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Why Newton's bucket cannot account for Earth's rotation

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Abstract

This note illustrates, at an undergraduate level, how an *a priori* legitimate assumption of uniform gravity can lead to intuitively satisfying but false results. Considering the simple situation of a liquid contained in a bowl at rest with respect to the ground at the North Pole, we look for the shape of its surface, taking Earth's rotation into account. The classical problem of a rotating fluid is converted to that of the determination of Earth's shape.

Keywords: fluid mechanics, Earth's rotation, Huygens, Newton

1. Introduction

This short contribution originates from a legitimate question after a talk on Newton's mechanics when a colleague (Professor D Geiger) suggested that I ask students what should the size of a bowl resting on the North Pole be in order for the concavity of the free surface of a liquid in it to be measurable. My immediate answer was that, since the bowl would need to be very large, one should have to take the non-uniformity of Earth's gravitational field into account, and thus the free surface shape could be convex.

We will see in a very simple way that this is indeed the case, and logically independent of the bowl's size. Since the correct result can seem counter-intuitive at first sight in a classical fluid mechanics context, it follows from Huygens' arguments on centrifugal force and Earth's shape [1], contemporaneous to Newton's *Principia* [2].

The famous thought experiment known as Newton's Bucket appears in his *Scholium* to the *Definitions* [2], where Newton considers the concavity of the free surface of water in a bucket to bring out a rotation relative to its surroundings. That is his *absolute space*, or at least an heliocentric world with respect to the so-called fixed stars, but not Earth itself since

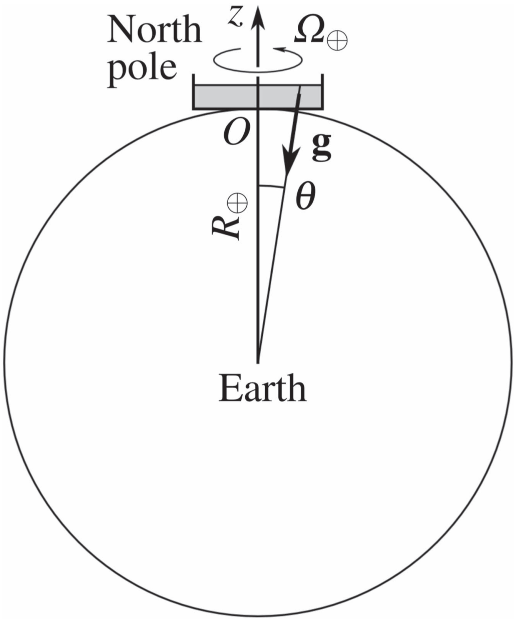

Figure 1. Liquid in equilibrium in a bowl at the North Pole.

Newton's law of gravitation and his shell theorems for a spherically symmetric planet lead to a central gravitational force, whether or not it rotates upon itself.

We can also note that, for a body in the vicinity of its axis of symmetry, a lightly flattened ellipsoidal Earth (see below) only induces a slight increase of the distance to the center of attraction, associated with the direction of *apparent* gravity. In our configuration (figure 1), we can thus consider a spherical Earth as a good approximation for our purpose, and a liquid in a bucket fixed to the ground at one pole should *a priori* reveal Earth's rotation about its axis with respect to other bodies in the Universe.

We will see that, in spite of its practical unmeasurability, this experiment logically exhibits Earth's shape rather than accounting for its spin.

2. Resolution

Let us consider an incompressible liquid at relative rest in a cylindrical bowl attached to Earth and centered at the North Pole. With respect to an assumed inertial frame $\mathcal{R}$ attached to Earth's center and whose axes point to the so-called fixed stars, conservation of the momentum is written as (one could alternatively write the equilibrium of the free surface in the rotating frame with inertial forces to avoid the gradient)

$$-\mathbf{grad}\, p + \rho\, \mathbf{g} = \rho\, \boldsymbol{\Gamma}_{\mathcal{R}}, \quad (1)$$

where p is the local pressure, ρ the liquid's mass density, $\mathbf{g}$ the local gravitational field of modulus g, and $\boldsymbol{\Gamma}_{\mathcal{R}}$ the local acceleration of the fluid with respect to $\mathcal{R}$.

Denoting by θ the colatitude and $R_\oplus$ the mean radius of Earth, we have in cylindrical coordinates $(O;\, r,\, \varphi,\, z)$, where the origin O is the North Pole and z is the ascending Earth's axis (figure 1):

$$\mathbf{g} = -g(\sin\theta\, \hat{\mathbf{e}}_r + \cos\theta\, \hat{\mathbf{e}}_z) \approx -g\left(\frac{r}{R_\oplus}\hat{\mathbf{e}}_r + \hat{\mathbf{e}}_z\right), \tag{2}$$

at first order, where in the vicinity of the pole $r \ll R_\oplus$ and the value of g can be considered constant, corrections due to the cosine being second order. (While these approximations are unnecessary, exact expressions for a spherical or ellipsoidal Earth lead to the same result.)

With $\Omega_\oplus$ as Earth's angular velocity about its axis, (1) thus becomes

$$\begin{cases} -\dfrac{\partial p}{\partial r} - \rho g \dfrac{r}{R_\oplus} = -\rho\, \Omega_\oplus^2\, r \\ -\dfrac{\partial p}{\partial z} - \rho g = 0 \end{cases} \quad \Leftrightarrow \quad p(r,\, z) = -\rho g\, z - \tfrac{1}{2}\rho\left(\frac{g}{R_\oplus} - \Omega_\oplus^2\right) r^2 + p_\circ, \tag{3}$$

so that the free isobaric surface is given by

$$z(r) = -\tfrac{1}{2}\left(\frac{1}{R_\oplus} - \frac{\Omega_\oplus^2}{g}\right) r^2 + z_\circ, \tag{4}$$

where $p_\circ$ and $z_\circ$ are constants whose values are of no importance here.

With the numerical values $\Omega_\oplus = 2\pi\, /\, 86\,164 \approx 7.3 \times 10^{-5}$ rad s^{-1} and $g\, /\, R_\oplus = Gm_\oplus \big/ R_\oplus^3 \approx 0.15 \times 10^{-5}\ \mathrm{s}^{-2}$, it appears that, whatever the value of r, that is the distance from the pole (edge effects being ignored), the radial centripetal attraction is 289 times greater than the centrifugal force due to Earth's rotation (which would result in the classical concave paraboloidal shape that can be found in any fluid mechanics textbook, as well as by Clairaut [3], part I, section XXVII) and that, since the factor of r^2 in (4) is thus negative, the free surface of the liquid has a *convex* shape.

This non-surprising result, here restricted to the vicinity of the pole, gives a curvature $\kappa = \mathrm{d}^2 z/\mathrm{d}r^2 \approx -1/R_\oplus$ which is obviously that of the ocean's surface and corresponds to Huygens' [1] calculations for a rotating Earth's flattened shape by considering a liquid surface at equilibrium under central forces. We also again find here that the bucket (or Earth) should turn $\sqrt{289} = 17$ times faster in order for the liquid's free surface to be flat.

Considering a liquid Earth surface and a problem not restricted to the vicinity of the pole, that is without the approximation (2), leads to an ellipsoidal figure of flattening $f = (R_{\text{equator}} - R_{\text{pole}})/R_{\text{equator}}$. The first calculations gave $f = 1/578$ for Huygens [1] who did not use Newton's law of gravitation and $f = 1/230$ for Newton [2] with a clever reasoning. A more modern and rigorous treatment of the problem, allowed by progresses in calculus, was performed by Clairaut [3] 50 years later and gave the correct value of about $1/300$.

3. Conclusion

This inquiring situation illustrates, independently of the measurability of a phenomenon, how an *a priori* legitimate assumption can lead to a qualitatively false result when one is not careful with the orders of magnitude. Despite a slightly ellipsoidal Earth, the problem of determining the liquid's surface at the North Pole can be solved satisfyingly with a spherical Earth assumption, as done here. With a twist from the *a priori* nature of the problem, it brings back the early calculations for the figure of Earth. With hindsight and since the shape of the liquid does not depend on the bucket itself but must be normal to the apparent gravity, this result appears previsible and is a necessary condition for a non-solid spinning planet to exist, viz. a centripetal attraction greater than the centrifugal force [1].

This problem also shows that using Newton's Bucket for exhibiting a (slow) rotation of a fluid relative to an inertial frame of reference is also sensitive to gravitational non-uniformity, here due to the non parallelism of the gravitational field, and could only be used for high angular velocities (as Newton's true observations) or very far from every attractive body, and thus in a straight accelerating rocket in order to have an equivalent descendent force.

The same remarks hold for two more interesting variants that also rest on the presence of a centrifugal force to bring out a rotation: Newton's measure of the tension in a cord connecting two globes [2], or Mach's astronaut feeling its arms and legs opening if rotating [4]. Both also need a *uniform* (but now possibly zero) gravitational field.

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