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Damien Lamberton, Giulia Terenzi

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Abstract

We give an analytical characterization of the price function of an American option in Heston-type models. Our approach is based on variational inequalities and extends recent results of Daskalopoulos and Feehan (2011). We study the existence and uniqueness of a weak solution of the associated degenerate parabolic obstacle problem. Then, we use suitable estimates on the joint distribution of the log-price process and the volatility process in order to characterize the analytical weak solution as the solution to the optimal stopping problem. We also rely on semi-group techniques and on the affine property of the model.

Keywords: American options; degenerate parabolic obstacle problem; optimal stopping problem.

1 Introduction

The model introduced by S. Heston in 1993 ([9]) is one of the most widely used stochastic volatility models in the financial world and it was the starting point for several more complex models which extend it. The great success of the Heston model is due to the fact that the dynamics of the underlying asset can take into account the non-lognormal distribution of the asset returns and the observed mean-reverting property of the volatility. Moreover, it remains analytically tractable and provides a closed-form valuation formula for European options using Fourier transform.

These features have called for an extensive literature on numerical methods to price derivatives in Heston-type models. In this framework, besides purely probabilistic methods such as standard Monte Carlo and tree approximations, there is a large class of algorithms which exploit numerical analysis techniques in order to solve the standard PDE (resp. the obstacle problem) formally associated with the European (resp. American) option price function. However, these algorithms have, in general, little mathematical support and in particular, as far as we know, a rigorous and complete study of the analytic characterization of the American price function is not present in the literature.

The main difficulties in this sense come from the degenerate nature of the model. In fact, the infinitesimal generator associated with the two dimensional diffusion given by the log-price process and the volatility process is not uniformly elliptic: it degenerates on the boundary of the domain, that is when the volatility variable vanishes. Moreover, it has unbounded coefficients with linear growth. Therefore, the existence and the uniqueness of the solution to the pricing PDE and obstacle problem do not follow from the classical theory, at least in the case in which the boundary of the state space is reached with positive probability, as happens in many cases of practical importance (see [3]). Moreover, the probabilistic representation of the solution, that is the identification with the price function, is far from trivial in the case of non regular payoffs.

It should be emphasized that a clear analytic characterization of the price function allows not only to formally justify the theoretical convergence of some classical pricing algorithms but also to investigate the regularity properties of the price function (see [11] for the case of the Black and Scholes models).

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Concerning the existing literature, E. Ekstrom and J. Tysk in [6] give a rigorous and complete analysis of these issues in the case of European options, proving that, under some regularity assumptions on the payoff functions, the price function is the unique classical solution of the associated PDE with a certain boundary behaviour for vanishing values of the volatility. However, the payoff functions they consider do not include the case of standard put and call options.

Recently, P. Daskalopoulos and P. Feehan studied the existence, the uniqueness, and some regularity properties of the solution of this kind of degenerate PDE and obstacle problems in the elliptic case, introducing suitable weighted Sobolev spaces which clarify the behaviour of the solution near the degenerate boundary. Again, as regards the probabilistic representation, they only treat the case with heavy regularity assumptions on the payoff function (see [7]).

The aim of this paper is to give a precise analytical characterization of the American option price function for a large class of payoffs which includes the standard put and call options. In particular, we give a variational formulation of the American pricing problem using the weighted Sobolev spaces and the bilinear form introduced in [5]. The paper is organized as follows. In Section 2 we introduce our notations and we state our main results. Then, in section 3 we study the existence and uniqueness of the solution of the associated variational inequality, extending the results obtained in [5] in the elliptic case. The proof essentially relies on the classical penalization technique introduced by Bensoussan and Lions [4] with some technical devices due to the degenerate nature of the problem. We also establish a Comparison Theorem. Finally, in section 4, we prove that the solution of the variational inequality with obstacle function ψ is actually the American option price function with payoff ψ, with conditions on ψ which are satisfied, for example, by the standard call and put options. In order to do this, we use the affine property of the underlying diffusion given by the log price process X and the volatility process Y. Thanks to this property, we first identify the analytic semigroup associated with the bilinear form with a correction term and the transition semigroup of the pair (X, Y) with a killing term. Then, we prove regularity results on the solution of the variational inequality and suitable estimates on the joint law of the process (X, Y) and we deduce from them the analytical characterization of the solution of the optimal stopping problem, that is the American option price.

Contents

2 Notations and main results

2.1 The Heston model

We recall that in the Heston model the dynamics under the pricing measure of the asset price S and the volatility process Y are governed by the stochastic differential equation system

$$\begin{cases} \frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{Y_t}dB_t, & S_0 = s > 0, \\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t, & Y_0 = y \geq 0, \end{cases}$$

where B and W denote two correlated Brownian motions with

$$d\langle B, W\rangle_t = \rho dt, \qquad \rho \in (-1, 1).$$

Here $r \geq 0$ and $\delta > 0$ are respectively the risk free rate of interest and the continuous dividend rate. The dynamics of Y follows a CIR process with mean reversion rate $\kappa \geq 0$ and long run state $\theta \geq 0$. The parameter $\sigma > 0$ is called the volatility of the volatility. Note that we do not require the Feller condition $2\kappa\theta \geq \sigma^2$: the volatility process Y can hit 0 (see, for example, [2, Section 1.2.4]).

We are interested in studying the price of an American option with payoff function ψ. For technical reasons which will be clarified later on, hereafter we consider the process

$$X_t = \log S_t - \bar{c}t, \qquad \text{with } \bar{c} = r - \delta - \frac{\rho\kappa\theta}{\sigma}, \tag{2.1}$$

which satisfies

$$\begin{cases} dX_t = \left(\frac{\rho\kappa\theta}{\sigma} - \frac{Y_t}{2}\right)dt + \sqrt{Y_t}dB_t, \\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t. \end{cases} \tag{2.2}$$

Note that, in this framework, we have to consider payoff functions ψ which depend on both the time and the space variables. For example, in the case of a standard put option (resp. a call option) with strike price K we have $\psi(t,x) = (K - e^{x+\bar{c}t})_+$ (resp. $\psi(t,x) = (e^{x+\bar{c}t} - K)_+$). So, the natural price at time t of an American option with a nice enough payoff $(\psi(t, X_t, Y_t))_{0\leq t\leq T}$ is given by $P(t, X_t, Y_t)$, with

$$P(t,x,y) = \operatorname{esssup}_{\theta\in\mathcal{T}_{t,T}}\mathbb{E}[e^{-r(\theta-t)}\psi(\theta, X_\theta^{t,x,y}, Y_\theta^{t,y})],$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times with values in $[t,T]$ and $(X_s^{t,x,y}, Y_s^{t,y})_{t\leq s\leq T}$ denotes the solution to (2.2) with the starting condition $(X_t, Y_t) = (x, y)$.

Our aim is to give an analytical characterization of the price function P. We recall that the infinitesimal generator of the two dimensional diffusion (X, Y) is given by

$$\mathcal{L} = \frac{y}{2}\left(\frac{\partial^2}{\partial x^2} + 2\rho\sigma\frac{\partial^2}{\partial y\partial x} + \sigma^2\frac{\partial^2}{\partial y^2}\right) + \kappa(\theta - y)\frac{\partial}{\partial y} + \left(\frac{\rho\kappa\theta}{\sigma} - \frac{y}{2}\right)\frac{\partial}{\partial x},$$

which is defined on the open set $\mathcal{O} := \mathbb{R}\times(0,\infty)$. Note that $\mathcal{L}$ has unbounded coefficients and is not uniformly elliptic: it degenerates on the boundary $\partial\mathcal{O} = \mathbb{R}\times\{0\}$.

2.2 American options and variational inequalities

2.2.1 Heuristics

From the optimal stopping theory, we know that the discounted price process $\tilde{P}(t, X_t, Y_t) = e^{-rt}P(t, X_t, Y_t)$ is a supermartingale and that its finite variation part only decreases on the set $P = \psi$. We want to have an analytical interpretation of these features on the function $P(t,x,y)$. So, assume that $P \in \mathcal{C}^{1,2}((0,T)\times\mathcal{O})$. Then, by applying Ito's formula, the finite variation part of $\tilde{P}(t, X_t, Y_t)$ is

$$\left(\frac{\partial\tilde{P}}{\partial t} + \mathcal{L}\tilde{P}\right)(t, X_t, Y_t).$$

Since $\tilde{P}$ is a supermartingale, we can deduce the inequality

$$\frac{\partial \tilde{P}}{\partial t} + \mathcal{L}\tilde{P} \le 0$$

and, since its finite variation part decreases only on the set $P(t, X_t, Y_t) = \psi(t, X_t, Y_t)$, we can write

$$\left(\frac{\partial \tilde{P}}{\partial t} + \mathcal{L}\tilde{P} \right) (\psi - P) = 0.$$

This relation has to be satisfied $dt - a.e.$ along the trajectories of (t, X_t, Y_t). Moreover, we have the two trivial conditions $P(T, x, y) = \psi(T, x, y)$ and $P \ge \psi$.

The previous discussion is only heuristic, since the price function P is not regular enough to apply the Ito's formula. However, it suggests the following strategy:

1. Study the obstacle problem

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u \le 0, \qquad u \ge \psi, & in\ [0,T] \times \mathcal{O}, \\ \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(\psi - u) = 0, & in\ [0,T] \times \mathcal{O}, \\ u(T, x, y) = \psi(T, x, y). \end{cases} \tag{2.3}$$

2. Show that the discounted price function $\tilde{P}$ is equal to the solution of (2.3) where ψ is replaced by $\tilde{\psi}(t, x, y) = e^{-rt}\psi(t, x, y)$.

We will follow this program providing a variational formulation of system (2.3).

2.2.2 Weighted Sobolev spaces and bilinear form associated with the Heston operator

We consider the measure first introduced in [5]:

$$\mathfrak{m}_{\gamma,\mu}(dx, dy) = y^{\beta-1} e^{-\gamma|x| - \mu y} dx dy,$$

with $\gamma > 0$, $\mu > 0$ and $\beta := \frac{2\kappa\theta}{\sigma^2}$. It will be clear later on that this measure in some sense describes the qualitative behaviour of the process (X, Y) near the degenerate boundary. For $u \in \mathbb{R}^n$ we denote by $|u|$ the standard euclidean norm of u in $\mathbb{R}^n$. The relevant Sobolev spaces are defined as follows (see [5]).

Definition 2.1. *For every $p \ge 1$ let $L^p(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})$ be the space of all measurable functions $u : \mathcal{O} \to \mathbb{R}$ for which*

$$\|u\|^p_{L^p(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})} := \int_{\mathcal{O}} |u|^p d\mathfrak{m}_{\gamma,\mu} < \infty,$$

and denote $H^0(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}) := L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})$.

1. *If $\nabla u := (u_x, u_y)$ and u_x, u_y are defined in the sense of distributions, we set*

$$H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}) := \{u \in L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}) : \sqrt{1+y}u \text{ and } \sqrt{y}|\nabla u| \in L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})\},$$

and

$$\|u\|^2_{H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})} := \int_{\mathcal{O}} \left(y|\nabla u|^2 + (1+y)u^2 \right) d\mathfrak{m}_{\gamma,\mu}.$$

2. *If $D^2u := (u_{xx}, u_{xy}, u_{yx}, u_{yy})$ and all derivatives of u are defined in the sense of distributions, we set*

$$H^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}) := \{u \in L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}) : \sqrt{1+y}u,\ (1+y)|\nabla u|,\ y|D^2u| \in L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})\}$$

and

$$\|u\|^2_{H^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})} := \int_{\mathcal{O}} \left(y^2|D^2u|^2 + (1+y)^2|\nabla u|^2 + (1+y)u^2 \right) d\mathfrak{m}_{\gamma,\mu}.$$

For brevity and when the context is clear, we shall often denote

$$H := H^0(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}), \qquad V := H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})$$

and

$$\|u\|_H := \|u\|_{L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})}, \qquad \|u\|_V := \|u\|_{H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})}.$$

Note that the spaces $H^k(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})$, for $k = 0, 1, 2$ are Hilbert spaces with the inner products

$$(u, v)_H = (u, v)_{L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})} = \int_{\mathcal{O}} uv d\mathfrak{m}_{\gamma,\mu},$$

$$(u, v)_V = (u, v)_{H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})} = \int_{\mathcal{O}} (y (\nabla u, \nabla v) + (1 + y)uv)\, d\mathfrak{m}_{\gamma,\mu}$$

and

$$(u, v)_{H^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})} := \int_{\mathcal{O}} \left(y^2 \left(D^2 u, D^2 v\right) + (1 + y)^2 (\nabla u, \nabla v) + (1 + y)uv\right) d\mathfrak{m}_{\gamma,\mu},$$

where $(\cdot, \cdot)$ denotes the standard scalar product in $\mathbb{R}^n$. Moreover, note that

$$H^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}) \subset H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu}).$$

We can now introduce the bilinear form associated with the differential operator $\mathcal{L}$.

Definition 2.2. *For any $u, v \in H^1(\mathcal{O}, \mathfrak{m}_{\gamma,\mu})$ we define the bilinear form*

$$\begin{aligned} a_{\gamma,\mu}(u, v) =& \frac{1}{2} \int_{\mathcal{O}} y \left(u_x v_x(x, y) + \rho\sigma u_x v_y(x, y) + \rho\sigma u_y v_x(x, y) + \sigma^2 u_y v_y(x, y)\right) d\mathfrak{m}_{\gamma,\mu} \\ &+ \int_{\mathcal{O}} y \left(j_{\gamma,\mu}(x) u_x(x, y) + k_{\gamma,\mu}(x) u_y(x, y)\right) v(x, y) d\mathfrak{m}_{\gamma,\mu}, \end{aligned}$$

where

$$j_{\gamma,\mu} = \frac{1}{2} \left(1 - \gamma sgn(x) - \mu\rho\sigma\right), \qquad k_{\gamma,\mu} = \kappa - \frac{\gamma\rho\sigma}{2} sgn(x) - \frac{\mu\sigma^2}{2}.$$

We will prove that for every $u \in H^2(\mathcal{O}, \mathfrak{m})$ and for every $v \in H^1(\mathcal{O}, \mathfrak{m})$, we have

$$(\mathcal{L}u, v)_H = -a_{\gamma,\mu}(u, v).$$

In order to simplify the notation, from now on we fix γ and μ and we write $\mathfrak{m}$ and a instead of $\mathfrak{m}_{\gamma,\mu}$ and $a_{\gamma,\mu}$.

2.3 Variational formulation of the American price

Fix $T > 0$. We consider an assumption on the payoff function ψ which will be crucial in the discussion of the penalized problem.

Assumption $\mathcal{H}^1$. We say that a function ψ satisfies Assumption $\mathcal{H}^1$ if $\psi \in \mathcal{C}([0, T]; H)$, $\sqrt{1 + y}\psi \in L^2([0, T]; V)$, $\psi(T) \in V$ and there exists $\Psi \in L^2([0, T]; V)$ such that $\left|\frac{\partial\psi}{\partial t}\right| \leq \Psi$.

We will also need a domination condition on ψ by a function Φ which satisfies the following assumption.

Assumption $\mathcal{H}^2$. We say that a function $\Phi \in L^2([0, T]; H^2(\mathcal{O}, \mathfrak{m}))$ satisfies Assumption $\mathcal{H}^2$ if $(1 + y)^{\frac{3}{2}}\Phi \in L^2([0, T]; H)$, $\frac{\partial\Phi}{\partial t} + \mathcal{L}\Phi \leq 0$ and $\sqrt{1 + y}\Phi \in L^\infty([0, T]; L^2(\mathcal{O}, \mathfrak{m}_{\gamma,\mu'}))$ for some $0 < \mu' < \mu$.

The domination condition is needed to deal with the lack of corercivity of the bilinear form associated with our problem. Similar conditions are also used in [5].

The first step in the variational formulation of the problem is to introduce the associated variational inequality and to prove the following existence and uniqueness result.

Theorem 2.3. *Assume that ψ satisfies Assumption $\mathcal{H}^1$ together with $0 \le \psi \le \Phi$, where Φ satisfies Assumption $\mathcal{H}^2$. Then, there exists a unique function u such that $u \in \mathcal{C}([0,T];H) \cap L^2([0,T];V)$, $\frac{\partial u}{\partial t} \in L^2([0,T];H)$ and*

$$\begin{cases} -\left(\frac{\partial u}{\partial t}, v-u\right)_H + a(u, v-u) \ge 0, & a.e. \text{ in } [0,T] \quad v \in V,\ v \ge \psi, \\ u \ge \psi \text{ a.e. in } [0,T] \times \mathbb{R} \times (0,\infty), \\ u(T) = \psi(T), \\ 0 \le u \le \Phi. \end{cases} \tag{2.4}$$

The proof is presented in Section 3 and essentially relies on the penalization technique introduced by Bensoussan and Lions (see also [8]) with some technical devices due to the degenerate nature of the problem. We extend in the parabolic framework the results obtained in [5] for the elliptic case.

The second step is to identify the unique solution of the variational inequality (2.4) as the solution of the optimal stopping problem, that is the (discounted) American option price.

Recall that an adapted right continuous process $(Z_t)_{t\ge0}$ is said to be of class $\mathcal{D}$ if the family $(Z_\tau)_{\tau\in\mathcal{T}_{0,\infty}}$, where $\mathcal{T}_{0,\infty}$ is the set of all stopping times with values in $[0,\infty)$, is uniformly integrable. We introduce the following further assumption:

Assumption $\mathcal{H}^*$. We say that a function $\Phi : [0,T] \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfies Assumption $\mathcal{H}^*$ if Φ is continuous and, for all $(t,x,y) \in [0,T] \times \mathbb{R} \times [0,\infty)$, the process $(\Phi(t+s, X_s^{t,x,y}, Y_s^{t,y}))_{s\in[0,T-t]}$ is of class $\mathcal{D}$.

Assumption $\mathcal{H}^*$ is crucial in order to get the following identification result.

Theorem 2.4. *Fix $p > \beta + \frac{5}{2}$. Assume that, in addition to the assumptions of Theorem 2.3, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of continuous functions on $[0,T] \times \mathbb{R} \times [0,\infty)$ which converges uniformly to ψ and satisfies the following properties for each $n \in \mathbb{N}$:*

1. *ψ_n satisfies Assumption $\mathcal{H}^1$ and $0 \le \psi_n \le \Phi_n$ for some Φ_n satisfying Assumption $\mathcal{H}^2$, Assumption $\mathcal{H}^*$ and $(1+y)\Phi_n \in L^p([0,T]; L^p(\mathcal{O}, \mathfrak{m}))$;*

2. *$\psi_n \in L^2([0,T], H^2(\mathcal{O}, \mathfrak{m}))$ and $\frac{\partial \psi_n}{\partial t} + \mathcal{L}\psi_n \in L^p([0,T]; L^p(\mathcal{O}, \mathfrak{m}))$.*

Then, the solution u of the variational inequality (2.4) associated with ψ is continuous and coincides with the function u^* defined by

$$u^*(t,x,y) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[\psi(\tau, X_\tau^{t,x,y}, Y_\tau^{t,x,y})\right].$$

We conclude this overview with a natural remark. The assumptions on ψ in Theorem 2.3 and Theorem 2.4 seem to be very stringent but we will see that, by choosing γ large enough, they are satisfied by the class of payoff functions $\psi = \psi(t,x) = e^{-rt}\bar{\psi}(x + \bar{c}t)$, where $\bar{c} = r - \delta - \frac{\rho\kappa\theta}{\sigma}$ as defined in (2.1), $\bar{\psi}$ is continuous, positive and such that

$$|\bar{\psi}| + \left|\bar{\psi}'\right| \le C\left(e^x + 1\right),$$

with $C > 0$. Note that the standard call and put payoff functions fall into this category (see Remark 4.17).

3 Existence and uniqueness of solutions to the variational inequality

3.1 Integration by parts and energy estimates

The following result justifies the definition of the bilinear form a.

Proposition 3.1. *If $u \in H^2(\mathcal{O}, \mathfrak{m})$ and $v \in H^1(\mathcal{O}, \mathfrak{m})$, we have*

$$(\mathcal{L}u, v)_H = -a(u,v). \tag{3.1}$$

Before proving Proposition 3.1, we show some preliminary results. The first one is about the standard regularization of a function by convolution.

Lemma 3.2. *Let $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be a C^∞ function with compact support in $[-1,+1] \times [-1,0]$ and such that $\int\int \varphi(x,y)dxdy = 1$. For $j \in \mathbb{N}$ we set $\varphi_j(x,y) = j^2\varphi(jx,jy)$. Then, for every function u locally square-integrable on $\mathbb{R} \times (0,\infty)$ and for every compact set K, we have*

$$\lim_{j\to\infty} \int\int_K (\varphi_j * u - u)^2(x,y)dxdy = 0.$$

Proof. We first observe that

$$\begin{aligned}\int\int_K (\varphi_j * u)^2(x,y)dxdy &\le \int\int_K dxdy \int\int \varphi_j(\xi,\zeta)u^2(x-\xi,y-\zeta)d\xi d\zeta \\ &= \int\int \varphi_j(\xi,\zeta)d\xi d\zeta \int\int 1_K(x+\xi,y+\zeta)u^2(x,y)dxdy.\end{aligned}$$

We deduce, for j large enough,

$$\int\int_K (\varphi_j * u)^2(x,y)dxdy \le \int\int_{\bar{K}} u^2(x,y)dxdy,$$

where $\bar{K} = \{(x,y) \in \mathcal{O} | d_\infty\big((x,y),K\big) \le \frac{1}{j}\}$. Let ϵ be a positive constant and v be a continuous function such that $\int\int_{\bar{K}}(u(x,y)-v(x,y))^2dxdy \le \epsilon$. We have

$$\begin{aligned}&\int\int_K (\varphi_j * u - u)^2(x,y)dxdy \\ &\quad\le 3\int\int_K(\varphi_j * u - \varphi_j * v)^2(x,y)dxdy + 3\int\int_K(\varphi_j * v - v)^2(x,y)dxdy + 3\int\int_K (v-u)^2(x,y)dxdy \\ &\quad\le 3\int\int_{\bar{K}}(v-u)^2(x,y)dxdy + 3\int\int_K(\varphi_j * v - v)^2(x,y)dxdy + 3\int\int_{\bar{K}}(v-u)^2(x,y)dxdy \\ &\quad\le 6\epsilon + 3\int\int_K(\varphi_j * v - v)^2(x,y)dxdy.\end{aligned}$$

Since v is continuous, we have $|\varphi_j * v| \le \sup_{x,y\in\bar{K}}|v(x,y)|$ and $\varphi_j * v(x,y) \to v(x,y)$ on K. Therefore, by Lebesgue Theorem, we can pass to the limit in the above inequality and we get

$$\limsup_{j\to\infty} \int\int_K (\varphi_j * u - u)^2(x,y)dxdy \le 6\epsilon,$$

which completes the proof. □

Then, the following two propositions justify the integration by parts formulas with respect to the measure $\mathfrak{m}$.

Proposition 3.3. *Let us consider $u, v : \mathcal{O} \to \mathbb{R}$ locally square-integrable on $\mathcal{O}$, with derivatives u_x and v_x locally square-integrable on $\mathcal{O}$ as well. Moreover, assume that*

$$\int_{\mathcal{O}} \big(|u_x(x,y)v(x,y)| + |u(x,y)v_x(x,y)| + |u(x,y)v(x,y)|\big)d\mathfrak{m} < \infty.$$

Then, we have

$$\int_{\mathcal{O}} u_x(x,y)v(x,y)d\mathfrak{m} = -\int_{\mathcal{O}} u(x,y)\left(v_x(x,y) - \gamma sgn(x)v\right)d\mathfrak{m}. \tag{3.2}$$

Proof. First we assume that v has compact support in $\mathbb{R} \times (0,\infty)$. For any $j \in \mathbb{N}$ we consider the C^∞ functions $u_j = \varphi_j * u$ and $v_j = \varphi_j * v$, with φ_j as in Lemma 3.2. Note that supp $v_j \subset$ supp $g+$supp φ_j and so, for j large enough, supp $v_j \subset \mathbb{R} \times (0,\infty)$. For any $\epsilon > 0$, integrating by parts, we have

$$\int_{-\infty}^{\infty}(u_j)_x(x,y)v_j(x,y)e^{-\gamma\sqrt{x^2+\epsilon}}dx = -\int_{-\infty}^{\infty} u_j\left((v_j)_x(x,y) - \gamma\frac{x}{\sqrt{x^2+\epsilon}}v_j(x,y)\right)e^{-\gamma\sqrt{x^2+\epsilon}}dx,$$

and, letting $\epsilon \to 0$,

$$\int_{-\infty}^{\infty} (u_j)_x(x,y)v_j(x,y)e^{-\gamma|x|}dx = -\int_{-\infty}^{\infty} u_j\big((v_j)_x(x,y) - \gamma sgn(x)v_j(x,y)\big)e^{-\gamma|x|}dx.$$

Multiplying by $y^{\beta-1}e^{-\mu y}$ and integrating in y we obtain

$$\int_{\mathcal{O}} (u_j)_x(x,y)v_j(x,y)d\mathfrak{m} = -\int_{\mathcal{O}} u_j\big((v_j)_x(x,y) - \gamma sgn(x)v_j(x,y)\big)d\mathfrak{m}.$$

Recall that, for j large enough, v_j has compact support in $\mathbb{R}\times(0,\infty)$ and $\mathfrak{m}$ is bounded on this compact. By using Lemma 3.2, letting $j \to \infty$ we get

$$\int_{\mathcal{O}} u_x(x,y)v(x,y)d\mathfrak{m} = -\int_{\mathcal{O}} u\big(v_x(x,y) - \gamma sgn(x)v(x,y)\big)d\mathfrak{m}.$$

Now let us consider the general case of a function v without compact support. We introduce a $C^\infty-$function α with values in $[0,1]$, $\alpha(x,y)=0$ for all $(x,y)\notin[-2,+2]\times[-2,+2]$, $\alpha(x,y)=1$ for all $(x,y)\in[-1,+1]\times[-1,+1]$ and a $C^\infty-$function χ with values in $[0,1]$, $\chi(y)=0$ for all $y\in[0,\frac{1}{2}]$, $\chi(y)=1$ for all $y\in[+1,\infty)$. We set

$$A_j(x,y) = \alpha\left(\frac{x}{j},\frac{y}{j}\right)\chi(jy), \qquad j\in\mathbb{N}.$$

For every $j\in\mathbb{N}$, A_j has compact support in $\mathcal{O}$ and we have

$$\int_{\mathcal{O}} u_x(x,y)A_j(x,y)v(x,y)d\mathfrak{m}$$

$$= -\int_{\mathcal{O}} u(x,y)\big(u_x(x,y) - \gamma sgn(x)v(x,y)\big)A_j(x,y)d\mathfrak{m} - \int_{\mathcal{O}} u(x,y)v(x,y)(A_j)_x(x,y)d\mathfrak{m}.$$

The function A_j is bounded by $\|\alpha\|_\infty\|\chi\|_\infty$ and $\lim_{j\to+\infty}A_j(x,y) = 1$ for every $(x,y)\in\mathcal{O}$. Moreover $(A_j)_x(x,y) = \frac{1}{j}\alpha_x\left(\frac{x}{j},\frac{y}{j}\right)\chi(jy)$, so that

$$\left|\int_{\mathcal{O}} u(x,y)v(x,y)(A_j)_x(x,y)d\mathfrak{m}\right| \le \frac{C}{j}\int_{\mathcal{O}} 1_{\{|x|\ge j\}}|u(x,y)v(x,y)|d\mathfrak{m},$$

where $C = \|\alpha_x\|_\infty\|\chi\|_\infty$. Therefore, we obtain (3.2) letting $j\to\infty$. $\square$

Proposition 3.4. *Let us consider $u,v:\mathcal{O}\to\mathbb{R}$ locally square-integrable on $\mathcal{O}$, with derivatives u_y and v_y locally square-integrable on $\mathcal{O}$ as well. Moreover, assume that*

$$\int_{\mathcal{O}} y\big(|u_y(x,y)v(x,y)| + |u(x,y)v_xy(x,y)|\big) + |u(x,y)v(x,y)|d\mathfrak{m} < \infty.$$

Then, we have

$$\int_{\mathcal{O}} yu_y(x,y)v(x,y)d\mathfrak{m} = -\int_{\mathcal{O}} yu(x,y)v_yd\mathfrak{m} - \int_{\mathcal{O}} (\beta-\mu y)u(x,y)v(x,y)d\mathfrak{m}. \tag{3.3}$$

Proof. If v has compact support in $\mathcal{O}$, we obtain (3.3) as in the proof of Proposition 3.3. On the other hand, if v has not compact support,

$$\int_{\mathcal{O}} yu_y(x,y)v(x,y)A_j(x,y)d\mathfrak{m} = -\int_{\mathcal{O}} yu(x,y)v_y(x,y)A_j(x,y)d\mathfrak{m}$$
$$-\int_{\mathcal{O}} (\beta-\mu y)u(x,y)v(x,y)A_j(x,y)d\mathfrak{m} - \int_{\mathcal{O}} yu(x,y)v(x,y)(A_j)_y(x,y)d\mathfrak{m},$$

where $A_j(x,y)=\alpha(\frac{x}{j},\frac{y}{j})\chi(jy)$, as in the proof of Proposition 3.3 but choosing χ such that, moreover, $\|y\chi'(y)\|_\infty<\infty$. We have $(A_j)_y(x,y)=\frac{1}{j}\alpha_y(\frac{x}{j},\frac{y}{j})\chi(jy)+j\alpha(\frac{x}{j},\frac{y}{j})\chi'(jy)$. Note that

$$\left|\int_{\mathcal{O}} yu(x,y)v(x,y)j\alpha\left(\frac{x}{j},\frac{y}{j}\right)\chi'(jy)d\mathfrak{m}\right|\le\int_{\mathcal{O}}1_{y\le\frac{1}{j}}|u(x,y)v(x,y)|\|\alpha\|_\infty\sup_{\zeta>0}|\zeta\chi'(\zeta)|d\mathfrak{m}.$$

The last expression goes to 0 as $j\to\infty$ since $\int_{\mathcal{O}}|u(x,y)v(x,y)|d\mathfrak{m}<\infty$. The assertion follows by passing to the limit $j\to\infty$. $\square$

We can now prove Proposition 3.1.

Proof of Proposition 3.1. By using Proposition 3.3 and Proposition 3.4 we have

$$\int_{\mathcal{O}} y\frac{\partial^2 u}{\partial x^2}vd\mathfrak{m}=-\int_{\mathcal{O}} y\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\gamma sgn(x)v\right)d\mathfrak{m},$$

$$\int_{\mathcal{O}} y\frac{\partial^2 u}{\partial y^2}vd\mathfrak{m}=-\int_{\mathcal{O}} y\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}d\mathfrak{m}_{\gamma,\mu}+\int_{\mathcal{O}}(\mu y-\beta)\frac{\partial u}{\partial y}vd\mathfrak{m},$$

$$\int_{\mathcal{O}} y\frac{\partial^2 u}{\partial x\partial y}vd\mathfrak{m}_{\gamma,\mu}=-\int_{\mathcal{O}} y\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x}-\gamma sgn(x)v\right)d\mathfrak{m}$$

and

$$\int_{\mathcal{O}} y\frac{\partial^2 u}{\partial x\partial y}vd\mathfrak{m}_{\gamma,\mu}=-\int_{\mathcal{O}} y\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}d\mathfrak{m}+\int_{\mathcal{O}}(\mu y-\beta)\frac{\partial u}{\partial x}vd\mathfrak{m}.$$

Recalling that

$$\mathcal{L}=\frac{y}{2}\left(\frac{\partial^2}{\partial x^2}+2\rho\sigma\frac{\partial^2}{\partial x\partial y}+\sigma^2\frac{\partial^2}{\partial y^2}\right)+\left(\frac{\rho\kappa\theta}{\sigma}-\frac{y}{2}\right)\frac{\partial}{\partial x}+\kappa(\theta-y)\frac{\partial}{\partial y}$$

and using the equality $\beta=2\kappa\theta/\sigma^2$, we have

$$\begin{aligned}(\mathcal{L}u,v)_H &= -\int_{\mathcal{O}}\frac{y}{2}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\sigma^2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\rho\sigma\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\rho\sigma\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)d\mathfrak{m}\\ &\quad+\int_{\mathcal{O}}\frac{1}{2}\frac{\partial u}{\partial x}\left(y\gamma sgn(x)+\rho\sigma(\mu y-\beta)\right)vd\mathfrak{m}\\ &\quad+\int_{\mathcal{O}}\frac{1}{2}\frac{\partial u}{\partial y}\left(\mu\sigma^2 y-\beta\sigma^2+\rho\sigma y\gamma sgn(x)\right)vd\mathfrak{m}\\ &\quad+\int_{\mathcal{O}}\left[\left(\frac{\rho\kappa\theta}{\sigma}-\frac{y}{2}\right)\frac{\partial u}{\partial x}+\kappa(\theta-y)\frac{\partial u}{\partial y}\right]vd\mathfrak{m}\\ &= -a(u,v).\end{aligned}$$

$\square$

Remark 3.5. *It is now clear why we have considered the process $X_t=\log S_t-\bar{c}t$ instead of the standard log-price process $\log S_t$. Actually, the choice of $\bar{c}$ allows to avoid terms of the type $\int(u_x+u_y)d\mathfrak{m}$ in the associated bilinear form a. This trick will be crucial in order to obtain suitable energy estimates.*

Recall the well known inequality

$$bc=(\sqrt{\zeta}b)\left(\frac{1}{\sqrt{\zeta}c}\right)\le\frac{\zeta}{2}b^2+\frac{1}{2\zeta}c^2,\qquad b,c\in\mathbb{R},\ \zeta>0. \tag{3.4}$$

Hereafter we will often apply (3.4) in the proofs even if it is not explicitly recalled each time.

Proposition 3.6. *For every $u, v \in V$, the bilinear form $a(\cdot,\cdot)$ satisfies*

$$|a(u,v)| \leq C_1 \|u\|_V \|v\|_V, \tag{3.5}$$

$$a(u,u) \geq C_2 \|u\|_V^2 - C_3 \|(1+y)^{\frac{1}{2}} u\|_H^2, \tag{3.6}$$

where

$$C_1 = \delta_0 + K_1, \quad C_2 = \frac{\delta_1}{2}, \quad C_3 = \frac{\delta_1}{2} + \frac{K_1^2}{2\delta_1},$$

with

$$\delta_0 = \sup_{s_1^2+t_1^2>0,\ s_2^2+t_2^2>0} \frac{|s_1 s_2 + \rho\sigma s_1 t_2 + \rho\sigma s_2 t_1 + \sigma^2 t_1 t_2|}{2\sqrt{(s_1^2+t_1^2)(s_2^2+t_2^2)}}, \tag{3.7}$$

$$\delta_1 = \inf_{s^2+t^2>0} \frac{s^2 + 2\rho\sigma st + \sigma^2 t^2}{2(s^2+t^2)}, \tag{3.8}$$

and

$$K_1 = \sup_{(x,y)\in\mathbb{R}\times]0,+\infty[} \sqrt{j_{\gamma,\mu}^2(x,y) + k_{\gamma,\mu}^2(x,y)}. \tag{3.9}$$

Proof. Recall that

$$\begin{aligned} a(u,v) =& \frac{1}{2}\int_{\mathcal{O}} y\left(u_x v_x + \rho\sigma u_x v_y + \rho\sigma u_y v_x + \sigma^2 u_y v_y\right) d\mathfrak{m} \\ &+ \int_{\mathcal{O}} y\left(j_{\gamma,\mu}(x) u_x(x,y) + k_{\gamma,\mu}(x) u_y(x,y)\right) v(x,y) d\mathfrak{m}. \end{aligned}$$

We can easily see that

$$\left|\frac{1}{2}\int_{\mathcal{O}} y\left(u_x v_x + \rho\sigma u_x v_y + \rho\sigma u_y v_x + \sigma^2 u_y v_y\right) d\mathfrak{m}\right| \leq \delta_0 \int_{\mathcal{O}} y|\nabla u||\nabla v| d\mathfrak{m} \leq \delta_0 \|u\|_V \|v\|_V$$

and

$$\left|\int_{\mathcal{O}} y\left(j_{\gamma,\mu}(x) u_x(x,y) + k_{\gamma,\mu}(x) u_y(x,y)\right) v(x,y) d\mathfrak{m}\right| \leq K_1 \int_{\mathcal{O}} y|\nabla u||v| d\mathfrak{m} \leq K_1 \|u\|_V \|v\|_V.$$

Then (3.5) immediately follows. In order to prove (3.6), we note that

$$\frac{1}{2}\int_{\mathcal{O}} y\left(u_x v_x + \rho\sigma u_x v_y + \rho\sigma u_y v_x + \sigma^2 u_y v_y\right) d\mathfrak{m} \geq \delta_1 \int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m}.$$

Therefore

$$\begin{aligned} a(u,u) &\geq \delta_1 \int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m} - K_1 \int_{\mathcal{O}} y|\nabla u||u| d\mathfrak{m} \\ &\geq \delta_1 \int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m} - \frac{K_1\zeta}{2}\int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m} - \frac{K_1}{2\zeta}\int_{\mathcal{O}} (1+y)u^2 d\mathfrak{m} \\ &= \left(\delta_1 - \frac{K_1\zeta}{2}\right)\int_{\mathcal{O}} \left(y|\nabla u|^2 + (1+y)u^2\right) d\mathfrak{m} - \left(\delta_1 - \frac{K_1\zeta}{2} + \frac{K_1}{2\zeta}\right)\int_{\mathcal{O}} (1+y)u^2 d\mathfrak{m}. \end{aligned}$$

Choosing $\zeta = \delta_1/K_1$ we have

$$a(u,u) \geq \frac{\delta_1}{2}\|u\|_V^2 - \left(\frac{\delta_1}{2} + \frac{K_1^2}{2\delta_1}\right)\|\sqrt{1+y}\,u\|_H^2$$

and the assertion is proved. ∎

3.2 Proof of Theorem 2.3

Among the standard assumptions required in [4] for the penalization procedure, there are the coercivity and the symmetry of the bilinear form a and the boundedness of the coefficients. In the Heston-type models these assumptions are no longer satisfied and this leads to some technical difficulties. In order to overcome them, we introduce some auxiliary operators.

From now on, we set

$$a(u,v) = \bar{a}(u,v) + \tilde{a}(u,v),$$

where

$$\begin{aligned}
\bar{a}(u,v) &= \int_{\mathcal{O}} \frac{y}{2}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \rho\sigma\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \rho\sigma\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \sigma^2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\mathfrak{m},\\
\tilde{a}(u,v) &= \int_{\mathcal{O}} y\frac{\partial u}{\partial x} j_{\gamma,\mu} v d\mathfrak{m} + \int_{\mathcal{O}} y\frac{\partial u}{\partial y} k_{\gamma,\mu} v d\mathfrak{m}.
\end{aligned}$$

Note that $\bar{a}$ is symmetric. We have, for every $u, v \in V$,

$$|\bar{a}(u,v)| \leq \delta_0 \int_{\mathcal{O}} y|\nabla u||\nabla v| d\mathfrak{m},$$

$$\bar{a}(u,u) \geq \delta_1 \int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m},$$

and

$$|\tilde{a}(u,v)| \leq K_1 \int_{\mathcal{O}} y|\nabla u||v| d\mathfrak{m},$$

with δ_0, δ_1 and K_1 defined in Proposition 3.6. Then, we introduce for $\lambda \geq 0$ and $M > 0$,

$$\begin{aligned}
a_\lambda(u,v) &= a(u,v) + \lambda \int_{\mathcal{O}} (1+y) uv d\mathfrak{m},\\
\bar{a}_\lambda(u,v) &= \bar{a}(u,v) + \lambda \int_{\mathcal{O}} (1+y) uv d\mathfrak{m},\\
\tilde{a}^{(M)}(u,v) &= \int_{\mathcal{O}} y \wedge M \left(\frac{\partial u}{\partial x} j_{\gamma,\mu} + \frac{\partial u}{\partial y} k_{\gamma,\mu}\right) v d\mathfrak{m}
\end{aligned}$$

and

$$a_\lambda^{(M)}(u,v) = \bar{a}_\lambda(u,v) + \tilde{a}^{(M)}(u,v).$$

Lemma 3.7. *Let δ_0, δ_1, K_1 be defined as in (3.7), (3.8) and (3.9) respectively. For any fixed $\lambda \geq \frac{\delta_1}{2} + \frac{K_1^2}{2\delta_1}$ the bilinear forms a_λ and $a_\lambda^{(M)}$ are continuous and coercive. More precisely, we have*

$$|a_\lambda(u,v)| \leq C\|u\|_V\|v\|_V, \qquad u, v \in V, \tag{3.10}$$

$$a_\lambda(u,u) \geq \frac{\delta_1}{2}\|v\|_V^2, \qquad u \in V, \tag{3.11}$$

and

$$|a_\lambda^{(M)}(u,v)| \leq C\|u\|_V\|v\|_V, \qquad u, v \in V, \tag{3.12}$$

$$a_\lambda^{(M)}(u,u) \geq \frac{\delta_1}{2}\|v\|_V^2, \qquad u \in V. \tag{3.13}$$

where $C = \delta_0 + K_1 + \lambda$.

Proof. Note that, for every $u, v \in V$,

$$|\tilde{a}^{(M)}(u,v)| \leq K_1 \int_{\mathcal{O}} y|\nabla u||v| d\mathfrak{m},$$

so that

$$\begin{aligned}|a_\lambda^{(M)}(u,v)| &\leq |\bar{a}(u,v)| + \lambda \int_{\mathcal{O}} (1+y)|u||v| d\mathfrak{m} + K_1 \int_{\mathcal{O}} y|\nabla u||v| d\mathfrak{m} \\ &\leq \delta_0 \int_{\mathcal{O}} y|\nabla u||\nabla v| d\mathfrak{m} + \lambda \int_{\mathcal{O}} (1+y)|u||v| d\mathfrak{m} + K_1 \int_{\mathcal{O}} y|\nabla u||v| d\mathfrak{m} \\ &\leq (\delta_0 + \lambda + K_1)\|u\|_V \|v\|_V.\end{aligned}$$

On the other hand, for every $\zeta > 0$,

$$\begin{aligned}a_\lambda^{(M)}(u,u) &\geq \delta_1 \int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m} + \lambda \int_{\mathcal{O}} (1+y)u^2 dm_{\gamma,\mu} - K_1 \int_{\mathcal{O}} y|\nabla u||u| d\mathfrak{m} \\ &\geq \left(\delta_1 - \frac{K_1\zeta}{2}\right) \int_{\mathcal{O}} y|\nabla u|^2 d\mathfrak{m} + \left(\lambda - \frac{K_1}{2\zeta}\right) \int_{\mathcal{O}} (1+y)u^2 d\mathfrak{m}.\end{aligned}$$

By choosing $\zeta = \delta_1/K_1$, we get

$$a_\lambda^{(M)}(u,u) \geq \frac{\delta_1}{2} \int_{\mathcal{O}} y|\nabla u|^2 dm + \left(\lambda - \frac{K_1^2}{2\delta_1}\right) \int_{\mathcal{O}} (1+y)u^2 d\mathfrak{m}.$$

We deduce that, if $\lambda \geq \frac{\delta_1}{2} + \frac{K_1^2}{2\delta_1}$,

$$a_\lambda^{(M)}(u,u) \geq \frac{\delta_1}{2}\|u\|_V^2.$$

The same calculations hold for the bilinear form a_λ and the assertion is proved. □

Remark 3.8. *Let* $\|a\| = \sup_{u,v\in V, u,v\neq 0} \frac{|a(u,v)|}{\|u\|_V \|v\|_V}$ *be the norm of a bilinear form* $a : V \times V \to \mathbb{R}$. *Then we stress that Lemma 3.7 gives us*

$$\sup_{M>0} \|a_\lambda^{(M)}\| \leq C,$$

where $C = \delta_0 + K_1 + \lambda$.

From now on in the rest of this paper we assume $\lambda \geq \frac{\delta_1}{2} + \frac{K_1^2}{2\delta_1}$ as in Lemma 3.7.

Finally, we define

$$\mathcal{L}^\lambda := \mathcal{L} - \lambda(1+y)$$

the differential operator associated with the bilinear form a_λ, that is

$$(\mathcal{L}^\lambda u, v)_H = -a_\lambda(u,v), \qquad u \in H^2(\mathcal{O},\mathfrak{m}),\ v \in V.$$

3.2.1 Penalized problem

For any fixed $\varepsilon > 0$ we define the penalizing operator

$$\zeta_\varepsilon(u) = -\frac{1}{\varepsilon}(\psi - u)_+ = \frac{1}{\varepsilon}\zeta(u), \qquad u \in V. \tag{3.14}$$

Since the function $x \mapsto -(\psi - x)_+$ is nondecreasing, we easily get the following monotonicity result.

Lemma 3.9. *The penalizing operator* (3.14) *is monotone, in the sense that*

$$(\zeta_\varepsilon(u) - \zeta_\varepsilon(v), u - v)_H \geq 0, \qquad u, v \in V.$$

We now introduce the intermediate penalized coercive problem with a source term g. We consider the following assumption:

Assumption $\mathcal{H}^0$. We say that a function g satisfies Assumption $\mathcal{H}^0$ if $\sqrt{1+y}g \in L^2([0,T];H)$.

Theorem 3.10. *Assume that ψ satisfies Assumption $\mathcal{H}^1$ and g satisfies Assumption $\mathcal{H}^0$. Then, for every fixed $\varepsilon > 0$, there exists a unique function $u_{\varepsilon,\lambda}$ such that $u_{\varepsilon,\lambda} \in \mathcal{C}([0,T];H) \cap L^2([0,T];V)$, $\frac{\partial u_{\varepsilon,\lambda}}{\partial t} \in L^2([0,T];H)$ and*

$$\begin{cases} -\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}(t), v\right)_H + a_\lambda(u_{\varepsilon,\lambda}, v) + (\zeta_\varepsilon(u_{\varepsilon,\lambda}(t)), v)_H = (g(t), v)_H, & \text{a.e. in } [0,T], \quad v \in V, \\ u_{\varepsilon,\lambda}(T) = \psi(T). \end{cases} \tag{3.15}$$

Moreover, the following estimates hold:

$$\|u_{\varepsilon,\lambda}\|_{L^\infty([0,T],V)} \leq K, \tag{3.16}$$

$$\left\|\frac{\partial u_{\varepsilon,\lambda}}{\partial t}\right\|_{L^2([0,T];H)} \leq K, \tag{3.17}$$

$$\frac{1}{\sqrt{\varepsilon}}\left\|(\psi - u_{\varepsilon,\lambda})^+\right\|_{L^\infty([0,T],H)} \leq K, \tag{3.18}$$

where $K = C\left(\|\Psi\|_{L^2([0,T];V)} + \|\sqrt{1+y}g\|_{L^2([0,T];H)} + \|\sqrt{1+y}\psi\|_{L^2([0,T];V)} + \|\psi(T)\|_V^2\right)$, with $C > 0$ independent of ε, and Ψ is given in Assumption $\mathcal{H}^1$.

We first prove uniqueness of the penalized coercive problem.

Proof of Theorem 3.10: uniqueness. Assume that there exist two functions u_1 and u_2 satisfying (3.15) and set $w = u_1 - u_2$. If we choose $v = u_1 - u_2$ in the equation satisfied by u_1 and $v = u_2 - u_1$ in the one satisfied by u_2 and then we add the resulting equations, we get

$$-\left(\frac{\partial w}{\partial t}(t), w(t)\right)_H + a_\lambda(w(t), w(t)) + (\zeta_\varepsilon(u_1(t)) - \zeta_\varepsilon(u_2(t)), w(t))_H = 0.$$

By the coercivity of a_λ and the monotonicity of the penalized operator we deduce that

$$-\left(\frac{\partial w}{\partial t}(t), w(t)\right)_H \leq 0 \Rightarrow \left(\frac{\partial w}{\partial t}(t), w(t)\right)_H = \frac{1}{2}\frac{\partial}{\partial t}\|w(t)\|_H^2 \geq 0.$$

But $w(T) = \psi(T) - \psi(T) = 0$, so $w(t) = 0$ a.e. in $[0,T]$, which means $u_1 = u_2$. □

The proof of existence in Theorem 3.10 is quite long and technical, so we split it into two propositions. We first consider the truncated penalized problem, which requires less stringent conditions on ψ and g.

Proposition 3.11. *Let $\psi \in \mathcal{C}([0,T];H) \cap L^2([0,T];V)$ and $g \in L^2([0,T];H)$. Moreover, assume that $\psi(T) \in H^2(\mathcal{O}, \mathfrak{m})$, $\frac{\partial \psi}{\partial t} \in L^2([0,T];V)$ and $\frac{\partial g}{\partial t} \in L^2([0,T];H)$. Then, there exists a unique function $u_{\varepsilon,\lambda,M}$ such that $u_{\varepsilon,\lambda,M} \in \mathcal{C}([0,T];V) \cup L^2([0,T];V)$, $\frac{\partial u_{\varepsilon,\lambda,M}}{\partial t} \in L^2([0,T];V)$ and*

$$\begin{cases} -\left(\frac{\partial u_{\varepsilon,\lambda,M}}{\partial t}(t), v\right)_H + a_\lambda^{(M)}(u_{\varepsilon,\lambda,M}(t), v) + (\zeta_\varepsilon(u_{\varepsilon,\lambda,M})(t), v)_H = (g(t), v)_H, & \text{a.e. in } [0,T), \quad v \in V, \\ u_{\varepsilon,\lambda,M}(T) = \psi(T). \end{cases} \tag{3.19}$$

Proof. 1. **Finite dimensional problem** We use the classical Galerkin method of approximation, which consists in introducing a nondecreasing sequence $(V_j)_j$ of subspaces of V such that $dimV_j < \infty$ and, for every $v \in V$, there exists a sequence $(v_j)_{j\in\mathbb{N}}$ such that $v_j \in V_j$ for any $j \in \mathbb{N}$ and $\|v - v_j\|_V \to 0$ as $j \to \infty$. Moreover, we assume that $\psi(T) \in V_j$, for all $j \in \mathbb{N}$. Let P_j be the projection of V onto V_j

and $\psi_j(t) = P_j\psi(t)$. We have $\psi_j(t) \to \psi(t)$ strongly in V and $\psi_j(T) = \psi(T)$ for any $j \in \mathbb{N}$. The finite dimensional problem is, therefore, to find $u_j : [0,T] \to V_j$ such that

$$\begin{cases} -\left(\frac{\partial u_j}{\partial t}(t), v\right)_H + a_\lambda^{(M)}(u_j(t), v) - \frac{1}{\varepsilon}((\psi_j(t) - u_j(t))_+, v)_H = (g(t), v)_H, \qquad v \in V_j, \\ u_j(T) = \psi(T). \end{cases} \tag{3.20}$$

This problem can be interpreted as an ordinary differential equation in V_j (dim $V_j < \infty$) and we can easily deduce the existence and the uniqueness of a solution u_j of (3.20), continuous from $[0,T]$ into V_j, a.e. differentiable and with bounded derivatives.

2. **Estimates on the finite dimensional problem** First, we take $v = u_j(t) - \psi_j(t)$ in (3.20). We get

$$-\left(\frac{\partial u_j}{\partial t}(t), u_j(t) - \psi_j(t)\right)_H + a_\lambda^{(M)}(u_j(t), u_j(t) - \psi_j(t)) - \frac{1}{\varepsilon}((\psi_j(t) - u_j(t))_+, u_j(t) - \psi_j(t))_H$$
$$= (g(t), u_j(t) - \psi_j(t))_H,$$

which can be rewritten as

$$-\frac{1}{2}\frac{d}{dt}\|u_j(t) - \psi_j(t)\|_H^2 - \left(\frac{\partial \psi_j}{\partial t}(t), u_j(t) - \psi_j(t)\right)_H + a_\lambda^{(M)}(u_j(t) - \psi_j(t), u_j(t) - \psi_j(t))_H$$
$$+ \frac{1}{\varepsilon}((\psi_j(t) - u_j(t))_+, \psi_j(t) - u_j(t))_H + a_\lambda^{(M)}(\psi_j(t), u_j(t) - \psi_j(t)) = (g(t), u_j(t) - \psi_j(t))_H.$$

We integrate between t and T and we use coercivity and $u_j(T) = \psi_j(T)$ to obtain

$$\begin{aligned} &\frac{1}{2}\|u_j(t) - \psi_j(t)\|_H^2 + \frac{\delta_1}{2}\int_t^T \|u_j(s) - \psi_j(s)\|_V^2 ds + \frac{1}{\varepsilon}\int_t^T \|(\psi_j(s) - u_j(s))_+\|_H^2 ds \\ &\leq \int_t^T \left\|\frac{\partial \psi_j(s)}{\partial t}\right\|_H \|u_j(s) - \psi_j(s)\|_H ds + \int_t^T \|g(s)\|_H \|u_j(s) - \psi_j(s)\|_H ds \\ &\quad + \|a_\lambda^{(M)}\| \int_t^T \|\psi_j(s)\|_V \|u_j(s) - \psi_j(s)\|_V ds \\ &\leq \frac{1}{2\zeta}\int_t^T \left\|\frac{\partial \psi_j(s)}{\partial t}\right\|_H^2 ds + \frac{\zeta}{2}\int_t^T \|u_j(s) - \psi_j(s)\|_H^2 ds + \frac{1}{2\zeta}\int_t^T \|g(s)\|_H^2 ds + \frac{\zeta}{2}\int_t^T \|u_j(s) - \psi_j(s)\|_V^2 ds \\ &\quad + \frac{\|a_\lambda^{(M)}\|\zeta}{2}\int_t^T \|u_j(s) - \psi_j(s)\|_H^2 ds + \frac{\|a_\lambda^{(M)}\|}{2\zeta}\int_t^T \|\psi_j(s)\|_V^2 ds, \end{aligned}$$

for any $\zeta > 0$. Recall that $\psi_j = P_j\psi$, and so $\|\psi_j(t)\|_V^2 \leq \|\psi(t)\|_V^2$. In the same way $\|\frac{\partial \psi_j(t)}{\partial t}\|_H^2 \leq \|\frac{\partial \psi_j(t)}{\partial t}\|_V^2 \leq \|\frac{\partial \psi(t)}{\partial t}\|_V^2$. Choosing $\zeta = \frac{\delta_1}{4\left(1 + \frac{\|a_\lambda^{(M)}\|}{2}\right)}$ after simple calculations we deduce that there exists $C > 0$ independent of M, ε and j such that

$$\begin{aligned} &\tfrac{1}{2}\|u_j(t)\|_H^2 + \tfrac{\delta_1}{4}\textstyle\int_t^T \|u_j(s))\|_V^2 ds + \tfrac{1}{\varepsilon}\int_t^T \|(\psi_j(s) - u_j(s))_+\|_H^2 ds \\ &\quad \leq C\left(\left\|\tfrac{\partial \psi}{\partial t}\right\|_{L^2([t,T];V)}^2 + \|g\|_{L^2([t,T];H)}^2 + \|\psi\|_{L^2([t,T];V)}^2\right). \end{aligned} \tag{3.21}$$

We now go back to (3.20) and we take $v = \frac{\partial u_j}{\partial t}(t)$. We have

$$-\left(\frac{\partial u_j}{\partial t}(t), \frac{\partial u_j}{\partial t}(t)\right)_H + a_\lambda^{(M)}\left(u_j(t), \frac{\partial u_j}{\partial t}(t)\right) - \frac{1}{\varepsilon}\left((\psi_j(t) - u_j(t))_+, \frac{\partial u_j}{\partial t}(t)\right)_H = \left(g(t), \frac{\partial u_j}{\partial t}(t)\right)_H,$$

so that

$$-\frac{1}{2}\left\|\frac{\partial u_j}{\partial t}(t)\right\|_H^2+\bar{a}_\lambda\left(u_j(t),\frac{\partial u_j}{\partial t}(t)\right)+\tilde{a}^{(M)}\left(u_j(t),\frac{\partial u_j}{\partial t}(t)\right)-\frac{1}{\varepsilon}\left((\psi_j(t)-u_j(t))_+,\frac{\partial u_j}{\partial t}(t)\right)_H$$
$$=\left(g(t),\frac{\partial u_j}{\partial t}(t)\right)_H.$$

Note that

$$\begin{aligned}-\frac{1}{\varepsilon}\left((\psi_j(t)-u_j(t))_+,\frac{\partial u_j}{\partial t}(t)\right)_H&=\frac{1}{\varepsilon}\left((\psi_j-u_j)_+,\frac{\partial(\psi_j-u_j)}{\partial t}(t)\right)_H-\frac{1}{\varepsilon}\left((\psi_j(t)-u_j(t))_+,\frac{\partial\psi_j}{\partial t}(t)\right)_H\\&=\frac{1}{2\varepsilon}\frac{d}{dt}\|(\psi_j-u_j)_+(t)\|_H^2-\frac{1}{\varepsilon}\left((\psi_j(t)-u_j(t))_+,\frac{\partial\psi_j}{\partial t}(t)\right)_H.\end{aligned}$$

Therefore, using the symmetry of $\bar{a}_\lambda$, we have

$$-\frac{1}{2}\left\|\frac{\partial u_j}{\partial t}(t)\right\|_H^2+\frac{1}{2}\frac{d}{dt}\bar{a}_\lambda(u_j(t),u_j(t))+\tilde{a}^{(M)}\left(u_j(t),\frac{\partial u_j}{\partial t}(t)\right)+\frac{1}{2\varepsilon}\frac{\partial}{\partial t}\|(\psi_j(t)-u_j(t))_+\|_H^2$$
$$-\frac{1}{\varepsilon}\left((\psi_j(t)-u_j(t))_+,\frac{\partial\psi_j}{\partial t}(t)\right)_H=\left(g(t),\frac{\partial u_j}{\partial t}(t)\right)_H.$$

Integrating between t and T, we obtain

$$\begin{aligned}&\int_t^T\frac{1}{2}\left\|\frac{\partial u_j}{\partial t}(s)\right\|_H^2ds+\frac{1}{2}\bar{a}_\lambda(u_j(t),u_j(t))+\frac{1}{2\varepsilon}\|(\psi_j(t)-u_j(t))_+\|_H^2\\&=\int_t^T\tilde{a}^{(M)}\left(u_j(s),\frac{\partial u_j}{\partial s}(s)\right)ds+\frac{1}{2}\bar{a}_\lambda(\psi_j(T),\psi_j(T))-\int_t^T\frac{1}{\varepsilon}\left((\psi_j(s)-u_j(s)_+,\frac{\partial\psi_j}{\partial s}(s)\right)_Hds\\&-\int_t^T\left(g(s),\frac{\partial u_j}{\partial s}(s)\right)_Hds.\end{aligned}$$

Recall that $\bar{a}_\lambda(u_j(t),u_j(t))\geq\delta_1\|u_j(t)\|_V^2$, $\bar{a}_\lambda(\psi_j(T),\psi_j(T))=\bar{a}_\lambda(\psi(T),\psi(T))\leq\|\bar{a_\lambda}\|\|\psi(T)\|_V^2$ and $|\tilde{a}^{(M)}(u,v)|\leq K_1\int_{\mathcal{O}}y\wedge M|\nabla u||v|d\mathfrak{m}$, so that, for every $\zeta>0$,

$$\begin{aligned}&\frac{1}{2}\int_t^T\left\|\frac{\partial u_j}{\partial s}(s)\right\|_H^2ds+\frac{\delta_1}{4}\|u_j(t)\|_V^2+\frac{1}{2\varepsilon}\|(\psi_j(t)-u_j(t))_+\|_H^2\\&\leq K_1\int_t^Tds\int_{\mathcal{O}}y\wedge M|\nabla u_j(s,.)|\left|\frac{\partial u_j}{\partial t}(s,.)\right|d\mathfrak{m}+\frac{\|\bar{a}_\lambda\|}{2}\|\psi(T)\|_V^2+\frac{1}{\varepsilon}\int_t^T\|(\psi_j(s)-u_j(s))_+\|_H\left\|\frac{\partial\psi_j}{\partial s}(s)\right\|_Hds\\&\quad+\int_t^T\|g(s)\|_H\left\|\frac{\partial u_j}{\partial s}(s)\right\|_Hds\\&\leq\frac{K_1}{2\zeta}\int_t^T\|u_j(s)\|_V^2ds+\frac{K_1M}{2}\zeta\int_t^T\left\|\frac{\partial u_j}{\partial s}(s)\right\|_H^2ds+\frac{\|\bar{a}_\lambda\|}{2}\|\psi(T)\|_V^2\\&\quad+\frac{\zeta}{\varepsilon}\int_t^T\|(\psi_j(s)-u_j(s))_+\|_H^2ds+\frac{1}{2\zeta}\int_t^T\left\|\frac{\partial\psi_j}{\partial t}(s)\right\|_H^2ds+\frac{1}{2\zeta}\int_t^T\|g(s)\|_H^2ds+\frac{\zeta}{2}\int_t^T\left\|\frac{\partial u_j}{\partial s}(s)\right\|_H^2ds.\end{aligned}$$

From (3.21), we already know that

$$\int_t^T\|u_j(s)\|_V^2ds+\frac{1}{\varepsilon}\int_t^T\|(\psi_j(s)-u_j(s))_+\|_H^2ds\leq C\left(\left\|\frac{\partial\psi}{\partial t}\right\|_{L^2([t,T];V)}^2+\|g\|_{L^2([t,T];H)}^2+\|\psi\|_{L^2([t,T];V)}^2\right),$$

then we can finally deduce

$$\begin{aligned}&\int_t^T \left\|\frac{\partial u_j}{\partial t}(s)\right\|_H^2 ds + \|u_j(t)\|_V^2 + \frac{1}{2\varepsilon}\|(\psi_j(t)-u_j(t))_+\|_H^2\\ &\quad\le C_{\varepsilon,M}\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2_{L^2([t,T];V)} + \|g\|^2_{L^2([t,T];H)} + \|\psi\|^2_{L^2([t,T];V)} + \|\psi(T)\|_V^2\right),\end{aligned} \tag{3.22}$$

where $C_{\varepsilon,M}$ is a constant which depends on ε and M but not on j.

We will also need a further estimation. If we denote $\bar u_j = \frac{\partial u_j}{\partial t}$ and we differentiate the equation (3.20) with respect to t for a fixed v independent of t, we obtain that $\bar u_j$ satisfies

$$-\left(\frac{\partial \bar u_j}{\partial t}(t), v\right)_H + a_\lambda^{(M)}(\bar u_j(t), v) - \frac{1}{\varepsilon}\left(\left(\frac{\partial \psi_j}{\partial t}(t) - \bar u_j(t)\right)1_{\{\psi_j(t)\ge u_j(t)\}}, v\right)_H = \left(\frac{\partial g}{\partial t}(t), v\right)_H, \quad v\in V_j. \tag{3.23}$$

As regards the initial condition, from (3.20) computed in $t=T$, for every $v\in V_j$ we have

$$\begin{aligned}\left(\frac{\partial u_j(T)}{\partial t}, v\right)_H &= a_\lambda^{(M)}(\psi(T), v) - (g(T), v)_H.\\ &= -\left(\mathcal{L}\psi(T), v\right)_H + \lambda\left((1+y)\psi(T), v\right)_H + \left((y\wedge M - y)(j_{\gamma,\mu}u_x + k_{\gamma,\mu}u_y), v\right)_H + \left(g(T), v\right)_H.\end{aligned}$$

Choosing $v = \frac{\partial u_j(T)}{\partial t}$, we deduce that

$$\begin{aligned}\left\|\frac{\partial u_j(T)}{\partial t}\right\|_H^2 &\le C\left\|\frac{\partial u_j(T)}{\partial t}\right\|_H \left(\|\mathcal{L}\psi(T)\|_H + \|(1+y)\psi(T)\|_H + \|(y-M)_+\nabla\psi(T)\|_H + \|g(T)\|_H\right)\\ &\le C\left\|\frac{\partial u_j(T)}{\partial t}\right\|_H \left(\|\mathcal{L}\psi(T)\|_H + \|(1+y)\psi(T)\|_H + \|g(T)\|_H\right),\end{aligned}$$

that is, $\left\|\frac{\partial u_j(T)}{\partial t}\right\|_H \le C\left(\|\mathcal{L}\psi(T)\|_H + \|(1+y)\psi(T)\|_H + \|g(T)\|_H\right)$.

We can take $v = \bar u_j(t)$ in (3.23) and we obtain

$$-\left(\frac{\partial \bar u_j}{\partial t}(t), \bar u_j(t)\right)_H + a_\lambda^{(M)}(\bar u_j(t), \bar u_j(t)) - \frac{1}{\varepsilon}\left(\left(\frac{\partial \psi_j}{\partial t}(t) - \bar u_j(t)\right)1_{\{\psi_j(t)\ge u_j(t)\}}, \bar u_j(t)\right)_H = \left(\frac{\partial g}{\partial t}(t), \bar u_j(t)\right)_H,$$

so that

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\|\bar u_j(t)\|_H^2 + \frac{\delta_1}{2}\|\bar u_j(t)\|_V^2 &\le \frac{1}{\varepsilon}\left(\left(\frac{\partial \psi_j}{\partial t}(t) - \bar u_j(t)\right)1_{\{\psi_j(t)\ge u_j\}}, \bar u_j(t)\right)_H + \left(\frac{\partial g}{\partial t}(t), \bar u_j(t)\right)_H\\ &\le \frac{1}{\varepsilon}\left(\frac{\partial \psi_j}{\partial t}(t)1_{\{\psi_j(t)\ge u_j\}}, \bar u_j(t)\right)_H + \left(\frac{\partial g}{\partial t}(t), \bar u_j(t)\right)_H.\end{aligned}$$

Integrating between t and T, with the usual calculations, we obtain, in particular, that

$$\int_t^T \|\bar u_j(s)\|_V^2 ds \le C_\varepsilon\left(\|\mathcal{L}\psi(T)\|_H + \|(1+y)\psi(T)\|_H^2 + \|g(T)\|_H^2 + \left\|\frac{\partial \psi}{\partial t}\right\|_{L^2([t,T];H)} + \left\|\frac{\partial g}{\partial t}\right\|_{L^2([t,T];H)}\right), \tag{3.24}$$

where C_ε is a constant which depends on ε, but not on j.

3. **Passage to the limit**

Let ε and M be fixed. By passing to a subsequence, from (3.22) we can assume that $\frac{\partial u_j}{\partial t}$ weakly converges to a function $u'_{\varepsilon,\lambda,M}$ in $L^2([0,T];H)$. We deduce that, for any fixed $t\in[0,T]$, $u_j(t)$ weakly converges in H to

$$u_{\varepsilon,\lambda,M}(t) = \psi(T) - \int_t^T u'_{\varepsilon,\lambda,M}(s)ds.$$

Indeed, $u_j(t)$ is bounded in V, so the convergence is weakly in V. Passing to the limit in (3.24) we deduce that $\frac{\partial u_{\varepsilon,\lambda,M}}{\partial t} \in L^2([0,T];V)$. Moreover, from (3.22), we have that $(\psi_j - u_j(t))^+$ weakly converges in H to a certain function $\chi \in H$. Now, for any $v \in V$ we know that there exists a sequence $(v_j)_{j\in\mathbb{N}}$ such that $v_j \in V_j$ for all $j \in \mathbb{N}$ and $\|v - v_j\|_V \to 0$. We have

$$-\left(\frac{\partial u_j}{\partial t}(t), v_j\right)_H + a_\lambda^{(M)}(u_j(t), v_j)_H - \frac{1}{\varepsilon}((\psi_j(t) - u_j(t))_+, v_j)_H = (g(t), v_j)_H$$

so, passing to the limit as $j \to \infty$,

$$-\left(\frac{\partial u_{\varepsilon,\lambda,M}}{\partial t}(t), v\right)_H + a_\lambda(u_{\varepsilon,\lambda,M}(t), v)_H - \frac{1}{\varepsilon}(\chi(t), v)_H = (g(t), v)_H.$$

We only have to note that $\chi(t) = (\psi(t) - u_{\varepsilon,\lambda,M}(t))_+$. In fact, $\psi_j(t) \to \psi(t)$ in V and, up to a subsequence, $1_{\mathcal{U}} u_j(t) \to 1_{\mathcal{U}} u_{\varepsilon,\lambda,M}(t)$ in $L^2(\mathcal{U}, \mathfrak{m})$ for every open $\mathcal{U}$ relatively compact in $\mathcal{O}$. Therefore, there exists a subsequence which converges a.e. and this allows to conclude the proof.

□

We want now to get rid of the truncated operator, that is to pass to the limit for $M \to \infty$. In order to do this we need some estimates on the function $u_{\varepsilon,\lambda,M}$ which are uniform in M.

Lemma 3.12. *Assume that, in addition to the assumptions of Proposition 3.11,* $\sqrt{1+y}\psi \in L^2([0,T];V)$, $\left|\frac{\partial \psi}{\partial t}\right| \le \Psi$ *with* $\Psi \in L^2([0,T];V)$ *and* g *satisfies Assumption* $\mathcal{H}^0$. *Let* $u_{\varepsilon,\lambda,M}$ *be the solution of* (3.19). *Then,*

$$\begin{aligned}&\int_t^T \left\|\frac{\partial u_{\varepsilon,\lambda,M}}{\partial s}(s)\right\|_H^2 ds + \|u_{\varepsilon,\lambda,M}(t)\|_V^2 + \tfrac{1}{\varepsilon}\|(\psi(t) - u_{\varepsilon,\lambda,M}(t))_+\|_H^2 \\ &\le C\left(\|\Psi\|_{L^2([0,T];V)} + \|\sqrt{1+y}g\|_{L^2([0,T];H)} + \|\sqrt{1+y}\psi\|^2_{L^2([0,T];V)} + \|\psi(T)\|_V^2\right),\end{aligned} \tag{3.25}$$

where C *is a positive constant independent of* M *and* ε.

Proof. To simplify the notation we denote $u_{\varepsilon,\lambda,M}$ by u. For $n \ge 0$, define $\varphi_n(x,y) = 1 + y \wedge n$. Since φ_n and its derivatives are bounded, if $v \in V$, we have $v\varphi_n \in V$. Applying (3.19) with $v = (u_{\varepsilon,\lambda,M} - \psi)\varphi_n = (u-\psi)\varphi_n$, we get

$$-\left(\frac{\partial u}{\partial t}, (u-\psi)\varphi_M\right)_H + a_\lambda^{(M)}(u, (u-\psi)\varphi_n) + (\zeta_\varepsilon(u), (u-\psi)\varphi_n)_H = (g, (u-\psi)\varphi_n)_H,$$

so that

$$\begin{aligned}-\left(\frac{\partial(u-\psi)}{\partial t}, (u-\psi)\varphi_n\right)_H + a_\lambda^{(M)}((u-\psi), (u-\psi)\varphi_n) + (\zeta_\varepsilon(u), (u-\psi)\varphi_n)_H &= \left(\frac{\partial \psi}{\partial t} + g, (u-\psi)\varphi_n\right)_H \\ &\quad + a_\lambda^{(M)}(\psi, (u-\psi)\varphi_n).\end{aligned}$$

With the notation $\varphi_n' = \frac{\partial \varphi_n}{\partial y} = \mathbb{1}_{\{y\le n\}}$, we have

$$\begin{aligned}a_\lambda^{(M)}((u-\psi), (u-\psi)\varphi_n) &= \int_{\mathcal{O}} \frac{y}{2}\left[\left(\frac{\partial(u-\psi)}{\partial x}\right)^2 + 2\rho\sigma\frac{\partial(u-\psi)}{\partial x}\frac{\partial(u-\psi)}{\partial y} + \sigma^2\left(\frac{\partial(u-\psi)}{\partial y}\right)^2\right]\varphi_n d\mathfrak{m} \\ &\quad + \int_{\mathcal{O}} \frac{y}{2}\left(\rho\sigma\frac{\partial(u-\psi)}{\partial x} + \sigma^2\frac{\partial(u-\psi)}{\partial y}\right)(u-\psi)\varphi_n' d\mathfrak{m} \\ &\quad + \int_{\mathcal{O}} y\wedge M\left(\frac{\partial(u-\psi)}{\partial x}j_{\gamma,\mu} + \frac{\partial(u-\psi)}{\partial y}k_{\gamma,\mu}\right)(u-\psi)\varphi_n d\mathfrak{m} \\ &\quad + \lambda\int_{\mathcal{O}}(1+y)(u-\psi)^2\varphi_n d\mathfrak{m} \\ &\ge \delta_1\int_{\mathcal{O}} y\,|\nabla(u-\psi)|^2\varphi_n d\mathfrak{m} + \lambda\int_{\mathcal{O}}(1+y)(u-\psi)^2\varphi_n d\mathfrak{m} \\ &\quad - K_1\int_{\mathcal{O}} y\,|\nabla(u-\psi)|\,|u-\psi|\varphi_n d\mathfrak{m} - K_2\int_{\mathcal{O}} y\,|\nabla(u-\psi)|\,|u-\psi|\mathbb{1}_{\{y\le n\}} d\mathfrak{m},\end{aligned}$$

where $K_2 = \frac{\sqrt{\rho^2\sigma^2+\sigma^4}}{2}$. Note that, if $n=0$, the last term vanishes, and that, for all $n>0$,

$$\int_{\mathcal{O}} y\,|\nabla(u-\psi)|\,|(u-\psi)|\mathbb{1}_{\{y\le n\}}d\mathfrak{m} \le ||(u-\psi)||_V^2.$$

Therefore, for all $\zeta>0$,

$$\begin{aligned}
a_\lambda^{(M)}((u-\psi),(u-\psi)\varphi_n) \ &\ge\ \delta_1\int_{\mathcal{O}} y\,|\nabla(u-\psi)|^2\varphi_n d\mathfrak{m} + \lambda\int_{\mathcal{O}}(1+y)(u-\psi)^2\varphi_n d\mathfrak{m}\\
&\quad -K_1\left(\int_{\mathcal{O}} y\left(\frac{\zeta}{2}|\nabla(u-\psi)|^2+\frac{1}{2\zeta}|u-\psi|^2\right)\varphi_n d\mathfrak{m}\right) - K_2||(u-\psi)||_V^2\\
&\ge\ \left(\delta_1-\frac{K_1\zeta}{2}\right)\int_{\mathcal{O}} y\,|\nabla(u-\psi)|^2\varphi_n d\mathfrak{m} + \left(\lambda-\frac{K_1}{2\zeta}\right)\int_{\mathcal{O}}(1+y)(u-\psi)^2\varphi_n d\mathfrak{m}\\
&\quad -K_2||(u-\psi)||_V^2\\
&\ge\ \frac{\delta_1}{2}\int_{\mathcal{O}}\left(y\,|\nabla(u-\psi)|^2+(1+y)(u-\psi)^2\right)\varphi_n d\mathfrak{m} - K_2||(u-\psi)||_V^2,
\end{aligned}$$

where, for the last inequality, we have chosen $\zeta=\delta_1/K_1$ and used the inequality $\lambda\ge\frac{\delta_1}{2}+\frac{K_1^2}{2\delta_1}$. Again, in the case $n=0$ the last term on the righthand side can be omitted.

Hence, we have, with the notation $||v||_{V,n}^2=\int_{\mathcal{O}}\left(y\,|\nabla v|^2+(1+y)v^2\right)\varphi_n d\mathfrak{m}$,

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}}(u-\psi)^2\varphi_n d\mathfrak{m}+\frac{\delta_1}{2}||u-\psi||_{V,n}^2+\frac{1}{\varepsilon}\int_{\mathcal{O}}(\psi-u)_+^2\varphi_n d\mathfrak{m} \ &\le\ \left(g+\frac{\partial\psi}{\partial t},(u-\psi)\varphi_n\right)_H\\
&\quad -a_\lambda^{(M)}(\psi,(u-\psi)\varphi_n)+K_2||(u-\psi)||_V^2.
\end{aligned}$$

In the case $n=0$, the inequality reduces to

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}}(u-\psi)^2 d\mathfrak{m}+\frac{\delta_1}{2}||u-\psi||_V^2+\frac{1}{\varepsilon}\int_{\mathcal{O}}(\psi-u)_+^2 d\mathfrak{m} \ \le\ \left(g+\frac{\partial\psi}{\partial t},(u-\psi)\right)_H - a_\lambda^{(M)}(\psi,(u-\psi)).$$

Now, integrate from t to T and use $u(T)=\psi(T)$ to derive

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}}(u(t)-\psi(t))^2\varphi_n d\mathfrak{m}+\frac{\delta_1}{2}\int_t^T ds||u(s)-\psi(s)||_{V,n}^2+\frac{1}{\varepsilon}\int_t^T ds\int_{\mathcal{O}}(\psi(s)-u(s))_+^2\varphi_n d\mathfrak{m}\\
&\le\int_t^T\left(g(s)+\frac{\partial\psi}{\partial t}(s),(u-\psi)(s)\varphi_n\right)_H ds+\left|\int_t^T a_\lambda^{(M)}(\psi(s),(u-\psi)(s)\varphi_n)ds\right|+K_2\int_t^T||u(s)-\psi(s)||_V^2 ds,
\end{aligned}\tag{3.26}$$

and, in the case $n=0$,

$$\begin{aligned}
\frac{1}{2}||u(t)-\psi(t)||_H^2+\frac{\delta_1}{2}\int_t^T ds||u(s)-\psi(s)||_V^2+\frac{1}{\varepsilon}\int_t^T ds\int_{\mathcal{O}}(\psi(s)-u(s))_+^2 d\mathfrak{m} &\le\int_t^T\left(g(s)+\frac{\partial\psi}{\partial t}(s),(u-\psi)(s)\right)_H ds\\
&\quad+\int_t^T\left|a_\lambda^{(M)}(\psi(s),(u-\psi)(s))\right|ds.
\end{aligned}\tag{3.27}$$

We have, for all $\zeta_1>0$,

$$\begin{aligned}
\int_t^T\left(g(s)+\frac{\partial\psi}{\partial t}(s),(u(s)-\psi(s))\varphi_n\right)_H ds &\le\frac{\zeta_1}{2}\int_t^T ds\int_{\mathcal{O}}|u(s)-\psi(s)|^2\varphi_n d\mathfrak{m}+\frac{1}{2\zeta_1}\int_t^T ds\int_{\mathcal{O}}\left|g(s)+\frac{\partial\psi}{\partial t}(s)\right|^2\varphi_n d\mathfrak{m}\\
&\le\frac{\zeta_1}{2}\int_t^T ds\int_{\mathcal{O}}|u(s)-\psi(s)|^2\varphi_n d\mathfrak{m}+\frac{1}{\zeta_1}\|\sqrt{1+y}g\|_{L^2([t,T];H)}^2+\frac{1}{\zeta_1}\left\|\sqrt{1+y}\frac{\partial\psi}{\partial t}\right\|_{L^2([t,T];H)}^2.
\end{aligned}$$

Moreover, it is easy to check that, for all v, $w \in V$,

$$|a_\lambda^{(M)}(w, v\varphi_n)| \le K_3||w||_{V,n}||v||_{V,n}, \text{ with } K_3 = \delta_0 + K_1 + K_2 + \lambda,$$

so that, for any $\zeta_2 > 0$,

$$\begin{aligned}\int_t^T |a_\lambda^{(M)}(\psi(s),(u(s)-\psi(s))\varphi_n)|ds &\le K_3\int_t^T ds||\psi(s)||_{V,n}||u(s)-\psi(s)||_{V,n}\\ &\le \frac{K_3\zeta_2}{2}\int_t^T ds||u(s)-\psi(s)||_{V,n}^2 + \frac{K_3}{2\zeta_2}\int_t^T ds||\psi(s)||_{V,n}^2.\end{aligned}$$

Now, choosing $\zeta_1 = K_3\zeta_2 = \delta_1/4$ and going back to (3.26) and (3.27), we get

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{O}}(u(t)-\psi(t))^2\varphi_n d\mathfrak{m} + \frac{\delta_1}{4}\int_t^T ds||u(s)-\psi(s)||_{V,n}^2 + \frac{1}{\varepsilon}\int_t^T ds\int_{\mathcal{O}}(\psi(s)-u(s))_+^2\varphi_n d\mathfrak{m}\\ &\le \frac{4}{\delta_1}\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)}\right) + \frac{2K_3^2}{\delta_1}\int_t^T ds||\psi(s)||^2_{V,n} + K_2\int_t^T ||u(s)-\psi(s)||_V^2 ds,\\ &\le \frac{4}{\delta_1}\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)}\right) + \frac{4K_3^2}{\delta_1}\left\|\sqrt{1+y}\psi\right\|^2_{L^2([t,T];V)} + K_2\int_t^T ||u(s)-\psi(s)||_V^2 ds,\end{aligned} \tag{3.28}$$

where the last inequality follows from the estimate $||v||^2_{V,n} \le 2||\sqrt{1+y}v||^2_V$, and, in the case $n = 0$,

$$\begin{aligned}\frac{1}{2}||u(t)-\psi(t)||_H^2 + \frac{\delta_1}{4}\int_t^T ds||u(s)-\psi(s)||_V^2 + \frac{1}{\varepsilon}\int_t^T ds\int_{\mathcal{O}}(\psi(s)-u(s))_+^2 d\mathfrak{m}\\ \le \frac{4}{\delta_1}\left(\|g\|^2_{L^2([t,T];H)} + \|\Psi\|^2_{L^2([t,T];H)}\right) + \frac{2K_3^2}{\delta_1}\|\psi\|^2_{L^2([t,T];V)}.\end{aligned} \tag{3.29}$$

From (3.29) we deduce

$$\begin{aligned}\int_t^T \|u(s)\|_V^2 ds &\le 2\int_t^T \|u(s)-\psi(s)\|_V^2 ds + 2\int_t^T \|\psi(s)\|_V^2 ds\\ &\le \frac{32}{\delta_1^2}\left(\|g\|^2_{L^2([t,T];H)} + \|\Psi\|^2_{L^2([t,T];H)}\right) + \left(\frac{16K_3^2}{\delta_1^2}+2\right)\|\psi\|^2_{L^2([t,T];V)}.\end{aligned} \tag{3.30}$$

Moreover, combining (3.28) and (3.29), we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{O}}(u(t)-\psi(t))^2\varphi_n d\mathfrak{m} + \frac{\delta_1}{4}\int_t^T ds||u(s)-\psi(s)||_{V,n}^2 + \frac{1}{\varepsilon}\int_t^T ds\int_{\mathcal{O}}(\psi(s)-u(s))_+^2\varphi_n d\mathfrak{m}\\ &\le \left(\frac{4}{\delta_1}+\frac{16K_2}{\delta_1^2}\right)\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)}\right) + \frac{4K_3^2}{\delta_1}\left(1+\frac{2K_2}{\delta_1}\right)\|\sqrt{1+y}\psi\|^2_{L^2([t,T];V)}.\end{aligned}$$

In particular,

$$\begin{aligned}\int_t^T ds\int_{\mathcal{O}} y|\nabla u|^2\varphi_n d\mathfrak{m} \le \int_t^T ||u(s)||^2_{V,n}ds &\le 2\int_t^T ds||u(s)-\psi(s)||^2_{V,n} + 2\int_t^T ds||\psi(s)||^2_{V,n}ds\\ &\le \frac{8}{\delta_1}\left(\frac{4}{\delta_1}+\frac{16K_2}{\delta_1^2}\right)\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)}\right)\\ &\quad + \left(\frac{32K_3^2}{\delta_1^2}\left(1+\frac{2K_2}{\delta_1}\right)+4\right)\|\sqrt{1+y}\psi\|^2_{L^2([t,T];V)}\end{aligned}$$

and, by using the Monotone convergence theorem, we deduce

$$\int_t^T ds\|y|\nabla u|\|_H^2 \le K_4\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\psi\|^2_{L^2([t,T];V)}\right), \tag{3.31}$$

where $K_4 = \frac{8}{\delta_1}\left(\frac{4}{\delta_1} + \frac{16K_2}{\delta_1^2}\right) \vee \left(\frac{32K_3^2}{\delta_1^2}\left(1 + \frac{2K_2}{\delta_1}\right) + 4\right)$.

We are now in a position to prove (3.25). Taking $v = \frac{\partial u}{\partial t}$ in (3.19), we have

$$-\left\|\frac{\partial u}{\partial t}\right\|_H^2 + \bar{a}_\lambda\left(u, \frac{\partial u}{\partial t}\right) + \tilde{a}^{(M)}\left(u, \frac{\partial u}{\partial t}\right) - \frac{1}{\varepsilon}\left((\psi - u)_+, \frac{\partial u}{\partial t}\right)_H = \left(g, \frac{\partial u}{\partial t}\right)_H.$$

Note that, since $\bar{a}_\lambda$ is symmetric, $\frac{d}{dt}\bar{a}_\lambda(u,u) = 2\bar{a}_\lambda\left(u, \frac{\partial u}{\partial t}\right)$. On the other hand,

$$\left((\psi(t) - u(t))_+, \frac{\partial u}{\partial t}\right)_H = -\frac{1}{2}\frac{d}{dt}||(\psi(t) - u(t))_+||_H^2 + \left((\psi(t) - u(t))_+, \frac{\partial \psi}{\partial t}(t)\right)_H,$$

so that

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|_H^2 - \frac{1}{2}\frac{d}{dt}\bar{a}_\lambda(u,u) - \frac{1}{2\varepsilon}\frac{d}{dt}||(\psi - u)_+||_H^2 &= \tilde{a}^{(M)}\left(u, \frac{\partial u}{\partial t}\right) - \left(g, \frac{\partial u}{\partial t}\right)_H - \frac{1}{\varepsilon}\left((\psi - u)_+, \frac{\partial \psi}{\partial t}\right)_H \\ &\leq \left|\tilde{a}^{(M)}\left(u, \frac{\partial u}{\partial t}\right)\right| + ||g||_H\left\|\frac{\partial u}{\partial t}\right\|_H + \frac{1}{\varepsilon}\left((\psi - u)_+, \Psi\right)_H \\ &\leq \left(K_1\left\|y|\nabla u|\right\|_H + \|g||_H\right)\left\|\frac{\partial u}{\partial t}\right\|_H + \frac{1}{\varepsilon}\left((\psi - u)_+, \Psi\right)_H.\end{aligned}$$

Moreover, if we take $v = \Psi$ in (3.19), we get

$$-\left(\frac{\partial u}{\partial t}, \Psi\right)_H + a_\lambda^{(M)}(u, \Psi) - \frac{1}{\epsilon}((\psi - u)_+, \Psi)_H = (g, \Psi)_H,$$

so that

$$\frac{1}{\varepsilon}\left((\psi - u)_+, \Psi\right)_H \leq \left\|\frac{\partial u}{\partial t}\right\|_H \|\Psi\|_H + ||a_\lambda^{(M)}|| \|u\|_V \|\Psi\|_V + \|g\|_H \|\Psi\|_H. \tag{3.32}$$

Therefore,

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|_H^2 - \frac{1}{2}\frac{d}{dt}\bar{a}_\lambda(u,u) - \frac{1}{2\varepsilon}\frac{d}{dt}||(\psi - u)_+||_H^2 \leq &\left(K_1\left\|y|\nabla u|\right\|_H + \|g||_H + \|\Psi\|_H\right)\left\|\frac{\partial u}{\partial t}\right\|_H \\ &+ ||a_\lambda^{(M)}|| \|u\|_V \|\Psi\|_V + \|g\|_H \|\Psi\|_H,\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_H^2 - \frac{1}{2}\frac{d}{dt}\bar{a}_\lambda(u,u) - \frac{1}{2\varepsilon}\frac{d}{dt}||(\psi - u)_+||_H^2 \leq &\frac{1}{2}\left(K_1\left\|y|\nabla u|\right\|_H + \|g||_H + \|\Psi\|_H\right)^2 \\ &+ ||a_\lambda^{(M)}|| \|u\|_V^2 \|\Psi\|_V^2 + \|g\|_H \|\Psi\|_H.\end{aligned}$$

Integrating between t and T, we get,

$$\begin{aligned}\frac{1}{2}\left\|\frac{\partial u}{\partial s}\right\|_{L^2([t,T];H)}^2 + \frac{1}{2}\bar{a}_\lambda(u(t), u(t)) + \frac{1}{2\varepsilon}\|(\psi(t) - u(t))_+\|_H^2 \leq &\frac{1}{2}\bar{a}_\lambda(\psi(T), \psi(T)) + 2\|g||_{L^2([t,T];H)}^2 + 2\|\Psi\|_{L^2([t,T];H)}^2 \\ &+ \frac{3K_1^2}{2}\left\|y|\nabla u|\right\|_{L^2([t,T];H)}^2 + \frac{\|a_\lambda^{(M)}\|}{2}\|u\|_{L^2([t,T];V)} + \frac{\|a_\lambda^{(M)}\|}{2}\|\Psi\|_{L^2([t,T];V)},\end{aligned}$$

so, recalling that $\bar{a}_\lambda(u(t), u(t) \geq (\delta_1 + \lambda)\|u(t)\|_V^2$,

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u}{\partial s}\right\|^2_{L^2([t,T];H)} + \frac{\delta_1+\lambda}{2}\|u(t)\|_V^2 + \frac{1}{2\varepsilon}\|(\psi(t)-u(t))_+\|_H^2 \\
&\quad\leq \frac{\|\bar{a}_\lambda\|}{2}\|\psi(T)\|_V^2 + 2\|g\|^2_{L^2([t,T];H)} + 2\|\Psi\|^2_{L^2([t,T];H)} \\
&\quad\quad + \frac{3K_1^2}{2}\|y|\nabla u|\|^2_{L^2([t,T];H)} + \frac{||a_\lambda^{(M)}||}{2}\|u\|_{L^2([t,T];V)} + \frac{||a_\lambda^{(M)}||}{2}\|\Psi\|_{L^2([t,T];V)} \\
&\quad\leq \frac{\|\bar{a}_\lambda\|}{2}\|\psi(T)\|_V^2 + 2\|g\|^2_{L^2([t,T];H)} + 2\|\Psi\|^2_{L^2([t,T];H)} \\
&\quad\quad + \frac{3K_1^2}{2}K_4\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\psi\|^2_{L^2([t,T];V)}\right) \\
&\quad\quad + \frac{||a_\lambda^{(M)}||}{2}\left(\frac{32}{\delta_1^2}\left(\|g\|^2_{L^2([t,T];H)} + \|\Psi\|^2_{L^2([t,T];H)}\right) + \left(\frac{16K_3^2}{\delta_1^2}+2\right)\|\psi\|^2_{L^2([t,T];V)}\right) + \frac{||a_\lambda^{(M)}||}{2}\|\Psi\|_{L^2([t,T];V)},
\end{aligned}
$$

where the last inequality follows from (3.30) and (3.31). Rearranging the terms, we deduce that there exists a constant $C > 0$ independent of M and ε such that

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u}{\partial s}\right\|^2_{L^2([t,T];H)} + \frac{\delta_1+\lambda}{4}\|u(t)\|_V^2 + \frac{1}{2\varepsilon}\|(\psi(t)-u(t))_+\|_H^2 \\
&\qquad\leq C\left(\|\sqrt{1+y}g\|^2_{L^2([t,T];H)} + \|\sqrt{1+y}\Psi\|^2_{L^2([t,T];H)} + \left\|\sqrt{1+y}\psi\right\|^2_{L^2([t,T];V)} + \|\psi(T)\|_V^2\right),
\end{aligned}
$$

which concludes the proof. □

Proof of Theorem 3.10: existence. Assume for a first moment that we have the further assumptions $\psi(T) \in H^2(\mathcal{O}, \mathfrak{m})$, $\frac{\partial\psi}{\partial t} \in L^2([0,T];V)$ and $\frac{\partial g}{\partial t} \in L^2([0,T];H)$. Thanks to (3.25) we can repeat the same arguments as in the proof of Proposition 3.11 in order to pass to the limit in j, but this time as $M \to \infty$. In fact, up to pass to a subsequence, from (3.25) we can suppose that $\frac{\partial u_{\varepsilon,\lambda,M}}{\partial t}$ weakly converges to a function $u'_{\varepsilon,\lambda}$ in $L^2([0,T];H)$. We deduce that, for any fixed $t \in [0,T]$, $u_j(t)$ converges weakly in H to

$$u_{\varepsilon,\lambda}(t) = \psi(T) - \int_t^T u'_{\varepsilon,\lambda}(s)ds.$$

Indeed, $u_{\varepsilon,\lambda,M}(t)$ is bounded in V, so the convergence is weakly in V. Moreover, again from (3.25) and from the fact that there is a subsequence of $u_{\varepsilon,\lambda,M}(t)$ which converges a.e. to $u_{\lambda,M}(t)$, we get that $(\psi(t) - u_{\varepsilon,\lambda,M}(t))_+$ weakly converges in H to $(\psi(t) - u_{\varepsilon,\lambda}(t))_+$. We have

$$-\left(\frac{\partial u_{\varepsilon,\lambda,M}}{\partial t}(t), v\right)_H + a_\lambda^{(M)}(u_{\varepsilon,\lambda,M}(t), v) + (\zeta_\varepsilon(u_{\varepsilon,\lambda,M})(t), v)_H = (g(t), v)_H$$

and, passing to the limit as $M \to \infty$, we get

$$-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}(t), v\right)_H + a_\lambda(u_{\varepsilon,\lambda}(t), v)_H - \frac{1}{\varepsilon}(\chi, (\psi(t) - u_{\varepsilon,\lambda}(t))_+)_H = (g(t), v)_H.$$

Finally we can prove that $\chi = u_{\varepsilon,\lambda}(t)$ as in the proof of Proposition 3.11. The estimates (3.16), (3.17) and (3.18) directly follows from (3.25) as $M \to \infty$.

We have now to weaken the assumptions on g and ψ. This is a standard regularization procedure. In fact, for example for the function g, we can consider a sequence of functions $g_n = g * \varphi_n$, where $(\varphi_n)_n \in C_c^\infty([0,T] \times \mathcal{O})$, $\int_{[0,T]\times\mathcal{O}} \varphi_n = 1$ and $\lim_{n\to\infty} \varphi_n(x) = \delta_x$. Then $(g_n)_n \subset L^2([0,T];H)$, $\frac{\partial g_n}{\partial t} \in L^2([0,T];H)$ $\forall n \in \mathbb{N}$ and $\|g_n - g\|_{L^2([0,T];H)} \to 0$ as $n \to \infty$. In the same way, we can find a sequence ψ_n such that $\psi_n(T) \in H^2(\mathcal{O}, \mathfrak{m})$

and $\frac{\partial \psi_n}{\partial t} \in L^2([0,T];V)$ for every $n \in \mathbb{N}$ and $\int_0^T \|\psi_n(t)-\psi(t)\|_V ds \to 0$ as $n \to \infty$. Therefore, the solution $u^n_{\varepsilon,\lambda,M}$ of the equation (3.15) with source function g_n and obstacle function ψ_n satisfies

$$\begin{aligned}&\int_t^T \left\| \frac{\partial u^n_{\varepsilon,\lambda,M}}{\partial s}(s)\right\|_H^2 ds + \|u^n_{\varepsilon,\lambda,M}(t)\|_V^2 + \tfrac{1}{\varepsilon}\|(\psi_n(t)-u^n_{\varepsilon,\lambda,M}(t))_+\|_H^2 \\ &\le C\left(\|\sqrt{1+y}g_n\|_{L^2([0,T];H)} + \|\sqrt{1+y}\psi_n\|^2_{L^2([0,T];V)} + \|\Psi|^2_{L^2([0,T];V)} + \|\psi_n(T)\|_V^2\right).\end{aligned} \tag{3.33}$$

Then, we can take the limit for $n \to \infty$ in (3.33) and the assertion follows as in the first part of the proof. ☐

Moreover, we have the following Comparison principle for the coercive penalized problem.

Proposition 3.13. *1. Assume that ψ_i satisfies Assumption $\mathcal{H}^1$ for $i=1,2$ and g satisfies Assumption $\mathcal{H}^0$. Let $u^i_{\varepsilon,\lambda}$ be the unique solution of (3.15) with obstacle function ψ_i and source function g. If $\psi_1 \le \psi_2$, then $u^1_{\varepsilon,\lambda} \le u^2_{\varepsilon,\lambda}$.*

2. *Assume that ψ satisfies Assumption $\mathcal{H}^1$ and g_i satisfy Assumption $\mathcal{H}^0$ for $i=1,2$. Let $u^i_{\varepsilon,\lambda}$ be the unique solution of (3.15) with obstacle function ψ and source function g_i. If $g_1 \le g_2$, then $u^1_{\varepsilon,\lambda} \le u^2_{\varepsilon,\lambda}$.*

3. *Assume that ψ_i satisfies Assumption $\mathcal{H}^1$ for $i=1,2$ and g satisfies Assumption $\mathcal{H}^0$. Let $u^i_{\varepsilon,\lambda}$ be the unique solution of (3.15) with obstacle function ψ_i and source function g. If $\psi_1-\psi_2 \in L^\infty$, then $u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda} \in L^\infty$ and $\|u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda}\|_\infty \le \|\psi_1-\psi_2\|_\infty$.*

Proof. 1. We take $v=(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+$ in the variational equation satisfied by $u^1_{\varepsilon,\lambda}$ and $u^1_{\varepsilon,\lambda}$. Subtracting the second equation from the first one, we get

$$-\left(\frac{\partial(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})}{\partial t},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\right)_H + a_\lambda(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)$$

$$-\frac{1}{\varepsilon}((\psi_1-u^1_{\varepsilon,\lambda})_+-(\psi_2-u^2_{\varepsilon,\lambda})_+,(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)_H=0.$$

Now,

$$a_\lambda(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+) = a_\lambda((u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+,(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+) \ge 0,$$

and

$$-\frac{1}{\varepsilon}((\psi_1-u^1_{\varepsilon,\lambda})_+-(\psi_2-u^2_{\varepsilon,\lambda})_+,(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)_H \ge 0,$$

the last inequality following from Lemma 3.9. Therefore

$$\left(\frac{\partial(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+}{\partial t},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\right)_H = \frac{1}{2}\frac{d}{dt}\|(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\|_H^2 \ge 0.$$

But $(u^1_{\varepsilon,\lambda}(T)-u^2_{\varepsilon,\lambda}(T))_+ = (\psi_1(T)-\psi_2(T))_+=0$, so $\|(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\|_H^2 \equiv 0$ and the proof is completed.

2. Again we consider $v=(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+$ and we prove that $v \equiv 0$. With the same passages, this time we get

$$-\left(\frac{\partial(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})}{\partial t},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\right)_H + a_\lambda(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)$$

$$+(\zeta_\varepsilon(u^1_{\varepsilon,\lambda})-\zeta_\varepsilon(u^2_{\varepsilon,\lambda}),(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)_H = (g_1-g_2,(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)_H \le 0.$$

Again

$$a_\lambda(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+) = a_\lambda((u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+,(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+) \ge 0,$$

and

$$(\zeta_\varepsilon(u^1_{\varepsilon,\lambda})-\zeta_\varepsilon(u^2_{\varepsilon,\lambda}),(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+)_H \ge 0,$$

thanks to the monotonicity of the penalized operator. Therefore we obtain

$$\left(\frac{\partial(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+}{\partial t},(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\right)_H=\frac{1}{2}\frac{d}{dt}\|(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda})_+\|^2_H\geq 0,$$

and we can conclude the proof as before.

3. With the same procedure, we choose $v=(u^1_{\varepsilon,\lambda}-u^2_{\varepsilon,\lambda}-C)_+$, with $C=\|\psi_1-\psi_2\|_\infty$ and, with the usual passages, we get

$$-\left(\frac{\partial v}{\partial t},v\right)_H+a_\lambda(v,v)+\int_{\mathcal{O}}(r+\lambda(1+y))\,Cvd\mathfrak{m}+\frac{1}{\varepsilon}((\psi_2-u^2_{\varepsilon,\lambda})_+-(\psi_1-u^1_{\varepsilon,\lambda})_+,v)_H=0.$$

The last three terms are all positives so the assertion follows as in the other cases.

□

3.2.2 Coercive variational inequality

Proposition 3.14. *Assume that ψ satisfies Assumption $\mathcal{H}^1$ and g satisfies Assumption $\mathcal{H}^0$. Moreover, assume that $0\leq\psi\leq\Phi$ with $\Phi:[0,T]\to H^2(\mathcal{O},\mathfrak{m})$ such that $\frac{\partial\Phi}{\partial t}+\mathcal{L}\Phi\leq 0$ and $0\leq g\leq-\frac{\partial\Phi}{\partial t}-\mathcal{L}^\lambda\Phi$. Then, there exists a unique function u_λ such that $u_\lambda\in L^2([0,T];V)$, $\frac{\partial u_\lambda}{\partial t}\in L^2([0,T];H)$ and*

$$\begin{cases}-\left(\frac{\partial u_\lambda}{\partial t},v-u_\lambda\right)_H+a_\lambda(u_\lambda,v-u_\lambda)\geq(g,v-u_\lambda)_H, & a.e.\ in\ [0,T]\qquad v\in V,\ v\geq\psi,\\ u_\lambda(T)=\psi(T),\\ u_\lambda\geq\psi\ a.e.\ in\ [0,T]\times\mathbb{R}\times(0,\infty).\end{cases}\tag{3.34}$$

Moreover, $0\leq u_\lambda\leq\Phi$.

Proof of uniqueness in Proposition 3.14. Suppose that there are two functions u_1 and u_2 which satisfy (3.34). We can take $v=u_2$ in the equation satisfied by u_1 and $v=u_1$ in the one satisfied by u_2 and we get

$$-\left(\frac{\partial u_1}{\partial t},u_2-u_1\right)_H+a_\lambda(u_1,u_2-u_1)\geq(g,u_2-u_1)_H,$$

$$-\left(\frac{\partial u_2}{\partial t},u_1-u_2\right)_H+a_\lambda(u_2,u_1-u_2)\geq(g,u_1-u_2)_H.$$

Setting $w:=u_2-u_1$ and adding the second equation from the first one we obtain

$$\left(\frac{\partial w}{\partial t},w\right)_H-a_\lambda(w,w)\geq 0,$$

so that

$$\left(\frac{\partial w}{\partial t},w\right)_H=\frac{1}{2}\frac{d}{dt}\|w\|_H\geq 0.$$

But $w(T)=u_1(T)-u_2(T)=\psi(T)-\psi(T)=0$ and, therefore, $w\equiv 0$, that is $u_1=u_2$. □

Proof of existence in Proposition 3.14. For each fixed $\varepsilon>0$ we have the estimates (3.16) and (3.17), so, for every $t\in[0,T]$, we can extract a subsequence $u_{\varepsilon,\lambda}$ such that $u_{\varepsilon,\lambda}(t)\rightharpoonup u_\lambda(t)$ in V as $\varepsilon\to 0$ and $u'_\varepsilon(t)\rightharpoonup u'_\lambda(t)$ in H for some function $u_\lambda\in V$.

Note that $u=0$ is the unique solution of (3.15) when $\psi=g=0$, while $u=\Phi$ is the unique solution of (3.15) when $\psi=\Phi$ and $g=-\frac{\partial\Phi}{\partial t}-\mathcal{L}^\lambda\Phi=-\frac{\partial\Phi}{\partial t}-\mathcal{L}\Phi+\lambda(1+y)\Phi$. Therefore, Proposition 3.13 implies that $0\leq u_{\varepsilon,\lambda}\leq\Phi$. Recall that $u_{\varepsilon,\lambda}(t)\to u_\lambda(t)$ in $L^2(\mathcal{U},\mathfrak{m})$ for every relatively compact open $\mathcal{U}\subset\mathcal{O}$. This, together with the fact that $d\mathfrak{m}$ is a finite measure, allows to conclude that we have strong convergence of $u_{\varepsilon,\lambda}$ to u_λ in H. In fact, if $\delta>0$ and $\mathcal{O}_\delta:=(-\frac{1}{\delta},\frac{1}{\delta})\times(\delta,\frac{1}{\delta})$,

$$\int_{\mathcal{O}}|u_{\varepsilon,\lambda}-u_\lambda|^2d\mathfrak{m}\leq\int_{\mathcal{O}_\delta}|u_{\varepsilon,\lambda}-u_\lambda|^2d\mathfrak{m}+\int_{\mathcal{O}^c_\delta}|u_{\varepsilon,\lambda}-u_\lambda|^2d\mathfrak{m}\leq\int_{\mathcal{O}_\delta}|u_{\varepsilon,\lambda}-u_\lambda|^2d\mathfrak{m}+\int_{\mathcal{O}^c_\delta}4\Phi^2d\mathfrak{m}$$

and it is enough to let δ goes to 0.

From (3.18) we also have that $(\psi - u_{\varepsilon,\lambda})^+ \to 0$ strongly in H as $\varepsilon \to 0$. On the other hand $(\psi - u_{\varepsilon,\lambda})^+ \rightharpoonup \chi$ weakly in H and $\chi = (\psi - u_\lambda)^+$ since there exists a subsequence of $u_{\varepsilon,\lambda}$ which converges pointwise to u_λ. Therefore, $(\psi - u_\lambda)^+ = 0$, which means $u_\lambda \geq \psi$.

If we consider the penalized coercive equation in (3.15) replacing v by $v - u_{\varepsilon,\lambda}$, with $v \geq \psi$, we have

$$-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}, v - u_{\varepsilon,\lambda}\right)_H + a_\lambda(u_{\varepsilon,\lambda}, v - u_{\varepsilon,\lambda})_H + (\zeta_\varepsilon(u_{\varepsilon,\lambda}), v - u_{\varepsilon,\lambda})_H = (g, v - u_{\varepsilon,\lambda})_H.$$

Since $\zeta_\varepsilon(v) = 0$, we can write

$$-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}, v - u_{\varepsilon,\lambda}\right)_H + a_\lambda(u_{\varepsilon,\lambda}, v - u_{\varepsilon,\lambda})_H - \underbrace{(\zeta_\varepsilon(v) - \zeta_\varepsilon(u_{\varepsilon,\lambda}), v - u_{\varepsilon,\lambda})_H}_{\geq 0} = (g, v - u_{\varepsilon,\lambda})_H.$$

Therefore

$$-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}, v - u_{\varepsilon,\lambda}\right)_H + a_\lambda(u_{\varepsilon,\lambda}, v - u_{\varepsilon,\lambda}) \geq (g, v - u_{\varepsilon,\lambda})_H$$

and, letting ε goes to 0, we have

$$\begin{aligned}-\left(\frac{\partial u_\lambda}{\partial t}, v - u_\lambda\right)_H + a_\lambda(u_\lambda, v) &\geq (g, v - u_\lambda)_H + \liminf_{\varepsilon\to 0} a_\lambda(u_{\varepsilon,\lambda}, u_{\varepsilon,\lambda}) \\ &\geq (g, v - u_\lambda)_H + a_\lambda(u_\lambda, u_\lambda).\end{aligned}$$

Moreover, since $0 \leq u_{\varepsilon,\lambda} \leq \Phi$ for every $\varepsilon > 0$ and $u_\lambda = \lim_{\varepsilon\to 0} u_{\varepsilon,\lambda}$, we have $0 \leq u_\lambda \leq \Phi$ and the assertion follows. □

The following Comparison Principle is a direct consequence of Proposition 3.13,.

Proposition 3.15. *1. For $i = 1, 2$, assume that ψ_i satisfies Assumption $\mathcal{H}^1$, g satisfies Assumption $\mathcal{H}^0$ and $0 \leq \psi_i \leq \Phi$ with $\Phi : [0,T] \to H^2(\mathcal{O}, \mathfrak{m})$ such that $\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi \leq 0$ and $0 \leq g \leq -\frac{\partial \Phi}{\partial t} - \mathcal{L}^\lambda \Phi$. Let u_λ^i be the unique solution of (3.34) with obstacle function ψ_i and source function g. If $\psi_1 \leq \psi_2$, then $u_\lambda^1 \leq u_\lambda^2$.*

2. *For $i = 1, 2$, assume that ψ satisfies Assumption $\mathcal{H}^1$, g_i satisfy Assumption $\mathcal{H}^0$ and $0 \leq \psi \leq \Phi$ with $\Phi : [0,T] \to H^2(\mathcal{O}, \mathfrak{m})$ such that $\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi \leq 0$ and $0 \leq g_i \leq -\frac{\partial \Phi}{\partial t} - \mathcal{L}^\lambda \Phi$. Let u_λ^i be the unique solution of (3.34) with obstacle function ψ and source function g_i. If $g_1 \leq g_2$, then $u_\lambda^1 \leq u_\lambda^2$.*

3. *For $i = 1, 2$, assume that ψ_i satisfies Assumption $\mathcal{H}^1$, g satisfies Assumption $\mathcal{H}^0$ and $0 \leq \psi_i \leq \Phi$ with $\Phi : [0,T] \to H^2(\mathcal{O}, \mathfrak{m})$ such that $\frac{\partial \Phi}{\partial t} + \mathcal{L}\Phi \leq 0$ and $0 \leq g \leq -\frac{\partial \Phi}{\partial t} - \mathcal{L}^\lambda \Phi$. Let u_λ^i be the unique solution of (3.34) with obstacle function ψ_i and source function g. If $\psi_1 - \psi_2 \in L^\infty$, then $u_\lambda^1 - u_\lambda^2 \in L^\infty$ and $\|u_\lambda^1 - u_\lambda^2\|_\infty \leq \|\psi_1 - \psi_2\|_\infty$.*

3.2.3 Non-coercive variational inequality

We can finally prove Theorem 2.3. Again, we first study the uniqueness of the solution and then we deal with the existence.

Proof of uniqueness in Theorem 2.3. Suppose that there are two functions u_1 and u_2 which satisfies (2.4). As usual, we take $v = u_2$ in the equation satisfied by u_1 and $v = u_1$ in the one satisfied by u_2 and we add the resulting equations. Setting $w := u_2 - u_1$, we get

$$\left(\frac{\partial w}{\partial t}, w\right)_H - a(w, w) \geq 0.$$

From the energy estimate (3.6), we know that

$$a(u, u) \geq C_1 \|u\|_V^2 - C_2 \|(1 + y)^{\frac{1}{2}} u\|_H^2,$$

so that

$$\frac{1}{2}\frac{d}{dt}\|w\|_H^2 + C_2\|(1+y)^{\frac{1}{2}}w\|_H^2 \geq 0.$$

By integrating from t to T, since $w(T) = 0$,we have

$$\begin{aligned}
\|w(t)\|_H^2 &\leq C_2 \int_t^T ds\|(1+y)^{\frac{1}{2}}w\|_H^2 \\
&\leq C_2\bigg(\int_t^T ds\int_{\mathcal{O}} 1_{\{y\leq\lambda\}}(1+y)w^2 d\mathfrak{m} + \int_t^T ds\int_{\mathcal{O}} 1_{\{y>\lambda\}}(1+y)w^2 d\mathfrak{m}\bigg) \\
&\leq C\bigg(\int_t^T ds\int_{\mathcal{O}} (1+\lambda)w^2 y^{\beta-1}e^{-\gamma|x|}e^{-\mu y}dxdy + \int_t^T ds\int_{\mathcal{O}} 1_{\{y>\lambda\}}(1+y)w^2y^{\beta-1}e^{-\gamma|x|}e^{-(\mu-\mu')y}e^{-\mu' y}dxdy\bigg) \\
&\leq C\bigg(\int_t^T ds\int_{\mathcal{O}} dxdy(1+\lambda)w^2y^{\beta-1}e^{-\gamma|x|}e^{-\mu y} + e^{-(\mu-\mu')\lambda}\int_t^T ds\int_{\mathcal{O}} dxdy(1+y)\Phi^2y^{\beta-1}e^{-\gamma|x|}e^{-\mu' y}\bigg),
\end{aligned}$$

where $\mu' < \mu$ and $\lambda > 0$. Since $C_2 = \int_{\mathcal{O}} dxdy(1+y)\Phi^2 y^{\beta-1}e^{-\gamma|x|}e^{-\mu' y} < \infty$, we have

$$\|w(t)\|_H^2 \leq C(1+\lambda)\int_t^T \|w(s)\|_H^2 ds + C_2(T-t)e^{-(\mu-\mu')\lambda},$$

so, by using the Gronwall Lemma,

$$\|w(t)\|_H^2 \leq C_2 T e^{-(\mu-\mu')\lambda + C(T-t)(1+\lambda)}.$$

Sending $\lambda\to\infty$, we deduce that $w(t) = 0$ in $[T,t]$ for t such that $T - t < \frac{\mu-\mu'}{C}$. Then, we iterate the same argument: we integrate between t' and t with $t - t' < \frac{\mu-\mu'}{C}$ and we have $w(t) = 0$ in $[T,t']$ and so on. We deduce that $w(t) = 0$ for all $t\in[0,T]$ so the assertion follows. □

Proof of existence in Theorem 2.3. Given $u_0 = \Phi$, we can construct a sequence $(u_n)_n \subset V$ such that

$$u_n \geq \psi \text{ a.e. in } [0,T]\times\mathcal{O}, \qquad n\geq 1, \tag{3.35}$$

$$-\left(\frac{\partial u_n}{\partial t}, v-u_n\right)_H + a(u_n, v-u_n) + \lambda((1+y)u_n, v-u_n)_H \geq \lambda((1+y)u_{n-1}, v-u_n)_H, \\ v\in V, \quad v\geq\psi, \quad \text{a.e. on } [0,T]\times\mathcal{O}, \qquad n\geq 1, \tag{3.36}$$

$$u_n(T) = \psi(T), \qquad \text{in } \mathcal{O}, \tag{3.37}$$

$$\Phi \geq u_1\geq u_2 \geq\cdots\geq u_{n-1}\geq u_n\geq\cdots\geq 0, \qquad \text{a.e. on } [0,T]\times\mathcal{O}. \tag{3.38}$$

In fact, if we have $0\leq u_{n-1}\leq\Phi$ for all $n\in\mathbb{N}$, then the assumptions of Proposition 3.14 are satisfied with

$$g_n = \lambda(1+y)u_{n-1},$$

since g_n and $\sqrt{1+y}g_n$ belong to $L^2([0,T];H)$ and $0\leq g_n\leq\lambda(1+y)\Phi\leq -\frac{\partial\Phi}{\partial t} - \mathcal{L}_\lambda\Phi$. Therefore, step by step, we can deduce the existence and the uniqueness of a solution u_n to (3.36) such that $0\leq u_n\leq\Phi$. (3.38) is a simple consequence of Proposition 3.15. In fact, proceeding by induction, at each step we have

$$g_n = \lambda(1+y)u_{n-1}\leq\lambda(1+y)u_{n-2} = g_{n-1}$$

so that $u_n\leq u_{n-1}$. Now, recall that

$$\|u_n\|_{L^\infty([0,T],V)}\leq K,$$

$$\left\|\frac{\partial u_n}{\partial t}\right\|_{L^2([0,T];H)}\leq K,$$

where $K = C\left(\|\Phi\|_{L^2([0,T];V)} + \|\sqrt{1+y}g_n\|_{L^2([0,T];H)} + \|\sqrt{1+y}\psi\|_{L^2([0,T];V)} + \|\psi(T)\|_V^2\right)$. Note that the constant K is independent of n since $|g_n| = |\lambda(1+y)u_{n-1},|\leq\lambda(1+y)\Phi$, for every $n\in\mathbb{N}$. Therefore, by passing to a subsequence, we can assume that there exists a function u such that $u \in L^2([0,T];V)$, $\frac{\partial u}{\partial t} \in L^2([0,T];H)$ and for every $t \in [0,T]$, $u_n'(t) \rightharpoonup u'(t)$ in H and $u_n(t) \rightharpoonup u(t)$ in V. Indeed, again thanks to the fact that $0 \le u_n \le \Phi$, we can deduce that $u_n(t) \to u(t)$ in H. Therefore we can pass to the limit in

$$-\left(\frac{\partial u_n}{\partial t}, u_n - v\right)_H + a(u_n, v - u_n) + \lambda((1+y)u_n, v - u_n)_H \ge \lambda((1+y)u_{n-1}, v - u_n)_H$$

and the assertion follows. □

Remark 3.16. *Keeping in mind our purpose of identifying the solution of the variational inequality* (2.4) *with the American option price we have considered the case without source term ($g = 0$) in the variational inequality* (2.4). *However, under the same assumptions of Theorem 2.3, we can prove in the same way the existence and the uniqueness of a solution of*

$$\begin{cases} -\left(\frac{\partial u}{\partial t}, v - u\right)_H + a(u, v-u) \ge (g, v-u)_H, & a.e.\ in\ [0,T] \quad v \in V,\ v \ge \psi, \\ u \ge \psi\ a.e.\ in\ [0,T] \times \mathbb{R} \times (0,\infty), \\ u(T) = \psi(T), \\ 0 \le u \le \Phi, \end{cases}$$

where g satisfies Assumption $\mathcal{H}^0$ and $0 \le g \le -\frac{\partial \Phi}{\partial t} - \mathcal{L}\Phi$.

We conclude stating the following Comparison Principle, whose proof is a direct consequence of Proposition 3.15 and the proof of Proposition 2.3.

Proposition 3.17. *For $i = 1, 2$, assume that ψ_i satisfies Assumption $\mathcal{H}^1$ and $0 \le \psi_i \le \Phi$ with Φ satisfying Assumption $\mathcal{H}^2$. Let u_λ^i be the unique solution of* (3.34) *with obstacle function ψ_i. Then:*

1. *If $\psi_1 \le \psi_2$, then $u_\lambda^1 \le u_\lambda^2$.*
2. *If $\psi_1 - \psi_2 \in L^\infty$, then $u_\lambda^1 - u_\lambda^2 \in L^\infty$ and $\|u_\lambda^1 - u_\lambda^2\|_\infty \le \|\psi_1 - \psi_2\|_\infty$.*

4 Connection with the optimal stopping problem

Once we have the existence and the uniqueness of a solution u of the variational inequality (2.3), our aim is to prove that it matches the solution of the optimal stopping problem, that is

$$u(t,x,y) = u^*(t,x,y), \qquad \text{on } [0,T] \times \bar{\mathcal{O}},$$

where u^* is defined by

$$u^*(t,x,y) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\psi(\tau, X_\tau^{t,x,y}, Y_\tau^{t,x,y})\right],$$

$\mathcal{T}_{t,T}$ being the set of the stopping times with values in $[t,T]$. Since the function u is not regular enough to apply Ito's Lemma, we use another strategy in order to prove the above identification. So, we first show, by using the affine character of the underlying diffusion, that the semigroup associated with the bilinear form a_λ coincides with the transition semigroup of the two dimensional diffusion (X,Y) with a killing term. Then, we prove suitable estimates on the joint law of (X,Y) and L^p-regularity results on the solution of the variational inequality and we deduce from them the probabilistic interpretation.

4.1 Semigroup associated with the bilinear form

We introduce now the semigroup associated with the coercive bilinear form a_λ. With a natural notation, we define the following spaces

$$L^2_{loc}(\mathbb{R}^+;H) = \{f : \mathbb{R}^+ \to H : \forall t \ge 0 \int_0^t \|f(s)\|_H^2 ds < \infty\},$$

$$L^2_{loc}(\mathbb{R}^+;V) = \{f : \mathbb{R}^+ \to V : \forall t \geq 0 \int_0^t \|f(s)\|^2_V ds < \infty\}.$$

First of all, we state the following result:

Proposition 4.1. *For every $\psi \in V$, $f \in L^2_{loc}(\mathbb{R}^+;H)$ with $\sqrt{y}f \in L^2_{loc}(\mathbb{R}^+;H)$, there exists a unique function $u \in L^2_{loc}(\mathbb{R}^+;V)$ such that $\frac{\partial u}{\partial t} \in L^2_{loc}(\mathbb{R}^+;H)$, $u(0) = \psi$ and*

$$\left(\frac{\partial u}{\partial t}, v\right)_H + a_\lambda(u,v) = (f,v)_H, \quad v \in V. \tag{4.1}$$

Moreover we have, for every $t \geq 0$,

$$||u(t)||^2_H + \frac{\delta_1}{2}\int_0^t ||u(s)||^2_V ds \leq ||\psi||^2_H + \frac{2}{\delta_1}\int_0^t ||f(s)||^2_H ds \tag{4.2}$$

and

$$||u(t)||^2_V + \int_0^t ||u_t(s)||^2_H ds \leq C\left(||\psi||^2_V + \frac{1}{2}\int_0^t ||\sqrt{1+y}f(s)||^2_H ds\right),$$

with $C > 0$.

The proof follows the same lines as the proof of Proposition 3.14 so we omit it. Moreover, we can prove a Comparison Principle for the equation (4.1) as we have done for the variational inequality.

We denote $u(t) = \bar{P}^\lambda_t \psi$ the solution of (4.1) corresponding to $u(0) = \psi$ and $g = 0$. From (4.2) we deduce that the operator $\bar{P}^\lambda_t$ is a linear contraction on H and, from uniqueness, we have the semigroup property.

Proposition 4.2. *Let us consider $f : \mathbb{R}^+ \to H$ such that $\sqrt{1+y}f \in L^2_{loc}(\mathbb{R}^+,H)$. Then, the solution of*

$$\begin{cases} \left(\frac{\partial u}{\partial t}, v\right) + a_\lambda(u,v) = (f,v), & v \in V, \\ u(0) = 0, \end{cases}$$

is given by $u(t) = \int_0^t \bar{P}^\lambda_s f(t-s)ds = \int_0^t \bar{P}^\lambda_{t-s} f(s)ds$.

Proof. Note that V is dense in H and recall the estimate (4.2), so it is enough to prove the assertion for $f = \mathbb{1}_{(t_1,t_2]}\psi$, with $0 \leq t_1 < t_2$ and $\psi \in V$. If we set $u(t) = \int_0^t \bar{P}^\lambda_{t-s} f(s)ds$, we have

$$\begin{aligned} u(t) &= \mathbb{1}_{\{t \geq t_1\}} \int_{t_1}^{t \wedge t_2} \bar{P}^\lambda_{t-s}\psi ds \\ &= \begin{cases} \int_{t_1}^{t_2} \bar{P}^\lambda_{t-s}\psi ds = \int_{t-t_2}^{t-t_1} \bar{P}^\lambda_s \psi ds & \text{if } t \geq t_2 \\ \displaystyle\int_{t_1}^{t} \bar{P}^\lambda_{t-s}\psi ds = \int_0^{t-t_1} \bar{P}^\lambda_s \psi ds & \text{if } t \in [t_1, t_2) \end{cases}. \end{aligned}$$

Therefore, for every $v \in V$, we have $(u_t, v) + a_\lambda(u,v) = 0$ if $t \leq t_1$ and, if $t \geq t_1$,

$$\left(\frac{\partial u}{\partial t}, v\right) + a_\lambda(u(t), v) = \begin{cases} \left(\bar{P}^\lambda_{t-t_1}\psi - \bar{P}^\lambda_{t-t_2}\psi, v\right) + a_\lambda\left(\int_{t-t_2}^{t-t_1} \bar{P}^\lambda_s \psi ds, v\right) & \text{if } t \geq t_2 \\ \left(\bar{P}^\lambda_{t-t_1}\psi, v\right) + a_\lambda\left(\int_0^{t-t_1} \bar{P}^\lambda_s \psi ds, v\right) & \text{if } t \in [t_1, t_2) \end{cases}.$$

The assertion follows from $(\bar{P}^\lambda_t \psi, v) + \int_0^t a_\lambda(\bar{P}_s \psi, v)ds = (\psi, v)$. □

Remark 4.3. *It is not difficult to prove that $\bar{P}^\lambda_t : L^p(\mathcal{O}, \mathfrak{m}) \to L^p(\mathcal{O}, \mathfrak{m})$ is a contraction for every $p \geq 2$, and it is an analytic semigroup. This is not useful to our purposes so we omit the proof.*

4.2 Transition semigroup

We define $\mathbb{E}_{x_0,y_0}(\quad) = \mathbb{E}(\quad|X_0 = x_0, Y_0 = y_0)$. Fix $\lambda > 0$. For every measurable positive function f defined on $\mathbb{R} \times [0, +\infty)$, we define

$$P_t^\lambda f(x_0, y_0) = \mathbb{E}_{x_0,y_0}\left(e^{-\lambda \int_0^t (1+Y_s)ds} f(X_t, Y_t)\right).$$

The operator P_t^λ is the transition semigroup of the two dimensional diffusion (X, Y) with the killing term $e^{-\lambda \int_0^t (1+Y_s)ds}$.

Set $\mathbb{E}_{y_0}(\quad) = \mathbb{E}(\quad|Y_0 = y_0)$. We first prove some useful results about the Laplace transform of the pair $(Y_t, \int_0^t Y_s ds)$. These results rely on the affine structure of the model and have already appeared in slightly different forms in the literature (see, for example, [2, Section 4.2.1]). We include a proof for convenience.

Proposition 4.4. *Let α and β be two complex numbers with nonpositive real parts. The equation*

$$\psi'(t) = \frac{\sigma^2}{2}\psi^2(t) - \kappa\psi(t) + \beta \tag{4.3}$$

has a unique solution $\psi_{\alpha,\beta}$ defined on $[0, +\infty)$, such that $\psi_{\alpha,\beta}(0) = \alpha$. Moreover, for every $t \geq 0$,

$$\mathbb{E}_{y_0}\left(e^{\alpha Y_t + \beta \int_0^t Y_s ds}\right) = e^{y_0\psi_{\alpha,\beta}(t) + \theta\kappa\phi_{\alpha,\beta}(t)},$$

with $\phi_{\alpha,\beta}(t) = \int_0^t \psi_{\alpha,\beta}(s)ds$.

Proof. Let ψ be the solution of (4.3). We define ψ_1 (resp. β_1) and ψ_2 (resp. β_2) the real and the imaginary part of ψ (resp. β). We have

$$\begin{cases} \psi_1'(t) = \frac{\sigma^2}{2}\left(\psi_1^2(t) - \psi_2^2(t)\right) - \kappa\psi_1(t) + \beta_1, \\ \psi_2'(t) = \sigma^2\psi_1(t)\psi_2(t) - \kappa\psi_2(t) + \beta_2. \end{cases}$$

From the first equation we deduce that $\psi_1'(t) \leq \frac{\sigma^2}{2}\left(\psi_1(t) - \frac{2\kappa}{\sigma^2}\right)\psi_1(t) + \beta_1$ and, since $\beta_1 \leq 0$, the function $t \mapsto \psi_1(t)e^{-\int_0^t (\psi_1(s) - \frac{2\kappa}{\sigma^2})ds}$ is nonincreasing. Therefore $\psi_1(t) \leq 0$ if $\psi_1(0) \leq 0$. Multiplying the first equation by $\psi_1(t)$ and the second one by $\psi_2(t)$ and adding we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(|\psi(t)|^2\right) &= \left(\frac{\sigma^2}{2}\psi_1(t) - \kappa\right)|\psi(t)|^2 + \beta_1\psi_1(t) + \beta_2\psi_2(t) \\ &\leq \left(\frac{\sigma^2}{2}\psi_1(t) - \kappa\right)|\psi(t)|^2 + |\beta||\psi(t)| \\ &\leq \left(\frac{\sigma^2}{2}\psi_1(t) - \kappa\right)|\psi(t)|^2 + \epsilon|\psi(t)|^2 + \frac{|\beta|^2}{4\epsilon}. \end{aligned}$$

We deduce that $|\psi(t)|$ cannot explode in finite time and, therefore, $\psi_{\alpha,\beta}$ actually exists on $[0, +\infty)$. Moreover, note that we have $|\psi(t)|^2 \leq C_{\kappa'}e^{-\kappa' t}$ for every $\kappa' < \kappa$.

Now, let us define the function $F_{\alpha,\beta}(t, y) = e^{y\psi_{\alpha,\beta}(t) + \theta\kappa\phi_{\alpha,\beta}(t)}$. $F_{\alpha,\beta}$ is $C^{1,2}$ on $[0, +\infty) \times \mathbb{R}$ and it satisfies by construction the following equation

$$\frac{\partial F_{\alpha,\beta}}{\partial t} = \frac{\sigma^2}{2}y\frac{\partial^2 F_{\alpha,\beta}}{\partial y^2} + \kappa(\theta - y)\frac{\partial F_{\alpha,\beta}}{\partial y} + \beta y F_{\alpha,\beta}.$$

Therefore, for every $T > 0$, the process $(M_t)_{0 \leq t \leq T}$ defined by

$$M_t = e^{\beta \int_0^t Y_s ds} F_{\alpha,\beta}(T - t, Y_t) \tag{4.4}$$

is a local martingale. On the other hand, note that $|M_t| \leq 1$, so the process $(M_t)_t$ is a true martingale indeed. We deduce that $F_{\alpha,\beta}(T, y_0) = \mathbb{E}_{y_0}\left(e^{\beta \int_0^T Y_s ds} e^{\alpha Y_T}\right)$ and the assertion follows. □

Remark 4.5. If we take $\alpha = 0$ and $\beta = -s$, with $s > 0$, we get

$$\mathbb{E}_{y_0}\left(e^{-s\int_0^t Y_v dv}\right) = e^{y_0\psi_{0,-s}(t)+\theta\kappa\phi_{0,-s}(t)},$$

and, as we have seen in the proof of Proposition 4.4, $\psi'_{0,-s}$ has constant sign. Since $\psi'_{0,-s}(0) = -s < 0$, the function $\psi_{0,-s}$ is decreasing. Again from the proof of Proposition 4.4,

$$\psi_{0,-s}(t) \geq \frac{\kappa}{\sigma^2} - \sqrt{\left(\frac{\kappa}{\sigma^2}\right)^2 + 2\frac{s}{\sigma^2}} \geq -\sqrt{2s/\sigma^2}$$

and, integrating, $\phi_{0,-s}(t) \geq -t\sqrt{2s/\sigma^2}$.

We deduce that, for every $y_0 \geq 0$,

$$\mathbb{E}_{y_0}\left(e^{-s\int_0^t Y_v dv}\right) \leq e^{-t\kappa\theta\sqrt{2s/\sigma^2}} = e^{-t\sigma\beta\sqrt{s/2}},$$

and, for every $q > 0$,

$$\begin{aligned}
\mathbb{E}_{y_0}\left(\int_0^t Y_v dv\right)^{-q} &= \mathbb{E}_{y_0}\left(\frac{1}{\Gamma(q)}\int_0^\infty s^{q-1}e^{-s\int_0^t Y_v dv}ds\right)\\
&\leq \frac{1}{\Gamma(q)}\int_0^\infty s^{q-1}e^{-t\sigma\beta\sqrt{s/2}}ds\\
&= \frac{2^q}{t^{2q}(\sigma\beta)^{2q}\Gamma(q)}\int_0^\infty s^{q-1}e^{-\sqrt{s}}ds = \frac{2^{q+1}\Gamma(2q)}{t^{2q}(\sigma\beta)^{2q}\Gamma(q)}.
\end{aligned}$$

We also have the following result.

Proposition 4.6. *Let λ_1 and λ_2 be two real numbers such that*

$$\frac{\sigma^2}{2}\lambda_1^2 - \kappa\lambda_1 + \lambda_2 \leq 0.$$

Then, the equation

$$\psi'(t) = \frac{\sigma^2}{2}\psi^2(t) - \kappa\psi(t) + \lambda_2 \tag{4.5}$$

has a unique solution $\psi_{\lambda_1,\lambda_2}$ defined on $[0, +\infty)$ such that $\psi_{\lambda_1,\lambda_2}(0) = \lambda_1$. Moreover, for every $t \geq 0$, we have

$$\mathbb{E}_{y_0}\left(e^{\lambda_1 Y_t+\lambda_2\int_0^t Y_s ds}\right) \leq e^{y_0\psi_{\lambda_1,\lambda_2}(t)+\theta\kappa\phi_{\lambda_1,\lambda_2}(t)},$$

with $\phi_{\lambda_1,\lambda_2}(t) = \int_0^t \psi_{\lambda_1,\lambda_2}(s)ds$.

Proof. Let ψ be the solution of (4.5) with $\psi(0) = \lambda_1$. We have

$$\psi''(t) = (\sigma^2\psi(t) - \kappa)\psi'(t).$$

Therefore $\psi'(t)$ has constant sign and the assumption on λ_1 and λ_2 ensures that $\psi'(0) \leq 0$. We deduce that $\psi'(t) \leq 0$ and $\psi(t)$ remains between the solutions of the equation

$$\frac{\sigma^2}{2}\lambda^2 - \kappa\lambda + \lambda_2 = 0.$$

This proves that the solution is defined on the whole interval $[0, +\infty)$. Now the assertion follows as in the proof of Proposition 4.4: just note that the the process $(M_t)_t$ defined as in (4.4) is no more a martingale but it remains a positive local martingale, hence a supermartingale. $\square$

Remark 4.7. *Let us now consider two real numbers λ_1 and λ_2 such that*

$$\frac{\sigma^2}{2}\lambda_1^2 - \kappa\lambda_1 + \lambda_2 < 0.$$

From the proof of Proposition 4.6, by using the optimal stopping theorem we have

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_y\left(e^{\mu\int_0^\tau Y_s ds} e^{\psi_{\lambda,\mu}(T-\tau)Y_\tau+\theta\kappa\phi_{\lambda,\mu}(T-\tau)}\right) \quad\leq\quad e^{y\psi_{\lambda,\mu}(T)+\theta\kappa\phi_{\lambda,\mu}(T)}.$$

Consider now $\epsilon > 0$ and let $\lambda_\epsilon = (1+\epsilon)\lambda$ and $\mu_\epsilon = (1+\epsilon)\mu$. For ϵ small enough, we have $\frac{\sigma^2}{2}\lambda_\epsilon^2 - \kappa\lambda_\epsilon + \mu_\epsilon < 0$. Therefore

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_y\left(e^{\mu_\epsilon\int_0^\tau Y_s ds} e^{\psi_{\lambda_\epsilon,\mu_\epsilon}(T-\tau)Y_\tau+\theta\kappa\phi_{\lambda_\epsilon,\mu_\epsilon}(T-\tau)}\right) \quad\leq\quad e^{y\psi_{\lambda_\epsilon,\mu_\epsilon}(T)+\theta\kappa\phi_{\lambda_\epsilon,\mu_\epsilon}(T)}.$$

If we have $\psi_{\lambda_\epsilon,\mu_\epsilon} \geq (1+\epsilon)\psi_{\lambda,\mu}$, we can deduce that

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_y\left(e^{\mu(1+\epsilon)\int_0^\tau Y_s ds} e^{(1+\epsilon)(\psi_{\lambda,\mu}(T-\tau)Y_\tau+\theta\kappa\phi_{\lambda,\mu}(T-\tau))}\right) \quad\leq\quad e^{y\psi_{\lambda_\epsilon,\mu_\epsilon}(T)+\theta\kappa\phi_{\lambda_\epsilon,\mu_\epsilon}(T)},$$

and, therefore, that the family $\left(e^{\mu\int_0^\tau Y_s ds} e^{\psi_{\lambda,\mu}(T-\tau)Y_\tau+\theta\kappa\phi_{\lambda,\mu}(T-\tau)}\right)_{\tau\in\mathcal{T}_{0,T}}$ is uniformly integrable. As a consequence, the process $(M_t)_t$ is a true martingale and we have

$$\mathbb{E}_y\left(e^{\lambda Y_t+\mu\int_0^t Y_s ds}\right) = e^{y\psi_{\lambda,\mu}(t)+\theta\kappa\phi_{\lambda,\mu}(t)}.$$

So, it remains to show that $\psi_{\lambda_\epsilon,\mu_\epsilon} \geq (1+\epsilon)\psi_{\lambda,\mu}$. In order to do this we set $g_\epsilon(t) = \psi_{\lambda_\epsilon,\mu_\epsilon}(t) - (1+\epsilon)\psi_{\lambda,\mu}(t)$. From the equations satisfied by $\psi_{\lambda_\epsilon,\mu_\epsilon}$ and $\psi_{\lambda,\mu}$ we deduce that

$$\begin{aligned}
g'_\epsilon(t) &= \frac{\sigma^2}{2}\left(\psi^2_{\lambda_\epsilon,\mu_\epsilon}(t) - (1+\epsilon)\psi^2_{\lambda,\mu}(t)\right) - \kappa\left(\psi_{\lambda_\epsilon,\mu_\epsilon}(t) - (1+\epsilon)\psi_{\lambda,\mu}(t)\right)\\
&= \frac{\sigma^2}{2}\left(\psi^2_{\lambda_\epsilon,\mu_\epsilon}(t) - (1+\epsilon)^2\psi^2_{\lambda,\mu}(t)\right) - \kappa g_\epsilon(t) + \frac{\sigma^2}{2}\left((1+\epsilon)^2 - (1+\epsilon)\right)\psi^2_{\lambda,\mu}(t)\\
&= \frac{\sigma^2}{2}\left(\psi_{\lambda_\epsilon,\mu_\epsilon}(t) + (1+\epsilon)\psi_{\lambda,\mu}(t)\right)g_\epsilon(t) - \kappa g_\epsilon(t) + \frac{\sigma^2}{2}\epsilon(1+\epsilon)\psi^2_{\lambda,\mu}(t)\\
&= f_\epsilon(t)g_\epsilon(t) + \frac{\sigma^2}{2}\epsilon(1+\epsilon)\psi^2_{\lambda,\mu}(t),
\end{aligned}$$

where

$$f_\epsilon(t) = \frac{\sigma^2}{2}\left(\psi_{\lambda_\epsilon,\mu_\epsilon}(t) + (1+\epsilon)\psi_{\lambda,\mu}(t)\right) - \kappa.$$

Therefore, the function $g_\epsilon(t)e^{-\int_0^t f_\epsilon(s)ds}$ is nondecreasing and, since $g_\epsilon(0) = 0$, we have $g_\epsilon(t) \geq 0$.

Now recall that the diffusion (X, Y) evolves according to the following stochastic differential system

$$\begin{cases} dX_t = \left(\frac{\rho\kappa\theta}{\sigma} - \frac{Y_t}{2}\right)dt + \sqrt{Y_t}dB_t,\\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t.\end{cases}$$

If we set $\tilde{X}_t = X_t - \frac{\rho}{\sigma}Y_t$, we have

$$\begin{cases} d\tilde{X}_t = \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)Y_t dt + \sqrt{1-\rho^2}\sqrt{Y_t}d\tilde{B}_t,\\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t.\end{cases} \tag{4.6}$$

where $\tilde{B}_t = (1-\rho^2)^{-1/2}\left(B_t - \rho W_t\right)$. Note that $\tilde{B}$ is a standard Brownian motion with $\langle\tilde{B}, W\rangle_t = 0$.

Proposition 4.8. *For all u, $v \in \mathbb{R}$, for all $\lambda \geq 0$ and for all $(x_0, y_0) \in \mathbb{R} \times [0, +\infty)$ we have*

$$\mathbb{E}_{x_0,y_0}\left(e^{iuX_t+ivY_t}e^{-\lambda\int_0^t Y_s ds}\right) = e^{iux_0+y_0(\psi_{\lambda_1,\mu}(t)-iu\frac{\rho}{\sigma})+\theta\kappa\phi_{\lambda_1,\mu}(t)},$$

where $\lambda_1 = i(u\frac{\rho}{\sigma} + v)$, $\mu = iu\left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right) - \frac{u^2}{2}(1-\rho^2) - \lambda$ and the function $\psi_{\lambda_1,\mu}$ and $\phi_{\lambda_1,\mu}$ are defined in Proposition 4.6.

Proof. We have

$$\mathbb{E}_{x_0,y_0}\left(e^{iuX_t+ivY_t-\lambda\int_0^t Y_s ds}\right) = \mathbb{E}_{x_0,y_0}\left(e^{iu(\tilde{X}_t+\frac{\rho}{\sigma}Y_t)+ivY_t-\lambda\int_0^t Y_s ds}\right)$$

and

$$\tilde{X}_t = x_0 - \frac{\rho}{\sigma}y_0 + \int_0^t\left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)Y_s ds + \int_0^t\sqrt{(1-\rho^2)Y_s}d\tilde{B}_s.$$

Since $\tilde{B}$ and W are independent,

$$\mathbb{E}\left(e^{iu\tilde{X}_t} \mid W\right) = e^{iu\left(x_0-\frac{\rho}{\sigma}y_0+\int_0^t\left(\frac{\rho\kappa}{\sigma}-\frac{1}{2}\right)Y_s ds\right)-\frac{u^2}{2}(1-\rho^2)\int_0^t Y_s ds}$$

and

$$\mathbb{E}_{x_0,y_0}\left(e^{iuX_t+ivY_t-\lambda\int_0^t Y_s ds}\right) = e^{iu\left(x_0-\frac{\rho}{\sigma}y_0\right)}\mathbb{E}_{y_0}\left(e^{i\left(u\frac{\rho}{\sigma}+v\right)Y_t+\left(iu\left(\frac{\rho\kappa}{\sigma}-\frac{1}{2}\right)-\frac{u^2}{2}(1-\rho^2)-\lambda\right)\int_0^t Y_s ds}\right).$$

Then the assertion follows by using Proposition 4.4. □

4.3 Identification of the semigroups

We now show that the semigroup associated with the coercive bilinear form $\bar{P}_t^\lambda$ can be actually identified with the transition semigroup P_t^λ.

Proposition 4.9. *We have, for every function $f \in H$ and for every $t \geq 0$,*

$$\bar{P}_t^\lambda f(x,y) = P_t^\lambda f(x,y), \quad dxdy \ a.e.$$

Proof. We only need to prove the equality for $f(x,y) = e^{iux+ivy}$ with u, $v \in \mathbb{R}$. We then have, by using Proposition 4.8,

$$\begin{aligned} P_t^\lambda f(x,y) &= \mathbb{E}_{x,y}\left(e^{-\lambda\int_0^t(1+Y_s)ds}e^{iuX_t+ivY_t}\right) \\ &= e^{-\lambda t}e^{iux+y\left(\psi_{\lambda_1,\mu}(t)-iu\frac{\rho}{\sigma}\right)+\theta\kappa\phi_{\lambda_1,\mu}(t)}, \end{aligned}$$

with $\lambda_1 = i(u\frac{\rho}{\sigma} + v)$, $\mu = iu\left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right) - \frac{u^2}{2}(1-\rho^2) - \lambda$. The function $F(t,x,y)$ defined by $F(t,x,y) = e^{-\lambda t}e^{iux+y\left(\psi_{\lambda_1,\mu}(t)-iu\frac{\rho}{\sigma}\right)+\theta\kappa\phi_{\lambda_1,\mu}(t)}$ satisfies $F(0,x,y) = e^{iux+ivy}$ and

$$\frac{\partial F}{\partial t} = \left(\mathcal{L} - \lambda(1+y)\right)F.$$

Moreover, for every $t \geq 0$, we have $\int_{\mathcal{O}}\left(y|D^2F|^2(t,x,y) + |\nabla F|^2(t,x,y)\right)d\mathfrak{m} < \infty$, so that, for every $v \in V$, $(\mathcal{L}F(t,.,.), v) = -a(F(t,.,.), v)$. Therefore

$$\left(\frac{\partial F}{\partial t}, v\right) + a_\lambda(F(t,.,.), v) = 0 \quad v \in V,$$

and $F(t,.,.) = \bar{P}_t^\lambda f$. □

4.4 Estimates on the joint law

In this section we prove some estimates on the joint law of the diffusion (X,Y) which will be crucial in order to prove Proposition 2.4. With the notations

$$\nu=\beta-1=\frac{2\kappa\theta}{\sigma^2}-1,\quad y_t=y_0e^{-\kappa t},\quad L_t=\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa t}\right),$$

it is well known (see, for example, [13, Section 6.2.2]) that the transition density of the process Y is given by

$$p_t(y_0,y)=\frac{e^{-\frac{y_t}{2L_t}}}{2y_t^{\nu/2}L_t}e^{-\frac{y}{2L_t}}y^{\nu/2}I_\nu\left(\frac{\sqrt{yy_t}}{L_t}\right),$$

where I_ν is the first-order modified Bessel function with index ν, defined by

$$I_\nu(y)=\left(\frac{y}{2}\right)^\nu\sum_{n=0}^\infty\frac{(y/2)^{2n}}{n!\Gamma(n+\nu+1)}.$$

It is clear that near $y=0$ we have $I_\nu(y)\sim\frac{1}{\Gamma(\nu+1)}\left(\frac{y}{2}\right)^\nu$ while, for $y\to\infty$, we have the asymptotic behaviour $I_\nu(y)\sim e^y/\sqrt{2\pi y}$ (see [1, page 377]).

Proposition 4.10. *There exists a constant $C_\beta>0$ (which depends only on β) such that, for every $t>0$,*

$$p_t(y_0,y)\le\frac{C_\beta}{L_t^{\beta+\frac12}}e^{-\frac{(\sqrt{y}-\sqrt{y_t})^2}{2L_t}}y^{\beta-1}\left(L_t^{1/2}+(yy_t)^{1/4}\right),\qquad(y_0,y)\in[0,+\infty)\times]0,+\infty).$$

Proof. From the asymptotic behaviour of I_ν near 0 and ∞ we deduce the existence of a constant $C_\nu>0$ such that

$$I_\nu(x)\le C_\nu\left(x^\nu\mathbb{1}_{\{x\le1\}}+\frac{e^x}{\sqrt{x}}\mathbb{1}_{\{x>1\}}\right).$$

Therefore

$$\begin{aligned}p_t(y_0,y)&=\frac{e^{-\frac{y_t+y}{2L_t}}}{2y_t^{\nu/2}L_t}y^{\nu/2}I_\nu\left(\frac{\sqrt{yy_t}}{L_t}\right)\\&\le\frac{e^{-\frac{y_t+y}{2L_t}}}{2y_t^{\nu/2}L_t}y^{\nu/2}C_\nu\left(\frac{(yy_t)^{\nu/2}}{L_t^\nu}\mathbb{1}_{\{yy_t\le L_t^2\}}+\frac{e^{\frac{\sqrt{yy_t}}{L_t}}}{(yy_t)^{1/4}/L_t^{1/2}}\mathbb{1}_{\{yy_t>L_t^2\}}\right)\\&=\frac{C_\nu}{2}e^{-\frac{y_t+y}{2L_t}}\left(\frac{y^\nu}{L_t^{\nu+1}}\mathbb{1}_{\{yy_t\le L_t^2\}}+\frac{y^{\frac{\nu}{2}-\frac14}e^{\frac{\sqrt{yy_t}}{L_t}}}{(y_t)^{\frac{\nu}{2}+\frac14}L_t^{1/2}}\mathbb{1}_{\{yy_t>L_t^2\}}\right).\end{aligned}$$

On $\{yy_t>L_t^2\}$, we have $y_t^{-1}\le y/L_t^2$ and, since $\nu+1>0$,

$$\frac{y^{\frac{\nu}{2}-\frac14}}{(y_t)^{\frac{\nu}{2}+\frac14}}=y_t^{1/4}\frac{y^{\frac{\nu}{2}-\frac14}}{(y_t)^{\frac{\nu}{2}+\frac12}}\le y_t^{1/4}\frac{y^{\nu+\frac14}}{L_t^{\nu+1}}.$$

So

$$\begin{aligned}p_t(y_0,y)&\le\frac{C_\nu}{2}e^{-\frac{y_t+y}{2L_t}}\left(\frac{y^\nu}{L_t^{\nu+1}}\mathbb{1}_{\{yy_t\le L_t^2\}}+\frac{(yy_t)^{1/4}y^\nu e^{\frac{\sqrt{yy_t}}{L_t}}}{L_t^{\nu+\frac32}}\mathbb{1}_{\{yy_t>L_t^2\}}\right)\\&\le\frac{C_\nu}{2L_t^{\nu+\frac32}}e^{-\frac{y_t+y}{2L_t}}y^\nu e^{\frac{\sqrt{yy_t}}{L_t}}\left(L_t^{1/2}\mathbb{1}_{\{yy_t\le L_t^2\}}+(yy_t)^{1/4}\mathbb{1}_{\{yy_t>L_t^2\}}\right)\\&=\frac{C_\nu}{2L_t^{\nu+\frac32}}e^{-\frac{(\sqrt{y}-\sqrt{y_t})^2}{2L_t}}y^\nu\left(L_t^{1/2}\mathbb{1}_{\{yy_t\le L_t^2\}}+(yy_t)^{1/4}\mathbb{1}_{\{yy_t>L_t^2\}}\right),\end{aligned}$$

and the assertion follows. $\square$

Theorem 4.11. *For all $p>1$, $\gamma>0$ and $\mu>0$ there exists $\lambda>0$ such that, for every compact $K\subseteq\mathbb{R}\times[0,+\infty)$ and for every $T>0$, there is $C_{p,K,T}>0$ such that*

$$P^\lambda_t f(x_0,y_0)\le \frac{C_{p,K,T}}{t^{\frac{\beta}{p}+\frac{3}{2p}}}||f||_{L^p(m_{\gamma,\mu})},\qquad (x_0,y_0)\in K.$$

for every measurable positive function f on $\mathbb{R}\times[0,+\infty)$ and for every $t\in(0,T]$.

Proof. Note that

$$P^\lambda_t f(x_0,y_0) = \mathbb{E}_{x_0,y_0}\left(e^{-\lambda\int_0^t(1+Y_s)ds}\tilde{f}(\tilde{X}_t,Y_t)\right),$$

where

$$\tilde{f}(x,y)=f\left(x+\frac{\rho}{\sigma}y,y\right)\quad\text{and}\quad \tilde{X}_t=X_t-\frac{\rho}{\sigma}Y_t.$$

Recall that the dynamics of $\tilde{X}$ is given by (4.6) so we have

$$\tilde{X}_t=\tilde{x}_0+\bar{\kappa}\int_0^t Y_s ds+\bar{\rho}\int_0^t\sqrt{Y_s}d\tilde{B}_s,$$

with

$$\tilde{x}_0=x_0-\frac{\rho}{\sigma}y_0,\quad \bar{\kappa}=\frac{\rho\kappa}{\sigma}-\frac{1}{2},\quad \bar{\rho}=\sqrt{1-\rho^2}.$$

Recall that the Brownian motion $\tilde{B}$ is independent of the process Y. We set $\Sigma_t=\sqrt{\int_0^t Y_s ds}$ and $n(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Therefore

$$\begin{aligned}
P^\lambda_t f(x_0,y_0) &= \mathbb{E}_{y_0}\left(e^{-\lambda t-\lambda\Sigma_t^2}\int\tilde{f}\left(\tilde{x}_0+\bar{\kappa}\Sigma_t^2+\bar{\rho}\Sigma_t z,Y_t\right)n(z)dz\right)\\
&\le \mathbb{E}_{y_0}\left(e^{-\lambda\Sigma_t^2}\int\tilde{f}\left(\tilde{x}_0+\bar{\kappa}\Sigma_t^2+\bar{\rho}\Sigma_t z,Y_t\right)n(z)dz\right)\\
&= \mathbb{E}_{y_0}\left(e^{-\lambda\Sigma_t^2}\int\tilde{f}\left(\tilde{x}_0+z,Y_t\right)n\left(\frac{z-\bar{\kappa}\Sigma_t^2}{\bar{\rho}\Sigma_t}\right)\frac{dz}{\bar{\rho}\Sigma_t}\right).
\end{aligned}$$

Hölder inequality with respect to the measure $e^{-\gamma|z|-\bar{\mu}Y_t}dzd\mathbb{P}_{y_0}$, where $\gamma>0$ and $\bar{\mu}$ will be chosen later on gives, for every $p>1$

$$P^\lambda_t f(x_0,y_0)\le\left[\mathbb{E}_{y_0}\left(\int e^{-\gamma|z|-\bar{\mu}Y_t}\tilde{f}^p\left(\tilde{x}_0+z,Y_t\right)dz\right)\right]^{1/p}J_q,\tag{4.7}$$

with $q=p/(p-1)$ and

$$J_q^q=\mathbb{E}_{y_0}\left(\int e^{(q-1)\gamma|z|+(q-1)\bar{\mu}Y_t-q\lambda\Sigma_t^2}n^q\left(\frac{z-\bar{\kappa}\Sigma_t^2}{\bar{\rho}\Sigma_t}\right)\frac{dz}{(\bar{\rho}\Sigma_t)^q}\right).$$

Using Proposition 4.10 we can write, for every $z\in\mathbb{R}$,

$$\begin{aligned}
\mathbb{E}_{y_0}\left(e^{-\bar{\mu}Y_t}\tilde{f}^p\left(\tilde{x}_0+z,Y_t\right)\right) &= \int_0^\infty dy p_t(y_0,y)e^{-\bar{\mu}y}\tilde{f}^p\left(\tilde{x}_0+z,y\right)\\
&\le \frac{C_\beta(\frac{\sigma^2}{4\kappa}+y_0^{1/4})}{L_t^{\beta+\frac{1}{2}}}\int_0^\infty dy e^{-\frac{(\sqrt{y}-\sqrt{y_t})^2}{2L_t}-\bar{\mu}y}y^{\beta-1}\left(1+y^{1/4}\right)\tilde{f}^p\left(\tilde{x}_0+z,y\right).
\end{aligned}$$

If we set $L_\infty = \sigma^2/(4\kappa)$, for every $\epsilon \in (0,1)$ we have

$$
\begin{aligned}
e^{-\frac{(\sqrt{y}-\sqrt{y_t})^2}{2L_t}} &\le e^{-\frac{(\sqrt{y}-\sqrt{y_t})^2}{2L_\infty}} \\
&= e^{-\frac{y}{2L_\infty}} e^{\frac{\sqrt{yy_t}}{L_\infty}-\frac{y_t}{2L_\infty}} \\
&\le e^{-\frac{y}{2L_\infty}} e^{\epsilon\frac{y}{2L_\infty}} e^{\frac{y_t}{2\epsilon L_\infty}} e^{-\frac{y_t}{2L_\infty}} \\
&= e^{-(1-\epsilon)\frac{y}{2L_\infty}} e^{\frac{y_t}{2\epsilon L_\infty}(1-\epsilon)} \\
&\le e^{-(1-\epsilon)\frac{y}{2L_\infty}} e^{\frac{y_0}{2\epsilon L_\infty}(1-\epsilon)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}_{y_0}\left(e^{-\bar{\mu}Y_t}\tilde{f}^p\left(\tilde{x}_0+z,Y_t\right)\right) &\le \frac{C_\beta e^{\frac{y_0(1-\epsilon)}{2\epsilon L_\infty}}\left(\frac{\sigma^2}{4\kappa}+y_0^{1/4}\right)}{L_t^{\beta+\frac{1}{2}}}\int_0^\infty dy e^{-y\left(\bar{\mu}+\frac{1-\epsilon}{2L_\infty}\right)} y^{\beta-1}\left(1+y^{1/4}\right)\tilde{f}^p\left(\tilde{x}_0+z,y\right) \\
&\le \frac{C_{\beta,\sigma,\kappa,\epsilon} e^{\frac{y_0(1-\epsilon)}{\epsilon L_\infty}}}{L_t^{\beta+\frac{1}{2}}}\int_0^\infty dy e^{-y\left(\bar{\mu}+\frac{1-2\epsilon}{2L_\infty}\right)} y^{\beta-1}\tilde{f}^p\left(\tilde{x}_0+z,y\right).
\end{aligned}
$$

As regards J_q, setting $z' = \frac{z-\bar{\kappa}\Sigma_t^2}{\bar{\rho}\Sigma_t}$, we have

$$
\begin{aligned}
J_q^q &= \mathbb{E}_{y_0}\left(\int e^{(q-1)\gamma|z'\bar{\rho}\Sigma_t+\bar{\kappa}\Sigma_t^2|+(q-1)\bar{\mu}Y_t-q\lambda\Sigma_t^2} n^q\left(z'\right)\frac{dz'}{(\bar{\rho}\Sigma_t)^{q-1}}\right) \\
&\le \mathbb{E}_{y_0}\left(\int e^{(q-1)\gamma\bar{\rho}\Sigma_t|z|+(q-1)\bar{\mu}Y_t+((q-1)|\bar{\kappa}|\gamma-q\lambda)\Sigma_t^2} n^q\left(z\right)\frac{dz}{(\bar{\rho}\Sigma_t)^{q-1}}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int e^{(q-1)\gamma\bar{\rho}\Sigma_t|z|} n^q(z)\,dz &= \int e^{(q-1)\gamma\bar{\rho}\Sigma_t|z|} e^{-qz^2/2}\frac{dz}{\sqrt{2\pi}} \\
&\le 2\int e^{(q-1)\gamma\bar{\rho}\Sigma_t z} e^{-qz^2/2}\frac{dz}{\sqrt{2\pi}} \\
&= 2\int e^{(q-1)\gamma\bar{\rho}\Sigma_t z} e^{-qz^2/2}\frac{dz}{\sqrt{2\pi}} \\
&= \frac{2}{\sqrt{q}} e^{\frac{(q-1)^2}{q}\gamma^2\bar{\rho}^2\Sigma_t^2},
\end{aligned}
$$

so that

$$
J_q^q \le \frac{2}{\sqrt{q}}\mathbb{E}_{y_0}\left(e^{(q-1)\bar{\mu}Y_t+\bar{\lambda}_q\Sigma_t^2}\frac{1}{(\bar{\rho}\Sigma_t)^{q-1}}\right),
$$

with

$$
\bar{\lambda}_q = (q-1)|\bar{\kappa}|\gamma + \frac{(q-1)^2}{q}\gamma^2\bar{\rho}^2 - q\lambda = \frac{1}{p-1}\left(|\bar{\kappa}|\gamma + \frac{1}{p}\gamma^2\bar{\rho}^2 - p\lambda\right).
$$

Using Hölder's inequality again we get, for every $p_1 > 1$ and $q_1 = p_1/(p_1-1)$,

$$
\begin{aligned}
J_q^q &\le \frac{2}{\sqrt{q}}\left(\mathbb{E}_{y_0}\left(e^{p_1(q-1)\bar{\mu}Y_t+p_1\bar{\lambda}_q\Sigma_t^2}\right)\right)^{1/p_1}\left(\mathbb{E}_{y_0}\left(\frac{1}{(\bar{\rho}\Sigma_t)^{q_1(q-1)}}\right)\right)^{1/q_1} \\
&\le \frac{C_{q,q_1}}{t^{q-1}}\left(\mathbb{E}_{y_0}\left(e^{p_1(q-1)\bar{\mu}Y_t+p_1\bar{\lambda}_q\Sigma_t^2}\right)\right)^{1/p_1},
\end{aligned}
$$

where the last inequality follows from Remark 4.5.

We now apply Proposition 4.6 with $\lambda_1 = p_1(q-1)\bar{\mu}$ et $\lambda_2 = p_1\bar{\lambda}_q$. The assumption on λ_1 and λ_2 becomes

$$\frac{\sigma^2}{2}p_1(q-1)\bar{\mu}^2 - \kappa\bar{\mu} + |\bar{\kappa}|\gamma + \frac{1}{p}\gamma^2\bar{\rho}^2 - p\lambda \leq 0$$

or, equivalently,

$$\lambda \geq \frac{\sigma^2}{2p(p-1)}p_1\bar{\mu}^2 - \kappa\frac{\bar{\mu}}{p} + |\bar{\kappa}|\frac{\gamma}{p} + \frac{1}{p^2}\gamma^2\bar{\rho}^2.$$

Note that the last inequality is satisfied for at least a $p_1 > 1$ if and only if

$$\lambda > \frac{\sigma^2}{2p(p-1)}\bar{\mu}^2 - \kappa\frac{\bar{\mu}}{p} + |\bar{\kappa}|\frac{\gamma}{p} + \frac{1}{p^2}\gamma^2\bar{\rho}^2. \tag{4.8}$$

Going back to (4.7) under the condition (4.8), we have

$$\begin{aligned}
P_t^\lambda f(x_0,y_0) &\leq \frac{C_{p,\epsilon}}{L_t^{\frac{\beta}{p}+\frac{1}{2p}}t^{1/p}}e^{A_{p,\epsilon}y_0}\left(\int dz e^{-\gamma|z|}\int_0^\infty dy e^{-y\left(\bar{\mu}+\frac{1-2\epsilon}{2L_\infty}\right)}y^{\beta-1}\tilde{f}^p\left(\tilde{x}_0+z,y\right)\right)^{1/p} \\
&\leq \frac{C_{p,\epsilon}e^{A_{p,\epsilon}y_0}}{t^{\frac{\beta}{p}+\frac{3}{2p}}}\left(\int dz e^{-\gamma|z|}\int_0^\infty dy e^{-y\left(\bar{\mu}+\frac{1-2\epsilon}{2L_\infty}\right)}y^{\beta-1}f^p\left(\tilde{x}_0+z+\frac{\rho}{\sigma}y,y\right)\right)^{1/p} \\
&= \frac{C_{p,\epsilon}e^{A_{p,\epsilon}y_0}}{t^{\frac{\beta}{p}+\frac{3}{2p}}}\left(\int dz e^{-\gamma|z-\tilde{x}_0-\frac{\rho}{\sigma}y|}\int_0^\infty dy e^{-y\left(\bar{\mu}+\frac{1-2\epsilon}{2L_\infty}\right)}y^{\beta-1}f^p\left(z,y\right)\right)^{1/p} \\
&\leq \frac{C_{p,\epsilon}e^{A_{p,\epsilon}y_0+\gamma|\tilde{x}_0|}}{t^{\frac{\beta}{p}+\frac{3}{2p}}}\left(\int dz e^{-\gamma|z|}\int_0^\infty dy e^{-y\left(\bar{\mu}-\gamma\frac{|\rho|}{\sigma}+\frac{1-2\epsilon}{2L_\infty}\right)}y^{\beta-1}f^p\left(z,y\right)\right)^{1/p}.
\end{aligned}$$

If we choose $\epsilon = 1/2$ and $\bar{\mu} = \mu + \gamma\frac{|\rho|}{\sigma}$, the assertion follows provided λ satisfies

$$\lambda > \frac{\sigma^2}{2p(p-1)}\left(\mu + \gamma\frac{|\rho|}{\sigma}\right)^2 - \kappa\frac{\mu+\gamma\frac{|\rho|}{\sigma}}{p} + |\bar{\kappa}|\frac{\gamma}{p} + \frac{1}{p^2}\gamma^2\bar{\rho}^2.$$

□

Now, note that. we can easily prove the continuous dependence of the process X with respect to the initial state.

Lemma 4.12. *Fix $(x,y) \in \mathbb{R}\times[0,+\infty)$. Denote by $(X_t^{x,y}, Y_t^y)_{t\geq 0}$ the solution of the system*

$$\begin{cases} dX_t = \left(\frac{\rho\kappa\theta}{\sigma} - \frac{Y_t}{2}\right)dt + \sqrt{Y_t}dB_t, \\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t, \end{cases}$$

with $X_0 = x$, $Y_0 = y$ and $\langle B, W\rangle_t = \rho t$. We have, for every $t \geq 0$ and for every (x,y), $(x',y') \in \mathbb{R}\times[0,+\infty)$, $\mathbb{E}\left|Y_t^{y'} - Y_t^y\right| \leq |y'-y|$ and

$$\mathbb{E}\left|X_t^{x',y'} - X_t^{x,y}\right| \leq |x'-x| + \frac{t}{2}|y'-y| + \sqrt{t|y'-y|}.$$

The proof of Lemma 4.12 is straightforward so we omit the details: the inequality $\mathbb{E}\left|Y_t^{y'} - Y_t^y\right| \leq |y'-y|$ can be proved by using standard techniques for the CIR process introduced in [10, Section IV.3] and the other inequality easily follows.

Then, thanks to Theorem 4.11, we have the following result.

Proposition 4.13. *Fix $p > 1$ and λ as in Theorem 4.11. If $\varphi \in L^p(\mathcal{O}, \mathfrak{m})$ then, $(t,x,y) \to P_t^\lambda\varphi(x,y)$ is continuous on $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$.*

Proof. If $((t_n, x_n, y_n))_n$ converges to (t, x, y), we deduce from Lemma 4.12 that $X^{x_n,y_n}_{t_n} \to X^{x,y}_t$, $Y^{y_n}_{t_n} \to Y^y_t$ and $\int_0^{t_n} Y^{y_n}_s ds \to \int_0^t Y^y_s ds$ in probability. Therefore, the assertion is trivial if φ is bounded continuous. If $\varphi \in \mathcal{L}^p(\mathcal{O}, \mathfrak{m})$, φ is the limit in L^p of a sequence of bounded continuous functions $(\varphi_n)_n$. Moreover, thanks to Theorem 4.11, for every compact $K \subseteq \mathbb{R} \times [0, +\infty)$, there is $C_{p,K,T} > 0$ such that

$$P^\lambda_t |\varphi_n - \varphi|(x, y) \leq \frac{C_{p,K,T}}{t^{\frac{\beta}{p}+\frac{3}{2p}}} ||\varphi_n - \varphi||_{L^p(m_{\gamma,\mu})}, \qquad (x, y) \in K.$$

Therefore $P^\lambda_t \varphi_n(x, y)$ converges locally uniformly to $P^\lambda_t \varphi(x, y)$ and the assertion follows. □

4.5 Proof of Theorem 2.4

We are finally ready to prove the identification Theorem 2.4. We first prove the result under further regularity assumptions on the payoff function ψ, then we deduce the general statement by an approximation technique.

4.5.1 Case with a regular function ψ

The following two regularity results pave the way for the identification theorem in the case of a regular payoff function.

Proposition 4.14. *Assume that ψ satisfies Assumption $\mathcal{H}^1$ and $0 \leq \psi \leq \Phi$ with Φ satisfying Assumption $\mathcal{H}^2$. If moreover we assume $\psi \in L^2([0, T]; H^2(\mathcal{O}, \mathfrak{m}))$ and $\frac{\partial \psi}{\partial t} + \mathcal{L}\psi$, $(1 + y)\Phi \in L^p([0, T]; L^p(\mathcal{O}, \mathfrak{m}))$ for $p \geq 2$, then there exists $\lambda_0 > 0$ and $F \in L^p([0, T]; L^p(\mathcal{O}, \mathfrak{m}))$ such that for all $\lambda \geq \lambda_0$ the solution u of (2.4) satisfies*

$$-\left(\frac{\partial u}{\partial t}, v\right)_H + a_\lambda(u, v) = (F, v)_H, \qquad v \in V. \tag{4.9}$$

Proof. Note that, for λ large enough, u can be seen as the solution u_λ of an equivalent coercive variational inequality, that is

$$-\left(\frac{\partial u_\lambda}{\partial t}, v - u_\lambda\right)_H + a_\lambda(u_\lambda, v - u_\lambda) \geq (g, v - u_\lambda)_H,$$

where $g = \lambda(1 + y)u$ satisfies the assumptions of Theorem 3.14. Therefore, there exists a sequence $(u_{\varepsilon,\lambda})_\varepsilon$ of non negative functions such that $\lim_{\varepsilon\to 0} u_{\varepsilon,\lambda} = u_\lambda$ and

$$-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}, v\right)_H + a_\lambda(u_{\varepsilon,\lambda}, v) + (\zeta_\varepsilon(u_{\varepsilon,\lambda}), v)_H = (g, v)_H, \qquad v \in V.$$

Since both $u_{\varepsilon,\lambda}$ and ψ are positive and ψ belongs to $L^p([0, T]; L^p(\mathcal{O}, \mathfrak{m}))$, we have $(\psi - u_{\varepsilon,\lambda})_+ \in L^p([0, T]; L^p(\mathcal{O}, \mathfrak{m}))$. Taking $v = (\psi - u_{\varepsilon,\lambda})^{p-1}_+$ and assuming that ψ is bounded we observe that $v \in L^2([0, T]; V)$ and we can write

$$-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t}, (\psi - u_{\varepsilon,\lambda})^{p-1}_+\right)_H + a_\lambda(u_{\varepsilon,\lambda}, (\psi - u_{\varepsilon,\lambda})^{p-1}_+) - \frac{1}{\varepsilon}\|(\psi - u_{\varepsilon,\lambda})_+\|^p_{L^p(\mathcal{O},\mathfrak{m})} = \left(g, (\psi - u_{\varepsilon,\lambda})^{p-1}_+\right)_H,$$

so that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|(\psi - u_{\varepsilon,\lambda})_+\|^p_{L^p(\mathcal{O},\mathfrak{m})} - a_\lambda(\psi - u_{\varepsilon,\lambda}, (\psi - u_{\varepsilon,\lambda})^{p-1}_+) - \frac{1}{\varepsilon}\|(\psi - u_{\varepsilon,\lambda})_+\|^p_{L^p(\mathcal{O},\mathfrak{m})}\\
&\quad = \left(g, (\psi - u_{\varepsilon,\lambda})^{p-1}_+\right)_H - \left(\frac{\partial \psi}{\partial t}, (\psi - u_{\varepsilon,\lambda})^{p-1}_+\right)_H + a_\lambda(\psi, (\psi - u_{\varepsilon,\lambda})^{p-1}_+).
\end{aligned}$$

Integrating from 0 to T we get

$$\begin{aligned}
&-\frac{1}{p}\|(\psi - u_{\varepsilon,\lambda})_+(0)\|^p_{L^p(\mathcal{O},\mathfrak{m})} - \int_0^T a_\lambda((\psi - u_{\varepsilon,\lambda})(t), (\psi - u_{\varepsilon,\lambda})^{p-1}_+(t)dt - \frac{1}{\varepsilon}\int_0^T \|(\psi - u_{\varepsilon,\lambda})_+(t)\|^p_{L^p(\mathcal{O},\mathfrak{m})}dt\\
&= \int_0^T \left(g(t), (\psi - u_{\varepsilon,\lambda})^{p-1}_+(t)\right)_H dt - \int_0^T \left(\frac{\partial \psi}{\partial t}(t), (\psi - u_{\varepsilon,\lambda})^{p-1}_+(t)\right)_H dt + \int_0^T a_\lambda(\psi(t), (\psi - u_{\varepsilon,\lambda})^{p-1}_+(t))dt.
\end{aligned} \tag{4.10}$$

Now, with the usual integration by parts,

$$
\begin{aligned}
&a_\lambda((\psi-u_{\varepsilon,\lambda})_+,(\psi-u_{\varepsilon,\lambda})_+^{p-1})\\
&=\int_{\mathcal{O}}\frac{y}{2}(p-1)(\psi-u_{\varepsilon,\lambda})_+^{p-2}\left[\left(\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}\right)^2+2\rho\sigma\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}+\sigma^2\left(\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}\right)^2\right]d\mathfrak{m}\\
&\quad+\int_{\mathcal{O}}y\left(j_{\gamma,\mu}(x)\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}+k_{\gamma,\mu}(x)\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}\right)(\psi-u_{\varepsilon,\lambda})_+^{p-1}d\mathfrak{m}+\lambda\int_{\mathcal{O}}(1+y)(\psi-u_{\varepsilon,\lambda})_+^{p}d\mathfrak{m}\\
&\geq\delta_1(p-1)\int_{\mathcal{O}}y(\psi-u_{\varepsilon,\lambda})_+^{p-2}\left[\left(\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}\right)^2+\left(\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}\right)^2\right]d\mathfrak{m}\\
&\quad+\int_{\mathcal{O}}y\left(j_{\gamma,\mu}(x)\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}+k_{\gamma,\mu}(x)\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}\right)(\psi-u_{\varepsilon,\lambda})_+^{p-1}d\mathfrak{m}+\lambda\int_{\mathcal{O}}y(\psi-u_{\varepsilon,\lambda})_+^{p}d\mathfrak{m}\\
&=\int_{\mathcal{O}}y(\psi-u_{\varepsilon,\lambda})_+^{p-2}\left[\delta_1(p-1)\left(\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}\right)^2+j_{\gamma,\mu}(x)\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial x}(\psi-u_{\varepsilon,\lambda})_++\frac{\lambda}{2}(\psi-u_{\varepsilon,\lambda})_+^2\right]d\mathfrak{m}\\
&\quad+\int_{\mathcal{O}}y(\psi-u_{\varepsilon,\lambda})_+^{p-2}\left[\delta_1(p-1)\left(\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}\right)^2+k_{\gamma,\mu}(x)\frac{\partial(\psi-u_{\varepsilon,\lambda})_+}{\partial y}(\psi-u_{\varepsilon,\lambda})_++\frac{\lambda}{2}(\psi-u_{\varepsilon,\lambda})_+^2\right]d\mathfrak{m}\\
&\geq 0,
\end{aligned}
$$

since, for λ large enough, the quadratic forms $(a,b)\to\delta_1(p-1)a^2+j_{\gamma.\mu}ab+\frac{\lambda}{2}b^2$ and $(a,b)\to\delta_1(p-1)a^2+k_{\gamma.\mu}ab+\frac{\lambda}{2}b^2$ are both positive definite.

Recall that $\psi\in L^2([0,T];H^2(\mathcal{O},\mathfrak{m}))$, $\frac{\partial\psi}{\partial t}+\mathcal{L}\psi\in L^p([0,T],L^p(\mathcal{O},\mathfrak{m}))$, $(1+y)\psi\leq(1+y)\Phi\in L^p([0,T],L^p(\mathcal{O},\mathfrak{m}))$ and $g=(1+y)u\leq(1+y)\Phi\in L^p([0,T];L^p(\mathcal{O},\mathfrak{m}))$. Therefore, going back to (4.10) and using Holder's inequality,

$$
\frac{1}{\varepsilon}\int_0^T\|\zeta(u_{\varepsilon,\lambda})\|^p_{L^p(\mathcal{O},\mathfrak{m})}dt\leq\left[\left(\int_0^T\|g\|^p_{L^p(\mathcal{O},\mathfrak{m})}dt\right)^{\frac{1}{p}}+\left(\int_0^T\left\|\frac{\partial\psi}{\partial t}+\mathcal{L}^\lambda\psi\right\|^p_{L^p(\mathcal{O},\mathfrak{m})}dt\right)^{\frac{1}{p}}\right]\left(\int_0^T\|\zeta(u_{\varepsilon,\lambda})\|_p^pdt\right)^{1-\frac{1}{p}}.
$$

We deduce that

$$
\left\|\frac{1}{\varepsilon}\zeta(u_{\varepsilon,\lambda})\right\|_{L^p([0,T];L^p(\mathcal{O},\mathfrak{m}))}\leq C, \tag{4.11}
$$

for a positive constant C independent of ε. Note that the estimate does not involve the L^∞-norm of ψ (which we assumed to be bounded for the payoff) so that by a standard approximation argument, it remains valid for unbounded ψ. The assertion then follows passing to the limit for $\varepsilon\to0$ in

$$
-\left(\frac{\partial u_{\varepsilon,\lambda}}{\partial t},v\right)_H+a_\lambda(u_{\varepsilon,\lambda},v)=\left(\frac{1}{\varepsilon}\zeta(u_{\varepsilon,\lambda}),v\right)_H+(g,v)_H,\qquad v\in V.
$$

□

Proposition 4.15. *Fix $p>\beta+\frac{5}{2}$ and λ as in Theorem 4.11. Let us consider $u\in C([0,T];H)\cap L^2([0,T];V)$, with $\frac{\partial u}{\partial t}\in L^2([0,T];H)$ such that*

$$
\begin{cases}
\left(\frac{\partial u}{\partial t},v\right)+a_\lambda(u(t),v)=(f(t),v), & v\in V,\\
u(0)=\psi,
\end{cases}
$$

with ψ continuous, $\psi\in V$, $\sqrt{1+y}f\in L^2([0,T];H)$ and $f\in L^p([0,T];L^p(\mathcal{O},\mathfrak{m}))$. Then, if ψ and λ satisfy the assumptions of Proposition 4.13, we have

1. *For every $t\in[0,T]$, $u(t)=P_t^\lambda\psi+\int_0^tP_s^\lambda f(t-s)ds$.*
2. *The function $(t,x,y)\mapsto u(t,x,y)$ is continuous on $[0,T]\times\mathbb{R}\times[0,+\infty)$.*

3. If $\Lambda_t = \lambda\int_0^t(1+Y_s)ds$, the process $(M_t)_{0\le t\le T}$, defined by

$$M_t = e^{-\Lambda_t}u(T-t,X_t,Y_t) + \int_0^t e^{-\Lambda_s}f(T-s,X_s,Y_s)ds,$$

with $X_0 = x$, $Y_0 = y$ is a martingale for every $(x,y)\in\mathbb{R}\times[0,+\infty)$.

Proof. The first assertion follows from Proposition 4.2.

The continuity of $(t,x,y)\mapsto P_t^\lambda\psi(x,y)$ is given by Proposition 4.13, while the continuity of $(t,x,y)\mapsto \int_0^t P_s^\lambda f(t-s,.)(x,y)ds$ can be proved with the same arguments. In fact, it is trivial if $(t,x,y)\mapsto f(t,x,y)$ is bounded continuous. If $f\in L^p([0,T];L^p(\mathcal{O},\mathfrak{m}))$, f is the limit in L^p of a sequence of bounded continuous functions and we have $\int_0^t P_s^\lambda f_n(t-s,\cdot)ds \to \int_0^t P_s^\lambda f(t-s,\cdot)ds$ uniformly in $[0,T]\times K$ for every compact K of $\mathbb{R}\times[0,+\infty)$). In fact, thanks to Theorem 4.11, we can write for $t\in[0,T]$ and $(x,y)\in K$

$$\begin{aligned}
\int_0^t P_s^\lambda|f_n-f|(t-s,\cdot,\cdot)(x,y)ds &\le \int_0^t \frac{C_{p,K,T}}{s^{\frac{2\beta+3}{2p}}}ds||(f_n-f)(t-s,\cdot,\cdot)||_{L^p(\mathcal{O},\mathfrak{m})}\\
&\le C_{p,K,T}\left(\int_0^t ||(f_n-f)(t-s,\cdot,\cdot)||^p_{L^p(\mathcal{O},\mathfrak{m})}ds\right)^{1/p}\left(\int_0^t \frac{ds}{s^{\frac{2\beta+3}{2(p-1)}}}\right)^{1-\frac{1}{p}} \qquad (4.12)\\
&\le C_{p,K,T}\left(\int_0^T ||(f_n-f)(s,\cdot,\cdot)||^p_{L^p(\mathcal{O},\mathfrak{m})}ds\right)^{1/p}\left(\int_0^T \frac{ds}{s^{\frac{2\beta+3}{2(p-1)}}}\right)^{1-\frac{1}{p}}.
\end{aligned}$$

The assumption $p>\beta+\frac{5}{2}$ ensures the convergence of the integral in the right hand side.

For the last assertion, note that $M_T = e^{-\Lambda_T}\psi(X_T,Y_T)+\int_0^T e^{-\Lambda_s}f(T-s,X_s,Y_s)ds$. Then, we can prove that M_t is integrable with the same arguments that we used to show the continuity of $(t,x,y)\mapsto u(t,x,y)$. Moreover, by using the Markov property,

$$\begin{aligned}
\mathbb{E}_{x,y}\left(M_T\,|\,\mathcal{F}_t\right) &= e^{-\Lambda_t}P^\lambda_{T-t}\psi(X_t,Y_t)+\int_0^t e^{-\Lambda_s}f(T-s,X_s,Y_s)ds + e^{-\Lambda_t}\int_t^T P^\lambda_{s-t}f(T-s,.,.)(X_t,Y_t)ds\\
&= e^{-\Lambda_t}\left(P^\lambda_{T-t}\psi(X_t,Y_t)+\int_0^{T-t}P_s^\lambda f(T-t-s,.,.)(X_t,Y_t)ds\right)+\int_0^t e^{-\Lambda_s}f(T-s,X_s,Y_s)ds\\
&= e^{-\Lambda_t}u(T-t,X_t,Y_t)+\int_0^t e^{-\Lambda_s}f(T-s,X_s,Y_s)ds = M_t.
\end{aligned}$$

□

We are now ready to prove the following proposition.

Proposition 4.16. *Fix $p>\beta+\frac{5}{2}$. Assume that ψ satisfies Assumption $\mathcal{H}^1$ and $0\le\psi\le\Phi$, with Φ satisfying Assumption $\mathcal{H}^2$, Assumption $\mathcal{H}^*$ and $(1+y)\Phi\in L^p([0,T],L^p(\mathcal{O},\mathfrak{m}))$. Moreover, assume that $\frac{\partial\psi}{\partial t}+\mathcal{L}\psi\in L^p([0,T];L^p(\mathcal{O},\mathfrak{m}))$. Then, the solution u of the variational inequality (2.4) satisfies*

$$u(t,x,y) = u^*(t,x,y), \qquad \text{on } [0,T]\times\bar{\mathcal{O}}, \qquad (4.13)$$

where u^* is defined by

$$u^*(t,x,y) = \sup_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}\left[\psi(\tau,X_\tau^{t,x,y},Y_\tau^{t,x,y})\right].$$

Proof. Thanks to Proposition 4.14 we know that, for λ large enough, there exists $F\in L^p([0,T];L^p(\mathcal{O},\mathfrak{m}))$ such that u satisfies

$$-\left(\frac{\partial u}{\partial t},v\right)_H + a_\lambda(u,v) = (F,v)_H, \qquad v\in V,$$

that is

$$-\left(\frac{\partial u}{\partial t},v\right)_H + a(u,v) = (F-\lambda(1+y)u,v)_H, \qquad v\in V.$$

On the other hand we know that

$$\begin{cases} -\left(\frac{\partial u}{\partial t}, v-u\right)_H + a(u, v-u) \geq 0, & \text{a.e. in } [0,T] \qquad v \in V,\ v \geq \psi, \\ u(T) = \psi(T), \\ u \geq \psi \text{ a.e. in } [0,T] \times \mathbb{R} \times (0,\infty). \end{cases}$$

From the previous relations we easily deduce that $F - \lambda(1+y)u \geq 0$ a.e. and, taking $v = \psi$, that $(F - \lambda(1+y)u, \psi - u) = 0$. Moreover, from Proposition 4.15 we know that the process $(M_t)_{0 \leq t \leq T}$, defined by

$$M_t = e^{-\Lambda_t} u(t, X_t, Y_t) + \int_0^t e^{-\Lambda_s} F(s, X_s, Y_s) ds, \tag{4.14}$$

with $X_0 = x$, $Y_0 = y$ is a martingale for every $(x,y) \in \mathbb{R} \times [0, +\infty)$. Then, we deduce that the process

$$\tilde{M}_t = u(t, X_t, Y_t) + \int_0^t \left(F(s, X_s, Y_s) - \lambda(1+Y_s)u(s, X_s, Y_s)\right) ds$$

is a local martingale. In fact, from (4.14) we can write

$$\begin{aligned} d\tilde{M}_t &= d\left[e^{\Lambda_t} M_t - e^{\Lambda_t} \int_0^t e^{-\Lambda_s} F(s, X_s, Y_s) ds\right] + F(t, X_t, Y_t)dt - \lambda(1+Y_t)u(t, X_t, Y_t)dt \\ &= e^{\Lambda_t} dM_t + \Big[\lambda(1+Y_t)M_t - \lambda(1+Y_t)e^{+\Lambda_t} \int_0^t e^{-\Lambda_s} F(s, X_s, Y_s) ds \\ &\quad - e^{\Lambda_t} e^{-\Lambda_t} F(t, X_t, Y_t) + F(t, X_t, Y_t) - \lambda(1+Y_t)u(t, X_t, Y_t)\Big] dt \\ &= e^{\Lambda_t} dM_t. \end{aligned}$$

So, for any stopping time τ there exists an increasing sequence of stopping times $(\tau_n)_n$ such that $\lim_n \tau_n = \infty$ and

$$\mathbb{E}_{x,y}[u(\tau \wedge \tau_n, X_{\tau\wedge\tau_n}, Y_{\tau\wedge\tau_n})] = u(0,x,y) - \mathbb{E}_{x,y}\left[\int_0^{\tau\wedge\tau_n} (F(s, X_s, Y_s) - \lambda(1+Y_s)u(s, X_s, Y_s))ds\right].$$

Since $F - \lambda(1+y)u \geq 0$ we can pass to the limit in the right hand side thanks to the monotone convergence theorem. On the other hand, we have that $\lim_{n\to\infty} \mathbb{E}_{x,y}[u(\tau \wedge \tau_n, X_{\tau\wedge\tau_n}, Y_{\tau\wedge\tau_n})] = \mathbb{E}_{x,y}[u(\tau, X_\tau, Y_\tau)]$ since $0 \leq u(t,x,y) \leq \Phi(t,x,y)$ and $(\Phi(t+s, X_s^{t,x,y}, Y_s^{t,y}))_{s\in[t,T]}$ is of class $\mathcal{D}$ for every $(t,x,y) \in [0,T] \times \mathbb{R} \times [0,\infty)$ by assumption. Therefore, passing to the limit as $n \to \infty$, we get

$$\mathbb{E}_{x,y}[u(\tau, X_\tau, Y_\tau)] = u(0,x,y) - \mathbb{E}_{x,y}\left[\int_0^\tau e^{\Lambda_\tau - \lambda_s}(F(s, X_s, Y_s) - \lambda(1+Y_s)u(s, X_s, Y_s))ds\right],$$

for every $\tau \in \mathcal{T}_{[0,T]}$. Recall that $F - \lambda(1+y)u \geq 0$, so the process $u(t, X_t, Y_t)$ is actually a supermartingale. Since $u \geq \psi$, we deduce directly from the definition of Snell envelope that $u(t, X_t, Y_t) \geq u^*(t, X_t, Y_t)$ a.e. for $t \in [0,T]$.

In order to show the opposite inequality, we consider the so called continuation region

$$\mathcal{C} = \{(t,x,y) \in [0,T) \times \mathbb{R} \times [0,\infty) : u(t,x,y) > \psi(t,x,y)\},$$

its t-sections

$$\mathcal{C}_t = \{(x,y) \in \mathbb{R} \times [0,\infty) : (t,x,y) \in \mathcal{C}\}, \qquad t \in [0,T),$$

and the stopping time

$$\begin{aligned} \tau_t &= \inf\{s \geq t : (s, X_s, Y_s) \notin \mathcal{C}\} \\ &= \inf\{s \geq t : u(s, X_s, Y_s) = \psi(s, X_s, Y_s)\}. \end{aligned}$$

Note that $u(x, X_s, Y_s) > \psi(s, X_s, Y_s)$ for $t \le s < \tau_t$. Moreover, recall that $(F - \lambda(1+y)u, \psi - u) = 0$ a.e., so $Leb\{(x,y) \in \mathcal{C}_t : F - \lambda(1+y)u \neq 0\} = 0\, dt$ a.e.. Since the two dimensional diffusion (X, Y) has a density, we deduce that $\mathbb{E}\left[F(s, X_s, Y_s) - \lambda(1+Y_s)u(s, X_s, Y_s)\mathbf{1}_{\{(X_s,Y_s)\in\mathcal{C}_s\}}\right] = 0$, and so $F(s, X_s, Y_s) - \lambda(1 + Y_s)u(s, X_s, Y_s) = 0\ ds,\ d\mathbb{P} - a.e.$ on $\{s < \tau_t\}$. Therefore,

$$\mathbb{E}\left[u(\tau_t, X_{\tau_t}, Y_{\tau_t})\right] = \mathbb{E}\left[u(t, X_t, Y_t)\right],$$

and, since $u(\tau_t, X_{\tau_t}, Y_{\tau_t}) = \psi(\tau_t, X_{\tau_t}, Y_{\tau_t})$,

$$E\left[u(t, X_t, Y_t)\right] = \mathbb{E}\left[\psi(\tau_t, X_{\tau_t}, Y_{\tau_t})\right] \le \mathbb{E}\left[u^*(t, X_t, Y_t)\right],$$

so that $u(t, X_t, Y_t) = u^*(t, X_t, Y_t)$ a.e.. With the same arguments we can prove that $u(t, x, y) = u^*(t, x, y)$ and this concludes the proof. □

4.5.2 Weaker assumptions on ψ

The last step is to establish the equality $u = u^*$ under weaker assumptions on ψ, so proving Theorem 2.4.

Proof of Theorem 2.4. For every $n \in \mathbb{N}$, with natural notations, we have

$$u_n(t, x, y) = u_n^*(t, x, y) \qquad \text{on } [0, T] \times \bar{\mathcal{O}}.$$

The left hand side converges to $u(t, x, y)$ thanks to the Comparison Principle. As regards the right hand side,

$$\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[\psi_n(\tau, X_\tau^{t,x,y}, Y_\tau^{t,x,y})\right] \to \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau-t)}\psi(\tau, X_\tau^{t,x,y}, Y_\tau^{t,x,y})\right]$$

thanks to the uniform convergence of ψ_n to ψ. □

Remark 4.17. *It is natural to ask if the assumptions of Theorem 2.4 are not vacuous and to look for payoff functions for which they are indeed satisfied.*

Let us consider, for example, the class of payoff functions $\psi = \psi(t, x) = e^{-rt}\bar{\psi}(x + \bar{c}t)$, where $\bar{c} = r - \delta - \frac{\rho\kappa\theta}{\sigma}$ as defined in (2.1) *and $\bar{\psi}$ is continuous, positive and such that*

$$|\bar{\psi}| + \left|\bar{\psi}'\right| \le C\left(e^x + 1\right),$$

with $C > 0$. Note that the standard call and put options fall into this category.

We can show that ψ satisfies the assumptions of Theorem 2.4. In fact, fix $p > \beta + \frac{5}{2}$ and assume $\gamma > p$ in the definition of the measure $\mathfrak{m}$. Note that

$$|\psi| + \left|\frac{\partial\psi}{\partial t}\right| + \left|\frac{\partial\psi}{\partial x}\right| \le \Phi(t, x),$$

where

$$\Phi(t, x) = C_T\left(e^{x - \frac{\rho\kappa\theta}{\sigma}t} + 1\right), \qquad (t, x) \in [0, T] \times \mathbb{R},$$

for some $C_T > 0$. Then, ψ satisfies Assumption $\mathcal{H}^1$ since $\psi \in \mathcal{C}([0, T]; H)$, $\sqrt{1+y}\psi \in L^2([0, T]; V)$, $\psi(T) \in V$ and $\left|\frac{\partial\psi}{\partial t}\right| \le \Phi$ with $\Phi \in L^2([0, T]; V)$. Moreover, Φ satisfies Assumption $\mathcal{H}^2$ since Φ has values in $H^2(\mathcal{O}, \mathfrak{m})$, $(1 + y)^{\frac{3}{2}}\Phi \in L^2([0, T], H)$ and, by straightforward computations, $\frac{\partial\Phi}{\partial t} + \mathcal{L}\Phi = 0$. Therefore, ψ satisfies the assumptions of Theorem 2.3 and there exists a unique solution of the variational inequality (2.4).

It remains to prove that ψ satisfies the further assumptions required in Theorem 2.4 in order to have the probabilistic representation (4.13). *Note that $\Phi \in L^p([0, T]; L^p(\mathcal{O}, \mathfrak{m}))$ and it satisfies Assumption $\mathcal{H}^*$. In fact, recall that the discounted and dividend adjusted price process $(e^{-(r-\delta)t}S_t)_t = (e^{X_t - \frac{\rho\kappa\theta}{\sigma}t})_t$ is a martingale (we refer to [12] for an analysis of the martingale property in general affine stochastic volatility models). Therefore, the process $(\Phi(t, X_t))_{t\in[0,T]} = C_T(e^{X_t - \frac{\rho\kappa\theta}{\sigma}t} + 1)_{t\in[0,T]}$ is a martingale, so that $(\Phi(s, X_s))_{s\in[t,T]}$ is of class $\mathcal{D}$ for every $(t, x, y) \in [0, T] \times \mathbb{R} \times [0, \infty)$.*

Then, recall the following result:

Lemma 4.18. *Let $0 < \nu_1 < \nu_2$. If $f \in W^{1,p}(\mathbb{R}, e^{-\nu_1|x|})$ with $p > 1$, there exists a sequence (f_n) such that $f_n \in W^{2,p}(\mathbb{R}, e^{-\nu_2|x|})$ and f_n converges to f uniformly.*

We refer to [11] for a proof. Since $\bar{\psi} \in W^{1,p}(\mathbb{R}, e^{-\gamma'|x|})$ for every $\gamma' > \gamma$, from Lemma 4.18 we deduce the existence of a sequence $(\bar{\psi}_n)_n \subseteq W^{2,p}(\mathbb{R}, e^{-\gamma|x|})$ which uniformly converges to $\bar{\psi}$. Hence, there exists a sequence $(\psi_n)_n$ which converge uniformly to ψ and such that, for every $n \in \mathbb{N}$, $0 \leq \psi_n \leq \Phi$ and ψ_n satisfies Assumption $\mathcal{H}^1$, $\psi_n \in L^2([0,T], H^2(\mathcal{O}, \mathfrak{m}))$ and $\frac{\partial \psi_n}{\partial t} + \mathcal{L}\psi_n$, $\Phi \in L^p([0,T]; L^p(\mathcal{O}, \mathfrak{m}))$.

Therefore, ψ satisfies all the assumptions required in Theorem 2.4 and we can identify the solution of the variational inequality with the solution of the optimal stopping problem, that is the American option price.

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