Random walk in a stratified independent random environment

Julien Brémont

Random walk in a stratified independent random environment

Julien Brémont

Université Paris-Est Créteil, novembre 2018

Abstract

We study Markov chains on a lattice in a codimension-one stratified independent random environment, exploiting results established in [2]. First of all the random walk is transient in dimension at least three. Focusing on dimension two, both recurrence and transience can happen, but transience remains by far the most general situation. We identify the critical scale of the local drift along the strata corresponding to the frontier between the two regimes.

1 Introduction

A very first and important question on the asymptotic behaviour of Markov chains on a lattice in heterogeneous environment is the question of recurrence/transience. We focus in this paper on the situation where the transition laws depend on a single coordinate, continuing the line of research developed in [1, 2]. Historically, a PDE model has first been introduced by Matheron and de Marsily [8] at the beginning of the eighties and a probabilistic version was introduced by Campanino and Petritis [3] in 2003. Let us detail the extended version studied in [2].

Fixing $d \geq 1$, we consider a Markov chain $(S_n)_{n\geq 0}$ in $\mathbb{Z}^d \times \mathbb{Z}$, with $S_0 = 0$. Quantities relative to the first (resp. second) coordinate in $\mathbb{Z}^d$ (resp. $\mathbb{Z}$) are declared "horizontal" (resp. "vertical"). We assume that the transition laws are constant on each affine hyperplane $\mathbb{Z}^d \times \{n\}$, $n \in \mathbb{Z}$. To model this, for each vertical $n \in \mathbb{Z}$, let positive reals p_n, q_n, r_n, with $p_n + q_n + r_n = 1$, and a probability measure μ_n with support in $\mathbb{Z}^d$, satisfying the following conditions :

Hypothesis 1.1 *$\exists \eta > 0$, $\forall n \in \mathbb{Z}$, $\min\{p_n, q_n, r_n\} \geq \eta$, $\sum_{k\in\mathbb{Z}^d} \|k\|^{\max(d,3)}\mu_n(k) \leq 1/\eta$ and the spectrum of $\sum_{k\in\mathbb{Z}^d} kk^t\mu_n(k)$ is included in $[\eta, +\infty)$.*

Observe that when $d = 1$, the last condition reduces to $\mu_n(0) \leq 1 - \eta$. The local horizontal drift is by definition $\varepsilon_n = \sum_{k\in\mathbb{Z}^d} k\mu_n(k)$, the expectation of μ_n, $n \in \mathbb{Z}$. The transition laws are now defined, for all $(m, n) \in \mathbb{Z}^d \times \mathbb{Z}$ and $k \in \mathbb{Z}^d$, by :

$$(m, n) \xrightarrow{p_n} (m, n + 1),\ (m, n) \xrightarrow{q_n} (m, n - 1),\ (m, n) \xrightarrow{r_n\mu_n(k)} (m + k, n).$$

The Matheron-De Marsily or Campanino-Petritis model corresponds to taking $d = 1$, with $p_n = q_n = r_n = 1/3$ and $\mu_n = \delta_{\alpha_n}$, fixing some $(\alpha_n)_{n\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$, corresponding to orientating horizontal lines. It is shown in [3], for example, that the random walk is recurrent when $\alpha_n = (-1)^n$ and transient when (α_n) are independent and identically distributed (we say *i.i.d* for the sequel) random variables with $\mathbb{P}(\alpha_n = \pm 1) = 1/2$, for a.-e. realization. Several extensions around the orientation setting then followed; see Campanino-Petritis [4] for a review. In [1], for the model introduced in the above paragraph, with $d = 1$ and $p_n = q_n$, $\forall n \in \mathbb{Z}$, (vertically flat model) a characterization of the recurrence/transience of the random walk was given. The rather abstract form of this criterion directly comes from the computation of a Poisson kernel in the half-plane, as

AMS 2000 *subject classifications : 60G17, 60J10, 60K37.*

Key words and phrases : Markov chain, recurrence, stratification, random environment, independence.

appearing in the recurrence criterion for *i.i.d.* random walks on $\mathbb{Z}^d$ (see Spitzer [10]). The latter is an essential ingredient in the proof. The principal result of [1] is the main step in analyzing the recurrence/transience properties of the vertically flat model, but some (little) extra work is needed to treat concrete examples. This was done at the end of [1], allowing to prove accurate results about recurrence/transience for this model.

The same recurrence criterion as in [1] was next extended in [2] to the general setting introduced at the beginning of this section. The structure stays the same, but a new metrization of the environment has to introduced, informally corresponding to the way the random walk sees the environment. An interpretation of the criterion was given in relation with the level lines of some interesting function of two variables measuring horizontal dispersion. This will be detailed later. Several examples of recurrent and transient random walks were finally given, both in dimensions 2 and 3 (i.e. $d \in \{1, 2\}$). In dimension ≥ 4 (i.e. $d = 3$), the random walk appears to be always transient ([2], prop. 2.5, 1)*i*)).

The purpose of the present short article is to specifically study the important case when the stratifications of the environment are random and independent, with a quenched point of view. We show that the recurrence criterion of [2], once mastered the user's manual, gives sharp quantitative results, probably having some flavour of what should happen for much more general models of random walks in independent environments.

Let us present the results. We always assume hypothesis 1.1. In the sequel, randomness is for the environment; we never enter the mechanism of the random walk.

Proposition 1.2
If $(p_n/q_n)_{n\in\mathbb{Z}}$ *are i.i.d. random variables, with either* $\mathbb{E}(\log(p_0/q_0)) \neq 0$ *or* $d \geq 2$*, then for a.-e. realization the random walk is transient.*

This will follow rather readily from [2]. Indeed, a key observation, already made in [3] and direct consequence of the invariance of the environment under horizontal translations, is that the random walk restricted to vertical jumps is a Markov chain on $\mathbb{Z}$ (this is never true in general, for a non-stratified environment). Its transition laws are :

$$n \overset{p_n/(p_n+q_n)}{\longrightarrow} n+1, \; n \overset{q_n/(p_n+q_n)}{\longrightarrow} n-1.$$

We call it the "vertical random walk". Classically when $\mathbb{E}(\log(p_0/q_0)) \neq 0$, the vertical random walk is transient (see [9]), so is the initial random walk. Transience when $\mathbb{E}(\log(p_0/q_0)) = 0$ and $d = 2$ will also be shortly proved. Recall here in passing the related conjecture that any random walk in *i.i.d.* random environment on $\mathbb{Z}^3$, with ellipticity conditions on the data, is transient.

To complete the study of the independent case, we now focus on the planar case, i.e. $d = 1$, with $\mathbb{E}(\log(p_0/q_0)) = 0$. Recall that $\varepsilon_n = \sum_{k\in\mathbb{Z}} k\mu_n(k)$.

When $\mathbb{P}(p_0 = q_0) = 1$, i.e. simply $p_n = q_n$, $n \in \mathbb{Z}$, as a consequence of [1] prop. 1.4 *i*), the random walk is recurrent for any $(\varepsilon_n)_{n\in\mathbb{Z}}$ verifying :

$$|\sum_{-n\leq k\leq 0} \varepsilon_k| + |\sum_{0\leq k\leq n} \varepsilon_k| = O((\log n)^{1/2}).$$

In the other direction, cf [1] prop. 1.6, when the $(\varepsilon_n)_n$ are independent random variables such that for some $\varepsilon > 0$:

$$\liminf \frac{1}{N} \sum_{n=1}^{N} \mathbb{P}\left(|\sum_{0\leq k\leq n} \varepsilon_k| \geq (\log n)^{1+\varepsilon} \right) > 0,$$

then, for a.-e. realization, the random walk is transient. This extended former results of [3] and Devulder-Pène [5]. In the independent case, the critical size of the sums $\sum_{0\leq k\leq n} \varepsilon_k$ with respect to recurrence/transience is expected to be $\log n$, so corresponding to ε_n having order $1/n$.

We now consider the remaining case, reformulating hypothesis 1.1 in a slightly stronger form. We denote by $B(x, r)$ the interval $]x - r, x + r[\subset \mathbb{R}$.

Theorem 1.3
Let $d = 1$ and assume $\exists \eta > 0, \forall n \in \mathbb{Z}$, $\min\{p_n, q_n, r_n\} \geq \eta$, $Supp(\mu_n) \subset [-1/\eta, 1/\eta]$, $\mu_n(0) \leq 1-\eta$, with $(p_n/q_n)_{n\in\mathbb{Z}}$ non-constant i.i.d. random variables, $\mathbb{E}(\log(p_0/q_0)) = 0$.

i) For a.-e. realization : any $(\varepsilon_n)_{n\in\mathbb{Z}}$ such that $\varepsilon_n = O(\exp(-|n|^{1/2+\delta}))$, $\delta > 0$, imply that the random walk is recurrent; any $(\varepsilon_n)_{n\in\mathbb{Z}}$ such that for some $\delta > 0$ and large n, $\varepsilon_n \geq \exp(-|n|^{1/2-\delta})$ (hence > 0) imply that the random walk is transient.

ii) Suppose that the $(\varepsilon_n)_n$ are independent random variables, independent of the $(p_n/q_n)_{n\in\mathbb{Z}}$. Assume $\exists \delta > 0$ so that for large $n \in \mathbb{N}$ and all $y \in \mathbb{R}$, $\mathbb{P}(\varepsilon_n \notin B(y, \exp(-n^{1/2-\delta}))) \geq n^{-\delta/5}$. Then for a.-e. realization the random walk is transient.

Remark. — We shall prove versions a little stronger, as indicated at the beginning of the proofs below. Roughly when $\mathbb{P}(p_0 = q_0) < 1$, the critical scale for ϵ_n with respect to recurrence/transience is $\exp(-|n|^{1/2})$, hence much smaller than for the vertically flat case. The vertical random walk is now Sinaï's random walk, not simple random walk. It stays a long time in deep valleys, coming back to zero with very small frequency. For the original random walk, the landscape looks like a succession of horizontally invariant canyons, where it stays confined for a long time. Just a little of horizontal drift is enough to make the random walk transient.

Remark. — Item *ii*), when $\mathbb{P}(\mu_n = \delta_1) = \mathbb{P}(\mu_n = \delta_{-1}) = 1/2$, giving $\mathbb{P}(\varepsilon_n = \pm 1) = 1/2$, was first proved by Kochler in his Ph-D thesis [7], after a long and delicate analysis of the Brownian path.

2 Preliminaries

Let us begin with some notations. We first fix an integer $K > 2(1 + 1/\eta)$.

Definition 2.1
i) Set $a_n = q_n/p_n$ and :

$$\rho_n = \begin{cases} a_1 \cdots a_n, & n \geq 1, \\ 1, & n = 0, \\ a_{n+1} \cdots a_{-1} a_0 & n \leq -1. \end{cases}$$

For $n \geq 0$, let :

$$v_+(n) = \sum_{0 \leq k \leq n} \rho_k \text{ and } v_-(n) = a_0 \sum_{-n-1 \leq k \leq -1} \rho_k.$$

ii) For functions $f(n)$ and $g(n)$ defined on $\mathbb{N} = \{0, 1, \cdots\}$, we write $f \asymp g$ if there exists a constant $C > 0$ so that for large n, $(1/C)f(n) \leq g(n) \leq Cf(n)$. We write $g \preceq f$ if $g(n) \leq Cf(n)$ only.
iii) Let $f : \mathbb{N} \to \mathbb{R}_+$, increasing, with $\lim f(n) = +\infty$. For large enough $x \in \mathbb{R}_+$, let $f^{-1}(x) = \max\{n \in \mathbb{N} \mid f(n) \leq x\}$. Note that $f(f^{-1}(x)) \leq x < f(f^{-1}(x) + 1)$ and $f^{-1}(f(n)) = n$.

Notice that $\rho_{n+1}/\rho_n \in [\eta, 1/\eta]$, because $a_k \in [\eta, 1/\eta]$, $\forall k$. It is immediate to see that, a.-e., $v_+(n)$ and $v_-(n)$ are both increasing and tend to $+\infty$, when $\mathbb{E}(\log(p_0/q_0)) = 0$. For example, the *i.i.d.* random walk $\sum_{0\leq k\leq n} \log(q_k/p_k)$ has an integrable and centered step, hence is recurrent, so almost-surely ρ_n does not go to 0.

Let us now introduce the functions describing the average horizontal macrodispersion of the environment.

Definition 2.2
i) The structure function, depending only on the vertical, is defined for $n \geq 0$ by :

$$\Phi_{str}(n) = \left(n \sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} 1/\rho_k \right)^{1/2}.$$

2) For $d=1$ and $m,n \geq 0$, introduce :

$$\Phi(-m,n) = \left(\sum_{-v_-^{-1}(m) \leq k \leq l \leq v_+^{-1}(n)} \rho_k \rho_l \left[1/\rho_k^2 + 1/\rho_l^2 + \left(\sum_{s=k}^{l} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right] \right)^{1/2}.$$

For $n \geq 0$, set $\Phi(n) = \Phi(-n,n)$ and $\Phi_+(n) = \sqrt{\Phi^2(-n,0) + \Phi^2(0,n)}$.

Observe that for $n \geq 1$:

$$\begin{aligned} \sum_{0 \leq k \leq l \leq v_+^{-1}(n)} \rho_k \rho_l \left(1/\rho_k^2 + 1/\rho_l^2\right) &= \sum_{0 \leq k \leq l \leq v_+^{-1}(n)} (\rho_k/\rho_l + \rho_l/\rho_k) \\ &\asymp \sum_{0 \leq k \leq v_+^{-1}(n)} \rho_k \sum_{0 \leq l \leq v_+^{-1}(n)} 1/\rho_l. \end{aligned}$$

Since $\sum_{0 \leq k \leq v_+^{-1}(n)} \rho_k \asymp n$, proceeding similarly for $\sum_{-v_-^{-1}(n) \leq k \leq l \leq 0} \rho_k \rho_l \left(1/\rho_k^2 + 1/\rho_l^2\right)$, we get :

$$\Phi_+(n) \asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \leq k \leq 0 \leq l \leq v_+^{-1}(n)} \rho_k \rho_l \left(\sum_{s=k}^{l} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}. \tag{1}$$

In a very similar fashion :

$$\Phi(n) \asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \leq k \leq l \leq v_+^{-1}(n)} \rho_k \rho_l \left(\sum_{s=k}^{l} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}. \tag{2}$$

In particular, one always has $\Phi(n) \succeq \Phi_+(n) \succeq \Phi_{str}(n)$.

Finally an important property, called dominated variation, was proved for Φ^{-1}, Φ_+^{-1} and Φ_{str}^{-1} in [2] (lemmas 4.8 and 6.2) and will play an important role in the sequel. This condition is in fact necessitated by the structure of the main result in [2] (theorem 2.4). Namely, for any of these function f^{-1}, there exists $C > 0$ so that for large $x > 0$:

$$f^{-1}(2x) \leq C f^{-1}(x). \tag{3}$$

As a result, for any $A > 0$, for $x > 0$ large enough, $f^{-1}(Ax) \asymp f^{-1}(x)$.

3 Proof of the results

- *Proof of proposition 1.2.* Given the remarks of the introduction, it remains to check transience when $d = 2$ and $\mathbb{E}(\log(p_0/q_0)) = 0$. Following [2], prop. 2.5 1)ii), it is enough to verify the condition on the structure function $\Phi_{str}(n) \geq \sqrt{n}(\log n)^{1/2+\varepsilon}$, $\varepsilon > 0$.

When $\mathbb{P}(p_0 = q_0) = 1$, this is clear since $\Phi_{str}(n) \succeq n$. When $\mathbb{P}(p_0 = q_0) < 1$, introduce $S_n = \sum_{k=1}^{n} \log(q_k/p_k)$. Fixing $\varepsilon > 0$, it is classical that, a.-s. for n large enough :

$$\max_{1 \leq k \leq n} S_k \leq n^{1/2+\varepsilon} \text{ and } \min_{1 \leq k \leq n} S_k \leq -n^{1/2-\varepsilon}.$$

Hence for large n, $v_+(n) \leq \exp(n^{1/2+2\varepsilon})$, so $v_+^{-1}(n) \geq (\log n)^{2-9\varepsilon}$. As a result :

$$\sum_{0 \leq k \leq v_+^{-1}(n)} 1/\rho_k \geq \exp(\log^{(2-9\varepsilon)(1/2-\varepsilon)} n) \geq \exp(\log^{1-7\varepsilon} n).$$

This gives $\Phi_{str}(n) \succeq \sqrt{n}\exp(\log^{1-\varepsilon} n)$, $\forall \varepsilon > 0$, so the condition is also verified.

- *Proof of theorem 1.3 i).* Let $d = 1$. For the recurrence part, let $(\varepsilon_n)_{n\in\mathbb{Z}}$ be such that :

$$\sum_{n\in\mathbb{Z}} |\varepsilon_n|/\rho_n < +\infty.$$

This is true if $\varepsilon_n = O(\exp(-|n|^{1/2+\delta}))$, $\delta > 0$, since then $\varepsilon_n/\rho_n = O(\exp(-|n|^{1/2+\delta/2}))$, which is summable on $\mathbb{Z}$. Next, since $\sum_{u=k}^{l}(r_s\varepsilon_s)/(p_s\rho_s)$ is bounded in (k,l), $k \le l$, we deduce from (2):

$$\Phi^2(n) \preceq \Phi^2_{str}(n) + n^2, \text{ so } \Phi^{-1}(n) \succeq \min\{\Phi^{-1}_{str}(n), n\}.$$

Similary, by (1), $\Phi_+(n) \succeq \Phi_{str}(n)$, giving $\Phi_+^{-1}(n) \preceq \Phi^{-1}_{str}(n)$. In view of [2], theorem 2.4, recurrence is equivalent to :

$$\sum_{n\ge1} \frac{1}{n^2}\frac{(\Phi^{-1}(n))^2}{\Phi_+^{-1}(n)} = +\infty.$$

It is thus sufficient to show the divergence of :

$$\sum_{n\ge1} \frac{1}{n^2}\frac{(\min\{\Phi^{-1}_{str}(n), n\})^2}{\Phi^{-1}_{str}(n)} = \sum_{n\ge1}\frac{1}{n}\min\left\{\frac{\Phi^{-1}_{str}(n)}{n}, \frac{n}{\Phi^{-1}_{str}(n)}\right\}.$$

Since $\Phi^{-1}_{str}(n)$ verifies dominated variation (3), it is equivalent to check that :

$$\sum_{n\ge1}\min\left\{\frac{\Phi^{-1}_{str}(K^n)}{K^n}, \frac{K^n}{\Phi^{-1}_{str}(K^n)}\right\} = +\infty. \tag{4}$$

We shall in fact prove that the general term does not go to 0.

Lemma 3.1
Introduce for $n \ge 0$:

$$w_+(n) = \sum_{0\le k\le n} 1/\rho_k \ \textit{and}\ w_-(n) = (1/a_0)\sum_{-n-1\le k\le -1} 1/\rho_k.$$

A.-s., $\limsup \dfrac{w_+(n)}{v_+(n)} = +\infty$ *and* $\limsup \min\left\{\dfrac{v_+(n)}{w_+(n)}, \dfrac{v_-(n)}{v_+(n)}, \dfrac{w_+(n)}{w_-(n)}\right\} = +\infty.$

Proof of the lemma :
Let us prove the second point, the first one being easier. Fix a sequence (k_n) with $k_n/k^2_{n-1} \to +\infty$. For $n \in \mathbb{Z}$, let $S_n = \log \rho_n$. Let :

$$U_n = \max_{k\in[-k_n,-k_{n-1})}(S_k - S_{-k_{n-1}}),\ V_n = \max_{k\in(k_{n-1},k_n]}(S_k - S_{k_{n-1}}).$$

In the same way, introduce :

$$W_n = \min_{k\in(k_{n-1},k_n]}(S_k - S_{k_{n-1}}),\ X_n = \min_{k\in[-k_n,-k_{n-1})}(S_k - S_{-k_{n-1}}).$$

Set $\sigma^2 = \mathrm{Var}(\log(q_0/p_0)) > 0$ and let $c = 2+2|\log(1/\eta)|$. Using functional convergence to standard Brownian motion $(B_t)_{t\in[-1,1]}$, we have, as $n \to +\infty$:

$$\begin{aligned}
&\mathbb{P}\left(\frac{U_n}{\sigma\sqrt{k_n - k_{n-1}}} \ge 1 + \frac{V_n}{\sigma\sqrt{k_n - k_{n-1}}} \ge 2 - \frac{W_n}{\sigma\sqrt{k_n - k_{n-1}}} \ge 3 - \frac{X_n}{\sigma\sqrt{k_n - k_{n-1}}}\right)\\
\longrightarrow\ &\mathbb{P}\left(\max_{t\in[-1,0]} B_t \ge 1 + \max_{t\in[0,1]} B_t \ge 2 - \min_{t\in[0,1]} B_t \ge 3 - \min_{t\in[-1,0]} B_t\right) =: \alpha > 0.
\end{aligned}$$

Using independence and Borel-Cantelli 2, almost-surely the event appearing in the first probability is realized for infinitely many n. For such a n, we have :

$$
\begin{aligned}
v_+(k_n) &\geq \exp(-ck_{n-1} + \max_{k\in(k_{n-1},k_n]}(S_k - S_{k_{n-1}})) \\
&\geq \exp(-ck_{n-1} + \sigma\sqrt{k_n - k_{n-1}} - \min_{k\in(k_{n-1},k_n]}(S_k - S_{k_{n-1}})) \\
&\geq \exp(-3ck_{n-1} + \sigma\sqrt{k_n - k_{n-1}} - \min_{k\in[1,k_n]} S_k) \\
&\geq \frac{w_+(k_n)}{k_n}\exp(-3ck_{n-1} + \sigma\sqrt{k_n - k_{n-1}}).
\end{aligned}
$$

If n is large, $v_+(k_n)/w_+(k_n) \geq \exp(\sigma\sqrt{k_n}/2)/k_n$, which is arbitrary large. Similar lower bounds are proved for $v_-(k_n)/v_+(k_n)$ and $w_+(k_n)/w_-(k_n)$.

□

We conclude the proof of the recurrence part. Using the first point of the lemma, a.-s., for any $A > 0$, infinitely often, $w_+(n) \geq Av_+(n)$, i.e. $w_+(v_+^{-1}(v_+(n))) \geq Av_+(n)$. Taking l so that $l \leq v_+(n) < l+1$, we obtain :

$$\Phi_{str}(l+1) \geq \sqrt{l+1}\sqrt{Al} \geq l\sqrt{A}.$$

Hence, $\Phi_{str}^{-1}(l\sqrt{A}) \leq l+1$. This implies that, a.-s., $\liminf \Phi_{str}^{-1}(n)/n = 0$. Using now the second point of the lemma, a.-s., for any $A > 1$, i.o., both $v_-(n) \geq v_+(n) \geq Aw_+(n)$ and $w_+(n) \geq w_-(n)$. For such a n, first $v_-^{-1}(v_+(n)) \leq n$. Then :

$$
\begin{aligned}
\frac{v_+(n)}{\sum_{-v^{-1}(v_+(n))\leq k\leq v_+^{-1}(v_+(n))} 1/\rho_k} &\geq \frac{v_+(n)}{\sum_{-n\leq k\leq n} 1/\rho_k} \geq \frac{v_+(n)}{a_0 w_-(n) + w_+(n)} \\
&\geq \frac{v_+(n)}{w_+(n)}(1+a_0)^{-1} \geq \frac{A}{1+a_0}.
\end{aligned}
$$

Choosing l such that $l \leq v_+(n) < l+1$, we obtain :

$$\frac{l+1}{\Phi_{str}(l)} \geq \frac{A}{1+a_0}.$$

Hence a.-s., $\limsup n/\Phi_{str}(n) = +\infty$ and so, in the same way as before, $\limsup \Phi_{str}^{-1}(n)/n = +\infty$. Let now $b_n = \Phi_{str}^{-1}(K^n)/K^n$. Using dominated variation (3), a.-s., the previous results furnish that $\liminf b_n = 0$, $\limsup b_n = +\infty$. Dominated variation also implies that, a.-s. for a constant $H > 1$, for all n, we have $b_n/b_{n+1} \in [1/H, H]$. Thus $b_n \in [1, H]$ for infinitely many n. For such a n, $\min\{b_n, 1/b_n\} = 1/b_n \geq 1/H$ and this shows (4).

Turn now to the proof of the transience result of item i). By [2], prop. 2.5 1), it is enough to prove that $\sum_{n\geq 1} 1/\Phi(n) < +\infty$. We shall assume that the ε_n are non-negative for large $n \in \mathbb{Z}$, with $\sum_{n\in\mathbb{Z}} \varepsilon_n/\rho_n = +\infty$ and :

$$\sum_{n\geq N_0}\left(n\sum_{-v_-^{-1}(n)\leq k\leq v_+^{-1}(n)} \varepsilon_k/\rho_k\right)^{-1} < +\infty, \tag{5}$$

where N_0 is chosen large enough so that the denominator of the generic term above is > 0 for $n \geq N_0$. This is verified if there exists $\delta > 0$ so that $\varepsilon_n \geq \exp(-|n|^{1/2-\delta})$, for large $n \in \mathbb{Z}$. Indeed, as in the proof of prop. 1.2, a.-s., for any $\varepsilon > 0$, for n large enough :

$$\min_{0\leq k\leq v_+^{-1}(n)} \rho_k \leq \exp(-\log^{1-\varepsilon} n).$$

If u_n is the first point in $[0, v_+^{-1}(n)]$ realizing the minimum of ρ_k on this interval, then $u_n \to +\infty$ and, fixing $\varepsilon > 0$, $u_n \leq (\log n)^{2+\varepsilon}$, for large n. Thus for n large enough :

$$\varepsilon_{u_n}/\rho_{u_n} \geq \exp(-u_n^{1/2-\delta})\exp(\log^{1-\varepsilon} n) \geq \exp(\log^{1-2\varepsilon} n),$$

if $\varepsilon > 0$ was chosen small enough. Then $\forall \varepsilon > 0$, for large n :

$$\sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} \varepsilon_k/\rho_k \geq \frac{1}{2}\varepsilon_{u_n}/\rho_{u_n} \geq \exp(\log^{1-2\varepsilon} n).$$

This shows (5). Notice that because of monotonicity for large n, condition (5) is equivalent to :

$$\sum_{n \geq N_0} \left(\sum_{-v_-^{-1}(K^n) \leq k \leq v_+^{-1}(K^n)} \varepsilon_k/\rho_k \right)^{-1} < +\infty. \tag{6}$$

Now, because Φ is increasing, we have to check that $\sum_{n \geq 1} K^n/\Phi(K^n) < +\infty$. Observe that for large n :

$$\begin{aligned}
\Phi^2(K^n) &\geq \sum_{\substack{-v_-^{-1}(K^n) \leq k < -v_-^{-1}(K^{n-1}) \\ v_+^{-1}(K^{n-1}) < l \leq v_+^{-1}(K^n)}} \rho_k \rho_l \left(\sum_{-v_-^{-1}(K^{n-1}) \leq s \leq v_+^{-1}(K^{n-1})} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \\
&\succeq \left(\sum_{-v_-^{-1}(K^{n-1}) \leq s \leq v_+^{-1}(K^{n-1})} \frac{\varepsilon_s}{\rho_s} \right)^2 \sum_{\substack{-v_-^{-1}(K^n) \leq k < -v_-^{-1}(K^{n-1}) \\ v_+^{-1}(K^{n-1}) < l \leq v_+^{-1}(K^n)}} \rho_k \rho_l \\
&\succeq K^{2n} \left(\sum_{-v_-^{-1}(K^{n-1}) \leq s \leq v_+^{-1}(K^{n-1})} \frac{\varepsilon_s}{\rho_s} \right)^2,
\end{aligned}$$

since :

$$K^n \geq \sum_{k=v_+^{-1}(K^{n-1})+1}^{v_+^{-1}(K^n)} \rho_k = v_+(v_+^{-1}(K^n)) - v_+(v_+^{-1}(K^{n-1})) \geq \frac{K^n}{1+1/\eta} - K^{n-1} \geq \frac{K^n}{2(1+1/\eta)},$$

with similar inequalities for $\sum_{-v_-^{-1}(K^n) \leq k < -v_-^{-1}(K^{n-1})} \rho_k$. Hence $\sum_{n \geq 1} K^n/\Phi(K^n) < +\infty$, by (6).

- *Proof of theorem 1.3 ii).* We now assume that the $(\varepsilon_n)_{\in \mathbb{Z}}$ are independent random variables, independent from the $(p_n/q_n)_{n \in \mathbb{Z}}$, such that there exists $\delta > 0$ such that for large $n \in \mathbb{N}$ and all $y \in \mathbb{R}$, $\mathbb{P}(\varepsilon_n \notin B(y, \exp(-n^{1/2-\delta}))) \geq n^{-\delta/5}$.

First of all, as above :

$$\sum_{v_+^{-1}(K^{n-1}) < l \leq v_+^{-1}(K^n)} \rho_l \asymp K^n \text{ and } \sum_{-v_-^{-1}(K^n) \leq l \leq -1} \rho_l \asymp K^n.$$

Introducing then the probability measure on $\mathbb{Z}^2$:

$$\nu_n = \frac{1}{Z_n} \sum_{-v_-^{-1}(K^n) \leq k \leq -1, v_+^{-1}(K^{n-1}) < l \leq v_+^{-1}(K^n)} \rho_k \rho_l \delta_{(k,l)},$$

the normalizing constant Z_n verifies $Z_n \asymp K^{2n}$.

Let $\delta/2 > \gamma > 2\delta/5$. Next choose $\varepsilon > 0$ so that $\gamma + (2+\varepsilon)(1/2-\delta) \leq 1 - 3\varepsilon$ and $(2+\varepsilon)\delta/5 < \gamma < \delta/2$. Let u_n in $[0, v_+^{-1}(K^{n-1})]$ be the first minimum of ρ_k on this interval. A.-s. for large n, we have $n^{2+\varepsilon} \geq v_+^{-1}(K^n) \geq u_n \geq n^{2-\varepsilon}$ and :

$$\frac{1}{\rho_{u_n}} \geq \exp(n^{1-\varepsilon}).$$

The support of the law of ε_n is included in $[-1/\eta, 1/\eta]$. Let $C_0 = 2/\eta + 1$ and $l(n) = n^\gamma$. Starting from $u_{n,0} = u_n$, choose recursively in a decreasing order in the interval $[0, u_n]$, points $u_{n,0} > u_{n,1} > \cdots > u_{n,l(n)} > 0$, so that $u_{n,l+1}$ is the closest point on the left side of $u_{n,l}$ with :

$$\frac{1}{4}\frac{C_0}{\rho_{u_{n,l}}} \geq \frac{1}{4}\frac{\exp(-u_{n,l}^{1/2-\delta})}{\rho_{u_{n,l}}} \geq \frac{C_0}{\rho_{u_{n,l+1}}} \geq \frac{1}{4K}\frac{\exp(-u_{n,l}^{1/2-\delta})}{\rho_{u_{n,l}}},\ 0 \leq l < l(n).$$

Recall for this that $\rho_k/\rho_{k+1} \in [1/K, K]$ and notice that $\rho_{u_{n,l}}$ is increasing, $0 \leq l \leq l(n)$, and indeed well defined and staying inside the interval $[0, u_{n,0}]$ since uniformly in $0 \leq l < l(n)$:

$$\begin{aligned}
\rho_{u_{n,l+1}} \leq (4C_0K)\rho_{u_{n,l}} e^{u_{n,l}^{1/2-\delta}} &\leq (4C_0K)^{l(n)}\rho_{u_{n,0}} \exp(l(n)u_{n,0}^{1/2-\delta}) \\
&\leq \exp(n^\gamma \log(4C_0K) - n^{1-\varepsilon} + n^\gamma n^{(2+\varepsilon)(1/2-\delta)}) \\
&\leq \exp(-n^{1-2\varepsilon}),
\end{aligned} \tag{7}$$

for n large enough. In particular, $u_{n,l(n)} \to +\infty$.

We now reason conditionally to the $(p_n/q_n)_{n\in\mathbb{Z}}$ and make a measurable construction. The probability $\mathbb{P}$ hence only refers to the $(\varepsilon_n)_{n\in\mathbb{Z}}$. Consider the event A_n, where $(\varepsilon_i)_{i\in\mathbb{Z}} \in A_n$ if :

$$\nu_n\left\{(k,l) \in \mathbb{Z}^2,\ \left|\sum_{k\leq u\leq l}\frac{\varepsilon_u}{\rho_u}\right| \leq \frac{C_0}{\rho_{u_{n,l(n)}}}\right\} \geq 3/4. \tag{8}$$

Let us write :

$$\mathbb{P}(A_n) = \mathbb{E}\left(\mathbb{E}(1_{(\varepsilon_k)_{k\in\mathbb{Z}}\in A_n} \mid (\varepsilon_l)_{l\neq u_{n,s}, 0\leq s<l(n)})\right).$$

Fix $(\varepsilon_l)_{l\neq u_{n,s}, 0\leq s<l(n)}$ and suppose that there exists $(\varepsilon_{u_{n,s}})_{0\leq s<l(n)}$ so that $\varepsilon = (\varepsilon_i)_{i\in\mathbb{Z}} \in A_n$. Let :

$$(\varepsilon'_{u_{n,s}})_{0\leq s<l(n)} \notin B\left(\varepsilon_{u_{n,l(n)-1}}, \exp(-u_{n,l(n)-1}^{1/2-\delta})\right) \times \cdots \times B\left(\varepsilon_{u_{n,0}}, \exp(-u_{n,0}^{1/2-\delta})\right).$$

We call ε' the point obtained from ε by replacing $\varepsilon_{u_{n,l}}$ by $\varepsilon'_{u_{n,l}}$, $0 \leq l < l(n)$, without changing the other coordinates. We claim that $\varepsilon' \notin A_n$. Indeed, let $0 \leq p < l(n)$ be the smallest index so that :

$$\varepsilon'_{u_{n,p}} \notin B\left(\varepsilon_{u_{n,p}}, \exp(-u_{n,p}^{1/2-\delta})\right).$$

Take (k,l) in the set involved in (8). Notice that $k < 0$ and $v_+^{-1}(K^{n-1}) < l$. Using the hypothesis on the laws of the ε_n :

$$\begin{aligned}
\left|\sum_{k\leq u\leq l}\frac{\varepsilon'_u}{\rho_u}\right| &= \left|\frac{\varepsilon'_{u_{n,p}} - \varepsilon_{u_{n,p}}}{\rho_{u_{n,p}}} + \sum_{p<q<l(n)}\frac{\varepsilon'_{u_{n,q}} - \varepsilon_{u_{n,q}}}{\rho_{u_{n,q}}} + \sum_{k\leq u\leq l}\frac{\varepsilon_u}{\rho_u}\right| \\
&\geq \frac{\exp(-u_{n,p}^{1/2-\delta})}{\rho_{u_{n,p}}} - \sum_{p<q<l(n)}\frac{C_0}{\rho_{u_{n,q}}} - \frac{C_0}{\rho_{u_{n,l(n)}}} \\
&\geq 4\frac{C_0}{\rho_{u_{n,p+1}}} - \frac{C_0}{\rho_{u_{n,p+1}}}\sum_{r\geq 0}4^{-r} \geq 2\frac{C_0}{\rho_{u_{n,p+1}}} \geq 2\frac{C_0}{\rho_{u_{n,l(n)}}}.
\end{aligned} \tag{9}$$

This furnishes :

$$\nu_n \left\{ (k,l) \in \mathbb{Z}^2, \left| \sum_{k \le u \le l} \frac{\varepsilon'_u}{\rho_u} \right| \le \frac{C_0}{\rho_{u_{n,l(n)}}} \right\} \le 1/4, \tag{10}$$

hence proving that $\varepsilon' \notin A_n$. As a result :

$$\begin{aligned} \mathbb{E}(1_{(\varepsilon_k)_{k\in\mathbb{Z}} \in A_n} \mid (\varepsilon_l)_{l \neq u_{n,s}, 0 \le s < l(n)}) &\le (1 - u_{n,l(n)-1}^{-\delta/5}) \cdots (1 - u_{n,0}^{-\delta/5}) \\ &\le (1 - u_n^{-\delta/5})^{l(n)} \le \exp(-n^{\gamma - (2+\varepsilon)\delta/5}), \end{aligned}$$

for large n. This gives $\mathbb{P}(A_n) \le \exp(-n^{\gamma-(2+\varepsilon)\delta/5})$, so $\sum \mathbb{P}(A_n) < \infty$, since $\gamma - (2+\varepsilon)\delta/5 > 0$. Making use of Borel-Cantelli 1, for almost-all $(\varepsilon_i)_{i\in\mathbb{Z}}$, for n large enough :

$$\nu_n \left\{ (k,l) \in \mathbb{Z}^2, \left| \sum_{k \le u \le l} \frac{\varepsilon_u}{\rho_u} \right| \ge \frac{C_0}{\rho_{u_{n,l(n)}}} \right\} \ge 1/4. \tag{11}$$

We now conclude. Write $\mathcal{E}_n(f(X,Y))$ for $\sum_{(k,l)\in\mathbb{Z}^2} f(x,y) d\nu_n(k,l)$. Using [2], prop. 2.5, we prove that $\sum_{n\ge1} 1/\Phi(n) < +\infty$, or equivalently $\sum_{n\ge1} K^n/\Phi(K^n) < +\infty$, as Φ is increasing. We have :

$$\begin{aligned} \sum_{n\ge1} \frac{K^n}{\Phi(K^n)} &\le \sum_{n\ge1} K^n \left(\sum_{-v_-^{-1}(K^n) \le k \le l \le v_+^{-1}(K^n)} \rho_k \rho_l \left(\sum_{k\le u\le l} \frac{\varepsilon_u}{\rho_u} \right)^2 \right)^{-1/2} \\ &\preceq \sum_{n\ge1} \left(\mathcal{E}_n \left[\left(\sum_{X \le u \le Y} \frac{\varepsilon_u}{\rho_u} \right)^2 \right] \right)^{-1/2}. \end{aligned}$$

From the previous results, a.-s., there exists N_0 so that for $n \ge N_0$:

$$\mathcal{E}_n \left[\left(\sum_{X\le u\le Y} \frac{\varepsilon_u}{\rho_u} \right)^2 \right] \ge \frac{1}{4} \frac{C_0^2}{\rho^2_{u_{n,l(n)}}}.$$

From (7), $\sum_{n\ge1} \rho^{-1}_{u_{n,l(n)}} \preceq \sum_{n\ge1} \exp(-n^{1-2\varepsilon}) < +\infty$. This shows $\sum_{n\ge1} K^n/\Phi(K^n) < +\infty$ and completes the proof of the theorem. □

4 Concluding remarks

This is an informal section. Let us fix the context of theorem 1.3. As in [1, 2], introduce the random times $0 = \sigma_0 < \tau_0 < \sigma_1 < \tau_1 < \cdots$, where $\tau_k = \min\{n > \sigma_k \mid S_n \notin \mathbb{Z} \times \{0\}\}$ and $\sigma_{k+1} = \{n > \tau_k \mid S_n \in \mathbb{Z} \times \{0\}\}$. Set $D_n = S_{\sigma_n} - S_{\sigma_{n-1}}$ and recall that the stratification of the environment implies that the $(D_n)_{n\ge1}$ are independent and identically distributed.

i) The condition $\sum_{n\in\mathbb{Z}} |\varepsilon_n|/\rho_n < +\infty$, implying recurrence in theorem 1.3 *i*), looks like the condition $\sum_{n\in\mathbb{Z}} 1/\rho_n < +\infty$ defining the half-pipe of [2], section 7.3, but is in fact of different nature. This last condition, not true here, implied that D_1 is integrable and that the random walk in this case is transient if and only if :

$$\sum_{s\in\mathbb{Z}} \frac{r_s \varepsilon_s}{p_s \rho_s} \neq 0. \tag{12}$$

Here, under the condition $\sum_{n\in\mathbb{Z}} |\varepsilon_n|/\rho_n < +\infty$, whatever the value of $\sum_{s\in\mathbb{Z}} r_s\varepsilon_s/(p_s\rho_s)$, the random walk is recurrent.

ii) We do not expect that a complete characterization of transience in such a simple form as (12) is possible in general. When $\varepsilon_n > 0$, $\forall n \in \mathbb{Z}$, setting :

$$C(n) = \sum_{-v_-^{-1}(n)\le k\le v_+^{-1}(n)} \varepsilon_k/\rho_k,$$

mention without proof that the following condition is equivalent to recurrence :

$$\sum_{n\ge 1} \frac{1}{n^2} \min\{\Phi_{str}^{-1}(n), (C^{-1}(n))^2/\Phi_{str}^{-1}(n)\} = +\infty.$$

Notice that the formulation only has a meaning when the ε_n are positive (because of $C^{-1}(n)$).

iii) Standardly, see Gnedenko-Kolmogorov [6], the correct normalization for $D_1+\cdots+D_n$ is related to the behaviour at the origin of the characteristic function of D_1. Here, as a consequence of [2], almost-surely, the $(D_1+\cdots+D_n)_{1\le n\le N}$ have a standard empirical deviation d_N (the square root of the empirical variance) around its empirical mean m_N given by $\Phi_+(N)$ and a global size given by $\Phi(N)$ (i.e. $|m_N| + d_N$). Taking $\varepsilon_n = 1$, one may deduce that the correct normalization for σ_n has order $\tilde{\Phi}(n)$, where :

$$\tilde{\Phi}(n) \asymp \left(\sum_{-v_-^{-1}(n)\le k\le l\le v_+^{-1}(n)} \rho_k\rho_l \left(\sum_{s=k}^{l} \frac{1}{\rho_s} \right)^2 \right)^{1/2}.$$

From this it is not difficult to infer that $\tilde{\Phi}^{-1}(n) \asymp \Psi^{-1}(n)$, where $\Psi(n) = n\sum_{-v_-^{-1}(n)\le k\le v_+^{-1}(n)} 1/\rho_k$.

In the present context, the correct normalization for S_n should then be $(\Phi\circ\Psi^{-1}(n), (\log n)^2)$. Of course some work is required to make it rigourous, for example to prove tightness or some limit theorem in law (making some stationarity hypothesis on the $(\varepsilon_n)_{n\in\mathbb{Z}}$). Recall that in the vertically flat case, when the $(\varepsilon_n)_{n\in\mathbb{Z}}$ are *i.i.d.* centered and non-constant, then $\Phi(n) \asymp n^{3/2}$ and $\Psi(n) \asymp n^2$, hence recovering the size $n^{3/4}$ for the first coordinate, as shown in [5].

Acknowledgments. We would like to thank Jacques Printems and Alexis Devulder for useful discussions. We also thank Françoise Pène for bringing reference [7] to our attention.

References

[1] J. Brémont, *On planar random walks in environments invariant by horizontal translations*, Markov Processes and Related Fields 2016, issue 2, vol. 22, 267-310.

[2] J. Brémont, *Markov chains in a stratified environment*, ALEA, Lat. Am. J. Probab. Math. Stat. 14, 751-798 (2017).

[3] M. Campanino, D. Petritis, *Random walks on randomly oriented lattices*, Markov Processes and Related Fields 9 (2003), no. 3, 391-412.

[4] M. Campanino, D. Petritis, *Type transitions of simple random walks on randomly directed regular lattices*, Journal of Applied Probability 51, 1065-1080 (2014).

[5] A. Devulder, F. Pene, *Random walk in random environment in a two-dimensional stratified medium with orientations*, Electron. Journal of Probab. 18 (2013), no. 88, 1-23.

[6] B. V. Gnedenko, A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Addison-Wesley Publishing Company, revised edition, 1968.

[7] M. Kochler, *Random walks in random environment : random orientations and branching*, Ph-D thesis, Technische Universität München, 2012.

[8] M. Matheron, G. de Marsily, *Is transport in porous media always diffusive ? A counterexample.* Water resources Res. 16:901-907, 1980.

[9] F. Solomon, *Random walks in a random environment.* Ann. Probability 3 (1975), 1-31.

[10] F. Spitzer, *Principles of random walks.* Second edition. Graduate Texts in Mathematics, vol. 34. Springer-Verlag, New-York Heidelberg, 1976.

Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est, Faculté des Sciences et Technologies, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, FRANCE

E-mail address : julien.bremont@u-pec.fr