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Les moyennes de Minkowski deviennent-elles progressivement plus convexes ?

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A R T I C L E I N F O

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A B S T R A C T

Let us define, for a compact set $A \subset \mathbf{R}^n$, the Minkowski averages of A:

$$A(k) = \left\{ \frac{a_1 + \cdots + a_k}{k} : a_1, \ldots, a_k \in A \right\} = \frac{1}{k}\Big(\underbrace{A + \cdots + A}_{k \text{ times}}\Big).$$

We study the monotonicity of the convergence of $A(k)$ towards the convex hull of A, when considering the Hausdorff distance, the volume deficit and a non-convexity index of Schneider as measures of convergence. For the volume deficit, we show that monotonicity fails in general, thus disproving a conjecture of Bobkov, Madiman and Wang. For Schneider's non-convexity index, we prove that a strong form of monotonicity holds, and for the Hausdorff distance, we establish that the sequence is eventually nonincreasing.

R É S U M É

Pour tout ensemble compact $A \subset \mathbf{R}^n$, définissons ses moyennes de Minkowski par

$$A(k) = \left\{ \frac{a_1 + \cdots + a_k}{k} : a_1, \ldots, a_k \in A \right\} = \frac{1}{k}\Big(\underbrace{A + \cdots + A}_{k \text{ fois}}\Big).$$

Nous étudions la monotonie de la convergence de $A(k)$ vers l'enveloppe convexe de A, mesurée par la distance de Hausdorff, le déficit volumique et par l'indice de non-convexité

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de Schneider. Pour le déficit volumique, nous démontrons que la propriété de monotonie n'est pas satisfaite en général, réfutant ainsi une conjecture de Bobkov, Madiman et Wang. Pour l'index de non-convexité de Schneider, nous montrons une propriété renforcée de monotonie, tandis que, pour la distance de Hausdorff, nous établissons que la suite est décroissante à partir d'un certain rang.

Version française abrégée

L'objectif de cette note est d'annoncer et de démontrer une partie des résultats obtenus dans [3] qui portent sur l'étude de la monotonie de la suite $(A(k))_{k\geq 1}$ définie en (1), mesurée à travers différentes mesures de non-convexité. Intuitivement, les ensembles $A(k)$ deviennent de plus en plus convexes au fur et à mesure que k croît. Cette intuition est précisée dans [7,2] où il est démontré que la suite $(A(k))$ converge vers son enveloppe convexe en distance de Hausdorff d_H.

L'origine de notre étude provient d'une conjecture de Bobkov, Madiman et Wang [1], qui affirme que la suite $(\Delta(A(k)))_{k\geq 1}$ est décroissante, où

$$\Delta(A) := \mathrm{Vol}_n(\mathrm{conv}(A)\setminus A) = \mathrm{Vol}_n(\mathrm{conv}(A)) - \mathrm{Vol}_n(A)$$

désigne le déficit volumique d'un ensemble compact de $\mathbf{R}^n$. Ici, Vol_n représente la mesure de Lebesgue dans $\mathbf{R}^n$ et $\mathrm{conv}(A)$ désigne l'enveloppe convexe de A. Nous réfutons cette conjecture en exhibant un contre-exemple explicite en dimension supérieure ou égale à 12. Le contre-exemple est la réunion de deux ensembles convexes inclus dans des sous-espaces de dimension (presque) moitié de l'espace ambiant (voir Fig. 1). Nous démontrons aussi la validité de la conjecture en dimension 1 en adaptant une démonstration de [4] sur le cardinal de sommes d'entiers ; cela a aussi été observé indépendamment par F. Barthe. La conjecture reste ouverte en dimension n, pour $1 < n < 12$.

De manière analogue à la conjecture de Bobkov–Madiman–Wang, nous étudions la monotonie de la suite $(c(A(k)))_{k\geq 1}$, où c est l'index de non-convexité de Schneider [6] défini par

$$c(A) := \inf\{\lambda \geq 0 : A + \lambda\, \mathrm{conv}(A) \text{ est convexe}\}.$$

Contrairement au déficit volumique, la suite $(c(A(k)))$ est strictement décroissante, à moins que $A(k)$ soit déjà convexe. Plus précisément nous montrons que pour tout ensemble compact A de $\mathbf{R}^n$ et tout $k \in \mathbb{N}^*$

$$c\,(A(k+1)) \leq \frac{k}{k+1} c\,(A(k))\,.$$

En outre, nous étudions dans [3] la monotonie de $A(k)$, mesurée par d'autres mesures de non-convexité. Ainsi, nous montrons que si l'on pose

$$d(A) = \mathrm{d}_H(A, \mathrm{conv}(A)) = \inf\{r > 0 : \mathrm{conv}(A) \subset A + rB_2^n\},$$

où B_2^n est la boule euclidienne centrée en 0 de rayon 1, alors pour tout compact A de $\mathbf{R}^n$ et pour $k \geq c(A)$,

$$d\,(A(k+1)) \leq \frac{k}{k+1} d\,(A(k))\,.$$

1. Introduction

This note announces and proves some of the results obtained in [3]. Let us denote for a compact set $A \subset \mathbf{R}^n$ and for a positive integer k,

$$A(k) = \left\{\frac{a_1 + \cdots + a_k}{k} : a_1, \ldots, a_k \in A\right\} = \frac{1}{k}\Big(\underbrace{A + \cdots + A}_{k \text{ times}}\Big). \tag{1}$$

Denoting by $\mathrm{conv}(A)$ the convex hull of A, and by

$$d(A) := \inf\{r > 0 : \mathrm{conv}(A) \subset A + rB_2^n\}$$

the Hausdorff distance between a set A and its convex hull, it is a classical fact (proved independently by [7,2] in 1969, and often called the Shapley–Folkmann–Starr theorem) that $A(k)$ converges in Hausdorff distance to $\mathrm{conv}(A)$ as $k \to \infty$. Furthermore [7,2] also determined the rate of convergence: it turns out that $d(A(k)) = O(1/k)$ for any compact set A. For sets of nonempty interior, this convergence of Minkowski averages to the convex hull can also be expressed in terms of the volume deficit $\Delta(A)$ of a compact set A in $\mathbf{R}^n$, which is defined as:

$$\Delta(A) := \mathrm{Vol}_n(\mathrm{conv}(A) \setminus A) = \mathrm{Vol}_n(\mathrm{conv}(A)) - \mathrm{Vol}_n(A),$$

where Vol_n denotes Lebesgue measure in $\mathbf{R}^n$. It was shown by [2] that if A is compact with nonempty interior, then the volume deficit of $A(k)$ also converges to 0; more precisely, $\Delta(A(k)) = O(1/k)$ for any compact set A with nonempty interior.

Our original motivation came from a conjecture made by Bobkov, Madiman and Wang [1]:

Conjecture 1. *(See [1].) Let A be a compact set in $\mathbf{R}^n$ for some $n \in \mathbb{N}$, and let $A(k)$ be defined as in (1). Then the sequence $\Delta(A(k))$ is non-increasing in k, or equivalently, $\{\mathrm{Vol}_n(A(k))\}_{k\geq 1}$ is non-decreasing.*

We show that Conjecture 1 fails to hold in general, even for moderately high dimension.

Theorem 2. *Conjecture 1 is false in $\mathbf{R}^n$ for $n \geq 12$, and true for $\mathbb{R}^1$.*

Notice that Conjecture 1 remains open for $1 < n < 12$. In particular, the arguments presented in this note do not seem to work. In analogy with Conjecture 1, we also consider whether one can have monotonicity of $\{c(A(k))\}_{k\geq 1}$, where c is a non-convexity index defined by Schneider [6] as follows:

$$c(A) := \inf\{\lambda \geq 0 : A + \lambda\, \mathrm{conv}(A) \text{ is convex}\}.$$

A nice property of Schneider's index is that it is affine-invariant, i.e., $c(TA + x) = c(A)$ for any nonsingular linear map T on $\mathbf{R}^n$ and any $x \in \mathbf{R}^n$.

Contrary to the volume deficit, we prove that Schneider's non-convexity index c satisfies a strong kind of monotonicity in any dimension.

Theorem 3. *Let A be a compact set in $\mathbf{R}^n$ and $k \in \mathbb{N}^*$. Then*

$$c\left(A(k+1)\right) \leq \frac{k}{k+1} c\left(A(k)\right).$$

Finally, we also prove that eventually, for $k \geq c(A)$, the Hausdorff distance between $A(k)$ and $\mathrm{conv}(A)$ is also strongly decreasing.

Theorem 4. *Let A be a compact set in $\mathbf{R}^n$ and $k \geq c(A)$ be an integer. Then*

$$d\left(A(k+1)\right) \leq \frac{k}{k+1} d\left(A(k)\right).$$

Moreover, Schneider proved in [6] that $c(A) \leq n$ for every compact subset A of $\mathbf{R}^n$. It follows that the eventual monotonicity of the sequence $d\left(A(k)\right)$ holds true for $k \geq n$.

It is natural to ask what the relationship is in general between convergence of c, Δ and d to 0, for arbitrary sequences (C_k) of compact sets. In fact, none of these three notions of approach to convexity are comparable with each other in general. To see why, observe that while c is scaling-invariant, neither Δ nor d are; so it is easy to construct examples of sequences (C_k) such that $c(C_k) \to 0$ but $\Delta(C_k)$ and $d(C_k)$ remain bounded away from 0. The same argument enables us to construct examples of sequences (C_k) such that $c(C_k)$ remain bounded away from 0, whereas $\Delta(C_k)$ and $d(C_k)$ converge to 0. Furthermore, $\Delta(C_k)$ remains bounded away from 0 for any sequence C_k of finite sets, whereas $c(C_k)$ and $d(C_k)$ could converge to 0 if the finite sets form a finer and finer grid filling out a convex set. An example where $\Delta(C_k) \to 0$ but both $c(C_k)$ and $d(C_k)$ are bounded away from 0 is given by taking a 3-point set with 2 of the points getting arbitrarily closer but staying away from the third. One can obtain further relationships between these measures of non-convexity if further conditions are imposed on the sequence C_k; details may be found in [3].

The rest of this note is devoted to the examination of whether $A(k)$ becomes progressively more convex as k increases, when measured through the functionals Δ, d and c. The concluding section contains some additional discussion.

2. The behavior of volume deficit

We prove Theorem 2 in this section. We start by constructing a counterexample to the conjecture in $\mathbf{R}^n$, for $n \geq 12$. Let F be a p-dimensional subspace of $\mathbf{R}^n$, where $p \in \{1, \ldots, n-1\}$. Let us consider $A = I_1 \cup I_2$, where $I_1 \subset F$ and $I_2 \subset F^{\perp}$, where $F^{\perp}$ denotes the orthogonal complement of F. One has (see Fig. 1):

$$A + A = 2I_1 \cup (I_1 \times I_2) \cup 2I_2,$$

$$A + A + A = 3I_1 \cup (2I_1 \times I_2) \cup (I_1 \times 2I_2) \cup 3I_2.$$

Notice that

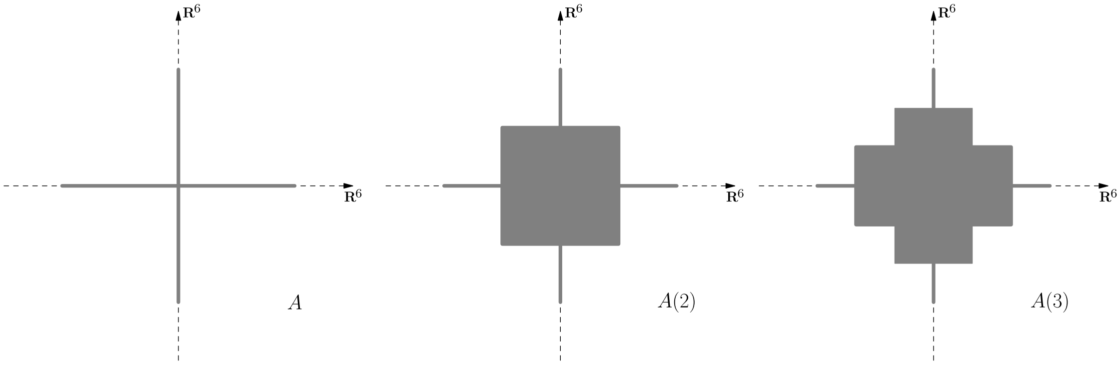

Fig. 1. A counterexample in $\mathbf{R}^{12}$.

$$\mathrm{Vol}_n(A+A)=\mathrm{Vol}_p(I_1)\mathrm{Vol}_{n-p}(I_2),$$
$$\mathrm{Vol}_n(A+A+A)=\mathrm{Vol}_p(I_1)\mathrm{Vol}_{n-p}(I_2)(2^p+2^{n-p}-1).$$

Thus, $\mathrm{Vol}_n(A(3))\geq \mathrm{Vol}_n(A(2))$ if and only if

$$2^p+2^{n-p}-1\geq\left(\frac{3}{2}\right)^n. \tag{2}$$

Notice that inequality (2) does not hold when $n\geq 12$ and $p=\lceil\frac{n}{2}\rceil$.

For $\mathbf{R}^1$, the conjecture may be proved by adapting a proof of [4] on cardinality of integer sumsets; this was also independently observed by F. Barthe. Let $k\geq 1$. Set $S=A_1+\cdots+A_k$ and for $i\in[k]$, let $a_i=\min A_i$, $b_i=\max A_i$,

$$S_i=\sum_{j\in[k]\setminus\{i\}}A_j,$$

$s_i=\sum_{j<i}a_j+\sum_{j>i}b_j$, $S_i^-=\{x\in S_i; x\leq s_i\}$ and $S_i^+=\{x\in S_i; x>s_i\}$. For all $i\in[k-1]$, one has

$$S\supset(a_i+S_i^-)\cup(b_{i+1}+S_{i+1}^+).$$

Since $a_i+s_i=\sum_{j\leq i}a_j+\sum_{j>i}b_j=b_{i+1}+s_{i+1}$, the above union is a disjoint union. Thus for $i\in[k-1]$

$$\mathrm{Vol}_1(S)\geq\mathrm{Vol}_1(a_i+S_i^-)+\mathrm{Vol}_1(b_{i+1}+S_{i+1}^+)=\mathrm{Vol}_1(S_i^-)+\mathrm{Vol}_1(S_{i+1}^+).$$

Notice that $S_1^-=S_1$ and $S_k^+=S_k\setminus\{s_k\}$, thus adding the above $k-1$ inequalities, we obtain

$$\begin{aligned}(k-1)\mathrm{Vol}_1(S)&\geq\sum_{i=1}^{k-1}\left(\mathrm{Vol}_1(S_i^-)+\mathrm{Vol}_1(S_{i+1}^+)\right)=\mathrm{Vol}_1(S_1^-)+\mathrm{Vol}_1(S_k^+)+\sum_{i=2}^{k-1}\mathrm{Vol}_1(S_i)\\&=\sum_{i=1}^{k}\mathrm{Vol}_1(S_i).\end{aligned}$$

Now taking all the sets $A_i=A$, and dividing through by $k(k-1)$, we see that we have established Conjecture 1 in dimension 1.

3. The behavior of Schneider's non-convexity index and the Hausdorff distance

We establish Theorems 3 and 4 in this section. This relies crucially on the elementary observations that $\mathrm{conv}(A+B)=\mathrm{conv}(A)+\mathrm{conv}(B)$ and $(t+s)\,\mathrm{conv}(A)=t\,\mathrm{conv}(A)+s\,\mathrm{conv}(A)$ for any $t,s>0$ and any compact sets A,B.

Proof of Theorem 3. Denote $\lambda=c(A(k))$. Since $\mathrm{conv}(A(k))=\mathrm{conv}(A)$, from the definition of c, one knows that $A(k)+\lambda\,\mathrm{conv}(A)=\mathrm{conv}(A)+\lambda\,\mathrm{conv}(A)=(1+\lambda)\,\mathrm{conv}(A)$. Using that $A(k+1)=\frac{A}{k+1}+\frac{k}{k+1}A(k)$, one has

$$\begin{aligned}A(k+1)+\frac{k}{k+1}\lambda\,\mathrm{conv}(A)&=\frac{A}{k+1}+\frac{k}{k+1}A(k)+\frac{k}{k+1}\lambda\,\mathrm{conv}(A)\\&=\frac{A}{k+1}+\frac{k}{k+1}\,\mathrm{conv}(A)+\frac{k}{k+1}\lambda\,\mathrm{conv}(A)\end{aligned}$$

$$
\begin{aligned}
&\supset \frac{\operatorname{conv}(A)}{k+1} + \frac{k}{k+1} A(k) + \frac{k}{k+1} \lambda \operatorname{conv}(A) \\
&= \frac{\operatorname{conv}(A)}{k+1} + \frac{k}{k+1} (1+\lambda) \operatorname{conv}(A) \\
&= \left(1 + \frac{k}{k+1}\lambda\right) \operatorname{conv}(A).
\end{aligned}
$$

Since the other inclusion is trivial, we deduce that $A(k+1) + \frac{k}{k+1}\lambda \operatorname{conv}(A)$ is convex, which proves that

$$
c(A(k+1)) \le \frac{k}{k+1}\lambda = \frac{k}{k+1} c(A(k)). \qquad \square
$$

Proof of Theorem 4. Let $k \ge c(A)$, then, from the definitions of $c(A)$ and $d(A(k))$, one has

$$
\begin{aligned}
\operatorname{conv}(A) = \frac{A}{k+1} + \frac{k}{k+1} \operatorname{conv}(A) &\subset \frac{A}{k+1} + \frac{k}{k+1} \left(A(k) + d(A(k)) B_2^n\right) \\
&= A(k+1) + \frac{k}{k+1} d(A(k)) B_2^n.
\end{aligned}
$$

We conclude that

$$
d(A(k+1)) \le \frac{k}{k+1} d(A(k)). \qquad \square
$$

4. Discussion

(i) By repeated application of Theorem 3, it is clear that the convergence of $c(A(k))$ is at a rate $O(1/k)$ for any compact set $A \subset \mathbf{R}^n$; this observation appears to be new. In [3], we study the question of the monotonicity of $A(k)$, as well as convergence rates, when considering several different ways to measure non-convexity, including some not mentioned in this note.

(ii) Some of the results in this note are of interest when one is considering Minkowski sums of different compact sets, not just sums of A with copies of itself. Indeed, the original conjecture of [1] was of this form, and would have provided a strengthening of the classical Brunn–Minkowski inequality for more than 2 sets; of course, that conjecture is false since the weaker Conjecture 1 is false. Nonetheless we do have some related observations in [3]; for instance, it turns out that in general dimension, for compact sets $A_1, \ldots, A_k$,

$$
\mathrm{Vol}_n\left(\sum_{i=1}^{k} A_i\right) \ge \frac{1}{k-1} \sum_{i=1}^{k} \mathrm{Vol}_n\left(\sum_{j\in[k]\setminus\{i\}} A_j\right).
$$

For convex sets B_i, an even stronger fact is true (that this is stronger may not be immediately obvious), but if follows from well-known results, see, e.g., [5]:

$$
\mathrm{Vol}_n(B_1 + B_2 + B_3) + \mathrm{Vol}_n(B_1) \ge \mathrm{Vol}_n(B_1 + B_2) + \mathrm{Vol}_n(B_1 + B_3).
$$

(iii) There is a variant of the strong monotonicity of Schneider's index when dealing with different sets. If A, B, C are subsets of $\mathbf{R}^n$, then it is shown in [3] (by a similar argument to that used for Theorem 3) that $c(A+B+C) \le \max\{c(A+B), c(B+C)\}$.

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