Upper bounds for the function solution of the homogenuous 2D Bolzmann equation with hard potential

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Abstract

We deal with $f_t(dv)$, solution of the homogenuous $2D$ Bolzmann equation without cutoff. An important point is that the initial condition $f_0(dv)$ may be any probability distribution (except a Dirac mass). However, for sufficientely hard potentials, the semigroup has a regularization property (see [5]): $f_t(dv) = f_t(v)dv$ for every $t > 0$. The aim of this paper is to give upper bounds for $f_t(v)$, the more significant one beeing $f_t(v) \leq Ct^{-\eta}e^{-|v|^\lambda}$ for some $\eta, \lambda > 0$.

Keywords: Bolzmann equation without cutoff, Hard potentials, Interpolation criterion, Integration by parts.

2010 MSC: 60H07, 60J75,82C40.

1 Introduction and main results

We are concerned with the solution of the two dimensional homogenous Boltzmann equation:

$$\partial_t f_t(v) = \int_{R^2} dv_* \int_{-\pi/2}^{\pi/2} d\theta \, |v - v_*|^\gamma \, b(\theta)(f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)). \tag{1.1}$$

Here $f_t(v)$ is a non negative measure on R^2 which represents the density of particles with velocity v in a model for a gas in dimension two, and, with R_θ being the rotation of angle θ,

$$v' = \frac{v + v_*}{2} + R_\theta \left(\frac{v - v_*}{2}\right), \quad v'_* = \frac{v + v_*}{2} - R_\theta \left(\frac{v - v_*}{2}\right).$$

The function $b : [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$ verifies the following hypothesis:

$$\begin{aligned} (H_\nu) \quad i) \quad & \exists 0 < c < C \quad s.t. \quad c\theta^{-(1+\nu)} \leq b(\theta) \leq C\theta^{-(1+\nu)} \\ ii) \quad & \forall k \in N, \exists C_k \quad s.t. \quad \left|b^{(k)}(\theta)\right| \leq C_k \theta^{-(k+1+\nu)} \end{aligned} \tag{1.2}$$

The rigorous sense of this equation is given by integrating against a test function - so one considers weak solutions of (1.1). In [13] one proves that, for every $\nu \in (0, \frac{1}{2})$ and $\gamma \in (0, 1]$, the above equation has a unique weak solution. More precisely: one assumes that (H_ν) holds and there exists $\lambda \in (\gamma, 2)$ such that $\int e^{|v|^\lambda} f_0(dv) < \infty$. Then there exists a unique solution f_t of (1.1) which starts from f_0. Furthermore the solution verifies $\sup_{t \leq T} \int e^{|v|^{\lambda'}} f_t(dv) < \infty$ for every $\lambda' < \lambda$. All over the paper these hypotheses are in force.

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Notice that $f_0(dv)$ is a probability distribution which is not assumed to be absolutely continuous with respect to the Lebesgue measure - we just assume that this is not a Dirac mass $\delta_{v_0}(dv)$ (in this trivial case the corresponding solution is $f_t(dv) = f_0(dv) = \delta_{v_0}(dv)$ for every $t > 0$). Our first aim is to give sufficient conditions under which, for every $t > 0$, one has $f_t(dv) = f_t(v)dv$, and to study the regularity of $f_t(v)$. This problem has already been addressed in [5] for the same equation, in [12] for the three dimensional Bolzmann equation and in [1] for the Bolzmann equation in an arbitrary dimension (however, in this last paper, $f_0(dv)$ is assumed to be absolutely continuous and to have finite entropy). For the Landau equation ($\gamma = 0$) this problem is addressed in [15] and [16] . Our second aim is to give upper bounds for $f_t(v)$ - this type of result seems to be new.
Let us first present our results. We will use the function

$$\varphi(\alpha) = \frac{(1-\nu)(1+\gamma+\alpha)}{1+\nu(\gamma+\alpha)} - 1 \tag{1.3}$$

and we construct the sequences

$$\alpha_{k+1} = \varphi(\alpha_k), \quad \kappa_{k+1} = \kappa_k - 1 + \frac{12(2+\nu)}{\nu}(1+\alpha_{k+1}) \tag{1.4}$$

with $\alpha_0 = 0$ and $\kappa_0 = 0$. We notice that φ is strictly increasing so that $\alpha_k \uparrow \alpha_*$ solution of $\varphi(\alpha_*) = \alpha_*$ (see (3.8) for the explicit value of α_*). And $\alpha_1 = \varphi(0) > 0$ is equivalent with $\nu < \frac{\gamma}{2\gamma+1}$.
In the following we assume that $\nu \in (0, \frac{1}{2})$ and $\gamma \in (0, 1]$. We also suppose that $f_0(dv)$ is not a Dirac mass and that for some $\lambda \in (\gamma, 2)$ one has $\int e^{|v|^\lambda} f_0(dv) < \infty$. Moreover, for every $\lambda' < \lambda$ we consider a function $\Phi_{\lambda'} \in C^\infty(R^2)$ such that $\Phi_{\lambda'}(v) = e^{|v|^{\lambda'}}$ for $|v| \geq 2$. The precise form of $\Phi_{\lambda'}$ is given in (4.31).

Theorem 1.1 ***A***.*Suppose that $\nu < \frac{\gamma}{2\gamma+1}$. Then $f_t(dv) = f_t(v)dv$.*
B. *Let $q \in \{0, 1, 2\}, k \in N$ and $p > 1$ be such that*

$$q + \frac{2}{p_*} < \phi(\alpha_k \wedge 2) = \alpha_{k+1} \wedge \varphi(2). \tag{1.5}$$

Then $\Phi_{\lambda'} f_t \in W^{q,p}(R^2)$ and for every $\lambda' < \lambda$ there exists a constant $C \geq 1$ such that

$$\|\Phi_{\lambda'} f_t\|_{q,p} \leq \frac{C}{t^{\kappa_{k+1}}}. \tag{1.6}$$

C a. *If $\nu < \frac{\gamma}{4\gamma+9}$ then $\Phi_{\lambda'} f_t \in L^p(R^2)$ for every $p > 1$ and*

$$\|\Phi_{\lambda'} \times f_t\|_p \leq \frac{C}{t^\eta} \quad with \quad \eta = \frac{2}{\varphi(2)-2}\left(\frac{48(2+\nu)}{\nu} - 1\right). \tag{1.7}$$

b. *If $\frac{\gamma}{4\gamma+9} \leq \nu < \frac{\gamma}{2\gamma+1}$ then $\alpha_* < 2$ and $\Phi_{\lambda'} f_t \in L^p(R^2)$ for every $1 < p < \frac{2}{2-\alpha_*}$.*
D.a *If $\nu < \frac{\gamma}{4\gamma+9}$ then $\Phi_{\lambda'} f_t \in W^{q,p}(R^2)$ for every $1 < p < p_q, q = 1, 2$ with*

$$p_1 = \frac{2(1+\nu(\gamma+2))}{1-\gamma+11\nu+5\nu\gamma} \quad and \quad p_2 = \frac{2(1+\nu(\gamma+2))}{2-\gamma+13\nu+6\nu\gamma} \tag{1.8}$$

Moreover for every $p < p_q$ one has (with η given in (1.7))

$$\|\Phi_{\lambda'} \times f_t\|_{q,p} \leq \frac{C}{t^\eta}. \tag{1.9}$$

b. *If $\frac{\gamma}{4\gamma+9} \leq \nu < \frac{\gamma}{3\gamma+4}$ then $\alpha_* < 3$ and $\Phi_{\lambda'} f_t \in W^{1,p}(R^2)$ for every $1 < p < \frac{2}{3-\alpha_*}$.*

In order to be able to compare this result with the ones in the papers which we quoted before we take $s>1$ and $\nu=\frac{2}{s-1},\gamma=\frac{s-5}{s-1}$: these are the values which are significant in the case of the three dimensional Bolzmann equation. Our condition $\gamma>0$ implies that $s>5$ and in the literature this case is known as the "hard potential" case. With this choice of ν and of γ we have $\nu<\frac{\gamma}{2\gamma+1}$ iff $s>9$ and $\nu<\frac{\gamma}{4\gamma+9}$ iff $s>16+\sqrt{193}\sim 30$. So, although they are not identical, the regularity result in the above theorem is analogous with the one in [5]. Notice that our result is less performing then the one in [12] where one deals with the real three dimensional equation and one obtains absolute continuity for a larger range for s. However our L^p estimates are stronger: we obtain $\Phi_{\lambda'}\times f_t\in L^p(R^2)$ instead of $f_t\in L^2(R^2)$, and bounds depending polynomialy on $t\downarrow 0$ are obtained. The result in [1] is stronger in the sense that it applies to equations in any dimension but it is supposed that the initial condition is already a function (so it is not really possible to compare them).
We give now some consequences of the previous result.

Corollary 1.2 *Suppose that* $\nu<\frac{\gamma}{2\gamma+1}$. *Then for every* $\lambda'<\lambda$ *there exists a constant* $C\geq 1$ *(depending on* λ'*) such that for every* $R>1, 0<t\leq 1$

$$f_t(\{v:|v|\geq R\})\leq \frac{C}{t^{\kappa}}e^{-R^{\lambda'}}\quad with \tag{1.10}$$

$$\kappa=\kappa_1=\frac{12(2+\nu)(1-\nu)(1+\gamma)}{\nu(1+\nu\gamma)}-1. \tag{1.11}$$

We give now the upper bound for f_t :

Theorem 1.3 *Suppose that* $\nu<\frac{\gamma}{4\gamma+9}$. *Then* $p_1>2$ *(given in (1.8)) and* $f_t\in C^{0,\chi}$ *with* $\chi=1-\frac{2}{p_1}$*.Moreover for every* $\lambda'<\lambda$

$$|f_t(v)|\leq \frac{C}{t^{\eta}}e^{-|v|^{\lambda'}} \tag{1.12}$$

with η *given in (1.7). Moreover, for every* $v,w\in R^2$ *with* $|w-v|\leq 1$

$$|f_t(w)-f_t(v)|\leq \frac{C}{t^{\eta}}e^{-|v|^{\lambda'}}\,|w-v|^{\chi}\,. \tag{1.13}$$

The estimate (1.12) seems to be new as well as the Hölder continuity of f_t and equation (1.13). However, in the case of the Landau equation (that is $\gamma=0$), some lower and upper bounds for f_t have been obtained in [16]. In the above paper one uses integration by parts techniques based on the Malliavin calculus for jump processes - this is not directly possible in our framework because of the indicator function which appears in equation (2.2).
Corollary 1.2 and Theorem 1.3 are the main contributions of our paper. The drawback of our approach is that our methodology allows to prove these properties for "very hard potentials" only ($s>9$ for (1.10) respectively $s>30$ for (1.12) and (1.13)). Moreover, η is not optimal - however this shows that the blow up is at most polynomial as $t\to 0$.
The proof is based on a "balance argument" which is interesting in itself so we give a hint here. Consider a family of random variables $F_\varepsilon\sim f_\varepsilon(v)dv,\varepsilon>0$ and a random variable F. We suppose that $d(F,F_\varepsilon)\to 0$ and $\|f_\varepsilon\|\to\infty$ as $\varepsilon\to 0$. Here d is some given distance and $\|f_\varepsilon\|$ is some Sobolev norm (see below). If the convergence to zero is sufficiently faster than the blow up then one may prove that $F\sim f(v)dv$ and obtains some regularity for f. This idea first appears in [14] and then has been used in several papers (see [9] for example). In these papers the "balance" between the speed of convergence to zero and the blow up is built by using Fourier analysis. Later on in [8] one introduced a new method based on a Besov space criterion, and this new method turns out to be significantly more powerful then the one based on Fourier analysis. This is the method used in [12] in the case of the three dimensional Bolzmann equaton (see also [7]). Finally, in [2] one introduced

a third method which is close to the interpolation theory. The criterion that we use in the present paper is an improvement of this last one.

In order to present this criterion we need to introduce some notation. For $f \in C^\infty(R^d), k, h \in N$ and $p > 1$ we define

$$\|f\|_{k,\infty} = \sum_{0 \leq |\alpha| \leq k} \sup_{x \in R^d} |\partial^\alpha f(x)|, \tag{1.14}$$

$$\|f\|_{k,h,p} = \sum_{0 \leq |\alpha| \leq k} (E(\int_{R^d} (1+|x|)^h |\partial^\alpha f(x)|^p \, dx))^{1/p} \tag{1.15}$$

$$\|f\|_{k,p} = \|f\|_{k,0,p} = \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha f\|_p. \tag{1.16}$$

Here $\alpha = (\alpha_1, ..., \alpha_m) \in \{1, ..., d\}^m, |\alpha| = m$ is the length of the multi-index α and ∂^α is the derivative associated to α. Moreover for two measures μ, ν we consider the distance

$$d_k(\mu, \nu) = \sup\{\left| \int f d\mu - \int f d\nu \right| : \|f\|_{k,\infty} \leq 1\}. \tag{1.17}$$

For $k = 0$ this is the total variation distance and for $k = 1$ this is the Fortèt Mourier distance. Our result is the following:

Theorem 1.4 *Let $q, k, d \in N$ and $p > 1$ be fixed. We consider a family of measures $\mu_\varepsilon(dx) = f_\varepsilon(x)dx, \varepsilon > 0$ with $f_\varepsilon \in C^\infty(R^d)$ and a finite measure μ on R^d which verify the following hypothesis. There exists $\varepsilon_* > 0, \beta > 0, a \geq 0, b \geq 0, C_0 \geq 1$ and $Q_h(q,p) \geq 1, h \in N$ such that*

$$i) \quad d_k(\mu_\varepsilon, \mu) \leq C_0 \varepsilon^\beta \quad \forall \varepsilon \in (0, \varepsilon_*) \tag{1.18}$$

$$ii) \quad \|f_\varepsilon\|_{2h+q,2h,p} \leq Q_h(q,p) \varepsilon^{-b(2h+q+a)} \quad \forall \varepsilon \in (0, \varepsilon_*), \forall h \in N \tag{1.19}$$

$$iii) \quad r := \beta - b(k + q + d/p_*) > 0. \tag{1.20}$$

We denote

$$h_* = \frac{1}{\varepsilon_*} \vee \frac{b(q+a)(k+q+d/p_*)}{r} \vee \frac{q+a}{2}. \tag{1.21}$$

Then, $\mu(dx) = f(x)dx$ with $f \in W^{q,p}(R^d)$. Moreover, for every $\delta > 0$, there exists a constant $C \geq 1$, depending on $q, k, d, p, \delta, \beta$ and a, b only, such that for every $h \geq h_*$ one has

$$\|f\|_{q,p} \leq C \times C_0 \times \left(h^{2b} Q_h^{1/2h}(q,p)\right)^{(1+\delta)(k+q+d/p_*)} \tag{1.22}$$

The upper bound given in (1.7) is based on (1.22).
The paper is organized as follows. In Section 2 we recall some results from [5] which represent the basic estimates that we use in the sequel (following Tanaka [17] we introduce a stochastic equation which represents the probabilistic representation of the Bolzmann equation and we construct some regularized version of this equation; then we estimate the error done by using such a regularization - this will be used in (1.18); moreover, we employ a Malliavin type calculus in order to build an integration by parts formula which permits to obtain (1.19). All these non trivial estimates have already been obtained in [5] and here we just use them. In Section 3 we use the results from Section 2 and Theorem 1.4 in order to prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. Finally, in the appendix we prove Theorem 1.4. We also develop a strategy based on integration by parts formulae which allows to obtain the absolute continuity of the law of a random variable as well as upper bounds for the density, in an abstract framework. This is done in Proposition 4.4 and Corollary 4.8.

2 Preliminary results

In this section we present some results from [5]. Throughout this section we fix $\nu \in [0, \frac{1}{2}), \gamma \in [0,1]$ and $\lambda \in (\gamma, 2)$ and the corresponding solution $f_t(dv)$ (which exists and is unique). Our first goal is to give the probabilistic interpretation of the equation (1.1). Using the Skorohod representation theorem we may find a measurable function $v_t : [0,1] \to R^2$ such that for every $\psi : R^2 \to R_+$

$$\int_0^1 \psi(v_t(\rho))d\rho = \int_{R^2} \psi(v) f_t(dv). \tag{2.1}$$

In [5] (following the ideas from [17]) one gives the probabilistic interpretation of the equation (1.1). We recall this now. Let $E = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0,1]$ and let $N(dt, d\theta, d\rho, du)$ be a Poisson point measure on $E \times R_+$ with intensity measure $b(\theta)d\theta \times d\rho \times du$. Consider also the matrix

$$A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix} = \frac{1}{2}(R_\theta - I).$$

We are interested in the equation

$$V_t = V_0 + \int_0^t \int_{E\times R_+} A(\theta)(V_{s-} - v_s(\rho)) 1_{\{u \le |V_{s-} - v_s(\rho)|^\gamma\}} N(ds, d\theta, d\rho, du) \tag{2.2}$$

with $P(V_0 \in dv) = f_0(dv)$. Proposition 2.1 in [5] asserts that the equation (2.2) has a unique càdlàg solution $(V_t)_{t\ge 0}$ and $P(V_t \in dv) = f_t(dv)$ (in this sense V_t represents the probabilistic representation for f_t).
In order to handle the equation (2.2) we face several difficulties: the derivatives of the function $v \to |v - v_s(\rho)|^\gamma$ blow up in the neighborhood of $v_s(\rho)$ - so we have to use a regularization procedure. Moreover, this function is unbounded and so we use a truncation argument. Finally, the measure $\theta^{-(1+\nu)}d\theta$ has infinite mass, and it is convenient to use a truncation argument also. We follow here the ideas and results from [5]. We fix $\eta_0 \in (1/\lambda, 1/(\gamma \vee \nu))$. Given $\varepsilon \in (0,1]$ we denote $\Gamma_\varepsilon = (\ln \frac{1}{\varepsilon})^{\eta_0}$ and we notice that since $\gamma\eta_0 < 1$ we have, for every $C \ge 1$ and $a > 0$

$$\overline{\lim_{\varepsilon\to 0}} \varepsilon^a e^{C\Gamma_\varepsilon^\gamma} = 0 \tag{2.3}$$

So $e^{C\Gamma_\varepsilon^\gamma} \le \varepsilon^{-a}$ for sufficiently small ε. Moreover, if $\kappa > 0$ is such that $\kappa\eta_0 > 1$, then for every $A \ge 1$

$$\overline{\lim_{\varepsilon\to 0}} \varepsilon^{-A} e^{-\Gamma_\varepsilon^\kappa} = 0 \tag{2.4}$$

So $e^{-\Gamma_\varepsilon^\kappa} \le \varepsilon^A$ for sufficiently small ε.
We construct the following approximation. We consider a C^∞ even non negative function χ supported by $[-1,1]$ and such that $\int_R \chi(x)dx = 1$ and we define

$$\varphi_\varepsilon(x) = \int_R ((y \vee 2\varepsilon) \wedge \Gamma_\varepsilon) \frac{\chi((x-y)/\varepsilon)}{\varepsilon} dy. \tag{2.5}$$

Observe that we have $2\varepsilon \le \varphi_\varepsilon(x) \le \Gamma_\varepsilon$ for every $x \in R, \varphi_\varepsilon(x) = x$ for $x \in (3\varepsilon, \Gamma_\varepsilon - 1), \varphi_\varepsilon(x) = 2\varepsilon$ for $x \in (0, \varepsilon)$ and $\varphi_\varepsilon(x) = \Gamma_\varepsilon$ for $x \in (\Gamma_\varepsilon, \infty)$. To the cut off function φ_ε one associates the equation

$$V_t^\varepsilon = V_0 + \int_0^t \int_{E\times R_+} A(\theta)(V_{s-}^\varepsilon - v_s(\rho)) 1_{\{u \le \varphi_\varepsilon^\gamma(|V_{s-}^\varepsilon - v_s(\rho)|)\}} N(ds, d\theta, d\rho, du). \tag{2.6}$$

We construct now a second approximation: for $\zeta > 0$ we consider a smooth cut off function I_ζ (which is a smooth version of $1_{\{|\theta| > \zeta\}}$) and the associated equation

$$V_t^{\varepsilon,\zeta} = V_0 + \int_0^t \int_{E\times R_+} A(\theta)(V_{s-}^{\varepsilon,\zeta} - v_s(\rho)) 1_{\{u \le \varphi_\varepsilon^\gamma(|V_{s-}^{\varepsilon,\zeta} - v_s(\rho)|)\}} I_\zeta(\theta) N(ds, d\theta, d\rho, du). \tag{2.7}$$

We state now a property which will be used in the following: given $\alpha \in [0,2]$ and $\kappa \geq 0$ there exists $K \geq 1$ such that for every $w \in R^2$ and every $0 < \varepsilon < 1$

$$(A_{\alpha,\kappa}) \qquad \sup_{t_0 \leq t \leq T} f_t(Ball(w,\varepsilon)) \leq \frac{K}{t_0^\kappa}\varepsilon^\alpha. \tag{2.8}$$

Since $f_t(dv)$ is a probability measure, this property is always verified with $K = 1, \alpha = 0$ and $\kappa = 0$. In Proposition 2.1 from [5] one proves that the equations (2.6) and (2.7) have a unique solution and

$$E\left|V_t^{\varepsilon,\zeta} - V_t^\varepsilon\right| \leq C_T e^{C\Gamma_\varepsilon^\gamma} \times \zeta^{1-\nu} \times t \quad \forall t \leq T. \tag{2.9}$$

Moreover, if $(A_{\alpha,\kappa})$ holds, then

$$E\left|V_t - V_t^\varepsilon\right| \leq C_T e^{C\Gamma_\varepsilon^\gamma} \times \varepsilon^{1+\gamma+\alpha} \times t^{1-\kappa} \quad \forall t \leq T. \tag{2.10}$$

We stress that in [5] the explicit dependence on the time t does not appear in the right hand side of the above estimates - but a quick glance to the proof shows that we have the dependence on t as in (2.9) and in (2.10) (and this is important if we look to short time behavior). Moreover, in the same proposition one proves that for every $0 < \lambda' < \lambda$ there exists some $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \leq \varepsilon_0} E(\sup_{t \leq T}(e^{|V_t|^{\lambda'}} + e^{|V_t^\varepsilon|^{\lambda'}} + e^{\left|V_t^{\zeta,\varepsilon}\right|^{\lambda'}}) =: C(\lambda') < \infty. \tag{2.11}$$

Finally in Theorem 4.1 in [5] one proves an integration by parts formula that we present now. One defines (see (4.1) and (4.2) in [5]) a random process $G_t^{\varepsilon,\zeta}$ which verifies

$$1_{\{\sup_{s\leq t}\left|V_s^{\zeta,\varepsilon}\right| \leq \Gamma_\varepsilon - 1\}} \leq G_t^{\varepsilon,\zeta} \leq 1_{\{\sup_{s\leq t}\left|V_s^{\zeta,\varepsilon}\right| \leq \Gamma_\varepsilon\}}. \tag{2.12}$$

The precise form of $G_t^{\varepsilon,\zeta}$ is not important here - the only property which we need is (2.12). Moreover one considers a two dimensional standard normal random variable Z and denotes

$$F_t^{\varepsilon,\zeta} = \sqrt{u_\zeta(t)}Z + V_t^{\varepsilon,\zeta} \qquad with \quad u_\zeta(t) = t\zeta^{4+\nu}.$$

Then one proves (see (4.3) and (4.4) in [5]) that for every multi-index $\beta \in \{1,2\}^q$ there exists a random variable $K_\beta(F_t^{\varepsilon,\zeta}, G_t^{\varepsilon,\zeta})$ such that for every function $\psi \in C^q(R^2)$

$$E(\partial^\beta \psi(F_t^{\varepsilon,\zeta})G_t^{\varepsilon,\zeta}) = E(\psi(F_t^{\varepsilon,\zeta})K_\beta(F_t^{\varepsilon,\zeta}, G_t^{\varepsilon,\zeta})). \tag{2.13}$$

One also proves that for every $q \in N$ and every $\kappa \in (\frac{1}{\eta_0}, \lambda)$ one may find a constant C (depending on q and κ only) such that for every $p \geq 1$

$$\left\|K_\beta(F_t^{\varepsilon,\zeta}, G_t^{\varepsilon,\zeta})\right\|_p \leq \frac{C}{t^{\frac{2+\nu}{\nu}(12q-4)}} e^{C\Gamma_\varepsilon^\gamma}(\varepsilon^{-q}\zeta^{-\nu q} + e^{-\Gamma_\varepsilon^\kappa}\zeta^{-2\nu q}). \tag{2.14}$$

In particular this gives for every function $\psi \in C^q(R^2)$ and every multi-index $\beta \in \{1,2\}^q$

$$\left|E(\partial^\beta \psi(F_t^{\varepsilon,\zeta})G_t^{\varepsilon,\zeta})\right| \leq \frac{C}{t^{\frac{2+\nu}{\nu}(12q-4)}} e^{C\Gamma_\varepsilon^\gamma}(\varepsilon^{-q}\zeta^{-\nu q} + e^{-\Gamma_\varepsilon^\kappa}\zeta^{-2\nu q}) \times \|\psi\|_\infty. \tag{2.15}$$

The only difference between (2.15) and the inequality proved in Theorem 4.1 in [5] is that here we give a precise dependence of the upper bound on t (which blows up as $t \downarrow 0$). We do not expect that this rate of blow up is optimal, but however it ensures that this rate is at most polynomial. In order to obtain this estimate one has to come back to the analysis done in [5]. There, in order to construct the weight $K_\beta(F_t^{\varepsilon,\zeta},G_t^{\varepsilon,\zeta})$, one uses a Malliavin type calculus (settled in [4]) and the crucial role in this calculus is played by the "Malliavin covariance matrix" of $F_t^{\varepsilon,\zeta}=\sqrt{u_\zeta(t)}Z+V_t^{\varepsilon,\zeta}$. We do not give here the precise form of this matrix but, if one denotes by d_t the determinant of this matrix, then we will need to estimate $E(d_t^{-p})$ for $p=3q-1$ (see (4.3) and (4.4) in [[5]). This is done in Proposition 4.4 of [5]. There the dependence of the constant $C_{t,p}$ on t is not given precisely but, looking to the proof of this proposition, one may see that

$$E(d_t^{-p})\leq C_pe^{C_p\Gamma_\varepsilon^\gamma}\left(\int_{\xi\in R^2}|\xi|^{8p-2}\exp(-ct\,|\xi|^{\nu/(2+\nu)})d\xi\right)^{1/2}.$$

Using a change of variable one obtains

$$E(d_t^{-p})\leq C_pe^{C_p\Gamma_\varepsilon^\gamma}\times\frac{1}{t^{4p\times\frac{2+\nu}{\nu}}}. \tag{2.16}$$

Now, by (4.4) in [5]

$$\left|K_\beta(F_t^{\varepsilon,\zeta},G_t^{\varepsilon,\zeta})\right|\leq\frac{C}{d_t^{3q-1}}\times\Lambda_q$$

with

$$\Lambda_q=\left|G_t^{\varepsilon,\zeta}\right|_q(1+\left|F_t^{\varepsilon,\zeta}\right|_{q+1})(1+\sum_{j=1}^q\sum_{k_1+...+k_j<q-j}\prod_{i=1}^j\left|LF_t^{\varepsilon,\zeta}\right|_{k_i}.$$

The precise significance of the quantities which appear in Λ_q is given in [5] but the only thing which is of interest for us is that for every $p\geq1$

$$\|\Lambda_q\|_p\leq Ce^{C\Gamma_\varepsilon^\gamma}(\varepsilon^{-q}\zeta^{-\nu q}+e^{-\Gamma_\varepsilon^\kappa}\zeta^{-2\nu q}).$$

This is done in the proof of Theorem 4.1 in [5]. There one takes $p=1$ but a short glance to the proof shows that the same reasoning and the same result holds for every $p\geq1$. So, using Schwarz inequality and (2.16) we get

$$\left\|K_\beta(F_t^{\varepsilon,\zeta},G_t^{\varepsilon,\zeta})\right\|_p\leq\left\|d_t^{-(3q-1)}\right\|_{2p}\|\Lambda_q\|_{2p}\leq C_pe^{C_p\Gamma_\varepsilon^\gamma}\times\frac{1}{t^{4(3q-1)\times\frac{2+\nu}{\nu}}}\|\Lambda_q\|_{2p}$$

and one obtains (2.14).

3 Proofs

In the following we adapt the results presented in the previous section to our specific goals. We suppose that $(A_{\alpha,\kappa})$ holds for some $\alpha\geq0$ and $\kappa\geq0$. In order to equilibrate the errors in (2.9) and (2.10) we take

$$\zeta=\zeta_\alpha(\varepsilon)=\varepsilon^{(1+\gamma+\alpha)/(1-\nu)}.$$

We recall that $\lambda\eta_0>1$, so we may choose (and fix) some $\lambda'\in(\frac{1}{\eta_0},\lambda)$. We work with a function $\Phi_{\lambda'}\in C^\infty(R^2)$ such that $\Phi_{\lambda'}(v)>0$ and $\Phi_{\lambda'}(v)=e^{|v|^{\lambda'}}$ for $|v|\geq2$ (the precise definition is given in (4.31)) Then we define

$$g_t(dv)=\Phi_{\lambda'}(v)f_t(dv).$$

And for $\varepsilon>0,\alpha\geq0$ we define $f_t^{\varepsilon,\alpha}$ and $g_t^{\varepsilon,\alpha}$ by

$$\int\psi(v)f_t^{\varepsilon,\alpha}(dv)=E(\psi(F_t^{\varepsilon,\zeta_\alpha(\varepsilon)})G_t^{\varepsilon,\zeta_\alpha(\varepsilon)}),\quad g_t^{\varepsilon,\alpha}(dv)=\Phi_{\lambda'}(v)f_t^{\varepsilon,\alpha}(dv).$$

In (1.3) we introduced the function φ which is strictly increasing and which solves the equation

$$1+\gamma+\alpha-(1+\varphi(\alpha))\frac{1+\nu(\gamma+\alpha)}{1-\nu}=0. \tag{3.1}$$

We know that $(A_{0,0})$ holds. Our aim now is to employ Corollary 4.8 in order to obtain $(A_{\alpha,\kappa})$ for a (as large as possible) α.

Lemma 3.1 ***A***. *Let $q\in N,\alpha\in[0,2]$ and $\kappa\geq 0$ be given. Suppose that $(A_{\alpha,\kappa})$ holds and $\varphi(\alpha)>q$. Then $g_t(dv)=g_t(v)dv$. Moreover, for every $p>1$ such that*

$$q+\frac{2}{p_*}<\varphi(\alpha) \tag{3.2}$$

there exists $C\geq 1$ such that

$$\left\|g_t\right\|_{q,p}\leq\frac{C}{t^{\kappa-1+\frac{12(2+\nu)}{\nu}(1+\varphi(\alpha))}}. \tag{3.3}$$

B *Suppose that $(A_{\alpha,\kappa})$ holds and $\varphi(\alpha)>0$. Then $(A_{\alpha',\kappa'})$ holds for every $\alpha'<\varphi(\alpha)\wedge 2$ with*

$$\kappa'=\kappa-1+\frac{12(2+\nu)}{\nu}(1+\varphi(\alpha)) \tag{3.4}$$

C. *Let $\alpha_k,\kappa_k,k\in N$ be the sequences defined in (1.4). Suppose that $\varphi(0)>0$. Then, for each $k\in N_*$ the property (A_{α,κ_k}) holds for $\alpha<\alpha_k\wedge 2$.*
D. *Suppose that $\varphi(0)>0$. Let $k,q\in N$ and let $p>1$ be such that*

$$q+\frac{2}{p_*}<\varphi(\alpha_k\wedge 2)=\alpha_{k+1}\wedge\varphi(2). \tag{3.5}$$

Then

$$\left\|g_t\right\|_{q,p}\leq\frac{C}{t^{\kappa_{k+1}}}. \tag{3.6}$$

Proof. We will use Corollary 4.8 with $d=2$, and $F_\varepsilon=F_t^{\varepsilon,\zeta_\alpha(\varepsilon)},G_\varepsilon=G_t^{\varepsilon,\zeta_\alpha(\varepsilon)}$. So we verify the hypothesis there.
Step 1. First, by (2.13) we know that the integration by parts formula (4.19) holds, with $H_{\beta,\varepsilon}=K_\beta(F_t^{\varepsilon,\zeta_\alpha(\varepsilon)},G_t^{\varepsilon,\zeta_\alpha(\varepsilon)})$. By (2.14) we obtain for every $\kappa\in(\frac{1}{\eta_0},\lambda)$ (with $\zeta=\zeta_\alpha(\varepsilon)$)

$$\sup_{|\beta|\leq q}\left\|H_{\beta,\varepsilon}\right\|_p\leq\frac{C}{t^{\frac{2+\nu}{\nu}(12q-4)}}e^{C\Gamma_\varepsilon^\gamma}(\varepsilon^{-q}\zeta^{-\nu q}+e^{-\Gamma_\varepsilon^\kappa}\zeta^{-2\nu q}).$$

We use (2.3) and (2.4) in order to obtain

$$e^{C\Gamma_\varepsilon^\gamma}(\varepsilon^{-q}\zeta^{-\nu q}+e^{-\Gamma_\varepsilon^\kappa}\zeta^{-2\nu(q+2)})\leq C\varepsilon^{-c}((\varepsilon\zeta^\nu)^{-q}+\varepsilon^A\varepsilon^{-q(4+\zeta)(1+\gamma+\alpha)/(1-\nu)})$$

for every $c>0$ and $A\geq 1$. It follows that

$$\sup_{|\beta|\leq q}\left\|H_{\beta,\varepsilon}\right\|_p\leq\frac{C}{t^{\frac{2+\nu}{\nu}(12q-4)}}\times\varepsilon^{-q\times\frac{1+\nu(\gamma+\alpha)}{1-\nu}-c}$$

and this means that (4.20) is verified with

$$\widehat{H}_{q,p}=\frac{C}{t^{\frac{2+\nu}{\nu}(12q-4)}},\quad b=\frac{1+\nu(\gamma+\alpha)}{1-\nu},\quad a=c.$$

Here we may take $c>0$ arbitrary small. Notice that for every $\delta>0$ we may find $h(\delta)$ such that for $h\geq h(\delta)$ we have

$$\widehat{H}_{2h+q+d,p}^{1/2h}\leq\frac{C}{t^{\frac{2+\nu}{\nu}\times 12(1+\delta)}}. \tag{3.7}$$

Step 2. Let us verify (4.25). Using (2.12) and (2.11)

$$\begin{aligned}\left\|1-G_t^{\varepsilon,\zeta_\alpha(\varepsilon)}\right\|_2 &\leq P^{1/2}(\sup_{s\leq t}\left|V_s^{\varepsilon,\zeta_\alpha(\varepsilon)}\right|\geq\Gamma_\varepsilon)\\ &\leq Ce^{-\frac{1}{2}\Gamma_\varepsilon^{\lambda'}}(E(\sup_{s\leq t}e^{\left|V_s^{\varepsilon,\zeta_\alpha(\varepsilon)}\right|^{\lambda'}}))^{1/2}\leq Ce^{-\frac{1}{2}\Gamma_\varepsilon^{\lambda'}}\leq C\varepsilon^{-A}.\end{aligned}$$

The last inequality is true for any $A\geq 1$. This is a consequence of $\lambda'\eta_0>1$ and of (2.4).
And by our choice of $\zeta_\alpha(\varepsilon)$ and as a consequence of (2.9),(2.10) and (2.3) we have (for every $c>0$)

$$E(\left|V_t-F_t^{\varepsilon,\zeta_\alpha(\varepsilon)}\right|)\leq\frac{C}{t^{\kappa-1}}\varepsilon^{1+\gamma+\alpha-c}.$$

We conclude that (4.25) holds with

$$C_0=\frac{C}{t^{\kappa-1}},\quad \beta=1+\gamma+\alpha-c.$$

Step 3. Now (3.2) ensures that, for sufficiently small $c>0$,

$$\begin{aligned}\beta-b(1+q+\frac{2}{p_*})&=1+\gamma+\alpha-c-(1+q+\frac{2}{p_*})\frac{1+\nu(\gamma+\alpha)}{1-\nu}\\ &>1+\gamma+\alpha-(1+\varphi(\alpha))\frac{1+\nu(\gamma+\alpha)}{1-\nu}=0\end{aligned}$$

so (4.28) holds (with $d=2$). Now we are able to use Corollary 4.8. Notice that for every $\theta\geq 1$ and every $\lambda'<\lambda''$ one may find C such that $\Phi_{\lambda'}^{\theta}\leq C\Phi_{\lambda''}$. So (2.11) gives $\widehat{C}_\theta(\lambda')<\infty$ (see (4.33) for the definition of $\widehat{C}_\theta(\lambda')$). So (4.35) and (3.7) give

$$\begin{aligned}\|\Phi_{\lambda'}f_t^\alpha\|_{q,p}&\leq C\times C_0\times\widehat{C}_\theta(\lambda')\times\left(h^{2b}\widehat{H}_{2h+q+d,p_*}^{1/2h}\right)^{(1+\delta)(1+q+2/p_*)}\\ &\leq\frac{C}{t^{\kappa-1+\frac{2+\nu}{\nu}\times 12(1+q+2/p_*)(1+\delta)}}\\ &\leq\frac{C}{t^{\kappa-1+\frac{2+\nu}{\nu}\times 12(1+\varphi(\alpha))(1+\delta)}}.\end{aligned}$$

In the previous inequality we have englobed $\widehat{C}_\theta(\lambda')$ and h in the constant C. So **A** is proved.
Proof of B. Let $p>0$ such that $\frac{2}{p_*}=\alpha'<\varphi(\alpha)\wedge 2$. We use **A** with $q=0$. By(3.3) $\|f_t^\alpha\|_p\leq\|\Phi_{\lambda'}f_t^\alpha\|_p\leq Ct^{-\kappa'}$ with κ' given in (3.4). Then, using Hölder's inequality

$$f_t(Ball(\varepsilon,v))\leq Ct^{-\kappa'}\times\varepsilon^{2/p_*}=Ct^{-\kappa'}\times\varepsilon^{\alpha'}.$$

Proof of C. Take first $k=1$. We know that $(A_{0,0})$ holds, and by hypothesis $\varphi(0)>0$. Then, by **B**, $(A_{\alpha',\kappa'})$ holds for every $\alpha'<\varphi(0)\wedge 2=\alpha_1\wedge 2$ with $\kappa'=0-1+\frac{12(2+\nu)}{\nu}(1+\varphi(0))=\kappa_1$. So our assertion holds for $k=1$.
Suppose now that the property is true for k and let us check it for $k+1$. Suppose first that $\alpha_k>2$. Then, the recurrence hypothesis says that (A_{α,κ_k}) holds for every $\alpha<\alpha_k\wedge 2=2$. Since $\kappa_k<\kappa_{k+1}$ the property $(A_{\alpha,\kappa_{k+1}})$ holds as well, and this is true for every $\alpha<2=\alpha_{k+1}\wedge 2$, so our assertion is proved. Suppose now that $\alpha_k\leq 2$ and take $\alpha'<\alpha_{k+1}\wedge 2=\varphi(\alpha_k)\wedge 2$. Since $\varphi(\alpha)\uparrow\varphi(a_k)$ as $\alpha\uparrow\alpha_k$,

we may find $\alpha < \alpha_k = \alpha_k \wedge 2$ such that $\alpha' < \varphi(\alpha) \wedge 2$. By the recurrence hypothesis we know that (A_{α,κ_k}) holds and then, using **B**, we obtain $(A_{\alpha',\kappa_{k+1}})$.
D. By (3.5) we may find $\alpha < \alpha_k \wedge 2$ such that $q + \frac{2}{p_*} < \varphi(\alpha)$. And by **C** we know that (A_{α,κ_k}) holds. Then we may use **A** and (3.3) gives (3.6). □
Proof of Theorem 1.1. We will work with the sequences α_k and κ_k given in (1.4). We recall that $\alpha_k \uparrow \alpha_*$ with $\alpha_* = \varphi(\alpha_*)$ and we have

$$\alpha_* = \frac{-(\gamma+2) + \sqrt{(\gamma+2)^2 + 4(\frac{\gamma}{\nu} - 2\gamma - 1)}}{2} \tag{3.8}$$

so that

$$\alpha_* > 1 \quad \Leftrightarrow \quad \nu < \frac{\gamma}{3\gamma+4}$$
$$\alpha_* > 2 \quad \Leftrightarrow \quad \nu < \frac{\gamma}{4\gamma+9}.$$

If $\nu < \frac{\gamma}{2\gamma+1}$ then $\varphi(0) > 0$ so we may use the point **A** in Lemma 3.1 with $q = 0$ and we obtain the point **A** in Theorem 1.1. The point **B** in Theorem 1.1 follows from the point **D** in Lemma 3.1.
Proof of C.b. If $\frac{\gamma}{4\gamma+9} < \nu < \frac{\gamma}{2\gamma+1}$ we have $\alpha_* \leq 2$ so that $\alpha_k < 2$ for every $k \in N$. And if $p < \frac{2}{2-\alpha_*}$ then $\frac{2}{p_*} < \alpha_*$ so we may find k such that $\frac{2}{p_*} < \alpha_{k+1} < 2$. So, using the point **B** in Theorem 1.1 we obtain $g_t \in L^p(R^2)$.
Proof of D.b. If $\frac{\gamma}{4\gamma+9} < \nu < \frac{\gamma}{3\gamma+4}$ we have $\alpha_* \in (1,2]$ so that $1 < \frac{2}{3-\alpha_*}$. We take $1 < p < \frac{2}{3-\alpha_*}$ and then $1 + \frac{2}{p_*} < \alpha_*$. We take k sufficiently large in order to have $1 + \frac{2}{p_*} < \alpha_{k+1}$ and then, as above, by **B** in Theorem 1.1 we obtain $g_t \in W^{1,p}(R^2)$.
C a and D a If $\nu < \frac{\gamma}{4\gamma+9}$ then $\alpha_* > 2$. We define $k_* = \min\{k : \alpha_k > 2\}$. By **C** in Lemma 3.1, for every $\alpha < 2$ the property $(A_{\alpha,\kappa_{k_*}})$ holds. We denote $\psi(\alpha) = \varphi(\alpha) - \alpha$. For $k < k_*$ we have $\alpha_{k+1} - \alpha_k = \psi(\alpha_k) > \psi(2)$ (this is because ψ is decreasing on $(0,2) \subset (0,\alpha_*)$). This implies that $k_* \leq 2/\psi(2)$ and then, since $1 + \varphi(0) \leq 4$,

$$\kappa_{k_*} \leq k_* \times (\frac{12(2+\nu)}{\nu}(1+\varphi(2)) - 1) \leq \frac{2}{\psi(2)}\left(\frac{48(2+\nu)}{\nu} - 1\right) = \eta$$

with η given in (1.7).
We use now the point **B** in Theorem 1.1. Take first $q = 0$. Since $\varphi(\alpha_{k_*} \wedge 2) = \varphi(2) > 2 > \frac{2}{p_*}$ (for every $p > 1$) we obtain $g_t \in \cap_{p>1} L^p(R^2)$. If $q \in \{1,2\}$ then we need $\frac{2}{p_*} < \varphi(2) - q$, which gives $p < 2/(q + 2 - \varphi(2)) = p_q$ with $p_q, q = 1,2$ given in (1.8). And we obtain

$$\|g_t\|_{q,p} \leq \frac{C}{t^\eta} \quad for \quad p < p_q.$$

□
Proof of Corollary 1.2 Since $\nu < \frac{\gamma}{2\gamma+1}$ we may find $p < \alpha_1$ such that $\|\Phi_{\lambda'} f_t\|_p \leq \frac{C}{t^{\kappa_1}}$. Then

$$\begin{aligned} f_t(B_R^c(0)) &= \int 1_{B_R^c(0)}(v)\Phi_{\lambda'}^{-1}(v)\Phi_{\lambda'}(v)f_t(v)dv \\ &\leq (\int 1_{B_R^c(0)}(v)e^{-p_*|v|^{\lambda'}}dv)^{1/p_*}\|\Phi_{\lambda'} f_t\|_p \\ &\leq e^{-\frac{1}{2}R^{\lambda'}}(\int 1_{B_R^c(0)}(v)e^{-\frac{p_*}{2}|v|^{\lambda'}}dv)^{1/p_*}\frac{C}{t^{\kappa_1}} \\ &\leq \frac{C}{t^{\kappa_1}}e^{-\frac{1}{2}R^{\lambda'}}. \end{aligned}$$

□
Proof of Theorem 1.3. This theorem is an immediate consequence of Corollary 4.8 (see (4.37) and (4.38)). □

4 Appendix: A regularity criterion based on interpolation

Let us first recall some results obtained in [2] concerning the regularity of a measure μ on $\mathbb{R}^d$. We fix $k, q, h \in \mathbb{N}$, with $h \geq 1$, and $p > 1$. Hereafter, we denote by $p_* = p/(p-1)$ the conjugate of p. Recall that in (1.14) and (1.15) we have defined $\|f\|_{k,\infty}$ and $\|f\|_{q,h,p}$ and in (1.17) we have defined $d_k(\mu, \nu)$ for two measures μ, ν on R^d.
For a signed finite measure μ and for a sequence of absolutely continuous signed finite measures $\mu_n(dx) = f_n(x)dx$ with $f_n \in C^{2h+q}(\mathbb{R}^d)$, we define

$$\pi_{k,q,h,p}(\mu, (\mu_n)_n) = \sum_{n=0}^{\infty} 2^{n(k+q+d/p_*)} d_k(\mu, \mu_n) + \sum_{n=0}^{\infty} \frac{1}{2^{2nh}} \|f_n\|_{2h+q,2h,p} \tag{4.1}$$

and

$$\overline{\pi}_{k,q,h,p}(\mu) = \inf\{\pi_{k,q,h,p}(\mu, (\mu_n)_n) : \mu_n(dx) = f_n(x)dx, \quad f_n \in C^{2h+q}(\mathbb{R}^d)\}.$$

Remark 4.1 *Notice that $\pi_{k,q,h,p}$ is a particular case of $\pi_{k,q,h,\mathbf{e}}$ treated in [2]: just choose the Young function $\mathbf{e}(x) \equiv \mathbf{e}_p(x) = |x|^p$, giving $\beta_{\mathbf{e}_p}(t) = t^{1/p_*}$ (see Example 1 in [2]). Moreover, $\pi_{k,q,h,p}$ is strongly related to interpolation spaces. More precisely, $\overline{\pi}_{k,q,h,p}$ is equivalent with the interpolation norm of order $\rho = \frac{k+q+d/p_*}{2h}$ between the spaces $W_*^{k,\infty}$ (the dual of $W^{k,\infty}$) and $W^{2h+q,2h,p} = \{f : \|f_n\|_{2h+q,2h,p} < \infty\}$ (see [6] for example). This is proved in [2], see Section 2.4 and Appendix B. So the inequality (4.2) below implies that the Sobolev space $W^{q,p}$ is included in the above interpolation space. However we prefer to stick to an elementary framework and to derive directly the consequences of (4.2) - see Lemma 4.3 and Lemma 4.2 below.*

The following result is the key point in our approach (this is Proposition 2.5 in [2]):

Lemma 4.2 *Let $p > 1$, $k, q \in \mathbb{N}$ and $h \in N_*$ be given. There exists a constant C_* (depending on k, q, h and p only) such that the following holds. Let μ be a finite measure for which $\overline{\pi}_{k,q,h,p}(\mu)$ is finite. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$ and*

$$\|f\|_{q,p} \leq C_* \times \overline{\pi}_{k,q,h,p}(\mu). \tag{4.2}$$

The proof of Lemma 4.2 is given in [2], being a particular case (take $\mathbf{e} = \mathbf{e}_p$) of Proposition A.1 in Appendix A (see also [3]). We will use the following consequence:

Lemma 4.3 *Let $p > 1$, $k, q \in \mathbb{N}$ and $h \in N_*$ be given and set*

$$\rho_h(q) := \frac{k + q + d/p_*}{2h}. \tag{4.3}$$

We also consider an increasing sequence $\theta(n) \geq 1, n \in \mathbb{N}$ such that $\lim_n \theta(n) = \infty$ and $\theta(n+1) \leq \Theta \times \theta(n)$ for some constant $\Theta \geq 1$. Let μ be a finite measure on R^d. Suppose that we may find a sequence of functions $f_n \in C^{2h+q}(\mathbb{R}^d), n \in \mathbb{N}$ such that

$$\|f_n\|_{2h+q,2h,p} \leq \theta(n) \tag{4.4}$$

and, with $\mu_n(dx) = f_n(x)dx$,

$$\limsup_n d_k(\mu, \mu_n) \times \theta^{\rho_h(q)+\eta}(n) < \infty \tag{4.5}$$

for some $\eta > 0$. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$.

Moreover, for $\delta, \eta > 0$ and $n_* \in \mathbb{N}$, we set

$$A(\delta) = |\mu|(\mathbb{R}^d) \times 2^{l(\delta)(1+\delta)(q+k+d/p_*)} \quad \textit{with } l(\delta) = \min\{l : 2^{l \times \frac{\delta}{1+\delta}} \geq l\}, \tag{4.6}$$

$$B(\eta) = \sum_{l=1}^{\infty} \frac{l^{2(q+k+d/p_*+\eta)}}{2^{2h\eta l}}, \tag{4.7}$$

$$C_{h,n_*}(\eta) = \sup_{n \geq n_*} d_k(\mu, \mu_n) \times \theta^{\rho_h(q)+\eta}(n). \tag{4.8}$$

Then

$$\|f\|_{q,p} \leq C_*(\Theta + A(\delta)\theta(n_*)^{\rho_h(q)(1+\delta)} + B(\eta)C_{h,n_*}(\eta)) \tag{4.9}$$

with C_* the constant in (4.2) and $\rho_h(q)$ given in (4.3).

Proof of Lemma 4.3. We will produce a sequence of measures $\nu_l(dx) = g_l(x)dx, l \in \mathbb{N}$ such that

$$\pi_{k,q,h,p}(\mu, (\nu_l)_l) \leq \Theta + A(\delta)\theta(n_*)^{\rho_h(q)(1+\delta)} + B(\eta)C_{h,n_*}(\eta) < \infty.$$

Then by Lemma 4.2 one gets $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$, and (4.9) follows from (4.2)). Let us stress that the ν_l's will be given by a suitable subsequence $\mu_{n(l)}$, $l \in \mathbb{N}$.

Step 1. We define

$$n(l) = \min\{n : \theta(n) \geq \frac{2^{2hl}}{l^2}\}$$

and we notice that

$$\frac{1}{\Theta}\theta(n(l)) \leq \theta(n(l)-1) < \frac{2^{2hl}}{l^2} \leq \theta(n(l)). \tag{4.10}$$

Moreover we recall that n_* is given and we define

$$l_* = \min\{l : \frac{2^{2hl}}{l^2} \geq \theta(n_*)\}.$$

Since

$$\theta(n(l_*)) \geq \frac{2^{2hl_*}}{l_*^2} \geq \theta(n_*)$$

it follows that $n(l_*) \geq n_*$.
We take now $\varepsilon(\delta) = \frac{h\delta}{1+\delta}$ which gives $\frac{2h}{2(h-\varepsilon(\delta))} = 1+\delta$. And we take $l(\delta) \geq 1$ such that $2^{l\delta/(1+\delta)} \geq l$ for $l \geq l(\delta)$ (see (4.6)). Since $h \geq 1$ it follows that $\varepsilon(\delta) \geq \frac{\delta}{1+\delta}$ so that, for $l \geq l(\delta)$ we also have $2^{l\varepsilon(\delta)} \geq l$. Now we check that

$$2^{2(h-\varepsilon(\delta))l_*} \leq 2^{2hl(\delta)}\theta(n_*). \tag{4.11}$$

If $l_* \leq l(\delta)$ then the inequality is evident (recall that $\theta(n) \geq 1$ for every n). And if $l_* > l(\delta)$ then $2^{l_*\varepsilon(\delta)} \geq l_*$. By the very definition of l_* we have

$$\frac{2^{2h(l_*-1)}}{(l_*-1)^2} < \theta(n_*)$$

so that

$$2^{2hl_*} \leq 2^{2h}(l_*-1)^2\theta(n_*) \leq 2^{2h} \times 2^{2l_*\varepsilon(\delta)}\theta(n_*)$$

and, since $l(\delta) \geq 1$, this gives (4.11).

Step 2. We define

$$\begin{aligned} \nu_l &= 0 \quad if \quad l < l_* \\ &= \mu_{n(l)} \quad if \quad l \geq l_* \end{aligned}$$

and we estimate $\pi_{k,q,h,p}(\mu,(\nu_l)_l)$. First, by (4.4) and (4.10)

$$\sum_{l=l_*}^{\infty}\frac{1}{2^{2hl}}\left\|f_{n(l)}\right\|_{q+2h,2h,p}\leq\sum_{l=l_*}^{\infty}\frac{1}{2^{2hl}}\theta(n(l))\leq\Theta\sum_{l=l_*}^{\infty}\frac{1}{l^2}\leq\Theta.$$

Then we write

$$\sum_{l=1}^{\infty}2^{(q+k+d/p_*)l}d_k(\mu,\nu_l)=S_1+S_2$$

with

$$S_1=\sum_{l=1}^{l_*-1}2^{(q+k+d/p_*)l}d_k(\mu,0),\quad S_2=\sum_{l=l_*}^{\infty}2^{(q+k+d/p_*)l}d_k(\mu,\mu_{n(l)}).$$

Since $d_k(\mu,0)\leq d_0(\mu,0)\leq|\mu|(\mathbb{R}^d)$ we use (4.11) and we obtain

$$\begin{aligned}S_1&\leq|\mu|(\mathbb{R}^d)\times 2^{(q+k+d/p_*)l_*}=|\mu|(\mathbb{R}^d)\times(2^{2(h-\varepsilon(\delta))l_*})^{(q+k+d/p_*)/2(h-\varepsilon(\delta))}\\&\leq|\mu|(\mathbb{R}^d)\times(2^{2hl(\delta)}\theta(n_*))^{\rho_h(q)(1+\delta)}=A(\delta)\theta(n_*)^{\rho_h(q)(1+\delta)}.\end{aligned}$$

If $l\geq l_*$ then $n(l)\geq n(l_*)\geq n_*$ so that, from (4.8),

$$d_k(\mu,\mu_{n(l)})\leq\frac{C_{h,n_*}(\eta)}{\theta^{\rho_h(q)+\eta}(n(l))}\leq C_{h,n_*}(\eta)\Big(\frac{l^2}{2^{2hl}}\Big)^{\rho_h(q)+\eta}=\frac{C_{h,n_*}(\eta)}{2^{(q+k+d/p_*)l}}\times\frac{l^{2(\rho_h(q)+\eta)}}{2^{2h\eta l}}.$$

We conclude that

$$S_2\leq C_{h,n_*}(\eta)\sum_{l=l_*}^{\infty}\frac{l^{2(\rho_h(q)+\eta)}}{2^{2\eta hl}}\leq C_{h,n_*}(\eta)\times B(\eta).$$

□

We give now a consequence of the above result which is more readable.

Proposition 4.4 *Let $q,k,d\in N$ and $p>1$ be fixed. We consider a family of measures $\mu_\varepsilon(dx)=f_\varepsilon(x)dx,\varepsilon>0$ with $f_\varepsilon\in C^\infty(R^d)$ and a finite measure μ on R^d which verify the following hypothesis. There exists $\varepsilon_*>0,\beta>0,a\geq 0,b\geq 0$ and $C_0\geq 1,Q_h(q,p)\geq 1,h\in N$ such that*

$$i)\quad d_k(\mu_\varepsilon,\mu)\leq C_0\varepsilon^\beta\quad\forall\varepsilon\in(0,\varepsilon_*)\tag{4.12}$$

$$ii)\quad\|f_\varepsilon\|_{2h+q,2h,p}\leq Q_h(q,p)\varepsilon^{-b(2h+q+a)}\quad\forall\varepsilon\in(0,\varepsilon_*),\forall h\in N\tag{4.13}$$

$$iii)\quad r:=\beta-b(k+q+d/p_*)>0\tag{4.14}$$

We denote

$$h_*=\frac{1}{\varepsilon_*}\vee\frac{b(q+a)(k+q+d/p_*)}{r}\vee\frac{q+a}{2}.\tag{4.15}$$

Then, $\mu(dx)=f(x)dx$ with $f\in W^{q,p}(R^d)$. Moreover, for every $\delta>0$, there exists a constant $C\geq 1$, depending on q,k,d,p,δ,β,r and a,b only (but not on h), such that for every $h\geq h_*$ one has

$$\|f\|_{q,p}\leq C\times C_0\times\left(h^{2b}Q_h^{1/2h}(q,p)\right)^{(1+\delta)(k+q+d/p_*)}\tag{4.16}$$

Proof. All over this proof C designs a constant which depends on q,k,d,p,δ,β,r and a,b only. We will use Lemma 4.3. We take

$$\eta=\frac{r}{2b(2h+q+a)}\wedge\frac{\delta(q+k+d/p_*)}{2h}.\tag{4.17}$$

For $\varepsilon \leq \varepsilon_*$, we have

$$d_k(\mu_\varepsilon, \mu) \left\| f_\varepsilon \right\|_{2h+q,2h,p}^{\rho_h(q)+\eta} \leq C_0 Q_h^{\rho_h(q)+\eta}(q,p) \varepsilon^{\beta-(\rho_h(q)+\eta)b(2h+q+a)}.$$

Notice that

$$\begin{aligned}\beta-(\rho_h(q)+\eta)b(2h+q+a) &= (\beta-2h\rho_h(q)b)-\rho_h(q)b(q+a)-\eta b(2h+q+a) \\ &\geq r-\frac{r}{2}-\frac{r}{2}=0\end{aligned}$$

the last inequality being a consequence of (4.17) and (4.15). So we obtain

$$d_k(\mu_\varepsilon, \mu) \left\| f_\varepsilon \right\|_{2h+q,2h,p}^{\rho_h(q)+\eta} \leq C_0 Q_h^{\rho_h(q)+\eta}(q,p) \leq C_0 Q_h^{\rho_h(q)(1+\delta)}(q,p). \tag{4.18}$$

We take now $\varepsilon_n = \frac{1}{n}$ and $n_* = h$ and we define

$$\begin{aligned} g_n &= 0 \quad if \quad n < n_* \\ &= f_{\varepsilon_n} \quad if \quad n \geq n_*. \end{aligned}$$

We will use Lemma 4.3 for $\nu_n(dx) = g_n(x)dx$ so we have to identify the quantities defined there. We define $\theta(n) = Q_h(q,p)n^{b(2h+q+a)}$ if $n \geq n_*$ and $\theta(n) = \theta(n_*)$ if $n \leq n_*$. By (4.13) we have $\left\| g_n \right\|_{2h+q,2h,p} \leq \theta(n)$ and moreover, for $n \geq n_* = h$

$$\frac{\theta(n+1)}{\theta(n)} = (1+\frac{1}{n})^{n\times\frac{b(2h+q+a)}{n}} \leq e^{2b+\frac{b(q+a)}{h}} \leq e^{3b}.$$

We conclude that $\Theta \leq e^{3b}$. We estimate now $B(\eta)$ defined in (4.7). Since

$$\frac{1}{\eta h} = \frac{2b(2h+q+a)}{rh} \vee \frac{2}{\delta(q+k+d/p_*)} \leq C$$

we obtain

$$\begin{aligned} B(\eta) &= \sum_{l=1}^{\infty} \frac{l^{2(q+k+d/p_*+\eta)}}{2^{2h\eta l}} \leq \int_0^\infty \frac{x^{2(q+k+d/p_*+1)}}{2^{2h\eta x}} dx \\ &= \frac{1}{(2\eta h)^{1+2(k+q+d/p_*+1)}} \int_0^\infty \frac{y^{2(q+k+d/p_*+1)}}{2^y} dy \leq C. \end{aligned}$$

Moreover, since $h \geq \frac{1}{2}(q+a)$ it follows that

$$\rho_h(q)(1+\delta)b(2h+q+a) \leq 2(1+\delta)b(k+q+d/p_*)$$

and consequently (recall that $n_* = h$)

$$\begin{aligned} \theta(n_*)^{\rho_h(q)(1+\delta)} &= Q_h^{\rho_h(q)(1+\delta)}(q,p) n_*^{\rho_h(q)(1+\delta)b(2h+q+a)} \\ &\leq Q_h^{\rho_h(q)(1+\delta)}(q,p) h^{2(1+\delta)b(k+q+d/p_*)}. \end{aligned}$$

Finally we notice that, by (4.18), the constant $C_{h,n_*}(\eta)$ defined in (4.8) verifies

$$C_{h,n_*}(\eta) \leq C_0 Q_h^{\rho_h(q)(1+\delta)}(q,p).$$

Now we use (4.9) and we obtain

$$\left\| f \right\|_{q,p} \leq C(1 + Q_h^{\rho_h(q)(1+\delta)}(q,p) h^{2b(1+\delta)(k+q+d/p_*)} + C_0 Q_h^{\rho_h(q)(1+\delta)}(q,p))$$

which gives (4.16). $\square$

4.1 Link with the integration by parts formula

We consider a family of random variables $F_\varepsilon \in R^d$ and $G_\varepsilon \in [0,1], \varepsilon > 0$ and we associate the measures μ_ε given by

$$\int \varphi d\mu_\varepsilon = E(\varphi(F_\varepsilon)G_\varepsilon).$$

We assume that μ_ε satisfy the integration by parts formula

$$E(\partial^\beta \varphi(F_\varepsilon)G_\varepsilon) = E(\varphi(F_\varepsilon)H_{\beta,\varepsilon}) \quad \forall \varphi \in C_b^\infty(R^d). \tag{4.19}$$

We also assume that for every $q \in N$ and $p > 1$ there exist some constants $\widehat{H}_{q,p}$ and $a, b, \varepsilon_* \geq 0$ such that for every $0 < \varepsilon < \varepsilon_*$

$$\sup_{|\beta| \leq q} \left\| H_{\beta,\varepsilon} \right\|_p \leq \widehat{H}_{q,p} \varepsilon^{b(q+a)}. \tag{4.20}$$

In particular this implies that $\mu_\varepsilon(dv) = p_\varepsilon(v)dv$.
For $y \in R$ we denote $I_y = (y, \infty)$ if $y \geq 0$ and $I_y = (-\infty, y)$ if $y < 0$. And for $v = (v_1, ..., v_d)$ we define

$$A_v = \prod_{i=1}^d I_{v_i} \tag{4.21}$$

Moreover we consider a random variable $F \in R^d$ and we denote μ the law of F (so $\int \phi d\mu = E(\phi(F))$). We also consider a function $\Phi \in C^\infty(R^d)$ such that $\Phi(v) > 0$, dv almost surely. Our aim is to give sufficient conditions which guarantee that $\mu(dv) = p(v)dv$ and to obtain estimates for Φp. In order to do this we give first estimates for Φp_ε, we estimate then $d_1(\Phi\mu, \Phi\mu_\varepsilon)$, and finally we use Proposition 4.4 in order to conclude.

Lemma 4.5 *Let $\Phi \in C^\infty(R^d)$. Assume that (4.19) and (4.20) hold. For every $q, h \in N$ and $p > 0$ there exists a constant C (depending on q, h, d and p only) such that, for every $\varepsilon \in (0, \varepsilon_*)$*

$$\left\| \Phi p_\varepsilon \right\|_{q,h,p} \leq C \widehat{H}_{q+d,p_*} (E(\Phi_{q,h,p}(F_\varepsilon)))^{1/p} \varepsilon^{b(q+a)} \tag{4.22}$$

with

$$\Phi_{q,h,p}(x) := \sup_{|\beta| \leq q} \int_{R^d} (1+|v|)^h \left| \partial^\beta \Phi(v) \right|^p 1_{A_v}(x) dv \tag{4.23}$$

Proof. By (4.19)

$$\begin{aligned}
\partial^\alpha(\Phi p_\varepsilon)(v) &= \sum_{(\beta,\gamma)=\alpha} \partial^\beta \Phi(v) \partial^\gamma p_\varepsilon(v) = \sum_{(\beta,\gamma)=\alpha} \partial^\beta \Phi(v) E(\partial^\gamma \delta_0(F_\varepsilon - v)G_\varepsilon) \\
&= \sum_{(\beta,\gamma)=\alpha} \partial^\beta \Phi(v) E(1_{A_v}(F_\varepsilon) H_{(\gamma,1,\ldots,d),\varepsilon})
\end{aligned}$$

so that

$$\begin{aligned}
\left| \partial^\alpha(\Phi p_\varepsilon)(v) \right| &\leq \sum_{(\beta,\gamma)=\alpha} \left| \partial^\beta \Phi(v) E(1_{A_v}(F_\varepsilon) H_{(\gamma,1,\ldots,d),\varepsilon}) \right| \\
&\leq \sum_{(\beta,\gamma)=\alpha} \left| \partial^\beta \Phi(v) \right| P^{1/p}(F_\varepsilon \in A_v) \left\| H_{(\gamma,1,\ldots,d),\varepsilon} \right\|_{p_*} \\
&\leq \sum_{|\beta| \leq q} \left| \partial^\beta \Phi(v) \right| P^{1/p}(F_\varepsilon \in A_v) \times \widehat{H}_{q+d,p_*} \varepsilon^{b(q+a)}.
\end{aligned}$$

This gives

$$\begin{aligned}\|\Phi p_\varepsilon\|_{q,h,p} &\leq C\widehat{H}_{q+d,p_*}\varepsilon^{b(q+a)}\sum_{|\beta|\leq q}(\int(1+|v|)^h\left|\partial^\beta\Phi(v)\right|^p E(1_{F_\varepsilon\in A_v})dv)^{1/p} \\ &= C\widehat{H}_{q+d,p_*}\varepsilon^{b(q+a)}(E(\Phi_{q,h,p}(F_\varepsilon)))^{1/p}.\end{aligned}$$

□

We denote $Q_\varepsilon=\sup_{\lambda\in(0,1)}(|\Phi|+|\nabla\Phi|)(\lambda F+(1-\lambda)F_\varepsilon)$ and we define the constants

$$\begin{aligned}C_1(\Phi,\theta) &= \sup_{\varepsilon>0}\|Q_\varepsilon\|_\theta \\ C_2(\Phi) &= \sup_{\varepsilon>0}\|\Phi(F_\varepsilon)\|_2\,, \quad C_3=\|F\|_2+\sup_{\varepsilon>0}\|F_\varepsilon\|_2\,.\end{aligned} \tag{4.24}$$

Lemma 4.6 *Suppose that the constants in (4.24) are finite and moreover suppose that*

$$\|1-G_\varepsilon\|_2+\|F-F_\varepsilon\|_1\leq C_0\varepsilon^\beta. \tag{4.25}$$

Then, for every $\delta>0$

$$d_1(\Phi\mu_\varepsilon,\Phi\mu)\leq C_\delta(\Phi)\varepsilon^{\beta(1-\delta)} \tag{4.26}$$

with

$$C_\delta(\Phi)=C_0(1+C_2(\Phi))+2C_3C_1^{1/\delta}(\Phi,\frac{2}{\delta})). \tag{4.27}$$

Proof. Let ϕ with $\|\phi\|_{1,\infty}\leq 1$. We estimate first

$$|E((\phi\Phi)(F_\varepsilon)(1-G_\varepsilon)|)\leq\|\phi\|_\infty\|\Phi(F_\varepsilon)\|_2\|1-G_\varepsilon\|_2\leq C_2(\Phi)C_0\varepsilon^\beta.$$

Then we write

$$\begin{aligned}|E((\phi\Phi)(F_\varepsilon)-(\phi\Phi)(F))| &\leq E\int_0^1|\nabla(\phi\Phi)(\lambda F+(1-\lambda)F_\varepsilon)(F-F_\varepsilon)|\,d\lambda \\ &\leq \|\phi\|_{1,\infty}E(Q_\varepsilon|F-F_\varepsilon|)\leq I_K+J_K\end{aligned}$$

with

$$I_K=E(Q_\varepsilon|F-F_\varepsilon|1_{\{Q\leq K\}})\leq KC_0\varepsilon^\beta$$

and

$$J_K=E(Q_\varepsilon|F-F_\varepsilon|1_{\{Q>K\}})\leq 2C_3(E(Q_\varepsilon^2 1_{\{Q>K\}}))^{1/2}\leq\frac{2C_3C_1^{\theta+1}(\Phi,2(\theta+1))}{K^\theta}.$$

In order to optimize we take $K=\varepsilon^{-\beta/(1+\theta)}$ and we obtain

$$|E((\phi\Phi)(F_\varepsilon)-(\phi\Phi)(F))|\leq(C_0+(2C_3C_1^{\theta+1}(\Phi,2(\theta+1))))\times\varepsilon^{\beta\times\frac{\theta}{1+\theta}}$$

Then taking $\theta=(1-\delta)/\delta$ we obtain (4.26). □

As an immediate consequence of Proposition 4.4 with $k=1$ and of the Lemma 4.5 and of Lemma 4.6 we obtain

Proposition 4.7 *Let $q\in N$ and $p>1$ be given. Suppose that (4.19),(4.20) and (4.25) hold and*

$$r=\beta-b(1+q+d/p_*)>0 \tag{4.28}$$

Then $\mu(dv)=p(v)dv$ and for every $\delta>0$ there exists C (depending on q,d,p,r,β and δ only) such that

$$\|\Phi p\|_{q,p}\leq C\times C_\delta(\Phi)\times\left(h^{2b}Q_h^{1/2h}(q,p)\right)^{(1+\delta)(1+q+d/p_*)} \tag{4.29}$$

with $C_\delta(\Phi)$ given in (4.27) and with (see (4.22))

$$Q_h(q,p)=\widehat{H}_{2h+q+d,p_*}(E(\Phi_{2h+q,2h,p}(F_\varepsilon)))^{1/p}. \tag{4.30}$$

This inequality holds for every $h\geq h_*$ with h_* given in (4.15).

We discuss now the particular case which appears in our framework: we consider a non decreasing function $\rho : R_+ \to R_+$ such that $\rho(u) = 1$ for $u \in (0,1), \rho(u) = u$ for $u \in (2,\infty)$ and $\rho \in C^\infty(R_+)$ and, for some $\lambda > 0$ we define

$$\Phi_\lambda(v) = e^{\rho(|v|^\lambda)}. \tag{4.31}$$

Then Φ_λ has the following property: for every $h, q \in N$ there exist some constants c_1, c_2 (depending on q and h) such that for every multi-index β with $|\beta| \le q$ one has

$$(1+|v|)^h \left| \partial^\beta \Phi_\lambda(v) \right| \le c_1 \Phi_\lambda^{c_2}(v). \tag{4.32}$$

For $\theta \ge 1$ we denote

$$\widehat{C}_\theta(\lambda) = E(\Phi_\lambda^\theta(F)) + \sup_{\varepsilon>0} E(\Phi_\lambda^\theta(F_\varepsilon)) \tag{4.33}$$

One easily verifies that the constants defined in (4.24) verify, for some universal constants C and θ'

$$C_1(\Phi_\lambda, \theta) + C_2(\Phi_\lambda) + C_3 \le C \times \widehat{C}_{\theta'}(\lambda)$$

and consequently (with C_0 from (4.25)), for every $\delta > 0$ there exist C, θ and θ' such that

$$C_\delta(\Phi_\lambda) \le C \times C_0 \times \widehat{C}_\theta^{\theta'}(\lambda) \tag{4.34}$$

Since $|x| \ge |v|$ for $x \in A_v$ it follows that the constant defined in (4.23) verifies (with C and θ depending on q,h,p) $\Phi_{q,h,p}(x) \le C\Phi_\lambda^\theta(x)$. So

$$E(\Phi_{2h+q,2h,p}(F_\varepsilon)) \le CE(\Phi_\lambda^\theta(F_\varepsilon)) \le C\widehat{C}_\theta(\lambda)$$

so finally the constant in (4.30) is

$$Q_h(q,p) = \widehat{H}_{2h+q+d,p_*} \times C \times \widehat{C}_\theta(\lambda).$$

Corollary 4.8 ***A***. *Let $q,h,d \in N$ and $p > 1, \delta > 0$ be given. Suppose that (4.19), (4.20), (4.25) and (4.28) hold (for these q,d,p and δ). Consider also some $\lambda \ge 0$ such that $\widehat{C}_\theta(\lambda) < \infty$ for every $\theta \ge 1$. There exist some constants $C \ge 1$ and $\theta, \theta' \ge 1$ (depending on q,h,d,p and δ) such that for $h \ge h_*$ (given in (4.15)) one has*

$$\left\|\Phi_\lambda p\right\|_{q,p} \le \Gamma_\lambda(q,h,p) \quad \text{with} \tag{4.35}$$

$$\Gamma_\lambda(q,h,p) := C \times C_0 \times \widehat{C}_\theta^{\theta'}(\lambda) \times \left(h^{2b}\widehat{H}^{1/2h}_{2h+q+d,p_*}\right)^{(1+\delta)(1+q+d/p_*)} \tag{4.36}$$

with C_0 given in (4.25) and $\widehat{C}_\theta(\lambda)$ given in (4.33).
B. *Suppose that the hypothesis from the point **A** holds for $q = 1$ and $p > d$. Then $p \in C^{0,\chi}(R^d)$ with $\chi = 1 - \frac{d}{p}$ and we have*

$$p(x) \le \left\|\Phi_\lambda p\right\|_{1,p} \times e^{-|x|^\lambda} \le \Gamma_\lambda(1,h,p) \times e^{-|x|^\lambda}. \tag{4.37}$$

Moreover, for every $x, y \in R^d$ with $|x-y| \le 1$ and every $\varepsilon > 0$,

$$\begin{aligned} |p(y)-p(x)| &\le \left\|\Phi_\lambda p\right\|_{1,p} \times e^{-(1-\varepsilon)|x|^\lambda} \times |x-y|^\chi \\ &\le \Gamma_\lambda(1,h,p) \times e^{-(1-\varepsilon)|x|^\lambda} \times |x-y|^\chi \end{aligned} \tag{4.38}$$

Proof. The point **A** is an immediate consequence of Proposition 4.7. Let us prove **B**. The fact that p is $\chi-$ Hölder continuous is a consequence of Morrey's theorem which also gives $\left\|\Phi_\lambda p\right\|_\infty \leq \left\|\Phi_\lambda p\right\|_{C^{0,\chi}} \leq C\left\|\Phi_\lambda p\right\|_{1,p}$ so we obtain (4.37).
Note that there exists C such that $|\nabla\Phi_\lambda(y)| \leq C\Phi_\lambda^{1+\varepsilon}(x)$ for every $x, y \in R^d$ such that $|x-y| \leq 1$. It follows that if $|x-y| \leq 1$ then $|\Phi_\lambda(y) - \Phi_\lambda(x)| \leq C\Phi_\lambda^{1+\varepsilon}(x)\,|x-y|$. We write now

$$(\Phi_\lambda p)(y) - (\Phi_\lambda p)(x) = \Phi_\lambda(x)(p(y) - p(x)) + (\Phi_\lambda(y) - \Phi_\lambda(x))p(x)$$

and this gives

$$\begin{aligned}\Phi_\lambda(x)\,|p(y)-p(x)| &\leq |\Phi_\lambda(y) - \Phi_\lambda(x)|\,p(x) + |(\Phi_\lambda p)(y) - (\Phi_\lambda p)(x)| \\ &\leq C\Phi_\lambda^{1+\varepsilon}(x)p(x)\,|x-y| + \Gamma_\lambda(q,h,p)\,|y|^\chi \\ &\leq C(\Phi_\lambda^{\varepsilon}(x) + \Gamma_\lambda(q,h,p))\,|x-y|^\chi\end{aligned}$$

and this gives (4.38). □

Remark 4.9 *The above estimates seem interesting even in the following simpler situation. Consider a random variable F for which the integration by parts formulae*

$$E(\partial^\alpha\varphi(F)) = E(\varphi(F)H_\alpha) \quad \forall\varphi \in C_b^\infty(R^d)$$

holds for every multi-index α and denote $\widehat{H}_{q,p} = \sup_{|\alpha|\leq q}\left\|H_\alpha\right\|_p$. Suppose that $\widehat{H}_{q,p} < \infty$ for everty $q \in N, p \geq 1$ and suppose also that $\widehat{C}_\theta(\lambda) := E(e^{\theta|F|^\lambda}) < \infty$ for every $\theta \geq 1$. Then $P(F \in dx) = p(x)dx$ and, for every $h \in N_*$, we have the estimate

$$p(x) \leq \Gamma_\lambda(1,h,p) \times e^{-|x|^\lambda}. \tag{4.39}$$

with $\Gamma_\lambda(q,h,p)$ defined in (4.36). Morover, using Morrey's theorem for arbitrary $q \in N$, we obtain, for every multi-index α with $|\alpha| \leq q$,

$$|\partial^\alpha p(x)| \leq \Gamma_\lambda(q,h,p) \times e^{-|x|^\lambda} \tag{4.40}$$

This imediately follows by taking $F_\varepsilon = F$ (so that $d_1(F,F_\varepsilon) = 0$ and so one may take $\beta = \infty$ in the above reasoning).

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