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TEMPERED DISTRIBUTIONS AND FOURIER TRANSFORM ON THE HEISENBERG GROUP

HAJER BAHOURI, JEAN-YVES CHEMIN, AND RAPHAEL DANCHIN

ABSTRACT. The final goal of the present work is to extend the Fourier transform on the Heisenberg group \mathbb{H}^d , to tempered distributions. As in the Euclidean setting, the strategy is to first show that the Fourier transform is an isomorphism on the Schwartz space, then to define the extension by duality. The difficulty that is here encountered is that the Fourier transform of an integrable function on \mathbb{H}^d is no longer a function on \mathbb{H}^d : according to the standard definition, it is a family of bounded operators on $L^2(\mathbb{R}^d)$. Following our new approach in [1], we here define the Fourier transform of an integrable function to be a mapping on the set $\widetilde{\mathbb{H}}^d = \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{R} \setminus \{0\}$ endowed with a suitable distance \widehat{d} . This viewpoint turns out to provide a user friendly description of the range of the Schwartz space on \mathbb{H}^d by the Fourier transform, which makes the extension to the whole set of tempered distributions straightforward. As a first application, we give an explicit formula for the Fourier transform of smooth functions on \mathbb{H}^d that are independent of the vertical variable. We also provide other examples.

Keywords: Fourier transform, Heisenberg group, frequency space, tempered distributions, Schwartz space.

AMS Subject Classification (2000): 43A30, 43A80.

1. INTRODUCTION

The present work aims at extending Fourier analysis on the Heisenberg group from integrable functions to tempered distributions. It is by now very classical that in the case of a commutative group, the Fourier transform is a function on the group of characters. In the Euclidean space \mathbb{R}^n the group of characters may be identified to the dual space $(\mathbb{R}^n)^\star$ of \mathbb{R}^n through the map $\xi \mapsto e^{i\langle \xi, \cdot \rangle}$, where $\langle \xi, \cdot \rangle$ designate the value of the one-form ξ when applied to elements of \mathbb{R}^n , and the Fourier transform of an integrable function f may be seen as a function on $(\mathbb{R}^n)^\star$, defined by the formula

$$(1.1) \quad \mathcal{F}(f)(\xi) = \widehat{f}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} f(x) dx.$$

A fundamental fact of the distribution theory on \mathbb{R}^n is that the Fourier transform is a bi-continuous isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ – the set of smooth functions whose derivatives decay at infinity faster than any power. Hence, one can define the transposed Fourier transform ${}^t\mathcal{F}$ on the so-called set of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, that is the topological dual of $\mathcal{S}(\mathbb{R}^n)$ (see e.g. [2, 3] for a self-contained presentation). Now, as the whole distribution theory on \mathbb{R}^n is based on identifying locally integrable functions with linear forms by means of the Lebesgue integral, it is natural to look for a more direct relationship between ${}^t\mathcal{F}$ and \mathcal{F} , by considering the following bilinear form on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$

$$(1.2) \quad \mathcal{B}_{\mathbb{R}}(f, \phi) \stackrel{\text{def}}{=} \int_{T^\star\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle} \phi(\xi) dx d\xi,$$

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where the cotangent bundle $T^*\mathbb{R}^n$ of \mathbb{R}^n is identified to $\mathbb{R}^n \times (\mathbb{R}^n)^*$. The above bilinear form allows to identify ${}^t\mathcal{F}|_{\mathcal{S}((\mathbb{R}^n)^*)}$ to $\mathcal{F}|_{\mathcal{S}(\mathbb{R}^n)}$, and still makes sense if f and ϕ are in $L^1(\mathbb{R}^n)$, because the function $f \otimes \phi$ is integrable on $T^*\mathbb{R}^n$. It is thus natural to define the extension of \mathcal{F} on $\mathcal{S}'(\mathbb{R}^n)$ to be ${}^t\mathcal{F}$. In other words,

$$(1.3) \quad \forall (T, \phi) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n), \quad \langle \widehat{T}, \phi \rangle_{\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)} \stackrel{\text{def}}{=} \langle T, \widehat{\phi} \rangle_{\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)}.$$

We aim at implementing that procedure on the Heisenberg group \mathbb{H}^d . As in the Euclidean case, to achieve our goal, it is fundamental to have a handy characterization of the range of the Schwartz space on \mathbb{H}^d by the Fourier transform. The first attempt in that direction goes back to the pioneering works by Geller in [4, 5] (see also [6, 7, 8] and the references therein), where asymptotic series are used. Whether the description of $\mathcal{F}(\mathcal{S}(\mathbb{H}^d))$ given therein allows to extend the Fourier transform to tempered distribution is unclear, though.

Before presenting our main results, we have to recall the definitions of the Heisenberg group \mathbb{H}^d and of the Fourier transform on \mathbb{H}^d . Throughout this paper we shall see \mathbb{H}^d as the set $T^*\mathbb{R}^d \times \mathbb{R}$ equipped with the product law

$$w \cdot w' \stackrel{\text{def}}{=} (Y + Y', s + s' + 2\sigma(Y, Y')) = (y + y', \eta + \eta', s + s' + 2\langle \eta, y' \rangle - 2\langle \eta', y \rangle)$$

where $w = (Y, s) = (y, \eta, s)$ and $w' = (Y', s') = (y', \eta', s')$ are generic elements of \mathbb{H}^d . In the above definition, the notation $\langle \cdot, \cdot \rangle$ designates the duality bracket between $(\mathbb{R}^d)^*$ and \mathbb{R}^d and σ is the canonical symplectic form on \mathbb{R}^{2d} seen as $T^*\mathbb{R}^d$. This gives on \mathbb{H}^d a structure of a non commutative group for which $w^{-1} = -w$. We refer for instance to the books [9, 10, 11, 12, 13, 14, 15, 16] and the references therein for further details.

In accordance with the above product formula, one can define the set of the dilations on the Heisenberg group to be the family of operators $(\delta_a)_{a>0}$ given by

$$(1.4) \quad \delta_a(w) = \delta_a(Y, s) \stackrel{\text{def}}{=} (aY, a^2s).$$

Note that dilations commute with the product law on \mathbb{H}^d , that is $\delta_a(w \cdot w') = \delta_a(w) \cdot \delta_a(w')$. Furthermore, as the determinant of δ_a (seen as an automorphism of \mathbb{R}^{2d+1}) is a^{2d+2} , it is natural to define the *homogeneous dimension* of \mathbb{H}^d to be $N \stackrel{\text{def}}{=} 2d + 2$.

The Heisenberg group is endowed with a smooth left invariant *Haar measure*, which, in the coordinate system (y, η, s) is just the Lebesgue measure on \mathbb{R}^{2d+1} . The corresponding Lebesgue spaces $L^p(\mathbb{H}^d)$ are thus the sets of measurable functions $f : \mathbb{H}^d \rightarrow \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{H}^d)} \stackrel{\text{def}}{=} \left(\int_{\mathbb{H}^d} |f(w)|^p dw \right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \leq p < \infty,$$

with the standard modification if $p = \infty$.

The convolution product of any two integrable functions f and g is given by

$$(1.5) \quad f \star g(w) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w \cdot v^{-1})g(v) dv = \int_{\mathbb{H}^d} f(v)g(v^{-1} \cdot w) dv.$$

As in the Euclidean case, the convolution product is an *associative* binary operation on the set of integrable functions. Even though it is no longer commutative, the following *Young inequalities* hold true:

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \text{whenever } 1 \leq p, q, r \leq \infty \text{ and } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

The *Schwartz space* $\mathcal{S}(\mathbb{H}^d)$ corresponds to the Schwartz space $\mathcal{S}(\mathbb{R}^{2d+1})$ (an equivalent definition involving the Heisenberg structure will be provided in Appendix A.3).

As the Heisenberg group is noncommutative, it is unfortunately not possible to define the Fourier transform of integrable functions on \mathbb{H}^d , by a formula similar to (1.1), just resorting to the characters of \mathbb{H}^d . Actually, the group of characters on \mathbb{H}^d is isometric to the group of characters on $T^*\mathbb{R}^d$ and, if one defines the Fourier transform according to Formula (1.1) then the information pertaining to the vertical variable s is lost. One has to use a more elaborate family of irreducible representations. As explained for instance in [15] Chapter 2, all irreducible representations of \mathbb{H}^d are unitary equivalent to the *Schrödinger representation* $(U^\lambda)_{\lambda \in \mathbb{R} \setminus \{0\}}$ which is the family of group homomorphisms $w \mapsto U_w^\lambda$ between \mathbb{H}^d and the unitary group $\mathcal{U}(L^2(\mathbb{R}^d))$ of $L^2(\mathbb{R}^d)$ defined for all $w = (y, \eta, s)$ in \mathbb{H}^d and u in $L^2(\mathbb{R}^d)$ by

$$U_w^\lambda u(x) \stackrel{\text{def}}{=} e^{-i\lambda(s+2\langle \eta, x-y \rangle)} u(x-2y).$$

The standard definition of the Fourier transform reads as follows.

Definition 1.1. For f in $L^1(\mathbb{H}^d)$ and λ in $\mathbb{R} \setminus \{0\}$, we define

$$\mathcal{F}^{\mathbb{H}}(f)(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w) U_w^\lambda dw.$$

The function $\mathcal{F}^{\mathbb{H}}(f)$ which takes values in the space of bounded operators on $L^2(\mathbb{R}^d)$, is by definition the Fourier transform of f .

As the map $w \mapsto U_w^\lambda$ is a homomorphism between \mathbb{H}^d and the unitary group $\mathcal{U}(L^2(\mathbb{R}^d))$ of $L^2(\mathbb{R}^d)$, it is clear that for any couple (f, g) of integrable functions, we have

$$(1.6) \quad \mathcal{F}^{\mathbb{H}}(f \star g)(\lambda) = \mathcal{F}^{\mathbb{H}}(f)(\lambda) \circ \mathcal{F}^{\mathbb{H}}(g)(\lambda).$$

An obvious drawback of Definition 1.1 is that $\mathcal{F}^{\mathbb{H}}f$ is *not* a complex valued function on some ‘frequency space’, but a much more complicated object. Consequently, with this viewpoint, one can hardly expect to have a characterization of the range of the Schwartz space by $\mathcal{F}^{\mathbb{H}}$, allowing for our extending the Fourier transform to tempered distributions.

To overcome that difficulty, we proposed in our recent paper [1] an alternative (equivalent) definition that makes the Fourier transform of any integrable function on \mathbb{H}^d , a continuous function on another (explicit and simple) set $\widehat{\mathbb{H}}^d$ endowed with some distance \widehat{d} .

Before giving our definition, we need to introduce some notation. Let us first recall that the Lie algebra of *left invariant* vector fields, that is vector fields commuting with any left translation $\tau_w(w') \stackrel{\text{def}}{=} w \cdot w'$, is spanned by the vector fields

$$S \stackrel{\text{def}}{=} \partial_s, \quad \mathcal{X}_j \stackrel{\text{def}}{=} \partial_{y_j} + 2\eta_j \partial_s \quad \text{and} \quad \Xi_j \stackrel{\text{def}}{=} \partial_{\eta_j} - 2y_j \partial_s, \quad 1 \leq j \leq d.$$

The *Laplacian* associated to the vector fields $(\mathcal{X}_j)_{1 \leq j \leq d}$ and $(\Xi_j)_{1 \leq j \leq d}$ is defined by

$$(1.7) \quad \Delta_{\mathbb{H}} \stackrel{\text{def}}{=} \sum_{j=1}^d (\mathcal{X}_j^2 + \Xi_j^2),$$

and may be alternately rewritten in terms of the usual derivatives as follows:

$$(1.8) \quad \Delta_{\mathbb{H}} f(Y, s) = \Delta_Y f(Y, s) + 4 \sum_{j=1}^d (\eta_j \partial_{y_j} - y_j \partial_{\eta_j}) \partial_s f(Y, s) + 4|Y|^2 \partial_s^2 f(Y, s).$$

The Laplacian plays a fundamental role in the Heisenberg group and in particular in the Fourier transform theory. The starting point is the following relation that holds true for functions on the Schwartz space (see e.g. [17, 18]):

$$(1.9) \quad \mathcal{F}^{\mathbb{H}}(\Delta_{\mathbb{H}}f)(\lambda) = 4\mathcal{F}^{\mathbb{H}}(f)(\lambda) \circ \Delta_{\text{osc}}^{\lambda} \quad \text{with} \quad \Delta_{\text{osc}}^{\lambda}u(x) \stackrel{\text{def}}{=} \sum_{j=1}^d \partial_j^2 u(x) - \lambda^2 |x|^2 u(x).$$

In order to take advantage of the spectral structure of the harmonic oscillator, it is natural to introduce the corresponding eigenvectors, that is the family of Hermite functions $(H_n)_{n \in \mathbb{N}^d}$ defined by

$$(1.10) \quad H_n \stackrel{\text{def}}{=} \left(\frac{1}{2^{|n|} n!} \right)^{\frac{1}{2}} C^n H_0 \quad \text{with} \quad C^n \stackrel{\text{def}}{=} \prod_{j=1}^d C_j^{n_j} \quad \text{and} \quad H_0(x) \stackrel{\text{def}}{=} \pi^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}},$$

where $C_j \stackrel{\text{def}}{=} -\partial_j + M_j$ stands for the *creation operator* with respect to the j -th variable and M_j is the multiplication operator defined by $M_j u(x) \stackrel{\text{def}}{=} x_j u(x)$. As usual, $n! \stackrel{\text{def}}{=}} n_1! \cdots n_d!$ and $|n| \stackrel{\text{def}}{=} n_1 + \cdots + n_d$.

Recall that $(H_n)_{n \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, and that we have

$$(1.11) \quad (-\partial_j^2 + M_j^2)H_n = (2n_j + 1)H_n \quad \text{and thus} \quad -\Delta_{\text{osc}}^1 H_n = (2|n| + d)H_n.$$

For λ in $\mathbb{R} \setminus \{0\}$, we finally introduce the rescaled Hermite function $H_{n,\lambda}(x) \stackrel{\text{def}}{=} |\lambda|^{\frac{d}{4}} H_n(|\lambda|^{\frac{1}{2}}x)$. It is obvious that $(H_{n,\lambda})_{n \in \mathbb{N}^d}$ is still an orthonormal basis of $L^2(\mathbb{R}^d)$ and that

$$(1.12) \quad (-\partial_j^2 + \lambda^2 M_j^2)H_{n,\lambda} = (2n_j + 1)|\lambda|H_{n,\lambda} \quad \text{and thus} \quad -\Delta_{\text{osc}}^{\lambda} H_{n,\lambda} = (2|n| + d)|\lambda|H_{n,\lambda}.$$

Remark 1.1. *The vector fields*

$$\tilde{\mathcal{X}}_j \stackrel{\text{def}}{=} \partial_{y_j} - 2\eta_j \partial_s \quad \text{and} \quad \tilde{\mathcal{X}}_j \stackrel{\text{def}}{=} \partial_{\eta_j} + 2y_j \partial_s$$

are right invariant and we have

$$\mathcal{F}^{\mathbb{H}}(\tilde{\Delta}_{\mathbb{H}}f)(\lambda) = 4\Delta_{\text{osc}}^{\lambda} \circ \mathcal{F}^{\mathbb{H}}(f)(\lambda).$$

Our alternative definition of the Fourier transform on \mathbb{H}^d reads as follows:

Definition 1.2. *Let $\tilde{\mathbb{H}}^d \stackrel{\text{def}}{=} \mathbb{N}^{2d} \times \mathbb{R} \setminus \{0\}$. We denote by $\hat{w} = (n, m, \lambda)$ a generic point of $\tilde{\mathbb{H}}^d$. For f in $L^1(\mathbb{H}^d)$, we define the map $\mathcal{F}_{\mathbb{H}}f$ (also denoted by $\hat{f}_{\mathbb{H}}$) to be*

$$\mathcal{F}_{\mathbb{H}}f : \begin{cases} \tilde{\mathbb{H}}^d & \longrightarrow \mathbb{C} \\ \hat{w} & \longmapsto (\mathcal{F}^{\mathbb{H}}(f)(\lambda) H_{m,\lambda} | H_{n,\lambda})_{L^2}. \end{cases}$$

To underline the similarity between that definition and the classical one in \mathbb{R}^n , one may further compute $(\mathcal{F}^{\mathbb{H}}(f)(\lambda) H_{m,\lambda} | H_{n,\lambda})_{L^2}$. One can observe that, after an obvious change of variable, the Fourier transform recasts in terms of the mean value of f modulated by some oscillatory functions which are closely related to Wigner transforms of Hermite functions, namely

$$(1.13) \quad \mathcal{F}_{\mathbb{H}}f(\hat{w}) = \int_{\mathbb{H}^d} \overline{e^{is\lambda} \mathcal{W}(\hat{w}, Y)} f(Y, s) dY ds \quad \text{with}$$

$$(1.14) \quad \mathcal{W}(\hat{w}, Y) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) dz.$$

Let us emphasize that with this new point of view, Formula (1.9) recasts as follows:

$$(1.15) \quad \mathcal{F}_{\mathbb{H}}(\Delta_{\mathbb{H}}f)(\widehat{w}) = -4|\lambda|(2|m| + d)\widehat{f}_{\mathbb{H}}(\widehat{w}).$$

Furthermore, if we endow the set $\widetilde{\mathbb{H}}^d$ with the measure $d\widehat{w}$ defined by the relation

$$(1.16) \quad \int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{w}) d\widehat{w} \stackrel{\text{def}}{=} \sum_{(n,m) \in \mathbb{N}^{2d}} \int_{\mathbb{R}} \theta(n, m, \lambda) |\lambda|^d d\lambda,$$

then the classical inversion formula and Fourier-Plancherel theorem recast as follows:

Theorem 1.1. *Let f be a function in $\mathcal{S}(\mathbb{H}^d)$. Then we have the inversion formula*

$$(1.17) \quad f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} e^{is\lambda} \mathcal{W}(\widehat{w}, Y) \widehat{f}_{\mathbb{H}}(\widehat{w}) d\widehat{w} \quad \text{for any } w \text{ in } \mathbb{H}^d.$$

Moreover, the Fourier transform $\mathcal{F}_{\mathbb{H}}$ can be extended into a bicontinuous isomorphism between $L^2(\mathbb{H}^d)$ and $L^2(\widetilde{\mathbb{H}}^d)$, which satisfies

$$(1.18) \quad \|\widehat{f}_{\mathbb{H}}\|_{L^2(\widetilde{\mathbb{H}}^d)}^2 = \frac{\pi^{d+1}}{2^{d-1}} \|f\|_{L^2(\mathbb{H}^d)}^2.$$

Finally, for any couple (f, g) of integrable functions, the following convolution identity holds true:

$$(1.19) \quad \begin{aligned} \mathcal{F}_{\mathbb{H}}(f \star g)(n, m, \lambda) &= (\widehat{f}_{\mathbb{H}} \cdot \widehat{g}_{\mathbb{H}})(n, m, \lambda) \quad \text{with} \\ (\widehat{f}_{\mathbb{H}} \cdot \widehat{g}_{\mathbb{H}})(n, m, \lambda) &\stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{N}^d} \widehat{f}_{\mathbb{H}}(n, \ell, \lambda) \widehat{g}_{\mathbb{H}}(\ell, m, \lambda). \end{aligned}$$

For the reader's convenience, we present a proof of Theorem 1.1 in the appendix.

2. MAIN RESULTS

As already mentioned, our main goal is to extend the Fourier transform to tempered distributions on \mathbb{H}^d . If we follow the standard approach of the Euclidean setting, that is described by (1.2) and (1.3), then we need a handy description of the range of $\mathcal{S}(\mathbb{H}^d)$ by the Fourier transform $\mathcal{F}_{\mathbb{H}}$ in order to guess what could be the appropriate bilinear form $\mathcal{B}_{\mathbb{H}}$ allowing for identifying ${}^t\mathcal{F}_{\mathbb{H}}$ with $\mathcal{F}_{\mathbb{H}}$. To characterize $\mathcal{F}(\mathcal{S}(\mathbb{H}^d))$, we shall just keep in mind the most obvious properties we expect the Fourier transform to have. The first one is that it should change regularity of functions on \mathbb{H}^d to decay of the Fourier transform. This is achieved in the following lemma (see the proof in [1]).

Lemma 2.1. *For any integer p , there exist an integer N_p and a positive constant C_p such that for all \widehat{w} in $\widetilde{\mathbb{H}}^d$ and all f in $\mathcal{S}(\mathbb{H}^d)$, we have*

$$(2.1) \quad (1 + |\lambda|(|n| + |m| + d) + |n - m|)^p |\widehat{f}_{\mathbb{H}}(n, m, \lambda)| \leq C_p \|f\|_{N_p, \mathcal{S}},$$

where $\|\cdot\|_{N, \mathcal{S}}$ denotes the classical family of semi-norms of $\mathcal{S}(\mathbb{R}^{2d+1})$, namely

$$\|f\|_{N, \mathcal{S}} \stackrel{\text{def}}{=} \sup_{|\alpha| \leq N} \|(1 + |Y|^2 + s^2)^{N/2} \partial_{Y, s}^{\alpha} f\|_{L^{\infty}}.$$

The decay inequality (2.1) prompts us to endow the set $\widetilde{\mathbb{H}}^d$ with the following distance \widehat{d} :

$$(2.2) \quad \widehat{d}(\widehat{w}, \widehat{w}') \stackrel{\text{def}}{=} |\lambda(n + m) - \lambda'(n' + m')|_1 + |(n - m) - (n' - m')|_1 + |\lambda - \lambda'|,$$

where $|\cdot|_1$ denotes the ℓ^1 norm on \mathbb{R}^d .

The second basic property we expect for the Fourier transform is that it changes decay properties into regularity. This is closely related to how it acts on suitable weight functions. As in the Euclidean case, we expect $\mathcal{F}_{\mathbb{H}}$ to transform multiplication by weight functions into a combination of derivatives, so we need a definition of differentiation for functions defined on $\widetilde{\mathbb{H}}^d$ that could fit the scope. This is the aim of the following definition (see also Proposition A.2 in Appendix):

Definition 2.1. For any function $\theta : \widetilde{\mathbb{H}}^d \rightarrow \mathbb{C}$ we define

$$(2.3) \quad \widehat{\Delta}\theta(\widehat{w}) \stackrel{\text{def}}{=} -\frac{1}{2|\lambda|}(|n+m|+d)\theta(\widehat{w}) \\ + \frac{1}{2|\lambda|} \sum_{j=1}^d \left(\sqrt{(n_j+1)(m_j+1)}\theta(\widehat{w}_j^+) + \sqrt{n_j m_j}\theta(\widehat{w}_j^-) \right)$$

and, if in addition θ is differentiable with respect to λ ,

$$(2.4) \quad \widehat{\mathcal{D}}_\lambda\theta(\widehat{w}) \stackrel{\text{def}}{=} \frac{d\theta}{d\lambda}(\widehat{w}) + \frac{d}{2\lambda}\theta(\widehat{w}) + \frac{1}{2\lambda} \sum_{j=1}^d \left(\sqrt{n_j m_j}\theta(\widehat{w}_j^-) - \sqrt{(n_j+1)(m_j+1)}\theta(\widehat{w}_j^+) \right)$$

where $\widehat{w}_j^\pm \stackrel{\text{def}}{=} (n \pm \delta_j, m \pm \delta_j, \lambda)$ and δ_j denotes the element of \mathbb{N}^d with all components equal to 0 except the j -th which has value 1.

The notation in the above definition is justified by the following lemma that will be proved in Subsection 3.2.

Lemma 2.2. Let M^2 and M_0 be the multiplication operators defined on $\mathcal{S}(\mathbb{H}^d)$ by

$$(2.5) \quad (M^2 f)(Y, s) \stackrel{\text{def}}{=} |Y|^2 f(Y, s) \quad \text{and} \quad M_0 f(Y, s) \stackrel{\text{def}}{=} -isf(Y, s).$$

Then for all f in $\mathcal{S}(\mathbb{H}^d)$, the following two relations hold true on $\widetilde{\mathbb{H}}^d$:

$$\mathcal{F}_{\mathbb{H}} M^2 f = -\widehat{\Delta}\mathcal{F}_{\mathbb{H}} f \quad \text{and} \quad \mathcal{F}_{\mathbb{H}}(M_0 f) = \widehat{\mathcal{D}}_\lambda \mathcal{F}_{\mathbb{H}} f.$$

The third important aspect of regularity for functions in $\mathcal{F}_{\mathbb{H}}(\mathcal{S}(\mathbb{H}^d))$ is the link between their values for positive λ and negative λ . That property, that has no equivalent in the Euclidean setting, is described by the following lemma:

Lemma 2.3. Let us consider on $\mathcal{S}(\mathbb{H}^d)$ the operator \mathcal{P} defined by

$$(2.6) \quad \mathcal{P}(f)(Y, s) \stackrel{\text{def}}{=} \frac{1}{2} \int_{-\infty}^s (f(Y, s') - f(Y, -s')) ds'.$$

Then \mathcal{P} maps continuously $\mathcal{S}(\mathbb{H}^d)$ to $\mathcal{S}(\mathbb{H}^d)$ and we have for any f in $\mathcal{S}(\mathbb{H}^d)$ and \widehat{w} in $\widetilde{\mathbb{H}}^d$,

$$(2.7) \quad 2i\mathcal{F}_{\mathbb{H}}(\mathcal{P}f) = \widehat{\Sigma}_0(\mathcal{F}_{\mathbb{H}}f) \quad \text{with} \quad (\widehat{\Sigma}_0\theta)(\widehat{w}) \stackrel{\text{def}}{=} \frac{\theta(n, m, \lambda) - (-1)^{|n+m|}\theta(m, n, -\lambda)}{\lambda}.$$

The above weird relation is just a consequence of the following property of the Wigner transform \mathcal{W} :

$$(2.8) \quad \forall (n, m, \lambda, Y) \in \widetilde{\mathbb{H}}^d \times T^*\mathbb{R}^d, \quad \mathcal{W}(n, m, \lambda, Y) = (-1)^{|n+m|}\mathcal{W}(m, n, -\lambda, Y).$$

In the case $m = n$, it means that the left and right limits at $\lambda = 0$ of functions in $\mathcal{F}_{\mathbb{H}}(\mathcal{S}(\mathbb{H}^d))$ must be the same.

Definition 2.2. We define $\mathcal{S}(\widetilde{\mathbb{H}}^d)$ to be the set of functions θ on $\widetilde{\mathbb{H}}^d$ such that:

- for any (n, m) in \mathbb{N}^{2d} , the map $\lambda \mapsto \theta(n, m, \lambda)$ is smooth on $\mathbb{R} \setminus \{0\}$,
- for any non negative integer N , the functions $\widehat{\Delta}^N \theta$, $\widehat{\mathcal{D}}_\lambda^N \theta$ and $\widehat{\Sigma}_0 \widehat{\mathcal{D}}_\lambda^N \theta$ decay faster than any power of $\widehat{d}_0(\widehat{w}) \stackrel{\text{def}}{=} |\lambda|(|n+m|_1 + d) + |m-n|_1$.

We equip $\mathcal{S}(\widetilde{\mathbb{H}}^d)$ with the family of semi-norms

$$\|\theta\|_{N, N', \mathcal{S}(\widetilde{\mathbb{H}}^d)} \stackrel{\text{def}}{=} \sup_{\widehat{w} \in \widetilde{\mathbb{H}}^d} (1 + \widehat{d}_0(\widehat{w}))^N \left(|\widehat{\Delta}^{N'} \theta(\widehat{w})| + |\widehat{\mathcal{D}}_\lambda^{N'} \theta(\widehat{w})| + |\widehat{\Sigma}_0 \widehat{\mathcal{D}}_\lambda^{N'} \theta(\widehat{w})| \right).$$

Let us first point out that an integer K exists such that

$$(2.9) \quad \|\theta\|_{L^1(\widetilde{\mathbb{H}}^d)} \leq C \|\theta\|_{K, 0, \mathcal{S}(\widetilde{\mathbb{H}}^d)}.$$

The main motivation of this definition is the following isomorphism theorem.

Theorem 2.1. *The Fourier transform $\mathcal{F}_{\mathbb{H}}$ is a bicontinuous isomorphism between $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(\widetilde{\mathbb{H}}^d)$, and the inverse map is given by*

$$(2.10) \quad \widetilde{\mathcal{F}}_{\mathbb{H}} \theta(w) \stackrel{\text{def}}{=} \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} e^{is\lambda} \mathcal{W}(\widehat{w}, Y) \theta(\widehat{w}) d\widehat{w}.$$

The definition of $\mathcal{S}(\widetilde{\mathbb{H}}^d)$ encodes a number of nontrivial hidden informations that are partly consequences of the sub-ellipticity of $\Delta_{\mathbb{H}}$. For instance, the stability of $\mathcal{S}(\widetilde{\mathbb{H}}^d)$ by the multiplication law defined in (1.19) is an obvious consequence of the stability of $\mathcal{S}(\mathbb{H}^d)$ by convolution and of Theorem 2.1. Another hidden information is the behavior of functions of $\mathcal{S}(\widetilde{\mathbb{H}}^d)$ when λ tends to 0. In fact, Achille's heel of the metric space $(\widetilde{\mathbb{H}}^d, \widehat{d})$ is that it is not complete. It turns out however that the Fourier transform of any integrable function on \mathbb{H}^d is *uniformly continuous* on $\widetilde{\mathbb{H}}^d$. Therefore, it is natural to extend it to the completion $\widehat{\mathbb{H}}^d$ of $\widetilde{\mathbb{H}}^d$. This is explained in greater details in the following statement that has been proved in [1].

Theorem 2.2. *The completion of $(\widetilde{\mathbb{H}}^d, \widehat{d})$ is the metric space $(\widehat{\mathbb{H}}^d, \widehat{d})$ defined by*

$$\widehat{\mathbb{H}}^d \stackrel{\text{def}}{=} \widetilde{\mathbb{H}}^d \cup \widehat{\mathbb{H}}_0^d \quad \text{with} \quad \widehat{\mathbb{H}}_0^d \stackrel{\text{def}}{=} \mathbb{R}_{\mp}^d \times \mathbb{Z}^d \quad \text{and} \quad \mathbb{R}_{\mp}^d \stackrel{\text{def}}{=} (\mathbb{R}_-)^d \cup (\mathbb{R}_+)^d.$$

Moreover, on $\widehat{\mathbb{H}}^d$, the extended distance (still denoted by \widehat{d}) is given for all $\widehat{w} = (n, m, \lambda)$ and $\widehat{w}' = (n', m', \lambda')$ in $\widetilde{\mathbb{H}}^d$, and for all (\dot{x}, k) and (\dot{x}', k') in $\widehat{\mathbb{H}}_0^d$ by

$$\begin{aligned} \widehat{d}(\widehat{w}, \widehat{w}') &= |\lambda(n+m) - \lambda'(n'+m')|_1 + |(m-n) - (m'-n')|_1 + |\lambda - \lambda'|, \\ \widehat{d}(\widehat{w}, (\dot{x}, k)) &= \widehat{d}((\dot{x}, k), \widehat{w}) \stackrel{\text{def}}{=} |\lambda(n+m) - \dot{x}|_1 + |m-n-k|_1 + |\lambda|, \\ \widehat{d}((\dot{x}, k), (\dot{x}', k')) &= |\dot{x} - \dot{x}'|_1 + |k - k'|_1. \end{aligned}$$

The Fourier transform $\widehat{f}_{\mathbb{H}}$ of any integrable function on \mathbb{H}^d may be extended continuously to the whole set $\widehat{\mathbb{H}}^d$. Still denoting by $\widehat{f}_{\mathbb{H}}$ (or $\mathcal{F}_{\mathbb{H}} f$) that extension, the linear map $\mathcal{F}_{\mathbb{H}} : f \mapsto \widehat{f}_{\mathbb{H}}$ is continuous from the space $L^1(\mathbb{H}^d)$ to the space $\mathcal{C}_0(\widehat{\mathbb{H}}^d)$ of continuous functions on $\widehat{\mathbb{H}}^d$ tending to 0 at infinity.

It is now natural to introduce the space $\mathcal{S}(\widehat{\mathbb{H}}^d)$.

Definition 2.3. *We denote by $\mathcal{S}(\widehat{\mathbb{H}}^d)$ the space of functions on $\widehat{\mathbb{H}}^d$ which are continuous extensions of elements of $\mathcal{S}(\widetilde{\mathbb{H}}^d)$.*

As an elementary exercise of functional analysis, the reader can prove that $\mathcal{S}(\widehat{\mathbb{H}}^d)$ endowed with the semi-norms $\|\cdot\|_{N,N',\mathcal{S}(\widehat{\mathbb{H}}^d)}$ is a Fréchet space. Those semi-norms will be denoted by $\|\cdot\|_{N,N',\mathcal{S}(\widehat{\mathbb{H}}^d)}$ in all that follows.

Note also that for any function θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$, having \widehat{w} tend to (\dot{x}, k) in (2.7) yields

$$(2.11) \quad \theta(\dot{x}, k) = (-1)^{|k|} \theta(-\dot{x}, -k).$$

As regards convolution, we obtain, after passing to the limit in (1.19), the following noteworthy formula, valid for any two functions f and g in $L^1(\widehat{\mathbb{H}}^d)$:

$$(2.12) \quad \begin{aligned} \mathcal{F}_{\mathbb{H}}(f \star g)_{\widehat{\mathbb{H}}_0^d} &= (\mathcal{F}_{\mathbb{H}}f)_{\widehat{\mathbb{H}}_0^d} \cdot_{\widehat{\mathbb{H}}_0^d} (\mathcal{F}_{\mathbb{H}}g)_{\widehat{\mathbb{H}}_0^d} \quad \text{with} \\ (\theta_1 \cdot_{\widehat{\mathbb{H}}_0^d} \theta_2)(\dot{x}, k) &\stackrel{\text{def}}{=} \sum_{k' \in \mathbb{Z}^d} \theta_1(\dot{x}, k') \theta_2(\dot{x}, k - k'). \end{aligned}$$

Remark 2.1. *Let us emphasize that the above product law (2.12) is commutative even though convolution of functions on the Heisenberg group is not (see (1.19)).*

A natural question then is how to extend the measure $d\widehat{w}$ to $\widehat{\mathbb{H}}^d$. In fact, we have for any positive real numbers R and ε ,

$$\begin{aligned} \int_{\widehat{\mathbb{H}}^d} \mathbf{1}_{\{|\lambda| |n+m| + |m-n| \leq R\}} \mathbf{1}_{|\lambda| \leq \varepsilon} d\widehat{w} &= \int_{-\varepsilon}^{\varepsilon} \left(\sum_{n,m} \mathbf{1}_{\{|\lambda| |n+m| + |m-n| \leq R\}} \right) |\lambda|^d d\lambda \\ &\leq CR^{2d} \int_{-\varepsilon}^{\varepsilon} d\lambda \\ &\leq CR^{2d} \varepsilon. \end{aligned}$$

Therefore, one can extend the measure $d\widehat{w}$ on $\widehat{\mathbb{H}}^d$ simply by defining, for any continuous compactly supported function θ on $\widehat{\mathbb{H}}^d$

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w}) d\widehat{w} \stackrel{\text{def}}{=} \int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w}) d\widehat{w}.$$

At this stage of the paper, pointing out nontrivial examples of functions of $\mathcal{S}(\widehat{\mathbb{H}}^d)$ is highly informative. To this end, we introduce the set \mathcal{S}_d^+ of smooth functions f on $[0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ such that for any integer p , we have

$$(2.13) \quad \sup_{\substack{(x_1, \dots, x_d, k, \lambda) \in [0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R} \\ |\alpha| \leq p}} (1 + x_1 + \dots + x_d + |k|)^p |\partial_{x, \lambda}^\alpha f(x_1, \dots, x_d, k, \lambda)| < \infty.$$

As may be easily checked by the reader, the space \mathcal{S}_d^+ is stable by derivation and multiplication by polynomial functions of (x, k) .

Theorem 2.3. *Let f be a function of \mathcal{S}_d^+ . Let us define for $\widehat{w} = (n, m, \lambda)$ in $\widehat{\mathbb{H}}^d$,*

$$\Theta_f(\widehat{w}) \stackrel{\text{def}}{=} f(|\lambda| R(n, m), m - n, \lambda) \quad \text{with} \quad R(n, m) \stackrel{\text{def}}{=} (n_j + m_j + 1)_{1 \leq j \leq d}.$$

Then Θ_f belongs to $\mathcal{S}(\widehat{\mathbb{H}}^d)$ if

- either f is supported in $[0, \infty[^d \times \{0\} \times \mathbb{R}$,

- or f is supported in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ for some positive real number r_0 , and satisfies

$$(2.14) \quad f(x, -k, \lambda) = (-1)^{|k|} f(x, k, \lambda).$$

An obvious consequence of Theorem 2.3 is that the fundamental solution of the heat equation in \mathbb{H}^d belongs to $\mathcal{S}(\mathbb{H}^d)$ (a highly nontrivial result that is usually deduced from the explicit formula established by B. Gaveau in [19]). Indeed, applying the Fourier transform with respect to the Heisenberg variable gives that if u is the solution of the heat equation with integrable initial data u_0 then

$$(2.15) \quad \widehat{u}_{\mathbb{H}}(t, n, m, \lambda) = e^{-4t|\lambda|(2|m|+d)} \widehat{u}_0(n, m, \lambda).$$

At the same time, we have

$$u(t) = u_0 \star h_t \quad \text{with} \quad h_t(y, \eta, s) = \frac{1}{t^{d+1}} h\left(\frac{y}{\sqrt{t}}, \frac{\eta}{\sqrt{t}}, \frac{s}{t}\right).$$

Hence combining the convolution formula (1.19) and Identity (2.15), we gather that

$$\widehat{h}_{\mathbb{H}}(\widehat{w}) = e^{-4|\lambda|(2|n|+d)} \mathbf{1}_{\{n=m\}}.$$

Then applying Theorem 2.3 to the function $e^{-4(x_1+\dots+x_d)} \mathbf{1}_{\{k=0\}}$ ensures that $\widehat{h}_{\mathbb{H}}$ belongs to $\mathcal{S}(\widehat{\mathbb{H}}^d)$, and the inversion theorem 2.1 thus implies that h is in $\mathcal{S}(\mathbb{H}^d)$.

Along the same lines, we recover Hulanicki's theorem [20] in the case of the Heisenberg group, namely if a belongs to $\mathcal{S}(\mathbb{R})$, then there exists a function h_a in $\mathcal{S}(\mathbb{H}^d)$ such that

$$(2.16) \quad \forall f \in \mathcal{S}(\mathbb{H}^d), \quad a(-\Delta_{\mathbb{H}})f = f \star h_a.$$

As already explained in the introduction, our final aim is to extend the Fourier transform to tempered distributions by adapting the Euclidean procedure described in (1.2)–(1.3). The purpose of the following definition is to specify what a *tempered distribution* on $\widehat{\mathbb{H}}^d$ is.

Definition 2.4. *Tempered distributions on $\widehat{\mathbb{H}}^d$ are elements of the set $\mathcal{S}'(\widehat{\mathbb{H}}^d)$ of continuous linear forms on the Fréchet space $\mathcal{S}(\widehat{\mathbb{H}}^d)$.*

We say that a sequence $(\mathcal{T}_n)_{n \in \mathbb{N}}$ of tempered distributions on $\widehat{\mathbb{H}}^d$ converges to a tempered distribution \mathcal{T} if

$$\forall \theta \in \mathcal{S}(\widehat{\mathbb{H}}^d), \quad \lim_{n \rightarrow \infty} \langle \mathcal{T}_n, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \langle \mathcal{T}, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

Let us now give some examples of elements of $\mathcal{S}'(\widehat{\mathbb{H}}^d)$ and present the most basic properties of this space. As a start, let us specify what are functions with moderate growth.

Definition 2.5. *Let us denote by $L_M^1(\widehat{\mathbb{H}}^d)$ the space of locally integrable functions f on $\widehat{\mathbb{H}}^d$ such that there exists an integer p satisfying*

$$\int_{\widehat{\mathbb{H}}^d} (1 + |\lambda|(n+m+d) + |n-m|)^{-p} |f(\widehat{w})| d\widehat{w} < \infty.$$

As in the Euclidean setting, functions of $L_M^1(\widehat{\mathbb{H}}^d)$ may be identified to tempered distributions:

Theorem 2.4. *Let us consider ι be the map defined by*

$$\iota : \begin{cases} L_M^1(\widehat{\mathbb{H}}^d) & \longrightarrow \mathcal{S}'(\widehat{\mathbb{H}}^d) \\ \psi & \longmapsto \iota(\psi) : \left[\theta \mapsto \int_{\widehat{\mathbb{H}}^d} \psi(\widehat{w}) \theta(\widehat{w}) d\widehat{w} \right]. \end{cases}$$

Then ι is a one-to-one linear map.

Moreover, if p is an integer such that the map

$$(n, m, \lambda) \mapsto (1 + |\lambda|(n + m) + d) + |n - m|)^{-p} f(n, m, \lambda)$$

belongs to $L^1(\widehat{\mathbb{H}}^d)$, then we have

$$(2.17) \quad |\langle \iota(f), \phi \rangle| \leq \left\| (1 + |\lambda|(n + m) + d) + |n - m|)^{-p} f \right\|_{L^1(\widehat{\mathbb{H}}^d)} \|\theta\|_{p,0,\mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

The following proposition provides examples of functions in $L^1_M(\widehat{\mathbb{H}}^d)$.

Proposition 2.1. For any $\gamma < d + 1$ the function f_γ defined on $\widehat{\mathbb{H}}^d$ by

$$f_\gamma(n, m, \lambda) \stackrel{\text{def}}{=} (|\lambda|(2|m| + d))^{-\gamma} \delta_{n,m}$$

belongs to $L^1_M(\widehat{\mathbb{H}}^d)$.

Remark 2.2. The above proposition is no longer true for $\gamma = d + 1$. If we look at the quantity $|\lambda|(2|n| + d)$ in $\widehat{\mathbb{H}}^d$ as an equivalent of $|\xi|^2$ for \mathbb{R}^d , then it means that the homogeneous dimension of $\widehat{\mathbb{H}}^d$ is $2d + 2$, as for \mathbb{H}^d (and as expected).

It is obvious that any Dirac mass on $\widehat{\mathbb{H}}^d$ is a tempered distribution. Let us also note that because

$$|\theta(n, n, \lambda)| \leq (1 + |\lambda|(2|n| + d))^{-d-3} \|\theta\|_{d+3,0,\mathcal{S}(\widehat{\mathbb{H}}^d)},$$

the linear form

$$(2.18) \quad \mathcal{I} : \begin{cases} \mathcal{S}(\widehat{\mathbb{H}}^d) & \longrightarrow \mathbb{C} \\ \theta & \longmapsto \sum_{n \in \mathbb{N}^d} \int_{\mathbb{R}} \theta(n, n, \lambda) |\lambda|^d d\lambda \end{cases}$$

is a tempered distribution on $\widehat{\mathbb{H}}^d$.

We now want to exhibit tempered distributions on $\widehat{\mathbb{H}}^d$ which are *not* measures. The following proposition states that the analogue on $\widehat{\mathbb{H}}^d$ of *finite part* distributions on \mathbb{R}^n , are indeed in $\mathcal{S}'(\widehat{\mathbb{H}}^d)$.

Proposition 2.2. Let γ be in the interval $]d + 1, d + 3/2[$ and denote by $\widehat{0}$ the element $(0, 0)$ of $\widehat{\mathbb{H}}_0^d$. Then for any function θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$, the function defined a.e. on $\widehat{\mathbb{H}}^d$ by

$$(n, m, \lambda) \mapsto \delta_{n,m} \left(\frac{\theta(n, n, \lambda) + \theta(n, n, -\lambda) - 2\theta(\widehat{0})}{|\lambda|^\gamma (2|n| + d)^\gamma} \right),$$

is integrable. Furthermore, the linear form defined by

$$\left\langle \text{Pf} \left(\frac{1}{|\lambda|^\gamma (2|n| + d)^\gamma} \right), \theta \right\rangle \stackrel{\text{def}}{=} \frac{1}{2} \int_{\widehat{\mathbb{H}}^d} \left(\frac{\theta(n, n, \lambda) + \theta(n, n, -\lambda) - 2\theta(\widehat{0})}{|\lambda|^\gamma (2|n| + d)^\gamma} \right) \delta_{n,m} d\widehat{w}$$

is in $\mathcal{S}'(\widehat{\mathbb{H}}^d)$, and its restriction to $\widetilde{\mathbb{H}}^d$ is the function

$$(n, m, \lambda) \mapsto \delta_{n,m} \frac{1}{|\lambda|^\gamma (2|n| + d)^\gamma}$$

in the sense that for any θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$ such that $\theta(n, n, \lambda) = 0$ for small enough $|\lambda|(2|n| + d)$, we have

$$\left\langle \text{Pf}\left(\frac{1}{|\lambda|^\gamma(2|n| + d)^\gamma}\right), \theta \right\rangle = \int_{\widehat{\mathbb{H}}^d} \frac{\theta(\widehat{w})}{|\lambda|^\gamma(2|n| + d)^\gamma} d\widehat{w}.$$

Another interesting example of tempered distribution on $\widehat{\mathbb{H}}^d$ is the measure $\mu_{\widehat{\mathbb{H}}_0^d}$ defined in Lemma 3.1 of [1] which, in our setting, recasts as follows:

Proposition 2.3. *Let the measure $\mu_{\widehat{\mathbb{H}}_0^d}$ be defined by*

$$(2.19) \quad \langle \mu_{\widehat{\mathbb{H}}_0^d}, \theta \rangle = \int_{\widehat{\mathbb{H}}_0^d} \theta(\dot{x}, k) d\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x}, k) \stackrel{\text{def}}{=} 2^{-d} \sum_{k \in \mathbb{Z}^d} \left(\int_{(\mathbb{R}_-)^d} \theta(\dot{x}, k) d\dot{x} + \int_{(\mathbb{R}_+)^d} \theta(\dot{x}, k) d\dot{x} \right)$$

for all functions θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$.

Then $\mu_{\widehat{\mathbb{H}}_0^d}$ is a tempered distribution on $\widehat{\mathbb{H}}^d$ and for any function ψ in $\mathcal{S}(\mathbb{R})$ with integral 1 we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \psi\left(\frac{\lambda}{\varepsilon}\right) = \mu_{\widehat{\mathbb{H}}_0^d} \quad \text{in } \mathcal{S}'(\widehat{\mathbb{H}}^d).$$

Let us finally explain how the Fourier transform may be extended to tempered distributions on \mathbb{H}^d , using an analog of Formulas (1.2) and (1.3). Let us define

$$(2.20) \quad \mathcal{B}_{\mathbb{H}} : \begin{cases} \mathcal{S}(\mathbb{H}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d) & \longrightarrow \mathbb{C} \\ (f, \theta) & \longmapsto \int_{\mathbb{H}^d \times \widehat{\mathbb{H}}^d} f(Y, s) \overline{e^{is\lambda\mathcal{W}(\widehat{w}, Y)}} \theta(\widehat{w}) dwd\widehat{w} \end{cases} \quad \text{and}$$

$$(2.21) \quad {}^t\mathcal{F}_{\mathbb{H}} : \begin{cases} \mathcal{S}(\widehat{\mathbb{H}}^d) & \longrightarrow \mathcal{S}(\mathbb{H}^d) \\ \theta & \longmapsto \int_{\widehat{\mathbb{H}}^d} \overline{e^{is\lambda\mathcal{W}(\widehat{w}, Y)}} \theta(\widehat{w}) d\widehat{w}. \end{cases}$$

Let us notice that for any θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$ and $w = (y, \eta, s)$ in \mathbb{H}^d , we have

$$(2.22) \quad ({}^t\mathcal{F}_{\mathbb{H}}\theta)(y, \eta, s) = \frac{\pi^{d+1}}{2^{d-1}} (\mathcal{F}_{\mathbb{H}}^{-1}\theta)(y, -\eta, -s).$$

Hence, Theorem 2.1 implies that ${}^t\mathcal{F}_{\mathbb{H}}$ is a continuous isomorphism between $\mathcal{S}(\widehat{\mathbb{H}}^d)$ and $\mathcal{S}(\mathbb{H}^d)$. Now, we observe that for any f in $\mathcal{S}(\mathbb{H}^d)$ and θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$, we have

$$(2.23) \quad \mathcal{B}_{\mathbb{H}}(f, \theta) = \int_{\mathbb{H}^d} f(w) ({}^t\mathcal{F}_{\mathbb{H}}\theta)(w) dw = \int_{\widehat{\mathbb{H}}^d} (\mathcal{F}_{\mathbb{H}}f)(\widehat{w}) \theta(\widehat{w}) d\widehat{w}.$$

This prompts us to extend $\mathcal{F}_{\mathbb{H}}$ on $\mathcal{S}'(\mathbb{H}^d)$ as follows:

Definition 2.6. *We define*

$$\mathcal{F}_{\mathbb{H}} : \begin{cases} \mathcal{S}'(\mathbb{H}^d) & \longrightarrow \mathcal{S}'(\widehat{\mathbb{H}}^d) \\ T & \longmapsto \left[\theta \mapsto \langle T, {}^t\mathcal{F}_{\mathbb{H}}\theta \rangle_{\mathcal{S}'(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)} \right]. \end{cases}$$

As a direct consequence of this definition, we have the following statement:

Proposition 2.4. *The map $\mathcal{F}_{\mathbb{H}}$ defined just above is continuous and one-to-one from $\mathcal{S}'(\mathbb{H}^d)$ onto $\mathcal{S}'(\widehat{\mathbb{H}}^d)$. Furthermore, its restriction to $L^1(\mathbb{H}^d)$ coincides with Definition 1.2.*

Just to compare with the Euclidean case, let us give some examples of simple computations of Fourier transform of tempered distributions on \mathbb{H}^d .

Proposition 2.5. *We have*

$$\mathcal{F}_{\mathbb{H}}(\delta_0) = \mathcal{I} \quad \text{and} \quad \mathcal{F}_{\mathbb{H}}(\mathbf{1}) = \frac{\pi^{d+1}}{2^{d-1}} \delta_{\hat{0}},$$

where \mathcal{I} is defined by (2.18) and $\hat{0}$ is the element of $\widehat{\mathbb{H}}_0^d$ corresponding to $\dot{x} = 0$ and $k = 0$.

One question that comes up naturally is to compute the Fourier transform of a function independent of the vertical variable. The answer to that question is given just below.

Theorem 2.5. *We have for any integrable function g on $T^*\mathbb{R}^d$,*

$$\mathcal{F}_{\mathbb{H}}(g \otimes \mathbf{1}) = (\mathcal{G}_{\mathbb{H}}g) \mu_{\widehat{\mathbb{H}}_0^d}$$

where $\mathcal{G}_{\mathbb{H}}g$ is defined by

$$\mathcal{G}_{\mathbb{H}}g : \begin{cases} \widehat{\mathbb{H}}_0^d & \longrightarrow \mathbb{C} \\ (\dot{x}, k) & \longmapsto \int_{T^*\mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y) g(Y) dY \end{cases} \quad \text{with}$$

$$\mathcal{K}_d(\dot{x}, k, Y) = \bigotimes_{j=1}^d \mathcal{K}(\dot{x}_j, k_j, Y_j) \quad \text{and}$$

$$\mathcal{K}(\dot{x}, k, y, \eta) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(2|\dot{x}|^{\frac{1}{2}}(y \sin z + \eta \operatorname{sgn}(\dot{x}) \cos z) + kz)} dz.$$

As we shall see, this result is just an interpretation of Theorem 1.4 of [1] in terms of tempered distributions.

The rest of the paper unfolds as follows. In Section 3, we prove Lemmas 2.2 and 2.3, and then Theorem 2.1. In Section 4, we establish Theorem 2.3. In Section 5, we study in full details the examples of tempered distributions on $\widehat{\mathbb{H}}^d$ given in Propositions 2.1–2.2, and Theorem 2.4. In Section 6, we prove Proposition 2.5 and Theorem 2.5. Further remarks as well as proofs (within our setting) of known results are postponed in the appendix.

3. THE RANGE OF THE SCHWARTZ CLASS BY THE FOURIER TRANSFORM

The present section aims at giving a handy characterization of the range of $\mathcal{S}(\mathbb{H}^d)$ by the Fourier transform. Our Ariadne thread throughout will be that we expect that, for the action of $\mathcal{F}_{\mathbb{H}}$, *regularity implies decay* and *decay implies regularity*. The answer to the first issue has been given in Lemma 2.1 (proved in [1]). Here we shall concentrate on the second issue, in connection with the definition of differentiation for functions on $\widetilde{\mathbb{H}}^d$, given in (2.3) and (2.4). To complete our analysis of the space $\mathcal{F}_{\mathbb{H}}(\mathcal{S}(\mathbb{H}^d))$, we will have to get some information on the behavior of elements of $\mathcal{F}_{\mathbb{H}}(\mathcal{S}(\mathbb{H}^d))$ for λ going to 0 (that is in the neighborhood of the set $\widehat{\mathbb{H}}_0^d$). This is Lemma 2.3 that points out an extra and fundamental relationship between *positive* and *negative* λ 's.

A great deal of our program will be achieved by describing the action of the weight function M^2 and of the differentiation operator ∂_{λ} on \mathcal{W} . This is the goal of the next paragraph.

3.1. Some properties for Wigner transform of Hermite functions. The following lemma describes the action of the weight function M^2 on \mathcal{W} .

Lemma 3.1. *For all \widehat{w} in $\widetilde{\mathbb{H}}^d$ and Y in $T^*\mathbb{R}^d$, we have*

$$|Y|^2 \mathcal{W}(\widehat{w}, Y) = -\widehat{\Delta} \mathcal{W}(\cdot, Y)(\widehat{w})$$

where Operator $\widehat{\Delta}$ has been defined in (2.3).

Proof. From the definition of \mathcal{W} and integrations by parts, we get

$$\begin{aligned} |Y|^2 \mathcal{W}(\widehat{w}, Y) &= \int_{\mathbb{R}^d} \left(|y|^2 - \frac{1}{4\lambda^2} \Delta_z \right) (e^{2i\lambda\langle \eta, z \rangle}) H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) dz \\ &= \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} |\lambda|^{\frac{d}{2}} \mathcal{I}(\widehat{w}, y, z) dz \\ \text{with } \mathcal{I}(\widehat{w}, y, z) &\stackrel{\text{def}}{=} \left(|y|^2 - \frac{1}{4\lambda^2} \Delta_z \right) (H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z))). \end{aligned}$$

From Leibniz formula, the chain rule and the following identity:

$$4|y|^2 = |y+z|^2 + |y-z|^2 + 2(y+z) \cdot (y-z),$$

we get

$$\begin{aligned} \mathcal{I}(\widehat{w}, y, z) &= -\frac{1}{4\lambda^2} ((\Delta_z - \lambda^2|y+z|^2) H_n(|\lambda|^{\frac{1}{2}}(y+z))) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \\ &\quad - \frac{1}{4\lambda^2} ((\Delta_z - \lambda^2|y-z|^2) H_m(|\lambda|^{\frac{1}{2}}(-y+z))) H_n(|\lambda|^{\frac{1}{2}}(y+z)) \\ &\quad - \frac{1}{2|\lambda|} \sum_{j=1}^d (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) (\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \\ &\quad - \frac{1}{2} (z+y) \cdot (z-y) H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)). \end{aligned}$$

Using (1.12), we end up with

$$\begin{aligned} \mathcal{I}(\widehat{w}, y, z) &= \frac{1}{2|\lambda|} (|n+m|+d) H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \\ &\quad - \frac{1}{2|\lambda|} \sum_{j=1}^d \left\{ (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) (\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \right. \\ &\quad \left. + (M_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) (M_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \right\}. \end{aligned}$$

Then, taking advantage of (A.4), we get Identity (2.3). \square

The purpose of the following lemma is to investigate the action of ∂_λ on \mathcal{W} .

Lemma 3.2. *We have, for all \widehat{w} in $\widetilde{\mathbb{H}}^d$, the following formula:*

$$\begin{aligned} (3.1) \quad \partial_\lambda \mathcal{W}(\widehat{w}, Y) &= -\frac{d}{2\lambda} \mathcal{W}(\widehat{w}, Y) \\ &\quad + \frac{1}{2\lambda} \sum_{j=1}^d \left\{ \sqrt{(n_j+1)(m_j+1)} \mathcal{W}(\widehat{w}_j^+, Y) - \sqrt{n_j m_j} \mathcal{W}(\widehat{w}_j^-, Y) \right\}. \end{aligned}$$

Proof. Let us write that

$$\begin{aligned}\partial_\lambda \mathcal{W}(\widehat{w}, Y) &= \int_{\mathbb{R}^d} \frac{d}{d\lambda} \left(|\lambda|^{\frac{d}{2}} e^{2i\lambda\langle \eta, z \rangle} H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \right) dz \\ &= \frac{d}{2\lambda} \mathcal{W}(\widehat{w}, Y) + \mathcal{W}_1(\widehat{w}, Y) + \mathcal{W}_2(\widehat{w}, Y) \quad \text{with} \\ \mathcal{W}_1(\widehat{w}, Y) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d} 2i\langle \eta, z \rangle e^{2i\lambda\langle \eta, z \rangle} |\lambda|^{\frac{d}{2}} H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) dz \quad \text{and} \\ \mathcal{W}_2(\widehat{w}, Y) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} |\lambda|^{\frac{d}{2}} \frac{d}{d\lambda} \left(H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \right) dz.\end{aligned}$$

As we have

$$2i\langle \eta, z \rangle e^{2i\lambda\langle \eta, z \rangle} = \frac{1}{\lambda} \sum_{j=1}^d z_j \partial_{z_j} e^{2i\lambda\langle \eta, z \rangle},$$

an integration by parts gives

$$(3.2) \quad \begin{aligned}\mathcal{W}_1(\widehat{w}, Y) &= -\frac{d}{\lambda} \mathcal{W}(\widehat{w}, Y) \\ &\quad - \frac{1}{\lambda} \sum_{j=1}^d \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} |\lambda|^{\frac{d}{2}} z_j \partial_{z_j} \left(H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \right) dz.\end{aligned}$$

Now let us compute

$$\mathcal{J}(\widehat{w}, y, z) \stackrel{\text{def}}{=} \left(\frac{d}{d\lambda} - \frac{1}{\lambda} \sum_{j=1}^d z_j \partial_{z_j} \right) \left(H_n(|\lambda|^{\frac{1}{2}}(y+z)) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \right).$$

From the chain rule we get

$$\begin{aligned}\mathcal{J}(\widehat{w}, y, z) &= \frac{|\lambda|^{\frac{1}{2}}}{2\lambda} \sum_{j=1}^d \left\{ (y_j + z_j) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) \right. \\ &\quad + (-y_j + z_j) H_n(|\lambda|^{\frac{1}{2}}(y+z)) (\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \\ &\quad - 2z_j H_m(|\lambda|^{\frac{1}{2}}(-y+z)) (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) \\ &\quad \left. - 2z_j H_n(|\lambda|^{\frac{1}{2}}(y+z)) (\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \right\}.\end{aligned}$$

This gives

$$\begin{aligned}\mathcal{J}(\widehat{w}, y, z) &= -\frac{1}{2\lambda} \sum_{j=1}^d \left\{ (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) |\lambda|^{\frac{1}{2}}(-y_j + z_j) H_m(|\lambda|^{\frac{1}{2}}(-y+z)) \right. \\ &\quad \left. + |\lambda|^{\frac{1}{2}}(y_j + z_j) H_n(|\lambda|^{\frac{1}{2}}(y+z)) (\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \right\}\end{aligned}$$

which writes

$$\begin{aligned}\mathcal{J}(\widehat{w}, y, z) &= -\frac{1}{2\lambda} \sum_{j=1}^d \left\{ (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) (M_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \right. \\ &\quad \left. + (M_j H_n)(|\lambda|^{\frac{1}{2}}(y+z)) (\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z)) \right\}.\end{aligned}$$

Using Relations (A.4) completes the proof of the Lemma. \square

3.2. Decay provides regularity. Granted with Lemmas 3.1 and 3.2, it is now easy to establish Lemma 2.2. Indeed, according to (1.13), we have

$$(\mathcal{F}_{\mathbb{H}} M^2 f)(\hat{w}) = \int_{\mathbb{H}^d} e^{-is\lambda} f(Y, s) |Y|^2 \overline{\mathcal{W}}(\hat{w}, Y) dY ds.$$

Therefore, Lemma 3.1 implies that

$$\begin{aligned} (\mathcal{F}_{\mathbb{H}} M^2 f)(\hat{w}) &= \frac{1}{2|\lambda|} (|n+m|+d) \int_{\mathbb{H}^d} f(Y, s) e^{-is\lambda} \overline{\mathcal{W}}(\hat{w}, Y) dY ds \\ &\quad - \frac{1}{2|\lambda|} \sum_{j=1}^d \left\{ \sqrt{(n_j+1)(m_j+1)} \int_{\mathbb{H}^d} f(Y, s) e^{-is\lambda} \overline{\mathcal{W}}(\hat{w}_j^+, Y) dY ds \right. \\ &\quad \left. + \sqrt{n_j m_j} \int_{\mathbb{H}^d} f(Y, s) e^{-is\lambda} \overline{\mathcal{W}}(\hat{w}_j^-, Y) dY ds \right\}. \end{aligned}$$

By the definition of the Fourier transform and of $\widehat{\Delta}$, this gives $\mathcal{F}_{\mathbb{H}} M^2 f = -\widehat{\Delta} \mathcal{F}_{\mathbb{H}} f$.

To establish (2.4), we start from (1.13) and get

$$\begin{aligned} \mathcal{F}_{\mathbb{H}}(M_0 f)(\hat{w}) &= \int_{\mathbb{H}^d} \frac{d}{d\lambda} (e^{-is\lambda}) f(Y, s) \overline{\mathcal{W}}(\hat{w}, Y) dY ds \\ &= \frac{d}{d\lambda} (\mathcal{F}_{\mathbb{H}} f)(\hat{w}) - \int_{\mathbb{H}^d} e^{-is\lambda} f(Y, s) \frac{d}{d\lambda} (\overline{\mathcal{W}}(\hat{w}, Y)) dY ds. \end{aligned}$$

Rewriting the last term according to Formula (3.1), we discover that

$$\begin{aligned} (\mathcal{F}_{\mathbb{H}} M_0 f)(\hat{w}) &= \frac{d}{d\lambda} (\mathcal{F}_{\mathbb{H}} f)(\hat{w}) + \frac{d}{2\lambda} \int_{\mathbb{H}^d} f(Y, s) e^{-is\lambda} \overline{\mathcal{W}}(\hat{w}, Y) dY ds \\ &\quad - \frac{1}{2\lambda} \sum_{j=1}^d \left\{ \sqrt{(n_j+1)(m_j+1)} \int_{\mathbb{H}^d} f(Y, s) e^{-is\lambda} \overline{\mathcal{W}}(\hat{w}_j^+, Y) dY ds \right. \\ &\quad \left. - \sqrt{n_j m_j} \int_{\mathbb{H}^d} f(Y, s) e^{-is\lambda} \overline{\mathcal{W}}(\hat{w}_j^-, Y) dY ds \right\}. \end{aligned}$$

By the definition of the Fourier transform, this concludes the proof of Lemma 2.2. \square

On the one hand, Lemmas 2.1 and 2.2 guarantee that decay in the physical space provides regularity in the Fourier space, and that regularity gives decay. On the other hand, the relations we established so far do not give much insight on the behavior of the Fourier transform near $\widehat{\mathbb{H}}_0^d$ even though we know from Theorem 2.2 that in the case of an integrable function, it has to be uniformly continuous *up to* $\lambda = 0$. Getting more information on the behavior of the Fourier transform of functions in $\mathcal{S}(\mathbb{H}^d)$ in a neighborhood of $\widehat{\mathbb{H}}_0^d$ is what we want to do now with the proof of Lemma 2.3.

Proof of Lemma 2.3. Fix some function f in $\mathcal{S}(\mathbb{H}^d)$, and observe that

$$\partial_s \mathcal{P} f(Y, s) = \frac{1}{2} (f(Y, s) - f(Y, -s)) \quad \text{with } \mathcal{P} \text{ defined in (2.6).}$$

Taking the Fourier transform with respect to the variable s gives

$$(3.3) \quad i\lambda \mathcal{F}_s(\mathcal{P} f)(Y, \lambda) = \frac{1}{2} (\mathcal{F}_s f(Y, \lambda) - \mathcal{F}_s f(Y, -\lambda)).$$

Let us consider a function χ in $\mathcal{D}(\mathbb{R})$ with value 1 near 0 and let us write

$$i\mathcal{F}_s(\mathcal{P}f)(Y, \lambda) = \frac{1 - \chi(\lambda)}{2\lambda} (\mathcal{F}_s f(Y, \lambda) - \mathcal{F}_s f(Y, -\lambda)) + \chi(\lambda) \int_0^1 (\partial_\lambda \mathcal{F}_s f)(Y, -\lambda + 2t\lambda) dt.$$

It is obvious that the two terms in the right-hand side belong to $\mathcal{S}(\mathbb{R}^{2d+1})$. Thus the operator

$$\phi \longmapsto \frac{\phi(Y, \lambda) - \phi(Y, -\lambda)}{2\lambda}$$

maps continuously $\mathcal{S}(\mathbb{R}^{2d+1})$ to $\mathcal{S}(\mathbb{R}^{2d+1})$. Hence \mathcal{P} maps continuously $\mathcal{S}(\mathbb{H}^d)$ to $\mathcal{S}(\mathbb{H}^d)$.

Note that in the case of a function g in $\mathcal{S}(\mathbb{H}^d)$, Formula (1.13) may be alternately written:

$$(3.4) \quad \mathcal{F}_{\mathbb{H}}g(\hat{w}) = \int_{T^*\mathbb{R}^d} \mathcal{F}_s g(Y, \lambda) \overline{\mathcal{W}}(\hat{w}, Y) dY \quad \text{for all } \hat{w} = (n, m, \lambda) \text{ in } \widetilde{\mathbb{H}}^d.$$

Relations (2.8) and (3.3) guarantee that

$$\begin{aligned} 2i\lambda \mathcal{F}_{\mathbb{H}}(\mathcal{P}f)(\hat{w}) &= \int_{T^*\mathbb{R}^d} 2i\lambda \mathcal{F}_s(\mathcal{P}f)(Y, \lambda) \overline{\mathcal{W}}(\hat{w}, Y) dY \\ &= \int_{T^*\mathbb{R}^d} (\mathcal{F}_s f(Y, \lambda) - \mathcal{F}_s f(Y, -\lambda)) \overline{\mathcal{W}}(\hat{w}, Y) dY \\ &= \int_{T^*\mathbb{R}^d} \mathcal{F}_s f(Y, \lambda) \overline{\mathcal{W}}(\hat{w}, Y) dY \\ &\quad - (-1)^{|n+m|} \int_{T^*\mathbb{R}^d} \mathcal{F}_s f(Y, -\lambda) \overline{\mathcal{W}}(m, n, -\lambda, Y) dY \\ &= \mathcal{F}_{\mathbb{H}}f(n, m, \lambda) - (-1)^{|n+m|} \mathcal{F}_{\mathbb{H}}f(m, n, -\lambda), \end{aligned}$$

which completes the proof of Lemma 2.3. \square

3.3. Proof of the inversion theorem in the Schwartz space. The aim of this section is to prove Theorem 2.1. To this end, let us first note that from Inequality (2.1) and Lemmas 2.2 and 2.3, we gather that $\mathcal{F}_{\mathbb{H}}$ maps $\mathcal{S}(\mathbb{H}^d)$ to $\mathcal{S}(\widehat{\mathbb{H}}^d)$. In addition, (2.9) guarantees that all elements of $\mathcal{S}(\widehat{\mathbb{H}}^d)$ are in $L^1(\widehat{\mathbb{H}}^d) \cap L^2(\widehat{\mathbb{H}}^d)$.

Hence Theorem 1.1 ensures that $\mathcal{F}_{\mathbb{H}} : \mathcal{S}(\mathbb{H}^d) \rightarrow \mathcal{S}(\widehat{\mathbb{H}}^d)$ is one-to-one, and that the inverse map has to be the functional $\widetilde{\mathcal{F}}_{\mathbb{H}}$ defined in (2.10). Therefore, there only remains to prove that $\widetilde{\mathcal{F}}_{\mathbb{H}}$ maps $\mathcal{S}(\widehat{\mathbb{H}}^d)$ to $\mathcal{S}(\mathbb{H}^d)$. To this end, it is convenient to introduce the following semi-norms:

$$(3.5) \quad \|f\|_{K, \mathcal{S}(\mathbb{H}^d)} \stackrel{\text{def}}{=} \sqrt{\|f\|_{L^2(\mathbb{H}^d)}^2 + \|M_{\mathbb{H}}^K f\|_{L^2(\mathbb{H}^d)}^2 + \|\Delta_{\mathbb{H}}^K f\|_{L^2(\mathbb{H}^d)}^2} \quad \text{with } M_{\mathbb{H}} \stackrel{\text{def}}{=} M^2 + M_0,$$

which are equivalent to the classical ones defined in Lemma 2.1 (see Prop. A.1).

Let us compute $M^2 \widetilde{\mathcal{F}}_{\mathbb{H}} \theta(Y, s)$. According to Lemma 3.1, we have for all $\hat{w} = (n, m, \lambda)$ in $\widetilde{\mathbb{H}}^d$,

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N}^{2d}} \theta(\hat{w}) |Y|^2 \mathcal{W}(\hat{w}, Y) &= \frac{1}{2|\lambda|} \sum_{(n,m) \in \mathbb{N}^{2d}} \left((|n+m| + d) \mathcal{W}(\hat{w}, Y) \theta(\hat{w}) \right. \\ &\quad \left. - \sum_{j=1}^d \sqrt{n_j m_j} \theta(\hat{w}) \mathcal{W}(\hat{w}_j^-, Y) - \sum_{j=1}^d \sqrt{(n_j + 1)(m_j + 1)} \theta(\hat{w}) \mathcal{W}(\hat{w}_j^+, Y) \right). \end{aligned}$$

Changing variable $(\tilde{n}, \tilde{m}) = (n + \delta_j, m + \delta_j)$ and $(\tilde{n}, \tilde{m}) = (n - \delta_j, m - \delta_j)$, respectively, gives

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N}^{2d}} \theta(\hat{w}) |Y|^2 \mathcal{W}(\hat{w}, Y) &= \frac{1}{2|\lambda|} \sum_{(n,m) \in \mathbb{N}^{2d}} \left((|n+m|+d)\theta(\hat{w})\mathcal{W}(\hat{w}, Y) \right. \\ &\quad \left. - \sum_{j=1}^d \left(\sqrt{(n_j+1)(m_j+1)}\theta(\hat{w}_j^+) + \sqrt{n_j m_j}\theta(\hat{w}_j^-) \right) \right) \mathcal{W}(\hat{w}, Y) \\ &= - \sum_{(n,m) \in \mathbb{N}^{2d}} \hat{\Delta}\theta(\hat{w})\mathcal{W}(\hat{w}, Y) \end{aligned}$$

where $\hat{\Delta}$ is the operator introduced in (2.3).

Multiplying by $2^{d-1}\pi^{-d-1}e^{is\lambda}$, integrating with respect to λ and remembering (2.10), we end up with

$$(3.6) \quad (M^2 \tilde{\mathcal{F}}_{\mathbb{H}}\theta)(Y, s) = -\tilde{\mathcal{F}}_{\mathbb{H}}(\hat{\Delta}\theta)(Y, s).$$

Understanding how M_0 acts on $\tilde{\mathcal{F}}_{\mathbb{H}}(\mathcal{S}(\hat{\mathbb{H}}^d))$ is more delicate. It requires our using the continuity property of Definition 2.2. Now, if θ is in $\mathcal{S}(\hat{\mathbb{H}}^d)$ then it is integrable. As obviously $|\mathcal{W}| \leq 1$, one may thus write for all $w = (Y, s)$ in \mathbb{H}^d , denoting $\mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \setminus [-\varepsilon, \varepsilon]$,

$$\begin{aligned} (M_0 \tilde{\mathcal{F}}_{\mathbb{H}}\theta)(w) &= \frac{2^{d-1}}{\pi^{d+1}} \lim_{\varepsilon \rightarrow 0} \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon(n, m, w) \quad \text{with} \\ \Psi_\varepsilon(n, m, w) &\stackrel{\text{def}}{=} - \int_{\mathbb{R}_\varepsilon} \left(\frac{d}{d\lambda} e^{is\lambda} \right) \theta(n, m, \lambda) \mathcal{W}(n, m, \lambda, Y) |\lambda|^d d\lambda. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} \Psi_\varepsilon(n, m, w) &= \Psi_\varepsilon^{(1)}(n, m, w) - \Psi_\varepsilon^{(2)}(n, m, w) \quad \text{with} \\ \Psi_\varepsilon^{(1)}(n, m, w) &\stackrel{\text{def}}{=} \int_{\mathbb{R}_\varepsilon} e^{is\lambda} \frac{d}{d\lambda} (\mathcal{W}(n, m, \lambda, Y) \theta(n, m, \lambda) |\lambda|^d) d\lambda \quad \text{and} \\ \Psi_\varepsilon^{(2)}(n, m, w) &\stackrel{\text{def}}{=} \varepsilon^d (e^{is\varepsilon} \mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) - e^{-is\varepsilon} \mathcal{W}(n, m, -\varepsilon, Y) \theta(n, m, -\varepsilon)). \end{aligned}$$

Let us compute

$$(3.7) \quad \Theta(\hat{w}, Y) \stackrel{\text{def}}{=} \frac{d}{d\lambda} (\mathcal{W}(n, m, \lambda, Y) \theta(n, m, \lambda) |\lambda|^d).$$

Leibniz formula gives

$$\Theta(\hat{w}, Y) = \partial_\lambda \mathcal{W}(\hat{w}, Y) \theta(\hat{w}) |\lambda|^d + \mathcal{W}(\hat{w}, Y) \frac{d}{d\lambda} (|\lambda|^d \theta(\hat{w})).$$

Hence, remembering Identity (3.1), we discover that

$$\begin{aligned} \Theta(\hat{w}, Y) &= \frac{d\theta}{d\lambda}(\hat{w}) \mathcal{W}(\hat{w}, Y) |\lambda|^d + \frac{d}{2\lambda} \theta(\hat{w}) \mathcal{W}(\hat{w}, Y) |\lambda|^d \\ &\quad - \frac{|\lambda|^d}{2\lambda} \sum_{j=1}^d \theta(\hat{w}) \left(\sqrt{n_j m_j} \mathcal{W}(\hat{w}_j^-, \lambda, Y) - \sqrt{(n_j+1)(m_j+1)} \mathcal{W}(\hat{w}_j^+, Y) \right). \end{aligned}$$

From the changes of variable $(n', m') = (n - \delta_j, m - \delta_j)$ and $(\tilde{n}, \tilde{m}) = (n + \delta_j, m + \delta_j)$, we infer that

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N}^{2d}} \theta(\hat{w}) \left(\sqrt{n_j m_j} \mathcal{W}(\hat{w}_j^-, Y) - \sqrt{(n_j+1)(m_j+1)} \mathcal{W}(\hat{w}_j^+, Y) \right) \\ = - \sum_{(n,m) \in \mathbb{N}^{2d}} \mathcal{W}(\hat{w}, Y) \left(\sqrt{n_j m_j} \theta(\hat{w}_j^-) - \sqrt{(n_j+1)(m_j+1)} \theta(\hat{w}_j^+) \right). \end{aligned}$$

Therefore, using the operator $\widehat{\mathcal{D}}_\lambda$ introduced in Lemma 2.2, we get

$$(3.8) \quad \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon^{(1)}(n, m, w) = \sum_{(n,m) \in \mathbb{N}^{2d}} \int_{\mathbb{R}^\varepsilon} e^{is\lambda} (\widehat{\mathcal{D}}_\lambda \theta)(n, m, \lambda) \mathcal{W}(n, m, \lambda, Y) |\lambda|^d d\lambda.$$

Now let us study the term $\Psi_\varepsilon^{(2)}(n, m, w)$. We have

$$\begin{aligned} e^{is\varepsilon} \mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) - e^{-is\varepsilon} \mathcal{W}(n, m, -\varepsilon, Y) \theta(n, m, -\varepsilon) \\ = (e^{is\varepsilon} - e^{-is\varepsilon}) \mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) \\ + e^{-is\varepsilon} (\mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) - \mathcal{W}(n, m, -\varepsilon, Y) \theta(n, m, -\varepsilon)). \end{aligned}$$

Hence, thanks to (2.8)

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon^{(2)}(n, m, w) = 2i\varepsilon^d \sin(s\varepsilon) \sum_{(n,m) \in \mathbb{N}^{2d}} \mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) \\ + \varepsilon^d e^{-is\varepsilon} \left(\sum_{(n,m) \in \mathbb{N}^{2d}} \mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) - \sum_{(n,m) \in \mathbb{N}^{2d}} (-1)^{|n+m|} \mathcal{W}(m, n, \varepsilon, Y) \theta(n, m, -\varepsilon) \right). \end{aligned}$$

Swapping indices n and m in the last sum gives

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon^{(2)}(n, m, w) = 2i\varepsilon^d \sin(s\varepsilon) \sum_{(n,m) \in \mathbb{N}^{2d}} \mathcal{W}(n, m, \varepsilon, Y) \theta(n, m, \varepsilon) \\ + \varepsilon^{d+1} e^{-is\varepsilon} \sum_{(n,m) \in \mathbb{N}^{2d}} \mathcal{W}(n, m, \varepsilon, Y) (\widehat{\Sigma}_0 \theta)(n, m, \varepsilon). \end{aligned}$$

Remembering that $|\mathcal{W}| \leq 1$, we thus get

$$(3.9) \quad \left| \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon^{(2)}(n, m, w) \right| \leq \varepsilon^{d+1} \left(2|s| \sum_{(n,m) \in \mathbb{N}^{2d}} |\theta(n, m, \varepsilon)| + \sum_{(n,m) \in \mathbb{N}^{2d}} |(\widehat{\Sigma}_0 \theta)(n, m, \varepsilon)| \right).$$

Now, let us use the fact that we have

$$\sum_{(n,m) \in \mathbb{N}^{2d}} |\theta(n, m, \varepsilon)| \leq \|\theta\|_{2d+2, 0, \mathcal{S}(\mathbb{H}^d)} \sum_{(n,m) \in \mathbb{N}^{2d}} (1 + \varepsilon(|n+m| + d) + |n-m|)^{-2d-2}.$$

We observe that

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N}^{2d}} (1 + \varepsilon(|n+m| + d) + |n-m|)^{-2d-2} &\leq \sum_{\ell \in \mathbb{N}^d} (1 + \varepsilon(|\ell| + d))^{-d-1} \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-d-1} \\ &\leq C\varepsilon^{-d}. \end{aligned}$$

Hence the first term of the right-hand side of (3.9) tends to 0 when ε goes to 0.

Employing the same argument with $\widehat{\Sigma}_0\theta$ guarantees that the last term of (3.9) tends to 0 when ε goes to 0. Therefore, we do have

$$\lim_{\varepsilon \rightarrow 0} \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon^{(2)}(n, m, w) = 0.$$

Using that $\widehat{\mathcal{D}}_\lambda\theta$ belongs to $\mathcal{S}(\widehat{\mathbb{H}}^d)$ and is thus integrable, we deduce from (3.8) that

$$\lim_{\varepsilon \rightarrow 0} \sum_{(n,m) \in \mathbb{N}^{2d}} \Psi_\varepsilon^{(1)}(n, m, w) = \int_{\widehat{\mathbb{H}}^d} e^{is\lambda}(\widehat{\mathcal{D}}_\lambda\theta)(\widehat{w})\mathcal{W}(\widehat{w}, Y)d\widehat{w}.$$

Thus this gives

$$(3.10) \quad M_0\widetilde{\mathcal{F}}_{\mathbb{H}}\theta = \widetilde{\mathcal{F}}_{\mathbb{H}}\widehat{\mathcal{D}}_\lambda\theta.$$

Together with (3.6), this implies that

$$(3.11) \quad M_{\mathbb{H}}\widetilde{\mathcal{F}}_{\mathbb{H}}\theta = \widetilde{\mathcal{F}}_{\mathbb{H}}((-\widehat{\Delta} + \widehat{\mathcal{D}}_\lambda)(\theta)) \quad \text{with} \quad M_{\mathbb{H}} \stackrel{\text{def}}{=} M^2 + M_0.$$

Hence we can conclude that for any integer K , there exist an integer N_K and a constant C_K so that

$$(3.12) \quad \|M_{\mathbb{H}}^K\widetilde{\mathcal{F}}_{\mathbb{H}}\theta\|_{L^2(\mathbb{H}^d)} \leq C_K\|\theta\|_{N_K, N_K, \mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

Finally, to study the action of the Laplacian on $\widetilde{\mathcal{F}}_{\mathbb{H}}(\mathcal{S}(\widehat{\mathbb{H}}^d))$, we write that by definition of \mathcal{X}_j and of \mathcal{W} , we have

$$\begin{aligned} \mathcal{X}_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) &= \int_{\mathbb{R}^d} \mathcal{X}_j(e^{is\lambda+2i\lambda\langle\eta, z\rangle}H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z))dz \\ &= \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta, z\rangle}(2i\lambda\eta_j + \partial_{y_j})(H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z))dz. \end{aligned}$$

As $2i\lambda\eta_j e^{2i\lambda\langle\eta, z\rangle} = \partial_{z_j}(e^{2i\lambda\langle\eta, z\rangle})$, integrating by parts yields

$$(3.13) \quad \mathcal{X}_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) = \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta, z\rangle}(\partial_{y_j} - \partial_{z_j})(H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z))dz.$$

The action of Ξ_j is simply described by

$$\begin{aligned} \Xi_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) &= \int_{\mathbb{R}^d} \Xi_j(e^{is\lambda+2i\lambda\langle\eta, z\rangle})H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)dz \\ &= \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta, z\rangle}2i\lambda(z_j - y_j)H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)dz. \end{aligned}$$

Together with (3.13) and the definition of $\Delta_{\mathbb{H}}$ in (1.7), this gives

$$\begin{aligned} \Delta_{\mathbb{H}}(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) &= 4 \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta, z\rangle}H_{n,\lambda}(y+z)(\Delta_{osc}^\lambda H_{m,\lambda})(-y+z)dz \\ &= -4|\lambda|(2|m| + d)e^{is\lambda}\mathcal{W}(\widehat{w}, Y). \end{aligned}$$

This implies that for all integer K , we have

$$(-\Delta_{\mathbb{H}})^K(\widetilde{\mathcal{F}}_{\mathbb{H}}\theta) = \widetilde{\mathcal{F}}_{\mathbb{H}}\widehat{M}^K\theta \quad \text{with} \quad \widehat{M}\theta(n, m, \lambda) \stackrel{\text{def}}{=} 4|\lambda|(2|m| + d)\theta(n, m, \lambda),$$

whence there exist an integer N_k and a constant C_K so that

$$(3.14) \quad \|(-\Delta_{\mathbb{H}})^K(\widetilde{\mathcal{F}}_{\mathbb{H}}\theta)\|_{L^2(\mathbb{H}^d)} \leq C_k\|\theta\|_{N_K, N_K, \mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

Putting (3.12) and (3.14) together and remembering the definition of the semi-norms on $\mathcal{S}(\mathbb{H}^d)$ given in (3.5), we conclude that for all integer K , there exist an integer N_K and a constant C_K so that

$$\|\tilde{\mathcal{F}}_{\mathbb{H}}\theta\|_{K,\mathcal{S}(\mathbb{H}^d)} \leq C_K \|\theta\|_{N_K, N_K, \mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

This completes the proof of Theorem 2.1.

4. EXAMPLES OF FUNCTIONS IN THE RANGE OF THE SCHWARTZ CLASS

The purpose of this section is to prove Theorem 2.3. Let us recall the notation

$$\Theta_f(\widehat{w}) \stackrel{\text{def}}{=} f(|\lambda|R(n, m), m - n, \lambda) \quad \text{with} \quad R(n, m) \stackrel{\text{def}}{=} (n_j + m_j + 1)_{1 \leq j \leq d}.$$

For any function f in \mathcal{S}_d^+ which is either supported in $[0, \infty[^d \times \{0\} \times \mathbb{R}$ or in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ for some positive real number r_0 and satisfies (2.14), the fact that $\|\Theta_f\|_{N,0,\mathcal{S}(\widehat{\mathbb{H}}^d)}$ is finite for all integer N is obvious. We next have to study the action of $\widehat{\Delta}$ and $\widehat{\mathcal{D}}_\lambda$ on Θ_f . To this end, we shall establish a Taylor type expansion of $\widehat{\Delta}\Theta_f$ and $\widehat{\mathcal{D}}_\lambda\Theta_f$ near $\lambda = 0$. To explain what kind of convergence we are looking for, we need the following definition.

Definition 4.1. *Let M be an integer. We say that two continuous functions θ and θ' on $\widehat{\mathbb{H}}^d$ are M -equivalent (denoted by $\theta \stackrel{M}{\equiv} \theta'$) if for all positive integer N , a constant $C_{N,M}$ exists such that*

$$\forall \widehat{w} \in \widehat{\mathbb{H}}^d, |\theta(\widehat{w}) - \theta'(\widehat{w})| \leq C_{N,M} |\lambda|^M (1 + |\lambda|(|n + m| + d) + |m - n|)^{-N}.$$

Let us first observe that, if $M \geq 1$ then

$$(4.1) \quad \theta \stackrel{M}{\equiv} 0 \implies \|\theta\|_{N,0,\mathcal{S}(\widehat{\mathbb{H}}^d)} < \infty \quad \text{for all integer } N.$$

Furthermore, whenever $0 \leq M_0 \leq M$, we have

$$(4.2) \quad \theta \stackrel{M}{\equiv} \theta' \implies |\lambda|^{-M_0} \theta \stackrel{M-M_0}{\equiv} |\lambda|^{-M_0} \theta',$$

and it is obvious that if P is a function bounded by a polynomial in (n, m) with total degree M_0 , then

$$(4.3) \quad \theta \stackrel{M}{\equiv} \theta' \implies P(n, m) \theta \stackrel{M-M_0}{\equiv} P(n, m) \theta'.$$

Finally, note that the definition of $\widehat{\Delta}$ in (2.3) implies that

$$(4.4) \quad \theta \stackrel{M}{\equiv} \theta' \implies \widehat{\Delta}\theta \stackrel{M-2}{\equiv} \widehat{\Delta}\theta'.$$

We have the following lemma.

Lemma 4.1. *For any positive integer M , we have*

$$\forall \widehat{w} \in \widehat{\mathbb{H}}^d, \Theta_f(\widehat{w}_j^\pm) \stackrel{M+1}{\equiv} \sum_{\ell=0}^M \frac{(\pm 2|\lambda|)^\ell}{\ell!} \Theta_{\partial_{x_j}^\ell f}(\widehat{w}).$$

Proof. Performing a Taylor expansion at order $M + 1$, we get

$$\begin{aligned} f(|\lambda|R(n \pm \delta_j, m \pm \delta_j), m - n, \lambda) &= \sum_{\ell=0}^M \frac{(\pm 2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n, m), m - n, \lambda) \\ &+ (\pm 2|\lambda|)^{M+1} \int_0^1 \frac{(1-t)^M}{M!} (\partial_{x_j}^{M+1} f)(|\lambda|R_j^\pm(n, m, t), m - n, \lambda) dt \end{aligned}$$

with $R_j^\pm(n, m, t) \stackrel{\text{def}}{=} (n_1 + m_1 + 1, \dots, n_j + m_j + 1 \pm 2t, \dots, n_d + m_d + 1)$. The fact that f belongs to \mathcal{S}_d^+ implies that for any positive integer N , we have

$$\left| \int_0^1 \frac{(1-t)^M}{M!} (\partial_{x_j}^{M+1} f)(|\lambda|R_j^+(n, m, t), m-n, \lambda) dt \right| \leq C_N (1 + |\lambda|(|n+m|+d) + |m-n|)^{-N}.$$

This gives the lemma. \square

One can now tackle the proof of Theorem 2.3. Let us first investigate the (easier) case when the support of f is included in $[0, \infty[^d \times \{0\} \times \mathbb{R}$. The first step consists in computing an equivalent (in the sense of Definition 4.1) of $\widehat{\Delta}\Theta_f$ at an order which will be chosen later on. For notational simplicity, we here set $R(n) \stackrel{\text{def}}{=} R(n, n)$ and omit the second variable of f . Now, by definition of the operator $\widehat{\Delta}$, we have

$$(4.5) \quad (-\widehat{\Delta}\Theta_f)(n, \lambda) = \frac{1}{2|\lambda|} \left((|2n|+d)f(|\lambda|R(n), \lambda) - \sum_{j=1}^d \widetilde{\Delta}_j(n, \lambda) \right) \quad \text{with}$$

$$\widetilde{\Delta}_j(n, \lambda) \stackrel{\text{def}}{=} (n_j+1)f(|\lambda|R(n+2\delta_j), \lambda) + n_j f(|\lambda|R(n-2\delta_j), \lambda).$$

Lemma 4.1, and Assertions (4.2) and (4.3) imply that

$$\begin{aligned} \frac{1}{2|\lambda|} \widetilde{\Delta}_j(n, \lambda) &\stackrel{2M-1}{\equiv} \frac{n_j+1}{2|\lambda|} \sum_{\ell=0}^{2M} \frac{(2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n), \lambda) \\ &\quad + \frac{n_j}{2|\lambda|} \sum_{\ell=0}^{2M} \frac{(-2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n), \lambda) \\ &\stackrel{2M-1}{\equiv} \frac{2n_j+1}{2|\lambda|} \sum_{\ell=0}^M \frac{(2\lambda)^{2\ell}}{(2\ell)!} (\partial_{x_j}^{2\ell} f)(|\lambda|R(n), \lambda) \\ &\quad + \frac{1}{2|\lambda|} \sum_{\ell=0}^{M-1} \frac{(2|\lambda|)^{2\ell+1}}{(2\ell+1)!} (\partial_{x_j}^{2\ell+1} f)(|\lambda|R(n), \lambda). \end{aligned}$$

Let us define

$$(4.6) \quad f_{2\ell}(x, \lambda) \stackrel{\text{def}}{=} \sum_{j=1}^d \frac{2^{2\ell-1}}{(2\ell)!} x_j \lambda^{2\ell-2} \partial_{x_j}^{2\ell} f(x, \lambda) \quad \text{and}$$

$$f_{2\ell+1}(x, \lambda) \stackrel{\text{def}}{=} \sum_{j=1}^d \frac{2^{2\ell}}{(2\ell+1)!} \lambda^{2\ell} \partial_{x_j}^{2\ell+1} f(x, \lambda).$$

Clearly, all functions f_ℓ are supported in $[0, \infty[^d \times \{0\} \times \mathbb{R}$ and belong to \mathcal{S}_d^+ , and the above equality rewrites

$$(4.7) \quad \widehat{\Delta}\Theta_f(n, \lambda) \stackrel{2M-1}{\equiv} - \sum_{\ell=1}^{2M} f_\ell(|\lambda|R(n), \lambda).$$

Arguing by induction, it is easy to establish that for any function f in \mathcal{S}_d^+ supported in $[0, \infty[\times \{0\} \times \mathbb{R}$ and any integers N and p , the quantity $\|\widehat{\Delta}^p \Theta_f\|_{N,0,\mathcal{S}(\widehat{\mathbb{H}}^d)}$ is finite. Indeed,

this is obvious for $p = 0$. Now, if the property holds true for some non negative integer p then, thanks to (4.7) and (4.4),

$$\widehat{\Delta}^{p+1}\Theta_f(n, \lambda) \stackrel{2M-1-2p}{\equiv} - \sum_{\ell=1}^{2M} \widehat{\Delta}^\ell \Theta_{f_\ell}(n, \lambda).$$

From (4.1), (4.7) and the induction hypothesis, it is clear that if we choose M greater than p then we get that $\|\widehat{\Delta}^p \Theta_f\|_{N,0,S(\widehat{\mathbb{H}}^d)}$ is finite for all integer N .

Let us next study the action of Operator $\widehat{\mathcal{D}}_\lambda$. From its definition in Lemma 2.2, we gather that

$$(\widehat{\mathcal{D}}_\lambda \Theta_f)(\widehat{w}) = \frac{d}{d\lambda}(f(|\lambda|R(n), \lambda)) + \frac{d}{2\lambda}f(|\lambda|R(n), \lambda) + \frac{1}{2\lambda} \sum_{j=1}^d \mathcal{D}_j(n, \lambda) \quad \text{with}$$

$$\mathcal{D}_j(n, \lambda) \stackrel{\text{def}}{=} n_j f(|\lambda|(R(n) - 2\delta_j), \lambda) - (n_j + 1)f(|\lambda|(R(n) + 2\delta_j), \lambda).$$

Lemma 4.1, and Assertions (4.2) and (4.3) imply that

$$(4.8) \quad \begin{aligned} \frac{1}{2\lambda} \mathcal{D}_j(n, \lambda) &\stackrel{2M-1}{\equiv} - \frac{n_j + 1}{2\lambda} \sum_{\ell=0}^{2M} \frac{(2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n), \lambda) \\ &\quad + \frac{n_j}{2\lambda} \sum_{\ell=0}^{2M} \frac{(-2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n), \lambda) \\ &\stackrel{2M-1}{\equiv} - \sum_{\ell=0}^M \frac{(2\lambda)^{2\ell-1}}{(2\ell)!} (\partial_{x_j}^{2\ell} f)(|\lambda|R(n), \lambda) \\ &\quad - \frac{2n_j + 1}{2\lambda} \sum_{\ell=0}^{M-1} \frac{(2|\lambda|)^{2\ell+1}}{(2\ell+1)!} (\partial_{x_j}^{2\ell+1} f)(|\lambda|R(n), \lambda). \end{aligned}$$

Applying the chain rule yields

$$(4.9) \quad \frac{d}{d\lambda}(f(|\lambda|R(n), \lambda)) = (\partial_\lambda f)(|\lambda|R(n), \lambda) + \text{sgn } \lambda \sum_{j=1}^d (2n_j + 1) (\partial_{x_j} f)(|\lambda|R(n), \lambda).$$

Defining for $\ell \geq 1$ the functions

$$\begin{aligned} \widetilde{f}_{2\ell}(x, \lambda) &\stackrel{\text{def}}{=} \sum_{j=1}^d \frac{2^{2\ell-1}}{(2\ell)!} \lambda^{2\ell-1} \partial_{x_j}^{2\ell} f(x, \lambda) \quad \text{and} \\ \widetilde{f}_{2\ell+1}(x, \lambda) &\stackrel{\text{def}}{=} \sum_{j=1}^d \frac{2^{2\ell}}{(2\ell+1)!} x_j \lambda^{2\ell-1} \partial_{x_j}^{2\ell+1} f(x, \lambda), \end{aligned}$$

we get, using (4.8) and (4.9),

$$(\widehat{\mathcal{D}}_\lambda \Theta_f)(\widehat{w}) \stackrel{2M-1}{\equiv} (\partial_\lambda f)(|\lambda|R(n), \lambda) - \sum_{\ell=2}^{2M} \widetilde{f}_\ell(|\lambda|R(n), \lambda).$$

From that relation, mimicking the induction proof for $\widehat{\Delta}$, we easily conclude that for any function f in \mathcal{S}_d^+ supported in $[0, \infty[^d \times \{0\} \times \mathbb{R}$, and any integer p , the quantity $\|\widehat{\mathcal{D}}_\lambda^p \Theta_f\|_{N,0,S(\widehat{\mathbb{H}}^d)}$ is finite for all integer N . This completes the proof Theorem 2.3 in that particular case.

Next, let us investigate the case when the function f of \mathcal{S}_d^+ is supported in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ for some positive r_0 and satisfies (2.14). Then, by definition of the operator $\widehat{\Delta}$, we have for all $\widehat{w} = (n, m, \lambda)$ in $\widetilde{\mathbb{H}}^d$, denoting $k \stackrel{\text{def}}{=} m - n$,

$$\begin{aligned} -\widehat{\Delta}\Theta_f(\widehat{w}) &\stackrel{\text{def}}{=} \frac{1}{2|\lambda|} \left((|n+m|+d)f(|\lambda|R(n, m), k, \lambda) - \sum_{j=1}^d \widetilde{\Delta}_j(\widehat{w}) \right) \quad \text{with} \\ \widetilde{\Delta}_j(\widehat{w}) &\stackrel{\text{def}}{=} \sqrt{(n_j+1)(m_j+1)} f(|\lambda|(R(n, m) + 2\delta_j), k, \lambda) \\ &\quad + \sqrt{n_j m_j} f(|\lambda|(R(n, m) - 2\delta_j), k, \lambda). \end{aligned}$$

Compared to (4.5), the computations get wilder, owing to the square roots in the above formula. Let M be an integer (to be suitably chosen later on). Lemma 4.1, and Assertions (4.2) and (4.3) imply that

$$\begin{aligned} \frac{1}{2|\lambda|} \widetilde{\Delta}_j(\widehat{w}) &\stackrel{2M-1}{\equiv} \frac{\sqrt{(n_j+1)(m_j+1)}}{2|\lambda|} \sum_{\ell=0}^{2M} \frac{(2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n, m), k, \lambda) \\ &\quad + \frac{\sqrt{n_j m_j}}{2|\lambda|} \sum_{\ell=0}^{2M} \frac{(-2|\lambda|)^\ell}{\ell!} (\partial_{x_j}^\ell f)(|\lambda|R(n, m), k, \lambda). \end{aligned}$$

Defining

$$(4.10) \quad \alpha^\pm(p, q) \stackrel{\text{def}}{=} \sqrt{(p+1)(q+1)} \pm \sqrt{pq},$$

for nonnegative integers p and q , we get

$$(4.11) \quad \frac{1}{2|\lambda|} \widetilde{\Delta}_j(\widehat{w}) \stackrel{2M-1}{\equiv} \widetilde{\Delta}_j^0(\widehat{w}) + \widetilde{\Delta}_j^1(\widehat{w})$$

$$\text{with } \widetilde{\Delta}_j^0(\widehat{w}) \stackrel{\text{def}}{=} \frac{\alpha^+(n_j, m_j)}{2|\lambda|} \sum_{\ell=0}^M \frac{(2\lambda)^{2\ell}}{(2\ell)!} (\partial_{x_j}^{2\ell} f)(|\lambda|R(n, m), k, \lambda)$$

$$\text{and } \widetilde{\Delta}_j^1(\widehat{w}) \stackrel{\text{def}}{=} \frac{\alpha^-(n_j, m_j)}{2|\lambda|} \sum_{\ell=0}^{M-1} \frac{(2|\lambda|)^{2\ell+1}}{(2\ell+1)!} (\partial_{x_j}^{2\ell+1} f)(|\lambda|R(n, m), k, \lambda).$$

Now let us compute an expansion of $\alpha_j^\pm(n, m)$ with respect to $n_j + m_j + 1$ and $n_j - m_j$. Let p and q be two integers and let us write

$$\begin{aligned} (p+1)(q+1) &= pq + p + q + 1 \quad \text{and} \\ pq &= \frac{1}{4}((p+q+1)^2 - 2(p+q+1) + 1 - (p-q)^2). \end{aligned}$$

We get that

$$\begin{aligned} \sqrt{(p+1)(q+1)} &= \frac{1}{2}(p+q+1) \sqrt{1 + \frac{2}{p+q+1} + \frac{1-(p-q)^2}{(p+q+1)^2}} \\ \text{and } \sqrt{pq} &= \frac{1}{2}(p+q+1) \sqrt{1 - \frac{2}{p+q+1} + \frac{1-(p-q)^2}{(p+q+1)^2}}. \end{aligned}$$

Let us introduce the notation $f(p, q) = \mathcal{O}_M(p, q)$ to mean that for some constant C , there holds

$$|f(p, q)| \leq C \left(\frac{1}{(p+q+1)^M} + \frac{|p-q|^{2M+2}}{(p+q+1)^{2M+1}} \right).$$

Using the following Taylor expansion with $K = 2M$:

$$\sqrt{1+u} = 1 + \sum_{\ell_1=1}^K a_{\ell_1} u^{\ell_1} + (K+1)a_{K+1}u^{K+1} \int_0^1 (1+tu)^{-K-\frac{1}{2}}(1-t)^K dt,$$

we gather that

$$\begin{aligned} \sqrt{(p+1)(q+1)} &= \frac{1}{2}(p+q+1) \left(1 + \sum_{\ell_1=1}^{2M} a_{\ell_1} \left(\frac{2}{p+q+1} + \frac{1-(p-q)^2}{(p+q+1)^2} \right)^{\ell_1} \right) + \mathcal{O}_{2M}(p, q) \\ \text{and } \sqrt{pq} &= \frac{1}{2}(p+q+1) \left(1 + \sum_{\ell_1=1}^{2M} a_{\ell_1} \left(-\frac{2}{p+q+1} + \frac{1-(p-q)^2}{(p+q+1)^2} \right)^{\ell_1} \right) + \mathcal{O}_{2M}(p, q). \end{aligned}$$

Now we can compute the expansion of $\alpha^\pm(p, q)$. Newton's formula gives

$$\begin{aligned} \alpha^+(p, q) &= p+q+1 + \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2} \frac{4^{\ell_2} (1-(p-q)^2)^{\ell_1-2\ell_2}}{(p+q+1)^{2\ell_1-2\ell_2-1}} + \mathcal{O}_{2M}(p, q) \\ \alpha^-(p, q) &= 2 \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2+1 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2+1} \frac{4^{\ell_2} (1-(p-q)^2)^{\ell_1-2\ell_2-1}}{(p+q+1)^{2\ell_1-2\ell_2-2}} + \mathcal{O}_{2M}(p, q). \end{aligned} \quad (4.12)$$

In the above expansion, some terms that turn out to be $\mathcal{O}_{2M}(p, q)$ are kept for notational simplicity. Now, one may check that for all functions θ and θ' supported in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ and any integers M_1 and M_2 , we have for all $j \in \{1, \dots, d\}$,

$$(f = \mathcal{O}_{M_1} \text{ and } \theta \stackrel{M_2}{\equiv} \theta') \implies f(n_j, m_j) \theta(\hat{w}) \stackrel{M_1 \pm M_2}{\equiv} f(n_j, m_j) \theta'(\hat{w}). \quad (4.13)$$

Then Assertion (4.12) implies that for any function g in \mathcal{S}_d^+ supported in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$, and any j in $\{1, \dots, d\}$, we have

$$\begin{aligned} \alpha^+(n_j, m_j) \Theta_g(\hat{w}) &\stackrel{2M-1}{\equiv} \left(n_j + m_j + 1 \right. \\ &\left. + \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2} \frac{4^{\ell_2} (1-(n_j-m_j)^2)^{\ell_1-2\ell_2}}{(n_j+m_j+1)^{2\ell_1-2\ell_2-1}} \right) \Theta_g(\hat{w}) \quad \text{and} \end{aligned} \quad (4.14)$$

$$\alpha^-(n_j, m_j) \theta_g(\hat{w}) \stackrel{2M-1}{\equiv} 2 \left(\sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2+1 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2+1} \frac{4^{\ell_2} (1-(n_j-m_j)^2)^{\ell_1-2\ell_2-1}}{(n_j+m_j+1)^{2\ell_1-2\ell_2-2}} \right) \Theta_g(\hat{w}). \quad (4.15)$$

Using (4.11), this gives

$$\begin{aligned} \tilde{\Delta}_j^{(0)}(\hat{w}) &\stackrel{2M-1}{\equiv} \left(\frac{n_j + m_j + 1}{2|\lambda|} \right) \Theta_f(\hat{w}) + \sum_{\ell=0}^M \Theta_{f_{j,2\ell}}(\hat{w}) \quad \text{with} \\ f_{j,0}(x, k, \lambda) &\stackrel{\text{def}}{=} \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2} 2^{2\ell_2-1} (1-k^2)^{\ell_1-2\ell_2} \frac{\lambda^{2(\ell_1-\ell_2-1)}}{x_j^{2\ell_1-2\ell_2-1}} f(x, k, \lambda) \end{aligned}$$

and, if $1 \leq \ell \leq M$,

$$f_{j,2\ell}(x, k, \lambda) \stackrel{\text{def}}{=} \frac{1}{(2\ell)!} \sum_{\substack{0 \leq \ell_1 \leq 2M \\ 2\ell_2 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2} 2^{2\ell_2-1+2\ell} (1 - k_j^2)^{\ell_1-2\ell_2} \frac{\lambda^{2(\ell+\ell_1-\ell_2-1)}}{x_j^{2\ell_1-2\ell_2-1}} \partial_{x_j}^{2\ell} f(x, k, \lambda).$$

Similarly,

$$\begin{aligned} \tilde{\Delta}_j^{(1)}(\hat{w}) &\stackrel{2M-1}{\equiv} \sum_{\ell=0}^{M-1} \Theta_{f_{j,2\ell+1}} \quad \text{with} \\ f_{j,2\ell+1}(x, k, \lambda) &\stackrel{\text{def}}{=} \frac{1}{(2\ell+1)!} \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2+1 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2+1} 4^{\ell_2+\ell} (1 - k_j^2)^{\ell_1-2\ell_2-1} \\ &\quad \times \frac{\lambda^{2(\ell+\ell_1-\ell_2-1)}}{x_j^{2\ell_1-2\ell_2-2}} \partial_{x_j}^{2\ell+1} f(x, k, \lambda). \end{aligned}$$

From the definition of Operator $\hat{\Delta}$, we thus infer that there exist functions f_ℓ of \mathcal{S}_d^+ supported in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ and satisfying (2.14), such that for all $M \geq 0$, we have

$$\hat{\Delta} \Theta_f \stackrel{2M-1}{\equiv} \sum_{\ell=0}^{2M} \Theta_{f_\ell}.$$

At this stage, one may prove by induction, as in the previous case, that $\|\hat{\Delta}^p \Theta_f\|_{N,0,S(\hat{\mathbb{H}}^d)}$ is finite for all integers N and p .

Let us finally study the action of $\hat{\mathcal{D}}_\lambda$. From its definition, setting $k = m - n$, we get

$$\begin{aligned} (\hat{\mathcal{D}}_\lambda \Theta_f)(\hat{w}) &= \frac{d}{d\lambda} (f(|\lambda|R(n, m), k, \lambda)) + \frac{d}{2\lambda} f(|\lambda|R(n, m), k, \lambda) + \frac{1}{2\lambda} \sum_{j=1}^d \mathcal{D}_j(\hat{w}) \quad \text{with} \\ \mathcal{D}_j(\hat{w}) &\stackrel{\text{def}}{=} \sqrt{n_j m_j} f(|\lambda|(R(n, m) - 2\delta_j), k, \lambda) - \sqrt{(n_j + 1)(m_j + 1)} f(|\lambda|(R(n, m) + 2\delta_j), k, \lambda). \end{aligned}$$

The chain rule implies that

$$(4.16) \quad \begin{aligned} \frac{d}{d\lambda} (f(|\lambda|R(n, m), k, \lambda)) &= (\partial_\lambda f)(|\lambda|R(n, m), k, \lambda) \\ &\quad + \text{sgn } \lambda \sum_{j=1}^d (n_j + m_j + 1) (\partial_{x_j} f)(|\lambda|R(n, m), k, \lambda). \end{aligned}$$

Combining Lemma 4.1, and Assertions (4.2) and (4.3) yields

$$\begin{aligned} -\frac{1}{2\lambda} \mathcal{D}_j(\hat{w}) &\stackrel{2M-1}{\equiv} \alpha^-(n_j, m_j) \sum_{\ell=0}^M \frac{(2\lambda)^{2\ell-1}}{(2\ell)!} (\partial_{x_j}^{2\ell} f)(|\lambda|R(n, m), k, \lambda) \\ &\quad + \alpha^+(n_j, m_j) \text{sgn } \lambda \sum_{\ell=0}^{M-1} \frac{(2\lambda)^{2\ell}}{(2\ell+1)!} (\partial_{x_j}^{2\ell+1} f)(|\lambda|R(n, m), k, \lambda). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
(\widehat{\mathcal{D}}_\lambda \Theta_f)(\widehat{w}) &\stackrel{2M-1}{\equiv} (\partial_\lambda f)(|\lambda|R(n, m), k, \lambda) + \frac{1}{2\lambda} \left(d - \sum_{j=1}^d \alpha^-(n_j, m_j) \right) f(|\lambda|R(n, m), k, \lambda) \\
&+ \operatorname{sgn} \lambda \sum_{j=1}^d (n_j + m_j + 1 - \alpha^+(n_j, m_j)) (\partial_{x_j} f)(|\lambda|R(n, m), k, \lambda) \\
&- \alpha^-(n_j, m_j) \sum_{\ell=1}^M \frac{(2\lambda)^{2\ell-1}}{(2\ell)!} (\partial_{x_j}^{2\ell} f)(|\lambda|R(n, m), k, \lambda) \\
&- \alpha^+(n_j, m_j) \operatorname{sgn} \lambda \sum_{\ell=1}^{M-1} \frac{(2\lambda)^{2\ell}}{(2\ell+1)!} (\partial_{x_j}^{2\ell+1} f)(|\lambda|R(n, m), k, \lambda).
\end{aligned}$$

Hence, using (4.14) and (4.15) and noticing that the coefficient a_{ℓ_1} involved in the expansion of $\alpha^\pm(n_j, m_j)$ is equal to $1/2$, we conclude that there exist some functions \widetilde{f}_j , f_j^b and $f_{j,\ell}^\#$ of \mathcal{S}_d^+ , supported in $[r_0, \infty[^d \times \mathbb{Z}^d \times \mathbb{R}$ and satisfying (2.14) so that

$$(\widehat{\mathcal{D}}_\lambda \Theta_f)(\widehat{w}) \stackrel{2M-1}{\equiv} (\Theta_{\partial_\lambda f})(\widehat{w}) + \sum_{j=1}^d (\Theta_{\widetilde{f}_j})(\widehat{w}) - \sum_{j=1}^d (\Theta_{f_j^b})(\widehat{w}) - \sum_{\ell=1}^{2M} \sum_{j=1}^d (\Theta_{f_{j,\ell}^\#})(\widehat{w}),$$

where

$$\widetilde{f}_j(x, k, \lambda) \stackrel{\text{def}}{=} \sum_{\substack{2 \leq \ell_1 \leq 2M \\ 2\ell_2 + 1 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2 + 1} \frac{4^{\ell_2} (1 - k_j^2)^{\ell_1 - 2\ell_2 - 1} \lambda^{2\ell_1 - 2\ell_2 - 3}}{x_j^{2\ell_1 - 2\ell_2 - 2}} f(x, k, \lambda),$$

$$f_j^b(x, k, \lambda) \stackrel{\text{def}}{=} \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2} \frac{4^{\ell_2} (1 - k_j^2)^{\ell_1 - 2\ell_2} \lambda^{2\ell_1 - 2\ell_2 - 1}}{x_j^{2\ell_1 - 2\ell_2 - 1}} \partial_{x_j} f(x, k, \lambda),$$

$$f_{j,2\ell}^\#(x, k, \lambda) \stackrel{\text{def}}{=} \left(\sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2 + 1 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2 + 1} \frac{2^{2\ell_2 + 2\ell} (1 - k_j^2)^{\ell_1 - 2\ell_2 - 1}}{x_j^{2\ell_1 - 2\ell_2 - 2}} \right) \frac{\lambda^{2\ell + 2\ell_1 - 2\ell_2 - 3}}{(2\ell)!} (\partial_{x_j}^{2\ell} f)(x, k, \lambda)$$

and

$$\begin{aligned}
f_{j,2\ell+1}^\#(x, k, \lambda) &\stackrel{\text{def}}{=} 2 \left(x_j + \sum_{\substack{1 \leq \ell_1 \leq 2M \\ 2\ell_2 \leq \ell_1}} a_{\ell_1} \binom{\ell_1}{2\ell_2} \frac{4^{\ell_2} (1 - k_j^2)^{\ell_1 - 2\ell_2} \lambda^{2\ell_1 - 2\ell_2}}{x_j^{2\ell_1 - 2\ell_2 - 1}} \right) \\
&\times \frac{(2\lambda)^{2\ell-1}}{(2\ell+1)!} (\partial_{x_j}^{2\ell+1} f)(x, k, \lambda).
\end{aligned}$$

At this stage, one can complete the proof as in the previous cases. \square

It will be useful to give the following asymptotic description of the operators $\widehat{\Delta}$ and $\widehat{\mathcal{D}}_\lambda$ when λ tends to 0:

Proposition 4.1. For any function f in \mathcal{S}_1^+ supported in $[r_0, \infty[\times \mathbb{Z} \times \mathbb{R}$ for some positive r_0 , the extension to $\widehat{\Delta}\Theta_f$ and $\widehat{\mathcal{D}}_\lambda\Theta_f$ to $\widehat{\mathbb{H}}_0^d$ is given by

$$\begin{aligned} (\widehat{\Delta}\Theta_f)(\dot{x}, k) &= \dot{x}\partial_{\dot{x}\dot{x}}^2 f(\dot{x}, k, 0) + \partial_{\dot{x}} f(\dot{x}, k, 0) - \frac{k^2}{4\dot{x}} f(\dot{x}, k, 0) \quad \text{and} \\ (\widehat{\mathcal{D}}_\lambda\Theta_f)(\dot{x}, k) &= \partial_\lambda f(\dot{x}, k, 0). \end{aligned}$$

Proof. For expository purpose, we omit the dependency on k , for f . Then we have by definition of Θ_f and $\widehat{\Delta}$, for all $(n, n+k, \lambda)$ in $\widetilde{\mathbb{H}}^d$ with positive λ ,

$$\begin{aligned} -2\lambda^2 \widehat{\Delta}\Theta_f(n, n+k, \lambda) &= \lambda(2n+k+1)f(\lambda(2n+k+1), \lambda) \\ &\quad - \lambda\sqrt{(n+1)(n+k+1)}f(\lambda(2n+k+3), \lambda) - \lambda\sqrt{n(n+k)}f(\lambda(2n+k-1), \lambda). \end{aligned}$$

Denoting $\dot{x} = 2\lambda n$, the above equality rewrites

$$\begin{aligned} (4.17) \quad -2\lambda^2 \widehat{\Delta}\Theta_f(\widehat{w}) &= \widetilde{\Delta}^1(\widehat{w}) - \widetilde{\Delta}^2(\widehat{w}) - \widetilde{\Delta}^3(\widehat{w}) \quad \text{with} \\ \widetilde{\Delta}^1(\widehat{w}) &\stackrel{\text{def}}{=} (\dot{x} + \lambda(k+1))f(\dot{x} + \lambda(k+1)), \\ \widetilde{\Delta}^2(\widehat{w}) &\stackrel{\text{def}}{=} \sqrt{\left(\frac{\dot{x}}{2} + \lambda\right)\left(\frac{\dot{x}}{2} + \lambda(k+1)\right)} f(\dot{x} + \lambda(k+3)), \\ \widetilde{\Delta}^3(\widehat{w}) &\stackrel{\text{def}}{=} \sqrt{\frac{\dot{x}}{2}\left(\frac{\dot{x}}{2} + \lambda k\right)} f(\dot{x} + \lambda(k-1)). \end{aligned}$$

In what follows, we shall use repeatedly the following asymptotic expansion for $y > 0$ and η in $] -y, y[$:

$$(4.18) \quad \sqrt{y+\eta} = \sqrt{y} + \frac{\eta}{2\sqrt{y}} - \frac{\eta^2}{8y\sqrt{y}} + \sqrt{y}\mathcal{O}\left(\left(\frac{\eta}{y}\right)^3\right).$$

Let us compute the second order expansions of $\widetilde{\Delta}^1(\widehat{w})$, $\widetilde{\Delta}^2(\widehat{w})$ and $\widetilde{\Delta}^3(\widehat{w})$ with respect to λ , for fixed (and positive) value of λn . We have

$$\begin{aligned} (4.19) \quad \widetilde{\Delta}^1(\widehat{w}) &= \dot{x}f(\dot{x}, 0) + \left((k+1)(f(\dot{x}, 0) + \dot{x}\partial_{\dot{x}}f(\dot{x}, 0)) + \dot{x}\partial_\lambda f(\dot{x}, 0) \right)\lambda \\ &\quad + \left(\frac{1}{2}\dot{x}\partial_{\lambda\lambda}^2 f(\dot{x}, 0) + \frac{(k+1)^2}{2}\left(\dot{x}\partial_{\dot{x}\dot{x}}^2 f(\dot{x}, 0) + 2\partial_{\dot{x}}f(\dot{x}, 0) \right) \right. \\ &\quad \left. + (k+1)\left(\partial_\lambda f(\dot{x}, 0) + \dot{x}\partial_{\dot{x}\lambda}^2 f(\dot{x}, 0) \right) \right)\lambda^2 + \mathcal{O}(\lambda^3). \end{aligned}$$

In order to find out the second order expansions of $\widetilde{\Delta}^2(\widehat{w})$ and $\widetilde{\Delta}^3(\widehat{w})$, we shall use the fact that, denoting $\dot{y} = \dot{x}/2$ and using (4.18),

$$\begin{aligned} \widetilde{\Delta}^2(\widehat{w}) &= \left(\sqrt{\dot{y}} + \frac{\lambda}{2\sqrt{\dot{y}}} - \frac{\lambda^2}{8\dot{y}\sqrt{\dot{y}}} \right) \left(\sqrt{\dot{y}} + \frac{(k+1)\lambda}{2\sqrt{\dot{y}}} - \frac{(k+1)^2\lambda^2}{8\dot{y}\sqrt{\dot{y}}} \right) \\ &\quad \times \left(f(\dot{x}, 0) + (\partial_\lambda f(\dot{x}, 0) + (k+3)\partial_{\dot{x}}f(\dot{x}, 0))\lambda \right. \\ &\quad \left. + \left(\frac{1}{2}\partial_{\lambda\lambda}^2 f(\dot{x}, 0) + (k+3)\partial_{\dot{x}\lambda}^2 f(\dot{x}, 0) + \frac{(k+3)^2}{2}\partial_{\dot{x}\dot{x}}^2 f(\dot{x}, 0) \right)\lambda^2 \right) + \mathcal{O}(\lambda^3). \end{aligned}$$

Hence, we get at the end, replacing \dot{y} by its value,

$$(4.20) \quad \begin{aligned} \tilde{\Delta}^2(\hat{w}) &= \frac{\dot{x}}{2}f(\dot{x}, 0) + \left(\left(1 + \frac{k}{2}\right)f(\dot{x}, 0) + \left(\frac{k+3}{2}\right)\dot{x}\partial_{\dot{x}}f(\dot{x}, 0) + \frac{1}{2}\dot{x}\partial_{\lambda}f(\dot{x}, 0) \right)\lambda \\ &\quad + \left(\frac{(k+3)^2}{4}\dot{x}\partial_{\dot{x}\dot{x}}^2f(\dot{x}, 0) + \left(\frac{k+3}{2}\right)\dot{x}\partial_{\dot{x}\lambda}^2f(\dot{x}, 0) + \frac{\dot{x}}{4}\partial_{\lambda\lambda}^2f(\dot{x}, 0) \right. \\ &\quad \left. + \left(1 + \frac{k}{2}\right)\left((k+3)\partial_{\dot{x}}f(\dot{x}, 0) + \partial_{\lambda}f(\dot{x}, 0)\right) - \frac{k^2}{4\dot{x}}f(\dot{x}, 0) \right)\lambda^2 + \mathcal{O}(\lambda^3). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \tilde{\Delta}^3(\hat{w}) &= \sqrt{\dot{y}}\left(\sqrt{\dot{y}} + \frac{k\lambda}{2\sqrt{\dot{y}}} - \frac{k^2\lambda^2}{8\dot{y}\sqrt{\dot{y}}}\right)\left(f(\dot{x}, 0) + (\partial_{\lambda}f(\dot{x}, 0) + (k-1)\partial_{\dot{x}}f(\dot{x}, 0))\lambda \right. \\ &\quad \left. + \left(\frac{1}{2}\partial_{\lambda\lambda}^2f(\dot{x}, 0) + (k-1)\partial_{\dot{x}\lambda}^2f(\dot{x}, 0) + \frac{(k-1)^2}{2}\partial_{\dot{x}\dot{x}}^2f(\dot{x}, 0)\right)\lambda^2\right) + \mathcal{O}(\lambda^3), \end{aligned}$$

whence,

$$(4.21) \quad \begin{aligned} \tilde{\Delta}^3(\hat{w}) &= \frac{\dot{x}}{2}f(\dot{x}, 0) + \left(\frac{k}{2}f(\dot{x}, 0) + \left(\frac{k-1}{2}\right)\dot{x}\partial_{\dot{x}}f(\dot{x}, 0) + \frac{1}{2}\dot{x}\partial_{\lambda}f(\dot{x}, 0)\right)\lambda \\ &\quad + \left(\frac{(k-1)^2}{4}\dot{x}\partial_{\dot{x}\dot{x}}^2f + \left(\frac{k-1}{2}\right)\dot{x}\partial_{\dot{x}\lambda}^2f(\dot{x}, 0) + \frac{\dot{x}}{4}\partial_{\lambda\lambda}^2f(\dot{x}, 0) \right. \\ &\quad \left. + \frac{k}{2}\left((k-1)\partial_{\dot{x}}f(\dot{x}, 0) + \partial_{\lambda}f(\dot{x}, 0)\right) - \frac{k^2}{4\dot{x}}f(\dot{x}, 0)\right)\lambda^2 + \mathcal{O}(\lambda^3). \end{aligned}$$

Inserting the above relations in (4.17), we discover that the zeroth and first order terms in the expansion cancel, and that

$$-2\lambda^2\widehat{\Delta}\Theta_f(\hat{w}) = \left(\frac{k^2}{2\dot{x}}f(\dot{x}, 0) - 2\partial_{\dot{x}}f(\dot{x}, 0) - 2\dot{x}\partial_{\dot{x}\dot{x}}^2f(\dot{x}, 0)\right)\lambda^2 + \mathcal{O}(\lambda^3),$$

which ensures that

$$\lim_{\lambda \rightarrow 0} \widehat{\Delta}\Theta_f(\hat{w}) = \dot{x}\partial_{\dot{x}\dot{x}}^2f(\dot{x}, 0) + \partial_{\dot{x}}f(\dot{x}, 0) - \frac{k^2}{4\dot{x}}f(\dot{x}, 0).$$

The proof for Operator $\widehat{\mathcal{D}}_{\lambda}$ is quite similar: from the definition of $\widehat{\mathcal{D}}_{\lambda}$ and the chain rule, we discover that for all $(n, n+k, \lambda)$ in $\widetilde{\mathbb{H}}^d$ with $\lambda > 0$,

$$\begin{aligned} \widehat{\mathcal{D}}_{\lambda}\Theta_f(\hat{w}) &= (2n+k+1)\partial_{\dot{x}}f(\lambda(2n+k+1), \lambda) + \partial_{\lambda}f(\lambda(2n+k+1), \lambda) \\ &\quad + \frac{1}{2\lambda}\left(f(\lambda(2n+k+1), \lambda) + \sqrt{n(n+k)}f(\lambda(2n+k-1), \lambda) - \sqrt{(n+1)(n+k+1)}f(\lambda(2n+k+3), \lambda)\right). \end{aligned}$$

Therefore, assuming that $\dot{x} \stackrel{\text{def}}{=} 2\lambda n > 0$, we get

$$(4.22) \quad \begin{aligned} \widehat{\mathcal{D}}_{\lambda}\Theta_f(\hat{w}) &= \partial_{\lambda}f(\dot{x} + (k+1)\lambda, \lambda) + \frac{1}{\lambda}(\dot{x} + (k+1)\lambda)\partial_{\dot{x}}f(\dot{x} + (k+1)\lambda, \lambda) \\ &\quad + \frac{1}{2\lambda^2}\left(\lambda f(\dot{x} + (k+1)\lambda, \lambda) + \tilde{\Delta}^3(\hat{w}) - \tilde{\Delta}^2(\hat{w})\right). \end{aligned}$$

Because

$$f(\dot{x} + (k+1)\lambda, \lambda) = f(\dot{x}, 0) + ((k+1)\partial_{\dot{x}}f(\dot{x}, 0) + \partial_{\lambda}f(\dot{x}, 0))\lambda + \mathcal{O}(\lambda^2)$$

and

$$\begin{aligned} (\dot{x} + (k+1)\lambda)\partial_{\dot{x}}f(\dot{x} + (k+1)\lambda, \lambda) &= \dot{x}\partial_{\dot{x}}f(\dot{x}, 0) \\ &\quad + ((k+1)(\partial_{\dot{x}}f(\dot{x}, 0) + \dot{x}\partial_{\dot{x}\dot{x}}^2f(\dot{x}, 0)) + \dot{x}\partial_{\dot{x}\lambda}^2f(\dot{x}, 0))\lambda + \mathcal{O}(\lambda^2), \end{aligned}$$

we get at the end, taking advantage of (4.20) and (4.21),

$$\widehat{\mathcal{D}}_\lambda\Theta_f(\widehat{w}) = \partial_\lambda f(\dot{x}, 0) + \mathcal{O}(\lambda),$$

which completes the proof. \square

5. EXAMPLES OF TEMPERED DISTRIBUTIONS

A first class of examples will be given by the functions belonging to the space $L_M^1(\widehat{\mathbb{H}}^d)$ of Definition 2.5. This is exactly what states Theorem 2.4 that we are going to prove now. Inequality (2.17) just follows from the definition of the semi-norms on $\widehat{\mathbb{H}}^d$. So let us focus on the proof of the first part of the statement. Let f be a function of $L_M^1(\widehat{\mathbb{H}}^d)$ such that $\iota(f) = 0$. We claim that $f = 0$ a.e. Clearly, it is enough to prove for all $K > 0$ and $b > a > 0$, we have

$$\int_{\widehat{\mathcal{C}}_{a,b,K}} |f(\widehat{w})| d\widehat{w} = 0$$

where $\widehat{\mathcal{C}}_{a,b,K} \stackrel{\text{def}}{=} \{(n, m, \lambda) \in \widehat{\mathbb{H}}^d : |\lambda|(|n+m|+d) \leq K, |n-m| \leq K \text{ and } a \leq |\lambda| \leq b\}$.

To this end, we introduce the bounded function :

$$g \stackrel{\text{def}}{=} \frac{\overline{f}}{|f|} \mathbf{1}_{f \neq 0} \mathbf{1}_{\widehat{\mathcal{C}}_{a,b,K}}$$

and smooth it out with respect to λ by setting

$$g_\varepsilon \stackrel{\text{def}}{=} \chi_\varepsilon \star_\lambda g$$

where $\chi_\varepsilon \stackrel{\text{def}}{=} \varepsilon^{-1}\chi(\varepsilon^{-1}\cdot)$ and χ stands for some smooth even function on \mathbb{R} , supported in the interval $[-1, 1]$ and with integral 1.

Note that by definition, g is supported in the set $\widehat{\mathcal{C}}_{a,b,K}$. Therefore, if $\varepsilon < a$ then g_ε is supported in $\widehat{\mathcal{C}}_{a-\varepsilon, b+\varepsilon, K}$. This readily ensures that $\|g_\varepsilon\|_{N,0,\mathcal{S}(\widehat{\mathbb{H}}^d)}$ is finite for all integer N (as regards the action of operator $\widehat{\Sigma}_0$, note that $g_\varepsilon(n, m, \lambda) = 0$ whenever $|\lambda| < a - \varepsilon$).

In order to prove that g_ε belongs to $\mathcal{S}(\widehat{\mathbb{H}}^d)$, it suffices to use the following lemma the proof of which is left to the reader:

Lemma 5.1. *Let h be a smooth function on $\widehat{\mathbb{H}}^d$ with support in $\{(n, m, \lambda) : |\lambda| \geq a\}$ for some $a > 0$. If h and all derivatives with respect to λ have fast decay, that is have finite semi-norm $\|\cdot\|_{N,0,\mathcal{S}(\widehat{\mathbb{H}}^d)}$ for all integer N , then the same properties hold true for $\widehat{\mathcal{D}}_\lambda h$ and $\widehat{\Delta} h$.*

Because g_ε is in $\mathcal{S}(\widehat{\mathbb{H}}^d)$ for all $0 < a < \varepsilon$, our assumption on f ensures that we have

$$I_\varepsilon \stackrel{\text{def}}{=} \int_{\widehat{\mathbb{H}}^d} f(\widehat{w}) g_\varepsilon(\widehat{w}) d\widehat{w} = 0.$$

Now, we notice that whenever $0 < \varepsilon \leq a/2$, we have for all $(n, m, \lambda) \in \widehat{\mathbb{H}}^d$ and $\lambda' \in \mathbb{R}$,

$$\frac{1}{\varepsilon}\chi\left(\frac{\lambda - \lambda'}{\varepsilon}\right)g(n, m, \lambda')f(n, m, \lambda) = \frac{1}{\varepsilon}\chi\left(\frac{\lambda - \lambda'}{\varepsilon}\right)g(n, m, \lambda')(\mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}}f)(n, m, \lambda),$$

which guarantees that

$$\int_{\widehat{\mathbb{H}}^d \times \mathbb{R}} \chi_\varepsilon(\lambda - \lambda') |g(n, m, \lambda')| |f(n, m, \lambda)| d\widehat{w} d\lambda' \leq \|\chi\|_{L^1} \|\mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}} f\|_{L^1} < \infty.$$

Therefore applying Fubini theorem, remembering that χ is an even function and exchanging the notation λ and λ' in the second line below,

$$\begin{aligned} I_\varepsilon &= \int_{\widehat{\mathbb{H}}^d} f(n, m, \lambda) \left(\int_{\mathbb{R}} \chi_\varepsilon(\lambda - \lambda') g(n, m, \lambda') d\lambda' \right) d\widehat{w} \\ &= \int_{\widehat{\mathbb{H}}^d} g(n, m, \lambda) \left(\int_{\mathbb{R}} \chi_\varepsilon(\lambda - \lambda') (\mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}} f)(n, m, \lambda') d\lambda' \right) d\widehat{w} \\ &= \int_{\widehat{\mathbb{H}}^d} (\chi_\varepsilon \star (\mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}} f))(\widehat{w}) g(\widehat{w}) d\widehat{w}. \end{aligned}$$

The standard density theorem for convolution in \mathbb{R} ensures that for all (n, m) in \mathbb{N}^{2d} , we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |\chi_\varepsilon \star (\mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}} f)(n, m, \lambda) - (\mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}} f)(n, m, \lambda)| d\lambda = 0.$$

Hence, because the supremum of g is bounded by 1, we get

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} I_\varepsilon \\ &= \int_{\widehat{\mathbb{H}}^d} \mathbf{1}_{\widehat{\mathcal{C}}_{a/2, b+a/2, K}} f(\widehat{w}) g(\widehat{w}) d\widehat{w} \\ &= \int_{\widehat{\mathcal{C}}_{a, b, K}} |f(\widehat{w})| d\widehat{w}, \end{aligned}$$

which completes the proof of Theorem 2.4. \square

Let us prove Proposition 2.1 which claims that the functions

$$f_\gamma(n, m, \lambda) \stackrel{\text{def}}{=} (|\lambda|(2|m| + d))^{-\gamma} \delta_{n, m}$$

are in L_M^1 in the case when γ is less than $d + 1$. As f_γ is continuous and bounded away from any neighborhood of $\widehat{0}$, it suffices to prove that

$$(5.1) \quad \sum_{n \in \mathbb{N}^d} \int_{|\lambda|(2|n| + d) \leq 1} (|\lambda|(2|n| + d))^{-\gamma} |\lambda|^d d\lambda < \infty.$$

Now, performing the change of variables $\lambda' = \lambda(2|n| + d)$, we find out that

$$\sum_{n \in \mathbb{N}^d} \int_{|\lambda|(2|n| + d) \leq 1} (|\lambda|(2|n| + d))^{-\gamma} |\lambda|^d d\lambda = \sum_{n \in \mathbb{N}^d} (2|n| + d)^{-d-1} \int_{|\lambda'| \leq 1} |\lambda'|^{d-\gamma} d\lambda'.$$

Because $\gamma < d + 1$, this implies that the last integral is finite. As $\sum_{n \in \mathbb{N}^d} (2|n| + d)^{-d-1}$ is finite,

one may conclude that f_γ is in L_M^1 . \square

In order to give an example of tempered distribution on the Heisenberg group that is not a function, let us finally prove Proposition 2.2. We start with the obvious observation that

$$\left| \int_{\widehat{\mathbb{H}}^d} \left(\frac{\theta(n, n, \lambda) + \theta(n, n, -\lambda) - 2\theta(\widehat{0})}{|\lambda|^\gamma (2|n| + d)^\gamma} \right) \delta_{n, m} d\widehat{w} \right| \leq \mathcal{I}_1 + \mathcal{I}_2$$

with

$$\mathcal{I}_1 \stackrel{\text{def}}{=} \int_{\widehat{\mathbb{H}}^d} \mathbf{1}_{\{|\lambda|(2|n|+d) \geq 1\}} \delta_{n,m} \frac{|\theta(n, n, \lambda) + \theta(n, n, -\lambda) - 2\theta(\widehat{0})|}{|\lambda|^\gamma (2|n|+d)^\gamma} d\widehat{w}$$

and $\mathcal{I}_2 \stackrel{\text{def}}{=} \int_{\widehat{\mathbb{H}}^d} \mathbf{1}_{\{|\lambda|(2|n|+d) < 1\}} \delta_{n,m} \frac{|\theta(n, n, \lambda) + \theta(n, n, -\lambda) - 2\theta(\widehat{0})|}{|\lambda|^\gamma (2|n|+d)^\gamma} d\widehat{w}.$

On the one hand, we have

$$\mathcal{I}_1 \leq 4\|\theta\|_{L^\infty(\widehat{\mathbb{H}}^d)} \sum_{n \in \mathbb{N}^d} \int_{\mathbb{R}} \mathbf{1}_{\{|\lambda|(2|n|+d) \geq 1\}} \frac{1}{|\lambda|^\gamma (2|n|+d)^\gamma} |\lambda|^d d\lambda.$$

Changing variable $\lambda' = \lambda(2|n|+d)$ gives

$$\mathcal{I}_1 \leq 4\|\theta\|_{L^\infty(\widehat{\mathbb{H}}^d)} \sum_{n \in \mathbb{N}^d} \frac{2}{(2|n|+d)^{d+1}} \int_1^\infty |\lambda'|^{d-\gamma} d\lambda'.$$

As γ is greater than $d+1$, the integral in λ' is finite and we get

$$(5.2) \quad \mathcal{I}_1 \leq C\|\theta\|_{L^\infty(\widehat{\mathbb{H}}^d)}.$$

On the other hand, changing again variable $\lambda' = \lambda(2|n|+d)$, we see that

$$\mathcal{I}_2 = \sum_{n \in \mathbb{N}^d} \frac{2}{(2|n|+d)^{d+1}} \int_0^1 \left(\theta\left(n, n, \frac{\lambda'}{2|n|+d}\right) + \theta\left(n, n, \frac{-\lambda'}{2|n|+d}\right) - 2\theta(\widehat{0}) \right) |\lambda'|^{d-\gamma} d\lambda'.$$

At this stage, we need a suitable bound of the integrand just above. This will be achieved thanks to the following lemma.

Lemma 5.2. *There exists an integer k such that for any function θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$, we have*

$$\forall (n, n, \lambda) \in \widehat{\mathbb{H}}^d, \quad |\theta(n, n, \lambda) - \theta(\widehat{0})| \leq C\|\theta\|_{k,k,\mathcal{S}(\widehat{\mathbb{H}}^d)} \sqrt{|\lambda|(2|n|+d)}.$$

Proof. Theorem 2.1 guarantees that θ is the Fourier transform of a function f of $\mathcal{S}(\mathbb{H}^d)$ (with control of semi-norms). Hence it suffices to prove that

$$(5.3) \quad \left| \widehat{f}_{\mathbb{H}}(\widehat{w}) - \delta_{n,m} \int_{\mathbb{H}^d} f(w) dw \right| \leq CN(f) \left(\sqrt{|\lambda|(|n+m|+d)} + |\lambda|\delta_{n,m} \right) \quad \text{with}$$

$$N(f) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} (1 + |Y| + |s + 2\langle \eta, y \rangle|) |f(Y, s)| dw.$$

According to (1.13), we have

$$\begin{aligned} & \widehat{f}_{\mathbb{H}}(\widehat{w}) - \delta_{n,m} \int_{\mathbb{H}^d} f(w) dw \\ &= \int_{\mathbb{H}^d} f(w) \left(e^{-i\lambda(s+2\langle \eta, y \rangle)} \int_{\mathbb{R}^d} e^{-2i\lambda\langle \eta, z \rangle} H_{n,\lambda}(z+2y) H_{m,\lambda}(z) dz - \int_{\mathbb{R}^d} H_{n,\lambda}(z) H_{m,\lambda}(z) dz \right) dw. \end{aligned}$$

The right-hand side may be decomposed into $I_1 + I_2 + I_3$ with

$$\begin{aligned} I_1 &= \int_{\mathbb{H}^d} e^{-i\lambda(s+2\langle \eta, y \rangle)} f(w) \left(\int_{\mathbb{R}^d} (e^{-2i\lambda\langle \eta, z \rangle} - 1) H_{n,\lambda}(z+2y) H_{m,\lambda}(z) dz \right) dw, \\ I_2 &= \int_{\mathbb{H}^d} e^{-i\lambda(s+2\langle \eta, y \rangle)} f(w) \left(\int_{\mathbb{R}^d} (H_{n,\lambda}(z+2y) - H_{n,\lambda}(z)) H_{m,\lambda}(z) dz \right) dw \quad \text{and} \\ I_3 &= \int_{\mathbb{H}^d} (e^{-i\lambda(s+2\langle \eta, y \rangle)} - 1) f(w) \left(\int_{\mathbb{R}^d} H_{n,\lambda}(z) H_{m,\lambda}(z) dz \right) dw. \end{aligned}$$

To bound I_1 , it suffices to use that

$$\left| \int_{\mathbb{R}^d} \left(e^{-2i\lambda\langle \eta, z \rangle} - 1 \right) H_{n,\lambda}(z+2y) H_{m,\lambda}(z) dz \right| \leq 2 \sum_{j=1}^d |\lambda \eta_j| \int_{\mathbb{R}^d} |H_{n,\lambda}(z+2y)| |z_j H_{m,\lambda}(z)| dz,$$

whence, combining Cauchy-Schwarz inequality and (A.4),

$$\left| \int_{\mathbb{R}^d} \left(e^{-2i\lambda\langle \eta, z \rangle} - 1 \right) H_{n,\lambda}(z+2y) H_{m,\lambda}(z) dz \right| \leq \sum_{j=1}^d |\eta_j| \sqrt{|\lambda|(4m_j+2)}.$$

This gives

$$(5.4) \quad |I_1| \leq \sqrt{|\lambda|(4|m|+2d)} \int_{\mathbb{H}^d} |\eta| |f(y, \eta, s)| dy d\eta ds.$$

To handle the term I_2 , we use the following mean value formula:

$$H_{n,\lambda}(z+2y) - H_{n,\lambda}(z) = 2y \cdot \int_0^1 \nabla H_{n,\lambda}(z+2ty) dt,$$

which implies, still using (A.4),

$$\left| \int_{\mathbb{R}^d} (H_{n,\lambda}(z+2y) - H_{n,\lambda}(z)) H_{m,\lambda}(z) dz \right| \leq \sum_{j=1}^d |y_j| \sqrt{|\lambda|(4n_j+2)},$$

and thus

$$(5.5) \quad |I_2| \leq \sqrt{|\lambda|(4|n|+2d)} \int_{\mathbb{H}^d} |y| |f(y, \eta, s)| dy d\eta ds.$$

Finally, it is clear that the mean value theorem (for the exponential function) and the fact that $(H_n)_{n \in \mathbb{N}^d}$ is an orthonormal family imply that

$$(5.6) \quad |I_3| \leq |\lambda| \delta_{n,m} \int_{\mathbb{H}^d} |s + 2\langle \eta, y \rangle| |f(y, \eta, s)| dy d\eta ds.$$

Putting (5.4), (5.5) and (5.6) together ends the proof of the lemma. \square

It is now easy to complete the proof of Proposition 2.2. Indeed, taking $\lambda = \pm \frac{\lambda'}{2|n|+d}$ in Lemma 5.2, we discover that

$$\left| \theta\left(n, n, \frac{\lambda'}{2|n|+d}\right) + \theta\left(n, n, -\frac{\lambda'}{2|n|+d}\right) - 2\theta(\widehat{0}) \right| \leq C \|\theta\|_{k,k,S(\widehat{\mathbb{H}}^d)} \sqrt{\lambda'}.$$

This implies that

$$\begin{aligned} \mathcal{I}_2 &\leq C \|\theta\|_{k,k,S(\widehat{\mathbb{H}}^d)} \sum_{n \in \mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}} \int_0^1 |\lambda'|^{d+\frac{1}{2}-\gamma} d\lambda' \\ &\leq C \|\theta\|_{k,k,S(\widehat{\mathbb{H}}^d)} \int_0^1 |\lambda'|^{d+\frac{1}{2}-\gamma} d\lambda'. \end{aligned}$$

As $\gamma < d + 3/2$, combining with (5.2) completes the proof of the proposition. \square

6. EXAMPLES OF COMPUTATIONS OF FOURIER TRANSFORMS

The present section aims at pointing out a few examples of computations of Fourier transform that may be easily achieved within our approach.

Let us start with Proposition 2.5. The first identity is easy to prove. Indeed, according to (1.14), we have

$$\begin{aligned} \langle \mathcal{F}_{\mathbb{H}}(\delta_0), \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} &= \langle \delta_0, {}^t \mathcal{F}_{\mathbb{H}}(\theta) \rangle_{\mathcal{S}'(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)} \\ &= \int_{\widehat{\mathbb{H}}^d} (H_{m,\lambda} | H_{n,\lambda})_{L^2} \theta(\widehat{w}) d\widehat{w}. \end{aligned}$$

As $(H_{n,\lambda})_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, we get

$$\langle \mathcal{F}_{\mathbb{H}}(\delta_0), \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \sum_{n \in \mathbb{N}^d} \int_{\mathbb{R}} \theta(n, n, \lambda) |\lambda|^d d\lambda$$

which is exactly the first identity.

For proving the second identity, we start again from the definition of the Fourier transform on $\mathcal{S}'(\mathbb{H}^d)$, and get

$$(6.1) \quad \langle \mathcal{F}_{\mathbb{H}}(\mathbf{1}), \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \int_{\mathbb{H}^d} ({}^t \mathcal{F}_{\mathbb{H}} \theta)(w) dw.$$

Let us underline that because ${}^t \mathcal{F}_{\mathbb{H}} \theta$ belongs to $\mathcal{S}(\mathbb{H}^d)$, the above integral makes sense. Besides, (2.22) implies that

$$\langle \mathcal{F}_{\mathbb{H}}(\mathbf{1}), \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} \int_{\mathbb{H}^d} (\mathcal{F}_{\mathbb{H}}^{-1}(\theta))(y, -\eta, -s) dy d\eta ds.$$

By Theorem 2.2 and Lemma 5.2 we have, for any integrable function f on \mathbb{H}^d ,

$$\widehat{f}_{\mathbb{H}}(\widehat{0}) = \int_{\mathbb{H}^d} f(w) dw.$$

Thus we get

$$\langle \mathcal{F}_{\mathbb{H}}(\mathbf{1}), \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} \mathcal{F}_{\mathbb{H}}(\mathcal{F}_{\mathbb{H}}^{-1}(\theta))(\widehat{0}) = \frac{\pi^{d+1}}{2^{d-1}} \theta(\widehat{0}).$$

This concludes the proof of the proposition. \square

In order to prove Theorem 2.5, we need to establish the following continuity property of the Fourier transform.

Proposition 6.1. *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of tempered distribution on \mathbb{H}^d which converges to T in $\mathcal{S}'(\mathbb{H}^d)$. Then the sequence $(\mathcal{F}_{\mathbb{H}} T_n)_{n \in \mathbb{N}}$ converges to $\mathcal{F}_{\mathbb{H}} T$ in $\mathcal{S}'(\widehat{\mathbb{H}}^d)$.*

Proof. By definition of the Fourier transform on \mathbb{H}^d , we have

$$\forall \theta \in \mathcal{S}(\widehat{\mathbb{H}}^d), \quad \langle \mathcal{F}_{\mathbb{H}} T_n, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \langle T_n, {}^t \mathcal{F}_{\mathbb{H}} \theta \rangle_{\mathcal{S}'(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)}$$

Since $(T_n)_{n \in \mathbb{N}}$ converges to T in $\mathcal{S}'(\mathbb{H}^d)$, we have

$$\forall \theta \in \mathcal{S}(\widehat{\mathbb{H}}^d), \quad \lim_{n \rightarrow \infty} \langle T_n, {}^t \mathcal{F}_{\mathbb{H}} \theta \rangle_{\mathcal{S}'(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)} = \langle T, {}^t \mathcal{F}_{\mathbb{H}} \theta \rangle_{\mathcal{S}'(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)}.$$

Therefore, putting the above two relations together eventually yields

$$\forall \theta \in \mathcal{S}(\widehat{\mathbb{H}}^d), \quad \lim_{n \rightarrow \infty} \langle \mathcal{F}_{\mathbb{H}} T_n, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)} = \langle T, {}^t \mathcal{F}_{\mathbb{H}} \theta \rangle_{\mathcal{S}'(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)} = \langle \mathcal{F}_{\mathbb{H}} T, \theta \rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d) \times \mathcal{S}(\widehat{\mathbb{H}}^d)}.$$

This concludes the proof of the proposition. \square

Now, proving Theorem 2.5 just amounts to recast Theorem 1.4 of [1] (and its proof) in terms of tempered distributions. We recall it here for the reader convenience.

Theorem 6.1. *Let χ be a function of $\mathcal{S}(\mathbb{R})$ with value 1 at 0, and compactly supported Fourier transform. Then for any function g in $L^1(T^*\mathbb{R}^d)$ and any sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ tending to 0, we have*

$$(6.2) \quad \lim_{n \rightarrow \infty} \mathcal{F}_{\mathbb{H}}(g \otimes \chi(\varepsilon_n \cdot)) = 2\pi(\mathcal{G}_{\mathbb{H}}g)\mu_{\widehat{\mathbb{H}}_0^d}$$

in the sense of measures on $\widehat{\mathbb{H}}^d$.

Because $g \otimes \chi(\varepsilon_n \cdot)$ tends to $g \otimes \mathbf{1}$ in $\mathcal{S}'(\mathbb{H}^d)$, Proposition 6.1 guarantees that

$$(6.3) \quad \mathcal{F}_{\mathbb{H}}(g \otimes \mathbf{1}) = \lim_{n \rightarrow \infty} \mathcal{F}_{\mathbb{H}}(g \otimes \chi(\varepsilon_n \cdot)).$$

Moreover, according to Theorem 1.4 of [1], we have, for any θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$,

$$\mathcal{I}_{\varepsilon_n}(g, \theta) = \int_{\widehat{\mathbb{H}}^d} \frac{1}{\varepsilon_n} \widehat{\chi}\left(\frac{\lambda}{\varepsilon_n}\right) G(\widehat{w}) \theta(\widehat{w}) d\widehat{w} \quad \text{with} \quad G(\widehat{w}) \stackrel{\text{def}}{=} \int_{T^*\mathbb{R}^d} \overline{\mathcal{W}}(\widehat{w}, Y) g(Y) dY.$$

As g is integrable on $T^*\mathbb{R}^d$, Proposition 2.1 of [1] implies that the (numerical) product $G\theta$ is a continuous function that satisfies

$$|G(\widehat{w})\theta(\widehat{w})| \leq C(1 + |\lambda|(|n+m| + d) + |n-m|)^{-2d+1}.$$

This matches the hypothesis of Lemma 3.1 in [1], and thus

$$\lim_{n \rightarrow \infty} \int_{\widehat{\mathbb{H}}^d} \frac{1}{\varepsilon_n} \widehat{\chi}\left(\frac{\lambda}{\varepsilon_n}\right) G(\widehat{w}) \theta(\widehat{w}) d\widehat{w} = \int_{\widehat{\mathbb{H}}_0^d} \theta(\dot{x}, k) (\mathcal{G}_{\mathbb{H}}g)(\dot{x}, k) d\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x}, k).$$

Together with (6.3), this proves the theorem. \square

APPENDIX A. USEFUL TOOLS AND MORE RESULTS

For the reader convenience, we here recall (and sometimes prove) some results that have been used repeatedly in the paper. We also provide one more result concerning the action of the Fourier transform on derivatives.

A.1. Hermite functions. In addition to the creation operator $C_j \stackrel{\text{def}}{=} -\partial_j + M_j$ already defined in the introduction, we used the following *annihilation operator*:

$$(A.1) \quad A_j \stackrel{\text{def}}{=} \partial_j + M_j.$$

It is very classical (see e.g. [18]) that

$$(A.2) \quad A_j H_n = \sqrt{2n_j} H_{n-\delta_j} \quad \text{and} \quad C_j H_n = \sqrt{2n_j + 2} H_{n+\delta_j}.$$

As, obviously,

$$(A.3) \quad 2M_j = C_j + A_j \quad \text{and} \quad 2\partial_j = A_j - C_j,$$

we discover that

$$(A.4) \quad \begin{aligned} M_j H_n &= \frac{1}{2} (\sqrt{2n_j} H_{n-\delta_j} + \sqrt{2n_j + 2} H_{n+\delta_j}) \quad \text{and} \\ \partial_j H_n &= \frac{1}{2} (\sqrt{2n_j} H_{n-\delta_j} - \sqrt{2n_j + 2} H_{n+\delta_j}). \end{aligned}$$

A.2. The inversion theorem. We here present the proof of Theorem 1.1. In order to establish the inversion formula, consider a function f in $\mathcal{S}(\mathbb{H}^d)$. Then we observe that if we make the change of variable $x' = x - 2y$ in the integral defining $(\mathcal{F}^{\mathbb{H}}(f)(\lambda)(u))(x)$ (for any u in $L^2(\mathbb{R}^d)$) and use the definition of the Fourier transform with respect to the variable s in \mathbb{R} , then we get

$$\begin{aligned} (\mathcal{F}^{\mathbb{H}}(f)(\lambda)(u))(x) &= \int_{\mathbb{H}^d} f(y, \eta, s) e^{-i\lambda s - 2i\lambda \langle \eta, x-y \rangle} u(x-2y) dy d\eta ds \\ (A.5) \quad &= 2^{-d} \int_{T^*\mathbb{R}^d} (\mathcal{F}_s f) \left(\frac{x-x'}{2}, \eta, \lambda \right) e^{-i\lambda \langle \eta, x+x' \rangle} u(x') dx' d\eta. \end{aligned}$$

This can be written

$$\begin{aligned} (A.6) \quad \mathcal{F}^{\mathbb{H}}(f)(\lambda)(u)(x) &= \int_{\mathbb{R}^d} K_f(x, x', \lambda) u(x') dx', \\ \text{with } K_f(x, x', \lambda) &\stackrel{\text{def}}{=} 2^{-d} \int_{(\mathbb{R}^d)^*} (\mathcal{F}_s f) \left(\frac{x-x'}{2}, \eta, \lambda \right) e^{-i\lambda \langle \eta, x+x' \rangle} d\eta \\ &= 2^{-d} (\mathcal{F}_{\eta, s} f) \left(\frac{x-x'}{2}, \lambda(x+x'), \lambda \right). \end{aligned}$$

This identity enables us to decompose $\mathcal{F}_{\mathbb{H}}$ into the product of three very simple operations, namely

$$\begin{aligned} (A.7) \quad \mathcal{F}_{\mathbb{H}} &= 2^{-d} P_H \circ \Phi \circ \mathcal{F}_{\eta, s} \quad \text{with} \\ \Phi(\phi)(x, x', \lambda) &\stackrel{\text{def}}{=} \phi \left(\frac{x-x'}{2}, \lambda(x+x'), \lambda \right) \quad \text{and} \\ (P_H \psi)(n, m, \lambda) &\stackrel{\text{def}}{=} (\psi(\cdot, \lambda) | H_{n, \lambda} \otimes H_{m, \lambda})_{L^2(\mathbb{R}^{2d})}. \end{aligned}$$

Let us point out that for all λ in $\mathbb{R} \setminus \{0\}$, the map

$$\phi(\cdot, \lambda) \longmapsto \Phi(\phi)(\cdot, \lambda)$$

is an automorphism of $L^2(\mathbb{R}^{2d})$ such that

$$(A.8) \quad \|\Phi(\phi)(\cdot, \lambda)\|_{L^2(\mathbb{R}^{2d})} = |\lambda|^{-\frac{d}{2}} \|\phi(\cdot, \lambda)\|_{L^2(\mathbb{R}^{2d})},$$

and that the inverse of Φ is explicitly given by

$$(A.9) \quad \Phi^{-1}(y, z, \lambda) = \psi \left(y + \frac{z}{2\lambda}, -y + \frac{z}{2\lambda}, \lambda \right).$$

Next, Operator P_H just associates to any vector of $L^2(\mathbb{R}^{2d})$ its coordinates with respect to the orthonormal basis $(H_{n, \lambda} \otimes H_{m, \lambda})_{(n, m) \in \mathbb{N}^{2d}}$. It is by definition an isometric isomorphism from $L^2(\mathbb{R}^{2d})$ to $\ell^2(\mathbb{N}^{2d})$, with inverse

$$(A.10) \quad (P_H^{-1} \theta)(x, x', \lambda) = \sum_{(n, m) \in \mathbb{N}^{2d}} \theta(n, m, \lambda) H_{n, \lambda}(x) H_{m, \lambda}(x').$$

Obviously, arguing by density, Formula (A.7) may be extended to $L^2(\mathbb{H}^d)$. Therefore, according to Identities (A.8)–(A.10), and thanks to the classical Fourier-Plancherel theorem in \mathbb{R}^{d+1} , the Fourier transform $\mathcal{F}_{\mathbb{H}}$ may be seen as the composition of three invertible and bounded operators on L^2 , and we have

$$\mathcal{F}_{\mathbb{H}}^{-1} = 2^d \mathcal{F}_{\eta, s}^{-1} \circ \Phi^{-1} \circ P_H^{-1}.$$

This gives (1.17) and (1.18). For the proof of (1.19), we refer for instance to [1]. This concludes the proof of Theorem 1.1. \square

A.3. Properties related to the sub-ellipticity of $\Delta_{\mathbb{H}}$. Let k be a nonnegative integer. Then setting

$$\|u\|_{\dot{H}^k(\mathbb{H}^d)}^2 \stackrel{\text{def}}{=} \sum_{\alpha \in \{1, \dots, 2d\}^k} \|\mathcal{X}^\alpha u\|_{L^2}^2,$$

we have the following well-known result (see the proof in e.g. [21, 22]):

Theorem A.1. *For any positive integer ℓ , we have for some constant $C_\ell > 0$,*

$$\|\Delta_{\mathbb{H}}^\ell u\|_{L^2(\mathbb{H}^d)} \leq \|u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)} \leq C_\ell \|\Delta_{\mathbb{H}}^\ell u\|_{L^2(\mathbb{H}^d)}.$$

This will enable us to establish the following proposition which states that the usual semi-norms on the Schwartz class and the semi-norms using the structure of \mathbb{H}^d are equivalent.

Proposition A.1. *Let us introduce the notation*

$$(M_{\mathbb{H}}f)(X, s) \stackrel{\text{def}}{=} (|X|^2 - is)f(X, s).$$

Next, for all $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{2d})$ in \mathbb{N}^{1+2d} , we define

$$w^\alpha \stackrel{\text{def}}{=} s^{\alpha_0} y_1^{\alpha_1} \dots y_d^{\alpha_d} \eta_1^{\alpha_{d+1}} \dots \eta_d^{\alpha_{2d}} \quad \text{and} \quad \tilde{|\alpha|} \stackrel{\text{def}}{=} 2\alpha_0 + \alpha_1 + \dots + \alpha_{2d}.$$

Then the two families of semi-norms defined on $\mathcal{S}(\mathbb{H}^d)$ by

$$\begin{aligned} \|f\|_{p, \mathcal{S}(\mathbb{H}^d)}^2 &\stackrel{\text{def}}{=} \|f\|_{L^2}^2 + \|M_{\mathbb{H}}^p f\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^p f\|_{L^2}^2 \quad \text{and} \\ N_p^2(f) &\stackrel{\text{def}}{=} \sum_{\tilde{|\alpha|+|\beta| \leq p} } \|w^\alpha \mathcal{X}^\beta f\|_{L^2}^2 \end{aligned}$$

are equivalent to the classical family of semi-norms on $\mathcal{S}(\mathbb{R}^{2d+1})$.

Proof. As obviously $\|f\|_{p, \mathcal{S}(\mathbb{H}^d)} \leq N_{2p}(f)$, showing that the two families of semi-norms are equivalent reduces to proving that

$$(A.11) \quad \forall p \in \mathbb{N}, \exists (C_p, M_p) / \forall f \in \mathcal{S}(\mathbb{H}^d), N_p(f) \leq C_p \|f\|_{M_p, \mathcal{S}(\mathbb{H}^d)}.$$

Now, integrating by parts yields

$$\int_{\mathbb{H}^d} w^\alpha \mathcal{X}^\beta f(w) w^\alpha \mathcal{X}^\beta \bar{f}(w) dw = (-1)^{|\beta|} \int_{\mathbb{H}^d} f(w) \mathcal{X}^\beta (w^{2\alpha} \mathcal{X}^\beta \bar{f}(w)) dw.$$

Observe that $\mathcal{X}^\gamma w^{\gamma'}$ is either null or an homogeneous polynomial (with respect to the dilations (1.4)) of degree $\gamma' - \gamma$, and equal to 0 if the length of γ is greater than the length of γ' . Thus, thanks to Leibniz' rule, we have

$$(A.12) \quad [\mathcal{X}^\beta, w^{2\alpha}]f(w) = \sum_{\substack{\tilde{|\alpha'| \leq 2\tilde{|\alpha|} - 1 \\ |\beta'| \leq |\beta| - 1}} a_{\alpha, \alpha', \beta', \beta} w^{\alpha'} \mathcal{X}^{\beta'} f(w).$$

Hence we get that

$$\int_{\mathbb{H}^d} f(w) \mathcal{X}^\beta (w^{2\alpha} \mathcal{X}^\beta \bar{f}(w)) dw = \sum_{\substack{\tilde{|\alpha'| \leq 2\tilde{|\alpha|} \\ |\beta'| \leq |\beta|}} a_{\alpha, \alpha', \beta', \beta} \int_{\mathbb{H}^d} w^{\alpha'} f(w) \mathcal{X}^{\beta'} \mathcal{X}^\beta \bar{f}(w) dw.$$

Thanks to Cauchy-Schwarz inequality and by definition of $M_{\mathbb{H}}$, we get, applying Theorem A.1 and taking p large enough,

$$\sum_{\substack{|\alpha'| \leq 2|\alpha| \\ |\beta'| \leq |\beta|}} a_{\alpha, \alpha', \beta, \beta'} \int_{\mathbb{H}^d} w^{\alpha'} f(w) \mathcal{X}^{\beta'} \mathcal{X}^{\beta} \bar{f}(w) dw \leq C(\|f\|_{L^2}^2 + \|M_{\mathbb{H}}^p f\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^p f\|_{L^2}^2).$$

This proves that the two families of semi-norms in the above statement are equivalent.

In order to establish that they are also equivalent to the classical family, one can observe that for all j in $\{1, \dots, d\}$,

$$S = \frac{1}{4}[\Xi_j, \mathcal{X}_j], \quad \partial_{y_j} = \mathcal{X}_j - \frac{\eta_j}{2}(\Xi_j \mathcal{X}_j - \mathcal{X}_j \Xi_j) \quad \text{and} \quad \partial_{\eta_j} = \Xi_j + \frac{y_j}{2}(\Xi_j \mathcal{X}_j - \mathcal{X}_j \Xi_j),$$

from which we easily infer that

$$\|f\|_{p, \mathcal{S}(\mathbb{R}^{2d+1})} \leq CN_{2p}(f) \quad \text{with} \quad \|f\|_{p, \mathcal{S}(\mathbb{R}^{2d+1})}^2 \stackrel{\text{def}}{=} \sum_{|\alpha|+|\beta| \leq p} \|x^\alpha \partial^\beta f\|_{L^2(\mathbb{R}^{2d+1})}^2.$$

This ends the proof of the proposition. \square

A.4. Derivations and multiplication in the space $\mathcal{S}(\widehat{\mathbb{H}}^d)$. In Section 2, we only considered the effect of the Laplacian $\Delta_{\mathbb{H}}$ or of the derivation ∂_s on Fourier transform. Those operations led to multiplication by $-4|\lambda|(2|m|+d)$ or $i\lambda$, respectively, of the Fourier transform. We also studied the effect of the multiplication by $|Y|^2$ or $-is$, and found out that they correspond to the ‘derivation operators’ $\widehat{\Delta}$ and $\widehat{\mathcal{D}}_\lambda$ for functions on $\widehat{\mathbb{H}}^d$.

Our purpose here is to study the effect of left invariant differentiations \mathcal{X}_j and Ξ_j and multiplication by $M_j^\pm \stackrel{\text{def}}{=} y_j \pm i\eta_j$ on the Fourier transform. This is described by the following proposition.

Proposition A.2. *For any function f in $\mathcal{S}(\mathbb{H}^d)$, we have, for λ different from 0,*

$$\begin{aligned} \mathcal{F}_{\mathbb{H}} \mathcal{X}_j f &= -\widehat{\mathcal{M}}_j^+ \widehat{f}_{\mathbb{H}} \quad \text{and} \quad (\mathcal{F}_{\mathbb{H}} \Xi_j f) = -\widehat{\mathcal{M}}_j^- \widehat{f}_{\mathbb{H}} \quad \text{with} \\ \widehat{\mathcal{M}}_j^+ \theta(\widehat{w}) &\stackrel{\text{def}}{=} |\lambda|^{\frac{1}{2}} (\sqrt{2m_j+2} \theta(n, m+\delta_j, \lambda) - \sqrt{2m_j} \theta(n, m-\delta_j, \lambda)) \quad \text{and} \\ \widehat{\mathcal{M}}_j^- \theta(\widehat{w}) &\stackrel{\text{def}}{=} \frac{i\lambda}{|\lambda|^{\frac{1}{2}}} (\sqrt{2m_j+2} \theta(n, m+\delta_j, \lambda) + \sqrt{2m_j} \theta(n, m-\delta_j, \lambda)). \end{aligned}$$

We also have $\mathcal{F}_{\mathbb{H}} M_j^\pm f = \widehat{\mathcal{D}}_j^\pm \widehat{f}_{\mathbb{H}}$ with

$$\begin{aligned} (\widehat{\mathcal{D}}_j^\pm \theta)(\widehat{w}) &\stackrel{\text{def}}{=} \frac{1_{\{\pm\lambda > 0\}}}{2|\lambda|^{\frac{1}{2}}} (\sqrt{2n_j} \theta(n-\delta_j, m, \lambda) - \sqrt{2m_j+2} \theta(n, m+\delta_j, \lambda)) \\ &\quad + \frac{1_{\{\pm\lambda < 0\}}}{2|\lambda|^{\frac{1}{2}}} (\sqrt{2n_j+2} \theta(n+\delta_j, m, \lambda) - \sqrt{2m_j} \theta(n, m-\delta_j, \lambda)). \end{aligned}$$

Proof. The main point is to compute

$$\partial_{y_j} \mathcal{W}(\widehat{w}, Y), \partial_{\eta_j} \mathcal{W}(\widehat{w}, Y), y_j \mathcal{W}(\widehat{w}, Y) \quad \text{and} \quad \eta_j \mathcal{W}(\widehat{w}, Y).$$

By the definition of \mathcal{W} and Leibniz formula, we have, using the notation $f_\lambda(x) \stackrel{\text{def}}{=} f(|\lambda|^{1/2}x)$,

$$\partial_{y_j} \mathcal{W}(\widehat{w}, Y) = \int_{\mathbb{R}^d} e^{2i\lambda \langle z, \eta \rangle} |\lambda|^{\frac{1}{2}} ((\partial_j H_n)_\lambda(y+z) H_{m,\lambda}(-y+z) - H_{n,\lambda}(y+z) (\partial_j H_m)_\lambda(-y+z)) dz.$$

From (A.4), we infer that

$$(A.13) \quad \partial_{y_j} \mathcal{W}(\widehat{w}, Y) = \frac{|\lambda|^{\frac{1}{2}}}{2} (\sqrt{2n_j} \mathcal{W}(n - \delta_j, m, \lambda, Y)) - \sqrt{2n_j + 2} \mathcal{W}(n + \delta_j, m, \lambda, Y) \\ - \sqrt{2m_j} \mathcal{W}(n, m - \delta_j, \lambda, Y) + \sqrt{2m_j + 2} \mathcal{W}(n, m + \delta_j, \lambda, Y).$$

Let us observe that

$$\begin{aligned} \partial_{\eta_j} \mathcal{W}(\widehat{w}, Y) &= \int_{\mathbb{R}^d} 2i\lambda z_j e^{2i\lambda\langle \eta, z \rangle} H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) dz \\ &= i\lambda \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} (y_j + z_j) H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) \\ &\quad + H_{n,\lambda}(y+z)(-y_j + z_j) H_{m,\lambda}(-y+z) dz \\ &= \frac{i\lambda}{|\lambda|^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} ((M_j H_n)_\lambda(y+z) H_{m,\lambda}(-y+z) \\ &\quad + H_{n,\lambda}(y+z)(M_j H_m)_\lambda(-y+z)) dz. \end{aligned}$$

Now, using again (A.4), we get

$$(A.14) \quad \partial_{\eta_j} \mathcal{W}(\widehat{w}, Y) = \frac{i\lambda}{2|\lambda|^{\frac{1}{2}}} (\sqrt{2n_j} \mathcal{W}(n - \delta_j, m, \lambda, Y)) + \sqrt{2n_j + 2} \mathcal{W}(n + \delta_j, m, \lambda, Y) \\ + \sqrt{2m_j} \mathcal{W}(n, m - \delta_j, \lambda, Y) + \sqrt{2m_j + 2} \mathcal{W}(n, m + \delta_j, \lambda, Y).$$

For multiplication by y_j , we proceed along the same lines. By definition of \mathcal{W} , we have

$$y_j \mathcal{W}(\widehat{w}, Y) = \frac{1}{2|\lambda|^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} ((M_j H_n)_\lambda(y+z) H_{m,\lambda}(-y+z) - H_{n,\lambda}(y+z)(M_j H_m)_\lambda(-y+z)) dz.$$

Still using (A.4), we deduce that

$$(A.15) \quad y_j \mathcal{W}(\widehat{w}, Y) = \frac{1}{4|\lambda|^{\frac{1}{2}}} (\sqrt{2n_j} \mathcal{W}(n - \delta_j, m, \lambda, Y)) + \sqrt{2n_j + 2} \mathcal{W}(n + \delta_j, m, \lambda, Y) \\ - \sqrt{2m_j} \mathcal{W}(n, m - \delta_j, \lambda, Y) - \sqrt{2m_j + 2} \mathcal{W}(n, m + \delta_j, \lambda, Y).$$

For the multiplication by η_j , let us observe that, performing an integration by parts, we can write

$$\begin{aligned} \eta_j \mathcal{W}(\widehat{w}, Y) &= \frac{1}{2i\lambda} \int_{\mathbb{R}^d} \partial_{z_j} (e^{2i\lambda\langle \eta, z \rangle}) H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) dz \\ &= \frac{i}{2\lambda} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} \partial_{z_j} (H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z)) dz. \end{aligned}$$

Leibniz formula implies that

$$\begin{aligned} \eta_j \mathcal{W}(\widehat{w}, Y) &= \frac{i|\lambda|^{\frac{1}{2}}}{2\lambda} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} ((\partial_j H_n)_\lambda(y+z) H_{m,\lambda}(-y+z) \\ &\quad + H_{n,\lambda}(y+z)(\partial_j H_m)_\lambda(-y+z)) dz. \end{aligned}$$

Using (A.4), we deduce that

$$(A.16) \quad \eta_j \mathcal{W}(\widehat{w}, Y) = \frac{i|\lambda|^{\frac{1}{2}}}{4\lambda} (\sqrt{2n_j} \mathcal{W}(n - \delta_j, m, \lambda, Y)) - \sqrt{2n_j + 2} \mathcal{W}(n + \delta_j, m, \lambda, Y) \\ + \sqrt{2m_j} \mathcal{W}(n, m - \delta_j, \lambda, Y) - \sqrt{2m_j + 2} \mathcal{W}(n, m + \delta_j, \lambda, Y).$$

As we have $e^{-is\lambda}\mathcal{X}_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) = 2i\eta_j\lambda\mathcal{W}(\widehat{w}, Y) + \partial_{y_j}\mathcal{W}(\widehat{w}, Y)$, we infer from (A.13) and (A.16) that

$$(A.17) \quad \begin{aligned} e^{-is\lambda}\mathcal{X}_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) &= |\lambda|^{\frac{1}{2}}(-\sqrt{2m_j}\mathcal{W}(n, m - \delta_j, \lambda, Y) \\ &\quad + \sqrt{2m_j + 2}\mathcal{W}(n, m + \delta_j, \lambda, Y)) \\ &= \mathcal{M}_j^+\mathcal{W}(\widehat{w}, Y). \end{aligned}$$

As we have

$$e^{-is\lambda}\Xi_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) = -2iy_j\lambda\mathcal{W}(\widehat{w}, Y) + \partial_{\eta_j}\mathcal{W}(\widehat{w}, Y),$$

we infer from (A.14) and (A.15) that

$$(A.18) \quad \begin{aligned} e^{-is\lambda}\Xi_j(e^{is\lambda}\mathcal{W}(\widehat{w}, Y)) &= \frac{i\lambda}{|\lambda|^{\frac{1}{2}}}(\sqrt{2m_j}\mathcal{W}(n, m - \delta_j, \lambda, Y) \\ &\quad + \sqrt{2m_j + 2}\mathcal{W}(n, m + \delta_j, \lambda, Y)) \\ &= \mathcal{M}_j^-\mathcal{W}(\widehat{w}, Y). \end{aligned}$$

It is obvious that (A.15) and (A.16) give

$$\begin{aligned} (y_j \pm i\eta_j)\mathcal{W}(\widehat{w}, Y) &= \frac{1}{4|\lambda|^{\frac{1}{2}}}\left(\sqrt{2n_j}\mathcal{W}(n - \delta_j, m, \lambda, Y) + \sqrt{2n_j + 2}\mathcal{W}(n + \delta_j, m, \lambda, Y) \right. \\ &\quad \left. - \sqrt{2m_j}\mathcal{W}(n, m - \delta_j, \lambda, Y) - \sqrt{2m_j + 2}\mathcal{W}(n, m + \delta_j, \lambda, Y) \right. \\ &\quad \left. \pm \operatorname{sgn}(\lambda)(\sqrt{2n_j}\mathcal{W}(n - \delta_j, m, \lambda, Y) - \sqrt{2n_j + 2}\mathcal{W}(n + \delta_j, m, \lambda, Y) \right. \\ &\quad \left. + \sqrt{2m_j}\mathcal{W}(n, m - \delta_j, \lambda, Y) - \sqrt{2m_j + 2}\mathcal{W}(n, m + \delta_j, \lambda, Y))\right). \end{aligned}$$

By definition of $\mathcal{F}_{\mathbb{H}}$, this gives the result. \square

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