvature) has been intensively studied by many authors (see [10] and reference therein for instance).
Here, we are interesting in the spectrum of the Paneitz operator and more generally of Paneitz-like operators (for which the classical Paneitz operator in dimension 4 is a particular case). In [20] we obtain general Reilly-type upper bounds generalizing previous estimates proved by Chen and Li in [8]. The Paneitz-like operators are defined for some constants a and b with $b \geqslant -\frac{n}{n-1}$ by

$$P_{a,b}u = \Delta^2 u - \operatorname{div}(a\mathrm{Scal}\,\nabla u + b\mathrm{Ric}\nabla u),$$

for any smooth function u on M. The fourth order operator $P_{a,b}$ is elliptic and self-adjoint so that it has a discrete real spectrum. In the sequel, we will restrict to the case where $P_{a,b}$ is positive. The positivity of $P_{a,b}$ is ensured under some curvature lower bounds (see [24] for more details). Here, we give upper bounds for the first eigenvalue of $P_{a,b}$ for hypersurfaces in products spaces $\mathbb{R} \times N$. Namely, we prove the following

⟨thm2⟩

Theorem 1.3. *Let (N^n, h) be a complete Riemannian manifold and (M^n, g) be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product $(\mathbb{R} \times N, dt^2 \oplus h)$. Let a and b two real constants with $b \geqslant -\frac{n}{n-1}$ and $na + b \geqslant 0$. Moreover, assume that M has nonnegative scalar curvature and that the Paneitz-like operator $P_{a,b}$ is positive. Then, the first eigenvalue Λ_1 of the Paneitz-like operator $P_{a,b}$ on M satisfies*

$$\Lambda_1 \leqslant n\kappa_+(M)\|H\|_\infty\Big(n\kappa_+(M)\|H\|_\infty + (a\mathrm{Scal}\,\mathrm{Id} + b\mathrm{Ric})_+\Big).$$

Remark 1.4. *This result is of interest only if the operator $P_{a,b}$ is positive. As mentioned, see* [24] *for details about the positivity of $P_{a,b}$.*

1.3. **Steklov-Wentzell problem.** Let Ω be a smooth domain of the Riemannian product $\mathbb{R} \times N$ with non-empty boundary $M = \partial\Omega$ and b a nonnegative constant. We will denote by g the induced metric on M and denote by $\overline{\Delta}$ and Δ the Laplacian on Ω and M respectively. We consider the following Steklov-type problem for the

Laplacian $\overline{\Delta}$ with the so-called Wentzell boundary condition. Namely, we consider

(SW) ?Wentzell?
$$\begin{cases} \overline{\Delta} f = 0 & \text{on } \Omega \\ \\ -b\Delta f - \dfrac{\partial f}{\partial \nu} = \alpha f & \text{on } M. \end{cases}$$

where $\dfrac{\partial f}{\partial \nu} =< \overline{\nabla} f, \nu >$ is the derivative of the function f with the respect to the inner unit normal ν. Here, $\overline{\nabla}$ is the gradient over Ω. Note that, if $b = 0$, then, we recover the classical Steklov problem. The spectrum of this problem is an increasing sequence (see [9, 22])

$$0 = \alpha_0 < \alpha_1 \leqslant \alpha_2 \cdots \leqslant \alpha_k \leqslant \cdots \longrightarrow +\infty.$$

The eigenvalue 0 is simple and the corresponding eigeinfunctions are the constant ones. Here again, we prove a Reilly-type upper bound for the first positive eigenvalue of this problem. Namely, we have

⟨thm3⟩

Theorem 1.5. *Let (N^n, h) be a complete Riemannian manifold and (M^n, g) be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product $(\mathbb{R} \times N, dt^2 \oplus h)$. Moreover, assume that M is mean-convex and bounds a domain Ω in $\mathbb{R} \times N$. Then, the first eigenvalue α_1 of the Steklov-Wentzell problem satisfies*

$$\alpha_1 \leqslant \kappa_+(M) \|H\|_\infty \left(\frac{1}{\inf\limits_M H} + bn \right).$$

1.4. **Biharmonic Steklov problem.** Let Ω be a smooth domain of the riemannian product $\mathbb{R} \times N$ with non-empty boundary $M = \partial\Omega$ and τ a positive constant. We consider the following biharmonic Steklov problem.

(BS) Steklovbih
$$\begin{cases} \overline{\Delta}^2 f - \tau \overline{\Delta} f = 0 & \text{on } \Omega, \\ \\ \dfrac{\partial^2 f}{\partial \nu^2} = 0 & \text{on } M, \\ \\ \tau \dfrac{\partial f}{\partial \nu} - \mathrm{div}_{\partial M}\big(P_{\partial M}((\nabla^2 f)\nu)\big) - \dfrac{\partial \Delta f}{\partial \nu} = \beta f & \text{on } M. \end{cases}$$

where $P_{\partial M}$ is the projection over the tangent space of ∂M.This problem has a discret spectrum consisting in an increasing sequence (see [7])

$$0 = \beta_0 < \beta_1 \leqslant \beta_2 \cdots \leqslant \beta_k \leqslant \cdots \longrightarrow +\infty.$$

The eigenvalue 0 is simple and the corresponding eigeinfunctions are the constant functions. In the next theorem, we prove an extrinsic upper for the first positive eigenvalue of this problem.

⟨thm4⟩

Theorem 1.6. *Let (N^n, h) be a complete Riemannian manifold and (M^n, g) be a closed oriented Riemannian manifold isometrically immersed into the Riemannian product $(\mathbb{R} \times N, dt^2 \oplus h)$. Moreover, assume that M is mean-convex and bounds a domain Ω in $\mathbb{R} \times N$. Then, the first eigenvalue of the biharmonic Steklov problem satisfies*

$$\beta_1 \leqslant \tau \kappa_+(M) \frac{\|H\|_\infty}{\inf\limits_M H}.$$

After giving the proof of these four theorems in Section 2, we will give some examples of their equality cases in Section 3.

2. Proofs of the results

⟨sec2⟩

2.1. **Proof of Theorem 1.1.** We recall that the variational characterization of λ_1 given by

$$\lambda_1 = \inf \left\{ \frac{\displaystyle\int_M \langle T\nabla f, \nabla f\rangle dv_g}{\displaystyle\int_M f^2 dv_g} \Big| u \neq 0,\ \int_M f dv_g = 0 \right\}.$$

Here, ∇u stands for the gradient of the function u over M and dv_g is the Riemannian volume form of M. Note that in the sequel, we will also use, without confusion, ∇ for the Levi-Civita connection of (M, g).

We will use as test function the function t which is the coordinate in the factor $\mathbb{R}$ of the product $\mathbb{R} \times N$. First, obviously, M is invariant by translation in the direction of $\mathbb{R}$, so we can assume that $\displaystyle\int_M t dv_g = 0$. Second, since the function H_T is positive, we deduce that t does not vanish identically. Indeed, if t vanishes identically over M, then M is included in the slice $\{0\} \times N$. Since M is a closed manifold, this is possible if and only if $M = N$ and so M is totally geodesic in the product $N \times \mathbb{R}$. This is a contradiction with the fact that $H_T > 0$. Hence, t does not vanish identically and can be used as a test function.Thus, we have

$$\lambda_1 \leqslant \frac{\displaystyle\int_M \langle T(\nabla t), \nabla t\rangle dv_g}{\displaystyle\int_M t^2 dv_g}.$$

Now, let us compute $L_T t$. For more convience, let $p \in M$ and consider $\{e_1, \cdots, e_n\}$ be a normal frame at p. We have

$$\begin{aligned}
L_T t &= -\mathrm{div}(T\nabla t) \\
&= -\sum_{i=1}^n \langle \nabla_{e_i}(T\nabla t), e_i\rangle \\
&= -\sum_{i,j=1}^n \langle \nabla_{e_i}(\langle \nabla t, e_j\rangle T e_j), e_i\rangle \\
&= -\sum_{i,j=1}^n e_i(\langle \nabla t, e_j\rangle T e_j, e_i\rangle + \sum_{i,j=1}^n \langle \nabla t, e_j\rangle \nabla_{e_i}(T e_j), e_i\rangle \\
&= -\sum_{i,j=1}^n e_i(\langle \nabla t, e_j\rangle T e_j, e_i\rangle,
\end{aligned}$$

where the second part of the right hand side vanishes in the last line since T is divergence free. Hence, denoting by $\overline{\nabla}$ the Levi-Civita connection of $(\mathbb{R}\times N, dt^2 \oplus h)$, and $T_{i,j} = \langle Te_i, e_j\rangle$, we have

$$\begin{aligned} L_T t &= -\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\nabla t, e_j\rangle T_{i,j} \\ &= -\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}(\partial_t - \langle \partial_t, \nu\rangle\nu), e_j\rangle T_{i,j}, \end{aligned}$$

where ν is a unit normal vector field. Moreover, since ∂_t is parallel for $\overline{\nabla}$ and $-\overline{\nabla}_{(\cdot)}\nu$ is the shape operator S, we get

$$\begin{aligned} L_T t &= -\sum_{i,j=1}^{n} \langle \partial_t, \nu\rangle \langle Se_i, e_j\rangle T_{i,j} \\ &= -\sum_{i=1}^{n} \langle \partial_t, \nu\rangle \langle Se_i, Te_i\rangle \\ &= -H_T u, \end{aligned}$$

where we have set $u = \langle \partial_t, \nu\rangle$. Then, we have

$$\begin{aligned} \lambda_1 &\leqslant \frac{\displaystyle\int_M tL_T t dv_g}{\displaystyle\int_M t^2 dv_g} \\ &\leqslant \frac{\left(\displaystyle\int_M tL_T t dv_g\right)^2}{\left(\displaystyle\int_M t^2 dv_g\right)\left(\displaystyle\int_M \langle T\nabla t, \nabla t\rangle dv_g\right)} \end{aligned}$$

But, since $L_T t = -H_T u$, we have from the Cauchy-Schwarz inequality

$$\left(\int_M tL_T t dv_g\right)^2 \leqslant \left(\int_M H_T^2 u^2 dv_g\right)\left(\int_M t^2 dv_g\right)$$

and so

$$\lambda_1 \leqslant \frac{\displaystyle\int H_T^2 dv_g}{\displaystyle\int_M \langle T\nabla t, \nabla t\rangle dv_g}.$$

On the other hand, we have $uL_T t = -H_T u^2$, which after integration gives

$$\int_M H_T u^2 dv_g = -\int_M \langle T\nabla u, \nabla t\rangle dv_g = \int_M \langle TS\nabla t, \nabla t\rangle dv_g, \tag{3}$$

intHT

since $\nabla u = \sum_{i=1}^n e_i(u)e_i = \sum_{i=1}^n e_i(\langle \nu, \partial t\rangle)e_i = -\sum_{i=1}^n \langle Se_i, \partial_t\rangle e_i = -S(\nabla t)$. FInally, we get

$$\lambda_1 \leqslant \|H_T\|_\infty \frac{\displaystyle\int_M \langle TS\nabla t, \nabla t\rangle dv_g}{\displaystyle\int_M \langle T\nabla t, \nabla t\rangle dv_g}$$

and so

$$\lambda_1 \leqslant \frac{(TS)_+}{T_-}\|H_T\|_\infty.$$

We recall that $T_- = \min\{T_-(x) | x \in M\}$ where $T_-(x)$ is the smallest eigenvalue of T at the point x. Note that T_- is a positive number since T is positive definite and M is compact. Also, $(TS)_+ = \max\{(TS)_+(x) | x \in M\}$ where $(TS)_+(x)$ is the largest eigenvalue of TS at the point x. Since H_T is a positive function, $(TS)_+$ is also a positive number.

Now, assume that T and S commute. Since T is positive definite and symmetric, there exists a square root of T, denoted U which is also symmetric, positive definite and which also commutes with S. Hence, we have

$$\begin{aligned}
\lambda_1 &\leqslant \|H_T\|_\infty \frac{\displaystyle\int_M \langle TS\nabla t, \nabla t\rangle dv_g}{\displaystyle\int_M \langle T\nabla t, \nabla t\rangle dv_g} \\
&\leqslant \|H_T\|_\infty \frac{\displaystyle\int_M \langle SU\nabla t, U\nabla t\rangle dv_g}{\displaystyle\int_M \langle U\nabla t, U\nabla t\rangle dv_g} \\
&\leqslant \|H_T\|_\infty \kappa_+(M).
\end{aligned}$$

This concludes the proof. $\square$

2.2. **Proof of Theorem 1.3.** From the variational characterization of Λ_1, we obtain, using t as test function

$$\begin{aligned}
\Lambda_1 \int_M t^2 dv_g &\leqslant \int_M t P_{a,b} t dv_g \\
&\leqslant \int_M \left(t\Delta^2 t - t\mathrm{div}\,(a\mathrm{Scal}\,\nabla t + b\mathrm{Ric}(\nabla t))\right) dv_g \\
&\leqslant \int_M \left(|\Delta t|^2 + a\mathrm{Scal}\,|\nabla t|^2 + b\langle \mathrm{Ric}(\nabla t), \nabla t\rangle\right) dv_g \\
&\leqslant \int_M \left(|\Delta t|^2 + (a\mathrm{Scal}\, + b\mathrm{Ric})_+ \|\nabla t\|^2\right) dv_g.
\end{aligned}$$

Note that from the assumption $na + b \geqslant 0$ and Scal $\geqslant 0$, then $(a\mathrm{Scal}\, + b\mathrm{Ric})_+$ is nonnegative. Moreover, as we have seen in the proof of Theorem 1.1 (with $T = \mathrm{Id}\,$), we have $\Delta t = -nHu$, with $u = \langle \partial_t, \nu\rangle$. Hence, we get

$$\begin{aligned}
\Lambda_1 &\leqslant \frac{n^2 \displaystyle\int_M H^2 u^2 dv_g}{\displaystyle\int_M t^2 dv_g} + (a\mathrm{Scal}\, + b\mathrm{Ric})_+ \frac{\displaystyle\int_M \|\nabla t\|^2 dv_g}{\displaystyle\int_M t^2 dv_g} \\
&\boxed{\texttt{inegLambdafinal}} \leqslant \left(\frac{n^2 \displaystyle\int_M H^2 u^2 dv_g}{\displaystyle\int_M \|\nabla t\|^2 dv_g} + (a\mathrm{Scal}\, + b\mathrm{Ric})_+ \right) \frac{\displaystyle\int_M \|\nabla t\|^2 dv_g}{\displaystyle\int_M t^2 dv_g}.
\end{aligned} \tag{4}$$

First, since $t\Delta t = -nHut$, we have

$$(5)\ \boxed{\texttt{Hut}} \qquad \begin{aligned} \int_M \|\nabla t\|^2 dv_g &= n\int_M Hutdv_g \\ &\leqslant n\left(\int_M H^2u^2dv_g\right)^{\frac{1}{2}}\left(\int_M t^2dv_g\right)^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality. Hence, we get

$$\begin{aligned} \frac{\displaystyle\int_M \|\nabla t\|^2dv_g}{\displaystyle\int_M t^2dv_g} &\leqslant \frac{n^2\displaystyle\int_M H^2u^2dv_g}{\displaystyle\int_M \|\nabla t\|^2dv_g} \\ &\leqslant n\|H\|_\infty \frac{n\displaystyle\int_M Hu^2dv_g}{\displaystyle\int_M \|\nabla t\|^2dv_g} \\ &\leqslant n\|H\|_\infty \frac{\displaystyle\int_M \langle S\nabla t, \nabla t\rangle dv_g}{\displaystyle\int_M \|\nabla t\|^2dv_g} \\ &\leqslant n\|H\|_\infty \kappa_+(M) \end{aligned}$$

where we have used (3) with $T = \mathrm{Id}$. Thus, reporting in (4), we obtain

$$\Lambda_1 \leqslant n\kappa_+(M)\|H\|_\infty\Big(n\kappa_+(M)\|H\|_\infty + (aS\mathrm{Id} + b\mathrm{Ric})_+\Big),$$

which concludes the proof of Theorem 1.3. ☐

2.3. **Proof of Theorem 1.5.** First, we recall that the first eigenvalue α_1 of Steklov-Wentzell problem has the following variational characterization (see [9, 22])

$$(6)\ \texttt{?characalpha1?}\ \alpha_1 = \inf\left\{\frac{\displaystyle\int_\Omega \|\overline{\nabla} f\|^2dv_{\overline{g}} + b\int_M \|\nabla f\|^2dv_g}{\displaystyle\int_M f^2dv_g}\ \Bigg|\ \int_{\partial M} fdv_g = 0\right\}.$$

As in the proof of Theorem 1.1, we may assume that the function t satisfies $\displaystyle\int_{\partial M} tdv_g = 0$ and thus use it as a test function. So, we get

$$\alpha_1 \leqslant \frac{\displaystyle\int_\Omega \|\overline{\nabla} t\|^2dv_{\overline{g}}}{\displaystyle\int_M t^2dv_g} + b\frac{\displaystyle\int_M \|\nabla t\|^2dv_g}{\displaystyle\int_M t^2dv_g}.$$

First, we have

$$\int_\Omega \|\overline{\nabla} t\|^2dv_{\overline{g}} = -\int_\Omega t\overline{\Delta} tdv_{\overline{g}} + \int_\Omega \mathrm{div}_{\overline{g}}(t\overline{\nabla} t)dv_{\overline{g}}$$

Since $\overline{\Delta} t = 0$, using the Stokes theorem, we get

$$\int_\Omega \|\overline{\nabla} t\|^2dv_{\overline{g}} = \int_M \langle t\overline{\nabla} t, \nu\rangle dv_g = \int_M tudv_g,$$

where u is defined as above by $u = \langle \partial_t, \nu \rangle = \langle \overline{\nabla} t, \nu \rangle$. Hence, by the Cauchy-Schwarz inequality, we obtain

$$\int_\Omega \|\overline{\nabla} t\|^2 dv_{\overline{g}} \leqslant \left(\int_M t^2 dv_g \right)^{\frac{1}{2}} \left(\int_M u^2 dv_g \right)^{\frac{1}{2}}$$

and thus

$$\frac{\displaystyle\int_\Omega \|\overline{\nabla} t\|^2 dv_{\overline{g}}}{\displaystyle\int_M t^2 dv_g} \leqslant \frac{\left(\displaystyle\int_M u^2 dv_g\right)^{\frac{1}{2}}}{\left(\displaystyle\int_M t^2 dv_g\right)^{\frac{1}{2}}}.$$

On the other hand, we have

$$\begin{aligned} n \inf_M (H) \int_M u^2 dv_g &\leqslant \int_M nHu^2 dv_g \\ &\leqslant \int_M \langle S\nabla t, \nabla t \rangle dv_g \\ &\leqslant \kappa_+(M) \int_M \|\nabla t\|^2 dv_g \\ &\leqslant \kappa_+(M) \int_M nHut dv_g \\ &\leqslant n\kappa_+(M) \|H\|_\infty \int_M ut dv_g \\ &\leqslant n\kappa_+(M) \|H\|_\infty \left(\int_M t^2 dv_g \right)^{\frac{1}{2}} \left(\int_M u^2 dv_g \right)^{\frac{1}{2}} \end{aligned}$$

where we have used (3) with $T = \mathrm{Id}$ and (5) successively. Finally, we get

$$\frac{\displaystyle\int_\Omega \|\overline{\nabla} t\|^2 dv_{\overline{g}}}{\displaystyle\int_M t^2 dv_g} \leqslant \frac{\kappa_+(M)\|H\|_\infty}{\inf_M (H)}. \tag{7}$$

lem1

Moreover, proceeding as in the proof of Theorem 1.1 with $T = \mathrm{Id}$, we obtain immediately that

$$\frac{\displaystyle\int_M \|\nabla t\|^2 dv_g}{\displaystyle\int_M t^2 dv_g} \leqslant n\kappa_+ \|H\|_\infty$$

which gives finally

$$\alpha_1 \leqslant \kappa_+(M)\|H\|_\infty \left(\frac{1}{\inf_M H} + bn \right).$$

$\square$

2.4. **Proof of Theorem 1.6.** The boundary conditions in the biharmonic Steklov problem (BS) are the natural one so that the weak formulation of this problem is the following (see [7]):

$$\int_\Omega \left(\left\langle \overline{\nabla}^2 f, \overline{\nabla}^2 \phi \right\rangle + \tau \left\langle \overline{\nabla} f, \overline{\nabla} \phi \right\rangle \right) dv_{\overline{g}} = \beta \int_M f\phi dv_g,$$

Hence, the first positive eigenvalue β_1 has the following variational characterization

(8) ?characbeta1? $$\beta_1 = \inf \left\{ \frac{\displaystyle\int_\Omega \left(\|\overline{\nabla}^2 f\|^2 + \tau \|\overline{\nabla} f\|^2 \right) dv_{\overline{g}}}{\displaystyle\int_M f^2 dv_g} \;\middle|\; \int_M f dv_g = 0 \right\}.$$

As previously, up to a possible translation, we use t as test function in the above variational characterization so that

$$\begin{aligned} \beta_1 \int_M t^2 dv_g &\leqslant \int_\Omega \left(\|\overline{\nabla}^2 t\|^2 + \tau \|\overline{\nabla} t\|^2 \right) dv_{\overline{g}} \\ &\leqslant \tau \int_\Omega \|\overline{\nabla} t\|^2 dv_{\overline{g}}, \end{aligned}$$

since $\overline{\nabla}^2 t = 0$. Thus, we have

$$\beta_1 \leqslant \tau \frac{\displaystyle\int_\Omega \|\overline{\nabla} t\|^2 dv_{\overline{g}}}{\displaystyle\int_M t^2 dv_g}$$

From the proof of Theorem 1.5, we have (7)

$$\frac{\displaystyle\int_\Omega \|\overline{\nabla} t\|^2 dv_{\overline{g}}}{\displaystyle\int_M t^2 dv_g} \leqslant \frac{\kappa_+(M)\|H\|_\infty}{\inf_M (H)},$$

which concludes the proof of Theorem 1.6. □

3. Examples of equality case

⟨sec3⟩

We finish this note by giving examples of equality cases for each of the four theorems.

Example 3.1. We consider here the well-known operators L_r associated to the higher order mean curvatures. The higher order mean curvatures are extrinsic quantities defined from the second fundamental form and generalizing the notion of mean curvature. Up to a normalisation constant the mean curvature H is the trace of the second fundamental form B:

$$H = \frac{1}{n} \mathrm{tr}\,(B).$$

In other words the mean curvature is

$$H = \frac{1}{n} S_1(\kappa_1, \ldots, \kappa_n),$$

where S_1 is the first elementary symmetric polynomial and $\kappa_1, \ldots, \kappa_n$ are the principal curvatures. Higher order mean curvatures are defined in a similar way for $r \in \{1, \ldots, n\}$ by

$$H_r = \frac{1}{\binom{n}{r}} S_r(\kappa_1, \cdots, \kappa_n),$$

where S_r is the r-th elementary symmetric polynomial, that is for any n-tuple $(x_1, \cdots, x_n)$,

$$S_r(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} x_{i_1} \cdots x_{i_r}.$$

By convention we set $H_0 = 1$ and $H_{n+1} = 0$. Finally, for convenience we also set $H_{-1} = -\langle X, \nu \rangle$.

To each H_r we associate a symmetric $(2,0)$-tensor, which is in coordinates given by

$$T_r = (T_r^{ij}) = \left(\frac{\partial S_{r+1}}{\partial B_{ij}} \right),$$

where S_{r+1} is now understood to depend on the second fundamental form and the metric. The relation between these two notions can be found in [4] for example. These tensors T_r are divergence-free (see [14] for instance) and satisfy the following relations:

$$\operatorname{tr}(T_r) = c(r) H_r \quad \text{and} \quad H_{T_r} = -c(r) H_{r+1} \nu,$$

where $c(r) = (n-r)\binom{n}{r}$ and H_{T_r} is given by the relation (2). The operator L_r is defined as the operator L_{T_r} associated with the tensor T_r. Note that in space forms, if $H_{r+1} > 0$, then L_r is a positive operator (see [4]).

Now, we consider the sphere $\mathbb{S}^n(R)$ of radius R into $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$. The first eigenvalue of L_r is $\frac{c(r)}{R^{n+2}}$ (see [1]). On the other hand, since T_r and B commute, the bound of Theorem 1.1 is

$$\kappa_+(M) \|H_{T_r}\|_\infty^2 = \frac{c(r)}{R^{r+2}},$$

since all the principal curvature are $\frac{1}{R}$ and $\|H_{T_r}\| = c(r) H_{r+1} = \frac{c(r)}{R^{r+1}}$. Hence equality occurs in Theorem 1.1.

Example 3.2. Now, we consider the sphere $\mathbb{S}^4(R)$ into $\mathbb{R}^5 = \mathbb{R} \times \mathbb{R}^4$ and $P = P_{\frac{2}{3}, -2}$ the Paneitz operator on $\mathbb{S}^4(R)$. The upper bounds of Theorem 1.3 is

$$4\kappa_+(M)\|H\|_\infty \Big(4\kappa_+(M)\|H\|_\infty + (\frac{2}{3}\mathrm{Scal}\,\mathrm{Id} - 2\mathrm{Ric})_+ \Big) \quad = \quad \frac{24}{R^4},$$

which is the first eigenvalue of the Paneitz operator on $\mathbb{S}^4(R)$ (see [8]). Hence equality in Theorem 1.3 is attained.

Example 3.3. For the Steklov-Wentzell problem, the same example provide the sharpness of Theorem 1.5. Indeed, for the sphere $\mathbb{S}^n(R)$ of radius R into $\mathbb{R}^{n+1}$, the upper bound of Theorem 1.5 is

$$\kappa_+(M)\|H\|_\infty \left(\frac{1}{\inf\limits_M H} + bn \right) = \frac{R + bn}{R^2}.$$

On the other hand, in [20, Theorem 3.2], we prove that for the sphere

$$\alpha_1 V(\mathbb{S}^n(R)) = \frac{1}{R^2}\Big(nV(B^{n+1}(R)) + bnV(\mathbb{S}^n(R)) \Big).$$

An immediate computation using the fact that $V(\mathbb{S}^n(R) = \omega_n R^2$ and $V(B^{n+1}(R)) = \frac{\omega_n R^{n+1}}{n+1}$ where ω_n is the volume $\mathbb{S}^n$, we obtain that $\alpha_1 = \frac{R+bn}{R^2}$ and equality occurs in Theorem 1.5.

Example 3.4. Finally, for the biharmonic Steklov problem, we consider again $\mathbb{S}^n(R)$ into $\mathbb{R}^{n+1}$. In that case, the first eigenvalue is (see [20, Theorem 3.3])

$$\beta_1 = \frac{(n+1)\tau}{R^2} \frac{V(B^{n+1}(R))}{V(\mathbb{S}^n(R))} = \frac{\tau}{R}$$

which coincides with the upper bound of Theorem 1.6, that is, $\tau\kappa_+(M)\frac{\|H\|_\infty}{\inf_M(H)}$.

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