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THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN VACUUM

RAPHAËL DANCHIN AND PIOTR BOGUSŁAW MUCHA

Abstract. We are concerned with the existence and uniqueness issue for the inhomogeneous incompressible Navier-Stokes equations supplemented with H^1 initial velocity and *only bounded nonnegative density.* In contrast with all the previous works on that topics, we do not require regularity or positive lower bound for the initial density, or compatibility conditions for the initial velocity, and still obtain *unique* solutions. Those solutions are global in the two-dimensional case for general data, and in the three-dimensional case if the velocity satisfies a suitable scaling invariant smallness condition. As a straightforward application, we provide a complete answer to Lions' question in [25], page 34, concerning the evolution of a drop of incompressible viscous fluid in the vacuum.

1. Introduction

Since the pioneering works by Lichtenstein [23], Wolibner [30] and Leray [19] at the beginning of the XXth century, studying fluid mechanics models has generated important advances in the development of mathematical analysis. Very schematically, classical fluid mechanics is divided into two types of models corresponding to whether the fluid is homogeneous or not. On the one hand, the incompressible Navier-Stokes equations

$$(NS)\qquad \begin{aligned} &v_t + v\cdot\nabla v - \mu\Delta v + \nabla P = 0 && \text{in}\quad \mathbb{R}_+\times\Omega,\\ &\operatorname{div} v = 0 && \text{in}\quad \mathbb{R}_+\times\Omega,\end{aligned}$$

govern the evolution of the velocity field $v = v(t,x) \in \mathbb{R}^d$ and pressure function $P = P(t,x) \in \mathbb{R}$ of a homogeneous incompressible viscous fluid with constant viscosity $\mu > 0$ (here $t \geq 0$ stands for the time variable and $x \in \Omega$, for the position in the fluid domain $\Omega \subset \mathbb{R}^d$). On the other hand, the evolution of compressible viscous flows obeys the following system

$$(CNS)\qquad \begin{aligned} &\rho_t + \operatorname{div}(\rho v) = 0 && \text{in}\quad \mathbb{R}_+\times\Omega,\\ &\rho v_t + \rho v\cdot\nabla v - \mu\Delta v - \mu'\nabla\operatorname{div} v + \nabla P = 0 && \text{in}\quad \mathbb{R}_+\times\Omega,\end{aligned}$$

where $\rho = \rho(t,x) \geq 0$ stands for the density of the fluid and $P = P(\rho)$ is a given pressure function. In between (NS) and (CNS), we find the *inhomogeneous* Navier-Stokes system that governs the evolution of incompressible viscous flows with *nonconstant* density. That system founds his place in the theory of geophysical flows, where fluids are incompressible but with variable density, like in oceans or rivers. In the present paper, we are concerned with that latter system that reads:

$$(INS)\qquad \begin{aligned} &\rho_t + v\cdot\nabla\rho = 0 && \text{in}\quad \mathbb{R}_+\times\Omega,\\ &\rho v_t + \rho v\cdot\nabla v - \mu\Delta v + \nabla P = 0 && \text{in}\quad \mathbb{R}_+\times\Omega,\\ &\operatorname{div} v = 0 && \text{in}\quad \mathbb{R}_+\times\Omega.\end{aligned}$$

The unknowns are the velocity field $v = v(t,x)$, the density $\rho = \rho(t,x)$ and the pressure $P = P(t,x)$. We shall assume that the fluid domain Ω is either the torus $\mathbb{T}^d$ (that is the fluid domain is $]0,1[^d$ and (INS) is supplemented with periodic boundary conditions), or a $\mathcal{C}^2$

simply connected bounded domain of $\mathbb{R}^d$ with $d=2,3$. In that latter case, System (INS) is supplemented with homogeneous Dirichlet boundary conditions for the velocity.

It is well known that sufficiently smooth solutions to (INS) fulfill for all $t\geq 0$:

- The energy balance:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|v|^2\,dx+\mu\int_\Omega |\nabla v|^2\,dx=0. \tag{1.1}$$

- The conservation of total momentum (in the case $\Omega=\mathbb{T}^d$):

$$\int_\Omega (\rho v)(t,x)\,dx=\int_\Omega (\rho_0 v_0)(x)\,dx. \tag{1.2}$$

- The conservation of total mass:

$$\int_\Omega \rho(t,x)\,dx=\int_\Omega \rho_0(x)\,dx. \tag{1.3}$$

- Any Lebesgue norm of ρ_0 is preserved through the evolution, and

$$\inf_{x\in\Omega}\rho(t,x)=\inf_{x\in\Omega}\rho_0(x)\quad\text{and}\quad \sup_{x\in\Omega}\rho(t,x)=\sup_{x\in\Omega}\rho_0(x). \tag{1.4}$$

The constant density case, that is System (NS), has been intensively investigated for the last 80 years. Since the works by J. Leray [19] in 1934 and O. Ladyzhenskaya [17] in 1959 (see also [24]), it is known that:

- In dimension $d=2,3$, for any v_0 in $L_2(\Omega)$ with $\operatorname{div} v_0=0$, there exist global weak solutions (the so-called turbulent solutions) to (NS), satisfying

$$\|v(t)\|_2^2+2\mu\int_0^t\|\nabla v\|_2^2\,d\tau\leq\|v_0\|_2^2.$$

- In the 2D case, turbulent solutions are unique, and additional regularity is preserved. In particular, if v_0 is in $H^1_0(\Omega)$, then $v\in\mathcal{C}_b(\mathbb{R}_+;H^1_0(\Omega))$.
- In the 3D case, if in addition v_0 in $H^1_0(\Omega)$ and

$$\mu^{-2}\|v_0\|_2\|\nabla v_0\|_2\quad\text{is small enough} \tag{1.5}$$

  then there exists a unique global solution $(v,\nabla P)$ with v in $\mathcal{C}_b(\mathbb{R}_+;H^1_0(\Omega))$.

For a large three-dimensional v_0 in $H^1_0(\Omega)$, we have a unique local-in-time smooth solution, but proving that smoothness persists for all time is essentially the global regularity issue of one of the Millennium problems (see *http://www.claymath.org/millennium-problems*).

As regards the inhomogeneous Navier-Stokes equations, the state-of-the-art says that the weak solution theory is similar to the one of the homogeneous case, and so is the strong solution theory if, beside smoothness, *the density is bounded away from zero.* More precisely, the following results are available:

- *Global weak solutions with finite energy:* If $d=2,3$, whenever $0\leq\rho_0\leq\rho^*$ for some $\rho^*>0$, and $\sqrt{\rho_0}\,v_0$ is in L_2, there exists a global distributional solution (ρ,v,P) to (INS) satisfying (1.3), and such that for all $t\geq0$,

$$\|\sqrt{\rho(t)}\,v(t)\|_2^2+2\int_0^t\|\nabla v\|_2^2\,d\tau\leq\|\sqrt{\rho_0}\,v_0\|_2^2,\qquad\text{and}\quad 0\leq\rho(t)\leq\rho^*. \tag{1.6}$$

  The constant viscosity case with $\inf\rho_0>0$ has been solved by A. Kazhikhov [16], then J. Simon [29] removed the lower bound assumption on ρ_0, and P.-L. Lions [25] proved that ρ is in fact a renormalized solution of the mass equation, which enabled him to consider also the case where μ depends on ρ (see also [11]).

- *Global strong solutions in the 2D case:* They have been first constructed by O. Ladyzhenskaya and V. Solonnikov in [18] in the bounded domain case, whenever v_0 is in $H^1_0(\Omega)$ and ρ_0 is in W^1_∞ with $\inf \rho_0 > 0$.
- *Strong solutions in the 3D case:* Under the hypotheses of the 2D case, there exists a unique local-in-time maximal strong solution, and if v_0 is small enough, then that solution is global (see [18]).

After the work of O. Ladyzhenskaya and V. Solonnikov [18], a number of papers have been devoted to the study of strong solutions to (INS) and, more particularly, to classes of data generating regular unique solutions. Recent developments involve two directions:

- Finding minimum assumptions for uniqueness in the nonvacuum case: Here one can mention the critical regularity approach of [4, 5] where density has to be continuous, bounded and bounded away from 0, and relatively new works like [6, 8] (further improved in [3, 9, 15, 28]), relying on the use of Lagrangian coordinates, and where the density need not be continuous. Recently, in connection with Lions' question, lots of attention has been brought to the case where the initial density is given by

$$\rho_0 = \eta_1 1_{D_0} + \eta_2 1_{^cD_0}, \qquad \eta_1, \eta_2 > 0, \qquad D_0 \subset \Omega.$$

  The main issue is whether the smoothness of D_0 is preserved through the time evolution (see [10, 13, 21, 22]).
- Smooth data with allowance of vacuum: As pointed out in [2] (see also [3], and [14] as regards the weak-strong uniqueness issue), one can solve (INS) uniquely in presence of vacuum if ρ_0 is smooth enough *and v_0 satisfies the compatibility condition*

$$-\Delta v_0 + \nabla P_0 = \sqrt{\rho_0}\, g \quad \text{for some} \quad g \in L_2(\Omega) \quad \text{and} \quad P_0 \in H^1(\Omega). \tag{1.7}$$

  Very recently, in [20], J. Li pointed out that Condition (1.7) can be removed but, still, the density must be regular and only local-in-time solutions are produced.

If it is not assumed that the density is bounded away from zero then the analysis gets wilder, since the system degenerates (in vacuum regions, the term ρv_t in the momentum equation vanishes), and the general strong solution theory is still open, even in the 2D case. Our main goal is to show existence and uniqueness for (INS) supplemented with general initial data satisfying the following 'minimal' assumptions:

$$v_0 \in H^1_0(\Omega) \ \text{ with } \ \operatorname{div} v_0 = 0, \quad \text{and} \quad 0 \le \rho_0 \le \rho^* < +\infty. \tag{1.8}$$

In other words, we aim at completing the program initiated by J. Leray, *for general inhomogeneous fluids with just bounded initial density*, establishing that

- in the 2D case, for arbitrary initial data fulfilling just (1.8) there exists a unique global-in-time solution with regular velocity;
- in the 3D case: for arbitrary initial data fulfilling (1.8) there exist a unique local-in-time solution with regular velocity, that is global if (1.5) is fulfilled.

Let us emphasize that, since we do not require any regularity or positive lower bound for the density, one can consider 'patches of density', that is initial densities that are characteristic functions of subsets of Ω (this corresponds for instance to a drop of incompressible fluid in vacuum or the opposite: a bubble of vacuum embedded in the fluid). As a consequence of our results, we show the persistence of the interface regularity through the evolution, which constitutes a complete answer to the question raised by P.-L. Lions in [25] at page 34.

In order to prove the above existence and uniqueness results for (INS) supplemented with data satisfying just (1.8), the main difficulty is to propagate enough regularity for the velocity to ensure uniqueness, while the density is rough and likely to vanish in some parts of the fluid domain. Recall that in most evolutionary fluid mechanics models, the uniqueness issue is closely connected to the Lipschitz control of the flow of the velocity field v, hence to the fact that ∇v is in $L_1(0,T;L_\infty(\Omega))$. The main breakthrough of our paper is that we manage to keep the H^1 norm of the velocity under control for all time *despite vacuum* and to exhibit a parabolic gain of regularity which is slightly weaker than the standard one, but still sufficient to eventually get ∇v in $L_1(0,T;L_\infty(\Omega))$. To achieve it, we combine time weighted energy estimates in the spirit of [20], classical Sobolev embedding and *shift of integrability* from time variable to space variable (more details are provided in the next section).

We shall assume throughout that the fluid domain is either the torus $\mathbb{T}^d$ or a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^d$, the generalization to unbounded domains (even the whole space) within our approach being unclear as regards global-in-time results. For simplicity, we only consider H^1 initial velocity fields, even though it should be possible to have less regular data, like in [28]. As regards the domain Ω, we do not strive for minimal regularity assumptions either.

We follow the standard notation for the evolutionary PDEs. By ∇ we denote the gradient with respect to space variables, and by $\partial_t u$ or u_t, the time derivative of function u. By $\|\cdot\|_p$, we mean p-power Lebesgue norms over Ω; $L_p(Q)$ is the p-Lebesgue space over a set Q; we denote by H^s and W^s_p the Sobolev (Slobodeckij for s not integer) space, and put $H^s=W^s_2$. Generic constants are denoted by C, and $A\lesssim B$ means that $A\leq CB$.

Finally, as a great part of our analysis will concern H^1 regularity and will work indistinctly in a bounded domain or in the torus, we shall adopt slightly abusively the notation $H^1_0(\Omega)$ to designate the set of $H^1(\Omega)$ functions that vanish at the boundary if Ω is a bounded domain, or general $H^1(\mathbb{T}^d)$ functions if $\Omega=\mathbb{T}^d$.

The rest of the paper unfolds as follows. In the next section, we state the main results. In Section 3, we concentrate on a priori estimates and on the proof of existence for (INS) while Section 4 concerns the uniqueness issue. A few technical results (in particular a key logarithmic interpolation inequality) are postponed in the appendix.

2. Results

Here we state of our results and give a overview of the strategy that we used to achieve them. Let us first write out our main result in the two dimensional case.

Theorem 2.1. *Let Ω be a $\mathcal{C}^2$ bounded subset of $\mathbb{R}^2$, or the torus $\mathbb{T}^2$. Consider any data (ρ_0,v_0) in $L_\infty(\Omega)\times H^1_0(\Omega)$ satisfying for some constant $\rho^*>0$,*

$$0\leq\rho_0\leq\rho^*,\quad \operatorname{div} v_0=0\quad \text{and}\quad M:=\int_\Omega\rho_0\,dx>0. \tag{2.1}$$

Then System (INS) supplemented with data (ρ_0,v_0) admits a unique global solution $(\rho,v,\nabla P)$ satisfying (1.1), (1.2) (in the case $\Omega=\mathbb{T}^d$), (1.3), (1.4), the following properties of regularity:

$$\rho\in L_\infty(\mathbb{R}_+;L_\infty(\Omega)),\quad v\in L_\infty(\mathbb{R}_+;H^1_0(\Omega)),\quad \sqrt{\rho}v_t,\nabla^2v,\nabla P\in L_2(\mathbb{R}_+;L_2(\Omega))$$

and also, for all $1\leq r<2$ and $1\leq m<\infty$,

$$\nabla(\sqrt{t}P),\ \nabla^2(\sqrt{t}v)\in L_\infty(0,T;L_r(\Omega))\cap L_2(0,T;L_m(\Omega))\quad \text{for all } T>0.$$

Furthermore, we have $\sqrt{\rho}v \in \mathcal{C}(\mathbb{R}_+; L_2(\Omega))$, $\rho \in \mathcal{C}(\mathbb{R}_+; L_p(\Omega))$ *for all finite* p, *and* $v \in H^\eta(0,T; L_p(\Omega))$ *for all* $\eta < 1/2$ *and* $T > 0$.

In the three dimensional case we have:

Theorem 2.2. *Let* Ω *be a* $\mathcal{C}^2$ *bounded subset of* $\mathbb{R}^3$ *or the torus* $\mathbb{T}^3$. *There exists a constant* $c > 0$ *such that for any data* (ρ_0, v_0) *in* $L_\infty(\Omega) \times H_0^1(\Omega)$ *satisfying* (2.1) *and*

$$(\rho^*)^{\frac{3}{2}} \|\sqrt{\rho_0}\, v_0\|_2 \|\nabla v_0\|_2 \leq c\mu^2, \tag{2.2}$$

System (INS) *supplemented with data* (ρ_0, v_0) *admits a unique global solution* $(\rho, v, \nabla P)$ *satisfying Identities* (1.1), (1.2) *(in the case* $\Omega = \mathbb{T}^d$*),* (1.3), (1.4), *and such that*

$$\rho \in L_\infty(\mathbb{R}_+; L_\infty(\Omega)), \quad v \in L_\infty(\mathbb{R}_+; H_0^1(\Omega)), \quad \sqrt{\rho}v_t, \nabla^2 v, \nabla P \in L_2(\mathbb{R}_+; L_2(\Omega))$$
$$\text{and} \quad \nabla(\sqrt{t}P),\ \nabla^2(\sqrt{t}v) \in L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; L_6(\Omega)) \quad \text{for all } T > 0.$$

Furthermore, we have $\sqrt{\rho}v \in \mathcal{C}(\mathbb{R}_+; L_2(\Omega))$, $\rho \in \mathcal{C}(\mathbb{R}_+; L_p(\Omega))$ *for all finite* p, *and* $v \in H^\eta(0,T; L_6(\Omega))$ *for all* $\eta < 1/2$ *and* $T > 0$.

As a by-product, we obtain the following answer to Lions' question ([25], page 34):

Corollary 2.1. *Let* Ω *be a* $\mathcal{C}^2$ *bounded domain of* $\mathbb{R}^d$ *(or the torus* $\mathbb{T}^d$*) with* $d = 2, 3$. *Assume that* $\rho_0 = 1_{D_0}$ *for some open subset* D_0 *of* Ω *with* $\mathcal{C}^{1,\alpha}$ *regularity (*$0 < \alpha < 1$ *if* $d = 2$ *and* $0 < \alpha < 1/2$ *if* $d = 3$*). Then for any divergence free initial velocity* v_0 *in* $H_0^1(\Omega)$ *(satisfying* (2.2) *with* $\rho^* = 1$ *if* $d = 3$*), the unique global solution* $(\rho, v, \nabla P)$ *provided by the above theorems is such that for all* $t \geq 0$,

$$\rho(t,\cdot) = 1_{D_t} \quad \text{with} \quad D_t := X(t, D_0),$$

where $X(t,\cdot)$ *stands for the flow of* v, *that is the unique solution of*

$$\frac{dX}{dt} = v(t,X), \qquad X|_{t=0} = y, \qquad y \in \Omega. \tag{2.3}$$

Furthermore, D_t *has* $\mathcal{C}^{1,\alpha}$ *regularity with a control of the Hölder norm in terms of the initial data.*

Proof. Assume that D_0 corresponds to the level set $\{f_0 = 0\}$ of some function $f_0 : \Omega \to \mathbb{R}$ with $\mathcal{C}^{1,\alpha}$ regularity. Then we have $D_t := f_t^{-1}(\{0\})$ with $f_t := f \circ X(t,\cdot)$. Fix some $T > 0$.

In the 2D case, Theorem 2.1 and interpolation imply that we have

$$\sqrt{t}\nabla^2 v \in L_{2+\varepsilon}(0,T; L_{1/\varepsilon}(\Omega)) \ \text{ for all small enough } \varepsilon > 0.$$

By Sobolev embedding with respect to the space variable, and Hölder inequality with respect to the time variable, one can conclude that ∇v is in $L^1(0,T; \mathcal{C}^{0,\beta})$ for all $\beta \in (0,1)$. Consequently the flow $X(t,\cdot)$ is in $\mathcal{C}^{1,\beta}$ for all $\beta \in (0,1)$, which implies that f_t is in $\mathcal{C}^{1,\alpha}$ provided that $\alpha < 1$.

Similarly, in the 3D case, Theorem 2.2 ensures that $\sqrt{t}\nabla^2 v$ is in $L_{2+\varepsilon}(0,T; L_{6-\varepsilon}(\Omega))$, and thus ∇v is in $L_1(0,T; W_r^1)$ for all $r < 6$. This implies that the flow $X(t,\cdot)$ is in $\mathcal{C}^{1,\beta}$ for all $\beta \in (0,1/2)$, and thus f_t is in $\mathcal{C}^{1,\alpha}$ if $\alpha < 1/2$. □

Remark 2.1. *In the 3D case, there exists a constant* $c = c(\Omega)$ *such that if Condition* (2.2) *is not satisfied, then Theorem 2.2 and Corollary 2.1 hold true on the time interval* $[0, T]$ *with* $T := \left(\frac{\mu}{\rho^*}\right)^7 \frac{c\rho^*}{\|\sqrt{\rho_0}v_0\|_2^2 \|\nabla v_0\|_2^6}$.

Remark 2.2. *The time regularity issue is rather subtile. In fact, unless the density is bounded away from* 0, *we do not have* v *in* $\mathcal{C}(\mathbb{R}_+; L_2)$. *At the same time, as the kinetic energy satisfies the energy balance* (1.1), *we do have* $\sqrt{\rho}v \in \mathcal{C}(\mathbb{R}_+; L_2)$. *It follows that, whenever the initial kinetic energy vanishes (that is* $\int_{\mathbb{T}^d} \rho_0 |v_0|^2\,dx = 0$*), the unique solution provided by Theorems 2.1 and 2.2 is* $v \equiv 0$ *for* $t > 0$, even though v_0 need not be 0. *This is consistent with physics, and with the time regularity properties exhibited above.*

Let us give some insight on the proof of Theorems 2.1 and 2.2. As in the constant density case, in order to get uniqueness, one has to propagate enough regularity of the velocity. In the present situation, starting from H^1 regularity for the velocity and implementing a basic energy method on the momentum equation, we will succeed in extracting some parabolic smoothing effect *even if the density is rough and vanishes.* At the end, we will have a control on $v_t, \nabla^2 v, \nabla P$ in $L_2(0, T \times \mathbb{T}^d)$ in terms of the initial data. Let us make it more precise : after testing $(INS)_2$ by v_t, it appears that the only troublemaker is the convection term $\rho v \cdot \nabla v$. In the case $\inf \rho_0 > 0$, the usual approach in dimension $d = 2$ (that goes back to [18]) is to combine Hölder inequality and the following special case of Gagliardo-Nirenberg inequalities, first pointed out by O. Ladyzhenskaya in [17]:

$$\|z\|_4^2 \leq C\|z\|_2 \|\nabla z\|_2, \tag{2.4}$$

to eventually get for all $\varepsilon > 0$,

$$\begin{aligned}\|\rho v \cdot \nabla v\|_2 &\leq C\rho^* \|v\|_2^{\frac{1}{2}} \|\nabla v\|_2 \|\nabla^2 v\|_2^{1/2}\\ &\leq \frac{C}{\varepsilon^{1/3}}(\rho^*)^2 \|v\|_2^2 \|\nabla v\|_2^2 \|\nabla v\|_2^2 + \varepsilon \|\nabla^2 v\|_2^2.\end{aligned}$$

The last term may be ‘absorbed’ if ε is chosen small enough, and the first one may be handled by Gronwall inequality. From it, one gets a global-in-time control on $\|v\|_{H^1}$, provided one can bound

$$\int_0^t \|v\|_2^2 \|\nabla v\|_2^2\,dt$$

in terms of the data. If ρ *is bounded away from* 0, then this is a consequence of the basic energy balance (1.6), as $\|v\|_2 \leq (\inf \rho)^{-1/2} \|\sqrt{\rho}\, v\|_2$.

To handle the case where we just have $\rho_0 \geq 0$, we shall take advantage of the following Desjardins' interpolation inequality (proved in [12] and in the appendix):

$$\left(\int \rho |v|^4\,dx\right)^{\frac{1}{2}} \leq C\|\sqrt{\rho}v\|_2 \|\nabla v\|_2 \log^{\frac{1}{2}}\left(e + \frac{\|\rho - M\|_2^2}{M^2} + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2}\right). \tag{2.5}$$

Note that (2.5) has just an additional logarithmic term compared to Ladyzhenskaya inequality, hence using a suitable generalized Gronwall inequality gives us a chance to get a global control on the solution for all time. Note also that in (2.5) the log correction involves just $\|\nabla v\|_2$, not higher norms of v.

The three-dimensional case turns out to be more direct, if we assume either smallness of v, or restrict to local-in-time results (the global existence issue for large data being open, as in the constant density case).

Looking back at what we obtained so far, we see that we have just $v \in L_2(0,T;H^2)$, hence we miss (by a little in dimension 2 and half a derivative in dimension 3) the property that

$$\nabla v \quad \text{is in} \quad L_1(0,T;L_\infty(\Omega)), \tag{2.6}$$

which, in most fluid mechanics models, is (almost) a necessary condition for uniqueness, and is also strongly connected to the existence and uniqueness of a Lipschitz flow for the velocity (and thus to the possibility of reformulate System (INS) in Lagrangian coordinates). At this stage, the idea is *to shift integrability from time to space variables*, that is

$$v \in L_2(0,T;H^2) \rightsquigarrow v \in L_{2-\sigma}(0,T;W^2_{2+\delta}) \quad \text{for suitable } \sigma, \delta > 0. \tag{2.7}$$

Indeed, it is clear that if (2.7) holds true (with $\delta > 1$ in dimension 3) then using Sobolev embedding gives (2.6). Getting (2.7) will follow from *time-weighted* estimates, a technique originating from the theory of parabolic equations that has been effectively applied to (INS) recently, in [20, 28]. In fact, we prove by means of a standard energy method that

$$\sup_{0\leq t\leq T}\biggl(t\int_{\mathbb{T}^d}\rho|v_t|^2\,dx\biggr)+\int_0^T\biggl(t\int_{\mathbb{T}^d}|\nabla v_t|^2\,dx\biggr)dt\leq C_{0,T} \tag{2.8}$$

with $C_{0,T}$ depending only on ρ^*, $\|\sqrt{\rho_0}v_0\|_2$, $\|\nabla v_0\|_2$ and T.

Now, one may bootstrap the regularity provided by (2.8) by rewriting the velocity equation of (INS) in the following elliptic form, treating the time variable as a parameter:

$$\begin{aligned} &-\Delta\sqrt{t}\,v+\nabla\sqrt{t}\,P=-\rho\sqrt{t}\,v_t-\sqrt{t}\,\rho v\cdot\nabla v && \text{in} \quad \Omega,\\ &\operatorname{div}\sqrt{t}\,v=0 && \text{in} \quad \Omega. \end{aligned} \tag{2.9}$$

In the 2D case, taking advantage of the classical maximal regularity properties of the Stokes system, as in [8] (or in [26, 27] in the context of the compressible Stokes system), we readily get for all $2\leq p\leq\infty$ and $\varepsilon>0$,

$$\nabla^2\sqrt{t}\,v, \nabla\sqrt{t}P \in L_p(0,T;L_{p^*-\varepsilon}), \qquad p^* := \frac{2p}{p-2}\cdot \tag{2.10}$$

Similarly, in the 3D case, we end up with

$$\nabla^2\sqrt{t}\,v, \nabla\sqrt{t}\,P \in L_p(0,T;L_r) \quad \text{with} \quad 2\leq r\leq \frac{6p}{3p-4}\cdot \tag{2.11}$$

In both cases, this implies that $\sqrt{t}\nabla v$ is in $L_3(0,T;L_\infty)$, and thus, by Hölder inequality,

$$\int_0^T\|\nabla v\|_\infty\,dt=\int_0^T\|\sqrt{t}\nabla v\|_\infty\,\frac{dt}{\sqrt{t}}\leq CT^{1/6}\|\sqrt{t}\nabla v\|_{L_3(0,T;L_\infty)}.$$

Finally, having (2.6) enables us to reformulate System (INS) in Lagrangian coordinates (see the beginning of Section 4) without requiring more regularity on the data than (2.1). This is the key to uniqueness (the direct method based on stability estimates for (INS) is bound to fail owing to the hyperbolicity of the mass equation: one lose one derivative, and one cannot afford any loss as ρ is not regular enough). As already pointed out in [6, 8], this loss does not occur if one looks at the difference between two solutions of (INS) originating from the same initial data, *in Lagrangian coordinates.* Estimating that difference may be done by means of basic energy arguments. The only difficulty is that the divergence is no longer 0 and one thus first has to solve a 'twisted' divergence equation to remove the non-divergence free part. Then, ending up with a Gronwall lemma, we get uniqueness on a small enough time interval, and arguing by induction yields uniqueness on the existence time interval.

3. The proof of existence in Theorems 2.1 and 2.2

This section mainly concerns the proof of a priori estimates for smooth solutions of (INS). Those estimates will eventually enable us to prove the existence part of Theorems 2.1 and 2.2 for any data satisfying (1.8).

For notational simplicity, we shall assume throughout that $\mu = 1$. This is not restrictive since $(\rho, v, P)(t,\cdot)$ satisfies (INS) with viscosity μ if and only if $(\rho, \mu v, \mu^2 P)(\mu t, \cdot)$ satisfies (INS) with viscosity 1. Finally, for expository purpose, we shall focus on the torus case. As the proof follows from energy arguments, functional embeddings and a Poincaré inequality which is the standard one in the bounded domain case, and is not obvious only in the torus case (see Lemma 5.1), the case of bounded domain may be treated along the same lines.

3.1. The persistence of Sobolev regularity.

In the 2D case, the first step is to prove the following a priori estimate.

Proposition 3.1. *Let (ρ, v) be a smooth solution to System (INS) on $[0,T)\times\mathbb{T}^2$, satisfying $0 \leq \rho \leq \rho^*$. There exists a constant C_0 depending only on M, $\|\rho_0\|_2$, $\|\sqrt{\rho_0}v_0\|_2$ and ρ^* so that for all $t \in [0,T)$, we have*

$$\|\nabla v(t)\|_2^2 + \frac{1}{2}\int_0^t \Big(\|\sqrt{\rho}v_t\|_2^2 + \frac{1}{\rho^*}\|\nabla^2 v, \nabla P\|_2^2\Big)d\tau \leq \big(e + \|\nabla v_0\|_2^2\big)^{\exp\{C_0\|\sqrt{\rho_0}\,v_0\|_{L_2}^2\}} - e. \tag{3.1}$$

Furthermore, for all $p \in [1,\infty)$ and $t \in [0,T)$, we have

$$\|v(t)\|_p \leq \frac{1}{M}\bigg|\int_{\mathbb{T}^2}(\rho_0 v_0)(x)\,dx\bigg| + C_p\bigg(1 + \frac{\|M-\rho_0\|_2}{M}\bigg)\|\nabla v(t)\|_2. \tag{3.2}$$

Proof. It is based on the following improvement of the Ladyzhenskaya inequality that has been pointed out by B. Desjardins in [12] (see also the Appendix of the present paper).

Lemma 3.1. *There exists a constant C so that for all $z \in H^1(\mathbb{T}^2)$ and function $\rho \in L_\infty(\mathbb{T}^2)$ with $0 \leq \rho \leq \rho^*$, we have*

$$\bigg(\int \rho z^4\,dx\bigg)^{\frac{1}{2}} \leq C\|\sqrt{\rho}z\|_2\|\nabla z\|_2 \log^{\frac{1}{2}}\bigg(e + \frac{\|\rho - M\|_2^2}{M^2} + \frac{\rho^*\|\nabla z\|_2^2}{\|\sqrt{\rho}z\|_2^2}\bigg)\cdot \tag{3.3}$$

Now, testing the momentum equation of (INS) by v_t yields:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla v|^2dx + \int_{\mathbb{T}^2}\rho|v_t|^2\,dx &= -\int_{\mathbb{T}^2}(\rho v\cdot\nabla v)\cdot v_t\,dx\\ &\leq \frac{1}{2}\int_{\mathbb{T}^2}\rho|v_t|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\rho|v\cdot\nabla v|^2\,dx.\end{aligned}$$

Hence

$$\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla v|^2dx + \int_{\mathbb{T}^2}\rho|v_t|^2\,dx \leq \int_{\mathbb{T}^2}\rho|v\cdot\nabla v|^2\,dx. \tag{3.4}$$

In order to estimate the second derivatives of v and the gradient of the pressure, we look at $(INS)_2$ in the form

$$\begin{aligned}&-\Delta v + \nabla P = -\rho v_t - \rho v\cdot\nabla v \quad \text{in} \quad (0,T)\times\mathbb{T}^2,\\ &\operatorname{div} v = 0 \qquad\qquad\qquad\qquad\quad\;\; \text{in} \quad (0,T)\times\mathbb{T}^2.\end{aligned} \tag{3.5}$$

Then, from the Helmholtz decomposition on the torus, we get

$$\|\nabla^2 v\|_2^2 + \|\nabla P\|_2^2 = \|\rho(v_t + v\cdot\nabla v)\|_2^2 \leq \frac{\rho^*}{2}\bigg(\int_{\mathbb{T}^2}\rho|v_t|^2\,dx + \int_{\mathbb{T}^2}\rho|v\cdot\nabla v|^2\,dx\bigg)\cdot$$

Putting together with (3.4) thus yields

$$\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla v|^2dx+\frac12\int_{\mathbb{T}^2}\rho|v_t|^2\,dx+\frac{1}{\rho^*}\big(\|\nabla^2v\|_2^2+\|\nabla P\|_2^2\big)\leq\frac32\int_{\mathbb{T}^2}\rho|v\cdot\nabla v|^2\,dx. \tag{3.6}$$

In order to bound the last term, we write that, thanks to (2.4),

$$\int_{\mathbb{T}^2}\rho|v\cdot\nabla v|^2\,dx\leq\|\sqrt{\rho}|v|^2\|_2\|\sqrt{\rho}|\nabla v|^2\|_2\leq C\sqrt{\rho^*}\|\sqrt{\rho}|v|^2\|_2\|\nabla v\|_2\|\nabla^2v\|_2. \tag{3.7}$$

To bound the term $\|\sqrt{\rho}|v|^2\|_2$ despite the fact that ρ vanish, it suffices to combine Inequality (3.3) (after observing that the function $z\mapsto z\log(e+1/z)$ is increasing), the energy balance (1.1) and (1.3), to get

$$\|\sqrt{\rho}\,|v|^2\|_2^2\leq C\|\sqrt{\rho_0}\,v_0\|_2^2\|\nabla v\|_2^2\log\bigg(e+\frac{\|\rho_0-M\|_2^2}{M^2}+\rho^*\frac{\|\nabla v\|_2^2}{\|\sqrt{\rho_0}\,v_0\|_2^2}\bigg). \tag{3.8}$$

Reverting to (3.7), we end up with

$$\begin{aligned}\int_{\mathbb{T}^2}\rho|v\cdot\nabla v|^2dx&\leq\frac{1}{3\rho^*}\|\nabla^2v\|_2^2+C(\rho^*)^2\|\sqrt{\rho}|v|^2\|_2^2\|\nabla v\|_2^2\\&\leq\frac{1}{3\rho^*}\|\nabla^2v\|_2^2+C(\rho^*)^2\|\sqrt{\rho_0}\,v_0\|_2^2\|\nabla v\|_2^4\log\bigg(e+\frac{\|\rho_0-M\|_2^2}{M^2}+\rho^*\frac{\|\nabla v\|_2^2}{\|\sqrt{\rho_0}\,v_0\|_2^2}\bigg).\end{aligned}$$

Then combining with (3.6) yields

$$\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla v|^2dx+\frac12\int_{\mathbb{T}^2}\Big(\rho|v_t|^2+\frac{1}{\rho^*}\big(|\nabla^2v|^2+|\nabla P|^2\big)\Big)dx\leq C_0\|\nabla v\|_2^2\log(e+\|\nabla v\|_2^2)\|\nabla v\|_2^2$$

with C_0 depending only on ρ^*, $\|\sqrt{\rho_0}\,v_0\|_2$, M and $\|\rho_0-M\|_2$.

Denoting $f(t):=C_0\|\nabla v(t)\|_{L_2}^2$ and

$$X(t):=\int_{\mathbb{T}^2}|\nabla v(t)|^2\,dx+\frac12\int_{\mathbb{T}^2}\Big(\rho|v_t|^2+\frac{1}{\rho^*}\big(|\nabla^2v|^2+|\nabla P|^2\big)\Big)dx,$$

the above inequality rewrites

$$\frac{d}{dt}X\leq fX\log(e+X),$$

from which we get, for all $t\geq0$,

$$e+X(t)\leq\big(e+X(0)\big)^{\exp\int_0^tf(\tau)\,d\tau}.$$

Hence, by virtue of (1.1), we have (3.1).

In order to prove (3.2), we observe that for all $p\in[1,\infty)$, denoting by $\bar v(t)$ the average of $v(t)$ on $\mathbb{T}^2$, we have by Sobolev embedding,

$$\|v(t)\|_p\leq|\bar v(t)|+\|v(t)-\bar v(t)\|_p\leq|\bar v(t)|+C_p\|\nabla v(t)\|_2. \tag{3.9}$$

Now, from the mass conservation (1.3), we get

$$M\,\bar v(t)=\int_{\mathbb{T}^2}(\rho v)(t,x)\,dx+\int_{\mathbb{T}^2}(M-\rho(t,x))(v(t,x)-\bar v(t))\,dx.$$

Hence, using (1.2), the conservation of the L^2 norm of the density, then Cauchy-Schwarz and Poincaré inequalities,

$$M\,|\bar v(t)|\leq\bigg|\int_{\mathbb{T}^2}(\rho_0v_0)(x)\,dx\bigg|+\|M-\rho_0\|_2\|\nabla v(t)\|_2.$$

Plugging that latter inequality in (3.9) yields (3.2). □

The adaptation of Proposition 3.1 to the 3D case reads as follows.

Proposition 3.2. *Let (ρ, v) satisfy System (INS) with $0 \le \rho_0 \le \rho^*$ and divergence free v_0 in $H^1(\mathbb{T}^3)$. There exists a universal constant c such that if*

$$(\rho^*)^{\frac{3}{2}}\|\sqrt{\rho_0}\, v_0\|_2\|\nabla v_0\|_2 \le c \tag{3.10}$$

then for all $t \in [0, T)$, we have

$$\|\nabla v(t)\|_2^2 + \frac{1}{2}\int_0^t \|\sqrt{\rho}\, v_t\|_2^2\, d\tau + \frac{1}{2\rho^*}\int_0^t \big(\|\nabla^2 v\|_2^2 + \|\nabla P\|_2^2\big)\, d\tau \le 2\|\nabla v_0\|_2^2. \tag{3.11}$$

Furthermore, if (3.10) *is not satisfied then* (3.11) *holds true on* $[0, T]$, *if*

$$T \le \frac{c}{(\rho^*)^6\|\sqrt{\rho_0} v_0\|_2^2\|\nabla v_0\|_2^6}. \tag{3.12}$$

Finally, Inequality (3.2) *holds true for all* $p \in [1, 6]$.

Proof. Compared to the 2D case, the only difference is when bounding the r.h.s. of (3.6): we write that owing to the Hölder inequality and Sobolev embedding $\dot H^1(\mathbb{T}^3) \hookrightarrow L_6(\mathbb{T}^3)$,

$$\begin{aligned}
\int_{\mathbb{T}^3} \rho|v\cdot\nabla v|^2\, dx &\le (\rho^*)^{1/2}\|\rho^{1/4}v\|_4^2\|\nabla v\|_4^2\\
&\le (\rho^*)^{3/4}\|\sqrt{\rho}\, v\|_2^{1/2}\|v\|_6^{3/2}\|\nabla v\|_2^{1/2}\|\nabla v\|_6^{3/2}\\
&\le C(\rho^*)^{3/4}\|\sqrt{\rho}\, v\|_2^{1/2}\|\nabla v\|_2^{3/2}\|\nabla v\|_2^{1/2}\|\nabla^2 v\|_2^{3/2}\\
&\le \frac{1}{3\rho^*}\|\nabla^2 v\|_2^2 + C(\rho^*)^6\|\sqrt{\rho}\, v\|_2^2\|\nabla v\|_2^8.
\end{aligned}$$

Therefore, using (3.6), we see that $\frac{d}{dt}X \le fX^3$ with

$$\begin{aligned}
X(t) &:= \|\nabla v(t)\|_2^2 + \frac{1}{2}\int_0^t \|\sqrt{\rho}\, v_t\|_2^2\, d\tau + \frac{1}{2\rho^*}\int_0^t \big(\|\nabla^2 v\|_2^2 + \|\nabla P\|_2^2\big)\, d\tau\\
\text{and}\quad f(t) &:= C(\rho^*)^6\|\sqrt{\rho}\, v\|_2^2\|\nabla v\|_2^2.
\end{aligned}$$

Hence, whenever T satisfies $2X^2(0)\int_0^T f(t)\, dt < 1$, we have

$$X^2(t) \le \frac{X^2(0)}{1 - 2X^2(0)\int_0^t f(\tau)\, d\tau} \quad \text{for all}\quad t \in [0, T].$$

Now, the basic energy conservation (1.1) tells us that

$$\int_0^T f(t)\, dt \le C(\rho^*)^6\|\sqrt{\rho_0} v_0\|_2^4.$$

Hence, if $(\rho^*)^{\frac{3}{2}}\|\sqrt{\rho_0}\, v_0\|_2\|\nabla v_0\|_2$ is small enough, then we have (3.11) for all value of T. If that condition is not satisfied, then we observe that

$$2X^2(0)\int_0^T f(t)\, dt \le \frac{1}{2} \quad \text{implies} \quad \sup_{t\in[0,T]} X^2(t) \le 2X^2(0).$$

Of course, one can use the fact that

$$\int_0^T f(t)\, dt \le C(\rho^*)^6\|\sqrt{\rho_0} v_0\|_2^2\, T \sup_{t\in[0,T]} X(t),$$

which, together with a bootstrap argument, ensures the second part of the statement.

The proof of the last part of statement goes exactly as in the 2D case. The only difference is that Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L_p(\mathbb{T}^3)$ holds true only for $p \leq 6$. $\square$

3.2. **Estimates on time derivatives.** Our next aim is to bound $\sqrt{\rho t}\, v_t$ and $\sqrt{t}\,\nabla v_t$ in $L_\infty([0,T];L_2)$ and $L_2([0,T];L_2)$, respectively, in terms of the data. To achieve it, let us differentiate $(INS)_2$ with respect to t:

$$\rho v_{tt} + \rho_t v_t + \rho_t v\cdot\nabla v + \rho v_t\cdot\nabla v + \rho v\cdot\nabla v_t - \Delta v_t + \nabla P_t = 0. \tag{3.13}$$

Then, multiplying (3.13) by $\sqrt{t}$ yields

$$\rho(\sqrt{t}\,v_t)_t - \frac{1}{2\sqrt{t}}\rho v_t + \sqrt{t}\,\rho_t v_t + \sqrt{t}\,\rho_t v\cdot\nabla v + \sqrt{t}\,\rho v_t\cdot\nabla v + \sqrt{t}\,\rho v\cdot\nabla v_t - \Delta(\sqrt{t}\,v_t) + \nabla(\sqrt{t}\,P_t) = 0.$$

Taking the L^2 scalar product with $\sqrt{t}\,v_t$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \rho t|v_t|^2\,dx + \int_{\mathbb{T}^d} t|\nabla v_t|^2\,dx = \sum_{i=1}^5 I_i, \tag{3.14}$$

with

$$I_1 = \frac{1}{2}\int_{\mathbb{T}^d} \rho|v_t|^2\,dx, \tag{3.15}$$

$$I_2 = -\int_{\mathbb{T}^d} t\rho_t|v_t|^2\,dx, \tag{3.16}$$

$$I_3 = -\int_{\mathbb{T}^d} \big(\sqrt{t}\,\rho_t v\cdot\nabla v\big)\cdot(\sqrt{t}\,v_t)\,dx, \tag{3.17}$$

$$I_4 = -\int_{\mathbb{T}^d} \big(\sqrt{t}\,\rho v_t\cdot\nabla v\big)\cdot(\sqrt{t}\,v_t)\,dx, \tag{3.18}$$

$$I_5 = -\int_{\mathbb{T}^d} \big(\sqrt{t}\,\rho v\cdot\nabla v_t\big)\cdot(\sqrt{t}\,v_t)\,dx. \tag{3.19}$$

To bound I_2, we write that

$$\begin{aligned} I_2 = \int_{\mathbb{T}^d} t\,\mathrm{div}\,(\rho v)|v_t|^2\,dx &\leq 2\int_{\mathbb{T}^d} t\rho|v|\,|\nabla v_t|\,|v_t|\,dx \\ &\leq C\bigg(\int_{\mathbb{T}^d} \rho t|v_t|^2\,dx\bigg)^{1/2}\bigg(\int_{\mathbb{T}^d} t\rho|v|^2|\nabla v_t|^2\,dx\bigg)^{1/2} \\ &\leq C\|\sqrt{\rho t}\,v_t\|_2\|v\|_\infty\|\sqrt{t}\,\nabla v_t\|_2 \\ &\leq \frac{1}{10}\|\sqrt{t}\,\nabla v_t\|_2^2 + C\|\sqrt{\rho t}\,v_t\|_2^2\|v\|_\infty^2. \end{aligned}$$

The last term may be controlled in terms of the data thanks to Propositions 3.1 and 3.2. Indeed, from (3.1) or (3.11), we get a bound on v in $L_4(\mathbb{R}_+;L_\infty)$, since

$$\|v\|_\infty^4 \lesssim \|v\|_2^2\|\nabla^2 v\|_2^2 \quad \text{if } d=2; \qquad \|v\|_\infty^4 \lesssim \|\nabla v\|_2^2\|\nabla^2 v\|_2^2 \quad \text{if } d=3.$$

To handle I_3, we use the continuity equation and perform an integration by parts:

$$I_3 = -\int_{\mathbb{T}^d} (\sqrt{t}\,\rho_t v\cdot\nabla v)\cdot(\sqrt{t}\,v_t)\,dx = -\int_{\mathbb{T}^d} t\rho v\cdot\nabla[(v\cdot\nabla v)\cdot v_t]\,dx.$$

Hence

$$I_3 \leq \int_{\mathbb{T}^d} t\rho|v|\big(|\nabla v|^2|v_t| + |v|\,|\nabla^2 v|\,|v_t| + |v|\,|\nabla v|\,|\nabla v_t|\big)\,dx =: I_{31} + I_{32} + I_{33}. \tag{3.20}$$

To bound I_{31} in the 2D case, we just write that for all $t\in[0,T]$,

$$I_{31}=\int_{\mathbb{T}^d}\sqrt{\rho t}\,|v||\nabla v||\nabla v|\,|\sqrt{\rho t}\,v_t|\,dx\leq\|v\|_\infty^2\|\sqrt{\rho t}v_t\|_2^2+CT\rho^*\|\nabla v\|_4^4,\tag{3.21}$$

and one can use again that v is bounded in $L_2(0,T;L_\infty)$, and that ∇v is bounded in $L_4(0,T;L_4)$, owing to Proposition 3.1 and Inequality (2.4).

This argument fails in the 3D case, but one can combine Hölder inequality and Sobolev embedding $\dot H^1(\mathbb{T}^3)\hookrightarrow L_6(\mathbb{T}^3)$ to get for some constant C_{T,ρ^*} depending only on T and ρ^*,

$$\begin{aligned}I_{31}&\leq\sqrt{\rho^*T}\|\sqrt{\rho t}v_t\|_4\|v\|_6\|\nabla v\|_{24/7}^2\\&\leq\sqrt{\rho^*T}\|\sqrt{\rho t}v_t\|_2^{1/4}\|\sqrt{t}\,v_t\|_6^{3/4}\|v\|_6\|\nabla v\|_{24/7}^2\\&\leq\frac{1}{10}\|\nabla\sqrt{t}\,v_t\|_2^2+C_{T,\rho^*}\|\sqrt{\rho t}\,v_t\|_2^{2/5}\|\nabla v\|_{24/7}^{16/5}\|\nabla v\|_2^{8/5}.\end{aligned}$$

Then we observe that

$$\|\nabla v\|_{24/7}^{16/5}\leq C\|\nabla v\|_2^{6/5}\|\nabla^2 v\|_2^2,$$

whence

$$I_{31}\leq\frac{1}{10}\|\nabla\sqrt{t}\,v_t\|_2^2+C_{T,\rho^*}\|\sqrt{\rho t}v_t\|_2^{2/5}\|\nabla v\|_2^{14/5}\|\nabla^2 v\|_2^2.\tag{3.22}$$

For I_{32}, we have

$$I_{32}=\int_{\mathbb{T}^d}t\rho|v|^2|\nabla^2 v|\,|v_t|\,dx\leq\rho^*T\|\nabla^2 v\|_2^2+\|v\|_\infty^4\|\sqrt{\rho t}\,v_t\|_2^2,$$

and one can use that $\nabla^2 v\in L_2(\mathbb{R}_+;L_2)$ and that $v\in L_4(\mathbb{R}_+;L_\infty)$, as already seen before.

Finally, for I_{33}, we just write that

$$I_{33}=\int_{\mathbb{T}^d}t\rho|v|^2|\nabla v|\,|\nabla v_t|\,dx\leq C\int_{\mathbb{T}^d}t\rho^2|v|^4|\nabla v|^2\,dx+\frac{1}{10}\int_{\mathbb{T}^d}|\nabla\sqrt{t}\,v_t|^2\,dx.$$

The first term of the r.h.s. is under control since $v\in L_4(\mathbb{R}_+;L_\infty)$ and $\nabla v\in L_\infty(\mathbb{R}_+;L_2)$.

To handle the term I_4, we write that

$$\begin{aligned}I_4&\leq\|\nabla v\|_2\|\sqrt{\rho t}\,v_t\|_4^2\\&\leq(\rho^*)^{3/4}\|\nabla v\|_2\|\sqrt{\rho t}\,v_t\|_2^{\frac12}\|\sqrt{t}\,v_t\|_6^{\frac32}\\&\leq C(\rho^*)^{3/4}\|\nabla v\|_2\|\sqrt{\rho t}\,v_t\|_2^{\frac12}\|\sqrt{t}\,\nabla v_t\|_2^{\frac32}\\&\leq\frac{1}{10}\|\sqrt{t}\,\nabla v_t\|_2^2+C(\rho^*)^3T\|\sqrt{\rho}v_t\|_2^2\|\nabla v\|_2^4.\end{aligned}$$

Finally, for I_5, we have to observe that

$$I_5=\int_{\mathbb{T}^d}|\rho v|\,|\sqrt{t}\,\nabla v_t|\,|\sqrt{t}\,v_t|\,dx\leq\frac{1}{10}\|\nabla\sqrt{t}\,v_t\|_2^2+C\rho^*\|v\|_\infty^2\|\sqrt{\rho t}\,v_t\|_2^2.$$

So altogether, if $d=2$, we get for some constant C_{T,ρ^*} depending only on ρ^* and T,

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho t}\,v_t\|_2^2+\|\nabla\sqrt{t}\,v_t\|_2^2&\leq C\big((1+\rho^*)\|v\|_\infty^2+\|v\|_\infty^4\big)\|\sqrt{\rho t}\,v_t\|_2^2\\&\quad+C_{T,\rho^*}\big(\|\nabla v\|_4^4+\|\nabla^2 v\|_2^2+\|v\|_\infty^4\|\nabla v\|_2^2+\|\sqrt{\rho}v_t\|_2^2(1+\|\nabla v\|_2^4)\big),\end{aligned}$$

and if $d=3$,

$$\frac{d}{dt}\|\sqrt{\rho t}\,v_t\|_2+\|\nabla\sqrt{t}\,v_t\|_2^2\leq C\big((1+\rho^*)\|v\|_\infty^2+\|v\|_\infty^4\big)\|\sqrt{\rho t}\,v_t\|_2^2$$
$$+C_{T,\rho^*}\big(\|\sqrt{\rho t}\,v_t\|_2^{2/5}\|\nabla v\|_2^{14/5}\|\nabla^2 v\|_2^2+\|\nabla^2 v\|_2^2+\|v\|_\infty^4\|\nabla v\|_2^2+\|\sqrt{\rho}v_t\|_2^2(1+\|\nabla v\|_2^4)\big).$$

The above two inequalities rewrite

$$\frac{d}{dt}\bigg(\|\sqrt{\rho t}\,v_t\|_2^2+\int_0^t\tau\|\nabla v_t\|_2^2\,d\tau\bigg)\leq h(t)\big(1+\|\sqrt{\rho t}\,v_t\|_2^2\big) \tag{3.23}$$

with $h\in L_{1,loc}(\mathbb{R}_+)$ depending only on ρ^*, $\|\sqrt{\rho_0}\,v_0\|_2$ and $\|\nabla v_0\|_2$.

Obviously, if the solution is smooth with density bounded away from zero, then we have

$$\lim_{t\to0^+}\int_{\mathbb{T}^d}\rho t|v_t(t,x)|^2\,dx=0.$$

Therefore, an obvious time integration in (3.23) yields

$$\|\sqrt{\rho t}\,v_t\|_2+\int_0^t\|\nabla\sqrt{\tau}\,v_t\|_2^2\,d\tau\leq\exp\bigg\{\int_0^t h(\tau)\,d\tau\bigg\}-1. \tag{3.24}$$

Using the same argument starting from time t_0, one arrives at the following lemma:

Lemma 3.2. *Assume $d=2,3$ and that the solution is smooth with no vacuum. Then for all $t_0,T\geq0$, we have*

$$\sup_{t_0\leq t\leq t_0+T}\int_{\mathbb{T}^d}\rho(t-t_0)|v_t|^2\,dx+\int_{t_0}^{t_0+T}\int_{\mathbb{T}^d}(t-t_0)|\nabla v_t|^2\,dx\,dt\leq c(T) \tag{3.25}$$

with $c(T)$ going to zero as $T\to0$.

From (3.24), one can deduce that for all $p<\infty$ if $d=2$ (for all $p\leq6$ if $d=3$), we have

$$\|\sqrt{t}v_t\|_{L_2(0,T;L_p)}\leq c(T)\quad\text{with } c(T)\to0 \text{ for } T\to0.$$

Indeed, denoting by $\overline{(v_t)}$ the average of v_t, one can write that

$$\int_{\mathbb{T}^d}\rho v_t\,dx=M\overline{(v_t)}+\int_{\mathbb{T}^d}\rho(v_t-\overline{(v_t)})\,dx.$$

Hence,

$$M|\overline{(v_t)}|\leq\|\rho\|_2\|\nabla v_t\|_2+M^{1/2}\|\sqrt{\rho}\,v_t\|_2.$$

Consequently, by Sobolev embedding, and because $\|\rho\|_2$ and M are time independent,

$$\|v_t\|_p\leq\|v_t-\overline{(v_t)}\|_p+|\overline{(v_t)}|\leq\bigg(C_p+\frac{\|\rho_0\|_2}{M}\bigg)\|\nabla v_t\|_2+\frac{1}{M^{1/2}}\|\sqrt{\rho}\,v_t\|_2.$$

This implies that, for all $p<\infty$ if $d=2$ and for all $p\leq6$ if $d=3$, we have

$$\|\sqrt{t}\,v_t\|_{L_2(0,T;L_p)}\leq\bigg(C_p+\frac{\|\rho_0\|_2}{M}\bigg)\|\sqrt{t}\,\nabla v_t\|_{L_2(0,T;L_2)}+\frac{1}{M^{1/2}}\|\sqrt{\rho t}\,v_t\|_{L_2(0,T;L_2)}. \tag{3.26}$$

Another consequence of (3.24) is that we have some control on the regularity of v with respect to the time variable. This is given by the following lemma.

Lemma 3.3. *Let $p\in[1,\infty]$ and $z:(0,T)\times\mathbb{T}^d\to\mathbb{R}$ satisfy $z\in L_2(0,T;L_p)$ and $\sqrt{t}\,z_t\in L_2(0,T;L_p)$. Then z is in $H^{\frac12-\alpha}(0,T;L_p)$ for all $\alpha\in(0,1/2)$ and we have*

$$\|z\|^2_{H^{\frac12-\alpha}(0,T;L_p)}\leq\|z\|^2_{L_2(0,T;L_p)}+C_{\alpha,T}\|\sqrt{t}\,z_t\|^2_{L_2(0,T;L_p)},\tag{3.27}$$

with $C_{\alpha,T}$ depending only on α and on T.

Proof. The proof relies on the definition of Sobolev norms in terms of finite differences. Indeed, we have

$$\|z\|^2_{H^{\frac12-\alpha}(0,T;L_p)}=\|z\|^2_{L_2(0,T;L_p)}+\int_0^T\biggl(\int_0^{T-h}\frac{\|v(t+h)-v(t)\|_p^2}{h^{2-2\alpha}}\,dt\biggr)dh.$$

Now, we observe that

$$\begin{aligned}\int_0^{T-h}\|v(t+h)-v(t)\|_p^2\,dt&\leq\int_0^{T-h}\biggl\|\int_t^{t+h}\sqrt{s}v_t(s)\,\frac{ds}{\sqrt{s}}\biggr\|_p^2\,dt\\&\leq\int_0^{T-h}\biggl(\int_t^{t+h}s^{-1}\,ds\biggr)\biggl(\int_t^{t+h}s\|v_t(s)\|_p^2\,ds\biggr)dt\\&\leq\|\sqrt{t}\,v_t\|^2_{L_2(0,T;L_p)}\int_0^{T-h}\biggl(\int_t^{t+h}s^{-1}\,ds\biggr)dt.\end{aligned}$$

From Fubini theorem, it is not difficult to see that, if $0<\alpha<1/2$,

$$\int_0^T h^{2\alpha-2}\biggl(\int_0^{T-h}\biggl(\int_t^{t+h}\frac{ds}{s}\biggr)dt\biggr)dh=\frac{1}{1-2\alpha}\int_0^T\biggl(\int_t^T\bigl((s-t)^{2\alpha-1}-T^{2\alpha-1}\bigr)\,\frac{ds}{s}\biggr)dt.$$

Therefore,

$$\int_0^T\biggl(\int_0^{T-h}\frac{\|v(t+h)-v(t)\|_p^2}{h^{2-2\alpha}}\,dt\biggr)dh\leq C_{\alpha,T}\|\sqrt{t}\,v_t\|^2_{L_2(0,T;L_p)},$$

which completes the proof of the lemma. □

3.3. **Shift of integrability.** The results that we proved so far will enable us to bound ∇v in $L_1(0,T;L_\infty)$, in terms of the data and of T. This will be achieved thanks to the 'shift of integrability' method alluded to in the introduction. Let us first examine the 2D case.

Lemma 3.4. *If $d=2$ then for all $T>0$, $p\in[2,\infty]$ and ε small enough, we have*

$$\|\nabla^2\sqrt{t}\,v\|_{L_p(0,T;L_{p^*-\varepsilon})}+\|\nabla\sqrt{t}\,P\|_{L_p(0,T;L_{p^*-\varepsilon})}\leq C_{0,T}\tag{3.28}$$

where $p^*:=\frac{2p}{p-2}$, and $C_{0,T}$ depends only on ρ^*, $\|\sqrt{\rho_0}\,v_0\|_2$, $\|\nabla v_0\|_2$, p and ε.

Furthermore, for all $1\leq s<2$, there exists $\beta>0$ such that

$$\int_0^T\|\nabla v(t)\|_\infty^s\,dt\leq C_{0,T}\,T^\beta.\tag{3.29}$$

Proof. From $(INS)_2$, we gather that

$$\begin{aligned}&-\Delta\sqrt{t}\,v+\nabla\sqrt{t}\,P=-\sqrt{\rho t}\,v_t-\sqrt{t}\,\rho v\cdot\nabla v &&\text{in}\quad(0,T)\times\mathbb{T}^2,\\&\operatorname{div}\sqrt{t}\,v=0&&\text{in}\quad(0,T)\times\mathbb{T}^2.\end{aligned}\tag{3.30}$$

As ρ is bounded and $\sqrt{\rho t}\,v_t$ is in $L_\infty(0,T;L_2)$, we have $\rho\sqrt{t}\,v_t$ in $L_\infty(0,T;L_2)$, too. Furthermore, according to (3.26), $\sqrt{t}\,v_t$ (and thus $\rho\sqrt{t}\,v_t$) is in $L_2(0,T;L_q)$ for all finite q. Therefore, by Hölder inequality, we have

$$\|\rho\sqrt{t}\,v_t\|_{L_p(0,T;L_r)}\leq C_{0,T}\quad\text{for all}\ \ p\in[2,\infty]\ \ \text{and}\ \ 2\leq r<p^*.\tag{3.31}$$

Similarly, the bounds for ∇v in $L_\infty(0,T;L_2)\cap L_2(0,T;H^1)$ imply that

$$\|\nabla v\|_{L_p(0,T;L_r)} \leq C_{0,T} \quad \text{for all } \ p\geq 2 \ \text{ and } \ r<p^*, \tag{3.32}$$

and, as obviously, v (and thus $\sqrt{t}\,\rho v$) is bounded in all spaces $L_q(0,T;L_r)$ (except $q=r=\infty$), one can conclude that

$$\|\sqrt{t}\,\rho v\cdot\nabla v\|_{L_p(0,T;L_r)} \leq C_{0,T} \quad \text{for all } \ p\in[2,\infty] \ \text{ and } 2\leq r<p^*. \tag{3.33}$$

Then, the maximal regularity estimate for the Stokes system implies that

$$\|\nabla^2\sqrt{t}\,v, \nabla\sqrt{t}\,P\|_{L_p(0,T;L_r)} \leq C_{0,T} \quad \text{for all } \ 2\leq p\leq\infty \ \text{ and } \ 2\leq r<p^*. \tag{3.34}$$

To prove (3.29), fix $p\in[2,\infty)$ so that $ps<2(p-s)$ and $1\leq s<2$, then take $r\in]2,p_*[$ (so that we have the embedding $W^1_r\hookrightarrow L_\infty$). We get, remembering (3.32),

$$\begin{aligned}\left(\int_0^T \|\nabla v\|_\infty^s\,dt\right)^{\frac1s} &\lesssim \left(\int_0^T \big(t^{-1/2}\,\|\sqrt{t}\,\nabla v\|_{W^1_r}\big)^s\,dt\right)^{\frac1s}\\ &\lesssim \left(\int_0^T t^{-\frac{ps}{2p-2s}}\,dt\right)^{\frac1s-\frac1p} \|\nabla\sqrt{t}\,v\|_{L_p(0,T;W^1_r)} \leq C_{0,T}\,T^{\frac{2p-2s-ps}{2ps}},\end{aligned}$$

whence the desired result. □

In the 3D case, Lemma 3.4 becomes:

Lemma 3.5. *For all $T>0$ and $p\in[2,\infty]$, we have*

$$\|\nabla^2\sqrt{t}\,v\|_{L_p(0,T;L_r)} + \|\nabla\sqrt{t}\,P\|_{L_p(0,T;L_r)} \leq C_{0,T} \quad \textit{for} \quad 2\leq r\leq \frac{6p}{3p-4}, \tag{3.35}$$

where $C_{0,T}$ depends only on ρ^*, $\|\sqrt{\rho_0}\,v_0\|_2$, $\|\nabla v_0\|_2$, p and ε.

Furthermore, if $1\leq s<4/3$, then we have for some $\beta>0$,

$$\int_0^T \|\nabla v(t)\|_\infty^s\,dt \leq C_{0,T}\,T^\beta. \tag{3.36}$$

Proof. From Lemma 3.2 and the embedding of $\dot H^1(\mathbb{T}^3)$ in $L_6(\mathbb{T}^3)$, we readily get that

$$\|\rho\sqrt{t}\,v_t\|_{L_p(0,T;L_r)} \leq C_{0,T} \quad \text{for all } \ p\in[2,\infty] \ \text{ and } \ 2\leq r\leq\frac{6p}{3p-4}. \tag{3.37}$$

We claim that we have the same type of bound for $\sqrt{t}\,\rho v\cdot\nabla v$. However, one has to proceed in two steps to get the full range of indices. As a start, we observe that (both properties being just consequences of Theorem 3.2 and obvious embedding):

$$\sqrt{t}\,\rho v\in L_4(0,T;L_\infty) \quad \text{and} \quad \nabla v\in L_4(0,T;L_3) \quad \text{implies} \quad \sqrt{t}\,\rho v\cdot\nabla v\in L_2(0,T;L_3),$$

and that, similarly

$$\sqrt{t}\,\rho v\in L_\infty(0,T;L_6) \quad \text{and} \quad \nabla v\in L_\infty(0,T;L_2) \quad \text{implies} \quad \sqrt{t}\,\rho v\cdot\nabla v\in L_\infty(0,T;L_{3/2}).$$

Therefore, interpolating and using Hölder inequality yields

$$\sqrt{t}\,\rho v\cdot\nabla v\in L_p(0,T;L_r) \quad \text{for all} \quad (p,r) \ \text{ such that } \ p\geq 2 \ \text{ and } \ \frac2p+\frac3r=2.$$

Then using the maximal regularity properties of (3.30) yields

$$\|\nabla^2\sqrt{t}\,v,\ \nabla\sqrt{t}\,P\|_{L_p(0,T;L_r)} \leq C_{0,T} \quad \text{for all} \quad (p,r) \ \text{ such that } \ p\geq 2 \ \text{ and } \ \frac2p+\frac3r=2. \tag{3.38}$$

From this, using the bound for ρv in $L_\infty(0,T;L_6)$ and the embedding $W^1_r(\mathbb{T}^3)\hookrightarrow L_q(\mathbb{T}^3)$ with $\frac{3}{q}=\frac{3}{r}-1$ if $1\leq r<3$ (which implies that $\nabla\sqrt{t}\,v$ is bounded in $L_p(0,T;L_q)$ with $\frac{2}{p}+\frac{3}{r}=2$), one gets (3.38) for the full range of indices.

To prove (3.36), fix $p\in(2,4)$ such that $ps<2p-2s$ (our condition on s makes it possible), and take $r=\frac{6p}{3p-4}\cdot$ Using that $W^1_r\hookrightarrow L_\infty$ (because $r>3$ for $2<p<4$), one can write that

$$\biggl(\int_0^T\|\nabla v\|_\infty^s\,dt\biggr)^{\frac1s}\lesssim\biggl(\int_0^T\|\sqrt{t}\,\nabla v\|_{W^1_r}^s\,\frac{dt}{\sqrt{t}}\biggr)^{\frac1s}\lesssim\biggl(\int_0^T t^{-\frac{ps}{2p-2s}}\,dt\biggr)^{\frac1s-\frac1p}\|\nabla\sqrt{t}\,v\|_{L_p(0,T;W^1_r)}$$

whence the desired result. □

3.4. **The proof of existence.** Here we briefly explain the issue of existence. For expository purpose, we focus on global results (that is either $d=2$ or $d=3$ and the velocity is small), and leave to the reader the construction of local solutions in the case of large v_0, if $d=3$.

The general idea is to take advantage of classical results to construct smooth solutions corresponding to smoothed out approximate data with no vacuum, then to pass to the limit. More precisely, consider

$$v_0^\varepsilon\in C^\infty(\mathbb{T}^d)\ \text{ with }\ \operatorname{div}v_0^\varepsilon=0,\quad\text{ and }\ \rho_0^\varepsilon\in C^\infty(\mathbb{T}^d)\ \text{ with }\ \varepsilon\leq\rho_0^\varepsilon\leq\rho^*,$$

such that

$$v_0^\varepsilon\to v_0 \text{ in } H^1,\qquad \rho_0^\varepsilon\rightharpoonup\rho_0 \text{ in } L_\infty \text{ weak * and } \rho_0^\varepsilon\to\rho_0 \text{ in } L_p,\ \text{if } p<\infty. \tag{3.39}$$

Then, in light of the classical strong solution theory for (INS) (see [18] and more recent developments in [4]), there exists a unique global smooth solution $(\rho^\varepsilon,v^\varepsilon,P^\varepsilon)$ corresponding to data $(\rho_0^\varepsilon,v_0^\varepsilon)$, and satisfying $\varepsilon\leq\rho^\varepsilon\leq\rho^*$.

Being smooth, the triplet $(\rho^\varepsilon,v^\varepsilon,P^\varepsilon)$ satisfies all the priori estimates of the previous subsection, and thus in particular,

$$\|\sqrt{\rho^\varepsilon}v_t^\varepsilon,\nabla P^\varepsilon\|_{L_2(\mathbb{R}_+\times\mathbb{T}^d)}+\|v^\varepsilon\|_{L_\infty(\mathbb{R}_+;H^1)\cap L_2(\mathbb{R}_+;H^2)}\leq C(\|\sqrt{\rho}v\|_2,\|\nabla v_0\|_2,\rho^*), \tag{3.40}$$

and also, thanks to (3.24),

$$\sup_{t\in[t_0,t_0+T)}\Bigl((t-t_0)\|\sqrt{\rho^\varepsilon(t)}v_t^\varepsilon(t)\|_2^2\Bigr) +\int_{t_0}^{t_0+T}(t-t_0)\|\nabla v_t^\varepsilon\|_2^2\,dt\leq C(T,\|\sqrt{\rho_0}v_0\|_2,\|\nabla v_0\|_2,\rho^*), \tag{3.41}$$

and, according to Inequality (3.27), for $\alpha\in(0,1/2)$,

$$\|v^\varepsilon\|_{H^{\frac12-\alpha}(0,T;L_2)}\leq C(T,\|\sqrt{\rho_0}v_0\|_2,\|\nabla v_0\|_2,\rho^*).$$

Interpolating with (3.40) yields for small enough $\eta>0$,

$$\|v^\varepsilon\|_{H^\eta(0,T\times\mathbb{T}^d)}\leq C(T,\|\sqrt{\rho_0}v_0\|_2,\|\nabla v_0\|_2,\rho^*). \tag{3.42}$$

The bounds on v^ε and standard compact embedding imply that, up to subsequence, $v^\varepsilon\to v$ in $L_{2,loc}(\mathbb{R}_+;H^1)$ for some v that, in addition, satisfies (3.40) (as regards ∇P^ε and v^ε) and (3.42). For the density, we have $\rho^\varepsilon\rightharpoonup\rho$ in $L_\infty(\mathbb{R}_+;L_\infty)$ and $0\leq\rho\leq\rho^*$. All those informations are more than enough to justify that (ρ,v) is a weak solution to (INS), namely

$$(\rho(t)v(t),\phi(t))-(\rho_0v_0,\phi_0)-\int_0^t(\rho v,\phi_t)\,d\tau-\int_0^t(\rho v\otimes v,\nabla\phi)\,d\tau+\int_0^t(\nabla v,\nabla\phi)\,d\tau=0 \tag{3.43}$$

for all smooth compactly supported divergence-free vector function $\phi \in C^\infty([0,\infty)\times\mathbb{T}^d;\mathbb{R}^d)$, and that the continuity equation is fulfilled in a distributional meaning:

$$\rho_t + \mathrm{div}\,(\rho v) = 0 \text{ in } \mathcal{D}'([0,\infty)\times\mathbb{T}^d). \tag{3.44}$$

Now, arguing as in e.g. [9], page 2405, one can show that $(\rho^\varepsilon)^2 \rightharpoonup (\rho)^2$ in $L_\infty(\mathbb{R}_+;L_\infty)$, which eventually implies that

$$\rho^\varepsilon \to \rho \quad\text{in}\quad \mathcal{C}(\mathbb{R}_+;L_p) \quad\text{for all finite}\quad p. \tag{3.45}$$

Therefore (3.41) is satisfied by (ρ, v). Furthermore, (3.40) and (3.44) applied to the formulation (3.43) yields that the momentum equation is fulfilled in the strong sense, i.e.

$$\rho v_t + \rho v\cdot\nabla v - \Delta v + \nabla P = 0 \text{ in } L_2(\mathbb{R}_+;L_2) \tag{3.46}$$

for some pressure function ∇P in $L_2(\mathbb{R}_+;L_2)$ that satisfies (3.40).

Of course, being smooth, the solution $(\rho^\varepsilon, v^\varepsilon, P^\varepsilon)$ fulfills (1.1), (1.2), (1.3), (1.4), and thus

$$\|\sqrt{\rho^\varepsilon(t)}\,v^\varepsilon(t)\|_2^2 + 2\int_{t_0}^t \|\nabla v^\varepsilon\|_2^2\,d\tau = \|\sqrt{\rho^\varepsilon(t_0)}\,v^\varepsilon(t_0)\|_2^2 \quad\text{for all}\quad t,t_0\geq 0. \tag{3.47}$$

By construction, in the case $t_0 = 0$, the last term tends to $\|\sqrt{\rho_0}v_0\|_2^2$, and the fact that $v^\varepsilon \to v$ in $L_{2,loc}(\mathbb{R}_+;H^1)$ guarantees that, for all $t\geq 0$, the second term converges to $\int_{t_0}^t \|\nabla v\|_2^2\,d\tau$. Next, (3.24) guarantees that ∇v_t^ε is bounded in $L_2(t_0,T\times\mathbb{T}^d)$ for all $0<t_0<T$. Then combining with (3.26) gives boundedness of v_t^ε in $L_2(t_0,T;H^1)$. Now, because v^ε is bounded in $L_\infty(t_0,T;H^1)$ thanks to Prop. 3.1 and 3.2, one can conclude by means of Ascoli theorem that, up to extraction, $v^\varepsilon \to v$ strongly in $\mathcal{C}([t_0,T];L_p)$ for all $p<6$ (and even $p<\infty$ if $d=2$). Combining with (3.45) ensures that $\sqrt{\rho}v$ is in $\mathcal{C}(]0,+\infty[;L_2)$ and that one can pass to the limit in all the terms of (3.47): we eventually get

$$\|\sqrt{\rho(t)}\,v(t)\|_2^2 + 2\int_{t_0}^t \|\nabla v\|_2^2\,d\tau = \|\sqrt{\rho(t_0)}\,v(t_0)\|_2^2 \quad\text{for all}\quad t,t_0\geq 0. \tag{3.48}$$

Finally, to get the strong continuity of $\sqrt{\rho}v$ at $t=0$, we notice that the uniform bounds ensure the weak continuity, and that (3.48) gives

$$\|\sqrt{\rho(t)}\,v(t)\|_2^2 \longrightarrow \|\sqrt{\rho_0}\,v_0\|_2^2 \quad\text{for}\quad t\to 0.$$

This completes the proof of the existence part of Theorems 2.1 and 2.2. □

4. The proof of uniqueness

As there is no hope to prove uniqueness of solutions at the level of the Eulerian coordinates system, owing to the lack of regularity of the density, we shall prove it for the solutions written in the Lagrangian coordinates.

To this end, we introduce the flow $X:\mathbb{R}_+\times\mathbb{T}^d\to\mathbb{T}^d$ of v, that is the unique solution to (2.3). In Lagrangian coordinates (t,y), a solution (ρ,v,P) to (INS) recasts in (η,u,Q) with

$$\eta(t,y) := \rho(t,X(t,y)), \qquad u(t,y) := v(t,X(t,y)) \quad\text{and}\quad Q(t,y) := P(t,X(t,y)). \tag{4.1}$$

Note that

$$X(t,y) = y + \int_0^t v(\tau,X(\tau,y))\,d\tau = \int_0^t u(\tau,y)\,d\tau,$$

and thus

$$\nabla_y X(t,y) = \mathrm{Id} + \int_0^t \nabla_y u(\tau,y)\,d\tau.$$

Let $A(t) := (\nabla_y X(t,\cdot))^{-1}$. In the (t,y)-coordinates, operators ∇, div and Δ translate into

$$\nabla_u := {}^T\!A\,\nabla_y, \quad \mathrm{div}_u := {}^T\!A : \nabla_y = \mathrm{div}_y\,(A\cdot) \ \text{ and } \ \Delta_u := \mathrm{div}_y\,(A\,{}^T\!A\nabla_y\cdot), \tag{4.2}$$

and the triplet (η, u, Q) thus satisfies

$$\begin{aligned} &\eta_t = 0 && \text{in } (0,T)\times\mathbb{T}^d,\\ &\eta u_t - \Delta_u u + \nabla_u Q = 0 && \text{in } (0,T)\times\mathbb{T}^d,\\ &\mathrm{div}_u\, u = 0 && \text{in } (0,T)\times\mathbb{T}^d. \end{aligned}\tag{4.3}$$

As pointed out in e.g. [6, 8], in our regularity framework, that latter system is equivalent to (INS) whenever, say,

$$\int_0^T \|\nabla u\|_\infty\,d\tau \le \frac12\cdot \tag{4.4}$$

Of course, if that condition is fulfilled then one may write that

$$A = (\mathrm{Id} + (\nabla_y X - \mathrm{Id}))^{-1} = \sum_{k=0}^{+\infty}(-1)^k\biggl(\int_0^t \nabla_y u(\tau,\cdot)\,d\tau\biggr)^k. \tag{4.5}$$

Let us tackle the proof of uniqueness in the torus case (the bounded domain case goes exactly the same : just change Lemma 5.2 to Lemma 5.3).

Consider two solutions (ρ^1, v^1, P^1) and (ρ^2, v^2, P^2) of (INS), emanating from the same initial data, and fulfilling the properties of Theorems 2.1 and 2.2, and denote by (η^1, u^1, Q^1) and (η^2, u^2, Q^2) the corresponding triplets in Lagrangian coordinates. Of course, we have $\eta^1 = \eta^2 = \rho_0$, which explains the choice of our approach here. In what follows, we shall use repeatedly the fact that for $i = 1, 2$, we have (recall (3.26))

$$\begin{aligned} &t^{1/2}\nabla u^i \in L_2(0,T;L_\infty), \quad \nabla u^i \in L_1(0,T;L_\infty)\cap L_4(0,T;L_3)\cap L_2(0,T;L_6),\\ &u^i \in L_4(0,T;L_\infty), \quad t^{1/2}\nabla Q^i \in L_2(0,T;L_4) \ \text{ and } \ t^{1/2}u^i_t \in L_{4/3}(0,T;L_6). \end{aligned}\tag{4.6}$$

Denoting $\delta u := u^2 - u^1$ and $\delta Q := Q^2 - Q^1$, we get

$$\begin{aligned} &\rho_0\delta u_t - \Delta_{u^1}\delta u + \nabla_{u^1}\delta Q = (\Delta_{u^2} - \Delta_{u^1})u^2 - (\nabla_{u^2} - \nabla_{u^1})Q^2,\\ &\mathrm{div}_{u^1}\delta u = (\mathrm{div}_{u^1} - \mathrm{div}_{u^2})u^2,\\ &\delta u|_{t=0} = 0. \end{aligned}\tag{4.7}$$

We claim that for sufficiently small $T > 0$, we have

$$\int_0^T\int_{\mathbb{T}^d}\bigl(|\delta u(t,y)|^2 + |\nabla\delta u(t,y)|^2\bigr)dy\,dt = 0.$$

To prove our claim, decompose δu into

$$\delta u = w + z, \tag{4.8}$$

where w is the solution given by Lemma 5.2 to the following problem:

$$\mathrm{div}_{u^1} w = (\mathrm{div}_{u^1} - \mathrm{div}_{u^2})u^2 = \mathrm{div}\,(\delta A\,u^2) = {}^T\delta A : \nabla u^2, \tag{4.9}$$

with $\delta A := A^2 - A^1$ and $A^i := A(u^i)$.

As a first step in the proof of our claim, let us establish the following lemma:

Lemma 4.1. *The solution w to* (4.9) *given by Lemma 5.2 satisfies*

$$\|w\|_{L_4(0,T;L_2)} + \|\nabla w\|_{L_2(0,T\times\mathbb{T}^d)} + \|w_t\|_{L_{4/3}(0,T;L_{3/2})} \leq c(T)\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)} \tag{4.10}$$

with $c(T)$ going to 0 *when T tends to* 0.

Proof. Lemma 5.2 and Identity (4.5) ensure that there exist two universal positive constants c and C such that if

$$\|\nabla u^1\|_{L_1(0,T;L_\infty)} + \|\nabla u^1\|_{L_2(0,T;L_6)} \leq c, \tag{4.11}$$

then the following inequalities hold true:

$$\begin{aligned} \|w\|_{L_4(0,T;L_2)} \leq C\|\delta A\, u^2\|_{L_4(0,T;L_2)}, \quad \|\nabla w\|_{L_2(0,T;L_2)} \leq C\|{}^T\delta A : \nabla u^2\|_{L_2(0,T;L_2)} \\ \text{and} \quad \|w_t\|_{L_{4/3}(0,T;L_{3/2})} \leq C\|\delta A\, u^2\|_{L_4(0,T;L_2)} + C\|(\delta A\, u^2)_t\|_{L_{4/3}(0,T;L_{3/2})}. \end{aligned} \tag{4.12}$$

In all that follows, $c(T)$ designates a nonnegative continuous increasing function of T, with $c(0) = 0$. Now, let us bound the r.h.s. of (4.12). Regarding ${}^T\delta A : \nabla u^2$, one can use the fact that if both u^1 and u^2 fulfill (4.11), then we have

$$\sup_{t\in[0,T]} \|t^{-1/2}\delta A\|_2 \leq C \sup_{t\in[0,T]} \left\|t^{-1/2}\int_0^t \nabla\delta u\, d\tau\right\|_2 \leq C\|\nabla\delta u\|_{L_2(0,T;L_2)}. \tag{4.13}$$

This stems from Hölder inequality and the following identity:

$$\delta A(t) = \left(\int_0^t \nabla\delta u\, d\tau\right)\cdot\left(\sum_{k\geq 1}\sum_{0\leq j<k} C_1^j C_2^{k-1-j}\right) \quad \text{with} \quad C_i(t) := \int_0^t \nabla u^i\, d\tau. \tag{4.14}$$

Therefore, thanks to (4.6) and (4.13), we have

$$\begin{aligned} \|{}^T\delta A : \nabla u^2\|_{L_2(0,T\times\mathbb{T}^d)} &\leq \sup_{t\in[0,T]} \|t^{-1/2}\delta A(t)\|_2 \|t^{1/2}\nabla u^2\|_{L_2(0,T;L_\infty)} \\ &\leq c(T)\|\nabla\delta u\|_{L_2(0,T;L_2)}. \end{aligned}$$

Similarly,

$$\|\delta A\, u^2\|_{L_4(0,T;L_2)} \leq \|t^{-1/2}\delta A\|_{L_\infty(0,T;L_2)}\|t^{1/2}u^2\|_{L_4(0,T;L_\infty)},$$

whence, using (4.6), (4.12) and (4.13) gives

$$\|w\|_{L_4(0,T;L_2)} \leq c(T)\|\nabla\delta u\|_{L_2(0,T;L_2)}. \tag{4.15}$$

In order to bound w_t, it suffices to derive an appropriate estimate in $L_{4/3}(0,T;L_{3/2})$ for

$$(\delta A\, u^2)_t = \delta A\, u_t^2 + (\delta A)_t\, u^2.$$

Thanks to (4.6) and (4.13)

$$\begin{aligned} \|\delta A\, u_t^2\|_{L_{4/3}(0,T;L_{3/2})} &\leq \|t^{-1/2}\delta A\|_{L_\infty(0,T;L_2)}\|t^{1/2}u_t^2\|_{L_{4/3}(0,T;L_6)} \\ &\leq c(T)\|\nabla\delta u\|_{L_2(0,T;L_2)}. \end{aligned}$$

One can bound the other term as follows:

$$\|\delta A_t\, u^2\|_{L_{4/3}(0,T;L_{3/2})} \leq \|\delta A_t\|_{L_2(0,T\times\mathbb{T}^d)}\|u^2\|_{L_4(0,T;L_6)}.$$

Differentiating (4.14) with respect to t and using (4.11) for u^1 and u^2, we see that

$$\|\delta A_t\|_2 \leq C\left(\|\nabla\delta u\|_2 + \left\|t^{-1/2}\int_0^t \nabla\delta u\, d\tau\right\|_2 \left(\|t^{1/2}\nabla u^1\|_\infty + \|t^{1/2}\nabla u^2\|_\infty\right)\right).$$

Therefore
$$\|\delta A_t\|_{L_2(0,T\times\mathbb{T}^d)} \leq C\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)},$$
and thus, owing to (4.6),
$$\|\delta A_t\, u^2\|_{L_{4/3}(0,T;L_{3/2})} \leq c(T)\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)}.$$
Altogether, this gives (4.10). □

Next, let us restate the equations for $(\delta u, \delta Q)$ as the following system for $(z, \delta Q)$:
$$\begin{aligned}&\rho_0 z_t - \Delta_{u^1} z + \nabla_{u^1}\delta Q = (\Delta_{u^2} - \Delta_{u^1})u^2 + (\nabla_{u^1} - \nabla_{u^2})Q^2 - \rho_0 w_t + \Delta_{u^1} w,\\ &\mathrm{div}_{u^1} z = 0.\end{aligned}\tag{4.16}$$
Let us test the equation by z. We first notice the following crucial property thanks to which one does not have to care about the difference of the pressures:
$$\int_{\mathbb{T}^d} (\nabla_{u^1}\delta Q)\cdot z\, dx = -\int_{\mathbb{T}^d} \mathrm{div}_{u^1} z\, \delta Q\, dx = 0.\tag{4.17}$$
So we have
$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \rho_0 |z|^2\, dx + \int_{\mathbb{T}^d} |\nabla_{u^1} z|^2 dx = \sum_{k=1}^4 I_k,\tag{4.18}$$
where
$$\begin{aligned}I_1 &:= \int_{\mathbb{T}^d} \big((\Delta_{u^2} - \Delta_{u^1})u^2\big)\cdot z\, dx, \qquad & I_2 &:= \int_{\mathbb{T}^d} \big((\nabla_{u^1} - \nabla_{u^2})Q^2\big)\cdot z\, dx,\\ I_3 &:= -\int_{\mathbb{T}^d} \rho_0\, w_t\cdot z\, dx, & I_4 &:= \int_{\mathbb{T}^d} (\Delta_{u^1} w)\cdot z\, dx.\end{aligned}$$
We have, using (4.2), (4.5) and (4.11),
$$\begin{aligned}I_1 &= \int_{\mathbb{T}^d} \mathrm{div}\,\big((\delta A^T A_2 + A_1{}^T\delta A)\nabla u^2\big)\cdot z\, dx\\ &\leq \int_{\mathbb{T}^d} \big|\delta A^T A_2 + A_1{}^T\delta A\big|\,|\nabla u^2|\,|\nabla z|\, dx \leq C\|t^{-1/2}\delta A\|_2\|t^{1/2}\nabla u^2\|_\infty\|\nabla z\|_2.\end{aligned}$$
Therefore, thanks to (4.6) and (4.13),
$$\begin{aligned}\int_0^T I_1(t)\, dt &\leq C\|t^{-1/2}\delta A\|_{L_\infty(0,T;L_2)}\|t^{1/2}\nabla u^2\|_{L_2(0,T;L_\infty)}\|\nabla z\|_{L_2(0,T\times\mathbb{T}^d)}\\ &\leq c(T)\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)}\|\nabla z\|_{L_2(0,T\times\mathbb{T}^d)}.\end{aligned}\tag{4.19}$$
Next,
$$I_2 \leq \left|\int_{\mathbb{T}^d} \delta A\nabla Q^2\cdot z\, dx\right| \leq C\|t^{-1/2}\delta A\|_2\|t^{1/2}\nabla Q^2\|_4\|z\|_4,\tag{4.20}$$
whence, according to (4.13) and Sobolev embedding
$$\begin{aligned}\int_0^T I_2(t)\, dt &\leq \|t^{-1/2}\delta A\|_{L_\infty(0,T;L_2)}\|t^{1/2}\nabla Q^2\|_{L_2(0,T;L_4)}\|z\|_{L_2(0,T;L_4)}\\ &\leq c(T)\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)}\|z\|_{L_2(0,T;H^1)}.\end{aligned}$$
At this stage, one may use Lemma 5.1 to bound z in H^1: we get, for some constant C depending only on ρ_0,
$$\|z\|_{H^1} \leq C\big(\|\sqrt{\rho_0}z\|_2 + \|\nabla z\|_2\big).\tag{4.21}$$

Therefore, one concludes that

$$\int_0^T I_2(t)\,dt \leq c(T)\left(\|\sqrt{\rho_0}\,z\|_{L_\infty(0,T;L_2)} + \|\nabla z\|_{L_2(0,T\times\mathbb{T}^d)}\right)\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)}.$$

Next, using Hölder inequality, one can write that

$$\int_0^T I_3(t)\,dt \leq \|\rho_0\|_\infty^{3/4}\|w_t\|_{L_{4/3}(0,T;L_{3/2})}\|{\rho_0}^{1/4}z\|_{L_4(0,T;L_3)}.$$

Note that from Hölder inequality and the Sobolev embedding $H^1(\mathbb{T}^d) \hookrightarrow L_6(\mathbb{T}^d)$, we have

$$\|{\rho_0}^{1/4}z\|_{L_4(0,T;L_3)} \leq \|\sqrt{\rho_0}\,z\|_{L_\infty(0,T;L_2)}^{1/2}\|z\|_{L_2(0,T;L_6)}^{1/2} \leq C\|\sqrt{\rho_0}\,z\|_{L_\infty(0,T;L_2)}^{1/2}\|z\|_{L_2(0,T;H^1)}^{1/2}.$$

Then, taking advantage of (4.21) and (4.10), we conclude that

$$\int_0^T I_3(t)\,dt \leq c(T)\left(\|\sqrt{\rho_0}\,z\|_{L_\infty(0,T;L_2)}+\|\nabla z\|_{L_2(0,T\times\mathbb{T}^d)}\right)^{1/2}\|\sqrt{\rho_0}\,z\|_{L_\infty(0,T;L_2)}^{1/2}\|\nabla\delta u\|_{L_2(0,T\times\mathbb{T}^d)}.$$

Finally, integrating by parts, and using (4.10) and (4.11),

$$\begin{aligned}\int_0^T I_4(t)\,dt &\leq \int_0^T\int_{\mathbb{T}^d}|\nabla_{u^1}w|\,|\nabla_{u^1}z|\,dx\,dt\\ &\leq \frac12\int_0^T\|\nabla_{u^1}z\|_2^2\,dt + \frac12\int_0^T\|\nabla_{u^1}w\|_2^2\,dt\\ &\leq \frac12\int_0^T\|\nabla_{u^1}z\|_2^2\,dt + c(T)\int_0^T\|\nabla\delta u\|_2^2\,dt.\end{aligned}$$

So altogether, this gives for all small enough $T>0$,

$$\sup_{t\in[0,T]}\|\sqrt{\rho_0}z(t)\|_2^2 + \int_0^T\|\nabla z\|_2^2\,dt \leq c(T)\int_0^T\|\nabla\delta u\|_2^2\,dt. \tag{4.22}$$

Combining with (4.10), we conclude that

$$\int_0^T\|\nabla\delta u\|_2^2\,dt \leq c(T)\int_0^T\|\nabla\delta u\|_2^2\,dt.$$

Hence $\nabla\delta u\equiv 0$ on $[0,T]\times\mathbb{T}^d$ if T is small enough.

Then, plugging that information in (4.22) yields

$$\|\sqrt{\rho_0}\,z\|_{L_\infty(0,T;L_2)} + \|\nabla z\|_{L_2(0,T\times\mathbb{T}^d)} = 0.$$

Combining with Lemma 5.1 finally implies that $z\equiv 0$ on $[0,T]\times\mathbb{T}^d$, and (4.10) clearly yields $w\equiv 0$. Therefore we proved that for small enough $T>0$,

$$u^1\equiv u^2 \quad\text{on}\quad [0,T]\times\mathbb{T}^d.$$

Reverting to Eulerian coordinates, we conclude that the two solutions of (INS) coincide on $[0,T]\times\mathbb{T}^d$. Then standard connectivity arguments yield uniqueness on the whole $\mathbb{R}_+$.

5. Appendix

We here establish the weighted Poincaré inequalities and the Desjardins interpolation inequality (3.3) that have been used several times in the proof of existence, and results for the 'twisted' divergence equation that were the key to the proof of uniqueness.

Let us start with the Poincaré inequalities in the case where Ω is the unit torus $\mathbb{T}^d$.

Lemma 5.1. *Let $a:(0,1)^d\to\mathbb{R}$ be a nonnegative and nonzero measurable function. Then we have for all z in $H^1(\mathbb{T}^d)$,*

$$\|z\|_2\leq\frac{1}{M}\biggl|\int_{\mathbb{T}^d}az\,dx\biggr|+\biggl(1+\frac{1}{M}\|M-a\|_2\biggr)\|\nabla z\|_2\quad\textit{with}\quad M:=\int_{\mathbb{T}^d}a\,dx. \tag{5.1}$$

Furthermore, in dimension $d=2$, there exists an absolute constant C so that

$$\|z\|_2\leq\frac{1}{M}\biggl|\int_{\mathbb{T}^2}az\,dx\biggr|+C\log^{\frac12}\biggl(e+\frac{\|a-M\|_2}{M}\biggr)\|\nabla z\|_2. \tag{5.2}$$

Proof. To prove (5.1), we start with the obvious inequality

$$\|z\|_2\leq|\bar z|+\|\nabla z\|_2\quad\text{with}\quad \bar z:=\int_{\mathbb{T}^d}z\,dx, \tag{5.3}$$

then we use the fact that

$$M\bar z=\int_{\mathbb{T}^d}az\,dx+\int_{\mathbb{T}^d}(M-a)(z-\bar z)\,dx, \tag{5.4}$$

which, according to the classical Cauchy-Schwarz and Poincaré inequalities in the torus, implies that

$$M|\bar z|\leq\biggl|\int_{\mathbb{T}^d}az\,dx\biggr|+\|M-a\|_2\|\nabla z\|_2. \tag{5.5}$$

To prove Inequality (5.2), we decompose $\widetilde z:=z-\bar z$ into Fourier series:

$$\widetilde z(x)=\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}}\widehat z_k\,e^{2i\pi k\cdot x}.$$

Then, for any integer n, we set

$$\widetilde z_n(x):=\sum_{1\leq|k|\leq n}\widehat z_k\,e^{2i\pi k\cdot x}.$$

Because the average of $\widetilde z_n$ is 0, one may write for any integer n,

$$\int_{\mathbb{T}^2}(a-M)\widetilde z\,dx=\int_{\mathbb{T}^2}a\widetilde z_n\,dx+\int_{\mathbb{T}^2}(a-M)(\widetilde z-\widetilde z_n)\,dx.$$

Therefore, using Hölder inequality,

$$\biggl|\int_{\mathbb{T}^2}(a-M)\widetilde z\,dx\biggr|\leq M\|\widetilde z_n\|_\infty+\|a-M\|_2\|\widetilde z-\widetilde z_n\|_2. \tag{5.6}$$

By Cauchy-Schwarz inequality, we have for all $x \in \mathbb{T}^2$,

$$
\begin{aligned}
|\widetilde{z}_n(x)| &\leq \sum_{1\leq|k|\leq n} |2\pi k \widehat{z}_k| \, \frac{|e^{2i\pi k\cdot x}|}{2\pi|k|} \\
&\leq \|\nabla z\|_2 \Big(\sum_{1\leq|k|\leq n} \frac{1}{4\pi^2|k|^2} \Big)^{1/2} \\
&\leq C\sqrt{\log n}\, \|\nabla z\|_2. \qquad (5.7)
\end{aligned}
$$

Inserting that inequality in (5.6) yields

$$
\left| \int_{\mathbb{T}^2} (a-M)\widetilde{z}\,dx \right| \leq C\Big(\sqrt{\log n}\, M + n^{-1}\|a-M\|_2\Big)\|\nabla z\|_2.
$$

Then taking $n \approx \|a-M\|_2/M$ and mingling with (5.3) and (5.4) gives (5.2). □

Let us now prove Lemma 3.1. As above, it is based on decomposition into low and high frequency, while the original proof by B. Desjardins in [12] relies on Trudinger inequality. The starting point is that for any $n \in \mathbb{N}^*$, we have

$$
\begin{aligned}
\int_{\mathbb{T}^2} \rho z^4\,dx &= \int_{\mathbb{T}^2} \rho z^2 \big(\bar{z} + \widetilde{z}_n + (\widetilde{z} - \widetilde{z}_n)\big)^2\,dx \\
&\leq 3\bigg(\bar{z}^2 \|\sqrt{\rho} z\|_2^2 + \|\widetilde{z}_n\|_\infty^2 \|\sqrt{\rho} z\|_2^2 + \sqrt{\rho^*}\|\widetilde{z}-\widetilde{z}_n\|_4^2 \Big(\int_{\mathbb{T}^2} \rho z^4\,dx \Big)^{\frac12} \bigg).
\end{aligned}
$$

We thus have, using Young inequality and embedding $\dot{H}^{\frac12} \hookrightarrow L^4$,

$$
\int_{\mathbb{T}^2} \rho z^4\,dx \leq 6\|\sqrt{\rho} z\|_2^2 \big(\bar{z}^2 + \|\widetilde{z}_n\|_\infty^2\big) + C\rho^* \|\widetilde{z}-\widetilde{z}_n\|^4_{\dot{H}^{\frac12}}. \qquad (5.8)
$$

Obviously, we have

$$
\|\widetilde{z}-\widetilde{z}_n\|_{\dot{H}^{\frac12}} \leq \frac{1}{\sqrt{n}}\|\nabla z\|_2. \qquad (5.9)
$$

Hence, using (5.7) to bound the first term of the right-hand side of (5.8), we get

$$
\int_{\mathbb{T}^2} \rho z^4\,dx \lesssim \bar{z}^2\|\sqrt{\rho} z\|_2^2 + \big(\log n \|\sqrt{\rho} z\|_2^2 + n^{-2}\rho^*\|\nabla z\|_2^2\big)\|\nabla z\|_2^2.
$$

Taking for n the closest positive integer to $\frac{\sqrt{\rho^*}\|\nabla z\|_2}{\|\sqrt{\rho} z\|_2}$, we end up with

$$
\Big(\int_{\mathbb{T}^2} \rho z^4\,dx \Big)^{\frac12} \leq C\|\sqrt{\rho} z\|_2 \bigg(|\bar{z}| + \|\nabla z\|_2 \log^{\frac12}\Big(e + \frac{\rho^*\|\nabla z\|_2^2}{\|\sqrt{\rho} z\|_2^2} \Big) \bigg). \qquad (5.10)
$$

Then, bounding $\bar{z}$ as in the proof of Inequality (5.2) yields (3.3). □

Inequality (5.10) (and thus Lemma 3.1) may be easily adapted to any bounded domain Ω. Indeed, as the estimates therein are invariant by translation, one may assume with no loss of generality that $\Omega \subset (0,R)^d$ for some $R > 0$. Now, any function in $H^1_0(\Omega)$ may be extended by 0 to a function of $H^1_0((0,R)^d)$, thus to H^1 on the torus with size R. Then one can apply the above results, and get the same inequalities for some constant depending only on R. In order to track the dependency with respect to R, it suffices to make the change of function

$z'(x) := z(Rx)$ and to apply to z' the inequality for some domain included in $(0,1)^d$. For example, Inequality (5.1) becomes for all z in $H^1_0(\Omega)$ with Ω of diameter less than R,

$$\|z\|_2 \leq \frac{R^{d/2}}{M}\biggl|\int_{\mathbb{T}^d} az\,dx\biggr| + R\biggl(1+\frac{R^{d/2}}{M}\|M-a\|_2\biggr)\|\nabla z\|_2.$$

Finally, let us consider the twisted divergence equation. The following lemma (in the spirit of the corresponding result in [7]) is the key to the proof of uniqueness, in the torus case.

Lemma 5.2. *Let A be a matrix valued function on $[0,T]\times\mathbb{T}^d$ satisfying*

$$\det A \equiv 1. \tag{5.11}$$

There exists a constant c depending only on d, such that if

$$\|\operatorname{Id}-A\|_{L_\infty(0,T;L_\infty)} + \|A_t\|_{L_2(0,T;L_6)} \leq c \tag{5.12}$$

then for all function $g:[0,T]\times\mathbb{T}^d\to\mathbb{R}$ satisfying $g\in L^2(0,T\times\mathbb{T}^d)$ and

$$g = \operatorname{div} R \quad \text{with} \quad R\in L_4(0,T;L_2) \quad \text{and} \quad R_t\in L_{4/3}(0,T;L_{3/2}),$$

the equation

$$\operatorname{div}(Aw) = g \quad \text{in} \quad [0,T]\times\mathbb{T}^d$$

admits a solution w in the space

$$X_T := \Bigl\{v\in L_4(0,T;L_2(\mathbb{T}^d))\,,\ \nabla v\in L_2(0,T;L_2(\mathbb{T}^d)) \text{ and } v_t\in L_{4/3}(0,T;L_{3/2}(\mathbb{T}^d))\Bigr\}$$

satisfying the following inequalities for some constant $C=C(d)$:

$$\begin{aligned} &\|w\|_{L_4(0,T;L_2)} \leq C\|R\|_{L_4(0,T;L_2)}, \quad \|\nabla w\|_{L_2(0,T;L_2)} \leq C\|g\|_{L_2(0,T;L_2)} \\ &\text{and} \quad \|w_t\|_{L_{4/3}(0,T;L_{3/2})} \leq C\|R\|_{L_4(0,T;L_2)} + C\|R_t\|_{L_{4/3}(0,T;L_{3/2})}. \end{aligned} \tag{5.13}$$

Proof. Let g satisfy the conditions of the lemma. Then for any $v\in X_T$, we set

$$\Phi(v) := \nabla\Delta^{-1}\operatorname{div}\bigl((\operatorname{Id}-A)v + R - \overline{(\operatorname{Id}-A)v+R}\bigr)$$
$$\text{with} \quad \overline{(\operatorname{Id}-A)v+R} := \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d}\bigl((\operatorname{Id}-A)v+R\bigr)\,dx.$$

It is obvious that $\Phi(v)$ satisfies the linear equation

$$\operatorname{div} w = \operatorname{div}((\operatorname{Id}-A)v) + g \qquad \text{in} \quad (0,T)\times\mathbb{T}^d.$$

Furthermore the operator $\nabla\Delta^{-1}\operatorname{div}$ maps the set $L_{2,0}(\mathbb{T}^d)$ (of $L_2(\mathbb{T}^d)$ functions with zero average) to itself (and with norm 1), and $\bigl|\overline{(\operatorname{Id}-A)v+R}\bigr| \leq \|(\operatorname{Id}-A)v+R\|_2$. Hence we have

$$\begin{aligned} \|\Phi(v)\|_{L_4(0,T;L_2)} &\leq 2\|(\operatorname{Id}-A)v+R\|_{L_4(0,T;L_2)} \\ &\leq 2\|\operatorname{Id}-A\|_{L_\infty(0,T;L_\infty)}\|v\|_{L_4(0,T;L_2)} + 2\|R\|_{L_4(0,T;L_2)}. \end{aligned}$$

Then using the fact that (5.11) implies that (see e.g. the Appendix of [6])

$$\operatorname{div}(Az) = A^T : \nabla z, \tag{5.14}$$

we get

$$\begin{aligned} \|\nabla\Phi(v)\|_{L_2(0,T;L_2)} &\leq \|^T(\operatorname{Id}-A) : \nabla v\|_{L_2(0,T;L_2)} + \|g\|_{L_2(0,T;L_2)} \\ &\leq \|\operatorname{Id}-A\|_{L_\infty(0,T;L_\infty)}\|\nabla v\|_{L_2(0,T;L_2)} + \|g\|_{L_2(0,T;L_2)}. \end{aligned}$$

And finally, because $((\mathrm{Id}-A)v)_t = (\mathrm{Id}-A)v_t - A_t v$, we have

$$\begin{aligned}\|(\Phi(v))_t\|_{L_{4/3}(0,T;L_{3/2})} &\leq C\big(\|(\mathrm{Id}-A)v_t\|_{L_{4/3}(0,T;L_{3/2})}+\|A_t v\|_{L_{4/3}(0,T;L_{3/2})}+\|R_t\|_{L_{4/3}(0,T;L_{3/2})}\big)\\ &\leq C\big(\|\,\mathrm{Id}-A\|_{L_\infty(0,T;L_\infty)}\|v_t\|_{L_{4/3}(0,T;L_{3/2})}\\ &\qquad + \|A_t\|_{L_2(0,T;L_6)}\|v\|_{L_4(0,T;L_2)} + \|R_t\|_{L_{4/3}(0,T;L_{3/2})}\big).\end{aligned}$$

This proves that Φ maps X_T to X_T. Then, obvious variations on the above computations give for any couple (v_1, v_2) in X_T^2, if c in (5.12) is small enough,

$$\|\Phi(v_2)-\Phi(v_1)\|_{X_T} \leq C\big(\|\,\mathrm{Id}-A\|_{L_\infty(0,T;L_\infty)} + \|A_t\|_{L_2(0,T;L_6)}\big)\|v_2-v_1\|_{X_T} \leq \frac{1}{2}\|v_2-v_1\|_{X_T}.$$

Hence, applying the standard Banach fixed point theorem in X_T provides a solution to the equation $\Phi(v)=v$. Then looking back at the above computations in the case $\Phi(v)=v$ gives the desired inequalities. □

In the bounded domain case, the previous lemma can be adapted as follows.

Lemma 5.3. *Let Ω be a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^d$, and A, a matrix valued function on $[0,T]\times\Omega$ satisfying* (5.11)*. If* (5.12) *is fulfilled then for all function $R : [0,T]\times\Omega \to \mathbb{R}^d$ satisfying $\operatorname{div} R \in L^2(0,T\times\Omega)$, $R \in L_4(0,T;L_2)$, $R_t \in L_{4/3}(0,T;L_{3/2})$ and $R\cdot n \equiv 0$ on $(0,T)\times\partial\Omega$, the equation*

$$\operatorname{div}(Aw) = \operatorname{div} R =: g \quad in \quad [0,T]\times\Omega$$

admits a solution in the space

$$X_T := \Big\{v \in L_2(0,T;H^1_0(\Omega))\,,\ v\in L_4(0,T;L_2(\Omega)) \text{ and } v_t \in L_{4/3}(0,T;L_{3/2}(\Omega))\Big\},$$

that satisfies Inequalities (5.13)*.*

Proof. This is essentially Theorem 4.1 of [7] adapted to time dependent coefficients. Recall that Theorem 3.1 of [7] states that there exists a linear operator $\mathcal{B} : k \to u$ that is continuous on $L_p(\Omega;\mathbb{R}^d)$ (for all $1<p<\infty$) so that for all k in $L_p(\Omega;\mathbb{R}^d)$ the vector field u satisfies

$$\int_\Omega u\cdot\nabla\phi\,dx = \int_\Omega k\cdot\nabla\phi\,dx \quad \text{for all} \quad \phi\in\mathcal{C}^\infty(\overline{\Omega};\mathbb{R}), \tag{5.15}$$

and such that, if in addition $\operatorname{div} k$ is in $L_p(\Omega)$ and $k\cdot n|_{\partial\Omega}=0$ then u is in $W^1_{p,0}(\Omega)$ with

$$\|u\|_{W^1_p(\Omega)} \leq C\|\operatorname{div} k\|_{L_p(\Omega)}.$$

Then we define Φ on the set X_T (treating the time variable as a parameter) by

$$\Phi(v) := \mathcal{B}\big((\mathrm{Id}-A)v + R\big). \tag{5.16}$$

The above result ensures that

$$\begin{aligned}\|\nabla\Phi(v)\|_{L_2(0,T;L_2)} &\lesssim \|{}^T(\mathrm{Id}-A):\nabla v + g\|_{L_2(0,T;L_2)}\\ &\lesssim \|g\|_{L_2(0,T;L_2)} + \|\,\mathrm{Id}-A\|_{L_\infty(0,T;L_\infty)}\|\nabla v\|_{L_2(0,T;L_2)},\end{aligned}$$

as well as

$$\begin{aligned}\|\Phi(v)\|_{L_4(0,T;L_2)} &\lesssim \|(\mathrm{Id}-A)v + R\|_{L_4(0,T;L_2)}\\ &\lesssim \|\,\mathrm{Id}-A\|_{L_\infty(0,T;L_\infty)}\|v\|_{L_4(0,T;L_2)} + R\|_{L_4(0,T;L_2)}.\end{aligned}$$

Moreover, differentiating (5.16) with respect to time gives $(\Phi(v))_t = \mathcal{B}((\mathrm{Id} - A)v_t - A_t v + R_t)$ whence, using the continuity property of $\mathcal{B}$ on $L_{3/2}(\Omega)$, then performing a time integration,

$$\begin{aligned}\|(\Phi(v))_t\|_{L_{4/3}(0,T;L_{3/2})} &\lesssim \|(\mathrm{Id} - A)v_t - A_t v\|_{L_{4/3}(0,T;L_{3/2})} + \|R_t\|_{L_{4/3}(0,T;L_{3/2})}\\ &\lesssim \|\mathrm{Id} - A\|_{L_\infty(0,T;L_\infty)}\|v_t\|_{L_{4/3}(0,T;L_{3/2})}\\ &\qquad + \|A_t\|_{L_2(0,T;L_6)}\|v\|_{L_4(0,T;L_2)} + \|R_t\|_{L_{4/3}(0,T;L_{3/2})},\end{aligned}$$

and one can conclude by the Banach fixed point theorem, exactly as in Lemma 5.2. □

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