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► **To cite this version:**

Antti Kupiainen, Rémi Rhodes, Vincent Vargas. INTEGRABILITY OF LIOUVILLE THEORY: PROOF OF THE DOZZ FORMULA. *Annals of Mathematics*, Princeton University, Department of Mathematics, 2020, 191 (1), pp.81-166. hal-01587086

HAL Id: hal-01587086

<https://hal-upec-upem.archives-ouvertes.fr/hal-01587086>

Submitted on 13 Sep 2017

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INTEGRABILITY OF LIOUVILLE THEORY: PROOF OF THE DOZZ FORMULA

ANTTI KUPIAINEN¹, RÉMI RHODES², AND VINCENT VARGAS²

ABSTRACT. Dorn and Otto (1994) and independently Zamolodchikov and Zamolodchikov (1996) proposed a remarkable explicit expression, the so-called DOZZ formula, for the 3 point structure constants of Liouville Conformal Field Theory (LCFT), which is expected to describe the scaling limit of large planar maps properly embedded into the Riemann sphere. In this paper we give a proof of the DOZZ formula based on a rigorous probabilistic construction of LCFT in terms of Gaussian Multiplicative Chaos given earlier by F. David and the authors. This result is a fundamental step in the path to prove integrability of LCFT, i.e. to mathematically justify the methods of Conformal Bootstrap used by physicists. From the purely probabilistic point of view, our proof constitutes the first rigorous integrability result on Gaussian Multiplicative Chaos measures.

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¹Supported by the Academy of Finland,²Research supported in part by ANR grant Liouville (ANR-15-CE40-0013)

Key words and phrases. Liouville Quantum Gravity, quantum field theory, Gaussian multiplicative chaos, Ward identities, BPZ equations, DOZZ formula .

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1. INTRODUCTION

A. Polyakov introduced Liouville Conformal Field theory (LCFT hereafter) in his 1981 seminal paper [43] where he proposed a path integral theory of random two dimensional Riemannian metrics. Motivated by an attempt to solve LCFT Belavin, Polyakov and Zamolodchikov (BPZ hereafter) formulated in their 1984 paper [3] the general structure of Conformal Field Theory (CFT hereafter). In the BPZ approach the basic objects of CFT are correlation functions of random fields and solving CFT consists in deriving explicit expressions for them. BPZ proposed to construct the correlation functions of a CFT recursively from two inputs: the *spectrum* and the *three point structure constants*. The former summarizes the representation content of the CFT (under the Virasoro algebra) and the latter determine the three point correlation functions, see Section 1.1. The recursive procedure to find higher point correlation functions is called *Conformal Bootstrap*. Though BPZ were able to find spectra and structure constants for a large class of CFT's (e.g. the Ising model) LCFT was not one of them¹. The spectrum of LCFT was soon conjectured in [12, 8, 26] but the structure constants remained a puzzle.

A decade later, Dorn and Otto [15] and independently Zamolodchikov and Zamolodchikov [59] (DOZZ hereafter) proposed a remarkable formula for the structure constants of LCFT the so-called DOZZ formula. Even by the physicists' standards the derivation was lacking rigor. To quote [59]: "It should be stressed that the arguments of this section have nothing to do with a derivation. These are rather some motivations and we consider the expression proposed as a guess which we try to support in the subsequent sections." Ever since these papers the derivation of the DOZZ formula from the original (heuristic) functional integral definition of LCFT given by Polyakov has remained a controversial open problem, even on the physical level of rigor.

Recently the present authors together with F. David gave a rigorous probabilistic construction of Polyakov's LCFT functional integral [13]. This was done using the probabilistic theory of Gaussian Multiplicative Chaos (GMC). Subsequently in [36] we proved identities for these correlation functions postulated in the work of BPZ (conformal Ward identities and BPZ equations). This provided a probabilistic setup to address the conformal bootstrap and in particular the DOZZ formula. In this paper we address the second problem: we prove that the probabilistic expression given in [13] for the structure constants is indeed given by the DOZZ formula. This result should be considered as an integrability result for LCFT and in particular for GMC. As such it constitutes the first rigorous proof of integrability in GMC theory. Integrability of GMC theory was also conjectured in statistical physics in the study of disordered systems. In this context, a remarkable formula for GMC on the circle has been proposed by Fyodorov and Bouchaud [23], based on

¹Following their work [3], Polyakov qualified CFT as an "unsuccessful attempt to solve the Liouville model" and did not at first want to publish his work, see [44].

arguments similar to [15, 59] (soon after, numerous integrability results for GMC on different 1d geometries appeared in the physics literature: see [24] for instance in the case of the unit interval). It turns out that the Fyodorov-Bouchaud formula is a particular case of the conjectured one point bulk structure constant for LCFT on the unit disk with boundary (these formula can be found in Nakayama's review [39]). We believe that our approach can be adapted to this situation and hence will provide a rigorous derivation of the Fyodorov-Bouchaud formula (work in progress [46]).

It should be noted that the LCFT structure constants and the DOZZ formula have a wider relevance than the scaling limits of planar maps. It has been argued in Ribault's review [47] that LCFT seems to be a universal CFT: e.g. the minimal model structure constants (e.g. Ising model, tri-critical Ising model, 3 states Potts model, etc...) originally found by BPZ may be recovered from the DOZZ formula by analytic continuation. In another spectacular development the LCFT structure constants show up in a seemingly completely different setup of four dimensional gauge theories via the so-called AGT correspondence [1] (see the work by Maulik-Okounkov [38] for the mathematical implications of these ideas).

In the remaining part of this introduction we briefly review LCFT in the path integral and in the conformal bootstrap approach and state the DOZZ formula.

1.1. LCFT in the path integral. In the Feynman path integral formulation, LCFT on the Riemann sphere $\hat{\mathbb{C}}$ is the study of conformal metrics on $\hat{\mathbb{C}}$ of the form $e^{\gamma\phi(z)}|dz|^2$ where z is the standard complex coordinate and d^2z the Lebesgue measure. $\phi(z)$ is a random function (a distribution in fact) and one defines an "expectation"

$$(1.1) \quad \langle F \rangle := \int F(\phi) e^{-S_L(\phi)} D\phi$$

where S_L is the *Liouville Action functional*

$$(1.2) \quad S_L(\phi) = \frac{1}{\pi} \int_{\mathbb{C}} (|\partial_z \phi(z)|^2 + \pi\mu e^{\gamma\phi(z)}) d^2z$$

see Section 2.1 for the precise formulation².

LCFT has two parameters $\gamma \in (0, 2)$ and $\mu > 0$. The positive parameter μ in front of the exponential interaction term $e^{\gamma\phi}$ is essential for the existence of the theory (the case $\mu = 0$ corresponds to Gaussian Free Field theory, a completely different theory) but plays no specific role³. On the other hand, the parameter γ encodes the conformal structure of the theory; more specifically, one can show that the central charge⁴ of the theory is $c_L = 1 + 6Q^2$ with

$$(1.3) \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

The basic objects of interest in LCFT are in physics terminology *vertex operators*

$$(1.4) \quad V_\alpha(z) = e^{\alpha\phi(z)}$$

where α is a complex number and their correlation functions $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$. Their definition involves a regularization and renormalization procedure and they were constructed rigorously in [13] for $N \geq 3$ and for real α_i satisfying certain conditions. The construction of the correlations in [13] is probabilistic and based on interpreting $e^{-\frac{1}{\pi} \int_{\mathbb{C}} |\partial_z \phi(z)|^2 d^2z} D\phi$ in terms of a suitable Gaussian Free Field (GFF) probability measure: see subsection 2.1 below for precise definitions and an explicit formula for the correlations in terms of Gaussian Multiplicative Chaos.

²We use brackets and not \mathbb{E} for the linear functional (1.1) since it turns out the measure $e^{-S_L(\phi)} D\phi$ is not normalizable into a probability measure.

³LCFT has a scaling relation with respect to μ so that the precise value of this parameter does not matter.

⁴In this article, the central charge of LCFT will not appear in a transparent way hence we refer to the work [13] or [36] for an account on the central charge.

In particular it was proved in [13] that these correlation functions are *conformal tensors*. More precisely, if z_1, \dots, z_N are N distinct points in \mathbb{C} then for a Möbius map $\psi(z) = \frac{az+b}{cz+d}$ (with $a, b, c, d \in \mathbb{C}$ and $ad-bc=1$)

$$(1.5) \quad \left\langle \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \right\rangle = \prod_{k=1}^N |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle$$

where $\Delta_{\alpha} = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ is called the conformal weight. This global conformal symmetry fixes the three point correlation functions up to a constant:

$$(1.6) \quad \left\langle \prod_{k=1}^3 V_{\alpha_k}(z_k) \right\rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_{\gamma}(\alpha_1, \alpha_2, \alpha_3)$$

with $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$, etc. The constants $C_{\gamma}(\alpha_1, \alpha_2, \alpha_3)$ are called the three point structure constants and they have an explicit expression in terms of the Gaussian Multiplicative Chaos, see Section 2.3. They are also the building blocks of LCFT in the conformal bootstrap approach as we now review.

1.2. LCFT in the conformal bootstrap. The bootstrap approach to Conformal Field Theory goes back to the 70's. It is based on the *operator product expansion* (OPE) introduced by K. Wilson in quantum field theory. In a conformal field theory, the OPE is expected to take a particularly simple form as was observed in the 70's [21, 45, 37]. The simplest CFT's (like the Ising model) contain a finite number of primary fields Φ_i i.e. random fields whose correlation functions transform as (1.5). The OPE is the statement that in a correlation function $\langle \prod \Phi_{\alpha_i}(z_i) \rangle$ one may substitute

$$(1.7) \quad \Phi_i(z_i) \Phi_j(z_j) = \sum_k \mathcal{C}_{ij}^k(z_i, z_j) \Phi_k(z_k)$$

where $\mathcal{C}_{ij}^k(z_i, z_j)$ is an (infinite) sum of linear differential operators which are completely determined up to the three point structure constants C_{ijk} (these are defined in the same way as in (1.6) for LCFT). Furthermore it was argued that the resulting expansion should be convergent. A recursive application of the OPE would then allow in principle to express the N-point function in terms of the structure constants i.e. to “solve” the CFT.

The input in the bootstrap is thus the set of its primary fields, called the *spectrum* of the theory, and their structure constants. In *unitary* CFT's such as the ones describing scaling limits of reflection positive statistical mechanics models the spectrum is in principle determined by the spectral analysis of the representation of the generator of dilations acting in the *physical Hilbert space* of the CFT. This space can be constructed using the Osterwalder-Schrader reconstruction theorem [41, 42] and in case of LCFT this is rather straightforward given the results of [13]: see [35] for lecture notes on this.

There is also plenty of evidence what the spectrum of LCFT should be [12, 8, 26], see in particular Teschner's review [56] for a thorough discussion. It should consist of the vertex operators V_{Q+iP} with $P \in \mathbb{R}_+$ i.e. there is a continuum of primary fields (unlike say in the Ising model where there are three). Assuming this, one ends up with the following rather explicit formula for the 4 point correlation functions for α_i in the spectrum [47]:

$$\langle V_{\alpha_1}(z) V_{\alpha_2}(0) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle = \int_{Q+i\mathbb{R}_+} C_{\gamma}(\alpha_1, \alpha_2, \alpha) C_{\gamma}(2Q - \alpha, \alpha_3, \alpha_4) |\mathcal{F}_{\alpha, \{\alpha_i\}}(z)|^2 d\alpha$$

where $\mathcal{F}_{\alpha, \{\alpha_i\}}(z)$ are explicit meromorphic functions (the so-called universal conformal blocks) which depend only on the parameters α_i, α and the central charge of LCFT $c_L = 1 + 6Q^2$. The integral over α is here the standard Lebesgue integral over P (where $\alpha = Q + iP$) and corresponds to the sum in (1.7).

Note that the spectrum of LCFT consists of vertex operators with *complex* α whereas the probabilistic approach naturally deals with real α . Also, the main application of LCFT to Liouville Quantum Gravity involves real values for α . In the theory of Liouville Quantum Gravity, the scaling limits of e.g. Ising correlations on a random planar map are given in terms of Liouville correlations with real α 's and regular planar Ising CFT correlations via the celebrated KPZ relation [33]: for an explicit mathematical conjecture, see [14, 35]. Thus the probabilistic and bootstrap approaches are in an interesting way complementary. The basis for the bootstrap approach, the DOZZ formula for $C_{\gamma}(\alpha_1, \alpha_2, \alpha_3)$, has a unique meromorphic

extension to $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}^3$. In our probabilistic approach we prove that the probabilistic expressions for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ are analytic functions in the α_i around their real values and extend to meromorphic functions in \mathbb{C}^3 that coincide with the DOZZ expression.

One should note that the bootstrap approach can be turned to a tool to classify CFT's. By making an ansatz for the spectrum one can compute the four point function $\langle \Phi_{\alpha_1}(z_1)\Phi_{\alpha_2}(z_2)\Phi_{\alpha_3}(z_3)\Phi_{\alpha_4}(z_4) \rangle$ using the OPE by pairing the fields in two different ways. This leads to quadratic relations for the structure constants that one can attempt to solve. This was successfully carried out by BPZ in [3] for the minimal models and has led to spectacular progress even in three dimensions, e.g. in case of the 3d Ising model [19, 20]. In the case of LCFT one can check that the DOZZ expression indeed solves these quadratic equations: see the review [47]. Furthermore, given the DOZZ formula the bootstrap expressions are formally defined for all $\gamma \in \mathbb{C} \setminus i\mathbb{R}$ ⁵. Hence the values of γ for which the theory is defined (within the framework of theoretical physics) is quite general compared to the path integral formulation which makes sense for $\gamma \in (0, 2)$ ⁶. We consider the mathematical justification of the bootstrap approach to be a major challenge to probabilists and plan to pursue this in the case of LCFT in the future (see also [54, 10, 29, 6, 7, 11, 17, 9] for recent spectacular progress in relating the critical $2d$ Ising model to the predictions of the bootstrap approach).

1.3. The DOZZ formula. As mentioned above, an explicit expression for the LCFT structure constants was proposed in [15, 59]. Subsequently it was observed by Teschner [55] that this formula may be derived by applying the bootstrap framework to special four point functions (see section 6). He argued that this leads to the following remarkable periodicity relations for the structure constants:

$$(1.8) \quad C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) = -\frac{1}{\pi\mu} \mathcal{A}(\frac{\gamma}{2}) C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)$$

$$(1.9) \quad C_\gamma(\alpha_1 + \frac{2}{\gamma}, \alpha_2, \alpha_3) = -\frac{1}{\pi\tilde{\mu}} \mathcal{A}(\frac{2}{\gamma}) C_\gamma(\alpha_1 - \frac{2}{\gamma}, \alpha_2, \alpha_3)$$

with $\tilde{\mu} = \frac{(\mu\pi l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{\pi l(\frac{\gamma^2}{2})}$ and

$$(1.10) \quad \mathcal{A}(\chi) = \frac{l(-\chi^2)l(\chi\alpha_1)l(\chi\alpha_1 - \chi^2)l(\frac{\chi}{2}(\bar{\alpha} - 2\alpha_1 - \chi))}{l(\frac{\chi}{2}(\bar{\alpha} - \chi - 2Q))l(\frac{\chi}{2}(\bar{\alpha} - 2\alpha_3 - \chi))l(\frac{\chi}{2}(\bar{\alpha} - 2\alpha_2 - \chi))}$$

where $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and

$$(1.11) \quad l(x) = \Gamma(x)/\Gamma(1-x).$$

The equations (1.8), (1.9) have a meromorphic solution which is the DOZZ formula. It is expressed in terms of a special function $\Upsilon_{\frac{\gamma}{2}}(z)$ defined for $0 < \Re(z) < Q$ by the formula⁷

$$(1.12) \quad \ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z \right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4}) \sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}.$$

The function $\Upsilon_{\frac{\gamma}{2}}$ can be analytically continued to \mathbb{C} because it satisfies remarkable functional relations: see formula (9.4) in the appendix. It has no poles in \mathbb{C} and the zeros of $\Upsilon_{\frac{\gamma}{2}}$ are simple (if $\gamma^2 \notin \mathbb{Q}$) and given by the discrete set $(-\frac{\gamma}{2}\mathbb{N} - \frac{2}{\gamma}\mathbb{N}) \cup (Q + \frac{\gamma}{2}\mathbb{N} + \frac{2}{\gamma}\mathbb{N})$. With these notations, the DOZZ formula (or proposal) $C_\gamma^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ is the following expression

$$(1.13) \quad C_\gamma^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3) = (\pi\mu l(\frac{\gamma^2}{4}))^{\frac{2Q - \sum_i \alpha_i}{\gamma}} \frac{\Upsilon'_{\frac{\gamma}{2}}(0) \Upsilon_{\frac{\gamma}{2}}(\alpha_1) \Upsilon_{\frac{\gamma}{2}}(\alpha_2) \Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha} - 2Q}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_1) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_2) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_3)}.$$

⁵In fact, LCFT was defined for $\gamma \in i\mathbb{R}$ in the physics literature quite recently by Ribault-Santachiara [48] but the theory is very different than LCFT for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$ hence we will not discuss this case here. Let us just mention that Iklhef, Jacobsen and Saleur [31] have shown how 3 point correlation functions of Conformal Loop Ensembles (CLE) relate to the three point structure constants of LCFT with $\gamma \in i\mathbb{R}$ discovered by Kostov-Petkova [34] and Zamolodchikov [58].

⁶In fact, one can make sense of LCFT in the path integral for $\gamma = 2$ but we will not discuss this case here.

⁷The function has a simple construction in terms of standard double gamma functions: see the reviews [39, 47, 56] for instance.

The main result of the present paper is to show the first important equality between LCFT in the path integral formulation and the conformal bootstrap approach, namely to prove that for $\gamma \in (0, 2)$ and appropriate $\alpha_1, \alpha_2, \alpha_3$ the structure constants $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ in (1.6) are equal to $C_\gamma^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ defined by (1.13).

Our proof is based on deriving the equations (1.8), (1.9) for the probabilistically defined C_γ . An essential role in this derivation is an identification in probabilistic terms of the *reflection coefficient* of LCFT. It has been known for a long time that in LCFT the following reflection relation should hold in some sense:

$$(1.14) \quad V_\alpha = R(\alpha)V_{2Q-\alpha}.$$

Indeed the DOZZ formula is compatible with the following form of (1.14):

$$(1.15) \quad C_\gamma^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3) = R^{\text{DOZZ}}(\alpha_1)C_\gamma^{\text{DOZZ}}(2Q - \alpha_1, \alpha_2, \alpha_3).$$

with

$$(1.16) \quad R^{\text{DOZZ}}(\alpha) = -(\pi \mu \ell (\frac{\gamma^2}{4}))^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(-\frac{\gamma(Q-\alpha)}{2})}{\Gamma(\frac{\gamma(Q-\alpha)}{2})} \frac{\Gamma(-\frac{2(Q-\alpha)}{\gamma})}{\Gamma(\frac{2(Q-\alpha)}{\gamma})}.$$

The mystery of this relation lies in the fact that the probabilistically defined $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ *vanish* if any of the $\alpha_i \geq Q$ whereas they are nonzero for $\alpha_i < Q$, see Section 2.2.

In our proof $R(\alpha)$ emerges from the analysis of the *tail behavior* of a Gaussian Multiplicative Chaos observable. We prove that it is also given by the following limit

$$(1.17) \quad 4R(\alpha) = \lim_{\epsilon \rightarrow 0} \epsilon C_\gamma(\epsilon, \alpha, \alpha)$$

i.e. $R(\alpha)$ has an interpretation in terms of a renormalized two-point function. We will show that for those values of α such that $R(\alpha)$ makes sense from the path integral perspective, i.e. $\alpha \in (\frac{\gamma}{2}, Q)$,

$$R(\alpha) = R^{\text{DOZZ}}(\alpha).$$

It turns out that some material related to the coefficient $R(\alpha)$ already appears in the beautiful work by Duplantier-Miller-Sheffield [18] where they introduce what they call quantum spheres (and other related objects). Quantum spheres are equivalence classes of random measures on the sphere with two marked points 0 and ∞ . Within this framework, the reflection coefficient $R(\alpha)$ can naturally be interpreted as the partition function of the theory⁸.

Finally, let us stress that the DOZZ formula 1.13 is invariant under the substitution of parameters

$$\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}, \quad \mu \leftrightarrow \tilde{\mu} = \frac{(\mu \pi \ell (\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{\pi \ell (\frac{4}{\gamma^2})}.$$

This duality symmetry is at the core of the DOZZ controversy. Indeed this symmetry is not manifest in the Liouville action functional (1.2) though duality was axiomatically assumed by Teschner [56] in his argument, especially to get (1.9). It was subsequently argued that this duality could come from the presence in the action (1.2) of an additional “dual” potential of the form $e^{\frac{2}{\gamma}\phi}$ with cosmological constant $\tilde{\mu}$ in front of it. As observed by Teschner [56], this dual cosmological constant may take negative (even infinite) values, which makes clearly no sense from the path integral perspective. That is why the derivation of the DOZZ formula from the LCFT path integral has remained shrouded in mystery for so long⁹.

⁸We will not elaborate more on this point as no prior knowledge of the work by Duplantier-Miller-Sheffield [18] is required to understand the sequel (see [2] for an account of the relation between [13] and [18]). More precisely, the required background to understand $R(\alpha)$ will be introduced in subsection 2.7 below.

⁹Indeed, there are numerous reviews and papers within the physics literature on the path integral approach of LCFT and its relation with the bootstrap approach but they offer different perspectives and conclusions (see [28, 40, 52] for instance)

1.4. Organization of the paper. The organization of the paper is the following: in the next section, we introduce the necessary background and the main result, namely Theorem 2.11. The next sections are devoted to the proof of the main result. Section 3 is devoted to the study of tail estimates of GMC and their connection with the reflection coefficient. In section 4, we show that the correlation functions of vertex operators are analytic functions of their arguments $(\alpha_k)_k$. We prove technical lemmas on the reflection coefficient in section 5. In Section 6, we prove various OPEs statements with degenerate vertex operators, which are used to derive non trivial relations between three point structure constants and the reflection coefficient. These relations will serve prominently in the proof of the DOZZ formula in sections 7 and 8.

Acknowledgements. The authors wish to thank Francois David, Sylvain Ribault and Raoul Santachiara for fruitful discussions on Liouville Field theory and the conformal bootstrap approach.

2. PROBABILISTIC FORMULATION OF LCFT AND MAIN RESULTS

In this section, we recall the precise definition of the Liouville correlation functions in the path integral formulation as given in [13], introduce some related probabilistic objects and state the main results.

Conventions and notations. In what follows, z, x, y and z_1, \dots, z_N all denote complex variables. We use the standard notation for complex derivatives $\partial_x = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ and $\partial_{\bar{x}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ for $x = x_1 + ix_2$. The Lebesgue measure on \mathbb{C} (seen as \mathbb{R}^2) is denoted by d^2x . We will also denote $|\cdot|$ the norm in \mathbb{C} of the standard Euclidean (flat) metric and for all $r > 0$ we will denote by $B(x, r)$ the Euclidean ball of center x and radius r .

2.1. Gaussian Free Field and Gaussian Multiplicative Chaos. The probabilistic definition of the functional integral (1.1) goes by expressing it as a functional of the Gaussian Free Field (GFF). Since we want LCFT to have Möbius symmetry the proper setup is the Riemann Sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ equipped with a conformal metric $g(z)|dz|^2$. The correlation functions of LCFT will then depend on the metric but they have simple transformation properties under the change of g , the so-called Weyl anomaly formula. We refer the reader to [13] for this point and proceed here by just stating a formulation that will be useful for the purposes of this paper.

We define the GFF $X(z)$ as the centered Gaussian random field with covariance (see [16, 53] for background on the GFF)

$$(2.1) \quad \mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y|} + \ln |x|_+ + \ln |y|_+ := G(x, y)$$

where we use the notation $|z|_+ = |z|$ if $|z| \geq 1$ and $|z|_+ = 1$ if $|z| \leq 1$.

Remark 2.1. *In the terminology of [13], consider the metric $g(z) = |z|_+^{-4}$ with scalar curvature $R_g(z) := -4g^{-1}\partial_z\partial_{\bar{z}}\ln g(z) = 4\nu$ with ν the uniform probability measure on the equator $|z| = 1$. Then X is the GFF with zero average on the equator: $\int X(z)R_g(d^2z) = 0$.*

For LCFT we need to consider the exponential of X . Since X is distribution valued a renormalization procedure is needed. Define the circle average of X by

$$(2.2) \quad X_r(z) := \frac{1}{2\pi i} \oint_{|w|=e^{-r}} X(z+w) \frac{dw}{w}$$

and consider the measure

$$(2.3) \quad M_{\gamma,r}(d^2x) := e^{\gamma X_r(x) - \frac{\gamma^2}{2}\mathbb{E}[X_r(x)^2]} |x|_+^{-4} d^2x.$$

Then, for $\gamma \in [0, 2)$, we have the convergence in probability

$$(2.4) \quad M_\gamma = \lim_{r \rightarrow \infty} M_{\gamma,r}$$

and convergence is in the sense of weak convergence of measures. This limiting measure is non trivial and is a (up to a multiplicative constant) Gaussian multiplicative chaos (GMC) of the field X with respect to the measure $|x|_+^{-4} d^2x$ (see Berestycki's work [4] for an elegant and elementary approach to GMC and references).

Remark 2.2. For later purpose we state a useful property of the circle averages. First, $X_0(0) = 0$, the processes $r \in \mathbb{R}_+ \rightarrow X_r(0)$ and $r \in \mathbb{R}_+ \rightarrow X_{-r}(0)$ are two independent Brownian motions starting from 0. For z center of a unit ball contained in $B(0, 1)^c$ the process $r \in \mathbb{R}_+ \rightarrow X_r(z) - X_0(z)$ is also a Brownian motion starting at 0 and for disjoint balls $B(z_i, 1) \subset B(0, 1)^c$ the processes $r \mapsto X_r(z_i) - X_0(z_i)$ are mutually independent and independent of the sigma algebra $\sigma\{X(z); z \in [\cup_{k=1}^N B(z_k, 1)]^c\}$.

2.2. Liouville correlation functions. We may now give the probabilistic definition of the functional integral (1.1):

$$(2.5) \quad \langle F \rangle := \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[F(X - 2Q \ln|z|_+ + c) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{C})}] dc$$

where \mathbb{E} is expectation over the GFF. We refer the reader to [13] (or to [36] for a brief summary) for the explanation of the connection between (1.1) and (2.5). Briefly, the variable c is essential and stems from the fact that in (1.1) we need to integrate over all ϕ and not only the GFF X which is defined up to constants. The origin of the factor e^{-2Qc} is topological and depends on the fact that we work on the sphere $\hat{\mathbb{C}}$. The random variable $M_\gamma(\mathbb{C})$ is almost surely finite because $\mathbb{E}M_\gamma(\mathbb{C}) = \int_{\mathbb{C}} |z|_+^{-4} d^2z < \infty$. This implies that $\langle \cdot \rangle$ is not normalizable: $\langle 1 \rangle = \infty$.

The class of F for which (2.5) is defined includes suitable vertex operator correlation functions once these are properly renormalized. We set

$$(2.6) \quad V_{\alpha, \epsilon}(z) = e^{\alpha c} e^{\alpha X_\epsilon(z) - \frac{\alpha^2}{2} \mathbb{E}[X_\epsilon(z)^2]} |x|_+^{-4\Delta_\alpha}$$

where we recall $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$. Let $z_i \in \mathbb{C}$, $i = 1, \dots, N$ with $z_i \neq z_j$ for all $i \neq j$. It was shown in [13] that the limit¹⁰

$$(2.7) \quad \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle := 2 \lim_{\epsilon \rightarrow 0} \langle \prod_{k=1}^N V_{\alpha_k, \epsilon}(z_k) \rangle$$

exists, is finite and nonzero if and only if the following *Seiberg bounds* originally introduced in [52] hold:

$$(2.8) \quad \sum_{k=1}^N \alpha_k > 2Q, \quad \alpha_i < Q, \quad \forall i.$$

The first condition guarantees that the limit is finite and the second that it is nonvanishing. Indeed, if there exists i such that $\alpha_i \geq Q$ then the limit is zero. Note that these bounds imply that for a nontrivial correlation we need at least *three* vertex operators; therefore, we have $N \geq 3$ in the sequel. The correlation function (2.7) satisfies the conformal invariance property (1.5).

The correlation function can be further simplified by performing the c -integral (see [13]):

$$(2.9) \quad \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 2\mu^{-s} \gamma^{-1} \Gamma(s) \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\prod_{k=1}^N e^{\alpha_k X_\epsilon(z_k) - \frac{\alpha_k^2}{2} \mathbb{E}[X_\epsilon(z_k)^2]} |z_k|_+^{-4\Delta_{\alpha_k}} M_\gamma(\mathbb{C})^{-s} \right]$$

where

$$(2.10) \quad s = \frac{\sum_{k=1}^N \alpha_k - 2Q}{\gamma}.$$

Using the Cameron-Martin theorem (see [13]) we may trade the vertex operators to a shift of X to obtain an expression in terms of the multiplicative chaos:

$$(2.11) \quad \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 2\mu^{-s} \gamma^{-1} \Gamma(s) \prod_{i < j} \frac{1}{|z_i - z_j|^{\alpha_i \alpha_j}} \mathbb{E} \left[\left(\int_{\mathbb{C}} F(x, \mathbf{z}) M_\gamma(d^2x) \right)^{-s} \right]$$

¹⁰The global constant 2 is included to match with the physics literature normalization which is based on the DOZZ formula (1.13)

where

$$(2.12) \quad F(x, \mathbf{z}) = \prod_{k=1}^N \left(\frac{|x|_+}{|x - z_k|} \right)^{\gamma \alpha_k}$$

Thus, up to explicit factors the Liouville correlations are reduced to the study of the random variable $\int_{\mathbb{C}} F(x, \mathbf{z}) M_\gamma(d^2x)$. In particular, the Seiberg bounds $\alpha_k < Q$ for all k are the condition of integrability of F against the chaos measure M_γ (see [13]).

Finally, we remark that expression (2.11) makes sense beyond the Seiberg bounds i.e. for some $s < 0$. Indeed, it was shown in [13] that

$$(2.13) \quad 0 < \mathbb{E} \left[\left(\int_{\mathbb{C}} F(x, \mathbf{z}) M_\gamma(d^2x) \right)^{-s} \right] < \infty$$

provided

$$(2.14) \quad -s < \frac{4}{\gamma^2} \wedge \min_{1 \leq k \leq N} \frac{2}{\gamma} (Q - \alpha_k), \quad \alpha_k < Q, \quad \forall k$$

with s given by (2.10). Under condition (2.14), it is then natural to define the so-called *unit volume correlations* by

$$(2.15) \quad \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_{uv} = \frac{\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle}{\Gamma(s)}.$$

i.e. we divide by the Γ function which has poles at $\sum_{k=1}^N \alpha_k - 2Q \in -\gamma\mathbb{N}$. An important ingredient in our proof of the DOZZ formula is Theorem 4.1 which says that these correlation functions have an analytic continuation in the α_i 's to a complex neighbourhood of the region allowed by the bounds (2.14).

Remark 2.3. *The DOZZ formula for the structure constants is analytic not only in α_i but also in γ . A direct proof of analyticity of the probabilistic correlation functions in γ seems difficult. However, it is an easy exercise in Multiplicative Chaos theory to prove their continuity in γ , a fact we will need in our argument. Actually, it is not hard to prove that they are C^∞ in γ but we will omit this as it is not needed in our argument.*

2.3. Structure constants and four point functions. The structure constants C_γ in (1.6) can be recovered as the following limit

$$(2.16) \quad C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \lim_{z_3 \rightarrow \infty} |z_3|^{4\Delta_3} \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z_3) \rangle$$

Combining (2.11) with (2.16) we get

$$(2.17) \quad C_\gamma(\alpha_1, \alpha_2, \alpha_3) = 2\mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}(\rho(\alpha_1, \alpha_2, \alpha_3)^{-s})$$

where

$$\rho(\alpha_1, \alpha_2, \alpha_3) = \int_{\mathbb{C}} \frac{|x|_+^{\gamma(\alpha_1 + \alpha_2 + \alpha_3)}}{|x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}} M_\gamma(d^2x).$$

We will also have to work with the unit volume three point structure constants defined by the formula

$$(2.18) \quad \bar{C}_\gamma(\alpha_1, \alpha_2, \alpha_3) = \frac{C_\gamma(\alpha_1, \alpha_2, \alpha_3)}{\Gamma(s)}.$$

The four point function is fixed by the Möbius invariance (1.5) up to a single function depending on the cross ratio of the points. For later purpose we label the points from 0 to 3 and consider the weights $\alpha_1, \alpha_2, \alpha_3$ fixed:

$$(2.19) \quad \left\langle \prod_{k=0}^3 V_{\alpha_k}(z_k) \right\rangle = |z_3 - z_0|^{-4\Delta - \frac{\gamma}{2}} |z_2 - z_1|^{2(\Delta_3 - \Delta_2 - \Delta_1 - \Delta_0)} |z_3 - z_1|^{2(\Delta_2 + \Delta_0 - \Delta_3 - \Delta_1)}$$

$$(2.20) \quad \times |z_3 - z_2|^{2(\Delta_1 + \Delta_0 - \Delta_3 - \Delta_2)} G_{\alpha_0} \left(\frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_2 - z_1)} \right)$$

We can recover G_{α_0} as the following limit

$$(2.21) \quad G_{\alpha_0}(z) = \lim_{z_3 \rightarrow \infty} |z_3|^{4\Delta_3} \langle V_{\alpha_0}(z) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z_3) \rangle.$$

Combining with (2.11) we get

$$(2.22) \quad G_{\alpha_0}(z) = |z|^{\frac{\gamma\alpha_1}{2}} |z-1|^{\frac{\gamma\alpha_2}{2}} \mathcal{T}_{\alpha_0}(z)$$

where, setting $s = \frac{\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 2Q}{\gamma}$, $\mathcal{T}_{\alpha_0}(z)$ is given by

$$(2.23) \quad \mathcal{T}_{\alpha_0}(z) = 2\mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}[R_{\alpha_0}(z)^{-s}]$$

and

$$(2.24) \quad R_{\alpha_0}(z) = \int_{\mathbb{C}} \frac{|x|_{+}^{\sum_{k=0}^3 \alpha_k}}{|x-z|^{\gamma\alpha_0} |x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}} M_{\gamma}(d^2x).$$

In this paper we will study the structure constants (2.17) by means of four point functions (2.20) with special values of α_0 .

2.4. BPZ equations. There are two special values of α_0 for which the reduced four point function $\mathcal{T}_{\alpha_0}(z)$ satisfies a second order differential equation. That such equations are expected in Conformal Field Theory goes back to BPZ [3]. In the case of LCFE it was proved in [36] that, under suitable assumptions on $\alpha_1, \alpha_2, \alpha_3$, if $\alpha_0 \in \{-\frac{\gamma}{2}, -\frac{2}{\gamma}\}$ then \mathcal{T}_{α_0} is a solution of a PDE version of the Gauss hypergeometric equation

$$(2.25) \quad \partial_z^2 \mathcal{T}_{\alpha_0}(z) + \frac{(c-z(a+b+1))}{z(1-z)} \partial_z \mathcal{T}_{\alpha_0}(z) - \frac{ab}{z(1-z)} \mathcal{T}_{\alpha_0}(z) = 0$$

where a, b, c are given by

$$(2.26) \quad a = \frac{\alpha_0}{2}(Q - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3) - \frac{1}{2}, \quad b = \frac{\alpha_0}{2}(Q - \alpha_1 - \alpha_2 + \alpha_3) + \frac{1}{2}, \quad c = 1 + \alpha_0(Q - \alpha_1).$$

This equation has two holomorphic solutions defined on $\mathbb{C} \setminus [1, \infty[$:

$$(2.27) \quad F_{-}(z) = {}_2F_1(a, b, c, z), \quad F_{+}(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c, 2-c, z)$$

where ${}_2F_1(a, b, c, z)$ is given by the standard hypergeometric series. Using the facts that $\mathcal{T}_{\alpha_0}(z)$ is real, single valued and C^2 in $\mathbb{C} \setminus \{0, 1\}$ we proved in [36] (Lemma 4.4) that it is determined up to a multiplicative constant as

$$(2.28) \quad \mathcal{T}_{\alpha_0}(z) = \lambda(|F_{-}(z)|^2 + A_{\gamma}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)|F_{+}(z)|^2)$$

where the coefficient $A_{\gamma}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is given by

$$(2.29) \quad A_{\gamma}(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = -\frac{\Gamma(c)^2 \Gamma(1-a) \Gamma(1-b) \Gamma(a-c+1) \Gamma(b-c+1)}{\Gamma(2-c)^2 \Gamma(c-a) \Gamma(c-b) \Gamma(a) \Gamma(b)}$$

provided $c \in \mathbb{R} \setminus \mathbb{Z}$ and $c-a-b \in \mathbb{R} \setminus \mathbb{Z}$. Furthermore, the constant λ is found by using the expressions (2.17) and (2.23) (note that s has a different meaning in these two expressions):

$$(2.30) \quad \lambda = \mathcal{T}_{\alpha_0}(0) = C_{\gamma}(\alpha_1 + \alpha_0, \alpha_2, \alpha_3).$$

Hence for $\alpha_0 \in \{-\frac{\gamma}{2}, -\frac{2}{\gamma}\}$ \mathcal{T}_{α_0} is completely determined in terms of $C_{\gamma}(\alpha_1 + \alpha_0, \alpha_2, \alpha_3)$.

In the case $\alpha_0 = -\frac{\gamma}{2}$ we were able to determine in [36] the leading asymptotics of the expression (2.23) as $z \rightarrow 0$ provided $\frac{1}{\gamma} + \gamma < \alpha_1 + \frac{\gamma}{2} < Q$:

$$(2.31) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_{\gamma}(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) + B(\alpha_1) C_{\gamma}(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) |z|^{2(1-c)} + o(|z|^{2(1-c)})$$

where

$$(2.32) \quad B(\alpha) = -\mu \frac{\pi}{l(-\frac{\gamma^2}{4}) l(\frac{\gamma\alpha}{2}) l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha}{2})}.$$

Hence, in view of (2.28) and (2.30), relations (2.31), (2.32) lead to

$$(2.33) \quad B(\alpha_1) C_{\gamma}(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) = A_{\gamma}(-\frac{\gamma}{2}, \alpha_1, \alpha_2, \alpha_3) C_{\gamma}(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)$$

which yields the relation (1.8) (after some algebra!) in the case $\frac{1}{\gamma} + \gamma < \alpha_1 + \frac{\gamma}{2} < Q$. Hence

$$(2.34) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)|F_-(z)|^2 + B(\alpha_1)C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)|F_+(z)|^2.$$

The restriction $\frac{1}{\gamma} + \gamma < \alpha_1 + \frac{\gamma}{2}$ for α_1 was technical in [36] and will be removed in section 6. The restriction $\alpha_1 + \frac{\gamma}{2} < Q$ seems necessary due to the Seiberg bounds as the probabilistic $C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)$ vanishes then. Understanding what happens when $\alpha_1 + \frac{\gamma}{2} > Q$ is the key to our proof of the DOZZ formula. Before turning to this we draw a useful corollary from the results of this section.

2.5. Crossing relation. Let us suppose $\alpha_1 < Q$ and $\alpha_2 + \frac{\gamma}{2} < Q$. We have from the previous subsection

$$(2.35) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)(|F_-(z)|^2 + A_\gamma(-\frac{\gamma}{2}, \alpha_1, \alpha_2, \alpha_3)|F_+(z)|^2).$$

The hypergeometric equation (2.25) has another basis of holomorphic solutions defined on $\mathbb{C} \setminus (-\infty, 0)$:

$$(2.36) \quad G_-(z) = {}_2F_1(a, b, c', 1-z), \quad G_+(z) = (1-z)^{1-c'} {}_2F_1(1+a-c', 1+b-c', 2-c', 1-z)$$

where $c' = 1+a+b-c = 1 - \frac{\gamma}{2}(Q - \alpha_2)$ (i.e. these are obtained by interchanging α_1 and α_2 and replacing z by $1-z$). The two basis are linearly related

$$F_-(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}G_-(z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}G_+(z)$$

$$F_+(z) = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}G_-(z) + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}(1-z)^{c-a-b}G_+(z).$$

and we get

$$\mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)(D|G_-(z)|^2 + E|G_+(z)|^2)$$

with explicit coefficients D, E (see [36], Appendix). On the other hand by studying the asymptotics as $z \rightarrow 1$ we get

$$(2.37) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1, \alpha_2 - \frac{\gamma}{2}, \alpha_3) + B(\alpha_2)C_\gamma(\alpha_1, \alpha_2 + \frac{\gamma}{2}, \alpha_3)|1-z|^{2(1-c')} + o(|z|^{2(1-c')}).$$

In view of expression (2.34), exploiting the decomposition of $\mathcal{T}_{-\frac{\gamma}{2}}$ in the basis $|G_-(z)|^2, |G_+(z)|^2$ leads to the following crossing symmetry relation:

Proposition 2.4. *Let $\alpha_2 + \frac{\gamma}{2} < Q$ and $\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2} > 2Q$. Then*

$$(2.38) \quad C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) = T(\alpha_1, \alpha_2, \alpha_3)C_\gamma(\alpha_1, \alpha_2 + \frac{\gamma}{2}, \alpha_3)$$

where T is the given by the following formula

$$(2.39) \quad T(\alpha_1, \alpha_2, \alpha_3) = -\mu\pi \frac{l(a)l(b)}{l(c)l(a+b-c)} \frac{1}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_2}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_2}{2})}$$

Remark 2.5. *The relations (1.8) and (2.38) were derived in the physics literature [55] by assuming (i) BPZ equations, (ii) the diagonal form of the solution (2.28) (iii) crossing symmetry (an essential input in the bootstrap approach). We want to stress that our proof makes no such assumptions, in fact (i)-(iii) are theorems.*

2.6. Reflection relation. One of the key inputs in our proof of the DOZZ formula is the extension of (2.34) to the case $\alpha_1 + \frac{\gamma}{2} > Q$. In order to appreciate what is involved let us first explain what we should expect from the DOZZ solution. One can check from the DOZZ formula the following identity:

$$(2.40) \quad B(\alpha) = \frac{R^{\text{DOZZ}}(\alpha)}{R^{\text{DOZZ}}(\alpha + \frac{\gamma}{2})}.$$

Combining this with (1.15) we get that (2.34) for $\alpha_1 + \frac{\gamma}{2} > Q$ is compatible with

$$(2.41) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)|F_-(z)|^2 + R(\alpha_1)C_\gamma(2Q - \alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)|F_+(z)|^2$$

where

$$(2.42) \quad R(\alpha) = R^{\text{DOZZ}}(\alpha).$$

We will prove (2.41) when $\alpha_1 + \frac{\gamma}{2} > Q$ (and under suitable assumptions on $\alpha_1, \alpha_2, \alpha_3$) in Theorem 6.2 with a probabilistic expression for $R(\alpha_1)$. Once this is done we need to prove (2.42) and use these facts to derive the DOZZ formula.

2.7. Reflection coefficient. The identity (2.41) follows from a careful analysis of the small z behaviour of $\mathbb{E}[R_{-\frac{\gamma}{2}}(z)^{-s}]$. This in turn will be determined by the behavior of the integral (2.24) around the singularity at the origin. To motivate the definitions let us consider the random variable

$$I(\alpha) := \int_{B(0,1)} |x|^{-\gamma\alpha} M_\gamma(d^2x).$$

If $\alpha > \frac{2}{\gamma}$ we have $\mathbb{E}I(\alpha) = \infty$ and $I(\alpha)$ has a heavy tail. The reflection coefficient enters in the tail behaviour of $I(\alpha)$. To study this we recall basic material introduced in [18] and in particular we consider the polar decomposition of the chaos measure. Let $X_s := X_s(0)$ be the circle average (2.2). We have

$$X(e^{-s}e^{i\theta}) = X_s + Y(s, \theta)$$

where X_s is Brownian Motion starting from origin at $s = 0$ and $Y(z)$ is an independent field with covariance

$$(2.43) \quad \mathbb{E}[Y(s, \theta)Y(t, \theta')] = \ln \frac{e^{-s} \vee e^{-t}}{|e^{-s}e^{i\theta} - e^{-t}e^{i\theta'}|}.$$

We introduce the chaos measure with respect to Y

$$(2.44) \quad N_\gamma(ds d\theta) = e^{\gamma Y(s, \theta) - \frac{\gamma^2 E|Y(s, \theta)|^2}{2}} ds d\theta.$$

Then we get

$$(2.45) \quad I(\alpha) \stackrel{\text{law}}{=} \int_0^\infty \int_0^{2\pi} e^{\gamma(B_s - (Q-\alpha)s)} Z_s ds.$$

with

$$(2.46) \quad Z_s = \int_0^{2\pi} e^{\gamma Y(s, \theta) - \frac{\gamma^2 E|Y(s, \theta)|^2}{2}} d\theta.$$

This is a slight abuse of notation since the process Z_s is not a function (for $\gamma \geq \sqrt{2}$) but rather a generalized function. With this convention, notice that $Z_s ds$ is stationary i.e. for all t the quality $Z_{t+s} = Z_s$ holds in distribution.

It satisfies for all bounded intervals I

$$(2.47) \quad \mathbb{E} \left(\int_I Z_s ds \right)^p < \infty, \quad -\infty < p < \frac{4}{\gamma^2}.$$

The following decomposition lemma due to Williams (see [57]) will be useful in the study of $I(\alpha)$:

Lemma 2.6. *Let $(B_s - \nu s)_{s \geq 0}$ be a Brownian motion with negative drift, i.e. $\nu > 0$ and let $M = \sup_s (B_s - \nu s)$. Then conditionally on M the law of the path $(B_s - \nu s)_{s \geq 0}$ is given by the joining of two independent paths:*

- *A Brownian motion $((B_s^1 + \nu s))_{s \leq \tau_M}$ with positive drift $\nu > 0$ run until its hitting time τ_M of M .*
- *$(M + B_t^2 - \nu t)_{t \geq 0}$ where $B_t^2 - \nu t$ is a Brownian motion with negative drift conditioned to stay negative.*

Moreover, one has the following time reversal property for all $C > 0$ (where τ_C denotes the hitting time of C)

$$(B_{\tau_C - s}^1 + \nu(\tau_C - s) - C)_{s \leq \tau_C} \stackrel{\text{law}}{=} (\tilde{B}_s - \nu s)_{s \leq L - C}$$

where $(\tilde{B}_s - \nu s)_{s \geq 0}$ is a Brownian motion with drift $-\nu$ conditioned to stay negative and $L - C$ is the last time $(\tilde{B}_s - \nu s)$ hits $-C$.

Remark 2.7. As a consequence of the above lemma, one can also deduce that the process $(\tilde{B}_{L_{-C}+s} - \nu(L_{-C} + s + C))_{s \geq 0}$ is equal in distribution to $(\tilde{B}_s - \nu s)_{s \geq 0}$.

This lemma motivates defining the process \mathcal{B}_s^α

$$\mathcal{B}_s^\alpha = \begin{cases} B_s^\alpha & \text{if } s < 0 \\ \bar{B}_s^\alpha & \text{if } s > 0 \end{cases}$$

where $B_s^\alpha, \bar{B}_s^\alpha$ are two independent Brownian motions with negative drift $\alpha - Q$ and conditioned to stay negative. We may apply Lemma 2.6 to (2.45). Let $M = \sup_{s \geq 0} (B_s - (Q - \alpha)s)$ and L_{-M} be the last time B_s^α hits $-M$. Then

$$(2.48) \quad \int_0^\infty e^{\gamma(B_s - (Q - \alpha)s)} Z_s ds \stackrel{\text{law}}{=} e^{\gamma M} \int_{-L_{-M}}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_{s+L_{-M}} ds \stackrel{\text{law}}{=} e^{\gamma M} \int_{-L_{-M}}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds$$

where we used stationarity of the process Z_s (and independence of Z_s and B_s). We will prove in section 3 that the tail behaviour of $I(\alpha)$ coincides with that of

$$J(\alpha) = e^{\gamma M} \int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds.$$

The distribution of M is well known (see section 3.5.C in the textbook [32] for instance):

$$(2.49) \quad \mathbb{P}(e^{\gamma M} > x) = \frac{1}{x^{\frac{2(Q-\alpha)}{\gamma}}}$$

which implies

$$(2.50) \quad \mathbb{P}(J(\alpha) > x) \underset{x \rightarrow \infty}{\sim} \mathbb{E} \left(\int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^{\frac{2(Q-\alpha)}{\gamma}} x^{-\frac{2(Q-\alpha)}{\gamma}}.$$

This is the tail behaviour that we prove for $I(\alpha)$ and its generalizations in section 3. Define the *unit volume reflection coefficient* $\bar{R}(\alpha)$ for $\alpha \in (\frac{\gamma}{2}, Q)$ by the following formula

$$(2.51) \quad \bar{R}(\alpha) = \mathbb{E} \left[\left(\int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^{\frac{2}{\gamma}(Q-\alpha)} \right].$$

$\bar{R}(\alpha)$ is indeed well defined as can be seen from the following lemma

Lemma 2.8. *Let $\alpha \in (\frac{\gamma}{2}, Q)$. Then*

$$(2.52) \quad \mathbb{E} \left[\left(\int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right] < \infty$$

for all $-\infty < p < \frac{4}{\gamma^2}$.

The full reflection coefficient is now defined for all $\alpha \in (\frac{\gamma}{2}, Q) \setminus \cup_{n \geq 1} \{\frac{2}{\gamma} - \frac{n-1}{2}\gamma\}$ by

$$(2.53) \quad R(\alpha) = \mu^{\frac{2(Q-\alpha)}{\gamma}} \Gamma\left(-\frac{2(Q-\alpha)}{\gamma}\right) \frac{2(Q-\alpha)}{\gamma} \bar{R}(\alpha)$$

The function $R(\alpha)$ has a divergence at the points $\frac{2}{\gamma} - \frac{n-1}{2}\gamma$ with $n \geq 1$ because of the Γ function entering the definition. Its connection to the structure constants is the following:

Lemma 2.9. *For all $\alpha \in (\frac{\gamma}{2}, Q) \setminus \cup_{n \geq 1} \{\frac{2}{\gamma} - \frac{n-1}{2}\gamma\}$, the following limit holds*

$$\lim_{\epsilon \rightarrow 0} \epsilon C_\gamma(\epsilon, \alpha, \alpha) = 4R(\alpha)$$

Hence the reflection coefficient should be seen as a 2 point correlation function.

2.8. Main results. The two main results of this paper are exact formulas for the two point correlation function (reflection coefficient) and the three point structure constants of the theory. These formulas were suggested in the context of the conformal bootstrap.

We will first prove the following formula:

Theorem 2.10. *For all $\alpha \in (\frac{\gamma}{2}, Q)$ one has*

$$(2.54) \quad R(\alpha) = R^{\text{DOZZ}}(\alpha)$$

Finally, the main result of this paper is the following identity:

Theorem 2.11. *Let $\alpha_1, \alpha_2, \alpha_3$ satisfy the bounds (2.14) with $N = 3$. The following equality holds*

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = C_\gamma^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3).$$

From the purely probabilistic point of view, Theorem 2.11 can be interpreted as a far reaching integrability result on GMC on the Riemann sphere; indeed recall that $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ has an expression in terms of a fractional moment of some form of GMC: see formula (2.17). There are numerous integrability results on GMC in the physics literature (see the introduction); to the best of our knowledge, theorem 2.11 is the first rigorous non trivial integrability result on GMC and we believe the techniques of this paper will enable to prove many other integrability results for GMC.

3. TAIL ESTIMATES FOR MULTIPLICATIVE CHAOS

In this section, we prove the tail estimates needed in this paper and that involves the reflection coefficient.

3.1. Tail estimate around one insertion. Let $|z| > 2$ and consider the random variable

$$W := \int_{B(z,1)} \frac{F(x)}{|x-z|^{\gamma\alpha}} M_\gamma(d^2x)$$

for F bounded and C^1 in a neighborhood of z . We assume $\frac{\gamma}{2} < \alpha < Q$ and define auxiliary quantities $\beta = (\frac{2}{\gamma}(Q-\alpha) + \frac{2}{\gamma^2}) \wedge \frac{4}{\gamma^2}$ and $\bar{\eta}$ by $(1-\bar{\eta})\beta = \frac{2}{\gamma}(Q-\alpha) + \bar{\eta}$. Hence $\bar{\eta}$ is strictly positive. With these definitions we have

Lemma 3.1. *For all $\eta < \bar{\eta}$ we have*

$$|\mathbb{P}(W > x) - |z|^{4\alpha(Q-\alpha)} F(z)^{\frac{2}{\gamma}(Q-\alpha)} \frac{\bar{R}(\alpha)}{x^{\frac{2}{\gamma}(Q-\alpha)}}| \leq \frac{C}{x^{\frac{2}{\gamma}(Q-\alpha)+\eta}}$$

Proof. We will write the integral in polar coordinates of $B(z,1)$. Define

$$N = \frac{1}{2\pi} \int_0^{2\pi} X(z + e^{i\theta}) d\theta.$$

Then

$$B_s := \frac{1}{2\pi} \int_0^{2\pi} (X(z + e^{-s} e^{i\theta}) - X(z + e^{i\theta})) d\theta$$

is a Brownian motion with $B(0) = 0$ and we may decompose the field X as

$$X(z+x) = N + B_{-\ln|x|} + Y_z(x)$$

where Y_z is a lateral noise centered around z given by

$$Y_z(x) = X(z+x) - \frac{1}{2\pi} \int_0^{2\pi} X(z + |x|e^{i\theta}) d\theta.$$

Notice that Y_z has same distribution as the lateral noise Y (centered around 0), that Y_z and B are independent and N is independent of B . We have

$$(3.1) \quad |\mathbb{E}[Y_z(x)N]| = |\ln|z+x| - \ln|z|| \leq C|x|$$

and the variance of N is

$$\mathbb{E}[N^2] = 2 \ln|z|.$$

Hence, we get the following decomposition into independent components

$$(3.2) \quad X(z+x) = N + B_{-\ln|x|} + (Y_z(x) - \frac{\mathbb{E}[Y_z(x)N]}{\mathbb{E}[N^2]}N) + \frac{\mathbb{E}[Y_z(x)N]}{\mathbb{E}[N^2]}N.$$

We introduce a variable \bar{N} distributed as N but independent of N, B, Y_z . We can rewrite (3.2) as the following equality in distribution:

$$(3.3) \quad X(z+x) = \bar{N} + B_{-\ln|x|} + (Y_z(x) - \frac{\mathbb{E}[Y_z(x)N]}{\mathbb{E}[N^2]}N) + \frac{\mathbb{E}[Y_z(x)N]}{\mathbb{E}[N^2]}\bar{N}$$

Consider the random function $u(x)$ given by

$$u(x) = e^{\frac{\mathbb{E}[Y_z(x)N]}{\mathbb{E}[N^2]}\bar{N} - \frac{\mathbb{E}[Y_z(x)N]}{\mathbb{E}[N^2]}N - \gamma^2 \mathbb{E}[Y_z(x)N]} \frac{F(z+x)}{|z+x|^4}.$$

By (3.1) and since F is C^1 around z we get

$$|u(x) - \frac{F(z)}{|z|^4}| \leq (C + e^{C|N|+C|\bar{N}|})|x|.$$

We may thus write $W = W_1 + W_2$

$$(3.4) \quad W_1 \stackrel{\text{law}}{=} e^{\gamma\bar{N} - \frac{\gamma^2}{2}\mathbb{E}[\bar{N}^2]} \frac{F(z)}{|z|^4} \int_0^\infty e^{\gamma(B_s - (Q-\alpha)s)} Z_s ds$$

with Z_s and \bar{N} independent and

$$|W_2| \leq C e^{C(\bar{N}+N)} \int_{B(0,1)} \frac{e^{\gamma B_{-\ln|x|} + \gamma Y_z(x) - \frac{\gamma^2 \mathbb{E}[(Y_z(x) + B_{-\ln|x|})^2]}}{|x|^{\gamma\alpha-1}} d^2x.$$

One can notice that

$$\int_{B(0,1)} \frac{e^{\gamma B_{-\ln|x|} + \gamma Y_z(x) - \frac{\gamma^2 \mathbb{E}[(Y_z(x) + B_{-\ln|x|})^2]}}{|x|^{\gamma\alpha-1}} d^2x \stackrel{\text{law}}{=} \int_0^\infty e^{\gamma(B_s - (Q-\alpha + \frac{1}{\gamma})s)} Z_s ds.$$

Recall the Williams decomposition Lemma 2.6. Let $m = \sup_{s \geq 0} (B_s - (Q-\alpha + \frac{1}{\gamma})s)$ and let L_{-m} be the largest s s.t. $\mathcal{B}_{-s}^\alpha = -m$. Then

$$(3.5) \quad \begin{aligned} \int_0^\infty e^{\gamma(B_s - (Q-\alpha + \frac{1}{\gamma})s)} Z_s ds &\stackrel{\text{law}}{=} e^{\gamma m} \int_{-L_{-m}}^\infty e^{\gamma \mathcal{B}_s^{\alpha + \frac{1}{\gamma}}} Z_{s+L_{-m}} ds \\ &\stackrel{\text{law}}{=} e^{\gamma m} \int_{-L_{-m}}^\infty e^{\gamma \mathcal{B}_s^{\alpha + \frac{1}{\gamma}}} Z_s(ds) \leq e^{\gamma m} \int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^{\alpha + \frac{1}{\gamma}}} Z_s ds \end{aligned}$$

where we used stationarity of the process Z_s .

For all $p < (\frac{2}{\gamma}(Q-\alpha) + \frac{2}{\gamma^2}) \wedge \frac{4}{\gamma^2} = \beta$, we have

$$(3.6) \quad \mathbb{P}(|W_2| \geq x) \leq Cx^{-p}$$

Indeed, for all $p_1, q_1 > 1$ with $\frac{1}{p_1} + \frac{1}{q_1} = 1$ we have by using Hölder and (3.5) that

$$(3.7) \quad \mathbb{P}(|W_2| \geq x) \leq \frac{1}{x^p} \mathbb{E}[|W_2|^p] \leq \frac{C}{x^p} \mathbb{E}[e^{Cp p_1(\bar{N}+N)}]^{1/p_1} \mathbb{E}[(e^{\gamma m} \int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^{\alpha + \frac{1}{\gamma}}} Z_s ds)^{p q_1}]^{1/q_1} \leq \frac{C}{x^p}$$

provided q_1 is sufficiently close to 1 and where we used Lemma 2.8 which requires $p < \frac{4}{\gamma^2}$.

We first prove an upper bound for $\mathbb{P}(W > x)$. From (3.7) we get for $\eta \in (0, 1)$:

$$\mathbb{P}(W > x) = \mathbb{P}(W_1 + W_2 > x) \leq \mathbb{P}(W_1 > x - x^{1-\eta}) + Cx^{-p(1-\eta)}$$

Proceeding as in (3.5) we get

$$\mathbb{P}(W_1 > x - x^{1-\eta}) \leq \mathbb{P}(e^{\gamma\bar{N} - \frac{\gamma^2}{2}\mathbb{E}[\bar{N}^2]} \frac{F(z)}{|z|^4} e^{\gamma M} \int_{-\infty}^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds > x - x^{1-\eta})$$

where $M = \sup_{s \geq 0} (B_s - (Q - \alpha)s)$. Then (2.49) implies

$$\begin{aligned} \mathbb{P}(W > x) &\leq e^{(2(Q-\alpha)^2 - \gamma(Q-\alpha))\mathbb{E}[\bar{N}^2]} \left(\frac{F(z)}{|z|^4} \right)^{\frac{2}{\gamma}(Q-\alpha)} \frac{\bar{R}(\alpha)}{(x - x^{1-\eta})^{\frac{2}{\gamma}(Q-\alpha)}} + Cx^{-p(1-\eta)} \\ &\leq |z|^{-4\alpha(Q-\alpha)} F(z)^{\frac{2}{\gamma}(Q-\alpha)} \frac{\bar{R}(\alpha)}{x^{\frac{2}{\gamma}(Q-\alpha)}} + Cx^{-\frac{2}{\gamma}(Q-\alpha)-\eta} + Cx^{-p(1-\eta)} \end{aligned}$$

for $p < \beta$. Recall that we defined $\bar{\eta} > 0$ by $(1 - \bar{\eta})\beta = \frac{2}{\gamma}(Q - \alpha) + \bar{\eta}$. We conclude

$$(3.8) \quad \mathbb{P}(W > x) \leq |z|^{-4\alpha(Q-\alpha)} F(z)^{\frac{2}{\gamma}(Q-\alpha)} \frac{\bar{R}(\alpha)}{x^{\frac{2}{\gamma}(Q-\alpha)}} + Cx^{-\frac{2}{\gamma}(Q-\alpha)-\eta}$$

for all $\eta < \bar{\eta}$.

Now, we consider the lower bound. We have

$$(3.9) \quad \mathbb{P}(W > x) \geq \mathbb{P}(W_1 > x + x^{1-\bar{\eta}}) - \mathbb{P}(W_2 < -x^{1-\bar{\eta}}) \geq \mathbb{P}(W_1 > x + x^{1-\bar{\eta}}) - Cx^{-\frac{2}{\gamma}(Q-\alpha)-\eta}$$

for all $\eta < \bar{\eta}$. By the Williams decomposition we get as in (3.5)

$$W_1 \stackrel{(Law)}{=} e^{\gamma\bar{N} - \frac{\gamma^2}{2}\mathbb{E}[\bar{N}^2]} \frac{F(z)}{|z|^4} e^{\gamma M} \int_{-L-M}^{\infty} e^{\gamma B_s^\alpha} Z_s ds := W(L-M).$$

where $M = \sup_{s \geq 0} (B_s - (Q - \alpha)s)$ and M and B^α are independent of Z_s .

Let η' be such that $(1 - \eta')\frac{4}{\gamma^2} = \frac{2}{\gamma}(Q - \alpha) + \eta'$. One has $\eta' \geq \bar{\eta}$. Consider the event \mathcal{E} defined by

$$e^{\gamma\bar{N} - \frac{\gamma^2}{2}\mathbb{E}[\bar{N}^2]} \frac{F(z)}{|z|^4} \int_{-L-M}^{\infty} e^{\gamma B_s^\alpha} Z_s ds < x^{1-\eta'}.$$

We have trivially

$$\mathbb{P}(W_1 > x + x^{1-\bar{\eta}}) \geq \mathbb{P}(\{W_1 > x + x^{1-\bar{\eta}}\} \cap \mathcal{E}).$$

Under $\{W_1 > x + x^{1-\bar{\eta}}\} \cap \mathcal{E}$ we have $e^{\gamma M} \geq |x|^{\eta'}$. Indeed, if $e^{\gamma M} < |x|^{\eta'}$ then under \mathcal{E} we get $W_1 < x$ which is impossible. Thus $M \geq -\frac{\eta'}{\gamma} \ln |x|$ whereby $L-M \geq L - \frac{\eta'}{\gamma} \ln |x|$ and hence $W(L - \frac{\eta'}{\gamma} \ln |x|) \leq W(L-M)$. We conclude

$$\begin{aligned} \mathbb{P}(W_1 > x + x^{1-\bar{\eta}}) &\geq \mathbb{P}(\{W(L - \frac{\eta'}{\gamma} \ln |x|) > x + x^{1-\bar{\eta}}\} \cap \mathcal{E}) \\ &\geq \mathbb{P}(W(L - \frac{\eta'}{\gamma} \ln |x|) > x + x^{1-\bar{\eta}}) - Cx^{-(1-\eta')\frac{4}{\gamma^2} + \epsilon} \\ (3.10) \quad &\geq |z|^{-4\alpha(Q-\alpha)} F(z)^{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\left(\int_{-L - \frac{\eta'}{\gamma} \ln x}^{\infty} e^{\gamma B_s^\alpha} Z_s ds\right)^{\frac{2}{\gamma}(Q-\alpha)}\right] (x + x^{1-\bar{\eta}})^{-\frac{2}{\gamma}(Q-\alpha)} - \frac{C}{x^{\frac{2}{\gamma}(Q-\alpha) + \eta' - \epsilon}} \end{aligned}$$

for all $\epsilon > 0$ where in the second step we used Lemma 2.8.

We claim now that

$$(3.11) \quad \mathbb{E}\left[\left(\int_{-\infty}^{\infty} e^{\gamma B_s^\alpha} Z_s ds\right)^{\frac{2}{\gamma}(Q-\alpha)}\right] - \mathbb{E}\left[\left(\int_{-L - \frac{\eta'}{\gamma} \ln x}^{\infty} e^{\gamma B_s^\alpha} Z_s ds\right)^{\frac{2}{\gamma}(Q-\alpha)}\right] \leq Cx^{-\eta'}.$$

Combined with (3.10) and (3.9) this yields

$$\begin{aligned} \mathbb{P}(W > x) &\geq |z|^{-4\alpha(Q-\alpha)} F(z)^{\frac{2}{\gamma}(Q-\alpha)} \frac{\bar{R}(\alpha)}{(x - x^{1-\eta})^{\frac{2}{\gamma}(Q-\alpha)}} + Cx^{-p(1-\eta)} \\ (3.12) \quad &\leq |z|^{-4\alpha(Q-\alpha)} F(z)^{\frac{2}{\gamma}(Q-\alpha)} \frac{\bar{R}(\alpha)}{x^{\frac{2}{\gamma}(Q-\alpha)}} - Cx^{-\frac{2}{\gamma}(Q-\alpha)-\eta} \end{aligned}$$

for all $\eta < \bar{\eta}$. (3.12) and (3.8) then finish the proof.

It remains to prove (3.11). By Remark 2.7, the process \hat{B}_s^α defined for $s \leq 0$ by the relation $\hat{B}_s^\alpha = \mathcal{B}_{s-L-\frac{\eta'}{\gamma} \ln x}^\alpha + \frac{\eta'}{\gamma} \ln x$ is independent from everything and distributed like $(\mathcal{B}_s^\alpha)_{s \leq 0}$. We can then write

$$\int_{-\infty}^{\infty} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds = A + x^{-\eta'} B$$

where

$$A = \int_{-L-\frac{\eta'}{\gamma} \ln x}^{\infty} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds$$

and

$$B = \int_{-\infty}^0 e^{\gamma \hat{B}_s^\alpha} Z_{s-L-\frac{\eta'}{\gamma} \ln x} ds.$$

We now distinguish two cases: $\frac{2}{\gamma}(Q-\alpha) \leq 1$ and $\frac{2}{\gamma}(Q-\alpha) > 1$.

$\frac{2}{\gamma}(Q-\alpha) \leq 1$. We use $(1+u)^{\frac{2}{\gamma}(Q-\alpha)} - 1 \leq \frac{2}{\gamma}(Q-\alpha)u$ for $u \geq 0$ to bound

$$\mathbb{E}[(A + x^{-\eta'} B)^{\frac{2}{\gamma}(Q-\alpha)} - A^{\frac{2}{\gamma}(Q-\alpha)}] \leq \frac{2}{\gamma}(Q-\alpha)x^{-\eta'} \mathbb{E}[BA^{\frac{2}{\gamma}(Q-\alpha)-1}]$$

By Hölder's inequality with $p \in (1, \frac{4}{\gamma^2})$, we get

$$\mathbb{E}[BA^{\frac{2}{\gamma}(Q-\alpha)-1}] \leq \mathbb{E}[B^p]^{1/p} \mathbb{E}[A^{q(\frac{2}{\gamma}(Q-\alpha)-1)}]^{\frac{1}{q}} < \infty$$

since B is equal in distribution to $\int_{-\infty}^0 e^{\gamma \hat{B}_s^\alpha} Z_s ds$ and $A \geq \int_0^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds$ which has negative moments of all order by Lemma 2.8.

$\frac{2}{\gamma}(Q-\alpha_2) > 1$. Let $p := \frac{2}{\gamma}(Q-\alpha)$. By triangle inequality we have

$$\begin{aligned} \mathbb{E}[(A + x^{-\eta'} B)^p - A^p] &\leq \left((\mathbb{E}[A^p])^{1/p} + x^{-\eta'} (\mathbb{E}[B^p])^{1/p} \right)^p - \mathbb{E}[A^p] \\ &\leq \left((\mathbb{E}[A^p])^{1/p} + Cx^{-\eta'} \right)^p - \mathbb{E}[A^p] \leq Cx^{-\eta'} \mathbb{E}[A^p]^{1-1/p} \leq Cx^{-\eta'} \end{aligned}$$

where again we used that A and B have moment of order p . □

Remark 3.2. A simple variation of the proof yields the result (2.50).

3.2. Tail estimate around two insertions. Let for $i = 2, 3$ ¹¹

$$W_i := \int_{B(z_i, 1)} \frac{F_i(x)}{|x - z_i|^{\gamma \alpha_i}} M_\gamma(d^2 x).$$

We will suppose that $|z_2| \geq 2$, $|z_3| \geq 2$ and $|z_2 - z_3| \geq 3$ so that the balls $B_i = B(z_i, 1)$ are well separated. We denote by $\bar{\eta}_2$ and $\bar{\eta}_3$ the exponents occurring in the tail estimates of Lemma 3.1 applied to W_2 and W_3 . Set

$$\tilde{\eta}_2 = \bar{\eta}_2 \wedge \frac{1}{\gamma}(Q - \alpha_3) \wedge \frac{1}{2}, \quad \tilde{\eta}_3 = \bar{\eta}_3 \wedge \frac{1}{\gamma}(Q - \alpha_2) \wedge \frac{1}{2}.$$

Then we have

Lemma 3.3. For all $\beta < \bar{\beta} := (\frac{2}{\gamma}(Q - \alpha_2) + \tilde{\eta}_2) \wedge (\frac{2}{\gamma}(Q - \alpha_3) + \tilde{\eta}_3)$

$$|\mathbb{P}(W_2 + W_3 > x) - \sum_{i=2}^3 |z_i|^{4\alpha_i(\alpha_i - Q)} F_i(z_i)^{\frac{2}{\gamma}(Q - \alpha_i)} \frac{\bar{R}(\alpha_i)}{x^{\frac{2}{\gamma}(Q - \alpha_i)}}| \leq Cx^{-\beta}$$

Remark 3.4. The above theorem is useful when $\bar{\beta} > \frac{2}{\gamma}(Q - \alpha_2) \vee \frac{2}{\gamma}(Q - \alpha_3)$. This is the case when α_2 and α_3 are sufficiently close to each other.

¹¹The indices 2, 3 occur in the applications of this estimate in the main text.

Remark 3.5. *The proof of Theorem 3.3 is based on the fact that the two variables W_2 and W_3 are “nearly” independent. Along the same lines as the proof of Theorem 3.3, one can in fact show that for all $p_2, p_3 > 0$ there exists some constant $C > 0$ such that*

$$\mathbb{E}[W_2^{p_2} W_3^{p_3}] \leq C \mathbb{E}[W_2^{p_2}] \mathbb{E}[W_3^{p_3}].$$

Proof. The strategy here is to apply the previous lemma with one insertion. We start with the upper bound. We have

$$(3.13) \quad \mathbb{P}(W_2 + W_3 > x) \leq \mathbb{P}(W_2 + W_3 > x, W_2 > \frac{x}{2}) + \mathbb{P}(W_2 + W_3 > x, W_3 > \frac{x}{2})$$

The variables W_2 and W_3 are nearly independent as we now argue. We consider the circle of radius $\frac{3}{2}$ centered at z_2 . By the Markov property of the GFF, we have the following decomposition inside $B(z_2, \frac{3}{2})$

$$X(x) = \tilde{X} + \mathcal{P}(X)(x)$$

where $\mathcal{P}(X)(x)$ is the Poisson kernel of the ball $B(z_2, \frac{3}{2})$ applied to X and \tilde{X} is a GFF with Dirichlet boundary conditions on $B(z_2, \frac{3}{2})$ independent of the outside of $B(z_2, \frac{3}{2})$. On the smaller ball $B(z_2, 1)$, the process $\mathcal{P}(X)(x)$ is a smooth Gaussian process hence for all $p > 0$

$$\mathbb{E}[e^{p \sup_{|x-z_2| \leq 1} \mathcal{P}(X)(x)}] < \infty.$$

We set $H = \sup_{|x-z_2| \leq 1} \mathcal{P}(X)(x)$. Of course, we have

$$W_2 \leq e^{\gamma H} \tilde{W}_2$$

where \tilde{W}_2 is computed with the chaos measure of \tilde{X} . \tilde{W}_2, W_3 have moments less than orders $\frac{2}{\gamma}(Q - \alpha_i)$ respectively so that for all $u, v > 0$ and all $\epsilon' > 0$ that

$$\begin{aligned} \mathbb{P}(W_2 > u, W_3 > v) &\leq \mathbb{P}(e^{\gamma H} \tilde{W}_2 > u, W_3 > v) \\ &\leq \frac{1}{u^{\frac{2}{\gamma}(Q-\alpha_2)-\epsilon'}} \mathbb{E}[\tilde{W}_2^{\frac{2}{\gamma}(Q-\alpha_2)-\epsilon'}] \mathbb{E}[e^{(2(Q-\alpha_2)-\gamma\epsilon')H} \mathbf{1}_{W_3 > v}] \\ &\leq \frac{1}{u^{\frac{2}{\gamma}(Q-\alpha_2)-\epsilon'}} \mathbb{E}[\tilde{W}_2^{\frac{2}{\gamma}(Q-\alpha_2)-\epsilon'}] \mathbb{E}[e^{p(2(Q-\alpha_2)-\gamma\epsilon')H}]^{1/p} \mathbb{P}(W_3 > v)^{1/q} \\ &\leq \frac{C}{u^{\frac{2}{\gamma}(Q-\alpha_2)-\epsilon'} v^{\frac{1}{q}(\frac{2}{\gamma}(Q-\alpha_3)-\epsilon')}} \end{aligned}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By taking q close to 1 we conclude

$$(3.14) \quad \mathbb{P}(W_2 > u, W_3 > v) \leq \frac{C}{u^{\frac{2}{\gamma}(Q-\alpha_2)-\epsilon} v^{\frac{2}{\gamma}(Q-\alpha_3)-\epsilon}}$$

for all $\epsilon > 0$. Therefore, exploiting (3.14) we have for all $\epsilon > 0$

$$\begin{aligned} \mathbb{P}(W_2 + W_3 > x, W_2 > \frac{x}{2}) &\leq \mathbb{P}(W_2 + W_3 > x, W_2 > \frac{x}{2}, W_3 \leq \sqrt{x}) + \mathbb{P}(W_2 > \frac{x}{2}, W_3 > \sqrt{x}) \\ &\leq \mathbb{P}(W_2 > x - \sqrt{x}) + \frac{C}{x^{\frac{2}{\gamma}(Q-\alpha_2)} x^{\frac{1}{\gamma}(Q-\alpha_3)-\epsilon}} \end{aligned}$$

We get a similar bound by interchanging 2 and 3. Inserting to (3.13) we obtain

$$\mathbb{P}(W_2 + W_3 > x) \leq \mathbb{P}(W_2 > x - \sqrt{x}) + \mathbb{P}(W_3 > x - \sqrt{x}) + \frac{C}{x^{\frac{2}{\gamma}(Q-\alpha_2)} x^{\frac{1}{\gamma}(Q-\alpha_3)-\epsilon}} + \frac{C}{x^{\frac{2}{\gamma}(Q-\alpha_3)} x^{\frac{1}{\gamma}(Q-\alpha_2)-\epsilon}}$$

and then we use Lemma 3.1 on one insertion.

Now, we proceed with the lower bound. We have, exploiting (3.14), that for all $\epsilon > 0$

$$\begin{aligned} \mathbb{P}(W_2 + W_3 > x) &\geq \mathbb{P}(\{W_2 > x\} \cup \{W_3 > x\}) \geq \mathbb{P}(W_2 > x) + \mathbb{P}(W_3 > x) - \mathbb{P}(W_2 > x, W_3 > x) \\ &\geq \mathbb{P}(W_2 > x) + \mathbb{P}(W_3 > x) - \frac{C}{x^{\frac{2}{\gamma}(Q-\alpha_2) + \frac{2}{\gamma}(Q-\alpha_3) - \epsilon}} \end{aligned}$$

and then we use again Lemma 3.1. □

4. ANALYTIC CONTINUATION OF LIOUVILLE CORRELATION FUNCTIONS

In this section we study the analytic continuation of the unit volume correlations (2.15). These are defined for real weights $\alpha = (\alpha_1, \dots, \alpha_N)$ satisfying the extended Seiberg bounds

$$(4.1) \quad U_N := \{\alpha \in \mathbb{R}^N : \frac{1}{\gamma}(2Q - \sum_{k=1}^N \alpha_k) < \frac{4}{\gamma^2} \wedge \min_{1 \leq k \leq N} \frac{2}{\gamma}(Q - \alpha_k), \quad \forall k : \alpha_k < Q\}.$$

We will prove

Theorem 4.1. *Fix $N \geq 3$ and disjoint points $z_1, \dots, z_N \in \mathbb{C}^N$. The unit volume correlation function (2.15) extends to an analytic function of α defined in a complex neighborhood of U_N in \mathbb{C}^N .*

Proof. By Möbius invariance we may assume $|z_i| > 2$ and $|z_i - z_j| > 2$. We use (2.9) to write the unit volume correlation functions as the limit

$$(4.2) \quad \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{uv} = 2\mu^{-s}\gamma^{-1} \prod_{k=1}^N |z_k|_+^{-4\Delta_{\alpha_k}} \lim_{r \rightarrow \infty} F_r(\alpha)$$

where

$$(4.3) \quad F_r(\alpha) = \mathbb{E} \left[\prod_{k=1}^N e^{\alpha_k X_r(z_k) - \frac{\alpha_k^2}{2} \mathbb{E} X_r(z_k)^2} M_\gamma(\mathbb{C}_r)^{-s} \right]$$

where $\mathbb{C}_r := \mathbb{C} \setminus \cup_{k=1}^N B(z_k, e^{-r})$. F_r is defined for all $\alpha \in \mathbb{C}^N$ and is complex differentiable in α_i , hence defining an entire function in the α_i . We show that there is an open $V \subset \mathbb{C}^N$ containing U_N s.t. F_r converges uniformly on compacts of V . Note that this is nontrivial since for $\alpha_k = a_k + ib_k$ we have

$$|e^{\alpha_k X_r(z_k) - \frac{\alpha_k^2}{2} \mathbb{E} X_r(z_k)^2}| = e^{\alpha_k X_r(z_k) - \frac{a_k^2}{2} \mathbb{E} X_r(z_k)^2} e^{\frac{b_k^2}{2} \mathbb{E} X_r(z_k)^2}$$

and $e^{\frac{b_k^2}{2} \mathbb{E} X_r(z_k)^2} \propto e^{\frac{b_k^2}{2} r}$ blows up as $r \rightarrow \infty$.

By Remark 2.2, we know that $t \in \mathbb{R}_+ \rightarrow B_{r+t}^k := X_{r+t}(z_k) - X_r(z_k)$ are mutually independent Brownian motions and they are independent of $\sigma\{X(x); x \in \mathbb{C}_r\}$. Hence

$$F_{r+1}(\alpha) - F_r(\alpha) = \mathbb{E} \left[\prod_{k=1}^N e^{\alpha_k X_{r+1}(z_k) - \frac{\alpha_k^2}{2} \mathbb{E} X_{r+1}(z_k)^2} (M_\gamma(\mathbb{C}_{r+1})^{-s} - M_\gamma(\mathbb{C}_r)^{-s}) \right].$$

Now we apply the Cameron-Martin theorem as in (2.11) to the real parts of the vertex insertions to get

$$(4.4) \quad |F_{r+1}(\alpha) - F_r(\alpha)| \leq C e^{(r+1) \sum_{k=1}^N \frac{b_k^2}{2}} \left| \mathbb{E} \left(\int_{\mathbb{C}_{r+1}} f_r(x) M_\gamma(d^2x) \right)^{-s} - \mathbb{E} \left(\int_{\mathbb{C}_r} f_r(x) M_\gamma(d^2x) \right)^{-s} \right|$$

where $f_r(x) = e^{\sum_{k=1}^N \gamma \alpha_k G_{r+1}(x, z_k)}$ and we have defined $G_{r+1}(z, z') := \mathbb{E}[X(z)X_{r+1}(z')]$. We get from (2.1)

$$f(x) := \sup_r f_r(x) \leq C \prod_k \left(\frac{|x|_+ |z_k|_+}{|x - z_k|} \right)^{\gamma \alpha_k}.$$

We need to estimate the difference of expectations in (4.4). Let

$$Y_r := \int_{\mathbb{C}_{r+1} \setminus \mathbb{C}_r} f_r(x) M_\gamma(d^2x).$$

and set also $Z_r := \int_{\mathbb{C}_r} f_r(x) M_\gamma(d^2x)$. Then

$$\begin{aligned} & \left| \mathbb{E} \left(\int_{\mathbb{C}_{r+1}} f_r(x) M_\gamma(d^2x) \right)^{-s} - \mathbb{E} \left(\int_{\mathbb{C}_r} f_r(x) M_\gamma(d^2x) \right)^{-s} \right| = \left| \mathbb{E} \left((Z_r + Y_r)^{-s} - Z_r^{-s} \right) \right| \\ & \leq \mathbb{E} [\mathbb{1}_{Y_r < \epsilon} |(Z_r + Y_r)^{-s} - Z_r^{-s}|] + \mathbb{E} [\mathbb{1}_{Y_r \geq \epsilon} |(Z_r + Y_r)^{-s} - Z_r^{-s}|] \end{aligned}$$

where $\epsilon > 0$ will be fixed later. The first expectation on the RHS is bounded by

$$\mathbb{E} \mathbb{1}_{Y_r < \epsilon} |(Z_r + Y_r)^{-s} - Z_r^{-s}| \leq C\epsilon \sup_{t \in [0,1]} \mathbb{E} (Z_r + tY_r)^{-\Re s - 1} \leq C\epsilon$$

uniformly in r . The last bound follows by noting that for $-\Re s - 1 > 0$ the expectation is bounded uniformly in r by $C\mathbb{E} \left(\int f(x) M_\gamma(d^2x) \right)^{-\Re s - 1}$ which is finite due to (2.14) whereas for $-\Re s - 1 < 0$ we may bound it for example by $C\mathbb{E} \left(\int_{\mathbb{C}_1 \setminus \mathbb{C}_2} M_\gamma(d^2x) \right)^{-\Re s - 1}$ which is finite as well.

For the second expectation we use the Hölder inequality

$$\mathbb{E} \mathbb{1}_{Y_r \geq \epsilon} |Z_{r+1}^{-s} - Z_r^{-s}| \leq C P(Y_r \geq \epsilon)^{1/p} \left((\mathbb{E} Z_{r+1}^{-q\Re s})^{1/q} + (\mathbb{E} Z_r^{-q\Re s})^{1/q} \right).$$

Taking $q > 1$ s.t. $-q\Re(s) < \min_j \frac{2}{\gamma} (Q - \alpha_j) \wedge \frac{4}{\gamma^2}$ we may bound the two expectations uniformly in r as in the previous paragraph and then using Markov inequality we get

$$\mathbb{E} \mathbb{1}_{Y_r \geq \epsilon} |Z_{r+1}^{-s} - Z_r^{-s}| \leq C\epsilon^{-m/p} (\mathbb{E} Y_r^m)^{1/p}.$$

It remains to bound $\mathbb{E} Y_r^m$ for suitable $m > 0$. We note that $\mathbb{C}_{r+1} \setminus \mathbb{C}_r = \cup_i A_r^i$ where A_r^i is the annulus centred at z_i with radii e^{-r-1}, e^{-r} . Then we obtain for $m < \frac{4}{\gamma^2}$

$$(4.5) \quad \mathbb{E} Y_r^m \leq C\mathbb{E} \left(\sum_k \int_{A_r^k} f(x) M_\gamma(d^2x) \right)^m \leq C \max_k e^{-r(\gamma(Q - a_k)m - \frac{\gamma^2 m^2}{2})} := C e^{-r\theta}$$

where in the second step we used the estimate (9.2). Now, let us fix $\mathbf{a}^0 \in U_N$. Then we can find $m > 0$ and $\delta > 0$ s.t. $\theta > 0$ for all \mathbf{a} with $\min_k |a_k - a_k^0| \leq \delta$. Hence, for $\boldsymbol{\alpha} \in \mathbb{C}^M$ with $\alpha_k = a_k + ib_k$ and $\epsilon > 0$

$$|F_{r+1}(\boldsymbol{\alpha}) - F_r(\boldsymbol{\alpha})| \leq C e^{(r+1)\sum_{k=1}^N \frac{b_k}{2}} (\epsilon + \epsilon^{-m/p} e^{-\frac{\theta}{p}\theta}).$$

Taking $\epsilon = e^{-\delta r}$ with $\delta = \frac{\theta}{p+m}$ we then have

$$|F_{r+1}(\boldsymbol{\alpha}) - F_r(\boldsymbol{\alpha})| \leq C e^{-(\delta - \sum_{k=1}^N \frac{b_k}{2})r}.$$

Hence, $F_r(\boldsymbol{\alpha})$ converges uniformly in a ball around \mathbf{a}^0 in \mathbb{C}^N . □

5. PROOF OF LEMMAS 2.8 AND 2.9 ON THE REFLECTION COEFFICIENT

5.1. Proof of lemma 2.8. By symmetry, it is enough to show that

$$\mathbb{E} \left[\left(\int_0^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right] < \infty.$$

Let first $p > 0$. If $0 < p \leq 1$ we have by subadditivity

$$\mathbb{E} \left[\left(\int_0^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right] \leq \sum_{n=1}^\infty \mathbb{E} \left[\left(\int_n^{n+1} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right]$$

and for $1 < p < \frac{4}{\gamma^2}$ by convexity

$$\left[\mathbb{E} \left(\int_0^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right]^{1/p} \leq \sum_{n=1}^\infty \left[\mathbb{E} \left(\int_n^{n+1} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right]^{1/p}$$

We set $\nu = Q - \alpha$. The process \mathcal{B}_s^α is stochastically dominated by a Brownian motion with drift $-\nu$ starting from origin and conditioned to stay below 1 (see Appendix); hence we have if B_s is a standard Brownian motion starting from 0

$$\begin{aligned} \mathbb{E} \left[\left(\int_n^{n+1} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^p \right] &\leq C \mathbb{E} \left[\left(1_{B_n - \nu n \leq 1} \int_n^{n+1} e^{\gamma(B_s - \nu s)} Z_s ds \right)^p \right] \\ &\leq C \mathbb{E} \left[\left(\int_n^{n+1} Z_s ds \right)^p \right] \mathbb{E} \left[e^{\gamma p \sup_{s \in [n, n+1]} (B_s - B_n)} \right] \mathbb{E} \left[1_{B_n - \nu n \leq 1} e^{\gamma p (B_n - \nu n)} \right] \\ &\leq C \mathbb{E} \left[1_{B_n - \nu n \leq 1} e^{\gamma p (B_n - \nu n)} \right] = C n^{-\frac{1}{2}} \int_{-\infty}^1 e^{\gamma p y} e^{-\frac{(y + \nu n)^2}{2n}} dy \end{aligned}$$

where we used (2.47). Considering separately $y < -\frac{\nu n}{2}$ and $y \in [-\frac{\nu n}{2}, 1]$ the last integral is seen to be exponentially small in n and the claim follows.

Let now $p = -q < 0$. Set $\tau_{-1} = \inf\{s \geq 0, \mathcal{B}_s^\alpha = -1\}$. The process $\mathcal{B}_{s+\tau_{-1}}^\alpha + 1$ is a Brownian motion with drift $-\nu$ starting from 0 and conditioned to stay below 1. Therefore, we have if B_s is a standard Brownian motion starting from 0 and $\beta := \sup_{s \geq 0} (B_s - \nu s)$

$$\mathbb{E} \left[\left(\int_0^\infty e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^{-q} \right] \leq \mathbb{E} \left[\left(\int_{\tau_{-1}}^{\tau_{-1}+1} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds \right)^{-q} \right] = C \mathbb{E} \left[1_{\beta \leq 1} \left(\int_0^1 e^{\gamma(B_s - \nu s)} Z_s ds \right)^{-q} \right]$$

since Z_s is stationary and \mathcal{B}_s^α is independent from Z . Finally, we conclude by

$$\mathbb{E} \left[1_{\beta \leq 1} \left(\int_0^1 e^{\gamma(B_s - \nu s)} Z_s ds \right)^{-q} \right] \leq \mathbb{E} \left[1_{\beta \leq 1} e^{-q \inf_{s \in [0, 1]} (B_s - \nu s)} \right] \mathbb{E} \left(\int_0^1 Z_s ds \right)^{-q} < \infty$$

where (2.47) was used. \square

5.2. Proof of lemma 2.9. We use formula (1.6) to write $C_\gamma(\epsilon, \alpha, \alpha)$ in terms of $\langle V_\epsilon(0) V_\alpha(z_2) V_\alpha(z_3) \rangle$ where we take $|z_2|, |z_3| > 2$ with $|z_2 - z_3| > 2$. Then

$$2\Delta_{12} = -2\Delta_\epsilon = -\epsilon(Q - \frac{\epsilon}{2}) \xrightarrow{\epsilon \rightarrow 0} 0.$$

and

$$2\Delta_{23} = 2\Delta_\epsilon - 4\Delta_\alpha \xrightarrow{\epsilon \rightarrow 0} \alpha(\alpha - 2Q)$$

Therefore, we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon C_\gamma(\epsilon, \alpha, \alpha) &= \lim_{\epsilon \rightarrow 0} \epsilon |z_2 - z_3|^{\alpha(2Q - \alpha)} \langle V_\epsilon(0) V_\alpha(z_2) V_\alpha(z_3) \rangle \\ &= |z_2 - z_3|^{2\alpha(Q - \alpha)} 2\mu^{\frac{2}{\gamma}(Q - \alpha)} \gamma^{-1} \Gamma(-\frac{2}{\gamma}(Q - \alpha)) \lim_{\epsilon \rightarrow 0} \epsilon \mathbb{E} \left[\left(\int_{\mathbb{C}} F_\epsilon(x) M_\gamma(d^2x) \right)^{\frac{2}{\gamma}(Q - \alpha) - \frac{\epsilon}{\gamma}} \right] \end{aligned}$$

where

$$F_\epsilon(x) = \frac{|x|_+^{\gamma(\epsilon + 2\alpha)}}{|x|^{\gamma\epsilon} |x - z_2|^{\gamma\alpha} |x - z_3|^{\gamma\alpha}}.$$

Let $W_{i,\epsilon} = \int_{B(z_i, 1)} F_\epsilon(x) M_\gamma(d^2x)$ for $i = 2, 3$ and $A_\epsilon = \int_{(B(z_1, 1) \cup B(z_2, 1))^c} F_\epsilon(x) M_\gamma(d^2x)$ so that

$$\int_{\mathbb{C}} F_\epsilon(x) M_\gamma(d^2x) = A_\epsilon + W_{2,\epsilon} + W_{3,\epsilon}.$$

Now, we have the following inequality if $\frac{2}{\gamma}(Q - \alpha) \leq 1$

$$\begin{aligned} \mathbb{E} \left[(W_{2,\epsilon} + W_{3,\epsilon})^{\frac{2}{\gamma}(Q - \alpha) - \frac{\epsilon}{\gamma}} \right] &\leq \mathbb{E} \left[(A_\epsilon + W_{2,\epsilon} + W_{3,\epsilon})^{\frac{2}{\gamma}(Q - \alpha) - \frac{\epsilon}{\gamma}} \right] \\ &\leq \mathbb{E} \left[A_\epsilon^{\frac{2}{\gamma}(Q - \alpha) - \frac{\epsilon}{\gamma}} \right] + \mathbb{E} \left[(W_{2,\epsilon} + W_{3,\epsilon})^{\frac{2}{\gamma}(Q - \alpha) - \frac{\epsilon}{\gamma}} \right] \end{aligned}$$

By the double tail estimate Lemma 3.3 we have

$$\mathbb{P}(W_{2,\epsilon} + W_{3,\epsilon} > x) = 2|z_2 - z_3|^{-2\alpha(Q - \alpha)} \bar{R}(\alpha) x^{-\frac{2}{\gamma}(Q - \alpha)} (1 + \mathcal{O}(x^{-\eta}))$$

for $\eta > 0$, uniformly in ϵ . Since $\epsilon \mathbb{E}[A_\epsilon^{\frac{2}{\gamma}(Q-\alpha)-\frac{\epsilon}{\gamma}}]$ converges to 0, this yields

$$\lim_{\epsilon \rightarrow 0} \epsilon \mathbb{E}\left[\left(\int_{\mathbb{C}} F_\epsilon(x) M_\gamma(d^2x)\right)^{\frac{2}{\gamma}(Q-\alpha)-\frac{\epsilon}{\gamma}}\right] = 2\gamma|z_2 - z_3|^{-2\alpha(Q-\alpha)} \bar{R}(\alpha)$$

and then

$$\lim_{\epsilon \rightarrow 0} \epsilon C_\gamma(\epsilon, \alpha, \alpha) = 4\mu^{\frac{2}{\gamma}(Q-\alpha)} \frac{2(Q-\alpha)}{\gamma} \Gamma\left(-\frac{2(Q-\alpha)}{\gamma}\right) \bar{R}(\alpha) = 4R(\alpha)$$

If $\frac{2}{\gamma}(Q-\alpha) > 1$ we have by triangle inequality and ϵ small enough so that $p = \frac{2}{\gamma}(Q-\alpha) - \frac{\epsilon}{\gamma} > 1$

$$[\mathbb{E}(W_{2,\epsilon} + W_{3,\epsilon})^p]^{1/p} \leq [\mathbb{E}(A_\epsilon + W_{2,\epsilon} + W_{3,\epsilon})^p]^{1/p} \leq [\mathbb{E}(A_\epsilon)^p]^{1/p} + [\mathbb{E}(W_{2,\epsilon} + W_{3,\epsilon})^p]^{1/p}$$

and we can conclude similarly as the previous case.

6. THE BPZ EQUATIONS AND ALGEBRAIC RELATIONS

This section is devoted to the study of the small z asymptotics of the four point functions $\mathcal{T}_{-\frac{\gamma}{2}}$ and $\mathcal{T}_{-\frac{2}{\gamma}}$ leading to the proof of (2.34) and (2.41). The proof of the latter is the technical core of the paper and the key input in the probabilistic identification of the reflection coefficient.

6.1. Fusion without reflection. As mentioned in Section 2.4 the relation (2.34) was proven in [36, Theorem 2.3] with the assumption $\frac{1}{\gamma} + \gamma < \alpha_1 + \frac{\gamma}{2} < Q$ or in other words $\frac{\gamma}{2} + \frac{1}{\gamma} < \alpha_1 < \frac{2}{\gamma}$. This interval is non empty if and only if $\gamma^2 < 2$. In this section we will remove this constraint. The reason for the restriction $\frac{\gamma}{2} + \frac{1}{\gamma} < \alpha_1$ was the following. In order to prove (2.34), one must perform the asymptotic expansion of $\mathcal{T}_{-\frac{\gamma}{2}}(z)$ around $z \rightarrow 0$ (2.31) as explained in section 2.4. In the case $\frac{1}{\gamma} + \frac{\gamma}{2} < \alpha_1$, the exponent $2(1-c)$ which is equal to $\gamma(Q-\alpha_1)$ is strictly less than 1 hence there are no polynomial terms in z and \bar{z} in the expansion (2.31) to that order (such terms are present in the small z expansion of $|F_-(z)|^2$). In the case $\alpha_1 < \frac{1}{\gamma} + \frac{\gamma}{2}$, the asymptotic expansion of $\mathcal{T}_{-\frac{\gamma}{2}}(z)$ around 0 is more involved. Nonetheless, we prove here:

Theorem 6.1. *We assume the Seiberg bounds for $(-\frac{\gamma}{2}, \alpha_1, \alpha_2, \alpha_3)$, i.e. $\sum_{k=1}^3 \alpha_k > 2Q + \frac{\gamma}{2}$ and $\alpha_k < Q$ for all k . If $\frac{\gamma}{2} < \alpha_1 < \frac{2}{\gamma}$ then*

$$(6.1) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) |F_-(z)|^2 - \mu \frac{\pi}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2})} C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) |F_+(z)|^2$$

and the relation (1.8) holds.

Proof. Let first $\gamma^2 < 2$. (6.1) was proven in [36, Theorem 2.3] in the case $\frac{\gamma}{2} + \frac{1}{\gamma} < \alpha_1 < \frac{2}{\gamma}$. This result extends to the interval $\frac{\gamma}{2} < \alpha_1 < \frac{2}{\gamma}$ by analyticity. Indeed, for fixed $\gamma \in (0, \sqrt{2})$, the interval $\frac{\gamma}{2} + \frac{1}{\gamma} < \alpha_1 < \frac{2}{\gamma}$ is non empty. Furthermore, by Theorem 4.1 both sides of eq. (6.1) are analytic in α_1 (with other parameters fixed) in a neighborhood of the interval $\frac{\gamma}{2} < \alpha_1 < \frac{2}{\gamma}$ seen as a subset of \mathbb{C} . Uniqueness of analytic continuation thus establishes (6.1) for $\gamma^2 < 2$. $\gamma^2 = 2$ is obtained by continuity in γ (see Remark 2.3 on this).

Let now $\gamma^2 > 2$ and $\frac{\gamma}{2} < \alpha_1 < \frac{2}{\gamma}$: The proof of (6.1) follows from the study of the function $\mathcal{T}_{-\frac{\gamma}{2}}(z)$ as z tends to 0. Thus by (2.23) we need to study the function (2.24) with $\alpha_0 = -\frac{\gamma}{2}$. To streamline notation let us set

$$(6.2) \quad K(z, x) = \frac{|x-z|^{\frac{\gamma^2}{2}} |x|_+^{\gamma(\sum_{k=1}^3 \alpha_k - \frac{\gamma}{2})}}{|x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}}$$

and for any Borel set $B \subset \mathbb{C}$

$$(6.3) \quad \mathcal{K}_B(z) := \int_B K(z, x) M_\gamma(d^2x).$$

Then $\mathcal{R}_{-\frac{\gamma}{2}}(z) = \mathcal{K}_{\mathbb{C}}(z)$. We will also write $\mathcal{K}(z)$ for $\mathcal{K}_{\mathbb{C}}(z)$. We set $p := \frac{1}{\gamma}(\sum_{k=1}^3 \alpha_k - \frac{\gamma}{2} - 2Q)$. Taylor expansion yields the relation

$$\mathbb{E}[\mathcal{K}(z)^{-p}] = \mathbb{E}[\mathcal{K}(0)^{-p}] + z \partial_z \mathbb{E}[\mathcal{K}(z)^{-p}]|_{z=0} + \bar{z} \partial_{\bar{z}} \mathbb{E}[\mathcal{K}(z)^{-p}]|_{\bar{z}=0} + \mathcal{R}(z)$$

where

$$\mathcal{R}(z) := \frac{1}{2} \int_0^1 \left(z^2 \partial_z^2 \mathbb{E}[\mathcal{K}(tz)^{-p}] + 2z \bar{z} \partial_z \partial_{\bar{z}} \mathbb{E}[\mathcal{K}(tz)^{-p}] + \bar{z}^2 \partial_{\bar{z}}^2 \mathbb{E}[\mathcal{K}(tz)^{-p}] \right) dt.$$

Notice that the term $\partial_z \mathbb{E}[\mathcal{K}(z)^{-p}]|_{z=0}$ is well defined. Indeed we have

$$\partial_z \mathbb{E}[\mathcal{K}(z)^{-p}]|_{z=0} = -p \frac{\gamma^2}{4} \mathbb{E} \left[\int_{\mathbb{C}} \frac{1}{x} K(0, x) M_\gamma(d^2 x) \mathcal{K}(0)^{-p-1} \right].$$

Split the the x -integral over \mathbb{C} in two parts, over $B_{1/2}$ and over $B_{1/2}^c$ where in this section we use the notation $B_r = B(0, r)$. Then

$$|\mathbb{E} \left(\int_{B_{1/2}^c} \frac{1}{|x|} K(0, x) M_\gamma(d^2 x) \mathcal{K}(0)^{-p-1} \right)| \leq 2 \mathbb{E} \mathcal{K}(0)^{-p} < \infty$$

as GMC measures possess negative moments of all orders. For the integral over $B_{\frac{1}{2}}$ we use the Cameron-Martin theorem to get

$$\begin{aligned} \mathbb{E} \left(\int_{B_{1/2}} \frac{1}{|x|} K(0, x) M_\gamma(d^2 x) \mathcal{K}(0)^{-p-1} \right) &\leq C \mathbb{E} \left(\int_{B_{1/2}} |x|^{-1-\gamma\alpha_1 + \frac{\gamma^2}{2}} M_\gamma(d^2 x) \mathcal{K}(0)^{-p-1} \right) \\ (6.4) \quad &= C \int_{B_{1/2}} |x|^{-1-\gamma\alpha_1 + \frac{\gamma^2}{2}} \mathbb{E} \left[\left(\int_{\mathbb{C}} K(0, u) e^{\gamma^2 G(x, u)} M_\gamma(d^2 u) \right)^{-p-1} \right] d^2 x. \end{aligned}$$

To bound the last expectation we note that the integrand is strictly positive on $x \in B_{\frac{1}{2}}$ and for example $u \in B(3, 1)$. This ball is far away from the singularities, hence on $u \in B(3, 1)$ the kernel $K(0, u) e^{\gamma^2 G(x, u)}$ is bounded from below away from 0. Thus

$$\mathbb{E} \left[\left(\int_{\mathbb{C}} K(0, u) e^{\gamma^2 G(x, u)} M_\gamma(d^2 u) \right)^{-p-1} \right] \leq C \mathbb{E} (M_\gamma(B(3, 1))^{-p-1}) < \infty$$

as the measure M_γ possesses moments of negative order. The final integral in (6.4) converges as the constraint $\alpha_1 < \frac{2}{\gamma}$ guarantees that $1 + \gamma\alpha_1 - \frac{\gamma^2}{2} < 3 - \frac{\gamma^2}{2} < 2$ since $\gamma^2 > 2$. The same argument shows that $\bar{z} \partial_{\bar{z}} \mathbb{E}[(\mathcal{K}_0(z))^{-p}]|_{\bar{z}=0}$ is well defined.

It remains to investigate the remainder $\mathcal{R}(z)$. All the three terms are treated in the same way and we focus on the first. It may be written as a sum of two terms:

$$\begin{aligned} \mathcal{R}^1(z) &:= -p \frac{\gamma^2}{4} \left(1 + \frac{\gamma^2}{4}\right) z^2 \int_0^1 (1-t) \mathbb{E} \left[\int_{\mathbb{C}} \frac{1}{(x-tz)^2} K(tz, x) M_\gamma(d^2 x) (\mathcal{K}(tz))^{-p-1} \right] dt \\ \mathcal{R}^2(z) &:= p(p+1) \frac{\gamma^4}{16} z^2 \int_0^1 (1-t) \mathbb{E} \left[\left(\int_{\mathbb{C}} \frac{1}{(x-tz)} K(tz, x) M_\gamma(d^2 x) \right)^2 (\mathcal{K}(tz))^{-p-2} \right] dt. \end{aligned}$$

The term \mathcal{R}^1 is the leading one contributing and explicit factor $c|z|^{\gamma(Q-\alpha_1)}$. It is analysed in the same way as a similar term in the proof of [36, Theorem 2.3] so we will be brief. First as above we may restrict the integral to $B_{1/2}$ up to an $\mathcal{O}(z^2)$ contribution from the integral over $B_{1/2}^c$. By Cameron-Martin and the change of variables $x \rightarrow ytz$ this term equals

$$\begin{aligned} &-p \frac{\gamma^2}{4} \left(1 + \frac{\gamma^2}{4}\right) |z|^{2 + \frac{\gamma^2}{2} - \gamma\alpha_1} \int_0^1 (1-t) t^{\frac{\gamma^2}{2} - \gamma\alpha_1} \int_{B_{\frac{1}{2|tz|}}} \frac{|y-1|^{\frac{\gamma^2}{2}}}{(y-1)^2 |y|^{\gamma\alpha_1} |yzt-1|^{\gamma\alpha_2}} \\ &\quad \times \mathbb{E} \left[\left(\int_{\mathbb{C}} K(tz, u) e^{\gamma^2 G(tyz, u)} M_\gamma(d^2 u) \right)^{-p-1} \right] dt d^2 y. \end{aligned}$$

Dominated convergence theorem then ensures that this term is equivalent as $z \rightarrow 0$ to

$$-p \frac{\gamma^2}{4} \left(1 + \frac{\gamma^2}{4}\right) |z|^{\gamma(Q-\alpha_1)} \int_0^1 (1-t) t^{\frac{\gamma^2}{2} - \gamma\alpha_1} dt \int_{\mathbb{C}} \frac{|y-1|^{\frac{\gamma^2}{2}}}{(y-1)^2 |y|^{\gamma\alpha_1}} d^2 y \mathbb{E} \left[\left(\int_{\mathbb{C}} K(0, u) e^{\gamma^2 G(0, u)} M_\gamma(d^2 u) \right)^{-p-1} \right].$$

The y integral can then be computed thanks to (9.7) and one can show that this term, combined with the corresponding term coming from the $\partial_{\bar{z}}^2$ derivative, gives rise to the structure constant $C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)$.

The rest of this proof is dedicated to showing that \mathcal{R}^2 is a $o(|z|^{\gamma(Q-\alpha_1)})$. We will bound the expectation occurring in \mathcal{R}^2 so let us denote

$$(6.5) \quad I(B, tz) := \mathbb{E} \left[\left(\int_B \frac{1}{|x-tz|} K(tz, x) M_\gamma(d^2x) \right)^2 \left(\int_{\mathbb{C}} K(tz, u) M_\gamma(d^2u) \right)^{-p-2} \right].$$

We shall prove

$$(6.6) \quad I(\mathbb{C}, tz) \leq C |tz|^{\gamma(Q-\alpha_1)-2+\eta}$$

with $\eta > 0$ which proves our claim since resulting t integral converges at 0 as $\gamma(Q-\alpha_1)-2 = \frac{\gamma^2}{2}-2 > -1$.

We can now put $t = 1$ and we will bound $I(\mathbb{C}, z)$ for z small. For z small enough $\frac{1}{|x-z|}$ is bounded in $B_{\frac{1}{2}}^c$ and we have $I(B_{\frac{1}{2}}^c, z) < C$. Since $I(\mathbb{C}, z) \leq 2(I(B_{\frac{1}{2}}, z) + I(B_{\frac{1}{2}}^c, z))$ it suffices to bound $I(B_{\frac{1}{2}}, z)$.

Next we bound $I(A, z)$ where A is the annulus centered at origin with radii $L|z|$ and $\frac{1}{2}$ and $L > 1$ will be chosen later. First, we use Jensen's inequality in the normalized measure $1_A(x)K(z, x)M_\gamma(d^2x)$ to get

$$I(A, z) \leq \mathbb{E} \left[\int_A \frac{1}{|x-z|^2} K(z, x) M_\gamma(d^2x) \mathcal{K}_A(z)^{-p-1} \right].$$

Next we observe that on A the covariance kernel of X coincides with that of the exact scale invariant kernel constructed in [51] (in the Gaussian case) up to a global additive constant. In particular, up to an additive independent Gaussian random variable, the restriction of X to $B_{\frac{1}{2}}$ can be constructed as an increasing function of some white noise in this ball and therefore we can make use of the FKG inequality (see [27, section 2.2] for the case of countable product) to get

$$\begin{aligned} & \mathbb{E} \left[\int_A \frac{1}{|x-z|^2} K(z, x) M_\gamma(d^2x) \mathcal{K}_A(z)^{-p-1} \right] \leq \mathbb{E} \left[\int_A \frac{1}{|x-z|^2} K(z, x) M_\gamma(d^2x) \right] \mathbb{E} [\mathcal{K}_A(z)^{-p-1}] \\ & \leq C \int_A \frac{|x-z|^{\frac{\gamma^2}{2}-2}}{|x|^{\gamma\alpha_1}} d^2x \leq C |z|^{\frac{\gamma^2}{2}-\gamma\alpha_1} \int_{|y|>L} \frac{|y-1|^{\frac{\gamma^2}{2}-2}}{|y|^{\gamma\alpha_1}} d^2y \leq C |z|^{-2} |z|^{\gamma(Q-\alpha_1)} L^{-\gamma(\alpha_1-\frac{\gamma}{2})} \end{aligned}$$

where the last integral was convergent due to $\alpha_1 > \frac{\gamma}{2}$. This fits to (6.6) provided we take $L = |z|^{-\delta}$ with $\delta > 0$.

We are left with estimating $I(B_{L|z|}, z)$. Let us first consider the part not too close to the singularity at z : set $S := B_{L|z|} \setminus B(z, |z|^{1+\epsilon})$ for some $\epsilon > 0$, to be fixed later. We have

$$\mathbb{E} \left[\left(\int_S \frac{1}{|x-z|} K(z, x) M_\gamma(d^2x) \right)^2 \mathcal{K}(z)^{-p-2} \right] \leq |z|^{-2-2\epsilon} \mathbb{E} [\mathcal{K}_S(z)^2 \mathcal{K}(z)^{-p-2}].$$

Then we get, for $r \in (0, 2)$ using the fact that $\mathcal{K}_S(z) \leq \mathcal{K}(z)$

$$\mathbb{E} [\mathcal{K}_S(z)^2 \mathcal{K}(z)^{-p-2}] \leq \mathbb{E} [\mathcal{K}_S(z)^r (\mathcal{K}(z))^{-p-r}] \leq C (\mathbb{E} \mathcal{K}_S(z)^{qr})^{1/q}$$

where in the second step we used Hölder inequality and bounded the negative GMC moment again by a constant. Finally, since $|x-z| \leq 2|Lz|$ on S we get

$$[\mathbb{E} (\mathcal{K}_S(z)^{qr})]^{1/q} \leq C |Lz|^{\frac{\gamma^2}{2}r} [\mathbb{E} [(\int_{B_{L|z|}} |x|^{-\gamma\alpha_1} M_\gamma(d^2x))^{qr}]]^{1/q} \leq C |Lz|^{\gamma(Q-\alpha_1+\frac{\gamma}{2})r-\frac{1}{2}\gamma^2qr^2}$$

where the last estimate was an easy consequence of the annulus bound (9.2) (see also [13, Lemma A.1]). Here we need to assume that $rq < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha_1)$. Notice that since we assume $\frac{\gamma}{2} < \alpha_1$ then $\frac{4}{\gamma^2} > \frac{2}{\gamma}(Q-\alpha_1)$ so that given q we need to have $0 < r < \frac{2}{\gamma q}(Q-\alpha_1)$. The optimal choice for r is $r^* = \frac{\frac{\gamma}{2}+Q-\alpha_1}{\gamma q}$ (this is less than $\frac{2}{\gamma q}(Q-\alpha_1)$ for $\alpha_1 < \frac{2}{\gamma}$), in which case

$$\mathbb{E} [\mathcal{K}_S(z)^{rq}]^{1/q} \leq C |Lz|^{\frac{1}{2q}(\frac{\gamma}{2}+Q-\alpha_1)^2}.$$

Gathering everything we conclude

$$\mathbb{E} \left[\left(\int_S \frac{1}{|x-z|} K(z, x) M_\gamma(d^2x) \right)^2 \mathcal{K}(z)^{-p-2} \right] \leq C L^{\frac{1}{2q}(\frac{\gamma}{2}+Q-\alpha_1)^2} |z|^{-2-2\epsilon+\frac{1}{2q}(\frac{\gamma}{2}+Q-\alpha_1)^2}.$$

We can now fix δ, q, ϵ . First notice that $\frac{1}{2}(\frac{\gamma}{2} + Q - \alpha_1)^2 - \gamma(Q - \alpha_1) = \frac{1}{2}(Q - \alpha_1 - \frac{\gamma}{2})^2 > 0$. Hence choosing q sufficiently close to 1 and then $\epsilon < \epsilon(q)$ and finally $\delta < \delta(\epsilon)$, $I(S, z)$ can be bounded by (6.6).

We are thus left with proving (6.6) for $I(B, z)$ where $B := B(z, |z|^{1+\epsilon})$. An application of the Cameron-Martin theorem gives

$$(6.7) \quad \begin{aligned} I(B, z) &= \int_{B^2} \frac{K(z, x)K(z, x')e^{\gamma^2 G(x, x')}}{(x-z)(x'-z)} \mathbb{E} \left[\left(\int_{\mathbb{C}} K(z, u)e^{\gamma^2 G(x, u) + \gamma^2 G(x', u)} M_\gamma(d^2 u) \right)^{-p-2} \right] d^2 x d^2 x' \\ &\leq C \int_{B^2} \frac{|x-z|^{\frac{\gamma^2}{2}-1} |x'-z|^{\frac{\gamma^2}{2}-1}}{|x|^{\gamma\alpha_1} |x'|^{\gamma\alpha_1} |x-x'|^{\gamma^2}} \mathbb{E} \left[\left(\int_{B_{\frac{1}{2}}} \frac{|u-z|^{\frac{\gamma^2}{2}}}{|u|^{\gamma\alpha_1} |u-x|^{\gamma^2} |u-x'|^{\gamma^2}} M_\gamma(d^2 u) \right)^{-p-2} \right] d^2 x d^2 x' \end{aligned}$$

where for upper bound we restricted the u integral to $B_{\frac{1}{2}}$. By a change of variables $x = zy, x' = zy'$ this becomes

$$(6.8) \quad I(B, z) \leq C |z|^{2-2\gamma\alpha_1} \int_{B(1, z^\epsilon)^2} \frac{|y-1|^{\frac{\gamma^2}{2}-1} |y'-1|^{\frac{\gamma^2}{2}-1}}{|y-y'|^{\gamma^2}} A(y, y', z) d^2 y d^2 y'$$

with

$$A(y, y', z) = \mathbb{E} \left[\left(\int_{B_{\frac{1}{2}}} \frac{|u-z|^{\frac{\gamma^2}{2}}}{|u|^{\gamma\alpha_1} |u-yz|^{\gamma^2} |u-y'z|^{\gamma^2}} M_\gamma(d^2 u) \right)^{-p-2} \right].$$

Note that the only potential divergence in the y, y' integral is at $y = y'$ since $\gamma^2 > 2$. Hence we need to show $A(y, y', z)$ vanishes at diagonal. The behaviour of $A(y, y', z)$ as $y \rightarrow y'$ is controlled by the fusion rules (see [36]). In the case at hand we have four insertions, located at $0, zy, zy', z$ that are all close to each other as $z \rightarrow 0$. Fusion estimates have been proven in [36] in the case of three insertions. A simple adaptation of that proof to the case of 4 insertions is stated in the Appendix, Lemma 9.1. The estimate for $A(y, y', z)$ depends on the relative positions of the four insertions. In our case we have $|zy - z| \vee |zy' - z| \vee |zy - zy'| \ll |z| \wedge |zy| \wedge |zy'|$. This means that the insertions zy, zy', z will merge together way before merging with 0. We will partition the integration region in (6.8) according to the relative positions of these three points or equivalently the relative positions of $y, y', 1$. By symmetry in y, y' we have then three integration regions in (6.8) to consider

- Let $A_1 := \{|y-1| \leq |y'-1| \leq |y-y'|\}$. Then on $B \cap A_1$ we have by Lemma 9.1

$$A(y, y', z) \leq C |1-y'|^{\frac{1}{2}(\frac{3\gamma}{2}-Q)^2} |z|^{\frac{1}{2}(\frac{3\gamma}{2}+\alpha_1-Q)^2}.$$

Since $2 - 2\gamma\alpha_1 + \frac{1}{2}(\frac{3\gamma}{2} + \alpha_1 - Q)^2 = -2 + \gamma(Q - \alpha_1) + \frac{1}{2}(\alpha - \frac{\gamma}{2})^2$ we get

$$(6.9) \quad I(B \cap A_1, z) \leq C |z|^{-2+\gamma(Q-\alpha_1)+\frac{1}{2}(\alpha-\frac{\gamma}{2})^2} \int_{B(1, |z|^\epsilon)^2} \frac{1}{|y-y'|^{2-\frac{1}{2}(\frac{3\gamma}{2}-Q)^2}} d^2 y d^2 y'.$$

The integral is convergent if $(\frac{3\gamma}{2} - Q)^2 > 0$ which is the case if $\gamma^2 \neq 2$.

- Let $A_2 := \{|y-1| \leq |y-y'| \leq |y'-1|\}$. Then on $B_1 \cap A_2$ we have by Lemma 9.1

$$A(y, y', z) \leq C |y-y'|^{\frac{1}{2}(\frac{3\gamma}{2}-Q)^2} |z|^{\frac{1}{2}(\frac{3\gamma}{2}+\alpha_1-Q)^2}.$$

Hence we end up with the bound (6.9) for $I(B \cap A_2)$ as well (since $\frac{\gamma^2}{2} - 1 > 0$).

- Let $A_3 := \{|y-y'| \leq |y-1| \leq |y'-1|\}$. Then on $B_1 \cap A_3$ we have by Lemma 9.1

$$A(y, y', z) \leq C |y-y'|^{\frac{1}{2}(2\gamma-Q)^2} |y-1|^{1-\frac{\gamma^2}{2}} |z|^{\frac{1}{2}(\frac{3\gamma}{2}+\alpha_1-Q)^2}.$$

Hence

$$I(B \cap A_3) \leq C |z|^{-2+\gamma(Q-\alpha_1)+\frac{1}{2}(\alpha-\frac{\gamma}{2})^2} \int_{B(1, |z|^\epsilon)^2} \frac{1}{|y-y'|^{\gamma^2-\frac{1}{2}(2\gamma-Q)^2}} d^2 y d^2 y'.$$

The integral converges since $\gamma^2 - \frac{1}{2}(2\gamma - Q)^2 = 4 - \frac{1}{2}Q^2 < 2$. \square

6.2. Fusion with reflection. In this section we uncover the probabilistic origin of the reflection relation (1.14), (1.15). We prove the following extension of Theorem 6.1 to the case $\alpha_1 + \frac{\gamma}{2} > Q$:

Theorem 6.2. *Let $\sum_i \alpha_i - \frac{\gamma}{2} > Q$ and $\alpha_i < Q$ for all i . There exists $\eta > 0$ s.t. if $Q - \alpha_1 < \eta$ then*

$$(6.10) \quad \mathcal{T}_{-\frac{\gamma}{2}}(z) = C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) |F_-(z)|^2 + R(\alpha_1) C_\gamma(2Q - \alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) |F_+(z)|^2.$$

Proof. We use the notations introduced in the proof of (6.1). We will prove

$$\mathbb{E}[\mathcal{K}(z)^{-p}] - \mathbb{E}[\mathcal{K}(0)^{-p}] = \mu^p \gamma \Gamma(p)^{-1} R(\alpha_1) C_\gamma(2Q - \alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) |z|^{\gamma(Q-\alpha_1)} + o(|z|^{\gamma(Q-\alpha_1)}).$$

Note that since now $\gamma(Q - \alpha_1) < 1$ we need Taylor series only to 0th order.

The leading asymptotics will result from the integral defining \mathcal{K} in a small ball at origin. Let us denote $B := B_{|z|^{1-\xi}}$ with $\xi \in (0, 1)$ to be fixed later. We define

$$(6.11) \quad T_1 := \mathbb{E}[\mathcal{K}_{B^c}(z)^{-p}] - \mathbb{E}[\mathcal{K}(0)^{-p}] \quad \text{and} \quad T_2 := \mathbb{E}[\mathcal{K}(z)^{-p}] - \mathbb{E}[\mathcal{K}_{B^c}(z)^{-p}]$$

so that

$$(6.12) \quad \mathbb{E}[\mathcal{K}(z)^{-p}] - \mathbb{E}[\mathcal{K}(0)^{-p}] = T_1 + T_2.$$

We first show that $T_1 = o(|z|^{\gamma(Q-\alpha_1)})$. Interpolating we get

$$(6.13) \quad \begin{aligned} |T_1| &\leq p \int_0^1 \mathbb{E} \left[|\mathcal{K}_{B^c}(z) - \mathcal{K}(0)| \left(t \mathcal{K}_{B^c}(z) + (1-t) \mathcal{K}(0) \right)^{-p-1} \right] dt \\ &\leq C \mathbb{E} \left[|\mathcal{K}_{B^c}(z) - \mathcal{K}(0)| \left(\mathcal{K}_{B^c}(0) \right)^{-p-1} \right] \end{aligned}$$

where we used $\mathcal{K}_{B^c}(z) \geq C\mathcal{K}(0) \geq \mathcal{K}_{B^c}(0)$ since $|x - z|^{\frac{\gamma}{2}} \geq C|x|^{\frac{\gamma}{2}}$ on B^c . Since $\mathcal{K}(0) = \mathcal{K}_B(0) + \mathcal{K}_{B^c}(0)$ we obtain $|T_1| \leq C(A_1 + A_2)$ where

$$A_1 = \mathbb{E}[\mathcal{K}_B(0) \mathcal{K}_{B^c}(0)^{-p-1}] \quad \text{and} \quad A_2 = \mathbb{E}[|\mathcal{K}_{B^c}(z) - \mathcal{K}_{B^c}(0)| \mathcal{K}_{B^c}(0)^{-p-1}].$$

Using Cameron-Martin theorem, we get for A_1 that

$$A_1 \leq C \int_{|x| \leq |z|^{1-\xi}} |x|^{\frac{\gamma}{2} - \gamma\alpha_1} \mathbb{E} \left[\left(\int_{|u| > |z|^{1-\xi}} K(0, u) |u - x|^{-\gamma^2} M_\gamma(du) \right)^{-p-1} \right] d^2x.$$

Since $|u - x| \leq 2|u|$ we may bound the expectation by

$$(6.14) \quad \mathbb{E} \left[\left(\int_{|u| > |z|^{1-\xi}} |u|^{-\gamma\alpha_1 - \frac{\gamma^2}{2}} M_\gamma(d^2u) \right)^{-p-1} \right] \leq C |z|^{(1-\xi) \frac{(\alpha_1 + \frac{\gamma}{2} - Q)^2}{2}}$$

where we used the GMC estimate (9.3) in the Appendix. We conclude that

$$(6.15) \quad A_1 \leq C |z|^{(1-\xi) \left(\frac{(\alpha_1 + \frac{\gamma}{2} - Q)^2}{2} + \gamma(Q - \alpha_1) \right)}.$$

Hence $A_1 = o(|z|^{\gamma(Q-\alpha_1)})$ if e.g. $\xi < \frac{1}{2}$ and η is small enough.

Next we bound A_2 . Let A be the annulus $A := \{x \in \mathbb{C}; |z|^{1-\xi} \leq |x| \leq 1/2\}$. We can split the numerator in A_2 into $|\mathcal{K}_{B_{1/2}^c}(z) - \mathcal{K}_{B_{1/2}^c}(0)|$ and $|\mathcal{K}_A(z) - \mathcal{K}_A(0)|$ by means of the triangular inequality. On $B_{1/2}^c$ we can use $\| |x - z|^{\frac{\gamma}{2}} - |x|^{\frac{\gamma}{2}} \| \leq C|x|^{\frac{\gamma}{2}}|z|$ to get

$$\mathbb{E}[|\mathcal{K}_{B_{1/2}^c}(z) - \mathcal{K}_{B_{1/2}^c}(0)| \mathcal{K}_{B^c}(0)^{-p-1}] \leq C |z| \mathbb{E}[\mathcal{K}_{B_{1/2}^c}(0) \mathcal{K}_{B^c}(0)^{-p-1}] \leq C |z| \mathbb{E}[\mathcal{K}_{B^c}(0)^{-p}] \leq C |z|.$$

Finally, using $\| |x - z|^{\frac{\gamma}{2}} - |x|^{\frac{\gamma}{2}} \| \leq C|x|^{\frac{\gamma}{2}-1}|z|$ on A and then applying Cameron-Martin, we get

$$\mathbb{E}[|\mathcal{K}_A(z) - \mathcal{K}_A(0)| \mathcal{K}_{B^c}(0)^{-p-1}] \leq C |z| \int_A |x|^{\frac{\gamma}{2}-1-\gamma\alpha_1} \mathbb{E} \left[\left(\int_{|u| > |z|^{1-\xi}} K(0, u) |u - x|^{-\gamma^2} M_\gamma(d^2u) \right)^{-p-1} \right] d^2x.$$

Since $|z|^{1-\xi} \leq |x|$ we can bound

$$\int_{|u| > |z|^{1-\xi}} K(0, u) |u - x|^{-\gamma^2} M_\gamma(d^2u) \geq C \int_{|u| > |x|} |u|^{-\gamma\alpha_1 - \frac{\gamma^2}{2}} M_\gamma(d^2u)$$

and then the GMC estimate (9.3) in the Appendix gives

$$(6.16) \quad \mathbb{E}[|\mathcal{K}_A(z) - \mathcal{K}_A(0)|\mathcal{K}_{B^c}(0)^{-p-1}] \leq C|z| \int_A |x|^{\frac{\gamma^2}{2}-1-\gamma\alpha_1+\frac{1}{2}(\alpha_1+\frac{\gamma}{2}-Q)^2} d^2x$$

$$(6.17) \quad \leq C|z|^{\xi+(1-\xi)(\gamma(Q-\alpha_1)+\frac{1}{2}(\alpha_1-\frac{\gamma}{2}))^2} = o(|z|^{\gamma(Q-\alpha_1)})$$

for $\xi < \frac{1}{2}$ and η small enough (since $\alpha_1 - \frac{\gamma}{2} > \frac{\gamma}{2} - \eta$). Hence $T_1 = o(|z|^{\gamma(Q-\alpha_1)})$.

Now we focus on T_2 . First we show that it suffices to restrict \mathcal{K} to the complement of the annulus $A_h := \{x \in \mathbb{C}; e^{-h}|z| \leq |x| \leq |z|^{1-\xi}\}$ where $h > 0$ is fixed: it will serve as a buffer zone to decorrelate the regions $\{x \in \mathbb{C}; |x| \leq e^{-h}|z|\}$ and $\{x \in \mathbb{C}; |x| > |z|^{1-\xi}\}$. Interpolating as in (7.11) we deduce

$$|\mathbb{E}[\mathcal{K}(z)^{-p} - \mathcal{K}_{A_h^c}(z)^{-p}]| \leq \mathbb{E}[\mathcal{K}_{A_h}(z)\mathcal{K}_{B^c}(0)^{-p-1}].$$

Using the Cameron-Martin theorem we get

$$(6.18) \quad |\mathbb{E}[\mathcal{K}(z)^{-p} - \mathcal{K}_{A_h^c}(z)^{-p}]| \leq C \int_{A_h} |x - z|^{\frac{\gamma^2}{2}} |x|^{-\gamma\alpha_1} \mathbb{E}\left[\left(\int_{B^c} K(0, u) |u - x|^{-\gamma^2} M_\gamma(du)\right)^{-p-1}\right] d^2x.$$

The expectation was estimated in (6.14) so that we get

$$(6.19) \quad |\mathbb{E}[\mathcal{K}(z)^{-p} - \mathcal{K}_{A_h^c}(z)^{-p}]| \leq C|z|^{(1-\xi)(\gamma(Q-\alpha_1)+\frac{1}{2}(Q-\alpha_1-\frac{\gamma}{2}))^2}.$$

For $\xi < \frac{1}{2}$ and η small this yields $|\mathbb{E}[\mathcal{K}(z)^{-p} - \mathcal{K}_{A_h^c}(z)^{-p}]| = o(|z|^{\gamma(Q-\alpha_1)})$.

Therefore, we just need to evaluate the quantity $\mathbb{E}[\mathcal{K}_{A_h^c}(z)^{-p}] - \mathbb{E}[\mathcal{K}_{B^c}(z)^{-p}]$ where we recall the definitions $B^c = \{|x| \geq |z|^{1-\xi}\}$ and $A_h^c = B^c \cup B_{e^{-h}|z|}$. Hence $\mathcal{K}_{A_h^c}(z) = \mathcal{K}_{B^c}(z) + \mathcal{K}_{B_{e^{-h}|z|}}(z)$. We use the polar decomposition of the chaos measure introduced in Section 2.7. Let $|z| = e^{-t}$. Then

$$\begin{aligned} \mathcal{K}_{B^c}(z) &= \int_0^{2\pi} \int_{-\infty}^{(1-\xi)t} e^{\gamma(B_s - (Q-\alpha_1)s)} \frac{|e^{-s+i\theta} - z|^{\frac{\gamma^2}{2}}}{|1 - e^{-s+i\theta}|^{\gamma\alpha_2}} (e^{-s(\gamma(\alpha_1+\alpha_a+\alpha_3-\frac{\gamma}{2})) \vee 1}) N_\gamma(ds d\theta) := \mathcal{K}^1 \\ \mathcal{K}_{B_{e^{-h}|z|}}(z) &= \int_0^{2\pi} \int_{t+h}^{\infty} e^{\gamma(B_s - (Q-\alpha_1)s)} \frac{|e^{-s+i\theta} - z|^{\frac{\gamma^2}{2}}}{|1 - e^{-s+i\theta}|^{\gamma\alpha_2}} N_\gamma(ds d\theta) := \mathcal{K}^2. \end{aligned}$$

The lateral noises in \mathcal{K}^1 and \mathcal{K}^2 are weakly correlated. Indeed, from (2.43) we get

$$(6.20) \quad -e^{-\xi t} \leq E[Y(s, \theta)Y(s', \theta')] \leq 2e^{-\xi t}.$$

for all $s < (1-\xi)t, s' > t+h$ and $\theta, \theta' \in [0, 2\pi]$. Define then the process

$$P(s, \theta) := Y(s, \theta)\mathbf{1}_{\{s < (1-\xi)t\}} + Y(s, \theta)\mathbf{1}_{\{s > t+h\}}.$$

Let \tilde{Y} be independent of everything with the same law as Y and define the process

$$\tilde{P}(s, \theta) := Y(s, \theta)\mathbf{1}_{\{s < (1-\xi)t\}} + \tilde{Y}(s, \theta)\mathbf{1}_{\{s > t+h\}}.$$

Then we get

$$\mathbb{E}[\tilde{P}(s, \theta)\tilde{P}(s', \theta')] - e^{-\xi t} \leq \mathbb{E}[P(s, \theta)P(s', \theta')] \leq \mathbb{E}[\tilde{P}(s, \theta)\tilde{P}(s', \theta')] + 2e^{-\xi t}.$$

Let N be a unit normal variable independent of everything. Then the above means that the covariance of $\tilde{P} + e^{-\xi t}N$ dominates that of P and the covariance of $P + \sqrt{2}e^{-\frac{1}{2}\xi t}N$ dominates that of \tilde{P} . Since $\mathcal{K}|_{P+aN} = e^{\gamma aN - \frac{1}{2}\gamma^2 a^2} \mathcal{K}|_P$ we get by Kahane's convexity inequality (see [49, Theorem 2.1]) with the convex function $x \in \mathbb{R}_+ \mapsto x^{-p}$

$$e^{-C|z|^{2\xi}} \mathbb{E}[(\mathcal{K}^1 + \tilde{\mathcal{K}}^2)^{-p}] \leq \mathbb{E}[(\mathcal{K}^1 + \mathcal{K}^2)^{-p}] \leq e^{C|z|^\xi} \mathbb{E}[(\mathcal{K}^1 + \tilde{\mathcal{K}}^2)^{-p}]$$

where $\tilde{\mathcal{K}}^2$ is computed with \tilde{Y} . Let

$$(6.21) \quad \epsilon := e^{\gamma B_{t+h} - \gamma(Q-\alpha_1)(t+h) - \frac{\gamma^2}{2}t}.$$

Then by the Markov property of Brownian motion

$$(6.22) \quad \tilde{\mathcal{K}}^2 = \epsilon \int_0^{2\pi} \int_0^\infty e^{\gamma(\tilde{B}_s - (Q - \alpha_1)s)} \frac{|e^{-s-h+i\theta} - \frac{z}{|z|}|^{\frac{\gamma^2}{2}}}{|1 - |z|e^{-s-h+i\theta}|^{\gamma\alpha_2}} \tilde{N}_\gamma(d(h+t+s), d\theta)$$

where \tilde{B} is a Brownian motion independent of everything. Moreover, by stationarity of \tilde{Y} and its independence of everything we may replace $N_\gamma(d(h+t+s), d\theta)$ by $N_\gamma(ds, d\theta)$. As a consequence

$$(6.23) \quad \mathbb{E}[(\mathcal{K}^1 + \epsilon c_- \mathcal{K}^3)^{-p}] \leq \mathbb{E}[(\mathcal{K}^1 + \tilde{\mathcal{K}}^2)^{-p}] \leq \mathbb{E}[(\mathcal{K}^1 + \epsilon c_+ \mathcal{K}^3)^{-p}]$$

where

$$\mathcal{K}^3 = \int_0^\infty e^{\gamma(\tilde{B}_s - (Q - \alpha_1)s)} \tilde{Z}_s ds$$

and

$$(6.24) \quad c_\pm := \frac{(1 \pm e^{-h})^{\frac{\gamma^2}{2}}}{(1 \mp |z|e^{-h})^{\gamma\alpha_2}}.$$

By the Williams path decomposition Lemma 2.6 and (2.48) we deduce

$$(6.25) \quad \mathcal{K}^3 \stackrel{\text{law}}{=} e^{\gamma M} \int_{-L-M}^\infty e^{\gamma \mathcal{B}_s^{\alpha_1}} \tilde{Z}_s ds$$

where we recall $M = \sup_s(\tilde{B}_s - (Q - \alpha_1)s)$ and L_{-M} is the last time $\mathcal{B}_s^{\alpha_1}$ hits $-M$. We discuss the lower and upper bounds in (6.23) in turn.

Lower bound. Let us use the notation $J_B = \int_B e^{\gamma \mathcal{B}_s^{\alpha_1}} \tilde{Z}_s ds$ and J for $J_{\mathbb{R}}$. We have

$$\mathbb{E}[(\mathcal{K}^1 + \epsilon c_- \mathcal{K}^3)^{-p}] \geq \mathbb{E}[(\mathcal{K}^1 + \epsilon c_- e^{\gamma M} J)^{-p}]$$

Now we can use the standard fact that M has exponential law with parameter $2(Q - \alpha_1)$ to get

$$\begin{aligned} & \mathbb{E}[(\mathcal{K}^1 + \epsilon c_- \mathcal{K}^3)^{-p}] - \mathbb{E}[(\mathcal{K}^1)^{-p}] \geq \frac{2(Q - \alpha_1)}{\gamma} \int_1^\infty \mathbb{E}\left[(\mathcal{K}^1 + \epsilon c_- v J)^{-p} - (\mathcal{K}^1)^{-p}\right] v^{-1 - \frac{2}{\gamma}(Q - \alpha_1)} dv \\ & \geq \frac{2(Q - \alpha_1)}{\gamma} c_-^{\frac{2}{\gamma}(Q - \alpha_1)} \mathbb{E}\left[(\epsilon J)^{\frac{2}{\gamma}(Q - \alpha_1)} (\mathcal{K}^1)^{-p - \frac{2}{\gamma}(Q - \alpha_1)}\right] \int_0^\infty \left((1 + w)^{-p} - 1\right) w^{-1 - \frac{2}{\gamma}(Q - \alpha_1)} dw \\ & = \frac{2(Q - \alpha_1)}{\gamma} c_-^{\frac{2}{\gamma}(Q - \alpha_1)} \frac{\Gamma(-\frac{2}{\gamma}(Q - \alpha_1)) \Gamma(p + \frac{2}{\gamma}(Q - \alpha_1))}{\Gamma(p)} \mathbb{E}[J^{\frac{2}{\gamma}(Q - \alpha_1)}] \mathbb{E}\left[\epsilon^{\frac{2}{\gamma}(Q - \alpha_1)} (\mathcal{K}^1)^{-p - \frac{2}{\gamma}(Q - \alpha_1)}\right] \end{aligned}$$

where in the second step we made a change of variables $w = \frac{\epsilon c_- J}{\mathcal{K}^1} v$ and for lower bound took the integration over $w \geq 0$. In the last step we used Lemma 9.3 in the appendix to compute the integral and independence of J from everything. We end up with

$$\mathbb{E}[(\mathcal{K}^1 + \epsilon c_- \mathcal{K}^3)^{-p}] - \mathbb{E}[(\mathcal{K}^1)^{-p}] \geq W c_-^{\frac{2}{\gamma}(Q - \alpha_1)} \mathbb{E}\left[\epsilon^{\frac{2}{\gamma}(Q - \alpha_1)} (\mathcal{K}^1)^{-p - \frac{2}{\gamma}(Q - \alpha_1)}\right]$$

where we have have set

$$W := \mu^{-\frac{2}{\gamma}(Q - \alpha_1)} R(\alpha_1) \frac{\Gamma(p + \frac{2}{\gamma}(Q - \alpha_1))}{\Gamma(p)}$$

and $R(\alpha_1)$ is the reflection coefficient defined in (2.53). The remaining expectation can be computed thanks to the Cameron-Martin theorem applied to the term $\epsilon^{\frac{2}{\gamma}(Q - \alpha_1)}$. Since $t + h > (1 - \xi)t$ we get

$$\mathbb{E}\left[\epsilon^{\frac{2}{\gamma}(Q - \alpha_1)} (\mathcal{K}^1)^{-p - \frac{2}{\gamma}(Q - \alpha_1)}\right] = |z|^{\gamma(Q - \alpha_1)} \mathbb{E}\left[\hat{\mathcal{K}}_{B^c}(z)^{-p - \frac{2}{\gamma}(Q - \alpha_1)}\right]$$

where we defined for $D \subset \mathbb{C}$

$$(6.26) \quad \hat{\mathcal{K}}_D(z) := \int_D \frac{|x - z|^{\frac{\gamma^2}{2}}}{|x|^{\gamma(2Q - \alpha_1)} |x - 1|^{\gamma\alpha_2}} |x|_+^{\gamma(2Q - \alpha_1 - \frac{\gamma}{2} + \alpha_2 + \alpha_3)} M_\gamma(d^2x)$$

and in the case $D = \mathbb{C}$ we will write $\widehat{\mathcal{K}}(z)$ for $\widehat{\mathcal{K}}_{\mathbb{C}}(z)$. Next, we claim

$$(6.27) \quad \mathbb{E}[\widehat{\mathcal{K}}_{B^c}(z)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] - \mathbb{E}[\widehat{\mathcal{K}}_{B^c}(0)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] = o(|z|^{\gamma(Q-\alpha_1)}).$$

Indeed, the LHS is just T_1 in (6.11) computed with a larger p and $|x|^{\gamma\alpha_1}$ replaced by $|x|^{\gamma(2Q-\alpha_1)}$. It is readily checked from (6.15) and (6.16) that (6.27) holds.

Summarizing all of our considerations related to the lower bound, we have shown that

$$\begin{aligned} & \mathbb{E}[(\mathcal{K}^1)^{-p}] - \mathbb{E}[(\mathcal{K}^2)^{-p}] \\ & \geq e^{-C|z|^\xi} \left(|z|^{\gamma(Q-\alpha_1)} c_+^{\frac{2}{\gamma}(Q-\alpha_1)} W \mathbb{E}[\widehat{\mathcal{K}}(0)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] + o(|z|^{\gamma(Q-\alpha_1)}) \right) - (1 - e^{-C|z|^\xi}) \mathbb{E}[\mathcal{K}_{B^c}(z)^{-p}]. \end{aligned}$$

The second term on the RHS is $\mathcal{O}(|z|^\xi) = o(|z|^{\gamma(Q-\alpha_1)})$ provided we take $\xi > \gamma(Q-\alpha_1)$ (this is the condition that fixes ξ) so that recalling (6.24), we deduce

$$\liminf_{z \rightarrow 0} |z|^{-\gamma(Q-\alpha_1)} \left(\mathbb{E}[\mathcal{K}_{B_{e^{-h}|z|}}(z)^{-p} - \mathcal{K}_{B^c}(z)^{-p}] \right) \geq (1 + e^{-h})^{\gamma(Q-\alpha_1)} W \mathbb{E}[\widehat{\mathcal{K}}(0)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}].$$

Since h is arbitrary, it can be chosen arbitrarily large so as to get

$$\begin{aligned} \liminf_{z \rightarrow 0} |z|^{-\gamma(Q-\alpha_1)} \left(\mathbb{E}[\mathcal{K}_{B_{e^{-h}|z|}}(z)^{-p} - \mathcal{K}_{B^c}(z)^{-p}] \right) & \geq W \mathbb{E}[\widehat{\mathcal{K}}(0)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] \\ & = \mu^p \gamma \Gamma(p)^{-1} R(\alpha_1) |C_\gamma(2Q - \alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) \end{aligned}$$

which is the desired lower bound.

Upper bound: For the upper bound we go back to the formula (6.25) where we need to face the integration region lower value L_{-M} . For $A > 0$ fixed, we consider first the quantity

$$L(z) := \mathbb{E} \left[\left(\mathcal{K}^1 + \epsilon c_+ e^{\gamma M} \int_{-L_{-M}}^{\infty} e^{\gamma \mathcal{B}_s^{\alpha_1}} \widetilde{Z}_s(ds) \right)^{-p} - (\mathcal{K}^1)^{-p} \right] \mathbf{1}_{\{M \leq A\}}$$

and we want to show that $L(z) = o(|z|^{\gamma(Q-\alpha_1)})$.

Indeed, interpolating and using Cameron-Martin for the ϵ we get

$$\begin{aligned} |L(z)| & \leq c_+ \mathbb{E} \left[\epsilon e^{\gamma M} \int_{-L_{-M}}^{\infty} e^{\gamma \mathcal{B}_s^{\alpha_1}} \widetilde{Z}_s ds \mathcal{K}_{B^c}(z)^{-p-1} \mathbf{1}_{\{M \leq A\}} \right] \leq C e^{\gamma A} \mathbb{E} \left[\int_{\mathbb{R}} e^{\gamma \mathcal{B}_s^{\alpha_1}} \widetilde{Z}_s ds \right] \mathbb{E} \left[\epsilon \mathcal{K}_{B^c \cap B(0, \frac{1}{2})}(z)^{-p-1} \right] \\ & \leq C |z|^{\gamma(Q-\alpha_1)} e^{\gamma A} \mathbb{E} \left[\int_{\mathbb{R}} e^{\gamma \mathcal{B}_s^{\alpha_1}} \widetilde{Z}_s ds \right] \mathbb{E} \left[\left(\int_{|z|^{1-\epsilon} \leq |z| \leq \frac{1}{2}} |x|^{-\gamma\alpha_1 - \frac{\gamma}{2}} M_\gamma(d^2x) \right)^{-p-1} \right] \\ & \leq C e^{\gamma A} |z|^{\gamma(Q-\alpha_1)} |z|^{\frac{1-\epsilon}{2}(Q-\alpha_1 - \frac{\gamma}{2})^2} \end{aligned}$$

where we used the GMC estimate (9.3) and Lemma 2.8

It remains to investigate the quantity

$$\begin{aligned} U(z) & := \mathbb{E} \left[\left(\mathcal{K}^1 + \epsilon c_+ e^{\gamma M} \int_{-L_{-M}}^{\infty} e^{\gamma \mathcal{B}_s^{\alpha_1}} \widetilde{Z}_s(ds) \right)^{-p} - (\mathcal{K}^1)^{-p} \right] \mathbf{1}_{\{M \geq A\}} \\ & \leq \mathbb{E} \left[\left(\mathcal{K}^1 + \epsilon c_+ e^{\gamma M} J(A) \right)^{-p} - (\mathcal{K}^1)^{-p} \right] \mathbf{1}_{\{M > A\}} \end{aligned}$$

where we set $J(A) = \int_{-L_{-A}}^{\infty} e^{\gamma \mathcal{B}_s^{\alpha_1}} \widetilde{Z}_s ds$. Using again the law of M , which is exponential with parameter $2(Q-\alpha_1)$, and making the change change of variables $\frac{\epsilon c_+ J_A}{\mathcal{K}^1} e^{\gamma v} = y$ we get

$$\begin{aligned} U(z) & \leq \frac{2(Q-\alpha_1)}{\gamma} \mathbb{E} \left[\int_A^{\infty} \left(\left(\mathcal{K}^1 + \epsilon c_+ e^{\gamma v} J_A \right)^{-p} - (\mathcal{K}^1)^{-p} \right) e^{-2(Q-\alpha_1)v} dv \right] \\ & = \frac{2(Q-\alpha_1)}{\gamma} c_+^{\frac{2}{\gamma}(Q-\alpha_1)} \mathbb{E} \left[J_A^{\frac{2}{\gamma}(Q-\alpha_1)} \epsilon^{\frac{2}{\gamma}(Q-\alpha_1)} \int_{e^{\gamma A} \frac{\epsilon c_+ J_A}{\mathcal{K}^1}}^{\infty} \left((1+y)^{-p} - 1 \right) (\mathcal{K}^1)^{-p-\frac{2}{\gamma}(Q-\alpha_1)} y^{-\frac{2}{\gamma}(Q-\alpha_1)-1} dy \right]. \end{aligned}$$

Now we can use Cameron-Martin as in the case of the lower bound to get that the above expectation can be rewritten as (recall (6.26))

$$\begin{aligned} & \mathbb{E} \left[J_A^{\frac{2}{\gamma}(Q-\alpha_1)} \epsilon^{\frac{2}{\gamma}(Q-\alpha_1)} \int_{e^{\gamma A} \frac{\epsilon c + J_A}{\kappa^1}}^{\infty} \left((1+y)^{-p} - 1 \right) (\kappa^1)^{-p - \frac{2}{\gamma}(Q-\alpha_1)} y^{-\frac{2}{\gamma}(Q-\alpha_1) - 1} dy \right] \\ &= |z|^{\gamma(Q-\alpha_1)} \mathbb{E} \left[J_A^{\frac{2}{\gamma}(Q-\alpha_1)} \int_{e^{\gamma A} \frac{\epsilon(|z|e^{-h})^{-2\gamma(Q-\alpha_1)c + J_A}}{\widehat{\kappa}_{B^c}(z)}}^{\infty} \left((1+y)^{-p} - 1 \right) \widehat{\mathcal{K}}_{B^c}(z)^{-p - \frac{2}{\gamma}(Q-\alpha_1)} y^{-\frac{2}{\gamma}(Q-\alpha_1) - 1} dy \right]. \end{aligned}$$

Recalling (6.21) we have $\epsilon(|z|e^{-h})^{-2\gamma(Q-\alpha_1)} = e^{\gamma B_{t+h} + \gamma(Q-\alpha_1)(t+h) - \frac{\gamma^2}{2}t}$ and thus $\epsilon(|z|e^{-h})^{-2\gamma(Q-\alpha_1)} \rightarrow 0$ almost surely as $z \rightarrow 0$ provided $\alpha_1 + \frac{\gamma}{2} > Q$ which is the case. Dominated convergence theorem then ensures that the latter expectation converges to

$$\mathbb{E} [J_A^{\frac{2}{\gamma}(Q-\alpha_1)}] \mathbb{E} [\widehat{\mathcal{K}}_{B^c}(z)^{-p - \frac{2}{\gamma}(Q-\alpha_1)}] \int_0^{\infty} \left((1+y)^{-p} - 1 \right) y^{-\frac{2}{\gamma}(Q-\alpha_1) - 1} dy.$$

We can then conclude as for the lower bound by letting $h, A \rightarrow \infty$. \square

6.3. The 4 point function with $-\frac{2}{\gamma}$ insertion. In this section, we prove an analogue of Theorem 6.2 for the other degenerate insertion with weight $-\frac{2}{\gamma}$:

Theorem 6.3. *We assume the Seiberg bounds for $(-\frac{2}{\gamma}, \alpha_1, \alpha_2, \alpha_3)$, i.e. $\sum_{k=1}^3 \alpha_k > 2Q + \frac{2}{\gamma}$ and $\alpha_k < Q$ for all k . There exists $\eta > 0$ such that for all $\alpha_1, \alpha_2, \alpha_3 \in (Q - \eta, Q)$*

$$(6.28) \quad \mathcal{T}_{-\frac{2}{\gamma}}(z) = C_{\gamma}(\alpha_1 - \frac{2}{\gamma}, \alpha_2, \alpha_3) |F_{-}(z)|^2 + R(\alpha_1) C_{\gamma}(2Q - \alpha_1 - \frac{2}{\gamma}, \alpha_2, \alpha_3) |F_{+}(z)|^2.$$

Proof. The proof follows the proof of Theorem 6.2 almost word by word and we keep the same notation with the following obvious modifications. The function K in (6.29) is replaced by

$$(6.29) \quad K(z, x) = \frac{|x - z|^2 |x|_+^{\gamma(\sum_{k=1}^3 \alpha_k - \frac{2}{\gamma})}}{|x|^{\gamma \alpha_1} |x - 1|^{\gamma \alpha_2}}$$

i.e. most important, the factor $|x - z|^{\frac{\gamma^2}{2}}$ is replaced by $|x - z|^2$. Furthermore the exponent p is now given by $p = (\alpha_1 + \alpha_2 + \alpha_3 - \frac{2}{\gamma} - 2Q)/\gamma$ and it is positive.

We will fix $\eta > 0$ and $\xi > 0$ so that the following conditions hold for all $\alpha_1 \in (Q - \eta, Q)$

$$(6.30) \quad \frac{4}{\gamma}(Q - \alpha_1) < (1 - \xi)(4 - \gamma \alpha_1 - 2\gamma\eta)$$

$$(6.31) \quad \frac{4}{\gamma}(Q - \alpha_1) < \xi.$$

Note that for $\xi = \eta = 0$ (6.30) holds since $4 - \gamma Q = 2 - \frac{\gamma^2}{2} > 0$ and therefore by continuity for small enough η and small enough $\xi > \frac{4}{\gamma}\eta$ they hold as well.

As in the proof of Theorem 6.2 we start with the splitting (6.12) to T_1 and T_2 given by (6.11) and we first show that $T_1 = o(|z|^{\frac{4}{\gamma}(Q-\alpha_1)})$. We obtain again $|T_1| \leq C(A_1 + A_2)$ with the same definitions for A_i .

The Cameron-Martin bound for A_1 becomes

$$A_1 \leq C \int_{|x| \leq |z|^{1-\xi}} |x|^{2-\gamma \alpha_1} \mathbb{E} \left[\left(\int_{|u| > |z|^{1-\xi}} K(0, u) |u - x|^{-\gamma^2} M_{\gamma}(du) \right)^{-p-1} \right] d^2x$$

and as the expectation is bounded by a constant we conclude that

$$(6.32) \quad A_1 \leq C |z|^{(1-\xi)(4-\gamma \alpha_1)} = o(|z|^{\frac{4}{\gamma}(Q-\alpha_1)})$$

by (6.30).

Next, for A_2 the bound (6.16) is replaced by

$$(6.33) \quad \mathbb{E} [|\mathcal{K}_A(z) - \mathcal{K}_A(0)| \mathcal{K}_{B^c}(0)^{-p-1}] \leq C |z| \int_A |x|^{1-\gamma \alpha_1} d^2x \leq C |z|^{1+(1-\xi)(3-\gamma \alpha_1)} = o(|z|^{\frac{4}{\gamma}(Q-\alpha_1)})$$

again by (6.30). Hence $T_1 = o(|z|^{\frac{4}{\gamma}(Q-\alpha_1)})$.

Now we proceed with T_2 , again with the obvious changes (e.g. $\frac{\gamma^2}{2}$ in the definitions for $\mathcal{K}^1, \mathcal{K}^2$ and c_{\pm} replaced by 2). Hence replacing (6.26) by

$$\widehat{\mathcal{K}}_D(z) := \int_D \frac{|x-z|^2}{|x|^{\gamma(2Q-\alpha_1)}|x-1|^{\gamma\alpha_2}} |x|_+^{\gamma(2Q-\alpha_1-\frac{2}{\gamma}+\alpha_2+\alpha_3)} M_{\gamma}(d^2x)$$

we obtain instead of (6.27) the bound

$$(6.34) \quad \mathbb{E}[\widehat{\mathcal{K}}_{B^c}(z)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] - \mathbb{E}[\widehat{\mathcal{K}}_{B^c}(0)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] = o(|z|^{\frac{4}{\gamma}(Q-\alpha_1)}).$$

Indeed, the LHS is T_1 computed with a larger p and $|x|^{\gamma\alpha_1}$ replaced by $|x|^{\gamma(2Q-\alpha_1)}$. Hence from (6.32) and (6.33) we get the bound

$$\mathbb{E}[\widehat{\mathcal{K}}_{B^c}(z)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] - \mathbb{E}[\widehat{\mathcal{K}}_{B^c}(0)^{-p-\frac{2}{\gamma}(Q-\alpha_1)}] \leq C|z|^{(1-\xi)(4-\gamma(2Q-\alpha_1))}$$

Since $4-\gamma(2Q-\alpha_1) = 4-\gamma\alpha_1-2\gamma(Q-\alpha_1) \leq 4-\gamma\alpha_1-2\gamma\eta$ and so (6.34) holds. The rest of the arguments for the lower and the upper bounds for T_2 follow then word by word. \square

6.4. Crossing relations. Proposition 2.4 now follows from Theorem 6.1 as explained in Section 2.5. Let us state it in the form we will apply it and also for the unit volume structure constants:

Proposition 6.4. *Let $\epsilon \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ and $\alpha, \alpha' < Q$ s.t. $\alpha + \alpha' + \epsilon - \frac{\gamma}{2} > 2Q$. Then*

$$(6.35) \quad C_{\gamma}(\alpha' - \frac{\gamma}{2}, \epsilon, \alpha) = T(\alpha', \epsilon, \alpha) C_{\gamma}(\alpha', \epsilon + \frac{\gamma}{2}, \alpha)$$

where T is the given by the following formula

$$(6.36) \quad T(\alpha', \epsilon, \alpha) = -\mu\pi \frac{l(a)l(b)}{l(c)l(a+b-c)} \frac{1}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\epsilon}{2})l(2+\frac{\gamma^2}{4}-\frac{\gamma\epsilon}{2})}$$

where

$$(6.37) \quad a = \frac{\gamma}{2}(\frac{\alpha'}{2} - \frac{Q}{2}) + \frac{\gamma}{2}(\frac{\epsilon}{2} + \frac{\alpha}{2} - \frac{\gamma}{2}) - \frac{1}{2} \quad b = \frac{\gamma}{2}(\frac{\alpha'}{2} - \frac{Q}{2}) + \frac{\gamma}{2}(\frac{\epsilon}{2} - \frac{\alpha}{2}) + \frac{1}{2}$$

and

$$(6.38) \quad c = 1 + \frac{\gamma}{2}(\alpha' - Q).$$

where a, b, c are given by

$$(6.39) \quad a = \frac{\gamma}{4}(\alpha' + \alpha + \epsilon - Q - \frac{\gamma}{2}) - \frac{1}{2} \quad b = \frac{\gamma}{4}(\alpha' - \alpha + \epsilon - Q) + \frac{1}{2} \quad c = 1 - \frac{\gamma}{2}(Q - \alpha').$$

The above relation can be rewritten under the following form for the unit volume correlations (see (2.18) for the definition)

$$(6.40) \quad \bar{C}_{\gamma}(\alpha' - \frac{\gamma}{2}, \epsilon, \alpha) = \bar{T}(\alpha', \epsilon, \alpha) \bar{C}_{\gamma}(\alpha', \epsilon + \frac{\gamma}{2}, \alpha)$$

where \bar{T} is given by

$$(6.41) \quad \bar{T}(\alpha', \epsilon, \alpha) = \frac{\Gamma(\frac{1}{\gamma}(\alpha + \alpha' + \epsilon + \frac{\gamma}{2} - 2Q))}{\Gamma(\frac{1}{\gamma}(\alpha + \alpha' + \epsilon - \frac{\gamma}{2} - 2Q))} T(\alpha', \epsilon, \alpha).$$

Along the same lines as Proposition 6.4, by exploiting Theorem 6.3 with the $-\frac{2}{\gamma}$ insertion, one can show the following crossing symmetry relations:

Proposition 6.5. *Let when α, ϵ and α' close but strictly less than Q with $\alpha + \alpha' + \epsilon > 2Q + \frac{2}{\gamma}$. Then*

$$(6.42) \quad C_{\gamma}(\alpha - \frac{2}{\gamma}, \epsilon, \alpha') = \tilde{T}(\alpha, \epsilon, \alpha') R(\epsilon) C_{\gamma}(\alpha, 2Q - \epsilon - \frac{2}{\gamma}, \alpha')$$

where \tilde{T} is the given by the following formula

$$(6.43) \quad \tilde{T}(\alpha, \epsilon, \alpha') = \frac{l(a)l(b)}{l(c)l(a+b-c)}$$

where

$$(6.44) \quad a = \frac{2}{\gamma} \left(\frac{\alpha}{2} - \frac{Q}{2} \right) + \frac{2}{\gamma} \left(\frac{\epsilon}{2} + \frac{\alpha'}{2} - \frac{2}{\gamma} \right) - \frac{1}{2} \quad b = \frac{2}{\gamma} \left(\frac{\alpha}{2} - \frac{Q}{2} \right) + \frac{2}{\gamma} \left(\frac{\epsilon}{2} - \frac{\alpha'}{2} \right) + \frac{1}{2}$$

and

$$(6.45) \quad c = 1 + \frac{2}{\gamma} (\alpha - Q).$$

7. PROOF OF FORMULA (2.54) ON THE REFLECTION COEFFICIENT

We will suppose that $\gamma^2 \notin \mathbb{Q}$. This is no restriction since the general case can be deduced from this case by continuity in γ (Remark 2.3). The proof of formula (2.54) for the reflection coefficient is made of several steps and relies on proving that R satisfies the same shift equations (9.5) and (9.6) as R^{DOZZ} . For the benefit of the reader we give here the summary of the structure of the argument (recall (2.53) and (2.52)):

Subsection 7.1: We prove that \bar{R} is analytic in the interval $(\frac{\gamma}{2}, Q)$. More specifically, we show that for all compact intervals $I \subset (\frac{\gamma}{2}, Q)$ there exists $\beta > 0$ such that \bar{R} can be extended to a holomorphic function in $I \times (-\beta, \beta)$.

Subsection 7.2: We prove that R satisfies the following shift equation for $\alpha \in (\gamma, Q)$

$$(7.1) \quad R(\alpha - \frac{\gamma}{2}) = -\mu\pi \frac{R(\alpha)}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{4})l(2 + \frac{\gamma^2}{2} - \frac{\gamma\alpha}{2})}$$

In fact, we will show this relation in a small interval of the form $(Q - \eta, Q)$ with $\eta > 0$. By analyticity, this leads to the same relation for $\alpha \in (\gamma, Q)$. One can choose an interval $I \subset (\frac{\gamma}{2}, Q)$ of length greater than $\frac{\gamma}{2}$ such that \bar{R} can be extended to a holomorphic function in $I \times (-\beta, \beta)$. The functional relation (7.1) enables to extend R to $\mathbb{R} \times (-\beta, \beta)$; the extension that we also denote R is meromorphic with simple poles on the real line located at $\{\frac{2}{\gamma} - \frac{\gamma}{2}\mathbb{N}\} \cup \{\frac{\gamma}{2} - \frac{2}{\gamma}\mathbb{N}\}$.

Subsection 7.4: We prove that R satisfies the following shift equation (inversion relation)

$$(7.2) \quad R(\alpha)R(2Q - \alpha) = 1$$

One key ingredient is the analytic continuation result of subsection 7.3, the so-called gluing lemma, i.e. lemma 7.3. In fact, we will prove this relation for α in a small interval of the form $(Q - \eta, Q)$ with $\eta > 0$. By analyticity, the condition will extend everywhere.

Subsection 7.5: We prove that R (as a meromorphic function on $\mathbb{R} \times (-\beta, \beta)$) satisfies the following shift equation

$$(7.3) \quad R(\alpha) = -c_\gamma \frac{R(\alpha + \frac{2}{\gamma})}{l(-\frac{4}{\gamma^2})l(\frac{2\alpha}{\gamma})l(2 + \frac{4}{\gamma^2} - \frac{2\alpha}{\gamma})}$$

where $c_\gamma = \frac{\gamma^2}{4} \mu\pi R(\gamma) \neq 0$ is an unknown constant depending on γ (and μ). Recall that from the DOZZ solution we expect that

$$(7.4) \quad c_\gamma = \frac{(\mu\pi l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{l(\frac{4}{\gamma^2})}.$$

This indeed is the case since, quite miraculously, we show that the three equations (7.1), (7.2), (7.3) fully determine R and in particular lead to the above relation for c_γ . Now R^{DOZZ} satisfies (7.1) and (7.3) with (7.4); these shift equations fully determine R^{DOZZ} and therefore we conclude R is equal to R^{DOZZ} .

7.1. Proof of analyticity of \bar{R} in the interval $(\frac{\gamma}{2}, Q)$. We start with the consequence (6.40) of the BPZ equations (with $\alpha = \alpha'$)

$$(7.5) \quad \bar{C}_\gamma(\alpha - \frac{\gamma}{2}, \epsilon, \alpha) = \bar{T}(\alpha, \epsilon, \alpha) \bar{C}_\gamma(\alpha, \epsilon + \frac{\gamma}{2}, \alpha)$$

which holds for α close but strictly less than Q and $\epsilon \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ with $2\alpha + \epsilon - \frac{\gamma}{2} > 2Q$. In the same way as in the proof of lemma 2.9, we get for $\alpha \in (\frac{\gamma}{2}, Q)$

$$\lim_{\epsilon \downarrow \frac{\gamma}{2}} (\epsilon - \frac{\gamma}{2}) \bar{C}_\gamma(\alpha - \frac{\gamma}{2}, \epsilon, \alpha) = \mu^{\frac{2}{\gamma}(Q-\alpha)} \frac{4(Q-\alpha)}{\gamma} \bar{R}(\alpha).$$

By Theorem 4.1, for $\epsilon > \frac{\gamma}{2}$, $\bar{C}_\gamma(\alpha - \frac{\gamma}{2}, \epsilon, \alpha)$ is analytic in $\alpha \in (\frac{\gamma}{2}, Q)$ and for $\epsilon \geq \frac{\gamma}{2}$, $\bar{C}_\gamma(\alpha, \epsilon + \frac{\gamma}{2}, \alpha)$ is analytic in $\alpha \in (\frac{\gamma}{4} + \frac{\epsilon}{2}, Q)$. Hence the relation (7.5) holds for all $\epsilon > \frac{\gamma}{2}$ and $\alpha \in (\frac{\gamma}{4} + \frac{\epsilon}{2}, Q)$. Since

$$\lim_{\epsilon \downarrow \frac{\gamma}{2}} (\epsilon - \frac{\gamma}{2}) \bar{T}(\alpha, \epsilon, \alpha) = \mu\pi \frac{2}{\gamma} \frac{l(\frac{\gamma}{2}\alpha - \frac{\gamma^2}{4} - 1)}{l(1 + \frac{\gamma}{2}(\alpha - Q))l(-\frac{\gamma^2}{4})l(\frac{\gamma^2}{4})} \frac{\Gamma(\frac{2}{\gamma}(\alpha - Q) + 1)}{\Gamma(\frac{2}{\gamma}(\alpha - Q))}$$

we conclude that for all $\alpha \in (\frac{\gamma}{2}, Q)$

$$(7.6) \quad \bar{R}(\alpha) = \mu^{1 - \frac{2}{\gamma}(Q-\alpha)} \frac{\pi}{\gamma} \frac{l(\frac{\gamma}{2}\alpha - \frac{\gamma^2}{4} - 1)}{l(1 + \frac{\gamma}{2}(\alpha - Q))l(-\frac{\gamma^2}{4})l(\frac{\gamma^2}{4})} \frac{\Gamma(\frac{2(\alpha-Q)}{\gamma} + 1)}{\Gamma(\frac{2(\alpha-Q)}{\gamma})} \bar{C}_\gamma(\alpha, \gamma, \alpha)$$

which proves our claim since $\bar{C}_\gamma(\alpha, \gamma, \alpha)$ is analytic in $\alpha \in (\frac{\gamma}{2}, Q)$.

7.2. Proof of the $\frac{\gamma}{2}$ shift equation. We use again (6.40)

$$(7.7) \quad \bar{C}_\gamma(\alpha' - \frac{\gamma}{2}, \epsilon, \alpha) = \bar{T}(\alpha', \epsilon, \alpha) \bar{C}_\gamma(\alpha', \epsilon + \frac{\gamma}{2}, \alpha)$$

which holds for $\epsilon \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ and $\alpha + \alpha' + \epsilon - \frac{\gamma}{2} > 2Q$ with $\alpha, \alpha' < Q$.

Let now

$$\alpha' = Q - \eta, \quad \alpha = \frac{2}{\gamma} + \eta$$

where we fix η such that

$$(7.8) \quad \eta \in (0, \frac{2}{\gamma} - \frac{\gamma}{2}) \wedge \frac{\gamma}{8}$$

and consider both sides of (7.7) as a function of ϵ . From Theorem 4.1 we infer that both sides are analytic in ϵ in the region $\epsilon > 2\eta$ and hence the identity (7.7) extends to this region. As in the proof of Lemma 2.9 we have

$$\lim_{\epsilon \downarrow 2\eta} (\epsilon - 2\eta) \bar{C}_\gamma(\alpha' - \frac{\gamma}{2}, \epsilon, \alpha) = \mu^{\frac{2}{\gamma}(Q-\alpha)} \frac{4(Q-\alpha)}{\gamma} \bar{R}(\alpha).$$

This indicates $\bar{C}_\gamma(\alpha' - \frac{\gamma}{2}, \epsilon, \alpha)$ has a pole at $\epsilon = 2\eta$. We will now extract this pole.

Fix points $z_2, z_3 \in \mathbb{C}$ such that $|z_2| \geq 2$, $|z_3| \geq 2$ and $|z_2 - z_3| \geq 3$. We have

$$\bar{C}_\gamma(\alpha' - \frac{\gamma}{2}, \epsilon, \alpha) = G_{\epsilon, \alpha, \alpha'}(z_2, z_3) \mathbb{E} Z_{\mathbb{C}}(\epsilon)^{1 - \frac{\epsilon}{\gamma}}$$

where

$$G_{\epsilon, \alpha, \alpha'}(z_2, z_3) = 2\mu^{1 - \frac{\epsilon}{\gamma}} \gamma^{-1} \prod_{i < j} \frac{1}{|z_i - z_j|^{\alpha_i \alpha_j + 2\Delta_{ij}}}$$

with $z_1 = 0$, $(\alpha_1, \alpha_2, \alpha_3) = (\epsilon, \alpha, \alpha' - \frac{\gamma}{2}) = (\epsilon, \frac{2}{\gamma} + \eta, \frac{2}{\gamma} - \eta)$ and for $A \subset \mathbb{C}$

$$(7.9) \quad Z_A(\epsilon) := \int_A \frac{|x|_+^{\gamma(\epsilon + \alpha_2 + \alpha_3)}}{|x|^{\gamma\epsilon} |x - z_2|^{\gamma\alpha_2} |x - z_3|^{\gamma\alpha_3}} M_\gamma(d^2x).$$

Define next

$$F(\epsilon) := \mathbb{E}(Z_{\mathbb{C}}(\epsilon)^{1 - \frac{\epsilon}{\gamma}} - (Z_{B_1(z_2)}(\epsilon) + Z_{B_1(z_3)}(\epsilon))^{1 - \frac{\epsilon}{\gamma}}).$$

Importantly, notice that $Z_{B_1(z_2)}(\epsilon)$ and $Z_{B_1(z_3)}(\epsilon)$ do not depend on ϵ since for $x \in B_1(z_2)$ or $x \in B_1(z_3)$ we have $\frac{|x|_+^{\gamma\epsilon}}{|x|^{\gamma\epsilon}} = 1$. Hence we denote them by $Z_{B_1(z_2)}$.

We want to show that $F(\epsilon)$ is analytic over a neighborhood of the pole at $\epsilon = 2\eta$ (observe that (7.8) entails $2\eta < \frac{1}{\gamma}$).

Lemma 7.1. $F(\epsilon)$ is analytic in $\epsilon \in (-2\eta - \delta, \frac{1}{\gamma})$ for some $\delta > 0$.

Proof. Let us fix $\delta > 0$ such that

$$(7.10) \quad 2\eta + \delta < \frac{4}{\gamma} - \gamma \quad \text{and} \quad 4\eta + \delta < \gamma$$

which is possible because of (7.8). As in the proof of Theorem 4.1 we construct F as the uniform limit as $t \rightarrow \infty$ of analytic functions F_t in a neighborhood of $(-2\eta - \delta, \frac{1}{\gamma})$. Let us denote $\mathbb{C}_t = \{z : |z| \geq e^{-t}\}$ and define (recall that B_r stands for the ball $B_r(0)$)

$$F_t(\epsilon) = \mathbb{E} \left[e^{\epsilon X_t(0) - \frac{t\epsilon^2}{2}} \left(Z_t(0)^{1-\frac{\epsilon}{\gamma}} - (Z_{B_1(z_2)} + Z_{B_1(z_3)})^{1-\frac{\epsilon}{\gamma}} \right) \right].$$

Let us first show that for each t , $\epsilon \mapsto F_t(\epsilon)$ is an analytic function of ϵ in an open neighborhood of $(-2\eta - \delta, \frac{1}{\gamma})$. Let $R_1 := Z_{B_1(z_2)} + Z_{B_1(z_3)}$ and $R_2 := Z_t(0) - R_1$. By (2.13) and (2.14) R_1 admits moments of order q for $q < \frac{2}{\gamma}(Q - \eta - \frac{2}{\gamma})$ and R_2 has moments of order q for $q < \frac{4}{\gamma^2}$. We interpolate

$$(7.11) \quad \mathbb{E}[e^{\epsilon X_t(0)} ((R_2 + R_1)^{1-\frac{\epsilon}{\gamma}} - R_1^{1-\frac{\epsilon}{\gamma}})] = \int_0^1 \mathbb{E}[e^{\epsilon X_t(0)} R_2(sR_2 + R_1)^{-\frac{\epsilon}{\gamma}}] ds$$

Let $\epsilon = \epsilon_1 + i\epsilon_2$. If $\epsilon_1 > 0$ then since $\mathbb{E}e^{p\epsilon X_t(0)} < \infty$ for all $p < \infty$ and since chaos has negative moments by Hölder we can bound the integrand by $CE[R_2^q]^{\frac{1}{q}}$ for any $q > 1$.

If $\epsilon_1 < 0$ we bound

$$|\mathbb{E}[R_2(sR_2 + R_1)^{-\frac{\epsilon}{\gamma}}]| \leq C(\mathbb{E}[R_2^{1-\frac{\epsilon_1}{\gamma}}] + \mathbb{E}[R_2 R_1^{-\frac{\epsilon_1}{\gamma}}]).$$

Hence we need $1 - \frac{\epsilon_1}{\gamma} < \frac{4}{\gamma^2}$ and by (slight variant of) Remark 3.5 we need $-\frac{\epsilon_1}{\gamma} < \frac{2}{\gamma}(Q - \eta - \frac{2}{\gamma}) = 1 - \frac{2\eta}{\gamma}$. These conditions hold due to (7.10). F_t is easily seen to be complex differentiable in ϵ .

Let us show that the family F_t is Cauchy for the topology of uniform convergence over compact subsets. For this we will bound $F_{t+1} - F_t$. First observe that because $Z_{B_1(z_2)}(\epsilon)$ and $Z_{B_1(z_3)}(\epsilon)$ are independent of $X_t(0)$ these terms cancel out in $F_{t+1} - F_t$. Furthermore, Girsanov theorem gives

$$\mathbb{E}e^{\epsilon X_t(0) - \frac{t\epsilon^2}{2}} Z_t(0)^{1-\frac{\epsilon}{\gamma}} = \mathbb{E}e^{i\epsilon_2 X_t(0) + \frac{t\epsilon_2^2}{2}} Z_t(\epsilon_1)^{1-\frac{\epsilon_1}{\gamma}}$$

Hence as in the proof of Theorem 4.1 we get

$$|F_{t+1} - F_t| \leq e^{\frac{(t+1)\epsilon_2^2}{2}} \mathbb{E}|Z_{t+1}(\epsilon_1)^{1-\frac{\epsilon_1}{\gamma}} - Z_t(\epsilon_1)^{1-\frac{\epsilon_1}{\gamma}}|.$$

From now on, since ϵ_1 is fixed we suppress it in the notation and denote $Z_t(\epsilon_1)$ by Z_t . We proceed as in the proof of Theorem 4.1. Let $Y_t := Z_{t+1} - Z_t$. We fix $\theta > 0$ and write

$$\mathbb{E}|Z_{t+1}^{1-\frac{\epsilon_1}{\gamma}} - Z_t^{1-\frac{\epsilon_1}{\gamma}}| \leq \mathbb{E}\mathbf{1}_{Y_t \leq e^{-\theta t}} |Z_{t+1}^{1-\frac{\epsilon_1}{\gamma}} - Z_t^{1-\frac{\epsilon_1}{\gamma}}| + \mathbb{E}\mathbf{1}_{Y_t \geq e^{-\theta t}} |Z_{t+1}^{1-\frac{\epsilon_1}{\gamma}} - Z_t^{1-\frac{\epsilon_1}{\gamma}}|$$

Interpolating the first term is bounded by

$$\mathbb{E}\mathbf{1}_{Y_t < e^{-\theta t}} |(Z_t + Y_t)^{1-\frac{\epsilon_1}{\gamma}} - Z_t^{1-\frac{\epsilon_1}{\gamma}}| \leq Ce^{-\theta t} \sup_{s \in [0,1]} \mathbb{E}(Z_t + sY_t)^{-\frac{\epsilon_1}{\gamma}} \leq Ce^{-\theta t} \mathbb{E}(Z_{\mathbb{C}}(\epsilon_1))^{-\frac{\epsilon_1}{\gamma}}$$

The last expectation is finite since $-\frac{\epsilon_1}{\gamma} < \frac{2}{\gamma}(Q - \eta - \frac{2}{\gamma}) = 1 - \frac{2\eta}{\gamma}$ holds by (7.10).

For the second term we use in turn Hölder's inequality (with conjugate exponents p, q), the mean value theorem and the Markov inequality (for some $m \in (0, 1)$ to ensure finiteness of the expectation last expectation below) to get

$$\begin{aligned} \mathbb{E}\mathbf{1}_{Y_t \geq e^{-\theta t}} |Z_{t+1}^{1-\frac{\epsilon_1}{\gamma}} - Z_t^{1-\frac{\epsilon_1}{\gamma}}| &\leq [\mathbb{P}(Y_t \geq e^{-\theta t})]^{1/p} [\mathbb{E}|Z_{t+1}^{1-\frac{\epsilon_1}{\gamma}} - Z_t^{1-\frac{\epsilon_1}{\gamma}}|^q]^{\frac{1}{q}} \\ &\leq [\mathbb{P}(Y_t \geq e^{-\theta t})]^{1/p} \sup_{s \in [0,1]} [\mathbb{E}Y_t^q (Z_t + sY_t)^{-q\frac{\epsilon_1}{\gamma}}]^{\frac{1}{q}} \\ &\leq C_t(q) e^{-\frac{\theta m}{p}t} \mathbb{E}[Y_t^m]^{1/p} \leq C_t(q) e^{\frac{1}{p}(\gamma(Q - \epsilon_1 - \theta)m - \frac{m^2\gamma^2}{2})} \end{aligned}$$

where we used Remark 3.5 and defined

$$(7.12) \quad C_t(q) = \sup_{s \in [0,1]} [\mathbb{E}Y_t^q (Z_t + sY_t)^{-q\frac{\epsilon_1}{\gamma}}]^\frac{1}{q}.$$

Summarizing, we have shown

$$(7.13) \quad |F_{t+1} - F_t| \leq C e^{\frac{t\epsilon_2^2}{2}} (e^{-\theta t} + C_t(q) e^{\frac{1}{p}(\gamma(Q-\epsilon_1-\theta)m - \frac{m^2\gamma^2}{2})}).$$

Now we have to optimize with respect to the free parameters p, q, θ, m . Let us first fix q (hence p). We first fix q to make (7.12) finite. Let first $\epsilon_1 > 0$. By existence of negative moments of chaos we get for all $r > q$

$$C_t(q) \leq C(r) [\mathbb{E}Y_t^r]^\frac{1}{r}.$$

Hence $\sup_t C_t(q) < \infty$ if $q < \frac{2}{\gamma}(Q - \epsilon_1) \wedge \frac{4}{\gamma^2} = \frac{4}{\gamma^2}$.

If $\epsilon_1 < 0$ we bound $Y_t \leq Z_{B_1}(\epsilon_1)$ and $Z_{t+1} \leq Z_{B_1}(\epsilon_1) + Z_{B_1^c}(\epsilon_1)$ to get

$$C_t(q) \leq [\mathbb{E}Y_t^q Z_{t+1}^{-q\frac{\epsilon_1}{\gamma}}]^\frac{1}{q} \leq C [\mathbb{E}Z_{B_1}(\epsilon_1)^{q(1-\frac{\epsilon_1}{\gamma})} + \mathbb{E}Z_{B_1}(\epsilon_1)^q Z_{B_1^c}(\epsilon_1)^{-q\frac{\epsilon_1}{\gamma}}]^\frac{1}{q}.$$

The first expectation is finite if $q(1 + \frac{2\eta+\delta}{\gamma}) < \frac{4}{\gamma^2}$ and by Remark 3.5 the second one is finite if $q\frac{2\eta+\delta}{\gamma} < \frac{2}{\gamma}(Q - \eta - \frac{2}{\gamma}) = 1 - \frac{2\eta}{\gamma}$. Due to (7.10) we can find $q > 1$ such that this condition holds and hence $\sup_t C_t(q) < \infty$.

Next, we choose $\theta > 0$ such that $Q - \frac{2}{\gamma} - \theta > 0$ and then $m \in (0, 1)$ small enough such that

$$\kappa := p^{-1}(\gamma(Q - \frac{2}{\gamma} - \theta)m - \frac{\gamma^2}{2}m^2) > 0.$$

As we have $\epsilon_1 < \frac{2}{\gamma}$ we get from (7.13)

$$(7.14) \quad |F_{t+1} - F_t| \leq C e^{\frac{t\epsilon_2^2}{2}} (e^{-\theta t} + e^{-\kappa t}).$$

Hence for $\epsilon_2^2 \leq \min(\theta, \kappa)$ the sequence F_t converges uniformly in compacts of a neighborhood of $(-2\eta - \delta, \frac{1}{\gamma})$. Finally observe that $F(\epsilon) = \lim_{t \rightarrow \infty} F_t(\epsilon)$ for $\epsilon \in \mathbb{R}$ with $\epsilon \in (2\eta, Q)$. \square

Lemma 7.2. *We have*

$$G_{\epsilon, \alpha, \alpha'}(z_2, z_3) \mathbb{E}((Z_{B_1(z_2)}(\epsilon) + Z_{B_1(z_3)}(\epsilon))^{1-\frac{\epsilon}{\gamma}}) = \frac{\gamma - 2\eta}{\epsilon - 2\eta} \bar{R}(\frac{2}{\gamma} + \eta) + F_2(\epsilon, z_2, z_3)$$

where F_2 is analytic in ϵ on $(-2\eta, \frac{1}{\gamma})$ and

$$\lim_{\epsilon \rightarrow -2\eta} (\epsilon + 2\eta) F_2(\epsilon) = 2\mu^{1+\frac{2\eta}{\gamma}} (1 + \frac{2\eta}{\gamma}) \bar{R}(\frac{2}{\gamma} - \eta).$$

Proof. Let us first make the following general observation. Consider a random variable $Y \geq 0$ admitting a "maximal" moment of order $\beta > 0$, namely $\mathbb{E}[Y^\alpha] < +\infty$ for $\alpha < \beta$ and $\mathbb{E}[Y^\beta] = +\infty$. Then the mapping $s \in \mathbb{C} \mapsto \mathbb{E}[Y^s]$ is holomorphic on the set $\{s \in \mathbb{C}; 0 < \Re(s) < \beta\}$. Furthermore it diverges at $s = \beta$. The point is to find an analytic continuation and, for this, we need to know more about the shape of the tail of the random variable Y .

The simplest situation is when we know that $|\mathbb{P}(Y > t) - c_1 t^{-\beta}| \leq c_2 t^{-\beta-\delta}$ for some constants $c_1, c_2, \beta, \delta > 0$ and for all $t \geq t_0$. In this case, we start from the following standard decomposition for $s < \beta$

$$\mathbb{E}[Y^s] = \int_0^\infty \mathbb{P}(Y > t) t^{s-1} dt$$

that we rewrite as

$$\mathbb{E}[Y^s] = \int_0^\infty (\mathbb{P}(Y > t) - c_1 t^{-\beta} \mathbf{1}_{\{t \geq 1\}}) t^{s-1} dt - \frac{c_1}{s - \beta}.$$

In the above right-hand side, the parametrized integral is holomorphic on the set $\{s \in \mathbb{C}; 0 < \Re(s) < \beta + \delta\}$ as can be shown by the theorem of holomorphicity of such integrals and our tail estimate. This provides the desired analytic continuation for $\mathbb{E}[Y^s]$ as well as identification of the pole at $s = \beta$.

The argument can be extended to situations when we are able to give a more precise asymptotic expansion of the tail. In our lemma, this corresponds to the case when we have a tail estimate of the type

$$|\mathbb{P}(Y > t) - c_1 t^{-\beta_1} - c_2 t^{-\beta_2}| \leq c_3 t^{-\beta_2 - \delta}$$

for some constants $c_1, c_2, c_3, \delta > 0$ and $0 < \beta_1 < \beta_2$. In this case, the analytic continuation (still denoted by $\mathbb{E}[Y^s]$) is given by the expression

$$\mathbb{E}[Y^s] = \int_0^\infty (\mathbb{P}(Y > t) - (c_1 t^{-\beta_1} + c_2 t^{-\beta_2}) \mathbf{1}_{\{t \geq 1\}}) t^{s-1} dt - \frac{c_1}{s - \beta_1} - \frac{c_2}{s - \beta_1}.$$

One again the integral is holomorphic for $s < \beta_2 + \delta$ and we have two poles. The second pole at $s = \beta_2$ can be recovered by taking the limit

$$\lim_{s \rightarrow \beta_2} (s - \beta_2) (\mathbb{E}[Y^s] + \frac{c_1}{s - \beta_1}) = c_2.$$

The statement of our lemma is nothing but the above argument applied to the random variable $Y := Z_{B_1(z_2)}(\epsilon) + Z_{B_1(z_3)}(\epsilon)$. The shape of the tail of this random variable is given by Lemma 3.3 (when α_2, α_3 are both sufficiently close to Q as indicated in Remark (3.4)). \square

Combining we get that the function

$$f(\epsilon) := \bar{C}_\gamma(\frac{2}{\gamma} - \eta, \epsilon, \frac{2}{\gamma} + \eta) - \frac{\gamma - 2\eta}{\epsilon - 2\eta} \bar{R}(\frac{2}{\gamma} + \eta)$$

is analytic on $\epsilon \in (-2\eta, \frac{1}{\gamma})$. By (7.7) on $\epsilon \in (2\eta, \frac{1}{\gamma})$ $f = g$ where

$$g(\epsilon) := \bar{T}(Q - \eta, \epsilon, \frac{2}{\gamma} + \eta) \bar{C}_\gamma(Q - \eta, \epsilon + \frac{\gamma}{2}, \frac{2}{\gamma} + \eta) - \frac{\gamma - 2\eta}{\epsilon - 2\eta} \bar{R}(\frac{2}{\gamma} + \eta)$$

Thus by analytic continuation g is analytic in ϵ on $(-2\eta, \frac{1}{\gamma})$.

Consider now the limit $\epsilon \rightarrow -2\eta$. We get from Lemma 7.2

$$(7.15) \quad \lim_{\epsilon \rightarrow -2\eta} (\epsilon + 2\eta) f(\epsilon) = 2\mu^{1+\frac{2\eta}{\gamma}} (1 + \frac{2\eta}{\gamma}) \bar{R}(\frac{2}{\gamma} - \eta) = \frac{R(\frac{2}{\gamma} - \eta)}{\Gamma(-1 - \frac{2\eta}{\gamma})}$$

and then

$$(7.16) \quad \lim_{\epsilon \rightarrow -2\eta} (\epsilon + 2\eta) g(\epsilon) = \frac{2T(Q - \eta, -2\eta, \frac{2}{\gamma} + \eta) R(Q - \eta)}{\Gamma(-1 - \frac{2\eta}{\gamma})}.$$

Now, we have for $\epsilon = -2\eta = -\frac{\gamma}{2} + \alpha' - \alpha$ that

$$a = \frac{\gamma}{2} \alpha' - \frac{\gamma^2}{2} - 1 \quad b = \frac{\gamma}{2} (\alpha' - \alpha) - \frac{\gamma^2}{4} = \frac{\gamma\epsilon}{2}$$

and

$$c = 1 + \frac{\gamma}{2} (\alpha' - Q) \quad a + b - c = \frac{\gamma}{2} (\alpha' - \alpha) - \frac{\gamma^2}{2} - 1 = 1 - (2 + \frac{\gamma^2}{4} - \frac{\gamma\epsilon}{2})$$

Therefore, we have $l(b) = l(\frac{\gamma\epsilon}{2})$ and $l(a + b - c) l(2 + \frac{\gamma^2}{4} - \frac{\gamma\epsilon}{2}) = 1$ and

$$T(Q - \eta, -2\eta, \frac{2}{\gamma} + \eta) = -\mu\pi \frac{l(\frac{\gamma}{2} \alpha' - \frac{\gamma^2}{2} - 1)}{l(1 + \frac{\gamma}{2} (\alpha' - Q)) l(-\frac{\gamma^2}{4})}$$

In conclusion, we get the following relation from the fact that (7.15) and (7.16) are equal:

$$(7.17) \quad R(\alpha' - \frac{\gamma}{2}) = -\mu\pi \frac{l(\frac{\gamma}{2} \alpha' - \frac{\gamma^2}{2} - 1)}{l(1 + \frac{\gamma}{2} (\alpha' - Q)) l(-\frac{\gamma^2}{4})} R(\alpha')$$

We have proven (7.17) for α' close to Q but since by previous subsection \bar{R} is analytic on $(\frac{\gamma}{2}, Q)$ it extends to $\alpha' \in (\gamma, Q)$. Combined with expression (7.6) for \bar{R} , we can use the relation (7.17) to analytically continue R to a neighborhood of \mathbb{R} of the form $\mathbb{R} \times (-\beta, \beta)$ with $\beta > 0$. In the sequel, we will consider the meromorphic extension of R to $\mathbb{R} \times (-\beta, \beta)$. Since R^{DOZZ} also satisfies (7.17) and $0 < \frac{R(\alpha)}{R^{\text{DOZZ}}(\alpha)} < \infty$ for $\alpha \in (\frac{\gamma}{2}, Q)$, one

can see that R and R^{DOZZ} have their poles and zeros located at the same place. For instance, the poles of R are located at $\{\frac{2}{\gamma} - \frac{\gamma}{2}\mathbb{N}\} \cup \{\frac{\gamma}{2} - \frac{2}{\gamma}\mathbb{N}\}$.

7.3. The gluing lemma. We introduce the following condition:

$$(7.18) \quad 2Q + \frac{\gamma}{2} - \frac{2}{\gamma} - \alpha_2 - \alpha_3 < \frac{4}{\gamma} \wedge \gamma \wedge \min_{2 \leq i \leq 3} 2(Q - \alpha_i), \quad \forall i, \alpha_i < Q$$

Lemma 7.3. *We suppose that α_2, α_3 satisfies condition (7.18). Then the function*

$$S : \alpha \mapsto \begin{cases} C_\gamma(\alpha, \alpha_2, \alpha_3), & \text{if } \alpha < Q \\ R(\alpha)C_\gamma(2Q - \alpha, \alpha_2, \alpha_3), & \text{if } \alpha \geq Q \end{cases}$$

is the restriction on the real line of a holomorphic function defined on a neighborhood of Q . This holomorphic function is explicitly given for $\alpha = \alpha_1 + \frac{\gamma}{2}$ by

$$(7.19) \quad \bar{S} : \alpha_1 \mapsto \frac{C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)}{\pi\mu} \frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(\frac{\alpha_1\gamma}{2} - \frac{\gamma^2}{4})l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q))l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2}))l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2}))}$$

In particular, the function S is real analytic in the neighborhood of Q .

Proof. We will show that $\alpha_1 \mapsto S(\frac{\gamma}{2} + \alpha_1)$ is the restriction on the real line of a holomorphic function defined on the neighborhood of $\frac{2}{\gamma}$. Clearly, one can find $\epsilon > 0$ and $\epsilon < \frac{2}{\gamma} - \frac{\gamma}{2}$ such that for all $\alpha_1 \in [\frac{2}{\gamma} - \epsilon, \frac{2}{\gamma} + \epsilon]$

$$(7.20) \quad 2Q + \frac{\gamma}{2} - \alpha_1 - \alpha_2 - \alpha_3 < \frac{4}{\gamma} \wedge \gamma \wedge \min_{1 \leq i \leq 3} 2(Q - \alpha_i), \quad \forall i, \alpha_i < Q$$

As a consequence of Theorem 6.1, we get the crossing symmetry relation (1.8); more specifically, one can find $\eta > 0$ such that for all $\tilde{\alpha}_2, \tilde{\alpha}_3 \in (Q - \eta, Q)$ and all $\tilde{\alpha}_1$ in an interval $I \subset (\frac{\gamma}{2}, \frac{2}{\gamma})$ of size η we have $2Q + \frac{\gamma}{2} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3 < 0$ and

$$C_\gamma(\tilde{\alpha}_1 + \frac{\gamma}{2}, \tilde{\alpha}_2, \tilde{\alpha}_3) = -\frac{C_\gamma(\tilde{\alpha}_1 - \frac{\gamma}{2}, \tilde{\alpha}_2, \tilde{\alpha}_3)}{\pi\mu} \frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\tilde{\alpha}_1}{2})l(\frac{\tilde{\alpha}_1\gamma}{2} - \frac{\gamma^2}{4})l(\frac{\gamma}{4}(\tilde{\alpha} - 2\tilde{\alpha}_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\tilde{\alpha} - \frac{\gamma}{2} - 2Q))l(\frac{\gamma}{4}(\tilde{\alpha} - 2\tilde{\alpha}_3 - \frac{\gamma}{2}))l(\frac{\gamma}{4}(\tilde{\alpha} - 2\tilde{\alpha}_2 - \frac{\gamma}{2}))}$$

where $\tilde{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$. By analytic continuation from $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ to $\alpha_1, \alpha_2, \alpha_3$ (see Theorem 4.1), we get for all $\alpha_1 \in (\frac{2}{\gamma} - \epsilon, \frac{2}{\gamma})$

$$C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) = -\frac{C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)}{\pi\mu} \frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(\frac{\alpha_1\gamma}{2} - \frac{\gamma^2}{4})l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q))l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2}))l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2}))}$$

where $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$.

Now, recall that we are working with R extended to a strip $\mathbb{R} \times (-\beta, \beta)$ thanks to relation (7.17). As a consequence of Theorem 6.2 we get the following crossing symmetry relation (derived similarly to (1.8)): one can find $\eta > 0$ such that for all $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \in (Q - \eta, Q)$ we have $2Q + \frac{\gamma}{2} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3 < 0$ and

$$R(\tilde{\alpha}_1 + \frac{\gamma}{2})C_\gamma(2Q - \tilde{\alpha}_1 - \frac{\gamma}{2}, \tilde{\alpha}_2, \tilde{\alpha}_3) = -\frac{C_\gamma(\tilde{\alpha}_1 - \frac{\gamma}{2}, \tilde{\alpha}_2, \tilde{\alpha}_3)}{\pi\mu} \frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\tilde{\alpha}_1}{2})l(\frac{\tilde{\alpha}_1\gamma}{2} - \frac{\gamma^2}{4})l(\frac{\gamma}{4}(\tilde{\alpha} - 2\tilde{\alpha}_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\tilde{\alpha} - \frac{\gamma}{2} - 2Q))l(\frac{\gamma}{4}(\tilde{\alpha} - 2\tilde{\alpha}_3 - \frac{\gamma}{2}))l(\frac{\gamma}{4}(\tilde{\alpha} - 2\tilde{\alpha}_2 - \frac{\gamma}{2}))}$$

where $\tilde{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$. By analytic continuation from $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ to $\alpha_1, \alpha_2, \alpha_3$ (see Theorem 4.1 and use the fact that \bar{R} is analytic), we get for all $\alpha_1 \in (\frac{2}{\gamma}, \frac{2}{\gamma} + \epsilon)$

$$R(\alpha_1 + \frac{\gamma}{2})C_\gamma(2Q - \alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) = -\frac{C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)}{\pi\mu} \frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(\frac{\alpha_1\gamma}{2} - \frac{\gamma^2}{4})l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q))l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2}))l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2}))}$$

where $\bar{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$. This yields the result. \square

7.4. Proof of the inversion relation. We use the relation from Proposition 6.5:

$$(7.21) \quad C_\gamma(\alpha - \frac{2}{\gamma}, \epsilon, \alpha') = T(\alpha, \epsilon, \alpha')R(\epsilon)C_\gamma(\alpha, 2Q - \epsilon - \frac{2}{\gamma}, \alpha')$$

which was proven for α, ϵ and α' close but strictly less than Q with $\alpha + \alpha' + \epsilon > 2Q + \frac{2}{\gamma}$.

The case $\gamma < \sqrt{2}$. By analytic continuation of the unit volume three point structure constants $\bar{C}_\gamma(\alpha - \frac{2}{\gamma}, \epsilon, \alpha')$, $\bar{C}_\gamma(\alpha, 2Q - \epsilon - \frac{2}{\gamma}, \alpha')$ (given by (2.18)) and $R(\epsilon)$, the relation is valid for

$$\alpha = Q - \eta, \quad \epsilon = \frac{2}{\gamma} + \eta', \quad \alpha' = \frac{2}{\gamma}$$

for η' in the interval $(0, \eta)$. One can notice that (7.21) is also valid within this range of parameters with no poles since

$$\alpha - \frac{2}{\gamma} + \epsilon + \alpha' = \frac{\gamma}{2} + \frac{4}{\gamma} + \eta' - \eta > \frac{4}{\gamma} + 2\eta'$$

and

$$\alpha + 2Q - \epsilon - \frac{2}{\gamma} + \alpha' = Q - \eta + \gamma - \eta' + \frac{2}{\gamma} = 2Q + \frac{\gamma}{2} - \eta - \eta'$$

Now, we wish to analytically continue α to the value $Q + \eta$ by using the gluing lemma 7.3. We can indeed use the gluing lemma when the 4 point correlation function

$$\langle V_{-\frac{\gamma}{2}}(z)V_{\frac{2}{\gamma}+\eta''}(0)V_{\gamma-\eta'}(1)V_{\frac{2}{\gamma}}(\infty) \rangle$$

stays well defined for all $\eta' \in (0, \eta)$ and all $\eta'' \in (-\eta, \eta)$ (indeed, one can show that the existence of the above 4 point correlation function implies that one can apply the gluing lemma). More specifically, we must check that

$$2Q + \frac{\gamma}{2} - \frac{2}{\gamma} - \eta'' - \gamma + \eta' - \frac{2}{\gamma} < \gamma \wedge (\gamma - 2\eta'') \wedge (\frac{4}{\gamma} - \gamma + 2\eta')$$

This is possible as soon as $\frac{\gamma}{2} < \gamma \wedge (\frac{4}{\gamma} - \gamma)$, i.e. $\gamma < \sqrt{\frac{8}{3}}$.

Therefore, the relation (7.21) becomes for $\alpha = Q + \eta, \epsilon = \frac{2}{\gamma} + \eta', \alpha' = \frac{2}{\gamma}$ for η sufficiently small and $\eta' \in (0, \eta)$

$$(7.22) \quad C_\gamma(\alpha - \frac{2}{\gamma}, \epsilon, \alpha') = T(\alpha, \epsilon, \alpha')R(\epsilon)R(\alpha)C_\gamma(2Q - \alpha, 2Q - \epsilon - \frac{2}{\gamma}, \alpha')$$

$$(7.23) \quad C_\gamma(Q + \eta - \frac{2}{\gamma}, \epsilon, \frac{2}{\gamma}) = T(\alpha, \epsilon, \alpha')R(\epsilon)R(\alpha)C_\gamma(Q - \eta, 2Q - \epsilon - \frac{2}{\gamma}, \frac{2}{\gamma})$$

Now, we consider the limit of (7.23) as ϵ increases to $Q - \eta$. From (2.17) we infer

$$\lim_{\epsilon \uparrow Q - \eta} (Q - \eta - \epsilon)C_\gamma(\alpha - \frac{2}{\gamma}, \epsilon, \frac{2}{\gamma}) = -2, \quad \lim_{\epsilon \uparrow Q - \eta} (Q - \eta - \epsilon)C_\gamma(Q - \eta, 2Q - \epsilon - \frac{2}{\gamma}, \frac{2}{\gamma}) = 2$$

Indeed, one can notice that the two limits above correspond to insertions such that s goes to 0 in expression (2.17). Also we get

$$\lim_{\epsilon \uparrow Q - \eta} (Q - \eta - \epsilon)l(a) = -\gamma, \quad \lim_{\epsilon \uparrow Q - \eta} (Q - \eta - \epsilon)l(b) = \frac{1}{\gamma}$$

and

$$\lim_{\epsilon \uparrow Q - \eta} l(c)l(a + b - c) = l(1 + \frac{2\eta}{\gamma})l(-\frac{2\eta}{\gamma}) = 1$$

This yields $R(Q - \eta)R(Q + \eta) = 1$ for η small hence $R(2Q - \alpha)R(\alpha) = 1$ on the strip $\mathbb{R} \times (-\beta, \beta)$ where R admits a meromorphic extension.

The case $\gamma > \sqrt{2}$. By analytic continuation, the relation is valid for

$$\alpha = Q - \eta, \quad \epsilon = \gamma + \eta', \quad \alpha' = \frac{2}{\gamma}$$

for η' around 0. Indeed, in this case, we have

$$\alpha + 2Q - \epsilon - \frac{2}{\gamma} + \frac{2}{\gamma} = Q - \eta - \eta' + \frac{4}{\gamma} > 2Q$$

and (in $C_\gamma(\alpha - \frac{2}{\gamma}, \epsilon, \alpha')$ the biggest insertion is at $\epsilon = \gamma + \eta'$ since $\gamma > \sqrt{2}$)

$$\alpha - \frac{2}{\gamma} + \epsilon + \alpha' = \frac{\gamma}{2} - \eta + \gamma + \eta' + \frac{2}{\gamma} > 2\gamma$$

In fact, both relations above are valid for η and η' in a small interval. We want to analytically continue $\alpha = Q - \eta$ to $\alpha = Q + \eta$ by using the gluing lemma. In order to do so, we must check that the 4 point correlation function

$$\langle V_{-\frac{\gamma}{2}}(z) V_{\frac{\gamma}{2} + \eta''}(0) V_{\frac{\gamma}{2} - \eta'}(1) V_{\frac{\gamma}{2}}(\infty) \rangle$$

is well defined during the procedure, i.e. for $\eta'' \in [-\eta, \eta]$ and η' small. We have

$$-\frac{\gamma}{2} + \frac{2}{\gamma} + \eta'' + \frac{2}{\gamma} - \eta' + \frac{2}{\gamma} > \frac{4}{\gamma} + \eta$$

for η small. Hence, we can analytically continue to $\alpha = Q + \eta$ which leads to the following relation

$$C_\gamma(\alpha - \frac{2}{\gamma}, \epsilon, \alpha') = T(\alpha, \epsilon, \alpha') R(\epsilon) R(\alpha) C_\gamma(2Q - \alpha, 2Q - \epsilon - \frac{2}{\gamma}, \alpha')$$

or equivalently for all η' small

$$C_\gamma(\frac{\gamma}{2} + \eta, \gamma + \eta', \frac{2}{\gamma}) = T(\alpha = Q + \eta, \epsilon = \gamma + \eta', \alpha' = \frac{2}{\gamma}) R(\gamma + \eta') R(Q + \eta) C_\gamma(Q - \eta, \frac{2}{\gamma} - \eta', \frac{2}{\gamma})$$

Now, we can conclude as in the previous case by letting $\epsilon = \gamma + \eta'$ increase to $Q - \eta$.

7.5. Proof of the $\frac{2}{\gamma}$ shift equation. We get that for all $\epsilon, \alpha, \alpha'$ close to but strictly less than Q that

$$(7.24) \quad R(\epsilon) C_\gamma(2Q - \epsilon - \frac{2}{\gamma}, \alpha, \alpha') = \frac{l(c-1)l(c-a-b+1)}{l(c-a)l(c-b)} R(\alpha) C_\gamma(\epsilon, 2Q - \alpha - \frac{2}{\gamma}, \alpha')$$

where

$$(7.25) \quad a = \frac{2}{\gamma}(\frac{\epsilon}{2} - \frac{Q}{2}) + \frac{2}{\gamma}(\frac{\alpha + \alpha'}{2} - \frac{2}{\gamma}) - \frac{1}{2} \quad b = \frac{2}{\gamma}(\frac{\epsilon}{2} - \frac{Q}{2}) + \frac{2}{\gamma}(\frac{\alpha - \alpha'}{2}) + \frac{1}{2}$$

and

$$(7.26) \quad c = 1 + \frac{2}{\gamma}(\epsilon - Q) \quad c - a - b + 1 = 2 - \frac{2}{\gamma}(\alpha - \frac{2}{\gamma})$$

This leads to the following values

$$(7.27) \quad c - a = \frac{3}{2} + \frac{2}{\gamma}(\frac{\epsilon}{2} - \frac{Q}{2}) - \frac{2}{\gamma}(\frac{\alpha + \alpha'}{2} - \frac{2}{\gamma}) \quad c - b = \frac{1}{2} + \frac{2}{\gamma}(\frac{\epsilon}{2} - \frac{Q}{2}) - \frac{2}{\gamma}(\frac{\alpha - \alpha'}{2})$$

and

$$(7.28) \quad c - 1 = \frac{2}{\gamma}(\epsilon - Q) \quad c - a - b + 1 = 2 - \frac{2}{\gamma}(\alpha - \frac{2}{\gamma})$$

We fix η very small. Now, we can analytically continue equation (7.24) to the following values

$$(7.29) \quad \epsilon = \frac{\gamma}{2} + \eta', \quad \alpha = \frac{\gamma}{2} + \eta'', \quad \alpha' = \frac{2}{\gamma} + \eta$$

with $\eta' \in [\eta, 2\eta]$ and $\eta'' \in [6\eta, 7\eta]$. This is indeed possible if each side is well defined without taking out poles. Indeed, we have with these values that

$$2Q - \epsilon - \frac{2}{\gamma} + \alpha + \alpha' = 2Q - \eta' + \eta'' + \eta > 2Q$$

so $C_\gamma(2Q - \epsilon - \frac{2}{\gamma}, \alpha, \alpha')$ satisfies the standard Seiberg bounds. Also, we have

$$\epsilon + 2Q - \alpha - \frac{2}{\gamma} + \alpha' = 2Q + \eta' - \eta'' + \eta$$

therefore the quantity $C_\gamma(\epsilon, 2Q - \alpha - \frac{2}{\gamma}, \alpha')$ exists provided that the largest insertion does not blow up. The largest insertion is $2Q - \alpha - \frac{2}{\gamma} = Q - \eta''$ which has a moment of order $\frac{2}{\gamma}\eta''$. The moment of $C_\gamma(\epsilon, 2Q - \alpha - \frac{2}{\gamma}, \alpha')$ is given by

$$\frac{1}{\gamma}(2Q - (2Q + \eta' - \eta'' + \eta)) = \frac{1}{\gamma}(\eta'' - \eta' - \eta) < \frac{2}{\gamma}\eta''$$

Now, we want to let η' go to $-\eta$. To do so, we must use the gluing lemma which corresponds to the fact that the following 4 point correlation function is defined during the procedure ($2Q - \epsilon - \frac{2}{\gamma} = Q - \eta'$)

$$\langle V_{-\frac{\gamma}{2}}(z)V_{\frac{2}{\gamma}-\eta'}(0)V_{\frac{\gamma}{2}+\eta''}(1)V_{\frac{2}{\gamma}+\eta}(\infty) \rangle$$

This well defined if for all $\tilde{\eta} \in (-2\eta, \eta)$ we have (biggest insertion is $\frac{2}{\gamma} + \eta$)

$$2Q + \frac{\gamma}{2} - \frac{2}{\gamma} - \tilde{\eta} - \frac{\gamma}{2} - \eta'' - \frac{2}{\gamma} - \eta < \gamma - 2\eta$$

which is equivalent to $\tilde{\eta} + \eta'' + \eta > 2\eta$ which holds since $\eta'' \in [6\eta, 7\eta]$.

Therefore, we get for $\epsilon = \frac{\gamma}{2} - \eta$ that for all $\eta'' \in [6\eta, 7\eta]$

$$(7.30) \quad R(\epsilon)R(2Q - \epsilon - \frac{2}{\gamma})C_\gamma(\epsilon + \frac{2}{\gamma}, \alpha, \alpha') = \frac{l(c-1)l(c-a-b+1)}{l(c-a)l(c-b)}R(\alpha)C_\gamma(\epsilon, 2Q - \alpha - \frac{2}{\gamma}, \alpha')$$

Thanks to the inversion relation, we have

$$(7.31) \quad \frac{R(\epsilon)}{R(\epsilon + \frac{2}{\gamma})}C_\gamma(\epsilon + \frac{2}{\gamma}, \alpha, \alpha') = \frac{l(c-1)l(c-a-b+1)}{l(c-a)l(c-b)}R(\alpha)C_\gamma(\epsilon, 2Q - \alpha - \frac{2}{\gamma}, \alpha')$$

Now, by analytic continuation, we can let η'' converge to 0, i.e. α converge to $\frac{\gamma}{2}$. We get the following limit when η'' goes to 0

$$\begin{aligned} \frac{l(c-1)l(c-a-b+1)}{l(c-a)l(c-b)}R(\alpha) &\underset{\eta'' \rightarrow 0}{\sim} \frac{1}{l(-\frac{4}{\gamma^2})l(\frac{2\epsilon}{\gamma})l(2 + \frac{4}{\gamma^2} - \frac{2\epsilon}{\gamma})}(-\frac{\eta''}{\gamma})R(\alpha) \\ &\underset{\eta'' \rightarrow 0}{\sim} \frac{1}{l(-\frac{4}{\gamma^2})l(\frac{2\epsilon}{\gamma})l(2 + \frac{4}{\gamma^2} - \frac{2\epsilon}{\gamma})}(-\frac{\eta''}{\gamma})\left(\frac{2}{\gamma\eta''} \frac{\mu\pi R(\gamma)}{l(-\frac{\gamma^2}{4})l(\frac{\gamma^2}{4})}\right) \\ &= \frac{1}{l(-\frac{4}{\gamma^2})l(\frac{2\epsilon}{\gamma})l(2 + \frac{4}{\gamma^2} - \frac{2\epsilon}{\gamma})}C_\gamma \end{aligned}$$

with $\tilde{c}_\gamma = \frac{\gamma^2}{8}\mu\pi R(\gamma)$ and where we have used $l(x)l(-x) = -\frac{1}{x^2}$ and the first shift equation

$$R(\alpha) = -\mu\pi \frac{R(\alpha + \frac{\gamma}{2})}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha}{2})}$$

It is clear that $C_\gamma(\epsilon + \frac{\gamma}{2}, \alpha, \alpha') \underset{\eta'' \rightarrow 0}{\sim} \frac{2}{\eta''}$. Now, we can conclude by looking at the limit as η'' goes to 0 of $C_\gamma(\frac{\gamma}{2} - \eta, Q - \eta'', \frac{2}{\gamma} + \eta)$. We prove below that

$$(7.32) \quad \lim_{\eta'' \rightarrow 0} \mathbb{E} \left[\rho(Q - \eta'', \frac{\gamma}{2} - \eta, \frac{2}{\gamma} + \eta)^{\frac{\eta''}{\gamma}} \right] = 2.$$

This yields that $C_\gamma(\frac{\gamma}{2} - \eta, Q - \eta'', \frac{2}{\gamma} + \eta) \underset{\eta'' \rightarrow 0}{\sim} -\frac{4}{\eta''}$ and therefore we get the shift equation

$$(7.33) \quad \frac{R(\epsilon)}{R(\epsilon + \frac{2}{\gamma})} = -\frac{c_\gamma}{l(-\frac{4}{\gamma^2})l(\frac{2\epsilon}{\gamma})l(2 + \frac{4}{\gamma^2} - \frac{2\epsilon}{\gamma})}$$

with $c_\gamma = \frac{\gamma^2}{4} \mu \pi R(\gamma) \neq 0$.

Remark 7.4. A straightforward computation yields that

$$\frac{\gamma^2}{4} \mu \pi R^{\text{DOZZ}}(\gamma) = \frac{(\pi \mu l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{l(\frac{4}{\gamma^2})}$$

and therefore we expect that $c_\gamma = \frac{(\pi \mu l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{l(\frac{4}{\gamma^2})}$. However, at this stage of the proof, we can not determine the constant c_γ . It will be determined indirectly in subsection 7.6.

It remains prove the limit (7.32), i.e. we study (recall expression (2.17))

$$\mathbb{E}\left[\rho(Q - \eta'', \frac{\gamma}{2} - \eta, \frac{2}{\gamma} + \eta)^{\frac{\eta''}{\gamma}}\right]$$

as η'' goes to 0 with $(\alpha_1, \alpha_2, \alpha_3) = Q - \eta'', \frac{\gamma}{2} - \eta, \frac{2}{\gamma} + \eta$, i.e. we put the $Q - \eta''$ insertion at 0. We decompose

$$\rho(Q - \eta'', \frac{\gamma}{2} - \eta, \frac{2}{\gamma} + \eta) = R_{\eta''} + S_{\eta''}$$

where $R_{\eta''}$ is the reminder

$$R_{\eta''} = \int_{|x| \geq \frac{1}{2}} \frac{|x|_+^{\gamma(\alpha_1 + \alpha_2 + \alpha_3)}}{|x|^{\gamma\alpha_1} |x - 1|^{\gamma\alpha_2}} M_\gamma(d^2x)$$

and $S_{\eta''}$ is the rest. We have the following lemma:

Lemma 7.5. For all $p \in (1, 2)$, we have

$$(7.34) \quad \lim_{\eta'' \rightarrow 0} \mathbb{E}[R_{\eta''}^{\frac{\eta'' p}{\gamma}}] = \frac{2}{2 - p}$$

Proof. It is obvious that we can replace $R_{\eta''}$ with

$$\int_{|x| \leq 1} \frac{1}{|x|^{\gamma\alpha_1}} M_\gamma(d^2x)$$

We set $M = \sup_{s \geq 0} (B_s - (Q - \eta'')s)$ where B_s is the circle average of X of radius e^{-s} . We set L_{-M} the last time $\mathcal{B}_{-s}^{\alpha_1}$ hits $-M$. Recall that $\mathbb{P}(M > v) = e^{-2\eta''v}$ and therefore $\mathbb{E}[(e^{\gamma M})^{\frac{\eta'' p}{\gamma}}] = \frac{2}{2-p}$. We have in distribution

$$\int_{|x| \leq 1} \frac{1}{|x|^{\gamma\alpha_1}} M_\gamma(d^2x) \stackrel{\text{law}}{=} e^{\gamma M} \int_{-L_{-M}}^{\infty} e^{\gamma \mathcal{B}_s^{\alpha_1}} Z_s ds.$$

Therefore, we have the following inequality with $\alpha = \frac{2}{\gamma}$ for instance and R_s a two sided Bessel process by stochastic domination (see the section 9.2 in the appendix on diffusions)

$$e^{\gamma M} \int_0^1 e^{\gamma \mathcal{B}_s^{\alpha_1}} Z_s ds \leq e^{\gamma M} \int_{-S_{-M}}^{\infty} e^{\gamma \mathcal{B}_s^{\alpha_1}} Z_s ds \leq e^{\gamma M} \int_{-\infty}^{\infty} e^{-\gamma R_s} Z_s ds.$$

From these inequalities, we can conclude that $\lim_{\eta'' \rightarrow 0} \mathbb{E}[R_{\eta''}^{\frac{\eta'' p}{\gamma}}] = \frac{2}{2-p}$. \square

Now, we have

$$\mathbb{E}[R_{\eta''}^{\frac{\eta''}{\gamma}}] \leq \mathbb{E}[\rho(Q - \eta'', \frac{\gamma}{2} - \eta, \frac{2}{\gamma} + \eta)^{\frac{\eta''}{\gamma}}] \leq \mathbb{E}[R_{\eta''}^{\frac{\eta''}{\gamma}} (1 + \frac{S_{\eta''}}{R_{\eta''}})^{\frac{\eta''}{\gamma}}]$$

The left hand side in the above inequality converges to 2. The right hand side satisfies for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\mathbb{E}[R_{\eta''}^{\frac{p''}{\gamma}} (1 + \frac{S_{\eta''}}{R_{\eta''}})^{\frac{q''}{\gamma}}] \leq \mathbb{E}[R_{\eta''}^{\frac{p''}{\gamma}}]^{1/p} \mathbb{E}[(1 + \frac{S_{\eta''}}{R_{\eta''}})^{\frac{q''}{\gamma}}]^{1/q}$$

Now for all $q > 1$, $\mathbb{E}[(1 + \frac{S_{\eta''}}{R_{\eta''}})^{\frac{q''}{\gamma}}]^{1/q} \xrightarrow{\eta'' \rightarrow 0} 1$ hence we the result (7.32) follows from the Lemma since p can be taken arbitrary close to 1.

7.6. Proof that $R = R^{\text{DOZZ}}$. Let $\psi(\alpha) = \frac{R(\alpha)}{R^{\text{DOZZ}}(\alpha)}$. ψ is meromorphic in the strip $\mathbb{R} \times (-\beta, \beta)$. Since R and R^{DOZZ} obey the same $\frac{\gamma}{2}$ shift equation, the function ψ is $\frac{\gamma}{2}$ periodic. ψ is strictly positive in $(\frac{\gamma}{2}, Q)$ so by periodicity ψ is strictly positive on \mathbb{R} . By the $\frac{\gamma}{2}$ shift equation, one has for all $\alpha \in \mathbb{R}$

$$\psi(\alpha) = C_{\gamma} \psi(\alpha + \frac{2}{\gamma})$$

for some constant C_{γ} . If $\frac{\gamma}{2}$ and $\frac{2}{\gamma}$ are independent over the rationals i.e. if $\gamma^2 \notin \mathbb{Q}$ then we conclude that $C_{\gamma} = 1$ and $\psi(\alpha) = \psi$ is constant in α . From (2.51) we see that $\bar{R}(Q) = 1$ and from (2.53) since $\Gamma(-x)x \rightarrow -1$ as $x \rightarrow 0$ we get $R(Q) = -1$. On the other hand, from (1.16) follows $R^{\text{DOZZ}}(Q) = -1$ hence the constant $\psi = 1$. Hence $R(\alpha) = R^{\text{DOZZ}}(\alpha)$ for all α . The case $\gamma^2 \in \mathbb{Q}$ follows by continuity. This concludes the proof.

8. PROOF OF THE DOZZ FORMULA

We suppose that $\gamma^2 \notin \mathbb{Q}$; the general case follows by continuity. Let us fix α_2, α_3 in $(Q - \eta, Q)$ for η sufficiently small and consider the function $F : \alpha_1 \mapsto C_{\gamma}(\alpha_1, \alpha_2, \alpha_3)$. Let us collect what we have proven about F . By Theorem 4.1 F is analytic on $(2\eta, Q)$ and by Theorem 6.1 it satisfies the the $\frac{\gamma}{2}$ shift equation (1.8), for $\frac{\gamma}{2} + 2\eta < \alpha_1 < \frac{2}{\gamma}$. Therefore F extends to a meromorphic function on a strip of the form $\mathbb{R} \times (-\beta, \beta)$ with $\beta > 0$ satisfying (1.8). We call this extension F too.

Now, using the exact expression for R (or relation (7.3) with $c_{\gamma} = \mu\pi l(\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}} l(\frac{4}{\gamma^2})^{-1}$) Theorem 6.3 can be written as

$$\mathcal{T}_{-\frac{2}{\gamma}}(z) = C_{\gamma}(\alpha_1 - \frac{2}{\gamma}, \alpha_2, \alpha_3) |F_{-}(z)|^2 - \frac{(\mu\pi l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{l(\frac{4}{\gamma^2})} \frac{R(\alpha_1 + \frac{2}{\gamma})}{l(-\frac{4}{\gamma^2}) l(\frac{2\alpha_1}{\gamma}) l(2 + \frac{4}{\gamma^2} - \frac{2\alpha_1}{\gamma})} C_{\gamma}(2Q - \alpha_1 - \frac{2}{\gamma}, \alpha_2, \alpha_3) |F_{+}(z)|^2$$

By the gluing Lemma, the extension F is given in a neighborhood of $\alpha = Q$ by $F(\alpha) = R(\alpha)F(2Q - \alpha)$. Hence, one can infer from the above expression the shift equation (1.9) for $\alpha_1 \in \mathbb{R} \times (-\beta, \beta)$ (same argument as the one used to derive (1.8)). Hence F satisfies both (1.8) and (1.9).

Now, we consider the function $\psi_{\alpha_2, \alpha_3} : \alpha_1 \mapsto \frac{C_{\gamma}(\alpha_1, \alpha_2, \alpha_3)}{C_{\gamma}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)}$ in the strip $\mathbb{R} \times (-\beta, \beta)$. This function is holomorphic since C_{γ} and C_{γ}^{DOZZ} are meromorphic with the same simple poles and zeros (which can be read off the $\frac{\gamma}{2}$ shift equation (1.8)). Furthermore, $\psi_{\alpha_2, \alpha_3}$ is γ and $\frac{4}{\gamma}$ periodic since C_{γ} and C_{γ}^{DOZZ} both satisfy (1.8) and (1.9). Therefore $\psi_{\alpha_2, \alpha_3}(\alpha_1) = c_{\alpha_2, \alpha_3}$ for some constant c_{α_2, α_3} depending on α_2, α_3 .

Since C_{γ} and C_{γ}^{DOZZ} are symmetric in their arguments we obtain $\psi_{\alpha_2, \alpha_3}(\alpha_1) = \psi_{\alpha_1, \alpha_3}(\alpha_2) = \psi_{\alpha_1, \alpha_2}(\alpha_3)$ for $\alpha_1, \alpha_2, \alpha_3 \in (Q - \eta, Q)$. Hence c_{α_2, α_3} is constant in α_2, α_3 . Therefore $C_{\gamma}(\alpha_1, \alpha_2, \alpha_3) = a_{\gamma} C_{\gamma}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ for $\alpha_1, \alpha_2, \alpha_3$ satisfying (2.14) with $N = 3$ for some constant a_{γ} (by analyticity). Finally $a_{\gamma} = 1$ since both C_{γ} and C_{γ}^{DOZZ} satisfy Lemma 2.9. \square

9. APPENDIX

9.1. Chaos estimates. We list in this Appendix estimates for chaos integrals that are used frequently in the paper. Let $A(z, \epsilon)$ be the annulus with radi $\epsilon, 2\epsilon$ and centre z . Then, for $m \in [0, \frac{4}{\gamma^2})$

$$(9.1) \quad \mathbb{E}(M_{\gamma}(A(z, \epsilon))^m) \leq C\epsilon^{\gamma Q m - \frac{\gamma^2 m^2}{2}}$$

and as a corollary

$$(9.2) \quad \mathbb{E} \left(\int_{A(z, \epsilon)} |x - z|^{-\alpha} M_\gamma(d^2x) \right)^m \leq C \epsilon^{\gamma(Q-\alpha)m - \frac{\gamma^2 m^2}{2}}$$

For negative moments we have for $\alpha > Q$:

$$(9.3) \quad \mathbb{E} \left[\left(\int_{|x-z|>\epsilon} |x - z|^{-\alpha} M_\gamma(d^2u) \right)^{-p} \right] \leq C \epsilon^{\frac{1}{2}(\alpha-Q)^2}$$

Lemma 9.1. *Assume $(\alpha_i)_{i=1, \dots, 4}$ are real numbers satisfying $\alpha_i < Q$ and $p := \gamma^{-1}(\sum_i \alpha_i - 2Q) > 0$. Consider $y_1, y_2, y_3, y_4 \in \mathbb{C}$ are such that $|y_1 - y_2| \leq |y_1 - y_3| \leq |y_2 - y_3| \leq \min_{i \in \{1, 2, 3\}} |y_4 - y_i|$.*

1) *If $\alpha_1 + \alpha_2 < Q$, $\alpha_1 + \alpha_2 + \alpha_3 \geq 0$ and $\alpha_4 \geq 0$ then*

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{B(y_1, 10)} \prod_{i=1}^4 |u - y_i|^{-\gamma \alpha_i} M_\gamma(d^2u) \right)^{-p-2} \right] \\ & \leq C \left(\frac{|y_1 - y_3|}{|y_1 - y_4|} \right)^{\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 - Q)^2} |y_1 - y_4|^{\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - Q)^2}. \end{aligned}$$

2) *If $\alpha_1 + \alpha_2 > Q$, $\alpha_3 \leq 0$ and $\alpha_3 + \alpha_4 \geq 0$ then*

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{B(y_1, 10)} \prod_{i=1}^4 |u - y_i|^{-\gamma \alpha_i} M_\gamma(d^2u) \right)^{-p-2} \right] \\ & \leq C \left(\frac{|y_1 - y_2|}{|y_1 - y_3|} \right)^{\frac{1}{2}(\alpha_1 + \alpha_2 - Q)^2} \left(\frac{|y_1 - y_3|}{|y_1 - y_4|} \right)^{\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 - Q)^2 - \frac{\alpha_3^2}{2}} |y_1 - y_4|^{\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - Q)^2}. \end{aligned}$$

9.2. A reminder on diffusions. A drifted Brownian motion $(B_t + \mu t)$ with $\mu > 0$ is a diffusion with generator $\mathcal{G}_\mu = \frac{1}{2} \frac{d^2}{dx^2} + \mu \frac{d}{dx}$. When seen until hitting $b > 0$, the dual process Y_b of $B_t + \mu t$ is a diffusion with generator $\frac{1}{2} \frac{d^2}{dx^2} - \mu \coth(\mu(b-x)) \frac{d}{dx}$. Therefore, $b - Y_b$ has generator $\frac{1}{2} \frac{d^2}{dx^2} + \mu \coth(\mu x) \frac{d}{dx}$ which is the generator of $(B_t + \mu t)$ conditioned to be positive. We denote this process \mathcal{B}_t^μ .

We have the following comparison principle:

Lemma 9.2. *There exists a probability space such that for all $\mu < \mu'$, we have $\mathcal{B}_t^\mu \leq \mathcal{B}_t^{\mu'}$ almost surely.*

Proof. For all $x > 0$, we consider the drift $\varphi_x(\mu) = \mu \coth(\mu x)$. A straightforward computation yields

$$\varphi'_x(\mu) = \frac{e^{4\mu x} - 4\mu x e^{2\mu x} - 1}{(e^{2\mu x} - 1)^2}.$$

Therefore $\varphi'_x(\mu) \geq 0$ since $e^u - ue^{\frac{u}{2}} - 1 \geq 0$ for all $u \geq 0$.

We will need another comparison principle. We want to show that \mathcal{B}_t^μ starting from 0 stochastically dominates $(B_t + \mu t)$ starting from 0 and conditioned to be above $-A$ with $A > 0$. This can also be read off the drift. Indeed, for μ, x fixed, we consider $\psi_{\mu, x}(A) = \mu \coth(\mu(x+A))$. We have

$$\forall x \geq -A, \quad \psi'_{\mu, x}(A) = \mu^2(1 - \coth(\mu(x+A))^2) \leq 0$$

□

By taking μ to 0 in lemma 9.2, the above comparison principle can in fact be extended to \mathcal{B}_t^0 where \mathcal{B}_t^0 denotes the standard 3d Bessel process.

9.3. Functional relations on $\Upsilon_{\frac{\gamma}{2}}$ and R^{DOZZ} . The function $\Upsilon_{\frac{\gamma}{2}}$ defined by (1.12) can be analytically continued to \mathbb{C} and it satisfies the following remarkable functional relations for $z \in \mathbb{C}$

$$(9.4) \quad \Upsilon_{\frac{\gamma}{2}}\left(z + \frac{\gamma}{2}\right) = \frac{\Gamma\left(\frac{\gamma}{2}z\right)}{\Gamma\left(1 - \frac{\gamma}{2}z\right)} \left(\frac{\gamma}{2}\right)^{1-\gamma z} \Upsilon_{\frac{\gamma}{2}}(z), \quad \Upsilon_{\frac{\gamma}{2}}\left(z + \frac{2}{\gamma}\right) = \frac{\Gamma\left(\frac{2}{\gamma}z\right)}{\Gamma\left(1 - \frac{2}{\gamma}z\right)} \left(\frac{\gamma}{2}\right)^{\frac{4}{\gamma}z-1} \Upsilon_{\frac{\gamma}{2}}(z).$$

The function $\Upsilon_{\frac{\gamma}{2}}$ has no poles in \mathbb{C} and the zeros of $\Upsilon_{\frac{\gamma}{2}}$ are simple (if $\gamma^2 \notin \mathbb{Q}$) and given by the discrete set $(-\frac{\gamma}{2}\mathbb{N} - \frac{2}{\gamma}\mathbb{N}) \cup (Q + \frac{\gamma}{2}\mathbb{N} + \frac{2}{\gamma}\mathbb{N})$: for more on the function $\Upsilon_{\frac{\gamma}{2}}$ and its properties, see the reviews [39, 47, 56] for instance.

With definition (1.16) and a little algebra, one can show that $R^{\text{DOZZ}}(\alpha)$ satisfies the following shift equation for all $\alpha \in \mathbb{C}$

$$(9.5) \quad R^{\text{DOZZ}}\left(\alpha - \frac{\gamma}{2}\right) = -\mu\pi \frac{R^{\text{DOZZ}}(\alpha)}{l\left(-\frac{\gamma^2}{4}\right)l\left(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{4}\right)l\left(2 + \frac{\gamma^2}{2} - \frac{\gamma\alpha}{2}\right)}$$

as well as the dual shift equation for all $\alpha \in \mathbb{C}$

$$(9.6) \quad R^{\text{DOZZ}}(\alpha) = -\frac{(\mu\pi l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{l(\frac{4}{\gamma^2})} \frac{R^{\text{DOZZ}}(\alpha + \frac{2}{\gamma})}{l(-\frac{4}{\gamma^2})l(\frac{2\alpha}{\gamma})l(2 + \frac{4}{\gamma^2} - \frac{2\alpha}{\gamma})}$$

9.4. Derivation of R^{DOZZ} form C_γ^{DOZZ} . Recall that the function $\Upsilon_{\frac{\gamma}{2}}$ satisfies the shift equations (9.4). According to the DOZZ formula (1.13), since $\Upsilon_{\frac{\gamma}{2}}(0) = 0$, we get for $\alpha > \frac{\gamma}{2}$ and using the above relations

$$\begin{aligned} \epsilon C(\alpha, \epsilon, \alpha) &\underset{\epsilon \rightarrow 0}{\sim} 4(\pi \mu l(\frac{\gamma^2}{4})) \left(\frac{\gamma}{2}\right)^{2-\gamma^2/2} \frac{\epsilon^2 \Upsilon'_{\frac{\gamma}{2}}(0)^2 \Upsilon_{\frac{\gamma}{2}}(\alpha)^2}{\epsilon^2 \Upsilon'_{\frac{\gamma}{2}}(0)^2 \Upsilon_{\frac{\gamma}{2}}(\alpha - Q) \Upsilon_{\frac{\gamma}{2}}(\alpha)} \\ &= 4(\pi \mu l(\frac{\gamma^2}{4})) \left(\frac{\gamma}{2}\right)^{2-\gamma^2/2} \frac{\Upsilon_{\frac{\gamma}{2}}(\alpha)}{\Upsilon_{\frac{\gamma}{2}}(\alpha - Q)} \\ &= 4(\pi \mu l(\frac{\gamma^2}{4})) \left(\frac{\gamma}{2}\right)^{2-\gamma^2/2} \frac{\Gamma(\frac{\gamma(\alpha - Q + \frac{2}{\gamma})}{2})}{\Gamma(1 - \frac{\gamma(\alpha - Q + \frac{2}{\gamma})}{2})} \left(\frac{\gamma}{2}\right)^{1-\gamma(\alpha - Q + \frac{2}{\gamma})} \frac{\Gamma(\frac{2(\alpha - Q)}{\gamma})}{\Gamma(1 - \frac{2(\alpha - Q)}{\gamma})} \left(\frac{\gamma}{2}\right)^{\frac{4}{\gamma}(\alpha - Q) - 1} \\ &= 4\left(\frac{2}{\gamma}\right)^{-2} (\pi \mu l(\frac{\gamma^2}{4}))^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(1 - \frac{\gamma(Q-\alpha)}{2})}{\Gamma(\frac{\gamma(Q-\alpha)}{2})} \frac{\Gamma(-\frac{2(Q-\alpha)}{\gamma})}{\Gamma(1 + \frac{2(Q-\alpha)}{\gamma})} \\ &= -4(\pi \mu l(\frac{\gamma^2}{4}))^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(-\frac{\gamma(Q-\alpha)}{2})}{\Gamma(\frac{\gamma(Q-\alpha)}{2})} \frac{\Gamma(-\frac{2(Q-\alpha)}{\gamma})}{\Gamma(\frac{2(Q-\alpha)}{\gamma})} \\ &= 4R^{\text{DOZZ}}(\alpha). \end{aligned}$$

9.5. An integral formula. We have:

Lemma 9.3. *For all $p > 0$ and $a \in (1, 2)$ the following identity holds*

$$\int_0^\infty \left(\frac{1}{(1+v)^p} - 1 \right) \frac{1}{v^a} dv = \frac{\Gamma(-a+1)\Gamma(p+a-1)}{\Gamma(p)}$$

Proof. We set $\bar{a} = -a + 1$ and $\bar{b} = p + a - 1$. We have

$$\begin{aligned} &\int_0^1 \left(\frac{1}{(1+v)^p} - 1 \right) \frac{1}{v^a} dv - \int_1^\infty \frac{1}{v^a} dv \\ &= -\frac{1}{a-1} \sum_{k \geq 1} (-1)^k \frac{(p)_k (-a+1)_k}{k! (-a+2)_k} - \frac{1}{a-1} \\ &= -\frac{1}{a-1} \sum_{k \geq 0} (-1)^k \frac{(p)_k (-a+1)_k}{k! (-a+2)_k} \\ &= \frac{1}{a} {}_2F_1(\bar{a}, \bar{a} + \bar{b}, \bar{a} + 1, z = -1) \end{aligned}$$

Now, we have

$$\begin{aligned} \int_1^\infty \frac{1}{(1+v)^p} \frac{1}{v^a} dv &= \int_0^1 \frac{1}{(1+v)^p} v^{p+a-2} dv \\ &= \frac{1}{p+a-1} \sum_{k \geq 0} (-1)^k \frac{(p)_k (p+a-1)_k}{k! (p+a)_k} \\ &= \frac{1}{\bar{b}} {}_2F_1(\bar{b}, \bar{a} + \bar{b}, \bar{b} + 1, z = -1) \end{aligned}$$

Now, we use the following formula (see [30]):

$$\bar{b} {}_2F_1(\bar{a}, \bar{a} + \bar{b}, \bar{a} + 1, z = -1) + \bar{a} {}_2F_1(\bar{b}, \bar{a} + \bar{b}, \bar{b} + 1, z = -1) = \frac{\Gamma(\bar{a} + 1) \Gamma(\bar{b} + 1)}{\Gamma(\bar{a} + \bar{b})}$$

This yields the desired relation since $\Gamma(z + 1) = z\Gamma(z)$.

9.6. Some identities. We have the following identity for all z

$$\int_{\mathbb{C}} \frac{|u-z|^{\frac{\gamma^2}{2}} - |u|^{\frac{\gamma^2}{2}} - \frac{\gamma^2}{4} |u|^{\frac{\gamma^2}{2}} (\frac{z}{u} + \frac{\bar{z}}{\bar{u}})}{|u|^{\gamma\alpha_1}} d^2u = |z|^{\gamma(Q-\alpha_1)} \frac{\pi}{l(\frac{\gamma\alpha_1}{2}) l(-\frac{\gamma^2}{4}) l(2 - \frac{\gamma\alpha_1}{2} + \frac{\gamma^2}{4})}.$$

By taking the ∂_{zz} derivative, we get

$$\frac{\gamma^2}{4} (\frac{\gamma^2}{4} - 1) \int_{\mathbb{C}} \frac{|u-z|^{\frac{\gamma^2}{2}}}{(z-u)^2 |u|^{\gamma\alpha_1}} d^2u = \frac{\gamma(Q-\alpha_1)}{2} (\frac{\gamma(Q-\alpha_1)}{2} - 1) \frac{|z|^{\gamma(Q-\alpha_1)}}{z^2} \frac{\pi}{l(\frac{\gamma\alpha_1}{2}) l(-\frac{\gamma^2}{4}) l(2 - \frac{\gamma\alpha_1}{2} + \frac{\gamma^2}{4})}.$$

Hence for $z = 1$ this yields

$$(9.7) \quad \frac{\gamma^2}{4} (\frac{\gamma^2}{4} - 1) \int_{\mathbb{C}} \frac{|u-1|^{\frac{\gamma^2}{2}}}{(1-u)^2 |u|^{\gamma\alpha_1}} d^2u = (\frac{\gamma^2}{4} + 1 - \frac{\gamma\alpha_1}{2}) (\frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2}) \frac{\pi}{l(\frac{\gamma\alpha_1}{2}) l(-\frac{\gamma^2}{4}) l(2 - \frac{\gamma\alpha_1}{2} + \frac{\gamma^2}{4})}.$$

Similarly, we get

$$(9.8) \quad \frac{\gamma^2}{4} (\frac{\gamma^2}{4} - 1) \int_{\mathbb{C}} \frac{|u-1|^{\frac{\gamma^2}{2}}}{(1-\bar{u})^2 |u|^{\gamma\alpha_1}} d^2u = (\frac{\gamma^2}{4} + 1 - \frac{\gamma\alpha_1}{2}) (\frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2}) \frac{\pi}{l(\frac{\gamma\alpha_1}{2}) l(-\frac{\gamma^2}{4}) l(2 - \frac{\gamma\alpha_1}{2} + \frac{\gamma^2}{4})}.$$

Finally, by taking the $\partial_{z\bar{z}}$ derivative, we get

$$(9.9) \quad (\frac{\gamma^2}{4})^2 \int_{\mathbb{C}} \frac{|u-1|^{\frac{\gamma^2}{2}}}{|1-u|^2 |u|^{\gamma\alpha_1}} d^2u = (\frac{\gamma^2}{4} + 1 - \frac{\gamma\alpha_1}{2})^2 \frac{\pi}{l(\frac{\gamma\alpha_1}{2}) l(-\frac{\gamma^2}{4}) l(2 - \frac{\gamma\alpha_1}{2} + \frac{\gamma^2}{4})}.$$

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