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▶ **To cite this version:**

François Bouchut, Xavier Lhébrard. Convergence of the the kinetic hydrostatic reconstruction scheme for the Saint Venant system with topography. 2019. hal-01515256v2

HAL Id: hal-01515256
https://hal.science/hal-01515256v2

Preprint submitted on 6 Sep 2019 (v2), last revised 4 Sep 2020 (v3)

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Convergence of the the kinetic hydrostatic reconstruction scheme for the Saint Venant system with topography

François Bouchut*, Xavier Lhébrard*

Abstract

We prove the convergence of the hydrostatic reconstruction scheme with kinetic numerical flux for the Saint Venant system with continuous topography with locally integrable derivative. We use a recently derived fully discrete sharp entropy inequality with dissipation, that enables us to establish an estimate in the inverse of the square root of the space increment Δx of the L^2 norm of the gradient of approximate solutions. By Diperna's method we conclude the strong convergence towards bounded weak entropy solutions.

Keywords: Saint Venant system with topography, well-balanced scheme, hydrostatic reconstruction, convergence, entropy inequality, kinetic function.

Mathematics Subject Classification: 65M12, 76M12, 35L65

1 Introduction and main result

We consider the Saint Venant system

$$\begin{aligned} &\partial_t h + \partial_x(hu) = 0, \\ &\partial_t(hu) + \partial_x(hu^2 + g\frac{h^2}{2}) + gh\partial_x z = 0, \end{aligned} \tag{1.1}$$

for $t \geq 0$ and $x \in \mathbb{R}$, where the unknowns are $h(t,x) \geq 0$ and $u(t,x) \in \mathbb{R}$, $g > 0$ is the gravity constant, and the topography $z(x)$ is given. The system

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is completed with an entropy (energy) inequality

$$\partial_t \left(h\frac{u^2}{2} + g\frac{h^2}{2} + ghz \right) + \partial_x \left(\big(h\frac{u^2}{2} + gh^2 + ghz\big)u \right) \leq 0. \qquad (1.2)$$

We shall denote $U = (h, hu)$ with $h \geq 0$, and

$$\eta(U) = h\frac{u^2}{2} + g\frac{h^2}{2}, \quad G(U) = \Big(h\frac{u^2}{2} + gh^2\Big)u, \qquad (1.3)$$

the entropy and entropy fluxes without topography.

Existence and stability results for the shallow water system have been established in [27, 21, 23, 28]. Concerning approximation, many schemes have been proposed, see for example [25, 4, 3, 6, 12, 7, 11, 19, 20]. The hydrostatic reconstruction scheme and its variants [4, 24, 18, 17, 16, 15] is often used, and it is the subject of the present paper. Some results concerning consistency, stability and convergence of those schemes have been obtained in [9, 10, 13, 26, 8, 1, 2].

In this paper we prove the convergence of the hydrostatic reconstruction scheme [4] with kinetic flux [25]. Our result strongly uses the work [5], which establishes that the hydrostatic reconstruction scheme, used with the classical kinetic solver, satisfies a fully discrete entropy inequality with an error term. In the case without topography, the error term vanishes and under a CFL condition we have the inequality

$$\begin{aligned}\eta(U_i^{n+1}) \leq \eta(U_i) - \frac{\Delta t}{\Delta x}\Big(\widetilde{G}_{i+1/2} - \widetilde{G}_{i-1/2}\Big) \\ - \nu_\beta \frac{\Delta t}{\Delta x}\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6}\Big(\mathbb{1}_{\xi<0}\,(M_{i+1}+M_i)\,(M_{i+1}-M_i)^2 \\ + \mathbb{1}_{\xi>0}\,(M_i+M_{i-1})\,(M_i-M_{i-1})^2\Big) d\xi, \quad (1.4)\end{aligned}$$

where $\widetilde{G}_{i+1/2}$ is a numerical entropy flux, and where the dissipation term (the integral in ξ) will be explained further on, see (2.16). In the (time and space continuous) kinetic BGK case and without topography, the single energy inequality ensures the convergence [10]. The fully-discrete case (still without topography) is treated in [8] (a related work is [22]) and the convergence is established under a dissipation assumption, that F^+ or $-F^-$ (defined in (1.27)) is strictly η-dissipative. The η-dissipativity has been defined in [13], and analyzed in the multi-d context in [14]. Unfortunately this property that F^+ or $-F^-$ is strictly η-dissipative does not hold for the scheme we consider, there is a lack of dissipativity of the kinetic scheme. In the present work

we are able to bypass this difficulty and establish the convergence, which is overall obtained in the case with topography, using the weak dissipation estimate (2.16). To include the topography is not an easy task, even if it is Lipschitz continuous. Indeed a typical error term produced by the scheme, corresponding to the topography term in (1.1), is $(h_{i+1} - h_i)(z_{i+1} - z_i)/\Delta x$. This quantity is not small a priori, since h is not continuous. In order to make this small one would need the compactness of h, that we have to prove. A key point is to use the dissipation of the discrete form of the entropy inequality (1.2). Notice that a discrete entropy inequality that puts the topography as a source term as formulated in [11] would not be sufficient because of the eventual presence of shocks.

Let us give some of the main steps of our proof. The first thing is to have a weak dissipation property. We use that $F^+ - F^-$ is strictly η-dissipative, which corresponds to the inequality

$$\begin{aligned} &\int_{\mathbb{R}} |\xi| \left(H_0(M_2) - H_0(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \\ &\geq \alpha \left(\eta(U_2) - \eta(U_1) - \eta'(U_1) (U_2 - U_1) \right), \end{aligned} \tag{1.5}$$

for some $\alpha > 0$. A rigorous statement of this result can be found in Lemma A.3. We have to point out however that this estimate is only valid in a closed bounded convex set which does not contain vanishing heights, and the constant α is not obtained explicitly.

In order to get a priori estimates we sum up over the space and time indices i and n the discrete entropy inequality. Then we are able to use (1.5), and as a consequence we get gradient estimates of the form

$$\|\partial_t U_\Delta\|_{L^2_{tx}} \leq \frac{C}{\sqrt{\Delta x}}, \quad \|\partial_x U_\Delta\|_{L^2_{tx}} \leq \frac{C}{\sqrt{\Delta x}}, \tag{1.6}$$

where U_Δ is the numerical approximate solution. We conclude as in [8] by a compensated compactness argument. Indeed we recall that the compensated compactness theory [27, 23] gives the compactness of a bounded sequence of approximate solutions (U_ε) which satisfy that

$$\partial_t \eta_S(U_\varepsilon) + \partial_x G_S(U_\varepsilon) \quad \text{is compact in } H^{-1}_{loc}, \tag{1.7}$$

for a sufficiently large family of entropies η_S. According to the classical DiPerna approach [21], the estimates (1.6) are enough to establish (1.7) for all entropies.

1.1 Kinetic Maxwellian equilibrium

We recall here the classical kinetic Maxwellian equilibrium, used in [25] for example. It is given by

$$M(U,\xi) = \frac{1}{g\pi}\left(2gh - (\xi-u)^2\right)_+^{1/2}, \tag{1.8}$$

where $U = (h, hu)$, $\xi \in \mathbb{R}$ and $x_+ \equiv \max(0,x)$ for any $x \in \mathbb{R}$. It satisfies the moment relations

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi)d\xi = U, \qquad \int_{\mathbb{R}} \xi^2 M(U,\xi)d\xi = hu^2 + g\frac{h^2}{2}. \tag{1.9}$$

The interest of this particular form lies in its compatibility with a kinetic entropy given by

$$H(f,\xi,z) = \frac{\xi^2}{2}f + \frac{g^2\pi^2}{6}f^3 + gzf, \tag{1.10}$$

where $f \geq 0$, $\xi \in \mathbb{R}$ and $z \in \mathbb{R}$, and its version without topography

$$H_0(f,\xi) = \frac{\xi^2}{2}f + \frac{g^2\pi^2}{6}f^3. \tag{1.11}$$

Then one can check the relations

$$\int_{\mathbb{R}} H(M(U,\xi),\xi,z)d\xi = \eta(U) + ghz, \tag{1.12}$$

$$\int_{\mathbb{R}} \xi H(M(U,\xi),\xi,z)d\xi = G(U) + ghzu, \tag{1.13}$$

where η and G are given by (1.3). Moreover, for any function $f(\xi) \geq 0$, setting $h = \int f(\xi)d\xi$, $hu = \int \xi f(\xi)d\xi$ (assumed finite), one has the following entropy minimization principle [5],

$$\eta(U) = \int_{\mathbb{R}} H_0(M(U,\xi),\xi)d\xi \leq \int_{\mathbb{R}} H_0(f(\xi),\xi)d\xi. \tag{1.14}$$

Indeed this inequality is strongly related to the property (see (1.19) in [5])

$$\partial_f H_0\big(M(U,\xi),\xi\big) = \begin{cases} \eta'(U)\begin{pmatrix} 1 \\ \xi \end{pmatrix} & \text{if } M(U,\xi) > 0, \\ \geq \eta'(U)\begin{pmatrix} 1 \\ \xi \end{pmatrix} & \text{if } M(U,\xi) = 0. \end{cases} \tag{1.15}$$

Here $\eta'(U)$ denotes the derivative of η with respect to U,

$$\eta'(U) = (gh - u^2/2, u). \tag{1.16}$$

1.2 Hydrostatic reconstruction and kinetic flux

We consider a uniform grid $(x_{i+1/2})_{i\in\mathbb{Z}}$ with space increment $\Delta x = x_{i+1/2} - x_{i-1/2}$, and discrete times t_n with a constant timestep Δt, $t_{n+1} - t_n = \Delta t$, $t_0 = 0$. We consider initial data $U^0 = (h^0, h^0u^0)$, $h^0 \geq 0$, $h^0, u^0 \in L^\infty(\mathbb{R})$ and a topography $z(x)$ assumed continuous . We define the discretization of the initial data as

$$U_i^0 = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U^0(y)dy, \tag{1.17}$$

and

$$z_i \text{ an approximation of } z(x_i), \tag{1.18}$$

where $x_i = \left(x_{i+1/2} + x_{i-1/2}\right)/2$. The hydrostatic reconstruction scheme writes [4]

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2-} - F_{i-1/2+}\right), \tag{1.19}$$

with

$$F_{i+1/2-} = \mathcal{F}(U_{i+1/2-}, U_{i+1/2+}) - S_{i+1/2-}, \tag{1.20}$$

$$F_{i+1/2+} = \mathcal{F}(U_{i+1/2-}, U_{i+1/2+}) + S_{i+1/2+}, \tag{1.21}$$

with $\mathcal{F}$ is a numerical flux for the system without topography. The source terms $S_{i+1/2-}$, $S_{i+1/2+}$ are defined by

$$S_{i+1/2-} = \begin{pmatrix} 0 \\ \frac{g}{2}h_{i+1/2-}^2 - \frac{g}{2}h_i^2 \end{pmatrix}, \quad S_{i+1/2+} = \begin{pmatrix} 0 \\ \frac{g}{2}h_{i+1}^2 - \frac{g}{2}h_{i+1/2+}^2 \end{pmatrix}. \tag{1.22}$$

The reconstructed states

$$U_{i+1/2-} = (h_{i+1/2-}, h_{i+1/2-}u_i), \quad U_{i+1/2+} = (h_{i+1/2+}, h_{i+1/2+}u_{i+1}) \tag{1.23}$$

are defined by

$$h_{i+1/2-} = (h_i + z_i - z_{i+1/2})_+, \quad h_{i+1/2+} = (h_{i+1} + z_{i+1} - z_{i+1/2})_+ \tag{1.24}$$

and

$$z_{i+1/2} = \max(z_i, z_{i+1}). \tag{1.25}$$

The hydrostatic reconstruction scheme is defined for arbitrary numerical flux $\mathcal{F}$, but in the present paper we are only able to analyze the kinetic flux vector splitting given by

$$\mathcal{F}(U_l, U_r) = F^+(U_l) + F^-(U_r), \tag{1.26}$$

$$F^+(U) = \int_{\mathbb{R}} \xi \mathbb{1}_{\xi>0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi, \tag{1.27}$$
$$F^-(U) = \int_{\mathbb{R}} \xi \mathbb{1}_{\xi<0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi,$$

with $M(U,\xi)$ defined by (1.8).

We consider a velocity $v_m \geq 0$ such that for all i,

$$M(U_i,\xi) > 0 \Rightarrow |\xi| \leq v_m. \tag{1.28}$$

This means equivalently that $|u_i| + \sqrt{2gh_i} \leq v_m$. We consider a CFL condition strictly less than one,

$$v_m \frac{\Delta t}{\Delta x} \leq \beta < 1, \tag{1.29}$$

where β is a constant.

An estimate that will be useful later on is that with the definitions (1.23)-(1.25) one has

$$0 \leq h_i - h_{i+1/2-} \leq |z_{i+1} - z_i|, \tag{1.30}$$
$$0 \leq h_i - h_{i-1/2+} \leq |z_i - z_{i-1}|. \tag{1.31}$$

1.3 Convergence result

Let (U_i^n, z_i) be defined by the scheme (1.17)-(1.27). We define the approximate solution by

$$\begin{aligned} &U_\Delta(t,x) \\ &= \frac{1}{\Delta t}\left[\frac{U_{i+1}^{n+1} - U_i^{n+1} - U_{i+1}^n + U_i^n}{\Delta x}(x - x_i) + U_i^{n+1} - U_i^n\right](t - t_n) \\ &\qquad + \frac{U_{i+1}^n - U_i^n}{\Delta x}(x - x_i) + U_i^n, \\ &\text{for } x_i < x < x_{i+1} \text{ and } t_n \leq t < t_{n+1}, \end{aligned} \tag{1.32}$$

and we set

$$z_\Delta(x) = \frac{z_{i+1} - z_i}{\Delta x}(x - x_i) + z_i, \quad \text{for } x_i < x < x_{i+1}. \tag{1.33}$$

In this way U_Δ and z_Δ are continuous. We shall assume that z is continuous and bounded with L^1_{loc} derivative, and that the values z_i are well chosen, so that as $\Delta x \to 0$

$$z_\Delta \longrightarrow z \text{ locally uniformly in } \mathbb{R}, \qquad \frac{\mathrm{d}z_\Delta}{\mathrm{d}x} \longrightarrow \frac{\mathrm{d}z}{\mathrm{d}x} \text{ in } L^1_{loc}(\mathbb{R}), \tag{1.34}$$

and for any bounded interval $[a,b]$,

$$TV2_{[a,b]}((z_i)) \to 0, \tag{1.35}$$

where $TV2_{[a,b]}((z_i))$ is defined as

$$TV2_{[a,b]}((z_i)) \equiv \sum_{[x_i,x_{i+1}]\subset[a,b]} (z_{i+1}-z_i)^2. \tag{1.36}$$

The properties (1.34) and (1.35) hold in particular for the choice $z_i = z(x_i)$, see Lemma A.1.

Moreover, for $0 < h_m < h_M$ and $u_M > 0$, we set

$$\mathcal{U}_{h_m,h_M,u_M} = \{(h,hu) \in \mathbb{R}^2, \quad h_m \le h \le h_M, \quad |u| \le u_M\}, \tag{1.37}$$

which is a convex set. We state now the main result of this article, which is the convergence of the hydrostatic reconstruction scheme with kinetic numerical flux.

Theorem 1.1. *Let $U^0 = (h^0, h^0u^0)$, $h^0 \ge 0$, $h^0, u^0 \in L^\infty(\mathbb{R})$, be an initial data and let z be a continuous and bounded given topography satisfying $\partial_x z \in L^1_{loc}$. Define (U_i^n, z_i) by the scheme* (1.17)-(1.27)*, the approximate solution U_Δ by* (1.32) *and the approximate topography z_Δ by* (1.33)*. We assume that the values z_i are well chosen, i.e. satisfy* (1.34), (1.35)*. Then we assume to have uniform bounds far from the vacuum,*

$$\forall i,n, \quad U_i^n \in \mathcal{U}_{h_m,h_M,u_M}, \tag{1.38}$$

for some $0 < h_m < h_M$, $u_M > 0$, with $\mathcal{U}_{h_m,h_M,u_M}$ defined by (1.37)*.*

Then, under the CFL condition (1.29) *and the inverse CFL condition*

$$1 \le v^* \frac{\Delta t}{\Delta x}, \tag{1.39}$$

for some $v^* > 0$, we have that up to a subsequence, $U_\Delta \to U$ a.e. in $(0,T)\times\mathbb{R}$ and in $C_t([0,T], L^\infty_{w*}(\mathbb{R}))$ as $\Delta t \to 0$ and $\Delta x \to 0$, where U is a weak solution to (1.1) *with initial data U^0, that satisfies the entropy inequality* (1.2)*, and the family of entropy regularity conditions*

$$\partial_t \eta_S(U) + \partial_x G_S(U) \in \mathcal{M}_{loc}, \tag{1.40}$$

for all suitable couples entropy-entropy flux (η_S, G_S).

Some comments on this theorem are in order. At first, a main assumption is the boundedness away from vacuum (1.38). We are not able to treat the vacuum at the present time. Also, to have L^∞ bounds is not guaranteed a priori, since only L^2 type bounds are available, obtained by integration in time and space of the discrete entropy inequality. Indeed, L^∞ bounds can only be proved in the context of having a large family of entropy inequalities, as in [9], while here we have only one. Note that the bound (1.28) involving v_m can be seen as a consequence of the upper bounds h_M, u_M involved in (1.38). The inverse CFL condition (1.39) is a technical assumption that ensures the finite speed of propagation: since the information propagates of at most one cell per timestep, this condition ensures that the domain of dependency remains bounded as Δt and Δx tend to 0. Another main assumption is the continuity of the topography. A discontinuous topography is not allowed, indeed in that case it is known that several severe difficulties arise, in particular one has non-uniqueness of solutions to the Riemann problem. Numerical issues in this situation of discontinuous topography are studied in [3, 29]. We overall assume that the topography has a locally integrable derivative, but this is a minimal assumption that enables to give sense to the product $h\partial_x z$ in (1.1).

We have to mention that the boundedness away from vacuum assumption (1.38) allows to bound also the reconstructed states $U_{i+1/2\pm}$. Indeed according to (1.23)-(1.25) one has

$$h_i-(z_{i+1}-z_i)_+ \leq h_{i+1/2-} \leq h_i, \quad h_{i+1}-(z_i-z_{i+1})_+ \leq h_{i+1/2+} \leq h_{i+1}, \tag{1.41}$$

thus $U_{i+1/2\pm} \in \mathcal{U}_{\tilde{h}_m,h_M,u_M}$, where $\tilde{h}_m$ is such that

$$0 < \tilde{h}_m \leq h_m - \sup_i |z_{i+1} - z_i|, \tag{1.42}$$

which is possible since $\sup_i |z_{i+1} - z_i| \to 0$ and thus it is lower than h_m for Δx small enough.

The outline of the paper is as follows. In Section 2 we establish estimates on the gradient of the approximate solution as stated in (1.6). In Section 3 we prove some interpolation estimates. In Section 4 we finally prove Theorem 1.1. We obtain (1.7) by combining the gradient estimate and the interpolation estimate, then we apply compensated compactness. An appendix is devoted to the proof of various technical lemmas.

2 Estimate of the gradient of the approximate solution

This section is devoted to the proof of the following estimate on the approximate solution.

Proposition 2.1. *With the assumptions of Theorem 1.1, we define for all* $U = (h, hu)$,

$$|U|^2 = h^2 + \frac{u^2 h^2}{g h_M}. \tag{2.1}$$

Let $N \in \mathbb{N}^*$, $T = N\Delta t$, i_0, $i_1 \in \mathbb{Z}$ *such that* $i_0 \leq i_1$. *For all* $i \leq j \in \mathbb{Z}$, *we define the interval*

$$I_{i,j}^{v^*} = (x_{i-1/2} - v^* T, x_{j+1/2} + v^* T). \tag{2.2}$$

Then there exists some constants C_1, C_2, C_3 *such that*

$$\sum_{n=0}^{N} \sum_{i=i_0}^{i_1-1} \Delta t |U_{i+1}^n - U_i^n|^2 \leq C_1, \tag{2.3}$$

$$\sum_{n=0}^{N-1} \sum_{i=i_0}^{i_1} \Delta t |U_i^{n+1} - U_i^n|^2 \leq C_1 \frac{\Delta t^2}{\Delta x^2} v_m^2, \tag{2.4}$$

$$\left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |\partial_x U_\Delta|^2 dx dt \right)^{1/2} \leq \frac{C_2}{\sqrt{\Delta x}}, \tag{2.5}$$

$$\left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |\partial_t U_\Delta|^2 dx dt \right)^{1/2} \leq \frac{C_3}{\sqrt{\Delta x}}. \tag{2.6}$$

The constants C_1, C_2, C_3 *depend only on* g, h_m, h_M, u_M, v_m, β, *the final time* T, $\|z\|_{L^\infty}$, $TV2_{I_{i_0,i_1}^{v^*}}((z_i))$, $\|\eta(U^0)\|_{L^1(I_{i_0,i_1}^{v^*})}$ *and* $\|h^0\|_{L^1(I_{i_0,i_1}^{v^*})}$.

The proof of this proposition is given below in the remainder of this section. These estimates on $\partial_t U_\Delta$ and $\partial_x U_\Delta$ use recent results on discrete kinetic inequalities established in [5]. In Subsection 2.3 we use several technical lemmas which are put in the appendix for the sake of clarity of the presentation.

2.1 Estimate of bounded propagation for the space integral of the height

We here establish some bound on $\sum_{i=i_0}^{i_1} \Delta x h_i^N$. We have

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^h - F_{i-1/2}^h\right), \tag{2.7}$$

with

$$F_{i+1/2}^h = \int_{\mathbb{R}} \xi \mathbb{1}_{\xi>0} M(U_{i+1/2-}^n, \xi) d\xi + \int_{\mathbb{R}} \xi \mathbb{1}_{\xi<0} M(U_{i+1/2+}^n, \xi) d\xi. \tag{2.8}$$

We recall that under the CFL condition (1.29) one has $h_i^{n+1} \geq 0$, see [5]. We multiply by Δx and sum over index i and we obtain

$$\sum_{i=i_0}^{i_1} \Delta x h_i^{n+1} = \sum_{i=i_0}^{i_1} \Delta x h_i^n - \Delta t \left(F_{i_1+1/2}^h - F_{i_0-1/2}^h\right). \tag{2.9}$$

Then we notice that with (1.28) and (1.41)

$$-F_{i_1+1/2}^h \leq v_m h_{i_1+1/2+}^n \leq v_m h_{i_1+1}^n, \quad F_{i_0-1/2}^h \leq v_m h_{i_0-1/2-}^n \leq v_m h_{i_0-1}^n. \tag{2.10}$$

With the CFL condition (1.29) we obtain

$$\sum_{i=i_0}^{i_1} \Delta x h_i^{n+1} \leq \sum_{i=i_0-1}^{i_1+1} \Delta x h_i^n. \tag{2.11}$$

Denoting $T = N\Delta t$, using the previous inequality and (1.17) we get

$$\sum_{i=i_0}^{i_1} \Delta x h_i^N \leq \sum_{i=i_0-N}^{i_1+N} \Delta x h_i^0 = \int_{x_{i_0-N-1/2}}^{x_{i_1+N+1/2}} h^0(x) dx. \tag{2.12}$$

Moreover we have

$$x_{i_0-N-1/2} = x_{i_0-1/2} - N\Delta x = x_{i_0-1/2} - T\frac{\Delta x}{\Delta t}, \tag{2.13}$$

$$x_{i_1+N+1/2} = x_{i_1+1/2} + N\Delta x = x_{i_1+1/2} + T\frac{\Delta x}{\Delta t}. \tag{2.14}$$

Therefore using the inverse CFL condition (1.39) we get

$$\sum_{i=i_0}^{i_1} \Delta x h_i^N \leq \int_{x_{i_0-1/2}-Tv^*}^{x_{i_1+1/2}+Tv^*} h^0(x) dx = \|h^0\|_{L^1(I_{i_0,i_1}^{v^*})}, \tag{2.15}$$

with $I_{i_0,i_1}^{v^*}$ defined in (2.2).

2.2 From kinetic to macroscopic discrete entropy inequality

We use the notations introduced in Proposition 2.1. Under the CFL condition (1.29) we can integrate with respect to ξ the kinetic entropy inequality of [5, Theorem 3.6] as in [5, Corollary 3.7], and we obtain

$$\begin{aligned}\eta(U_i^{n+1}) + gz_i h_i^{n+1} \leq \eta(U_i) + gz_i h_i - \frac{\Delta t}{\Delta x}\left(\widetilde{G}_{i+1/2} - \widetilde{G}_{i-1/2}\right) \\ - \nu_\beta \frac{\Delta t}{\Delta x}\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6}\Big(\mathbb{1}_{\xi<0}\left(M_{i+1/2+} + M_{i+1/2-}\right)\left(M_{i+1/2+} - M_{i+1/2-}\right)^2 \\ + \mathbb{1}_{\xi>0}\left(M_{i-1/2+} + M_{i-1/2-}\right)\left(M_{i-1/2+} - M_{i-1/2-}\right)^2\Big) d\xi \\ + C_\beta \left(\frac{\Delta t}{\Delta x} v_m\right)^2 g\left(|z_{i+1} - z_i|^2 + |z_i - z_{i-1}|^2\right), \quad (2.16)\end{aligned}$$

with

$$\widetilde{G}_{i+1/2} = \int_{\xi<0} \xi H(M_{i+1/2+}, \xi, z_{i+1/2}) d\xi + \int_{\xi>0} \xi H(M_{i+1/2-}, \xi, z_{i+1/2}) d\xi, \quad (2.17)$$

$\nu_\beta > 0$ is a dissipation constant depending only on β, and $C_\beta \geq 0$ is a constant depending only on β. We use here the shorthand notation $M_i \equiv M(U_i, \xi)$, $M_{i+1/2+} \equiv M(U_{i+1/2+}, \xi)$, $M_{i+1/2-} \equiv M(U_{i+1/2-}, \xi)$.

Then we follow the computations done on the height in Subsection 2.1. We multiply by Δx, take the sum over i and obtain

$$\begin{aligned}\sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^{n+1}) + gz_i h_i^{n+1}\right) \leq \sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i) + \ gz_i h_i\right) \\ - \Delta t \widetilde{G}_{i_1+1/2} + \Delta t \widetilde{G}_{i_0-1/2} \\ - \nu_\beta \Delta t \sum_{i=i_0}^{i_1-1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6}\left(M_{i+1/2+} + M_{i+1/2-}\right)\left(M_{i+1/2+} - M_{i+1/2-}\right)^2 d\xi \\ - \nu_\beta \Delta t \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \mathbb{1}_{\xi<0}\left(M_{i_1+1/2+} + M_{i_1+1/2-}\right)\left(M_{i_1+1/2+} - M_{i_1+1/2-}\right)^2 d\xi \\ - \nu_\beta \Delta t \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \mathbb{1}_{\xi>0}\left(M_{i_0-1/2+} + M_{i_0-1/2-}\right)\left(M_{i_0-1/2+} - M_{i_0-1/2-}\right)^2 d\xi \\ + 2C_\beta \frac{\Delta t^2}{\Delta x} v_m^2 g \sum_{i=i_0-1}^{i_1} |z_{i+1} - z_i|^2. \quad (2.18)\end{aligned}$$

Since z is bounded, adding if necessary a sufficiently large constant to it, we can assume that $z \geq 0$. Then we notice that according to (2.17) we have

$$-\widetilde{G}_{i_1+1/2} \leq v_m \eta(U_{i_1+1/2+}) + v_m g h_{i_1+1/2+} z_{i_1+1/2}, \tag{2.19}$$

and

$$\widetilde{G}_{i_0-1/2} \leq v_m \eta(U_{i_0-1/2-}) + v_m g h_{i_0-1/2-} z_{i_0-1/2}. \tag{2.20}$$

According to the definitions (1.3) of η and (1.23), (1.24) of the reconstructed states, and to the inequalities (1.41), one has

$$\begin{aligned} &\eta(U_{i+1/2+}) + g h_{i+1/2+} z_{i+1/2} \\ &\leq \eta(U_{i+1}) + g\frac{h_{i+1/2+}^2}{2} - g\frac{h_{i+1}^2}{2} + g h_{i+1/2+} z_{i+1/2} \\ &\leq \eta(U_{i+1}) + g h_{i+1} z_{i+1}. \end{aligned} \tag{2.21}$$

Indeed, to get the last inequality, we observe that since $h_{i+1/2+} = (h_{i+1} + z_{i+1} - z_{i+1/2})_+$, either $h_{i+1} + z_{i+1} - z_{i+1/2} \leq 0$, which implies that $h_{i+1/2+} = 0$ and the desired inequality, or $h_{i+1} + z_{i+1} - z_{i+1/2} \geq 0$, which implies

$$\begin{aligned} &g\frac{h_{i+1/2+}^2}{2} - g\frac{h_{i+1}^2}{2} + g h_{i+1/2+} z_{i+1/2} \\ &\leq g h_{i+1/2+}(h_{i+1/2+} - h_{i+1} + z_{i+1/2}) \\ &\leq g h_{i+1} z_{i+1}. \end{aligned} \tag{2.22}$$

Similarly to (2.21) one has

$$\eta(U_{i-1/2-}) + g h_{i-1/2-} z_{i-1/2} \leq \eta(U_{i-1}) + g h_{i-1} z_{i-1}. \tag{2.23}$$

Using (2.21), (2.23) in (2.19), (2.20) leads to

$$-\Delta t \widetilde{G}_{i_1+1/2} \leq \Delta t v_m \big(\eta(U_{i_1+1}) + g h_{i_1+1} z_{i_1+1}\big), \tag{2.24}$$

$$\Delta t \widetilde{G}_{i_0-1/2} \leq \Delta t v_m \big(\eta(U_{i_0-1}) + g h_{i_0-1} z_{i_0-1}\big). \tag{2.25}$$

Neglecting in (2.18) the two boundary integrals and using (2.24),(2.25), we obtain

$$\begin{aligned} \sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^{n+1}) + g z_i h_i^{n+1}\right) \leq \sum_{i=i_0-1}^{i_1+1} \Delta x \left(\eta(U_i) + \ g z_i h_i\right) \\ - \nu_\beta \Delta t \sum_{i=i_0}^{i_1-1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi \\ + 2C_\beta \frac{\Delta t^2}{\Delta x} v_m^2 g \sum_{i=i_0-1}^{i_1} |z_{i+1} - z_i|^2 . \end{aligned} \tag{2.26}$$

Iterating (2.26) from $n = N-1$ to $n = 0$ and using that

$$\begin{aligned}&\sum_{n=0}^{N-1} 2C_\beta \frac{\Delta t^2}{\Delta x} v_m^2 g \sum_{i=i_0-N}^{i_1+N} |z_{i+1} - z_i|^2 \\ &\leq 2C_\beta T \frac{v_m \Delta t}{\Delta x} v_m g TV2_{[x_{i_0-N}, x_{i_1+N}]}((z_i)),\end{aligned} \tag{2.27}$$

and that according to (1.39) one has $N\Delta x \leq v^* N \Delta t = v^* T$, we get

$$\begin{aligned}&\sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^N) + g z_i h_i^N\right) \\ &+ \nu_\beta \sum_{n=0}^{N-1} \Delta t \sum_{i=i_0}^{i_1-1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+}^n + M_{i+1/2-}^n\right) (M_{i+1/2+}^n - M_{i+1/2-}^n)^2 d\xi \\ &\leq \sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U_i^0) + \ g z_i h_i^0\right) + C\, TV2_{[x_{i_0}-v^*T, x_{i_1}+v^*T]}((z_i)),\end{aligned} \tag{2.28}$$

with C depending on g, T, β, v_m.

We are going to show next that the integral in the LHS of (2.28) is underestimated by a term proportionnal to $\sum_{n=0}^{N-1} \sum_{i=i_0}^{i_1-1} \Delta t |U_{i+1/2+}^n - U_{i+1/2-}^n|^2$.

2.3 Lower estimate of dissipation terms

We first notice that

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+}^n + M_{i+1/2-}^n\right) (M_{i+1/2+}^n - M_{i+1/2-}^n)^2 d\xi \\ &\geq \frac{1}{2} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(2M_{i+1/2+}^n + M_{i+1/2-}^n\right) (M_{i+1/2+}^n - M_{i+1/2-}^n)^2 d\xi.\end{aligned} \tag{2.29}$$

Now according to Lemma A.5, there exists a constant $C > 0$ depending only on $g, \tilde{h}_m, h_M, u_M$ such that

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2) (M_1 - M_2)^2 d\xi \\ &\geq C \left(g \frac{(h_2 - h_1)^2}{2} + \tilde{h}_m \frac{(u_2 - u_1)^2}{2}\right)\end{aligned} \tag{2.30}$$

for all $U_1, U_2 \in \mathcal{U}_{\tilde{h}_m, h_M, u_M}$, where $M_1 = M(U_1, \xi)$, $M_2 = M(U_2, \xi)$. We notice that from (1.41), (1.42) we have

$$U_{i+1/2+}^n, U_{i+1/2-}^n \in \mathcal{U}_{\tilde{h}_m, h_M, u_M}. \tag{2.31}$$

Thus from (2.29) and applying the last estimate (2.30) with $U_1 = U^n_{i+1/2+}$ and $U_2 = U^n_{i+1/2-}$, we get

$$\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M^n_{i+1/2+} + M^n_{i+1/2-}\right) (M^n_{i+1/2+} - M^n_{i+1/2-})^2 d\xi \\ \geq C_5 |U^n_{i+1/2+} - U^n_{i+1/2-}|^2, \tag{2.32}$$

where $C_5 > 0$ depends only on g, h_m, $\tilde{h}_m$, h_M, u_M, and $|\cdot|$ is defined in (2.1).

2.4 Estimate of the discrete gradient

Now we use (2.32) in (2.28) and get

$$\begin{aligned} &\nu_\beta C_5 \sum_{n=0}^{N-1} \sum_{i=i_0}^{i_1-1} \Delta t |U^n_{i+1/2+} - U^n_{i+1/2-}|^2 \\ &\leq \sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U^0_i) + \ gz_i h^0_i\right) - \sum_{i=i_0}^{i_1} \Delta x \left(\eta(U^N_i) + gz_i h^N_i\right) \\ &\quad + C\, TV2_{[x_{i_0}-v^*T, x_{i_1}+v^*T]}((z_i)), \end{aligned} \tag{2.33}$$

with C depending on g, T, β, v_m. Next, using (1.17) we have

$$gz_i h^0_i \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} g\|z\|_\infty h^0(x)\, dx, \tag{2.34}$$

and by convexity of η

$$\eta(U^0_i) \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \eta(U^0(x))\, dx. \tag{2.35}$$

Summing over i we obtain

$$\sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U^0_i) + \ gz_i h^0_i\right) \ \leq \int_{x_{i_0-N-1/2}}^{x_{i_1+N+1/2}} \Big(\eta(U^0(x)) + g\|z\|_\infty h^0(x)\Big)\, dx. \tag{2.36}$$

We notice that $x_{i_0-N-1/2} = x_{i_0-1/2} - N\Delta x = x_{i_0-1/2} - T\frac{\Delta x}{\Delta t}$, and according to the finite propagation hypothesis (1.39) we deduce that

$$\begin{aligned} \sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U^0_i) + \ gz_i h^0_i\right) &\leq \int_{x_{i_0-1/2}-v^*T}^{x_{i_1+1/2}+v^*T} \Big(\eta(U^0(x)) + g\|z\|_\infty h^0(x)\Big)\, dx \\ &= \|\eta(U^0)\|_{L^1(I^{v^*}_{i_0,i_1})} + g\|z\|_\infty \|h^0\|_{L^1(I^{v^*}_{i_0,i_1})}, \end{aligned} \tag{2.37}$$

with $I^{v^*}_{i_0,i_1}$ defined in (2.2). In addition, according to the preliminary computation (2.15) we have

$$-\sum_{i=i_0}^{i_1}\Delta x\, g z_i h_i^N \le g\|z\|_\infty \sum_{i=i_0}^{i_1}\Delta x h_i^N \le g\|z\|_\infty \|h^0\|_{L^1(I^{v^*}_{i_0,i_1})}. \tag{2.38}$$

Using (2.37), (2.38) in (2.33) and noticing that $\eta(U_i^N)\ge 0$, we obtain that

$$\sum_{n=0}^{N-1}\sum_{i=i_0}^{i_1-1}\Delta t|U^n_{i+1/2+}-U^n_{i+1/2-}|^2 \le C, \tag{2.39}$$

where C depends on g, h_m, $\tilde h_m$, h_M, u_M, v_m, β, T, $\|z\|_{L^\infty}$, $TV2_{I^{v^*}_{i_0,i_1}}((z_i))$, $\|\eta(U^0)\|_{L^1(I^{v^*}_{i_0,i_1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0,i_1})}$. Moreover using the triangle inequality and (2.1), (1.23)-(1.25), (1.30), (1.31), we have

$$\begin{aligned}
&|U_{i+1}-U_i|^2\\
\le& 3\big(|U_{i+1/2+}-U_{i+1/2-}|^2+|U_{i+1/2+}-U_{i+1}|^2+|U_{i+1/2-}-U_i|^2\big)\\
\le& 3\big(|U_{i+1/2+}-U_{i+1/2-}|^2+(1+u_{i+1}^2/gh_M)|z_{i+1}-z_i|^2\\
&+(1+u_i^2/gh_M)|z_i-z_{i-1}|^2\big).
\end{aligned}\tag{2.40}$$

With (2.39) we get (2.3) of Proposition 2.1 (apply the inequality to the final time $T+\Delta t$ to get the sum until $n=N$). Moreover, using (1.19), (A.100), (A.101) and (2.3), we get (2.4) of Proposition 2.1.

2.5 End of the proof of Proposition 2.1: estimate of the gradient of the approximate solution

Now from (1.32) we compute for $t_n\le t<t_{n+1}$ and $x_i<x<x_{i+1}$

$$\partial_x U_\Delta = \frac{t-t_n}{\Delta t}\frac{U^{n+1}_{i+1}-U^{n+1}_i-U^n_{i+1}+U^n_i}{\Delta x}+\frac{U^n_{i+1}-U^n_i}{\Delta x}. \tag{2.41}$$

Thus we get

$$\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}}|\partial_x U_\Delta|^2 dxdt \le \frac{\Delta t}{\Delta x}\left[|U^{n+1}_{i+1}-U^{n+1}_i|^2+|U^n_{i+1}-U^n_i|^2\right]. \tag{2.42}$$

In consequence, by using (2.3) we get (2.5) by summing over i and n. Similarly, from (1.32) we compute for $t_n\le t<t_{n+1}$ and $x_i<x<x_{i+1}$

$$\partial_t U_\Delta = \frac{1}{\Delta t}\left[\frac{U^{n+1}_{i+1}-U^{n+1}_i-U^n_{i+1}+U^n_i}{\Delta x}(x-x_i)+U^{n+1}_i-U^n_i\right]. \tag{2.43}$$

Thus

$$\int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} |\partial_t U_\Delta|^2 dx dt \leq \frac{\Delta x}{\Delta t} \left[|U_{i+1}^{n+1} - U_{i+1}^n|^2 + |U_i^{n+1} - U_i^n|^2 \right]. \quad (2.44)$$

In consequence, by using (2.4) we get (2.6) by summing over i and n. This concludes the proof of Proposition 2.1.

3 Interpolation estimates

Before going into the proof of Theorem 1.1, we give some interpolation estimates.

3.1 Definition of interpolation functions $\widetilde{U}_\Delta$ and $\widetilde{F}_\Delta$

We define $\widetilde{U}_\Delta(t,x)$ a piecewise constant function in space by

$$\widetilde{U}_\Delta(t,x) = U_i^n - \frac{t-t_n}{\Delta x} \left(F_{i+1/2-} - F_{i-1/2+} \right) \quad (3.1)$$

for $t_n \leq t < t_{n+1}$, $x_{i-1/2} < x < x_{i+1/2}$, with $F_{i+1/2-}$, $F_{i-1/2+}$ defined in (1.20), (1.21). We remark that for $x_{i-1/2} < x < x_{i+1/2}$ and $n = 0, \ldots, N$,

$$\widetilde{U}_\Delta(t_n, x) = U_i^n, \quad (3.2)$$

and thus, with (1.19), $\widetilde{U}_\Delta$ is continuous with respect to time. We also define $\widetilde{F}_\Delta(t,x)$ for $x_{i-1/2} < x < x_{i+1/2}$, $t_n \leq t < t_{n+1}$, by

$$\begin{aligned} \widetilde{F}_\Delta(t,x) = & \frac{x - x_{i-1/2}}{\Delta x} \left(F^+(U_{i+1/2-}^n) + F^-(U_{i+1/2+}^n) \right) \\ & + \frac{x_{i+1/2} - x}{\Delta x} \left(F^+(U_{i-1/2-}^n) + F^-(U_{i-1/2+}^n) \right), \end{aligned} \quad (3.3)$$

with F^+, F^- defined in (1.27), $U_{i+1/2-}^n$, $U_{i+1/2+}^n$ defined in (1.23). Then $\widetilde{F}_\Delta$ is continuous with respect to x and we have

$$\forall i \in \mathbb{Z} \quad \widetilde{F}_\Delta(t, x_{i+1/2}) = F^+(U_{i+1/2-}^n) + F^-(U_{i+1/2+}^n) = \mathcal{F}\left(U_{i+1/2-}^n, U_{i+1/2+}^n \right). \quad (3.4)$$

Then because of (1.20), (1.21) we have the partial differential equation

$$\partial_t \widetilde{U}_\Delta + \partial_x \widetilde{F}_\Delta = \widetilde{S}_\Delta, \quad (3.5)$$

with $\widetilde{S}_\Delta$ piecewise constant in time and space defined by

$$\widetilde{S}_\Delta(t,x) = \frac{1}{\Delta x} \left(S_{i+1/2-} + S_{i-1/2+} \right) \quad (3.6)$$

for $t_n \leq t < t_{n+1}$ and $x_{i-1/2} < x < x_{i+1/2}$, with $S_{i+1/2-}$, $S_{i+1/2+}$ defined in (1.22).

3.2 Estimate of $\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - \widetilde{U}_\Delta|^2 \, dtdx$

Lemma 3.1. *With the assumptions of Theorem 1.1, let $N \in \mathbb{N}^*$, $T = N\Delta t$, i_0, $i_1 \in \mathbb{Z}$ such that $i_0 \leq i_1$. Let U_Δ be the approximate solution* (1.32) *and $\widetilde{U}_\Delta$ defined by* (3.1)*. Then*

$$\left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - \widetilde{U}_\Delta|^2 \, dtdx \right)^{1/2} \leq C\sqrt{\Delta x}, \tag{3.7}$$

with $|\cdot|$ defined by (2.1)*. The constant C depends only on g, h_m, h_M, u_M, v_m, β, T, $\|z\|_{L^\infty}$, $TV2_{I^{v^*}_{i_0-1,i_1+1}}((z_i))$, $\|\eta(U^0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$, with $I^{v^*}_{i_0-1,i_1+1}$ defined in* (2.2)*.*

Proof. We use the definition (1.32) of U_Δ and write for all $x_i < x < x_{i+1}$ and $t_n \leq t < t_{n+1}$

$$\begin{aligned} & U_\Delta - U_i^n \\ =& \frac{1}{\Delta t}\left[\frac{U_{i+1}^{n+1} - U_i^{n+1} - U_{i+1}^n + U_i^n}{\Delta x}(x - x_i) + U_i^{n+1} - U_i^n \right](t - t_n) \\ & + \frac{U_{i+1}^n - U_i^n}{\Delta x}(x - x_i). \end{aligned} \tag{3.8}$$

Using the triangle inequality, we obtain

$$|U_\Delta - U_i^n| \leq |U_{i+1}^{n+1} - U_i^{n+1}| + |U_{i+1}^n - U_i^n| + |U_i^{n+1} - U_i^n|. \tag{3.9}$$

It implies also

$$|U_\Delta - U_{i+1}^n| \leq |U_{i+1}^{n+1} - U_i^{n+1}| + 2|U_{i+1}^n - U_i^n| + |U_i^{n+1} - U_i^n|. \tag{3.10}$$

Thus

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} \left| U_\Delta - (U_i^n \mathbf{1}_{x_i < x < x_{i+1/2}} + U_{i+1}^n \mathbf{1}_{x_{i+1/2} < x < x_{i+1}}) \right|^2 dxdt \\ & \leq 3\Delta t \Delta x \Big(|U_{i+1}^{n+1} - U_i^{n+1}|^2 + 4|U_{i+1}^n - U_i^n|^2 + |U_i^{n+1} - U_i^n|^2 \Big). \end{aligned} \tag{3.11}$$

Next, we set

$$U_\Delta^1(t, x) = U_i^n, \tag{3.12}$$

for $x_{i-1/2} < x < x_{i+1/2}, t_n \leq t < t_{n+1}$. Taking the sum over n and i of (3.11) and doing translations of indices, we get

$$\begin{aligned} \int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dxdt \leq & \, 15\Delta x \sum_{n=0}^{N} \sum_{i=i_0-1}^{i_1} \Delta t |U_{i+1}^n - U_i^n|^2 \\ & + 3\Delta x \sum_{n=0}^{N-1} \sum_{i=i_0-1}^{i_1} \Delta t |U_i^{n+1} - U_i^n|^2. \end{aligned} \tag{3.13}$$

Then we use the discrete gradient estimates (2.3), (2.4) and the CFL condition (1.29) to get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dxdt \leq C_2 \Delta x, \tag{3.14}$$

with C_2 a constant depending on g, h_m, h_M, u_M, v_m, β, T, $\|z\|_{L^\infty}$, $TV2_{I^{v^*}_{i_0-1,i_1+1}}((z_i))$, $\|\eta(U^0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$.

Next we use the definition (3.1) of $\widetilde{U}_\Delta$ and the definitions (1.19)-(1.22) and get for all $x_{i-1/2} < x < x_{i+1/2}$ and $t_n \leq t < t_{n+1}$

$$\begin{aligned} &U_i^n - \widetilde{U}_\Delta \\ &= \frac{t-t_n}{\Delta x}\Bigg(F^+(U^n_{i+1/2-}) + F^-(U^n_{i+1/2+}) - F^+(U^n_{i-1/2-}) - F^-(U^n_{i-1/2+}) \\ &\qquad - \frac{g}{2}\begin{pmatrix} 0 \\ (h^n_{i+1/2-})^2 - (h^n_i)^2 + (h^n_i)^2 - (h^n_{i-1/2+})^2) \end{pmatrix}\Bigg), \end{aligned} \tag{3.15}$$

with F^+, F^- defined in (1.27), $U^n_{i+1/2-}$, $U^n_{i+1/2+}$ defined in (1.23), $h^n_{i+1/2+}$, $h^n_{i+1/2-}$ defined in (1.24). Then, using that F^+ and F^- are Lipschitz continuous, see (A.100) and (A.101), with the CFL condition (1.29) we obtain that there exists $C_3 > 0$, depending on g, h_m, h_M, u_M and v_m such that

$$\begin{aligned} |U_i^n - \widetilde{U}_\Delta| \leq & C_3\Bigg(|U^n_{i+1/2-} - U^n_{i-1/2-}| + |U^n_{i+1/2+} - U^n_{i-1/2+}| \\ &\qquad + |h^n_i - h^n_{i+1/2-}| + |h^n_i - h^n_{i-1/2+}|\Bigg). \end{aligned} \tag{3.16}$$

Then using an estimate similar to (2.40) we obtain

$$|U_i^n - \widetilde{U}_\Delta| \leq C_3\Bigg(|U_i^n - U_{i-1}^n| + |U_{i+1}^n - U_i^n| + |z_i - z_{i-1}| + |z_{i+1} - z_i|\Bigg). \tag{3.17}$$

Thus

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |U_i^n - \widetilde{U}_\Delta|^2 dtdx \\ \leq & 4C_3^2 \Delta t \Delta x \left(|U_i^n - U_{i-1}^n|^2 + |U_{i+1}^n - U_i^n|^2 + |z_i - z_{i-1}|^2 + |z_{i+1} - z_i|^2 \right). \end{aligned} \tag{3.18}$$

Taking the sum over n and i and doing translations of indices, we get

$$\begin{aligned}&\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta^1 - \widetilde{U}_\Delta|^2 dxdt \\ \leq & 8C_3^2 \Delta x \left(\sum_{n=0}^{N-1} \sum_{i=i_0-1}^{i_1} \Delta t |U_{i+1}^n - U_i^n|^2 + \sum_{n=0}^{N-1} \sum_{i=i_0-1}^{i_1} \Delta t |z_{i+1} - z_i|^2 \right), \end{aligned} \tag{3.19}$$

with U_Δ^1 defined in (3.12). Next, using the gradient estimate (2.3) we get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta^1 - \widetilde{U}_\Delta|^2 dxdt \leq C_2 \Delta x, \tag{3.20}$$

with C_2 a constant depending on g, h_m, h_M, u_M, v_m, β, T, $\|z\|_{L^\infty}$, $TV2_{I^{v^*}_{i_0-1,i_1+1}}((z_i))$, $\|\eta(U^0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$.

Finally, noticing that $U_\Delta - \widetilde{U}_\Delta = (U_\Delta - U_\Delta^1) + (U_\Delta^1 - \widetilde{U}_\Delta)$, we get

$$\begin{aligned}&\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - \widetilde{U}_\Delta|^2 dtdx \\ \leq & \, 2 \left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dtdx + \int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta^1 - \widetilde{U}_\Delta|^2 dtdx \right). \end{aligned} \tag{3.21}$$

With (3.14) and (3.20) we get (3.7), which concludes the proof. □

3.3 Estimate of $\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |F(U_\Delta) - \widetilde{F}_\Delta|^2 \, dtdx$

We will see later on that in order to prove compactness of the sequence $\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta)$ in H^{-1}_{loc}, we need an estimate on $F(U_\Delta) - \widetilde{F}_\Delta$.

Lemma 3.2. *With the assumptions of Theorem 1.1, let $N \in \mathbb{N}^*$, $T = N\Delta t$, i_0, $i_1 \in \mathbb{Z}$ such that $i_0 \leq i_1$. Let U_Δ be the approximate solution* (1.32) *and $\widetilde{F}_\Delta$ defined by* (3.3)*. Then*

$$\left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |F(U_\Delta) - \widetilde{F}_\Delta|^2 \, dtdx \right)^{1/2} \leq C\sqrt{\Delta x}, \tag{3.22}$$

with $|\cdot|$ defined by (2.1)*. The constant C depends only on g, h_m, h_M, u_M, v_m, β, T, $\|z\|_{L^\infty}$, $TV2_{I^{v^*}_{i_0-1,i_1+1}}((z_i))$, $\|\eta(U^0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$, $I^{v^*}_{i_0-1,i_1+1}$ defined in* (2.2)*.*

Proof. We recall here (3.3)

$$\widetilde{F}_\Delta(t,x) = \frac{x - x_{i-1/2}}{\Delta x}\left(F^+(U^n_{i+1/2-}) + F^-(U^n_{i+1/2+})\right) + \frac{x_{i+1/2} - x}{\Delta x}\left(F^+(U^n_{i-1/2-}) + F^-(U^n_{i-1/2+})\right), \tag{3.23}$$

for all $x_{i-1/2} < x < x_{i+1/2}$ and $t_n \le t < t_{n+1}$. Moreover, we have

$$F(U_\Delta) = F^+(U_\Delta) + F^-(U_\Delta). \tag{3.24}$$

Thus, using the triangle inequality, for all $x_{i-1/2} < x < x_{i+1/2}$, we get

$$\begin{aligned} &\left|\widetilde{F}_\Delta(t,x) - F(U_\Delta(t,x))\right| \\ &\le \left|F^+(U^n_{i+1/2-}) - F^+(U_\Delta)\right| + \left|F^-(U^n_{i+1/2+}) - F^-(U_\Delta)\right| \\ &\quad + \left|F^+(U^n_{i-1/2-}) - F^+(U_\Delta)\right| + \left|F^-(U^n_{i+1/2+}) - F^-(U_\Delta)\right|. \end{aligned} \tag{3.25}$$

Then, using that F^+ and F^- are Lipschitz continuous, see (A.100) and (A.101), we obtain that there exists $C > 0$, depending on g, h_m, h_M, u_M and v_m such that

$$\begin{aligned} &|\widetilde{F}_\Delta(t,x) - F(U_\Delta)| \\ &\le C\left(\left|U^n_{i+1/2-} - U_\Delta\right| + \left|U^n_{i+1/2+} - U_\Delta\right| + \left|U^n_{i-1/2-} - U_\Delta\right| + \left|U^n_{i-1/2+} - U_\Delta\right|\right). \end{aligned} \tag{3.26}$$

Moreover using (1.23)-(1.25), (1.30), (1.31), we get

$$\begin{aligned} |\widetilde{F}_\Delta(t,x) - F(U_\Delta)| \le C\Big(&2\left|U^n_i - U_\Delta\right| + \left|U^n_{i+1} - U_\Delta\right| + \left|U^n_{i-1} - U_\Delta\right| \\ &+ 2|z_{i+1} - z_i| + 2|z_i - z_{i-1}|\Big). \end{aligned} \tag{3.27}$$

Thus we get

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |\widetilde{F}_\Delta(t,x) - F(U_\Delta)|^2 dtdx \\ \le& C^2 \int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \left|U^n_i - U_\Delta\right|^2 dtdx \\ &+ C^2 \Delta t \Delta x \left(\left|U^n_{i+1} - U^n_i\right|^2 + \left|U^n_{i-1} - U^n_i\right|^2 + |z_{i+1} - z_i|^2 + |z_i - z_{i-1}|^2\right). \end{aligned} \tag{3.28}$$

Taking the sum over n and i and doing translations of indices, we get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |\widetilde{F}_\Delta(t,x) - F(U_\Delta(t,x))|^2 dxdt \leq C^2 \int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dxdt + C^2 \Delta x \left(\sum_{n=0}^{N-1} \sum_{i=i_0-1}^{i_1} \Delta t |U_{i+1}^n - U_i^n|^2 + \sum_{n=0}^{N-1} \sum_{i=i_0-1}^{i_1} \Delta t |z_{i+1} - z_i|^2 \right). \tag{3.29}$$

Using the previous estimate (3.14) involving $U_\Delta - U_\Delta^1$ and the gradient estimate (2.3), we get (3.22), which concludes the proof. □

4 Proof of Theorem 1.1

Using (3.5) we write

$$\partial_t U_\Delta + \partial_x F(U_\Delta) = \partial_t (U_\Delta - \widetilde{U}_\Delta) + \partial_x \left(F(U_\Delta) - \widetilde{F}_\Delta \right) + \widetilde{S}_\Delta, \tag{4.1}$$

with U_Δ defined in (1.32), $\widetilde{U}_\Delta$ defined in (3.1), $\widetilde{F}_\Delta$ defined in (3.3), and $\widetilde{S}_\Delta$ defined in (3.6). Note that in (4.1) all terms are locally bounded functions. We multiply (4.1) by $\eta'(U_\Delta)$ and get, for any entropy-entropy flux (η, G), the decomposition

$$\begin{aligned} \partial_t \eta(U_\Delta) + \partial_x G(U_\Delta) = & \ \eta'(U_\Delta) \cdot \partial_t (U_\Delta - \widetilde{U}_\Delta) \\ & + \eta'(U_\Delta) \cdot \partial_x \left(F(U_\Delta) - \widetilde{F}_\Delta \right) + \ \eta'(U_\Delta) \cdot \widetilde{S}_\Delta \\ \equiv & \ R_1 + M_1 + R_2 + M_2 + \eta'(U_\Delta) \cdot \widetilde{S}_\Delta, \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} R_1 = & \ \partial_t \left(\eta'(U_\Delta) \cdot (U_\Delta - \widetilde{U}_\Delta) \right), \\ M_1 = & \ -\eta''(U_\Delta) \cdot \partial_t U_\Delta \cdot \left(U_\Delta - \widetilde{U}_\Delta \right), \\ R_2 = & \ \partial_x \left(\eta'(U_\Delta) \cdot \left(F(U_\Delta) - \widetilde{F}_\Delta \right) \right), \\ M_2 = & \ -\eta''(U_\Delta) \cdot \partial_x U_\Delta \cdot \left(F(U_\Delta) - \widetilde{F}_\Delta \right). \end{aligned} \tag{4.3}$$

We have using (3.22)

$$\begin{aligned}
&\int_0^T \int_{-R}^R \left|\eta'(U_\Delta)\cdot\left(F(U_\Delta)-\widetilde{F}_\Delta\right)\right|^2 dxdt \\
&\leq \|\eta'(U_\Delta)\|^2_{L^\infty((0,T)\times(-R,R))} \int_0^T \int_{-R}^R \left|F(U_\Delta)-\widetilde{F}_\Delta\right|^2 dxdt \\
&\leq C_R\Delta x, \qquad (4.4)
\end{aligned}$$

thus R_2 goes to zero in H^{-1}_{loc} as $\Delta x \to 0$. Similarly, using (3.7), R_1 goes to zero in H^{-1}_{loc} as $\Delta x \to 0$. Furthermore, using (2.5) and (3.22), we have

$$\begin{aligned}
&\int_0^T \int_{-R}^R |M_2| dxdt \\
\leq &\|\eta''(U_\Delta)\|_{L^\infty} \left(\iint |\partial_x U_\Delta|^2 dxdt\right)^{1/2} \left(\int\int \left|F(U_\Delta)-\widetilde{F}_\Delta\right|^2 dxdt\right)^{1/2} \\
\leq &\|\eta''(U_\Delta)\|_{L^\infty} \frac{C_2}{\sqrt{\Delta x}} C\sqrt{\Delta x} \\
\leq &\, C_R. \qquad (4.5)
\end{aligned}$$

Thus M_2 is bounded in $\mathcal{M}_{loc}((0,T)\times\mathbb{R})$. Similarly, using (2.6) and (3.7), M_1 is bounded in $\mathcal{M}_{loc}((0,T)\times\mathbb{R})$.

Then, the definition (3.6) of $\widetilde{S}_\Delta$ and the definitions (1.22) of $S_{i+1/2-}$, $S_{i+1/2+}$ yield with (1.30), (1.31) and the L^1_{loc} condition (1.34) that $\widetilde{S}_\Delta$ is uniformly bounded in L^1_{loc}. According to (4.2) and (4.3) one has

$$\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta) - R_1 - R_2 = M_1 + M_2 + \eta'(U_\Delta)\cdot\widetilde{S}_\Delta. \qquad (4.6)$$

The right-hand side is bounded in $\mathcal{M}_{loc} \cap W^{-1,p}_{loc}$, $\forall p$, $1 < p < +\infty$, as a consequence it is compact in H^{-1}_{loc}. At this point, we know that $R_1 + R_2$ and $M_1 + M_2 + \eta'(U_\Delta)\cdot\widetilde{S}_\Delta$ are compact in H^{-1}_{loc}, therefore their sum, which is equal to $\partial_t\eta(U_\Delta) + \partial_x G(U_\Delta)$, is compact in H^{-1}_{loc}. This holds for any couple entropy-entropy flux (η, G). Furthermore, $(U_\Delta)_{\Delta>0}$ is bounded since we assume that $(U_i^n)_{i,n}$ is a bounded sequence. We are now able to apply the compensated compactness method [23] and we get that up to a subsequence $U_\Delta \to U$ a.e. and in $L^1_{loc,t,x}$ as $\Delta t \to 0$ and $\Delta x \to 0$.

Moreover, according to Lemma A.8, $\partial_t U_\Delta$ is bounded in $L^\infty_t(\mathcal{D}'_x)$ and therefore we deduce that $U_\Delta \to U$ in $C_t([0,T], L^\infty_{x,w*}(\mathbb{R}_{loc}))$, by the Arzelà Ascoli theorem. Then, knowing that U_Δ converges in L^p_{loc} to U, we can apply Lemma A.9, which concludes the convergence of the approximate source term $\widetilde{S}_\Delta$ to S.

Finally we pass to the limit in (4.1) using (3.7), (3.22), which enables us to get that the limit U is a weak solution to our system (1.1). Moreover passing to the limit weakly in (2.16) using (1.35), we get (1.2). Similarly, the weak limit of (4.6) yields (1.40). This ends the proof of Theorem 1.1.

Appendix: some technical lemmas

We prove here some technical results used throughout the paper. The notations are introduced in Section 1.

Lemma A.1. *Let $z_i = z(x_i)$ for all $i \in \mathbb{Z}$, where $z \in C(\mathbb{R})$ satisfies $\partial_x z \in L^1_{loc}(\mathbb{R})$, and (x_i) is a uniform grid of length Δx. Then for any bounded interval $[a,b]$,*

$$TV2_{[a,b]}((z_i)) \equiv \sum_{[x_i,x_{i+1}]\subset[a,b]} (z_{i+1}-z_i)^2 \tag{A.1}$$

verifies

$$TV2_{[a,b]}((z_i)) \leq \left(\int_a^b |\partial_x z(x)|dx\right)^2, \tag{A.2}$$

and

$$TV2_{[a,b]}((z_i)) \to 0 \quad \text{as } \Delta x \to 0. \tag{A.3}$$

Proof. We have

$$z_{i+1}-z_i = z(x_{i+1})-z(x_i) = \int_{x_i}^{x_{i+1}} \partial_x z(x)dx, \tag{A.4}$$

thus for $[x_i, x_{i+1}] \subset [a,b]$

$$|z_{i+1}-z_i| \leq \int_{x_i}^{x_{i+1}} |\partial_x z(x)|dx \leq \int_a^b |\partial_x z(x)|dx. \tag{A.5}$$

It follows that

$$\begin{aligned} \sum (z_{i+1}-z_i)^2 &\leq \sum \int_{x_i}^{x_{i+1}} |\partial_x z(x)|dx \times \int_a^b |\partial_x z(x)|dx \\ &\leq \left(\int_a^b |\partial_x z(x)|dx\right)^2, \end{aligned} \tag{A.6}$$

which proves (A.2). Next, when z is Lipschitz continuous one has $|z_{i+1}-z_i| \leq \mathrm{Lip}(z)\Delta x$, thus $TV2_{[a,b]}((z_i)) \leq \mathrm{Lip}(z)^2 \Delta x(b-a)$ and (A.3) holds. When we have only $\partial_x z \in L^1_{loc}$, for any $\varepsilon > 0$ one can find $z_\varepsilon \in \mathrm{Lip}(\mathbb{R})$ such that $\|\partial_x z - \partial_x z_\varepsilon\|_{L^1([a,b])} \leq \varepsilon$, and if follows that (A.3) also holds. ∎

Lemma A.2. *Let $U_k = (h_k, h_k u_k)$ for $k = 1$, 2 with $h_k \geq 0$. Then*

$$\begin{aligned}
&\frac{g^2\pi^2}{6}\left(2M_1 + M_2\right)\left(M_1 - M_2\right)^2 \\
=&H_0(M_2) - H_0(M_1) - \eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix}\left(M_2 - M_1\right) \\
&- \mathbb{1}_{(\xi-u_1)^2>2gh_1}\ M_2\left(\frac{(\xi-u_1)^2}{2} - gh_1\right),
\end{aligned} \tag{A.7}$$

where $M_k \equiv M_k(\xi) \equiv M(U_k, \xi)$ and $M(U, \xi)$ is defined in (1.8)*, $H_0(f) \equiv H_0(f, \xi)$ is defined in* (1.11)*.*

Proof. This lemma indeed gives the remainder in the inequality (1.15). Using the identity

$$b^3 - a^3 - 3a^2(b-a) = (b+2a)(b-a)^2, \tag{A.8}$$

one has

$$\frac{g^2\pi^2}{6}\left(2M_1 + M_2\right)\left(M_1 - M_2\right)^2 = H_0(M_2) - H_0(M_1) - H_0'(M_1)\left(M_2 - M_1\right), \tag{A.9}$$

where we donote $H_0'(f, \xi) \equiv \frac{\partial}{\partial f} H_0(f, \xi)$. Thus we have to prove that

$$\begin{aligned}
&\left(\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} - H_0'(M_1)\right)\left(M_2 - M_1\right) \\
&= -\mathbb{1}_{(\xi-u_1)^2>2gh_1}\ M_2\left(\frac{(\xi-u_1)^2}{2} - gh_1\right).
\end{aligned} \tag{A.10}$$

On the one hand we compute according to (1.16)

$$\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} = gh_1 - \frac{u_1^2}{2} + u_1\xi = gh_1 + \frac{\xi^2}{2} - \frac{(\xi-u_1)^2}{2}. \tag{A.11}$$

On the other hand we have using (1.8)

$$H_0'(M_1) = \frac{\xi^2}{2} + \frac{g^2\pi^2}{2}M_1^2 = \frac{\xi^2}{2} + \left(gh_1 - \frac{(\xi-u_1)^2}{2}\right)_+. \tag{A.12}$$

Subtracting (A.12) to (A.11) it follows that

$$\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} - H_0'(M_1) = -\mathbb{1}_{(\xi-u_1)^2>2gh_1}\left(\frac{(\xi-u_1)^2}{2} - gh_1\right), \tag{A.13}$$

and therefore that

$$\begin{aligned}&\left(\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} - H_0'(M_1)\right)(M_2 - M_1)\\ =&-\mathbb{1}_{(\xi-u_1)^2>2gh_1}\ \left(\frac{(\xi-u_1)^2}{2} - gh_1\right)(M_2 - M_1).\end{aligned} \tag{A.14}$$

Finally we notice that

$$(\xi - u_1)^2 \geq 2gh_1 \Longleftrightarrow M_1 = 0, \tag{A.15}$$

thus we get (A.10), which concludes the proof. □

Lemma A.3. *There exists some constant $\alpha > 0$, depending only on the gravity constant g and on the constants h_m, h_M, u_M involved in* (1.37)*, such that*

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| \left(H_0(M_2) - H_0(M_1) - \eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix}(M_2 - M_1)\right) d\xi\\ &\geq \alpha\left(\eta(U_2) - \eta(U_1) - \eta'(U_1)(U_2 - U_1)\right)\end{aligned} \tag{A.16}$$

for all U_1, $U_2 \in \mathcal{U}_{h_m,h_M,u_M}$ defined by (1.37) *and where $M_k \equiv M_k(\xi) \equiv M(U_k,\xi)$, with $M(U,\xi)$ defined in* (1.8)*, $H_0(f) \equiv H_0(f,\xi)$ is defined in* (1.11) *and $\eta(U)$ is defined in* (1.3)*.*

Proof. Note that without the factor $|\xi|$ in (A.16), the inequality would become an equality with $\alpha = 1$. Thus the difficulty is to show that small values of ξ do not make a significant contribution. The idea is to make a linear combination in the variable $\eta'(U)$ as in [14, Lemma 2.3], but the difficulty is that in this variable, the set where $h \geq 0$ is not convex. This is why we have to be far from vacuum. We set

$$\widehat{\mathcal{U}}_m = \left\{(h,hu) \in \mathbb{R}^2,\ h \geq h_m\right\}, \tag{A.17}$$

and we first deal with the case

$$U_1 = \begin{pmatrix}h_1\\ h_1u_1\end{pmatrix} \text{ and } U_2 = \begin{pmatrix}h_2\\ h_2u_2\end{pmatrix} \in \widehat{\mathcal{U}}_m, \text{ such that } |u_1 - u_2| \leq \sqrt{gh_m}. \tag{A.18}$$

In this case we have

$$\forall t \in [0,1],\ (1-t)\eta'(U_1) + t\eta'(U_2) \in \eta'(\widetilde{\mathcal{U}}_m). \tag{A.19}$$

with

$$\widetilde{\mathcal{U}}_m = \left\{(h,hu) \in \mathbb{R}^2,\ h \geq \frac{h_m}{2}\right\}. \tag{A.20}$$

Indeed we notice that using (1.16),

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \eta'(\widetilde{\mathfrak{U}}_m) \Longleftrightarrow V_1 \geq g\frac{h_m}{2} - \frac{V_2^2}{2}. \tag{A.21}$$

Thus (A.19) is equivalent to

$$\begin{aligned} &\forall t \in [0,1], \forall h_1, h_2 \geq h_m, \forall u_1, u_2 \in \mathbb{R}, \text{ such that } |u_1 - u_2| \leq \sqrt{gh_m}, \\ &(1-t)\left(gh_1 - \frac{u_1^2}{2}\right) + t\left(gh_2 - \frac{u_2^2}{2}\right) \geq g\frac{h_m}{2} - \frac{1}{2}\Big((1-t)u_1 + tu_2\Big)^2. \end{aligned} \tag{A.22}$$

Thus we have to check that

$$\begin{aligned} &\forall t \in [0,1], \forall u_1, u_2 \in \mathbb{R}, \text{ such that } |u_1 - u_2| \leq \sqrt{gh_m}, \\ &(1-t)\left(gh_m - \frac{u_1^2}{2}\right) + t\left(gh_m - \frac{u_2^2}{2}\right) \geq g\frac{h_m}{2} - \frac{1}{2}\Big((1-t)u_1 + tu_2\Big)^2. \end{aligned} \tag{A.23}$$

This inequality simplifies to

$$\frac{gh_m}{2} \geq \frac{t(1-t)}{2}\left(u_1 - u_2\right)^2, \tag{A.24}$$

which holds true when $t \in [0,1]$ and $|u_1 - u_2| \leq 2\sqrt{gh_m}$. This proves (A.19).

According to the property (A.19) we can now define a path $v(t) \in \widetilde{\mathfrak{U}}_m$, for $0 \leq t \leq 1$, connecting the two states U_1, U_2 satisfying (A.18), by

$$\eta'\left(v(t)\right) = (1-t)\eta'(U_1) + t\eta'(U_2). \tag{A.25}$$

Such a definition is possible because η' is a diffeormorphism, see (1.16). It enables us to set

$$\begin{aligned} \phi(t) = \int_{\mathbb{R}} |\xi| \Big(& H_0(M(v(t),\xi),\xi) - H_0(M(U_1,\xi),\xi) \\ & - \eta'(U_1)\begin{pmatrix} 1 \\ \xi \end{pmatrix}\left(M(v(t),\xi) - M(U_1,\xi)\right)\Big) d\xi \\ & - \alpha\left(\eta(v(t)) - \eta(U_1) - \eta'(U_1)\left(v(t) - U_1\right)\right). \end{aligned} \tag{A.26}$$

We notice that $\phi(0) = 0$, and the desired inequality (A.16) is equivalent to $\phi(1) \geq 0$. Thus it is sufficient to prove that ϕ is nondecreasing. Using the fact that

$$\eta'(U)\begin{pmatrix} 1 \\ \xi \end{pmatrix} = H_0'\left(M(U,\xi),\xi\right), \text{ for all } \xi \in \mathbb{R} \text{ such that } M(U,\xi) > 0, \tag{A.27}$$

we can compute

$$\begin{aligned}\phi'(t) = &\int_{\mathbb{R}} |\xi| \left(\eta'(v(t)) - \eta'(U_1)\right) \begin{pmatrix}1\\ \xi\end{pmatrix} M'(v(t),\xi)v'(t)d\xi \\ &- \alpha \left(\eta'(v(t)) - \eta'(U_1)\right) v'(t). \end{aligned} \tag{A.28}$$

Moreover, using that

$$\eta'(v(t)) - \eta'(U_1) = t(\eta'(U_2) - \eta'(U_1)) = t\eta''(v(t))v'(t), \tag{A.29}$$

we get

$$\begin{aligned}\phi'(t) = t&\int_{\mathbb{R}} |\xi|\eta''(v(t))v'(t) \begin{pmatrix}1\\ \xi\end{pmatrix} M'(v(t),\xi)v'(t)d\xi \\ &- \alpha\, t\, \eta''(v(t))v'(t)v'(t). \end{aligned} \tag{A.30}$$

This can be rewritten as

$$\begin{aligned}\phi'(t) = t&\int_{\mathbb{R}} |\xi| M'(v(t),\xi) \otimes \left(\eta''(v(t)) \begin{pmatrix}1\\ \xi\end{pmatrix}\right) \cdot v'(t) \cdot v'(t)d\xi \\ &- \alpha\, t\, \eta''(v(t)) \cdot v'(t) \cdot v'(t). \end{aligned} \tag{A.31}$$

Thus now it is sufficient for getting (A.16) to prove that

$$\begin{aligned}&\forall U \in \widetilde{\mathcal{U}}_m, \forall X \in \mathbb{R}^2 \\ &\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes \left(\eta''(U) \begin{pmatrix}1\\ \xi\end{pmatrix}\right) \cdot X \cdot X d\xi \geq \alpha\, \eta''(U) \cdot X \cdot X. \end{aligned} \tag{A.32}$$

For all $U \in \widetilde{\mathcal{U}}_m$ and $\xi \in \mathbb{R}$ such that $M(U,\xi) > 0$, we compute

$$\eta'(U) \begin{pmatrix}1\\ \xi\end{pmatrix} = H_0' \left(M(U,\xi),\xi\right) \tag{A.33}$$

and

$$\eta''(U) \begin{pmatrix}1\\ \xi\end{pmatrix} = H_0'' \left(M(U,\xi),\xi\right) M'(U,\xi). \tag{A.34}$$

Moreover one can check that

$$H_0'' \left(M(U,\xi)\right) = g^2\pi^2 M(U,\xi). \tag{A.35}$$

Thus we obtain

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes \left(\eta''(U) \begin{pmatrix}1\\ \xi\end{pmatrix}\right) d\xi \\ =&g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi)\, M'(U,\xi) \otimes M'(U,\xi)d\xi, \end{aligned} \tag{A.36}$$

and therefore the desired inequality (A.32) can be written

$$\begin{aligned}&\forall U \in \widetilde{\mathcal{U}}_m, \forall X \in \mathbb{R}^2\\ &g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi)\left(M'(U,\xi)X\right)^2 d\xi \geq \alpha\, \eta''(U)\cdot X \cdot X. \end{aligned} \tag{A.37}$$

According to (1.16), we have

$$\eta'(h,q) = \left(-\frac{1}{2}\frac{q^2}{h^2} + gh, \frac{q}{h}\right), \tag{A.38}$$

$$\eta''(h,q) = \begin{pmatrix} \frac{q^2}{h^3} + g & -\frac{q}{h^2} \\ -\frac{q}{h^2} & \frac{1}{h} \end{pmatrix} = \begin{pmatrix} \frac{u^2}{h} + g & -\frac{u}{h} \\ -\frac{u}{h} & \frac{1}{h} \end{pmatrix}. \tag{A.39}$$

Denoting $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we get

$$\begin{aligned} &\eta''(U)\cdot X \cdot X\\ =&\left(g + \frac{u^2}{h}\right)x_1^2 + \frac{1}{h}x_2^2 - \frac{2u}{h}x_1x_2\\ =&gx_1^2 + \frac{1}{h}\left(x_2 - ux_1\right)^2\\ =&gx_1^2 + hx_3^2, \end{aligned} \tag{A.40}$$

where we denote

$$x_3 = \frac{1}{h}\left(x_2 - ux_1\right). \tag{A.41}$$

In order to compute $M'(U,\xi)X$, with (1.8) we compute the partial derivatives where $M(U,\xi) > 0$,

$$\partial_h M(U,\xi) = \frac{1}{2g\pi}\left(2gh - (\xi-u)^2\right)^{-1/2}\left(2g - 2\frac{u}{h}(\xi-u)\right), \tag{A.42}$$

$$\partial_{hu} M(U,\xi) = \frac{1}{2g\pi}\left(2gh - (\xi-u)^2\right)^{-1/2}\frac{2}{h}(\xi-u). \tag{A.43}$$

It leads to the formula (where $M(U,\xi) > 0$)

$$\begin{aligned} M'(U,\xi)X =&\frac{M(U,\xi)^{-1}}{g^2\pi^2}\left(gx_1 + \frac{(\xi-u)}{h}\left(x_2 - ux_1\right)\right)\\ =&\frac{M(U,\xi)^{-1}}{g^2\pi^2}\left(gx_1 + (\xi-u)x_3\right). \end{aligned} \tag{A.44}$$

Using (A.44) in the integral of (A.37) we get

$$\begin{aligned}
&g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi)\left(M'(U,\xi)X\right)^2 d\xi \\
=&\frac{1}{g^2\pi^2} \int_{M(U,\xi)>0} |\xi|\ \frac{1}{M(U,\xi)}\left(gx_1+(\xi-u)x_3\right)^2 d\xi \\
=&\frac{1}{g\pi} \int_{(\xi-u)^2<2gh} |\xi|\ \frac{1}{(2gh-(\xi-u)^2)^{1/2}}\left(gx_1+(\xi-u)x_3\right)^2 d\xi := I. \quad \text{(A.45)}
\end{aligned}$$

As for (A.16), we notice that without the factor $|\xi|$, the inequality (A.37) would become an equality with $\alpha = 1$ (compute the integral (A.45) without the factor $|\xi|$). With the factor $|\xi|$, we use the substitution $v = \xi - u$ in (A.45) and the convention that if $u = 0$ then $\operatorname{sgn}(u) = 1$, to obtain

$$\begin{aligned}
I &\geq \frac{1}{g\pi\sqrt{2gh}} \int_{|v|<\sqrt{2gh}} |v+u|\left(gx_1+vx_3\right)^2 dv \\
&\geq \frac{1}{g\pi\sqrt{2gh}} \int_{|v|<\sqrt{2gh},\operatorname{sgn}(v)=\operatorname{sgn}(u)} (|v|+|u|)\left(gx_1+|v|\operatorname{sgn}(u)x_3\right)^2 dv \\
&\geq \frac{1}{g\pi\sqrt{2gh}} \int_0^{\sqrt{2gh}} v\left(gx_1+v\operatorname{sgn}(u)x_3\right)^2 dv \\
&\geq \frac{1}{2g\pi} \int_{\frac{\sqrt{2gh}}{2}}^{\sqrt{2gh}} \left(gx_1+v\operatorname{sgn}(u)x_3\right)^2 dv \\
&= \frac{\sqrt{h}}{\sqrt{2g}\pi} \int_{1/2}^1 \left(gx_1+\xi\sqrt{2gh}\operatorname{sgn}(u)x_3\right)^2 d\xi. \quad \text{(A.46)}
\end{aligned}$$

The last integral is a positive definite quadratic form with respect to $y_1 = gx_1$ and $y_3 = \sqrt{2gh}\operatorname{sgn}(u)x_3$. Thus we have for some absolute constant $C_0 > 0$ (one can check that $C_0 = 1/(8\times 13)$ works)

$$I \geq C_0 \frac{\sqrt{h}}{\sqrt{2g}\pi}\left((gx_1)^2+ghx_3^2\right) = C_0\frac{\sqrt{gh}}{\sqrt{2}\pi}\left(gx_1^2+hx_3^2\right). \quad \text{(A.47)}$$

Therefore by (A.45), (A.47) we get

$$g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi)\left(M'(U,\xi)X\right)^2 d\xi \geq C_0\frac{\sqrt{gh}}{\sqrt{2}\pi}\left(gx_1^2+hx_3^2\right). \quad \text{(A.48)}$$

Because of (A.40), this proves that (A.37) holds with $\alpha_1 = C_0\sqrt{gh_m}/2\pi$. We conclude that (A.16) holds for all $U_1, U_2 \in \widehat{\mathfrak{U}}_m$ such that $|u_1 - u_2| \leq \sqrt{gh_m}$, with the constant α_1.

Thus, to conclude the lemma it is now sufficient to prove that

$$\exists \alpha_2 > 0, \quad \forall U_1, U_2 \in \mathcal{U}_{h_m,h_M,u_M} \text{ such that } |u_1 - u_2| > \sqrt{gh_m},$$
$$\int_{\mathbb{R}} |\xi| \left(H_0(M_2) - H_0(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \geq \alpha_2. \tag{A.49}$$

Indeed, when $U_1, U_2 \in \mathcal{U}_{h_m,h_M,u_M}$ we have

$$\eta(U_2) - \eta(U_1) - \eta'(U_1)(U_2 - U_1) = g\frac{(h_2 - h_1)^2}{2} + h_2\frac{(u_2 - u_1)^2}{2} \leq C(g, h_M, u_M), \tag{A.50}$$

thus when (A.49) holds, we deduce (A.16). Proceeding by reductio ad absurdum, let us assume that (A.49) does not hold. Thus

$$\begin{aligned} &\forall n > 0, \quad \exists U_1^n, U_2^n \in \mathcal{U}_{h_m,h_M,u_M}, \text{ such that} \\ &|u_1^n - u_2^n| > \sqrt{gh_m} \\ &\text{and } \int_{\mathbb{R}} |\xi| \left(H_0(M_2^n) - H_0(M_1^n) - \eta'(U_1^n) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2^n - M_1^n) \right) d\xi \leq \frac{1}{n}, \end{aligned} \tag{A.51}$$

where $M_i^n = M(U_i^n, \xi)$. As $\mathcal{U}_{h_m,h_M,u_M}$ is a closed and bounded set, we can extract a subsequence (that we still denote U_1^n, U_2^n) such that

$$U_1^n \to U_1 \in \mathcal{U}_{h_m,h_M,u_M}, \quad U_2^n \to U_2 \in \mathcal{U}_{h_m,h_M,u_M}, \tag{A.52}$$

with

$$|u_1 - u_2| \geq \sqrt{gh_m}. \tag{A.53}$$

By Lebesgue'theorem, (A.51) implies that

$$\int_{\mathbb{R}} |\xi| \left(H_0(M_2) - H_0(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \leq 0. \tag{A.54}$$

We also know by (A.7) that

$$H_0(M_2) - H_0(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \geq \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2, \tag{A.55}$$

and therefore we get that

$$(2M_1 + M_2)(M_1 - M_2)^2 = 0 \quad \text{for almost every } \xi. \tag{A.56}$$

This implies that $M_1 = M_2$ a.e. and therefore that $U_1 = U_2$, in contradiction with (A.53). This concludes the proof of Lemma A.3. □

Lemma A.4. *One has*

$$\begin{aligned}&\int_{(\xi-u_1)^2>2gh_1}|\xi|M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2}-gh_1\right)d\xi\\&\leq\frac{4\left(|u_2|+\sqrt{2gh_2}\right)}{g\pi\sqrt{gh_2}}K^{\frac{3}{2}}\min(gh_2,K),\end{aligned}\tag{A.57}$$

for all $U_1=(h_1,h_1u_1)$, $h_1>0$ *and* $U_2=(h_2,h_2u_2)$, $h_2>0$, *where* $M(U,\xi)$ *is defined in* (1.8), *and*

$$K=g|h_1-h_2|+(|u_2|+\sqrt{2gh_2})|u_1-u_2|+\frac{1}{2}|u_1^2-u_2^2|.\tag{A.58}$$

Proof. We notice that for all $\xi\in\mathrm{supp}(M_2)$ one has $|\xi|\leq|u_2|+\sqrt{2gh_2}$, thus we have

$$\begin{aligned}&\left|gh_2-\frac{(\xi-u_2)^2}{2}-\left(gh_1-\frac{(\xi-u_1)^2}{2}\right)\right|\\&=\left|g(h_2-h_1)+\xi(u_2-u_1)-\frac{1}{2}(u_2^2-u_1^2)\right|\\&\leq K.\end{aligned}\tag{A.59}$$

Therefore, using that $\xi\in\mathrm{supp}(M_1)^c\cap\mathrm{supp}(M_2)$ iff $gh_2-\frac{(\xi-u_2)^2}{2}\geq 0$ and $\frac{(\xi-u_1)^2}{2}-gh_1>0$, we get

$$\left|gh_2-\frac{(\xi-u_2)^2}{2}\right|+\left|\frac{(\xi-u_1)^2}{2}-gh_1\right|\leq K.\tag{A.60}$$

Using (1.8) and (A.60), we get

$$\begin{aligned}&\int_{(\xi-u_1)^2>2gh_1}|\xi|M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2}-gh_1\right)d\xi\\&=\frac{\sqrt{2}}{g\pi}\int_{\mathrm{supp}(M_1)^c\cap\mathrm{supp}(M_2)}|\xi|\left(gh_2-\frac{(\xi-u_2)^2}{2}\right)^{1/2}\left(\frac{(\xi-u_1)^2}{2}-gh_1\right)d\xi\\&\leq\frac{\sqrt{2}}{g\pi}(|u_2|+\sqrt{2gh_2})\,|\mathrm{supp}(M_1)^c\cap\mathrm{supp}(M_2)|\,K^{3/2}.\end{aligned}\tag{A.61}$$

Thus it is now sufficient for getting (A.57) to prove that

$$|\mathrm{supp}(M_1)^c\cap\mathrm{supp}(M_2)|\leq\frac{4\min(gh_2,K)}{\sqrt{2gh_2}}.\tag{A.62}$$

We observe from (A.60) that for $\xi \in \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)$ one has

$$P(\xi) \leq 0, \tag{A.63}$$

where

$$P(\xi) \equiv gh_2 - \frac{(\xi - u_2)^2}{2} - K. \tag{A.64}$$

We notice that when $\xi = u_2$, P reaches a maximum equals to $gh_2 - K$, and we distinguish two cases:

- If $K \geq gh_2$ then

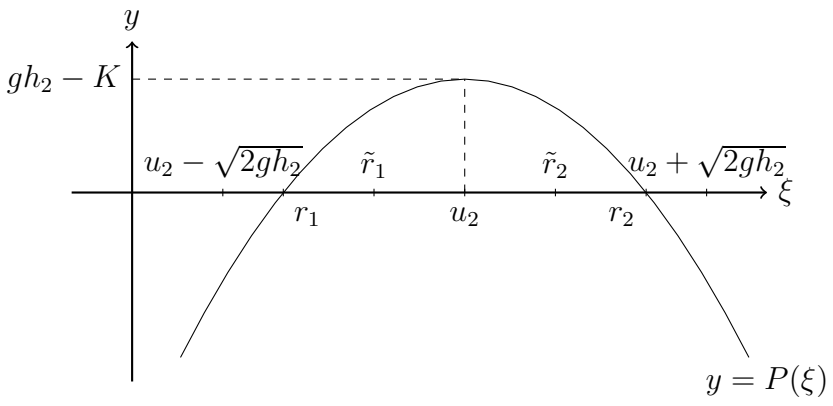
$$|\mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)| \leq |\mathrm{supp}(M_2)| = 2\sqrt{2gh_2}, \tag{A.65}$$

  which concludes (A.62).

- If $K < gh_2$, then the maximum of P is positive and using (A.63) we get that for $\xi \in \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)$ we have

$$\xi \in \left[u_2 - \sqrt{2gh_2}, r_1\right] \bigcup \left[r_2, u_2 + \sqrt{2gh_2}\right], \tag{A.66}$$

  with $r_1 < u_2 < r_2$ are such that $P(r_1) = P(r_2) = 0$. We have $u_2 - \sqrt{2gh_2} < r_1$ because $P(u_2 - \sqrt{2gh_2}) = -K < 0$, and $r_2 < u_2 + \sqrt{2gh_2}$ because $P(u_2 + \sqrt{2gh_2}) = -K < 0$. This configuration is illustrated on the following picture.



Graph of $\xi \mapsto P(\xi)$ when $K < gh_2$

Thus

$$|\mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)| \leq \left| r_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - r_2 \right|. \tag{A.67}$$

We set

$$\tilde{r}_1 = u_2 - \sqrt{2gh_2} + \frac{2K}{\sqrt{2gh_2}}, \tag{A.68}$$

and we notice that

$$\frac{2K}{\sqrt{2gh_2}} < \sqrt{2gh_2} \tag{A.69}$$

because of the assumption $K < gh_2$. Thus we obtain that

$$\tilde{r}_1 < u_2. \tag{A.70}$$

Moreover

$$\begin{aligned} P(\tilde{r}_1) =& gh_2 - \frac{(\tilde{r}_1 - u_2)^2}{2} - K \\ =& gh_2 - \frac{1}{2}\left(-\sqrt{2gh_2} + \frac{2K}{\sqrt{2gh_2}} \right)^2 - K \\ =& -\frac{K^2}{gh_2} + 2K - K = K\left(1 - \frac{K}{gh_2} \right) > 0. \end{aligned} \tag{A.71}$$

With (A.70) we deduce that

$$r_1 < \tilde{r}_1 < u_2. \tag{A.72}$$

Similarly we set

$$\tilde{r}_2 = u_2 + \sqrt{2gh_2} - \frac{2K}{\sqrt{2gh_2}}, \tag{A.73}$$

and by the same arguments we obtain that

$$u_2 < \tilde{r}_2 < r_2. \tag{A.74}$$

Putting together (A.72) and (A.74), we get

$$\begin{aligned} & \left| r_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - r_2 \right| \\ \leq & \left| \tilde{r}_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - \tilde{r}_2 \right| \\ = & \frac{4K}{\sqrt{2gh_2}}. \end{aligned} \tag{A.75}$$

With (A.67) we get (A.62) in the case $K < gh_2$, and this concludes the proof of Lemma A.4.

□

Lemma A.5. *There exists some $C > 0$ depending only on g, h_m, h_M, u_M such that*

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2) (M_1 - M_2)^2 d\xi \\ \geq & C \left(g \frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2} \right), \end{aligned} \tag{A.76}$$

for all U_1, $U_2 \in \mathcal{U}_{h_m, h_M, u_M}$ defined by (1.37) and where $M_k \equiv M_k(\xi) \equiv M(U_k, \xi)$, with $M(U, \xi)$ defined by (1.8).

Proof. Let $U_1, U_2 \in \mathcal{U}_{h_m, h_M, u_M}$. According to Lemma A.2 we have

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2) (M_1 - M_2)^2 d\xi \\ = & \int_{\mathbb{R}} |\xi| \left(H_0(M_2) - H_0(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \\ & - \int_{(\xi - u_1)^2 > 2gh_1} |\xi| M_2 \left(\frac{(\xi - u_1)^2}{2} - gh_1 \right) d\xi. \end{aligned} \tag{A.77}$$

Let us first consider the case of data

$$U_1, U_2 \text{ such that } |h_1 - h_2| \leq \frac{1}{4\widetilde{C}_1^2} \text{ and } |u_1 - u_2| \leq \frac{1}{4\widetilde{C}_2^2}, \tag{A.78}$$

for some positive constants $\widetilde{C}_1$, $\widetilde{C}_2$ depending on g, h_m, h_M, u_M such that $4\widetilde{C}_2^2 \geq \frac{1}{\sqrt{gh_m}}$. These constants will be chosen further on.

For data satisfying (A.78), we are going to estimate the right-hand side of (A.77). On the one hand, in order to estimate the first term in the RHS of (A.77), we apply Lemma A.3. Since $4\widetilde{C}_2^2 \geq \frac{1}{\sqrt{gh_m}}$ we are in the case (A.18) and we get

$$\begin{aligned}&\int_{\mathbb{R}} |\xi| \left(H_0(M_2) - H_0(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \\ \geq & \alpha_1 \left(\eta(U_2) - \eta(U_1) - \eta'(U_1) (U_2 - U_1) \right) \\ = & \alpha_1 \left(g \frac{(h_2 - h_1)^2}{2} + h_2 \frac{(u_2 - u_1)^2}{2} \right) \\ \geq & \alpha_1 \left(g \frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2} \right), \end{aligned} \tag{A.79}$$

with $\alpha_1 = C_0\sqrt{gh_m}/2\pi$ and $C_0 > 0$ is and absolute constant. On the other hand, in order to estimate the second term in the RHS of (A.77), we apply Lemma A.4 and obtain

$$\begin{aligned}
&\int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2} - gh_1\right) d\xi \\
&\leq \frac{4\left(|u_2| + \sqrt{2gh_2}\right)}{g\pi\sqrt{gh_2}}\left(g|h_1-h_2| + (|u_2| + \sqrt{2gh_2})|u_1-u_2| + \frac{1}{2}|u_1^2 - u_2^2|\right)^{\frac{5}{2}} \\
&\leq C_1(g,h_m,h_M,u_M)\left(g|h_1-h_2| + C_2(g,h_m,h_M,u_M)|u_1-u_2|\right)^{\frac{5}{2}}, \qquad \text{(A.80)}
\end{aligned}$$

with

$$C_1(g,h_m,h_M,u_M) = \frac{4\left(u_M + \sqrt{2gh_M}\right)}{g\pi\sqrt{gh_m}}, \qquad \text{(A.81)}$$

$$C_2(g,h_m,h_M,u_M) = 2u_M + \sqrt{2gh_M}. \qquad \text{(A.82)}$$

Using the Jensen inequality we have for $a, b \geq 0$,

$$(a+b)^{5/2} = 2^{5/2}\left(\frac{a+b}{2}\right)^{5/2} \leq 2^{5/2}\frac{a^{5/2}+b^{5/2}}{2}, \qquad \text{(A.83)}$$

we get

$$\begin{aligned}
&\int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2} - gh_1\right) d\xi \\
&\leq 2^{3/2} C_1(g,h_m,h_M,u_M)\left(g^{\frac{5}{2}}|h_1-h_2|^{\frac{5}{2}} + C_2(g,h_m,h_M,u_M)^{\frac{5}{2}}|u_1-u_2|^{\frac{5}{2}}\right). \qquad \text{(A.84)}
\end{aligned}$$

Thus, using the estimates (A.79) and (A.84) in the RHS of (A.77), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6}\left(2M_1 + M_2\right)(M_1 - M_2)^2 d\xi \\
&\geq \alpha_1\left(g\frac{(h_2-h_1)^2}{2} + h_m\frac{(u_2-u_1)^2}{2}\right) \\
&\quad - 2^{3/2}C_1\left(g^{\frac{5}{2}}|h_1-h_2|^{\frac{5}{2}} + C_2^{\frac{5}{2}}|u_1-u_2|^{\frac{5}{2}}\right) \\
&= \alpha_1\frac{g(h_2-h_1)^2}{2}\left(1 - \widetilde{C}_1|h_1-h_2|^{\frac{1}{2}}\right) \\
&\quad + \alpha_1\frac{h_m(u_2-u_1)^2}{2}\left(1 - \widetilde{C}_2|u_1-u_2|^{\frac{1}{2}}\right), \qquad \text{(A.85)}
\end{aligned}$$

with

$$\widetilde{C}_1 = \frac{2^{3/2+1} C_1 g^{\frac{5}{2}}}{\alpha_1 g}, \qquad \widetilde{C}_2 = \frac{2^{3/2+1} C_1 C_2^{\frac{5}{2}}}{\alpha_1 h_m}. \tag{A.86}$$

One can check that $\widetilde{C}_2 > (g h_m)^{-1/4}/2$. From (A.85), since we are dealing with U_1, U_2 satisfying (A.78), we get

$$\begin{aligned} &\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2 d\xi \\ &\geq \frac{\alpha_1}{2} \left(g \frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2} \right). \end{aligned} \tag{A.87}$$

At this point we have the result (A.76) for all $U_1, U_2 \in \mathcal{U}_{h_m, h_M, u_M}$ satisfying (A.78). Thus, since the right-hand side of (A.76) is bounded, it is now sufficient to prove that

$$\begin{aligned} &\exists \alpha_3 > 0, \quad \forall U_1, U_2 \in \mathcal{U}_{h_m, h_M, u_M} \text{ such that} \\ &|h_1 - h_2| > \frac{1}{4\widetilde{C}_1^2} \text{ or } |u_1 - u_2| > \frac{1}{4\widetilde{C}_2^2}, \\ &\text{we have } \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2 d\xi \geq \alpha_3. \end{aligned} \tag{A.88}$$

Using a reductio ad absurdum as in the proof of Lemma A.3, we suppose that (A.88) does not hold. Thus

$$\begin{aligned} &\forall n > 0, \quad \exists U_1^n, U_2^n \in \mathcal{U}_{h_m, h_M, u_M}, \text{ such that} \\ &4\widetilde{C}_1^2 |h_1^n - h_2^n| + 4\widetilde{C}_2^2 |u_1^n - u_2^n| > 1 \\ &\text{and } \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1^n + M_2^n)(M_1^n - M_2^n)^2 d\xi \leq \frac{1}{n}, \end{aligned} \tag{A.89}$$

where $M_k^n = M(U_k^n, \xi)$. As $\mathcal{U}_{h_m, h_M, u_M}$ is a closed and bounded set, we can extract a subsequence such that

$$U_1^n \to U_1 \in \mathcal{U}_{h_m, h_M, u_M}, \quad U_2^n \to U_2 \in \mathcal{U}_{h_m, h_M, u_M} \tag{A.90}$$

with

$$4\widetilde{C}_1^2 |h_1 - h_2| + 4\widetilde{C}_2^2 |u_1 - u_2| \geq 1, \tag{A.91}$$

and by Lebesgue's theorem

$$\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2 d\xi = 0. \tag{A.92}$$

Therefore we get

$$(2M_1 + M_2)\,(M_1 - M_2)^2 = 0 \quad \text{for almost all } \xi, \tag{A.93}$$

itself implying that $M_1 = M_2$ a.e. and therefore $U_1 = U_2$, in contradiction with (A.91). This concludes the proof of Lemma A.5. □

Lemma A.6. *Let $U_k = (h_k, h_k u_k)$, $k = 1, 2$ with $h_k \geq 0$. Then*

$$\begin{aligned}&\int_{\mathbb{R}} |M(U_1,\xi) - M(U_2,\xi)|\, d\xi \\ &\leq \frac{2\sqrt{3}}{\sqrt{g}} \left(g(h_2 - h_1)^2 + \min(h_1,h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}},\end{aligned} \tag{A.94}$$

with $M(U,\xi)$ defined by (1.8).

Proof. Let us recall that from [5, Lemma 3.11] one has

$$\begin{aligned}&\int_{\mathbb{R}} M(U_1,\xi)\,(M(U_1,\xi) - M(U_2,\xi))^2\, d\xi \\ &\leq \frac{3}{g^2\pi^2} \left(g(h_2 - h_1)^2 + \min(h_1,h_2)(u_2 - u_1)^2\right).\end{aligned} \tag{A.95}$$

Then using the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{\mathbb{R}} |M(U_1,\xi) - M(U_2,\xi)|\, d\xi \\ &\leq \int_{M_1>0} |M_1 - M_2|\, d\xi + \int_{M_2>0} |M_1 - M_2|\, d\xi \\ &\leq \left(\int_{M_1>0} \frac{1}{M_1} d\xi\right)^{1/2} \left(\int_{M_1>0} M_1\,(M_1 - M_2)^2\, d\xi\right)^{1/2} \\ &\quad + \left(\int_{M_2>0} \frac{1}{M_2} d\xi\right)^{1/2} \left(\int_{M_2>0} M_2\,(M_1 - M_2)^2\, d\xi\right)^{1/2}.\end{aligned} \tag{A.96}$$

Using the substitution $v = \frac{\xi - u}{\sqrt{2gh}}$ we get

$$\begin{aligned}&\int_{M(U,\xi)>0} \frac{1}{M(U,\xi)} d\xi = \int_{u-\sqrt{2gh}}^{u+\sqrt{2gh}} \frac{g\pi}{(2gh - (\xi - u)^2)^{1/2}} d\xi \\ &= \int_{-1}^{1} \frac{g\pi\sqrt{2gh}}{\sqrt{2gh}\,(1 - v^2)^{1/2}} dv = g\pi \Big[\arcsin(v)\Big]_{-1}^{1} = g\pi^2.\end{aligned} \tag{A.97}$$

Now from (A.96), using (A.95) and (A.97) we get (A.94), which concludes the proof. □

Lemma A.7. *Let $U_k = (h_k, h_k u_k)$, $k = 1, 2$ with $h_k \geq 0$, and set*

$$C_4 = \max_{v \in \left\{|u_1| + \sqrt{2gh_1}, |u_2| + \sqrt{2gh_2}\right\}} |v| \left(1 + \nu^2 v^2\right)^{\frac{1}{2}}, \tag{A.98}$$

for some given $\nu > 0$. Then one has denoting $|||(x_1, x_2)|||^2 = x_1^2 + \nu^2 x_2^2$,

$$|||F(U_1) - F(U_2)||| \leq \frac{2\sqrt{3}}{\sqrt{g}} C_4 \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}}, \tag{A.99}$$

$$|||F^+(U_1) - F^+(U_2)||| \leq \frac{2\sqrt{3}}{\sqrt{g}} C_4 \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}}, \tag{A.100}$$

$$|||F^-(U_1) - F^-(U_2)||| \leq \frac{2\sqrt{3}}{\sqrt{g}} C_4 \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}}. \tag{A.101}$$

Proof. We recall that from (1.27),

$$\begin{aligned} F^+(U) &= \int_{\mathbb{R}} \xi \mathbb{1}_{\xi > 0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U, \xi) d\xi, \\ F^-(U) &= \int_{\mathbb{R}} \xi \mathbb{1}_{\xi < 0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U, \xi) d\xi, \\ \text{and} \quad F(U) &= F^+(U) + F^-(U) = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U, \xi) d\xi. \end{aligned} \tag{A.102}$$

Thus the result is an immediate consequence of Lemma A.6 and of the fact that

$$\forall \xi \in \mathrm{supp} M_1 \cup \mathrm{supp} M_2, \quad |||\xi \begin{pmatrix} 1 \\ \xi \end{pmatrix}||| \leq C_4. \tag{A.103}$$

□

Lemma A.8. *With the assumptions of Theorem 1.1, let U_Δ be the approximate solution* (1.32) *and $\phi \in \mathcal{D}(\mathbb{R})$. Then there exists some $C > 0$ depending only on the available bounds and on ϕ such that*

$$\forall t \in [0, T], \quad < \partial_t U_\Delta(t, \cdot), \phi > \leq C. \tag{A.104}$$

Proof. Using (2.43) we get for any $t_n \leq t < t_{n+1}$

$$< \partial_t U_\Delta, \phi >= A + B, \tag{A.105}$$

with

$$A = \sum_i \frac{1}{\Delta t} \left[\frac{U_{i+1}^{n+1} - U_i^{n+1} - U_{i+1}^n + U_i^n}{\Delta x} \right] \int_{x_i}^{x_{i+1}} (x - x_i)\phi(x)dx \tag{A.106}$$

and

$$B = \sum_i \frac{1}{\Delta t} \left[U_i^{n+1} - U_i^n \right] \int_{x_i}^{x_{i+1}} \phi(x)dx. \tag{A.107}$$

First we notice that

$$\int_{x_i}^{x_{i+1}} (x - x_i)\phi(x)dx = \Delta x \psi(x_{i+1}) - \int_{x_i}^{x_{i+1}} \psi(x)dx, \tag{A.108}$$

where ψ is an antiderivative of ϕ. Thus we get

$$A = \sum_i \frac{1}{\Delta t} \left[\frac{U_{i+1}^{n+1} - U_i^{n+1} - U_{i+1}^n + U_i^n}{\Delta x} \right] \Delta x \Delta \psi_{i+1/2}, \tag{A.109}$$

with $\Delta\psi_{i+1/2} := \psi(x_{i+1}) - \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \psi(x)dx$. Moreover, by doing translations of indices we get

$$A = \sum_i \frac{1}{\Delta t} \left[U_i^{n+1} - U_i^n \right] \left[\Delta\psi_{i-1/2} - \Delta\psi_{i+1/2} \right]. \tag{A.110}$$

Next, using that U_i^n is bounded we get from (1.19) that

$$|U_i^{n+1} - U_i^n| \leq 2\frac{\Delta t}{\Delta x} \left(\|F^+(U)\|_\infty + \|F^-(U)\|_\infty + \|gh^2\|_\infty \right). \tag{A.111}$$

Moreover we notice that

$$\left| \Delta\psi_{i-1/2} - \Delta\psi_{i+1/2} \right| \leq \Delta x^2 \mathrm{Lip}(\phi), \tag{A.112}$$

which enables us to get

$$|A| \leq 2C\Delta x \mathrm{Lip}(\phi) \sum_i \mathbf{1}_{\mathrm{dist}(x_i, \mathrm{supp}\phi) \leq \Delta x} \leq C', \tag{A.113}$$

with $C' > 0$ a constant depending on ϕ. Next, from (A.107), we use (1.19) and get

$$B = -\sum_i \frac{1}{\Delta x} \left[F_{i+1/2-} - F_{i-1/2+} \right] \Delta x \phi_{i+1/2}, \tag{A.114}$$

with $\phi_{i+1/2} := \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \phi(x)dx$. Using (1.20)-(1.22) we get

$$\begin{aligned} B &= -\sum_i \left[\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right] \phi_{i+1/2} - \sum_i \left[-S_{i+1/2-} - S_{i-1/2+}\right] \phi_{i+1/2} \\ &= -\sum_i \mathcal{F}_{i+1/2} \left[\phi_{i+1/2} - \phi_{i+3/2}\right] + \sum_i \left[S_{i+1/2-} + S_{i-1/2+}\right] \phi_{i+1/2}. \end{aligned} \tag{A.115}$$

We have

$$|\phi_{i+1/2} - \phi_{i+3/2}| \leq \Delta x \mathrm{Lip}(\phi), \tag{A.116}$$

and

$$|S_{i+1/2-} + S_{i-1/2+}| \leq v_m^2(|z_{i+1} - z_i| + |z_i - z_{i-1}|). \tag{A.117}$$

We thus conclude that B is also bounded since $\partial_x z_\Delta$ is bounded in L^1_{loc}. □

Lemma A.9. *With the assumptions of Theorem 1.1, let $U_\Delta = (h_\Delta, h_\Delta u_\Delta)$ be the approximate solution* (1.32)*, and $\widetilde{S}_\Delta$ be the approximate source defined by* (3.6)*. We assume that there exists U such that U_Δ tends to U a.e. as $\Delta x, \Delta t \to 0$. Then*

$$\forall \phi(t,x) \in \mathcal{D}(\mathbb{R}^2), \quad \iint \widetilde{S}_\Delta(t,x)\phi(t,x)\,dtdx \underset{\Delta x, \Delta t \to 0}{\longrightarrow} \iint S(t,x)\phi(t,x)\,dtdx, \tag{A.118}$$

with $S(t,x) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix}$.

Proof. Let $\phi(t,x) \in \mathcal{D}(\mathbb{R}^2)$. We compute the integral

$$\iint \widetilde{S}_\Delta(t,x)\phi(t,x)\,dtdx = \sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty} \Delta t \left(S_{i+1/2-} + S_{i-1/2+}\right) \phi_i^n,$$

with $\phi_i^n = \dfrac{1}{\Delta t}\dfrac{1}{\Delta x}\displaystyle\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \phi(t,x)dtdx$. Then we perform a translation of the index i and get

$$\begin{aligned} &\sum_n \sum_i \Delta t \left(S_{i+1/2-} + S_{i-1/2+}\right) \phi_i^n \\ =& \sum_n \Delta t \sum_i S_{i+1/2-}\phi_i^n + \sum_n \Delta t \sum_i S_{i+1/2+}\phi_{i+1}^n. \end{aligned}$$

Then we notice that $|\phi^n_{i+1} - \phi^n_i| \leq \mathrm{Lip}(\phi)\Delta x$ and we obtain that

$$\iint \widetilde{S}_\Delta \phi \, dtdx - \sum_n \sum_i \Delta t \left(S_{i+1/2-} + S_{i+1/2+} \right) \phi^n_i \to 0. \tag{A.119}$$

Next, for $\Delta x, |z_{i+1} - z_i|$ small enough, we have on the one hand using (1.42)

$$\begin{aligned} S^{hu}_{i+1/2-} =& g\frac{h^2_{i+1/2-}}{2} - g\frac{h^2_i}{2} \\ =& g\frac{(h_i + z_i - z_{i+1/2})^2}{2} - g\frac{h^2_i}{2} \\ =& g(z_i - z_{i+1/2}) \left(h_i + \frac{z_i - z_{i+1/2}}{2} \right). \end{aligned} \tag{A.120}$$

On the other hand, we have similarly

$$S^{hu}_{i+1/2+} = g\frac{h^2_{i+1}}{2} - g\frac{h^2_{i+1/2+}}{2} = -g(z_{i+1} - z_{i+1/2}) \left(h_{i+1} + \frac{z_{i+1} - z_{i+1/2}}{2} \right). \tag{A.121}$$

Moreover noticing that $h_{i+1} = h_i + (h_{i+1} - h_i)$, with (A.120), (A.121) we get

$$\begin{aligned} &\sum_n \sum_i \Delta t \left(S^{hu}_{i+1/2-} + S^{hu}_{i+1/2+} \right) \phi^n_i \\ &= -\sum_n \sum_i \Delta t \, g(z_{i+1} - z_i) h^n_i \phi^n_i + \sum_n \sum_i \Delta t \Delta x R^n_i \phi^n_i, \end{aligned} \tag{A.122}$$

with

$$\Delta x R^n_i = -g(z_{i+1} - z_{i+1/2}) \left(h^n_{i+1} - h^n_i \right) + g\frac{(z_i - z_{i+1/2})^2}{2} \quad -g\frac{(z_{i+1} - z_{i+1/2})^2}{2}. \tag{A.123}$$

The last term in the RHS of (A.122) tends to 0 because of (1.35), the Cauchy Schwarz inequality and the bound (2.3). The first term in the RHS of (A.122) converges to the source term since

$$\begin{aligned} &-\sum_n \sum_i \Delta t g(z_{i+1} - z_i) h^n_i \phi^n_i \\ &\simeq \iint -g\frac{\mathrm{d}z_\Delta(x)}{\mathrm{d}x} h_\Delta(t,x)\phi(t,x) \, dtdx \to \iint -g\frac{\mathrm{d}z(x)}{\mathrm{d}x} h(t,x)\phi(t,x) \, dtdx. \end{aligned}$$

The convergence holds because we supposed $h_\Delta \to h$ a.e., and $\frac{\mathrm{d}z_\Delta}{\mathrm{d}x} \to \frac{\mathrm{d}z}{\mathrm{d}x}$ in L^1_{loc}. This concludes the proof of the lemma. □

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