SHOIQ with transitive closure of roles is decidable

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$\mathcal{SHOIQ}$ with transitive closure of roles is decidable

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Abstract. The Semantic Web makes an extensive use of the OWL DL ontology language, underlied by the $\mathcal{SHOIQ}$ description logic, to formalize its resources. In this paper, we propose a decision procedure for this logic extended with the transitive closure of roles in concept axioms, a feature needed in several application domains. To address the problem of consistency in this logic, we introduce a new structure for characterizing models which may have an infinite non-tree-like part.

1 Introduction

The ontology language OWL-DL [1] is widely used to formalize data resources on the Semantic Web. This language is mainly based on the description logic $\mathcal{SHOIN}$ which is known to be decidable [2]. Although $\mathcal{SHOIN}$ provides *transitive roles* to model transitivity of relations, we can find several applications in which the *transitive closure of roles*, that is more expressive than transitive roles, is needed. For instance, if we denote by R^- and R^+ the inverse and transitive closure of a role R respectively then it is obvious that the concept $\exists\mathsf{R}^+.\forall\mathsf{R}^-.\bot$ is unsatisfiable w.r.t. an empty TBox. If we now substitute R^+ for a transitive role R_t such that $\mathsf{R} \sqsubseteq \mathsf{R}_t$ (i.e. we substitute each occurrence of R^+ in axioms for R_t) then the concept $\exists\mathsf{R}_t.\forall\mathsf{R}^-.\bot$ is satisfiable. The point is that an instance of R^+ represents a sequence of instances of R but an instance of R_t corresponds to a sequence of instances of *itself*.

In this paper, we consider an extension of $\mathcal{SHOIQ}$ by enabling transitive closure of roles in concept axioms. In the general case, transitive closure is not expressible in the first order logic [3], the logic from which DL is a sublanguage, while the second order logic is sufficiently expressive to do so.

In the DL literature ([4]; [5]), there have been works dealing with transitive closure of roles. Recently, Ortiz [5] has proposed an algorithm for deciding consistency in the logic $\mathcal{ALCQI}b^+_{reg}$ which allows for transitive closure of roles. However, nominals are disallowed in this logic. It is known that reasoning with a DL including number restrictions, inverse roles, nominals and transitive closure of roles is hard. The reason for this is that there exists an ontology in that DL whose models have an *infinite* non-tree-like part. Calvanese *et al.* [6] have presented an automata-based technique for dealing with the logic $\mathcal{ZOIQ}$ that includes transitive closure of roles, and showed that the sublogics $\mathcal{ZIQ}$, $\mathcal{ZOQ}$ and $\mathcal{ZOI}$ are decidable. To obtain this result, the authors have introduced the *quasi-forest model property* to characterize models of ontologies in these sublogics.

Although they are very expressive, none of these sublogics includes $\mathcal{SHOIQ}$ with transitive closure of roles, namely $\mathcal{SHOIQ}_{(+)}$. The following example[3], noted $\mathcal{K}_1$, shows that there is an ontology in $\mathcal{SHOIQ}_{(+)}$ which does not enjoy the quasi-forest model property. We consider the following axioms:

(1) $\{o\} \sqsubseteq \mathsf{A}$; $\mathsf{A} \sqcap \mathsf{B} \sqsubseteq \bot$; $\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{A} \sqcap \exists \mathsf{R}'.\mathsf{B}$; $\mathsf{B} \sqsubseteq \exists \mathsf{S}^+.\{o\}$

(2) $\{o\} \sqsubseteq \forall \mathsf{X}^-.\bot$; $\top \sqsubseteq\ \leq 1\, \mathsf{X}.\top$; $\top \sqsubseteq\ \leq 1\, \mathsf{X}^-.\top$ where $\mathsf{X} \in \{\mathsf{R}, \mathsf{R}', \mathsf{S}\}$

Figure 1 shows an infinite non-tree-like model of $\mathcal{K}_1$. In fact, each individual x that satisfies $\exists \mathsf{S}^+.\{o\}$ must have two distinct paths from x to the individual satisfying nominal o. Intuitively, we can see that (i) such a x must satisfy $\exists \mathsf{S}^+.\{o\}$ and B, (ii) an individual satisfying B must connect to another individual satisfying A which must have a R-path to nominal o, and (iii) two concepts A and B are disjoint.

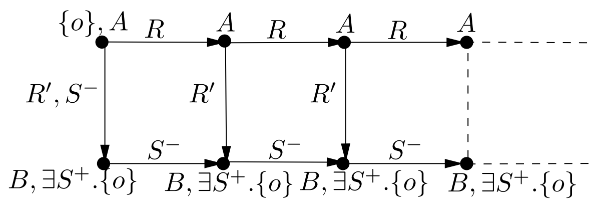

Fig. 1. An infinite non tree-like model of $\mathcal{K}_1$

This example shows that methods ([7], [8], [6]) based on the hypothesis which says that if an ontology is consistent it has a *quasi-forest model*, could fail to address the problem of consistency in a DL including simultaneously $\mathcal{O}$ (nominals), $\mathcal{I}$ (inverse roles), $\mathcal{Q}$ (number restrictions) and transitive closure of roles.

In this paper, we propose a decision procedure for the problem of consistency in $\mathcal{SHOIQ}$ with transitive closure of roles in concept axioms. The underlying idea of our algorithm is founded on the *star-type* and *frame* notions introduced by Pratt-Hartmann [9]. This technique uses star-types to represent individuals and "tiles" them together to form a frame for representing a model. For each star-type σ, we maintain a function $\delta(\sigma)$ which stores the number of individuals satisfying this star-type. To obtain termination condition, we introduce two additional structures into a frame : (i) the first one, namely *cycles*, describes duplicate parts of a model resulting from interactions of logic constructors in $\mathcal{SHOIQ}$, (ii) the second one, namely *blocking-blocked cycles*, describes parts of a model bordered by cycles which allow a frame to satisfy transitive closure of roles occurring in concepts of the form $\exists R^+.C$.

2 The Description Logic $\mathcal{SHOIQ}_{(+)}$

In this section, we present the syntax, the semantics and main inference problems of $\mathcal{SHOIQ}_{(+)}$. In addition, we introduce a tableau structure for $\mathcal{SHOIQ}_{(+)}$, which allows us to represent a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.

[3] This example is initially proposed by Sebastian Rudolph from an informal discussion

Definition 1. *Let $\mathbf{R}$ be a non-empty set of* role names *and $\mathbf{R}_+ \subseteq \mathbf{R}$ be a set of transitive role names. We use $\mathbf{R}_\mathsf{I} = \{P^- \mid P \in \mathbf{R}\}$ to denote a set of inverse roles, and $\mathbf{R}_\oplus = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$ to denote a set of transitive closure of roles. Each element of $\mathbf{R} \cup \mathbf{R}_\mathsf{I} \cup \mathbf{R}_\oplus$ is called a $\mathcal{SHOIQ}_{(+)}$-role. A* role inclusion axiom *is of the form $R \sqsubseteq S$ for two $\mathcal{SHOIQ}_{(+)}$-roles R and S such that $R \notin \mathbf{R}_\oplus$ and $S \notin \mathbf{R}_\oplus$. A* role hierarchy *$\mathcal{R}$ is a finite set of role inclusion axioms. An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty set $\Delta^\mathcal{I}$ (*domain*) and a function $\cdot^\mathcal{I}$ which maps each role name to a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ such that*

$$R^{-\mathcal{I}} = \{\langle x, y\rangle \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid \langle y, x\rangle \in R^\mathcal{I}\} \textit{ for all } R \in \mathbf{R},$$
$$\langle x, z\rangle \in S^\mathcal{I}, \langle z, y\rangle \in S^\mathcal{I} \textit{ implies } \langle x, y\rangle \in S^\mathcal{I} \textit{ for each } S \in \mathbf{R}_+, \textit{ and}$$
$$(Q^+)^\mathcal{I} = \bigcup_{n>0} (Q^n)^\mathcal{I} \textit{ with } (Q^1)^\mathcal{I} = Q^\mathcal{I},$$
$$(Q^n)^\mathcal{I} = \{\langle x, y\rangle \in (\Delta^\mathcal{I})^2 \mid \exists z \in \Delta^\mathcal{I}, \langle x, z\rangle \in (Q^{n-1})^\mathcal{I}, \langle z, y\rangle \in Q^\mathcal{I}\} \textit{ for } Q^+ \in \mathbf{R}_\oplus$$

$*$ *An interpretation $\mathcal{I}$ satisfies a role hierarchy $\mathcal{R}$ if $R^\mathcal{I} \subseteq S^\mathcal{I}$ for each $R \sqsubseteq S \in \mathcal{R}$. Such an interpretation is called a* model *of $\mathcal{R}$, denoted by $\mathcal{I} \models \mathcal{R}$. To simplify notations for nested inverse roles and transitive closures of roles, we define two functions $\cdot^\ominus$ and $\cdot^\oplus$ as follows:*

$$R^\ominus = \begin{cases} R^- & \textit{if } R \in \mathbf{R}; \\ S & \textit{if } R = S^- \textit{ and } S \in \mathbf{R}; \\ (S^-)^+ & \textit{if } R = S^+,\ S \in \mathbf{R}, \\ S^+ & \textit{if } R = (S^-)^+,\ S \in \mathbf{R} \end{cases} \quad R^\oplus = \begin{cases} R^+ & \textit{if } R \in \mathbf{R}; \\ S^+ & \textit{if } R = (S^+)^+ \textit{ and } S \in \mathbf{R}; \\ (S^-)^+ & \textit{if } R = S^- \textit{ and } S \in \mathbf{R}; \\ (S^-)^+ & \textit{if } R = (S^+)^- \textit{ and } S \in \mathbf{R} \end{cases}$$

$*$ *A relation $\underline{\underline{\circledast}}$ is defined as the transitive-reflexive closure $\mathcal{R}^+$ of $\sqsubseteq$ on $\mathcal{R} \cup \{R^\ominus \sqsubseteq S^\ominus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{R^\oplus \sqsubseteq S^\oplus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^\oplus \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$. We define a function* $\mathsf{Trans}(R)$ *which returns* true *iff there is some $Q \in \mathbf{R}_+ \cup \{P^\ominus \mid P \in \mathbf{R}_+\} \cup \{P^\oplus \mid P \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$ such that $Q \underline{\underline{\circledast}} R \in \mathcal{R}^+$. A role R is called* simple *w.r.t. $\mathcal{R}$ if* $\mathsf{Trans}(R) = \mathsf{false}$.

The reason for the introduction of two functions $\cdot^\ominus$ and $\cdot^\oplus$ in Definition 1 is that they avoid using R^{--} and R^{++}. Moreover, it remains a unique nested case $(R^-)^+$. According to Definition 1, axiom $R \sqsubseteq Q^\oplus$ is not allowed in a role hierarchy $\mathcal{R}$ since this may lead to undecidability [10]. Notice that the closure $\mathcal{R}^+$ may contain $R \sqsubseteq Q^\oplus$ if $R \sqsubseteq Q$ belongs to $\mathcal{R}^+$.

Definition 2 (terminology). *Let $\mathbf{C}$ be a non-empty set of* concept names *with a non-empty subset $\mathbf{C}_o \subseteq \mathbf{C}$ of nominals. The set of $\mathcal{SHOIQ}_{(+)}$-concepts is inductively defined as the smallest set containing all C in $\mathbf{C}$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n\, S.C)$ and $(\geq n\, S.C)$ where n is a positive integer, C and D are $\mathcal{SHOIQ}_{(+)}$-concepts, R is an $\mathcal{SHOIQ}_{(+)}$-role and S is a simple role w.r.t. a role hierarchy. We denote $\perp$ for $\neg\top$. The interpretation function $\cdot^\mathcal{I}$ of an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ maps each concept name to a subset of $\Delta^\mathcal{I}$ such that $\top^\mathcal{I} = \Delta^\mathcal{I}$, $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$, $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$, $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \backslash C^\mathcal{I}$, $|\{o^\mathcal{I}\}| = 1$ for all $o \in \mathbf{C}_o$, $(\exists R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I}, \langle x, y\rangle \in R^\mathcal{I} \wedge y \in C^\mathcal{I}\}$, $(\forall R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I}, \langle x, y\rangle \in R^\mathcal{I} \Rightarrow y \in C^\mathcal{I}\}$, $(\geq n\, S.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid |\{y \in C^\mathcal{I} \mid \langle x, y\rangle \in S^\mathcal{I}| \geq n\}$, $(\leq n\, S.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid |\{y \in C^\mathcal{I} \mid \langle x, y\rangle \in S^\mathcal{I}| \leq n\}$ where $|S|$ is denoted for the cardinality of a set S. An axiom $C \sqsubseteq D$ is called a general concept inclusion (GCI)* *where C, D are $\mathcal{SHOIQ}_{(+)}$-concepts (possibly complex), and a finite set of GCIs is called a terminology $\mathcal{T}$. An interpretation $\mathcal{I}$ satisfies a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $\mathcal{I}$ satisfies a terminology $\mathcal{T}$ if $\mathcal{I}$ satisfies each GCI in $\mathcal{T}$. Such an interpretation is called a* model *of $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$. A pair $(\mathcal{T}, \mathcal{R})$ is called a $\mathcal{SHOIQ}_{(+)}$ knowledge base where $\mathcal{R}$ is a $\mathcal{SHOIQ}_{(+)}$ role hierarchy and $\mathcal{T}$ is a $\mathcal{SHOIQ}_{(+)}$ terminology. A knowledge base $(\mathcal{T}, \mathcal{R})$ is said to be consistent if there is a model $\mathcal{I}$ of both $\mathcal{T}$ and $\mathcal{R}$, i.e., $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{R}$. A concept C is called* satisfiable *w.r.t. $(\mathcal{T}, \mathcal{R})$ iff there is some interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a* model *of C w.r.t. $(\mathcal{T}, \mathcal{R})$. A concept D* subsumes *a concept C w.r.t. $(\mathcal{T}, \mathcal{R})$, denoted by $C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in each model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{R})$.* $\lhd$

Since unsatisfiability, subsumption and consistency w.r.t. a $\mathcal{SHOIQ}_{(+)}$ knowledge base can be reduced to each other, it suffices to study knowledge base consistency. For the ease of construction, we assume all concepts to be in *negation normal form* (NNF), i.e., negation occurs only in front of concept names. Any $\mathcal{SHOIQ}_{(+)}$-concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [11]. According to [12], $\mathsf{nnf}(C)$ can be computed in polynomial time in the size of C. For a concept C, we denote the nnf of C by $\mathsf{nnf}(C)$ and the nnf of $\neg C$ by $\dot\neg C$. Let D be a $\mathcal{SHOIQ}_{(+)}$-concept in NNF. We define $\mathsf{cl}(D)$ to be the smallest set that contains all sub-concepts of D including D. For a knowledge base $(\mathcal{T}, \mathcal{R})$, we reuse $\mathsf{cl}(\mathcal{T}, \mathcal{R})$, which was introduced by Horrocks *et al.* [7], to denote all sub-concepts occurring in the axioms of $(\mathcal{T}, \mathcal{R})$. We have $\mathsf{cl}(\mathcal{T}, \mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ [7]. To translate *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for $\mathcal{C}^2$ into those for $\mathcal{SHOIQ}$, we need to add new sets of concepts, denoted $\mathsf{cl}_1(\mathcal{T}, \mathcal{R})$ and $\mathsf{cl}_2(\mathcal{T}, \mathcal{R})$, to the signature of a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$.

$$\begin{aligned}\mathsf{cl}_1(\mathcal{T}, \mathcal{R}) = \{&\leq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \mathsf{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n\} \cup \\ \{&\geq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \mathsf{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n\}\end{aligned}$$

For a generating concept $(\geq nS.C)$ and a set $I \subseteq \{0, \cdots, log\ n + 1\}$, we denote $\mathscr{C}^I_{(\geq nS.C)} = \prod_{i \in I} C^i_{(\geq nS.C)} \sqcap \prod_{j \notin I} \neg C^j_{(\geq nS.C)}$ where $C^i_{(\geq nS.C)}$ are new concept names for $0 \leq i \leq log\ n + 1$. We define $\mathsf{cl}_2(\mathcal{T}, \mathcal{R})$ as follows:

$$\begin{aligned}\mathsf{cl}_2(\mathcal{T}, \mathcal{R}) = \{&C^i_{(\geq S.C)} \mid (\geq nS.C) \in \mathsf{cl}(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T}, \mathcal{R}), 0 \leq i \leq log\ n + 1\} \cup \\ \{&\mathscr{C}^I_{(\geq nS.C)} \mid (\geq nS.C) \in \mathsf{cl}(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T}, \mathcal{R}), I \subseteq \{0, \cdots, log\ n + 1\}\}\end{aligned}$$

Remark 1. If numbers are encoded in binary then the number of new concept names $C^i_{(\geq nS.D)}$ for $0 \leq i \leq log\ n + 1$, is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ since n is bounded by $\mathcal{O}(2^{|(\mathcal{T}, \mathcal{R})|})$. This implies that $|\mathsf{cl}_2(\mathcal{T}, \mathcal{R})|$ is bounded by $\mathcal{O}(2^{|(\mathcal{T}, \mathcal{R})|})$. Note that two concepts $\mathscr{C}^I_{(\geq nS.C)}$ and $\mathscr{C}^J_{(\geq nS.C)}$ are disjoint for all $I, J \subseteq \{0, \cdots, log\ n + 1\}$, $I \neq J$. The concepts $\mathscr{C}_{(\exists S.C)}$ and $\mathscr{C}^I_{(\geq nS.C)}$ will be used for building chromatic star-types. This notion will be clarified after introducing the frame structure (Definition 5).

Finally, we denote $\mathbf{CL}(\mathcal{T}, \mathcal{R}) = \mathsf{cl}(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_1(\mathcal{T}, \mathcal{R}) \cup \mathsf{cl}_2(\mathcal{T}, \mathcal{R})$, and use $\mathbf{R}(\mathcal{T}, \mathcal{R})$ to denote the set of all role names occurring in $\mathcal{T}, \mathcal{R}$ with their inverse. The definition of $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ is inspired from the Fischer-Ladner closure that was introduced in [13]. The closure $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ contains not only sub-concepts syntactically obtained from $\mathcal{T}$

but also sub-concepts that are semantically derived from $\mathcal{T}$ w.r.t. $\mathcal{R}$. For instance, if $\forall S.C$ is a sub-concept from $\mathcal{T}$ and $R \,\underline{\circledast}\, S \in \mathcal{R}$ then $\forall R.C \in \mathbf{CL}(\mathcal{T}, \mathcal{R})$.

To describe a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base in a more intuitive way, we use a tableau structure that expresses semantic constraints resulting directly from the logic constructors in $\mathcal{SHOIQ}_{(+)}$. A tableau definition for $\mathcal{SHOIQ}_{(+)}$ can be found in [14].

3 A Decision Procedure For $\mathcal{SHOIQ}_{(+)}$

This section starts by translating *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for $\mathcal{C}^2$ into those for $\mathcal{SHOIQ}_{(+)}$.

Definition 3 (star-type). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. A* star-type *is a pair $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$, where $\lambda(\sigma) \in 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$ is called* core label, *$\xi(\sigma) = (\langle r_1, l_1 \rangle, \cdots, \langle r_d, l_d \rangle)$ is a d-tuple over $2^{\mathbf{R}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$. A pair $\langle r, l \rangle$ is a* ray *of σ if $\langle r, l \rangle = \langle r_i, l_i \rangle$ for some $1 \le i \le d$. We use $\langle r(\rho), l(\rho) \rangle$ to denote a ray $\rho = \langle r, l \rangle$ where $r(\rho) = r$ and $l(\rho) = l$.*

- *A star-type σ is* nominal *if $o \in \lambda(\sigma)$ for some $o \in \mathbf{C}_o$.*
- *A star-type σ is* chromatic *if $\rho \neq \rho'$ implies $l(\rho) \neq l(\rho')$ for two rays ρ, ρ' of σ. When a star-type σ is chromatic, $\xi(\sigma)$ can be considered as a set of rays.*
- *Two star-types σ, σ' are* equivalent *if $\lambda(\sigma) = \lambda(\sigma')$, and there is a bijection π between $\xi(\sigma)$ and $\xi(\sigma')$ such that $\pi(\rho) = \rho'$ implies $r(\rho') = r(\rho)$ and $l(\rho') = l(\rho)$.*

We denote Σ for the set of all star-types for $(\mathcal{T}, \mathcal{R})$. $\triangleleft$

Note that for a chromatic star-type σ, $\xi(\sigma)$ can be considered as a set of rays since rays are distinct and not ordered. We can think of a star-type σ as the set of individuals x satisfying all concepts in $\lambda(\sigma)$, and each ray ρ of σ corresponds to a "neighbor" individual x_i of x such that $r(\rho)$ is the label of the link between x and x_i; and x_i satisfies all concepts in $l(\rho)$. In this case, we say that x *satisfies* σ.

Definition 4 (valid star-type). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. Let σ be a star-type for $(\mathcal{T}, \mathcal{R})$ where $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$. The star-type σ is* valid *if σ is chromatic and the following conditions are satisfied:*

1. *If $C \sqsubseteq D \in \mathcal{T}$ then $\mathsf{nnf}(\neg C \sqcup D) \in \lambda(\sigma)$;*
2. *$\{A, \neg A\} \not\subseteq \lambda$ for every concept name A where $\lambda = \lambda(\sigma)$ or $\lambda = l(\rho)$ for each $\rho \in \xi(\sigma)$;*
3. *If $C_1 \sqcap C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \subseteq \lambda(\sigma)$;*
4. *If $C_1 \sqcup C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \cap \lambda(\sigma) \neq \emptyset$;*
5. *If $\exists R.C \in \lambda(\sigma)$ then there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $R \in r(\rho)$;*
6. *If $(\le nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $S \in r(\rho)$ then $C \in l(\rho)$ or $\dot\neg C \in l(\rho)$;*
7. *If $(\le nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $S \in r(\rho)$ then there is some $1 \le m \le n$ such that $\{(\le mS.C), (\ge mS.C)\} \subseteq \lambda(\sigma)$;*
8. *For each ray $\rho \in \xi(\sigma)$, if $R \in r(\rho)$ and $R \,\underline{\circledast}\, S$ then $S \in r(\rho)$;*

9. *If* $\forall R.C \in \lambda(\sigma)$ *and* $R \in r(\rho)$ *for some ray* $\rho \in \xi(\sigma)$ *then* $C \in l(\rho)$*;*
10. *If* $\forall R.D \in \lambda(\sigma)$*,* $S \mathrel{\underline{\boxast}} R$*,* $\mathsf{Trans}(S)$ *and* $R \in r(\rho)$ *for some ray* $\rho \in \xi(\sigma)$ *then* $\forall S.D \in l(\rho)$*;*
11. *If* $\forall Q^{\oplus}.C \in \lambda(\sigma)$*,* $R \mathrel{\underline{\boxast}} Q$ *and* $R \in r(\rho)$ *for some ray* $\rho \in \xi(\sigma)$ *then* $\forall Q^{\oplus}.C \in l(\rho)$*;*
12. *If* $\exists Q^{\oplus}.C \in \lambda(\sigma)$ *then* $(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \in \lambda(\sigma)$*;*
13. *If* $(\geq nS.C) \in \lambda(\sigma)$ *then there are* n *distinct rays* $\rho_1, \cdots, \rho_n \in \xi(\sigma)$ *such that* $\{C, \mathscr{C}^{I_i}_{(\geq nS.C)}\} \subseteq l(\rho_i)$*,* $S \in r(\rho_i)$ *for all* $1 \leq i \leq n$*; and* $I_j, I_k \subseteq \{0, \cdots, log\, n + 1\}$*,* $I_j \neq I_k$ *for all* $1 \leq j < k \leq n$*.*
14. *If* $(\leq nS.C) \in \lambda(\sigma)$ *and there do not exist* $n+1$ *rays* $\rho_0, \cdots, \rho_n \in \xi(\sigma)$ *such that* $C \in l(\rho_i)$ *and* $S \in r(\rho_i)$ *for all* $0 \leq i \leq n$*.* $\triangleleft$

Roughly speaking, a star-type σ is valid if each individual x satisfies *semantically* all concepts in $\lambda(\sigma)$. In fact, each condition in Definition 4 represents the semantics of a constructor in $\mathcal{SHOIQ}_{(+)}$ except for transitive closure of roles. From valid star-types, we can "tile" a model instead of using expansion rules for generating nodes as described in tableau algorithms. Before presenting how to "tile" a model from star-types, we need some notation that will be used in the remainder of the paper.

Notation 1 *We call* $\mathcal{P} = \langle(\sigma_1, \rho_1, d_1), \cdots, (\sigma_k, \rho_k, d_k)\rangle$ *a* sequence *where* $\sigma_i \in \Sigma$*,* $\rho_i \in \xi(\sigma_i)$ *and* $d_i \in \mathbb{N}$ *for* $1 \leq i \leq k$*.*

- $\mathsf{tail}(\mathcal{P}) = (\sigma_k, \rho_k, d_k)$*,* $\mathsf{tail}_\sigma(\mathcal{P}) = \sigma_k$*,* $\mathsf{tail}_\rho(\mathcal{P}) = \rho_k$*,* $\mathsf{tail}_\delta(\mathcal{P}) = d_k$ *and* $|\mathcal{P}| = k$*. We denote* $\mathcal{L}(\mathcal{P}) = \lambda(\mathsf{tail}_\sigma(\mathcal{P}))$*.*
- $\mathsf{p}^i(\mathcal{P}) = (\sigma_i, \rho_i, d_i)$*,* $\mathsf{p}^i_\sigma(\mathcal{P}) = \sigma_i$*,* $\mathsf{p}^i_\rho(\mathcal{P}) = \rho_i$ *and* $\mathsf{p}^i_\delta(\mathcal{P}) = d_i$ *for each* $1 \leq i \leq k$*.*
- *an operation* $\mathsf{add}(\mathcal{P}, (\sigma, \rho, d))$ *extends* $\mathcal{P}$ *to a new sequence with* $\mathsf{add}(\mathcal{P}, (\sigma, \rho, d)) = \langle\mathcal{P}, (\sigma, \rho, d)\rangle$*.*

Definition 5 (frame). *Let* $(\mathcal{T}, \mathcal{R})$ *be a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base. A* frame *for* $(\mathcal{T}, \mathcal{R})$ *is a tuple* $\mathcal{F} = \langle\mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$*, where*

1. $\mathcal{N}$ *is a set of valid star-types such that* σ *is not equivalent to* σ' *for all* $\sigma, \sigma' \in \mathcal{N}$*;*
2. $\mathcal{N}_o \subseteq \mathcal{N}$ *is a set of nominal star-types;*
3. Ω *is a function that maps each pair* (σ, ρ) *with* $\sigma \in \mathcal{N}$ *and* $\rho \in \xi(\sigma)$ *to a sequence* $\Omega(\sigma, \rho) = \langle(\sigma_1, \rho_1, d_1), \cdots, (\sigma_m, \rho_m, d_m)\rangle$ *with* $\sigma_i \in \mathcal{N}$*,* $\rho_i \in \xi(\sigma_i)$*,* $d_i \in \mathbb{N}$ *for* $1 \leq i \leq m$ *such that for each* σ_i *with* $1 \leq i \leq m$*, it holds that* $l(\rho) = \lambda(\sigma_i)$*,* $l(\rho_i) = \lambda(\sigma)$ *and* $r(\rho_i) = r^-(\rho)$ *where* $r^-(\rho) = \{R^\ominus \mid R \in r(\rho)\}$*.*
4. δ *is a function* $\delta : \mathcal{N} \to \mathbb{N}$*. By abuse of notation, we also use* δ *to denote a function which maps each pair* (σ, ρ) *with* $\sigma \in \mathcal{N}$ *and* $\rho \in \xi(\sigma)$ *into a number in* $\mathbb{N}$*, i.e.,* $\delta(\sigma, \rho) \in \mathbb{N}$*.* $\triangleleft$

The frame structure, as introduced in Definition 5, allows us to compress individuals of a model into star-types. For each star-type σ and each ray $\rho \in \xi(\sigma)$, a list $\Omega(\sigma, \rho)$ of triples (σ_i, ρ_i, d_i) with $\rho_i \in \xi(\sigma_i)$ is maintained where σ_i is a "neighbor" star-type of σ via $\rho \in \xi(\sigma)$, and d_i indicates the d_i-th "layer" of rays of σ_i. We can think a layer of rays of σ_i as an individual that connects to its neighbor individuals via the rays of σ_i. The following definition presents how to connect such layers to form paths in a frame.

Definition 6 (path). *Let* $\mathcal{F} = \langle\mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ *be a frame for a* $\mathcal{SHOIQ}_{(+)}$ *knowledge base* $(\mathcal{T}, \mathcal{R})$*. A* path *is inductively defined as follows:*

1. *A sequence $\langle \emptyset, (\sigma, \rho, 1)\rangle$ is a path if $\sigma \in \mathcal{N}_o$ and $\rho \in \xi(\sigma)$;*
2. *A sequence $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ with $\mathcal{P} \neq \emptyset$ and $\mathsf{tail}(\mathcal{P}) = (\sigma_0, \rho_0, d_0)$, is a path if $(\sigma, \rho, d) = \mathsf{p}^{d_0}(\Omega(\sigma_0, \rho'))$ for each $\rho' \neq \rho_0$. In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is the ρ'-*neighbor *of $\mathcal{P}$, and two paths $\mathcal{P}$, $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ are neighbors.*
   Additionally, if $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is a ρ'-neighbor of $\mathcal{P}$ and $Q \in r(\rho')$ then $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is a Q-neighbor of $\mathcal{P}$. In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$ is a Q-neighbor of $\mathcal{P}$, or $\mathcal{P}$ is a $Q^{\ominus}$-neighbor of $\langle \mathcal{P}, (\sigma, \rho, d)\rangle$.

We define $\mathcal{P} \sim \mathcal{P}'$ if $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ and $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$. Since $\sim$ is an equivalence relation over the set of all paths, we use $\mathscr{P}$ to denote the set of all equivalence classes $[\mathcal{P}]$ of paths in $\mathcal{F}$. For $[\mathcal{P}], [\mathcal{Q}] \in \mathscr{P}$, we define:

1. *$[\mathcal{P}]$ is a neighbor (ρ'-neighbor) of $[\mathcal{Q}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that $\mathcal{Q}'$ is a neighbor (ρ'-neighbor) of $\mathcal{P}'$;*
2. *$[\mathcal{Q}]$ is a reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \cdots, [\mathcal{P}_n] \in \mathscr{P}$ such that $[\mathcal{P}_{i+1}]$ is a neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$ where $[\mathcal{P}_1] = [\mathcal{P}]$ and $[\mathcal{Q}] = [\mathcal{P}_n]$.*
3. *$[\mathcal{Q}]$ is a Q-neighbor of $[\mathcal{P}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that $\mathcal{Q}'$ is a Q-neighbor of $\mathcal{P}'$, or $\mathcal{P}'$ is a $Q^{\ominus}$-neighbor of $\mathcal{Q}'$;*
4. *$[\mathcal{Q}]$ is a Q-reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \cdots, [\mathcal{P}_n] \in \mathscr{P}$ such that $[\mathcal{P}_{i+1}]$ is a Q-neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$ where $[\mathcal{P}_1] = [\mathcal{P}]$ and $[\mathcal{Q}] = [\mathcal{P}_n]$.* $\triangleleft$

Note that for two paths $\mathcal{P}, \mathcal{P}'$ with $\mathsf{tail}_\rho(\mathcal{P}) \neq \mathsf{tail}_\rho(\mathcal{P}')$, we have $\mathcal{P} \sim \mathcal{P}'$ if $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ and $\mathsf{tail}_\delta(\mathcal{P}) = \mathsf{tail}_\delta(\mathcal{P}')$. This does not allow for extending $\mathsf{tail}_\rho(\mathcal{P})$ to $\mathsf{tail}_\rho([\mathcal{P}])$. As a consequence, there may be several "predecessors" of an equivalence class $[\mathcal{P}]$. However, we can define $\mathsf{tail}_\sigma([\mathcal{P}]) = \mathsf{tail}_\sigma(\mathcal{P})$, $\mathsf{tail}_\delta([\mathcal{P}]) = \mathsf{tail}_\delta(\mathcal{P})$ and $\mathcal{L}([\mathcal{P}]) = \mathcal{L}(\mathcal{P})$. In the sequel, we use $\mathcal{P}$ instead of $[\mathcal{P}]$ whenever it is clear from the context.

Definition 7 (cycle). *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta\rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set $\mathscr{P}$ of paths in $\mathcal{F}$.*

1. *A* cycle *is a set Θ of triples $(\mathcal{P}, \rho, \mathcal{Q})$ with $\mathcal{P}, \mathcal{Q} \in \mathscr{P}$ and $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that for each $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ the following conditions are satisfied:*
   (a) $\mathsf{tail}_\delta(\mathcal{P}) > 1$ and $\mathsf{tail}_\delta(\mathcal{Q}) > 1$;
   (b) If $\mathcal{P}'$ is the ρ-neighbor of $\mathcal{P}$ then $\mathsf{tail}_\sigma(\mathcal{P}') = \mathsf{tail}_\sigma(\mathcal{Q})$;
   (c) for each sequence $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$ such that $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_2$ is not the ρ-neighbor of $\mathcal{P}$, and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $1 \leq i < n$, there is some $(\mathcal{P}'', \rho'', \mathcal{Q}'') \in \Theta$ such that
      i. either $\mathcal{Q}'' = \mathcal{P}_j$, $\mathsf{tail}_\sigma(\mathcal{P}_{j+1}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{j+1}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ and $\mathcal{P}_j$ is the ρ-neighbor of $\mathcal{P}_{j+1}$ for some $1 < j < n$,
      ii. or there are $\mathcal{P}_{n+1}, \cdots, \mathcal{P}_{n+m} \in \mathscr{P}$ with $\mathcal{Q}'' = \mathcal{P}_{n+m-1}$, $\mathsf{tail}_\sigma(\mathcal{P}_{n+m}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{n+m}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $n \leq i < n+m$.

   In this case, we say that $\mathcal{Q}$ is cycled *by $\mathcal{P}$ via ρ.*
2. *A cycle Θ' is a* reachable *cycle of Θ if for each $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ and for each sequence $\mathcal{P}_1, \cdots, \mathcal{P}_n \in \mathscr{P}$ such that $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_2$ is not a ρ-neighbor of $\mathcal{P}$, and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $1 \leq i < n$, there is some $(\mathcal{P}'', \rho'', \mathcal{Q}'') \in \Theta'$ such that*

(a) either $\mathcal{Q}'' = \mathcal{P}_j$, $\mathsf{tail}_\sigma(\mathcal{P}_{j+1}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{j+1}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ and $\mathcal{P}_j$ is a ρ-neighbor of $\mathcal{P}_{j+1}$ for some $1 < j < n$,

(b) or there are $\mathcal{P}_{n+1}, \cdots, \mathcal{P}_{n+m} \in \mathscr{P}$ with $\mathcal{Q}'' = \mathcal{P}_{n+m-1}$, $\mathsf{tail}_\sigma(\mathcal{P}_{n+m}) = \mathsf{tail}_\sigma(\mathcal{P}'')$, $\mathsf{tail}_\delta(\mathcal{P}_{n+m}) \geq \mathsf{tail}_\delta(\mathcal{P}'')$ and $\mathcal{P}_{i+1}$ is a neighbor of $\mathcal{P}_i$ for $n \leq i < n+m$. $\lhd$

Note that cycles may encapsulate loops if $\mathsf{tail}_\delta(\mathcal{P}_{j+1}) = \mathsf{tail}_\delta(\mathcal{P}'')$ holds in Conditions 1(c)i and 1(c)ii, Definition 7. Let Θ be a cycle in a frame. Definition 7 ensures that each reachable path of some path $\mathcal{P}$ with $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ goes through a star-type $\sigma = \mathsf{tail}_\sigma(\mathcal{Q}')$ with some $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta$. Such a cycle, which is similar to blocking-blocked nodes in completion graphs for $\mathcal{SHOIQ}$ [7], allows for "unravelling" infinitely the frame to obtain a model of a KB in $\mathcal{SHOIQ}$ (without transitive closure of roles). This means that we can extend the set $\mathscr{P}$ of paths by adding infinitely paths which lengthen $\mathcal{Q}$ such that $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ and $\mathcal{P}$ is not a neighbor of $\mathcal{Q}$. However, such a cycle structure is not sufficient to represent models of a KB with transitive closure of roles since a concept such as $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P})$ can be satisfied by a Q-reachable path $\mathcal{P}'$ of $\mathcal{P}$ which is arbitrarily far from $\mathcal{P}$. There are the following possibilities for an algorithm which builds a frame: (i) the algorithm stops building the frame as soon as a cycle Θ is detected such that each concept of the form $\exists Q^\oplus.D$ occurring in $\mathcal{L}(\mathcal{P})$ for each cycling path $\mathcal{P}$ of Θ is satisfied, i.e., $\mathcal{P}$ has a Q-reachable path $\mathcal{P}'$ with $\in \exists Q.D \in \mathcal{L}(\mathcal{P})$, (ii) despite of several detected cycles, the algorithm continues building the frame until each concept of the form $\exists Q^\oplus.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycling path $\mathcal{P}$ of Θ. If we adopt the first possibility, the completeness of such an algorithm cannot be established since there are models in which paths satisfying concepts of the form $\exists Q^\oplus.D$ can spread over several "iterative structures" such as cycles. For this reason, we adopt the second possibility by introducing into frames an additional structure, namely *blocking-blocked cycles*, which determines a sequence of cycles $\Theta_1, \cdots, \Theta_k$ such that Θ_{i+1} is a reachable cycle of Θ_i. Reachability of cycles allows for "unravelling" the frame between cycled paths $\mathcal{Q}'$ with $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta_k$ and cycling paths $\mathcal{P}$ with $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_1$.

Definition 8 (blocking). *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set $\mathscr{P}$ of all paths in $\mathcal{F}$.*

1. *A cycle Θ' is* blockable *by a cycle Θ if Θ' is a reachable cycle of Θ, and for each $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta'$ there is some $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ such that $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$, $\mathcal{L}(\mathcal{Q}) = \mathcal{L}(\mathcal{Q}')$ and $r(\rho) = r(\rho')$. In this case, we say that $\mathcal{Q}'$ is blockable by $\mathcal{P}$ via ρ.*
2. *A cycle Θ' is* blocked *by a cycle Θ if there are $\Theta_1, \cdots, \Theta_k$ with $\Theta = \Theta_1$, $\Theta' = \Theta_k$ such that Θ_{i+1} is blockable by Θ_i for $1 \leq i < k$, and for each $(\lambda, s, \lambda') \in 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{R}(\mathcal{T},\mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T},\mathcal{R})}$ with $\exists Q^\oplus.D \in \lambda$,*
   - *if there is some path $\mathcal{P}_k$ with $(\mathcal{P}_k, \rho_k, \mathcal{Q}_k) \in \Theta_k$, $\mathcal{L}(\mathcal{P}_k) = \lambda$, $\mathcal{L}(\mathcal{Q}_k) = \lambda'$, $s = r(\rho_k)$ such that $\mathcal{P}_k$ has no Q-reachable path $\mathcal{P}'_k$ with $\exists Q.D \in \mathcal{L}(\mathcal{P}'_k)$,*
   - *then there is some $\mathcal{P}_1$ with $(\mathcal{P}_1, \rho_1, \mathcal{Q}_1) \in \Theta_1$, $\mathcal{L}(\mathcal{P}_1) = \lambda$, $\mathcal{L}(\mathcal{Q}_1) = \lambda'$, $r(\rho_1) = s$ such that for each $\exists P^\oplus.C \in \mathcal{L}(\mathcal{P}_1)$,*
     - *there is a P-reachable path $\mathcal{Q}''$ of $\mathcal{P}_1$ with $\exists P.C \in \mathcal{L}(\mathcal{Q}'')$, and*

- *there are two triples $(\mathcal{P}_j, \rho_j, \mathcal{Q}_j) \in \Theta_j$ and $(\mathcal{P}_{j+1}, \rho_{j+1}, \mathcal{Q}_{j+1}) \in \Theta_{j+1}$ for some $1 \leq j < k$,*

which satisfy that $\mathcal{Q}''$ is a reachable path of $\mathcal{Q}_j$ and $\mathcal{Q}_{j+1}$ is a reachable path of $\mathcal{Q}''$.

In this case, we say that $\mathcal{P}_1$ blocks $\mathcal{P}_k$ via ρ_k. $\triangleleft$

According to Definition 8, there are a sequentially reachable cycles between a blocking cycle Θ_1 and a blocked cycle Θ_k, which allows for unravelling the frame between Θ_k and Θ_1. Condition 2 Definition 8 says that if $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_k$ and $\mathcal{L}(\mathcal{P})$ contains concepts of the form $\exists Q^{\oplus}.D$ which are not satisfied by reachable paths of $\mathcal{P}$ then there exists some $\mathcal{P}'$ with $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta_1$ which allows for satisfying these concepts $\exists Q^{\oplus}.D$ in $\mathcal{L}(\mathcal{P})$ by unravelling. We would like to note that a path $\mathcal{P}$ is blocked if there is some blocked cycle Θ_k such that $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_k$.

Definition 9 (valid frame). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}$ knowledge base. A frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ is* valid *if the following conditions are satisfied:*

1. *For each $o \in \mathbf{C}_o$ there is a unique $\sigma_o \in \mathcal{N}_o$ such that $o \in \lambda(\sigma_o)$ and $\delta(\sigma_o) = 1$;*
2. *For each $\sigma \in \mathcal{N}$, σ is valid;*
3. *If $\exists Q^{\oplus}.C \in \lambda(\mathsf{tail}_\sigma(\mathcal{P}_0))$ for some $\mathcal{P}_0 \in \mathscr{P}$ then there are $\mathcal{P}, \mathcal{P}' \in \mathscr{P}$ such that one of the following conditions is satisfied:*
   (a) *$\mathcal{P}_0 = \mathcal{P} = \mathcal{P}'$ and $\exists Q.C \in \mathcal{L}(\mathcal{P}_0)$;*
   (b) *$\mathcal{P}'$ is a Q-reachable of $\mathcal{P}$, and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or $\mathcal{P}$ blocks $\mathcal{P}_0$;*
   (c) *$\mathcal{P}$ is a $Q^{\ominus}$-reachable of $\mathcal{P}'$, and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or $\mathcal{P}$ blocks $\mathcal{P}_0$.* $\triangleleft$

Conditions 1-3 in Definition 9 ensure satisfaction of tableau properties [14]. In particular, Condition 3 takes into account satisfaction of transitive closure of roles. In fact, for each blocking path $\mathcal{P}$ with $\exists Q^{\oplus}.D \in \mathcal{L}(\mathcal{P})$, this condition says that $\mathcal{P}$ must be satisfied by a Q-reachable path $\mathcal{P}'$ of $\mathcal{P}$ between a blocking cycle Θ_1 and a blocked cycle Θ_k. If there is some path $\mathcal{Q}$ between Θ_1 and Θ_k with $\exists S^{\oplus}.C \in \mathcal{L}(\mathcal{Q})$ that is not satisfied, due to the construction of star-types and frames, a concept $\exists S^{\oplus}.C$ is propagated along S-reachable paths of $\mathcal{Q}$. This implies that $\mathcal{Q}$ has a S-reachable path $\mathcal{Q}'$ such that $\mathcal{Q}'$ is blocked and $\exists S^{\oplus}.C \in \mathcal{L}(\mathcal{Q}')$. By unravelling (details will be given in soundness proof), we can build an (extended) S-reachable path $\mathcal{Q}''$ of $\mathcal{Q}'$ such that $\exists S.C \in \mathcal{L}(\mathcal{Q}'')$.

We now present Algorithm 1 for building a frame which is valid if the conditions in Definition 9 are satisfied. This algorithm starts by adding nominal star-types to the frame. For each non blocked path $\mathcal{P}$ with a ray $\rho \in \xi(\mathsf{tail}_\sigma(\mathcal{P}))$ such that $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ is minimal and there is a difference between $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ and $\delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, the algorithm picks in a nondeterministic way a valid star-type ω that matches $\mathsf{tail}_\sigma(\mathcal{P})$ via ρ, and updates $\Omega(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, $\Omega(\omega, \rho')$, $\delta(\mathsf{tail}_\sigma(\mathcal{P}), \rho)$, $\delta(\omega, \rho')$, eventually, $\delta(\mathsf{tail}_\sigma(\mathcal{P}))$ and $\delta(\omega)$ by calling $\mathsf{updateFrame}(\cdots)$[14]. The algorithm terminates when a blocked cycle is detected.

Figure 2 depicts a frame when executing Algorithm 1 for $\mathcal{K}_1$ in the example presented in Section 1. The algorithm builds a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ where $\mathcal{N} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\mathcal{N}_o = \{\sigma_0\}$. The dashed arrows indicate how the function

Require: A $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$
Ensure: A frame $\langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ for $(\mathcal{T}, \mathcal{R})$

```
Let Σ be the set of all star-types for (T, R)
for all o ∈ C_o do
    if there is no σ ∈ N such that o ∈ λ(σ) then
        Choose a star-type σ_o ∈ Σ such that o ∈ λ(σ_o)
        Set δ(σ_o) = 1, N = N ∪ {σ_o} and N_o = N_o ∪ {σ_o}
        Set δ(σ_o, ρ) = 0, Ω(σ_o, ρ) = ∅ for all ρ ∈ ξ(σ_o)
    end if
end for
while there is a path P that is not blocked and a ray ρ ∈ ξ(tail_σ(P)) such that
        tail_δ(P) = δ(tail_σ(P), ρ) + 1 and δ(tail_σ(P)) ≤ δ(ω) for all ω ∈ N do
    Choose a star-type σ' ∈ Σ such that there is a ray ρ' ∈ ξ(σ') satisfying
            l(ρ) = λ(σ'), l(ρ') = λ(σ), r(ρ') = r⁻(ρ), and
            σ' ∈ N implies δ(σ') = δ(σ', ρ') + 1
    updateFrame(σ, ρ, σ', ρ')
end while
```

Algorithm 1: An algorithm for building a frame

$\Omega(\sigma, \rho)$ can be built. For example, $\Omega(\sigma_0, \rho_0) = \{(\sigma_1, \nu_0, 1)\}$, $\Omega(\sigma_0, \rho_1) = \{(\sigma_2, \rho'_0, 1)\}$ where ρ_0 and ρ_1 are the respective horizontal and vertical rays of σ_0; ν_0 is the left ray of σ_1; ρ'_0 is the vertical ray of σ_2. Moreover, the directed dashed arrow from σ_0 to σ_1 indicates that the ray ρ_0 of σ_0 can match the ray ν_0 on the left ray of σ_1 since $l(\rho_0) = \lambda(\sigma_1)$, $r(\nu_0) = \lambda(\sigma_0)$, $r(\nu_0) = r^-(\rho_0)$.

Then, the algorithm generates $\delta(\sigma_0) = 1$, $\delta(\sigma_1) = 1$, $\delta(\sigma_2) = 1$ and forms a cycle Θ consisting of the following triples : $((\sigma_3, 2), \rho_1, (\sigma_3, 3))$ (ρ_1 is the left ray of σ_3)), $((\sigma_3, 2), \rho_3, (\sigma_4, 1))$ (ρ_3 is the vertical ray of σ_3), $((\sigma_4, 1), \rho_4, (\sigma_4, 2))$ (ρ_4 is the left ray of σ_4) and $((\sigma_4, 1), \rho_5, (\sigma_3, 2))$ (ρ_5 is the vertical ray of σ_4). Note that for the sake of brevity, we use just $\mathsf{tail}_\sigma(\mathcal{P})$ and $\mathsf{tail}_\delta(\mathcal{P})$ to denote a path in the triples. We can check that any path that is an extension of a path $\mathcal{P}$ gets through a path $\mathcal{Q}$ where $\mathcal{P}$ is the first component of a triple and $\mathcal{Q}$ is the third component of a triple.

The algorithm may add some more paths that go through σ_3 and σ_4 to form a blocked cycle. A model of the ontology can be built by starting from σ_0 and getting (i) σ_4 via σ_1, (ii) σ_3 via σ_1, and (iii) σ_3 via σ_2. From σ_3 and σ_4, the model goes through σ_3 and σ_4 infinitely. Note that from any individual x satisfying σ_3 (or σ_4), i.e. the "label" of x contains $\exists Q^+.\{o\}$, there is a path containing S which goes back the individual satisfying σ_0. Thus, the concept $\exists Q^+.\{o\}$ is satisfied for each individual whose label contains $\exists Q^+.\{o\}$.

Lemma 1. *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base.*

1. *Algorithm 1 terminates.*
2. *If Algorithm 1 can build a valid frame for $(\mathcal{T}, \mathcal{R})$ then $(\mathcal{T}, \mathcal{R})$ is consistent.*
3. *If $(\mathcal{T}, \mathcal{R})$ is consistent then Algorithm 1 can build a valid frame $\mathcal{F}$ for $(\mathcal{T}, \mathcal{R})$.*

Proof (sketch). Since the functions $\delta(\sigma)$ and $\delta(\sigma, \langle r, l \rangle)$ is increased monotonously by Algorithm 1, termination of the algorithm can be proved if we can show that : (i) the

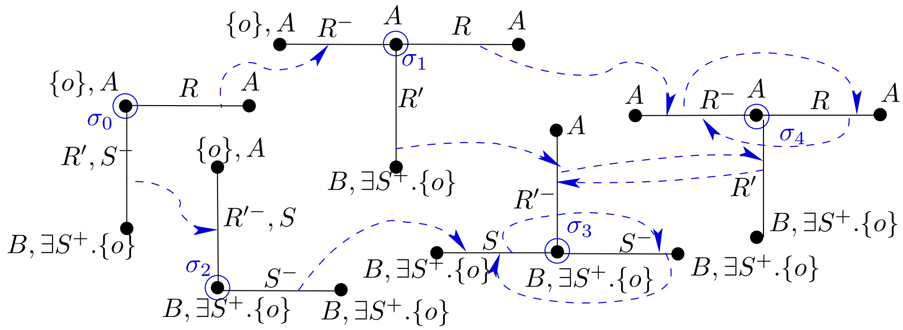

Fig. 2. A frame obtained by Algorithm 1 for $\mathcal{K}_1$ in the example in Section 1

number of different star-types is bounded; (ii) the detection of a blocked cycle according to Definition 8 terminates. For the soundness of Algorithm 1, we can extend the set $\mathscr{P}$ of paths to a set $\widehat{\mathscr{P}}$ of extended paths by "unravelling" the frame between blocking-blocked cycles. A tableau [14] can be built from $\widehat{\mathscr{P}}$. The main argument is that when extending a path $\mathcal{P}$ if $\mathsf{tail}_\sigma(\mathcal{P}) \neq \mathsf{tail}_\sigma(\mathcal{P}')$ for all blocked path $\mathcal{P}'$ then this extension process can be continued up to a star-type $\sigma' = \mathsf{tail}_\sigma(\mathcal{P}'')$ for some blocked path $\mathcal{P}''$. This holds due to the definition of cycles and blockable cycles. Otherwise, i.e., $\mathsf{tail}_\sigma(\mathcal{P}) = \mathsf{tail}_\sigma(\mathcal{P}')$ for some blocked path $\mathcal{P}'$ by $\mathcal{Q}$ then $\mathcal{P}$ can be extended by getting through $\mathsf{tail}_\sigma(\mathcal{Q})$.

Regarding completeness, a tableau can guide the algorithm (i) to choose valid star-types, (ii) to ensure that $\delta(\sigma) = 1$ for each nominal star-type σ, and (iii) to detect a pair (Θ_1, Θ_k) of blocking and blocked cycles as soon as each concept of the form $\exists Q^\oplus.D$ in Θ_1 is satisfied. We refer the readers to [14] for a complete proof of Lemma 1.

The following theorem is a consequence of Lemma 1.

Theorem 1. *$\mathcal{SHOIQ}_{(+)}$ is decidable.*

4 Conclusion

In this paper, we have presented a decision procedure for the description logic $\mathcal{SHOIQ}$ with transitive closure of roles in concept axioms, whose decidability was not known. The most significant feature of our contribution is to introduce a structure for characterizing models which have an infinite non-tree-like part. This structure would provide an insight into regularity of such models which would be enjoyed by a more expressive DL, such as $\mathcal{ZOIQ}$ [6], whose decidability remains open. In future work, we aim to improve the algorithm by making it more goal-directed and aim to investigate another open question about the hardness of $\mathcal{SHOIQ}_{(+)}$.

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