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Abstract

We prove the convergence of the hydrostatic reconstruction scheme with kinetic numerical flux for the Saint Venant system with Lipschitz continuous topography. We use a recently derived fully discrete sharp entropy inequality with dissipation, that enables us to establish an estimate in the inverse of the square root of the space increment Δx of the L^2 norm of the gradient of approximate solutions. By Diperna's method we conclude the strong convergence towards bounded weak entropy solutions.

Keywords: Saint Venant system with topography, well-balanced scheme, hydrostatic reconstruction, convergence, entropy inequality.

Mathematics Subject Classification: 65M12, 76M12, 35L65

1 Introduction and main result

We consider the Saint Venant system

$$
\begin{aligned}
&\partial_t h + \partial_x(hu) = 0,\\
&\partial_t(hu) + \partial_x(hu^2 + g\frac{h^2}{2}) + gh\partial_x z = 0,
\end{aligned}
\tag{1.1}
$$

for $t \geq 0$ and $x \in \mathbb{R}$, where the unknowns are $h(t,x) \geq 0$ and $u(t,x) \in \mathbb{R}$, $g > 0$ is the gravity constant, and the topography $z(x)$ is given. The system

*Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050), CNRS, UPEM, UPEC, F-77454, Marne-la-Vallée, France (`Francois.Bouchut@u-pem.fr`), (`Xavier.Lhebrard@u-pem.fr`)

is completed with an entropy inequality

$$\partial_t\left(h\frac{u^2}{2}+g\frac{h^2}{2}+ghz\right)+\partial_x\left((h\frac{u^2}{2}+gh^2+ghz)u\right)\leq 0. \tag{1.2}$$

We shall denote $U=(h,hu)$ with $h\geq 0$, and

$$\eta(U)=h\frac{u^2}{2}+g\frac{h^2}{2},\quad G(U)=\left(h\frac{u^2}{2}+gh^2\right)u, \tag{1.3}$$

the entropy and entropy fluxes without topography.

For this system we have some existence and stability results [27, 21, 23, 29]. Concerning the approximation of this system, several schemes have been invastigated [25, 4, 3, 6, 11, 15, 10] and we have some results concerning consistency, stability and convergence of those schemes [9, 12, 26, 8, 2].

In this context this paper gives a proof of convergence for the hydrostatic reconstruction scheme [4] with kinetic flux [25]. Our result is a consequence of the work [5], which states that the hydrostatic reconstruction scheme, used with the classical kinetic solver, satisfies a fully discrete entropy inequality with an error term. In the case without topography, the error terms vanish and we have the following inequality:

$$\begin{aligned}\eta(U_i^{n+1})\leq \eta(U_i^n)&-\frac{\Delta t}{\Delta x}\left(\widetilde{G}_{i+1/2}-\widetilde{G}_{i-1/2}\right)\\ -\nu_\beta\frac{\Delta t}{\Delta x}\int_{\mathbb{R}}|\xi|\frac{g^2\pi^2}{6}\Big(&\mathbb{1}_{\xi<0}\left(M_{i+1/2+}+M_{i+1/2-}\right)\left(M_{i+1/2+}-M_{i+1/2-}\right)^2\\ &+\mathbb{1}_{\xi>0}\left(M_{i-1/2+}+M_{i-1/2-}\right)\left(M_{i-1/2+}-M_{i-1/2-}\right)^2\Big)d\xi. \end{aligned} \tag{1.4}$$

In the time-only discrete case and without topography, this single energy inequality that holds for the kinetic scheme ensures the convergence [9]. The fully-discrete case (still without topography) is treated in [8] and the result is given under the dissipation assumption that F^+ or $-F^-$ (defined in (1.25)) are η dissipative . Unfortunatly this property does not hold for the scheme we considerer, there is a lack of dissipativity of the kinetic scheme. Thus the new contribution of this paper is to give a proof for the convergence in the case of non constant topography, under weaker dissipation assumptions.

Let us give here some of the main ideas of our proof. First thing had been to find a weaker dissipation property that enables us to prove the convergence. In that matter we found that F^+-F^- is η dissipative, which corresponds to the following inequality

$$\int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi$$
$$\geq \alpha \left(\eta(U_2) - \eta(U_1) - \eta'(U_1) \left(U_2 - U_1 \right) \right). \tag{1.5}$$

A rigorous statement of this result can be founded in lemma 5.2 and we can point out that it is only valid on an open bounded convex set which does not content zero value for the height, and the constant α is not explicit.

In order to go further in the proof we multiply by time increment Δt (1.4) and sum over indices i and n. Then after doing some additional computations we are able to use (1.5), as a consequence we get a gradient estimate and we conclude by a compensated compactness result. Indeed we recall that the compensated compactness theory [27] gives the compactness on a bounded sequence of approximated solutions of the system (U_ε) which satisfies the relations

$$\partial_t \eta(U_\varepsilon) + \partial_x G(U_\varepsilon) \tag{1.6}$$

to be compact in H^{-1}_{loc}, on a sufficiently large family of entropies. In our work, we found an estimate on the gradient of the approximate numerical solutions (U_Δ):

$$\|\partial_t U_\Delta\|_{L^2_{tx}} \leq \frac{C}{\sqrt{\Delta x}}, \quad \|\partial_x U_\Delta\|_{L^2_{tx}} \leq \frac{C}{\sqrt{\Delta x}}, \tag{1.7}$$

where Δx is the space increment of the scheme. This is the key point of the method. Indeed, the estimates (1.7) are enough as in Di Perna approximation technique [21] to control all entropies as (1.6).

We have non-uniqueness of Riemann solution for nonconstant topography, concerning numerical issues associated see [3, 30].

1.1 Saint Venant system

Before going into discretised models, we recall the classical kinetic Maxwellian equilibrium, used in [25] for example, at the coninuous level. The kinetic Maxwellian is given by

$$M(U,\xi) = \frac{1}{g\pi} \left(2gh - (\xi - u)^2 \right)_+^{1/2}, \tag{1.8}$$

where $\xi \in \mathbb{R}$ and $x_+ \equiv \max(0,x)$ for any $x \in \mathbb{R}$. It satisfies the following moment relation,

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U, \xi) d\xi = U. \tag{1.9}$$

The interest of this particular form lies in its link with a kinetic entropy. Consider the kinetic entropy,

$$H(f,\xi,z) = \frac{\xi^2}{2}f + \frac{g^2\pi^2}{6}f^3 + gzf, \tag{1.10}$$

where $f \geq 0$, $\xi \in \mathbb{R}$ and $z \in \mathbb{R}$, and its version without topography

$$H_0(f,\xi) = \frac{\xi^2}{2}f + \frac{g^2\pi^2}{6}f^3. \tag{1.11}$$

Then one can check the relations

$$\int_{\mathbb{R}} H(M(U,\xi),\xi,z)d\xi = \eta(U) + ghz, \tag{1.12}$$

$$\int_{\mathbb{R}} \xi H(M(U,\xi),\xi,z)d\xi = G(U) + ghzu. \tag{1.13}$$

Moreover, for any $f(\xi) \geq 0$, setting $h = \int f(\xi)d\xi$, $hu = \int \xi f(\xi)d\xi$ (assumed finite), one has the following entropy minimization principle [5],

$$\eta(U) = \int_{\mathbb{R}} H_0(M(U,\xi),\xi)d\xi \leq \int_{\mathbb{R}} H_0(f(\xi),\xi)d\xi. \tag{1.14}$$

1.2 Hydrostatic reconstruction and kinetic flux

We consider a time-step Δt and an uniform grid $(x_{i+1/2})_{i\in\mathbb{Z}}$ with space increment $\Delta x = x_{i+1/2} - x_{i-1/2}$, we set $x_{i-1/2} = i\Delta x$ and $t_n = n\Delta t$. Let $U^0 = (h^0, h^0u^0)$, $h^0 \geq 0$, $h^0, u^0 \in L^\infty(\mathbb{R})$ and $x \mapsto z(x)$, assumed Lipschitz continuous, be an initial data. We define the discretization of the initial data as

$$U_i^0 = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U^0(y)dy, \tag{1.15}$$

and

$$z_i \text{ an approximation of } z(x_i), \tag{1.16}$$

where $x_i = \left(x_{i+1/2} + x_{i-1/2}\right)/2$. Then the scheme writes

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2-} - F_{i-1/2+}\right), \tag{1.17}$$

with

$$F_{i+1/2-} = \mathcal{F}(U_{i+1/2-}, U_{i+1/2+}) - S_{i+1/2-}, \tag{1.18}$$

$$F_{i-1/2+} = \mathcal{F}(U_{i-1/2-}, U_{i-1/2+}) + S_{i-1/2+}, \tag{1.19}$$

with $\mathcal{F}$ is a numerical flux for the system without topography. The source terms $S_{i+1/2-}$, $S_{i-1/2+}$ are defined by

$$S_{i+1/2-} = \begin{pmatrix} 0 \\ g\frac{h_{i+1/2-}^2}{2} - g\frac{h_i^2}{2} \end{pmatrix}, \quad S_{i-1/2+} = \begin{pmatrix} 0 \\ g\frac{h_i^2}{2} - g\frac{h_{i-1/2+}^2}{2} \end{pmatrix}. \tag{1.20}$$

The reconstructed states

$$U_{i+1/2-} = (h_{i+1/2-}, h_{i+1/2-}u_i), \quad U_{i+1/2+} = (h_{i+1/2+}, h_{i+1/2+}u_i) \tag{1.21}$$

are defined by

$$h_{i+1/2-} = (h_i + z_i - z_{i+1/2})_+, \quad h_{i+1/2+} = (h_{i+1} + z_{i+1} - z_{i+1/2})_+ \tag{1.22}$$

and

$$z_{i+1/2} = \max(z_i, z_{i+1}). \tag{1.23}$$

We will use in this paper a kinetic numerical flux $\mathcal{F}$ introduced in [25]

$$\mathcal{F}(U_l, U_r) = F^+(U_l) + F^-(U_r), \tag{1.24}$$

$$\begin{aligned} F^+(U) &= \int_{\mathbb{R}} \xi \mathbb{1}_{\xi>0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi, \\ F^-(U) &= \int_{\mathbb{R}} \xi \mathbb{1}_{\xi<0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi, \end{aligned} \tag{1.25}$$

with $M(U,\xi)$ defined by (1.8).

We consider the velocity $v_m \geq 0$ such that for all i,

$$M(U_i, \xi) > 0 \Leftrightarrow |\xi| \leq v_m. \tag{1.26}$$

This means equivalently that $|u_i| + \sqrt{2gh_i} \leq v_m$. We consider a CFL condition strictly less than one,

$$v_m \frac{\Delta t}{\Delta x} \leq \beta < 1, \tag{1.27}$$

where β is a given constant.

1.3 Convergence result

Let (U_i^n, z_i) be the scheme defined by (1.15)-(1.8). We define the approximate solution, at fixed Δx, by

$$\begin{aligned} &U_\Delta(t,x) \\ &= \frac{1}{\Delta t}\left[\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1} - U_{i+1}^n + U_{i-1}^n}{2\Delta x}(x - x_{i-1/2}) + U_i^{n+1} - U_i^n\right](t - t_n) \\ &\qquad\quad + \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}(x - x_{i-1/2}) + U_i^n \\ &\text{for } x_{i-1/2} < x < x_{i+1/2} \text{ and } t_n \le t < t_{n+1}, \end{aligned} \tag{1.28}$$

and we set

$$z_\Delta(x) = \frac{z_{i+1} - z_i}{\Delta x}(x - x_i) + z_i, \quad \text{for } x_i < x < x_{i+1}. \tag{1.29}$$

Moreover, for $h_M > h_m > 0$ and $u_M \ge 0$, we set

$$\mathcal{U}_{h_m,h_M,u_M} = \{(h,u) \in \mathbb{R}^2, \quad h_m \le h \le h_M, \quad |u| \le u_M\} \tag{1.30}$$

which is a convex set. We state now the main result of this article, which is the proof of the convergence of the numerical scheme.

Theorem 1.1. *Let $U^0 = (h^0, h^0u^0)$, $h^0 \ge 0$, $h^0, u^0 \in L^\infty(\mathbb{R})$, be an initial data and let $z \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ and Lipschitz continuous be the given topography. Let (U_i^n, z_i) be the scheme defined by* (1.15)-(1.8). *Let U_Δ be the continuous approximate solution to* (1.1) *defined by* (1.28) *and z_Δ the approximate topography defined by* (1.29). *We assume that*

$$1 \le v^* \frac{\Delta t}{\Delta x}, \tag{1.31}$$

for some $v^* > 0$. Then let $h_m > 0$, $h_M > 0$ and $u_M > 0$. We assume that

$$\sup_i(|z_{i+1} - z_i|) < h_m, \tag{1.32}$$

which enables us to set $\tilde{h}_m = h_m - \sup_i(|z_{i+1} - z_i|) > 0$, and make the assumption

$$\forall i,n, \quad U_i^n \in \mathcal{U}_{\tilde{h}_m,h_M,u_M}, \tag{1.33}$$

with $\mathcal{U}_{h_m,h_M,u_M}$ defined by (1.30). *Moreover we assume*

$$\mathrm{Lip}(z_\Delta) \le C, \tag{1.34}$$

and that

$$z_\Delta \underset{\substack{\Delta t\to 0\\ \Delta x\to 0}}{\longrightarrow} z, \quad \frac{\mathrm{d}z_\Delta}{\mathrm{d}x} \underset{\substack{\Delta t\to 0\\ \Delta x\to 0}}{\longrightarrow} \frac{\mathrm{d}z}{\mathrm{d}x}, \text{ uniformly.} \tag{1.35}$$

Then, under the CFL condition (1.27)*, up to a subsequence, $U_\Delta \to U$ a.e. in* (t,x) *and in* $C_t([0,T], L^\infty_{x,w*}(\mathbb{R}))$ *as* $\Delta t \to 0$ *and* $\Delta x \to 0$ *where U is a weak solution to* (1.1) *with initial data U^0 satisfying the entropy condition*

$$\partial_t \eta(U) + \partial_x G(U) \in \mathcal{M}_{loc}, \tag{1.36}$$

for all suitable couple entropy-entropy flux (η, G) *and inequality* (1.2).

The outline of the parper is as follows. In section 2, we establish estimates on the gradient of the approximate solution as we mentionned in (1.7). In section 3, we introduce some interpolation functions and prove some regularity estimates on the approximate solution. In section 4 we prove theorem 1.1, first we obtain that (1.6) is compact in H^{-1}_{loc} by combining the gradient estimate and the regularity estimate, then we complete the proof by applying a compensated compactness result.

2 Estimate on the gradient of approximate solution

This section is devoted to the proof of proposition 2.1.

We have the following estimate on the approximate solution

Proposition 2.1. *Let $U^0 = (h^0, h^0u^0)$, $h^0 \geq 0$, $h^0, u^0 \in L^\infty(\mathbb{R})$, be an initial data and let $z \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ and Lipschitz continuous be the given topography. Let (U_i^n, z_i) be the scheme defined by* (1.15)-(1.8)*. Let U_Δ be the continuous approximate solution to* (1.1) *defined by* (1.28) *and z_Δ the approximate topography defined by* (1.29)*. Let $\beta > 0$ and v^*, h_m, h_M, u_M>0, involved respectively in assumption* (1.27) *and* (1.31)-(1.34)*. We define for all $U = (h, hu)$,*

$$|U|^2 = g\frac{h^2}{2} + \frac{u^2h^2}{2h_m}. \tag{2.1}$$

Let $N \in \mathbb{N}$, $T = N\Delta t$, i_0, $i_1 \in \mathbb{N}$ such that $i_0 < i_1$. For all $i < j \in \mathbb{N}$, we set

$$I^{v^*}_{i,j} = (x_{i-1/2} - v^*T, x_{j+1/2} + v^*T). \tag{2.2}$$

Then there exists some constants C_1, C_2, C_3 such that

$$\sum_{n=0}^{N-1} \sum_{i=i_0}^{i_1-1} \Delta t |U^n_{i+1} - U^n_i|^2 \leq C_1. \tag{2.3}$$

$$\sum_{n=0}^{N-1}\sum_{i=i_0}^{i_1-1}\Delta t|U_i^{n+1}-U_i^n|^2 \le C_1\frac{\Delta t^2}{\Delta x^2}v_m^2\left(1+v_m^2\right), \tag{2.4}$$

$$\left(\int_0^T\int_{x_{i_0-1/2}}^{x_{i_1+1/2}}|\partial_x U_\Delta|^2 dxdt\right)^{1/2} \le \frac{C_2}{\sqrt{\Delta x}}. \tag{2.5}$$

$$\left(\int_0^T\int_{x_{i_0-1/2}}^{x_{i_1+1/2}}|\partial_t U_\Delta|^2 dxdt\right)^{1/2} \le \frac{C_3}{\sqrt{\Delta x}}. \tag{2.6}$$

The constants C_1, C_2, C_3 depend only on gravity constant g, h_m, h_M, u_M, v_m, β, on final time T, on $|x_{i_0-1/2}-x_{i_1+1/2}|$, on $\|z\|_{L^\infty}$, $\|\eta(U_0)\|_{L^1(I^{v^*}_{i_0,i_1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0,i_1})}$.

We are able to find thoses estimates on $\partial_t U_\Delta$ and $\partial_x U_\Delta$ using recent results on discrete kinetic inequalities founded in [5]. The proof we will be developing is rather technical and we will use several lemmas in section 2.3. We put their demonstrations in the appendix in order to keep clarity of the demonstration.

2.1 Bounded propagation on the space integral of the height

Here we found some bound on $\sum_{i=i_0}^{i_1}\Delta x h_i^N$.

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(F^h_{i+1/2}-F^h_{i-1/2}\right), \tag{2.7}$$

with

$$F^h_{i+1/2} = \int_{\mathbb{R}}\xi\mathbb{1}_{\xi>0}M(U_i,\xi)d\xi + \int_{\mathbb{R}}\xi\mathbb{1}_{\xi<0}M(U_{i+1},\xi)d\xi. \tag{2.8}$$

We multiply by Δx and sum over index i and we obtain

$$\sum_{i=i_0}^{i_1}\Delta x h_i^{n+1} = \sum_{i=i_0}^{i_1}\Delta x h_i^n - \Delta t\left(F^h_{i_1+1/2}-F^h_{i_0-1/2}\right). \tag{2.9}$$

Then we notice that

$$-\Delta t F^h_{i_1+1/2} \le \Delta t v_m h_{i_1+1}, \quad \Delta t F^h_{i_0-1/2} \le \Delta t v_m h_{i_0-1}. \tag{2.10}$$

With CFL condition (1.27) we have

$$\sum_{i=i_0}^{i_1}\Delta x h_i^{n+1} \le \sum_{i=i_0-1}^{i_1+1}\Delta x h_i^n. \tag{2.11}$$

Let $N \in \mathbb{N}$ and $T = N\Delta t$, using the previous inequality we get

$$\sum_{i=i_0}^{i_1} \Delta x h_i^N \leq \sum_{i=i_0-N}^{i_1+N} \Delta x h_i^0 = \int_{x_{i_0-1/2-N}}^{x_{i_1+1/2+N}} h^0(x)dx. \tag{2.12}$$

The last equality holds because $h_i^0 = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} h^0(x)dx$. Moreover we have

$$x_{i_0-1/2-N} = x_{i_0-1/2} - N\Delta x = x_{i_0-1/2} - T\frac{\Delta x}{\Delta t}, \tag{2.13}$$

$$x_{i_1+1/2+N} = x_{i_1+1/2} + N\Delta x = x_{i_1+1/2} + T\frac{\Delta x}{\Delta t}. \tag{2.14}$$

Therefore by assumption (1.31) we get

$$\sum_{i=i_0}^{i_1} \Delta x h_i^N \leq \int_{x_{i_0-1/2-Tv^*}}^{x_{i_1+1/2+Tv^*}} h^0(x)dx = \|h^0\|_{L^1(I^{v^*}_{i_0,i_1})}, \tag{2.15}$$

with $I^{v^*}_{i_0,i_1}$ defined in (2.2).

2.2 From kinetic to macroscopic discrete entropy inequality

We use the notations introduced in proposition 2.1. Using CFL condition (1.27) we can use and integrate kinetic entropy inequality [5, Theorem 3.7] with respect to ξ and we obtain

$$\begin{aligned}
\eta(U_i^{n+1}) + gz_i h_i^{n+1} \leq\ & \eta(U_i^n) + gz_i h_i - \sigma_i\left(\widetilde{G}_{i+1/2} - \widetilde{G}_{i-1/2}\right)\\
& - \nu_\beta \frac{\Delta t}{\Delta x}\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6}\Big(\mathbb{1}_{\xi<0}\left(M_{i+1/2+} + M_{i+1/2-}\right)\left(M_{i+1/2+} - M_{i+1/2-}\right)^2\\
& \qquad + \mathbb{1}_{\xi>0}\left(M_{i-1/2+} + M_{i-1/2-}\right)\left(M_{i-1/2+} - M_{i-1/2-}\right)^2\Big)d\xi\\
& + C_\beta\left(\frac{\Delta t}{\Delta x}v_m\right)^2 \frac{g^2\pi^2}{6}\int_{\mathbb{R}} M_i\Big((M_i - M_{i+1/2-})^2 + (M_i - M_{i-1/2+})^2\Big)d\xi,
\end{aligned} \tag{2.16}$$

with

$$\widetilde{G}_{i+1/2} = \int_{\xi<0} \xi H(M_{i+1/2+}, \xi, z_{i+1/2})d\xi + \int_{\xi>0} \xi H(M_{i+1/2-}, \xi, z_{i+1/2})d\xi, \tag{2.17}$$

the constant $\nu_\beta > 0$ is a dissipation constant depending only on β, and $C_\beta \geq 0$ is a constant depending only on β. Using (1.9) and technical resultat over maxwellian functions (5.106), we get that

$$\begin{aligned}
\eta(U_i^{n+1}) + gz_i h_i^{n+1} \leq \eta(U_i^n) + gz_i h_i - \sigma_i \left(\widetilde{G}_{i+1/2} - \widetilde{G}_{i-1/2}\right) \\
- \nu_\beta \frac{\Delta t}{\Delta x} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \Big(\mathbb{1}_{\xi<0} \left(M_{i+1/2+} + M_{i+1/2-}\right) \left(M_{i+1/2+} - M_{i+1/2-}\right)^2 \\
+ \mathbb{1}_{\xi>0} \left(M_{i-1/2+} + M_{i-1/2-}\right) \left(M_{i-1/2+} - M_{i-1/2-}\right)^2 \Big) d\xi \\
+ C_\beta \left(\frac{\Delta t}{\Delta x} v_m\right)^2 \Big(g(h_i - h_{i+1/2-})^2 + g(h_i - h_{i-1/2+})^2 \Big). \quad (2.18)
\end{aligned}$$

Using the definition (1.22) we get that

$$0 \leq h_i - h_{i+1/2-} \leq |z_{i+1} - z_i|, \quad (2.19)$$

$$0 \leq h_i - h_{i-1/2+} \leq |z_i - z_{i-1}|, \quad (2.20)$$

and we deduce that

$$\begin{aligned}
\eta(U_i^{n+1}) + gz_i h_i^{n+1} \leq \eta(U_i^n) + gz_i h_i - \sigma_i \left(\widetilde{G}_{i+1/2} - \widetilde{G}_{i-1/2}\right) \\
- \nu_\beta \frac{\Delta t}{\Delta x} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \Big(\mathbb{1}_{\xi<0} \left(M_{i+1/2+} + M_{i+1/2-}\right) \left(M_{i+1/2+} - M_{i+1/2-}\right)^2 \\
+ \mathbb{1}_{\xi>0} \left(M_{i-1/2+} + M_{i-1/2-}\right) \left(M_{i-1/2+} - M_{i-1/2-}\right)^2 \Big) d\xi \\
+ gC_\beta \left(\frac{\Delta t}{\Delta x} v_m\right)^2 \left(|z_{i+1} - z_i|^2 + |z_i - z_{i-1}|^2 \right). \quad (2.21)
\end{aligned}$$

Then we follow the computations over height done in subsection 2.1. Thus we multiply by Δx, take the sum over i and make a translation over index i

in order to obtain

$$
\begin{aligned}
\sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^{n+1}) + g z_i h_i^{n+1}\right) \leq \sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^n) + \ g z_i h_i\right) \\
- \Delta t \widetilde{G}_{i_1+1/2} + \Delta t \widetilde{G}_{i_0-1/2} \\
- \nu_\beta \Delta t \sum_{i=i_0}^{i_1-1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi \\
- \nu_\beta \Delta t \int_{\mathbb{R}} |\xi| \mathbb{1}_{\xi<0} \frac{g^2\pi^2}{6} \left(M_{i_1+1/2+} + M_{i_1+1/2-}\right) (M_{i_1+1/2+} - M_{i_1+1/2-})^2 d\xi \\
- \nu_\beta \Delta t \int_{\mathbb{R}} |\xi| \mathbb{1}_{\xi>0} \frac{g^2\pi^2}{6} \left(M_{i_0-1/2+} + M_{i+1/2-}\right) (M_{i_0-1/2+} - M_{i_0-1/2-})^2 d\xi \\
+ \sum_{i=i_0-1}^{i_1} 2 g C_\beta \frac{\Delta t^2}{\Delta x} v_m \left|z_{i+1} - z_i\right|^2 .
\end{aligned}
\tag{2.22}
$$

We notice according to (2.17) that we have

$$
-\Delta t \widetilde{G}_{i_1+1/2} \leq v_m \Delta t \eta(U_{i_1+1/2+}) + v_m \Delta t g h_{i+1/2+} z_{i+1/2}, \tag{2.23}
$$

and

$$
-\Delta t \widetilde{G}_{i_0-1/2} \leq v_m \Delta t \eta(U_{i_0-1/2-}) + v_m \Delta t g h_{i-1/2-} z_{i-1/2}, \tag{2.24}
$$

with (1.27), $h_{i_1+1/2+} \leq h_{i+1}$ and $|z_{i_1+1/2} - z_{i_1+1}| \leq |z_{i_1+1} - z_{i_1}|$, it leads to

$$
-\Delta t \widetilde{G}_{i_1+1/2} \leq \Delta x \eta(U_{i_1+1}) + v_m \Delta t g h_{i_1+1} z_{i_1+1} + g h_M \Delta t |z_{i_1+1} - z_{i_1}|, \tag{2.25}
$$

and similarly we get

$$
-\Delta t \widetilde{G}_{i_0-1/2} \leq \Delta x \eta(U_{i_0-1}) + v_m \Delta t g h_{i_0-1} z_{i_0-1} + g h_M \Delta t |z_{i_0} - z_{i_0-1}|. \tag{2.26}
$$

From (2.22), noticing that the last two integrals are nonpositive and using (2.25),(2.26), we obtain

$$
\begin{aligned}
\sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^{n+1}) + g z_i h_i^{n+1}\right) \leq \sum_{i=i_0-1}^{i_1+1} \Delta x \left(\eta(U_i^n) + \ g z_i h_i\right) \\
- \nu_\beta \Delta t \sum_{i=i_0}^{i_1-1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi \\
+ g h_M \Delta t |z_{i_0} - z_{i_0-1}| + g h_M \Delta t |z_{i_1+1} - z_{i_1}| + \sum_{i=i_0}^{i_1} 2 g C_\beta \frac{\Delta t^2}{\Delta x} v_m \left|z_{i+1} - z_i\right|^2 .
\end{aligned}
\tag{2.27}
$$

Next, we sum now over index n and we use that $T = N\Delta t$ and that, by assumption (1.34) and (1.29), we have

$$|z_{i+1} - z_i| \leq C\Delta x, \tag{2.28}$$

and therefore we get

$$gTh_M|z_{i_0} - z_{i_0-1}| + gTh_M|z_{i_1+1} - z_{i_1}| + \sum_{n=0}^{N-1}\sum_{i=i_0}^{i_1} 2gC_\beta \frac{\Delta t^2}{\Delta x} v_m \left|z_{i+1} - z_i\right|^2 \leq C. \tag{2.29}$$

Thus we get

$$\begin{aligned}
&\sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^N) + gz_ih_i^N\right)\\
&+ \nu_\beta \sum_{n=0}^{N-1} \Delta t \sum_{i=i_0}^{i_1-1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi\\
&\qquad\qquad \leq \sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U_i^0) + \ gz_ih_i^0\right) + C
\end{aligned} \tag{2.30}$$

with C depending on g, T, h_M, β, v_m, $|x_{i_0-1/2} - x_{i_1+1/2}|$. Now we will see that the integral in LHS of (2.30) is underestimated by a term proportionnal to $\sum_{n=0}^{N-1} \sum_{i=i_0-1}^{i_1+1} \Delta t |U_{i+1/2+} - U_{i-1/2+}|^2$.

2.3 Lower estimate of dissipation terms

First we notice that

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi\\
&\geq \frac{1}{2} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(2M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi.
\end{aligned} \tag{2.31}$$

Now using lemma 5.4, we obtain that there exists some $C > 0$ depending only on g, h_m, h_M, u_M such that

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(2M_1 + M_2\right) (M_1 - M_2)^2 d\xi\\
\geq& C\left(g\frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2}\right)
\end{aligned} \tag{2.32}$$

for every U_1 , $U_2 \in \mathfrak{U}_{h_m,h_M,u_M}$.

Next we notice using the definitions (1.21) and the assumption (1.33) we get that

$$U_{i+1/2+}, U_{i+1/2-} \in \mathfrak{U}_{h_m,h_M,u_M}. \tag{2.33}$$

Thus from (2.31) and applying the last estimate (2.32) with $U_1 = U_{i+1/2+}$ and $U_2 = U_{i+1/2-}$, there exists some constant $C > 0$ depending only on g, h_m, h_M, u_M such that

$$\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(M_{i+1/2+} + M_{i+1/2-}\right) (M_{i+1/2+} - M_{i+1/2-})^2 d\xi \\ \geq C|U_{i+1/2+} - U_{i+1/2-}|^2 \tag{2.34}$$

where $|\cdot|$ is defined in (2.1).

2.4 Estimate of discrete gradient

Now we use (2.34) in (2.30) and we get

$$\nu_\beta C \sum_{n=0}^{N-1} \sum_{i=i_0}^{i_1-1} \Delta t |U_{i+1/2+} - U_{i+1/2-}|^2 \\ \leq \sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U_i^0) + \ g z_i h_i\right) - \sum_{i=i_0}^{i_1} \Delta x \left(\eta(U_i^N) + g z_i h_i^N\right) + C. \tag{2.35}$$

with C depending on g, T, h_M, β, v_m, $|x_{i_0-1/2} - x_{i_1+1/2}|$.

Then we notice that $\eta(U_i^N) \geq 0$ and we get

$$\nu_\beta C \sum_{n=0}^{N-1} \sum_{i=i_0}^{i=i_1-1} \Delta t |U_{i+1/2+} - U_{i+1/2-}|^2 \\ \leq \sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U_i^0) + \ g z_i h_i\right) + \sum_{i=i_0}^{i_1} \Delta x \left(-g z_i h_i^N\right) + C. \tag{2.36}$$

Next, using (1.16) and the fact we assumed $z \in L^\infty(\mathbb{R})$ we get $\forall i$, $z_i \leq \|z\|_\infty$, we have

$$g z_i h_i^0 \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} g\|z\|_\infty h^0(x)\ dx, \tag{2.37}$$

Moreover, by convexity of $(h, hu) \mapsto \eta(U)$, we have

$$\eta(U_i^0) \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \eta(U^0(x))\ dx. \tag{2.38}$$

Combining the last two results and summing over n we get

$$\sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U_i^0) + \ g z_i h_i^0\right) \ \le \int_{x_{i_0-1/2-N}}^{x_{i_1+1/2+N}} \eta\left(U^0(x)\right) + g\|z\|_\infty h^0(x)\ dx. \tag{2.39}$$

One notice that $x_{i_0-1/2-N} = x_{i_0} - N\Delta x = x_{i_0} - T\frac{\Delta x}{\Delta t}$ and by finite propagation hypothesis (1.31) one deduce that

$$\sum_{i=i_0-N}^{i_1+N} \Delta x \left(\eta(U_i^0) + \ g z_i h_i^0\right) \le \int_{x_{i_0-1/2-v^*T}}^{x_{i_1+1/2+v^*T}} \eta\left(U^0(x)\right) + g\|z\|_\infty h^0(x)\ dx \tag{2.40}$$

$$= \|\eta(U_i^0)\|_{L^1(I_{i_0,i_1}^{v^*})} + g\|z\|_\infty \|h^0\|_{L^1(I_{i_0,i_1}^{v^*})}, \tag{2.41}$$

with $I_{i_0,i_1}^{v^*}$ defined in (2.2). In addition, by preliminary computation (2.15), we have

$$\sum_{i=i_0}^{i_1} \Delta x \left(-g z_i h_i^N\right) \le g\|z\|_\infty \sum_{i=i_0}^{i_1} \Delta x h_i^N \le g\|z\|_\infty \|h^0\|_{L^1(I_{i_0,i_1}^{v^*})}. \tag{2.42}$$

Using together (2.40), (2.42) and (2.29) in (2.36), we get

$$\sum_{n=0}^{N-1}\sum_{i=i_0}^{i_1-1} \Delta t |U_{i+1/2+} - U_{i+1/2-}|^2 \le C \tag{2.43}$$

where C depends on g, h_m, h_M, u_M, v_m, β, T, on $|x_{i_0-1/2} - x_{i_1+1/2}|$, $\|z\|_{L^\infty}$, $\|\eta(U_0)\|_{L^1(I_{i_0,i_1}^{v^*})}$ and $\|h^0\|_{L^1(I_{i_0,i_1}^{v^*})}$. Moreover using triangle inequality and (1.21)-(1.23),(2.1)(2.20),(2.19) there exist some absolute constant C such that

$$\begin{aligned}
&|U_{i+1} - U_i|^2 \\
\le& C|U_{i+1/2+} - U_{i+1/2-}|^2 + C|U_{i+1/2+} - U_{i+1}|^2 + C|U_{i+1/2-} - U_i|^2, \\
\le& C|U_{i+1/2+} - U_{i+1/2-}|^2 + C|z_{i+1} - z_i|^2 + C|z_i - z_{i-1}|^2, \\
\le& C|U_{i+1/2+} - U_{i+1/2-}|^2 + 2C\Delta x^2.
\end{aligned} \tag{2.44}$$

Last inequality holds because of (2.28). With (2.43), we get (2.3) of proposition 2.1.

In addition, using (1.17), (5.112), (5.113) and (2.3), we get (2.4) of proposition 2.1.

2.5 End of the proof of proposition 2.1: estimate the gradient of the approximate solution

Now from (1.28) we compute

$$\partial_x U_\Delta = \frac{t-t_n}{\Delta t}\frac{U_{i+1}^{n+1}-U_{i-1}^{n+1}-U_{i+1}^n+U_{i-1}^n}{2\Delta x}+\frac{U_{i+1}^n-U_{i-1}^n}{2\Delta x} \tag{2.45}$$

and using the triangle inequality we obtain that

$$\begin{aligned}&\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}|\partial_x U_\Delta|^2 dxdt\\ &\le C\frac{\Delta t}{\Delta x}\left[|U_{i+1}^{n+1}-U_i^{n+1}|^2+|U_i^{n+1}-U_{i-1}^{n+1}|^2+|U_{i+1}^n-U_i^n|^2+|U_i^n-U_{i-1}^n|^2\right].\end{aligned} \tag{2.46}$$

with $C>0$ an absolute constant. In consequence, by using (2.3) we get (2.5) by summing over i and n. Similarly, from (1.28) we compute

$$\partial_t U_\Delta = \frac{1}{\Delta t}\left[\frac{U_{i+1}^{n+1}-U_{i-1}^{n+1}-U_{i+1}^n+U_{i-1}^n}{2\Delta x}(x-x_{i-1/2})+U_i^{n+1}-U_i^n\right] \tag{2.47}$$

thus

$$\begin{aligned}&\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}}|\partial_t U_\Delta|^2 dxdt\\ &\le \frac{\Delta x}{\Delta t}\left[|U_{i+1}^{n+1}-U_{i+1}^n|^2+|U_{i-1}^n-U_{i-1}^{n+1}|^2+|U_i^{n+1}-U_i^n|^2\right]\\ &= \frac{1}{\Delta x}\cdot\frac{\Delta x^2}{\Delta t}\left[|U_{i+1}^{n+1}-U_{i+1}^n|^2+|U_{i-1}^n-U_{i-1}^{n+1}|^2+|U_i^{n+1}-U_i^n|^2\right]\end{aligned} \tag{2.48}$$

in consequence, by using (2.4) we get (2.6) by summing over i and n.

This concludes the proof of proposition 2.1

3 Regularity estimates

Before going into the proof of theorem 1.1 , we give some regularity estimate.

3.1 Definition of interpolation functions $\widetilde{U}_\Delta$ and $\widetilde{F}_\Delta$

We define $\widetilde{U}_\Delta(t)$ a piecewise linear function by

$$\widetilde{U}_\Delta(t) = U_i - \frac{t-t_n}{\Delta x}\left(F_{i+1/2-}-F_{i-1/2+}\right) \tag{3.1}$$

for $t_n \le t < t_{n+1}$, with $F_{i+1/2-}$, $F_{i-1/2+}$ defined in (1.18).

Let us remark that

$$\forall n \in [|0,N|], \quad \widetilde{U}_\Delta(t^n) = U_i^n, \qquad \lim_{\substack{t\to t^{n+1}\\ t<t^{n+1}}} \widetilde{U}_\Delta(t) = U_i^{n+1} = \widetilde{U}_\Delta(t^{n+1}) \qquad (3.2)$$

i.e. $\widetilde{U}_\Delta(t)$ is continuous.

We also define $\widetilde{F}_\Delta(x) \in C(\mathbb{R})$ by

$$\begin{aligned}\widetilde{F}_\Delta(x) = {} & \frac{x - x_{i-1/2}}{\Delta x}\left(F^+(U_{i+1/2-}) + F^-(U_{i+1/2+})\right)\\ & + \frac{x_{i+1/2} - x}{\Delta x}\left(F^+(U_{i-1/2-}) + F^-(U_{i-1/2+})\right)\end{aligned} \qquad (3.3)$$

for $x_{i-1/2} \le x < x_{i+1/2}$, with F^+, F^- defined in (1.25), $U_{i-1/2-}$, $U_{i-1/2+}$ defined in (1.21).

Let us remark that

$$\forall i \in \mathbb{Z}, \quad \widetilde{F}_\Delta(x_{i-1/2}) = F\left(U_{i-1/2-}, U_{i-1/2+}\right). \qquad (3.4)$$

and

$$\lim_{\substack{x\to x_{i+1/2}\\ x<x_{i+1/2}}} \widetilde{F}_\Delta(x) = F\left(U_{i+1/2-}, U_{i+1/2+}\right) = \widetilde{F}_\Delta(x_{i+1/2}) \qquad (3.5)$$

They satisfy a partial differential equation

$$\partial_t \widetilde{U}_\Delta + \partial_x \widetilde{F}_\Delta = \widetilde{S}_\Delta \qquad (3.6)$$

with

$$\widetilde{S}_\Delta(t,x) = \frac{1}{\Delta x}\left(S_{i+1/2-} + S_{i-1/2+}\right) \qquad (3.7)$$

for $t_n \le t \le t_{n+1}$ and $x_{i-1/2} \le x \le x_{i+1/2}$, with $h_{i+1/2-}$, $h_{i+1/2+}$ defined in (1.22) and $S_{i+1/2-}$, $S_{i+1/2+}$ defined in (1.20).

3.2 Estimate of $\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - \widetilde{U}_\Delta|^2 \, dtdx$

We will see later on that, in order to prove compactness of the sequence $\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta)$ in a convenient space, we will need an estimate on $U_\Delta - \widetilde{U}_\Delta$. It is the following proposition:

Lemma 3.1. *Let $\beta > 0$ and v^*, h_m, h_M, u_M>0, involved respectively in assumption* (1.27) *and* (1.31)-(1.34)*. Let $N \in \mathbb{N}$, $T = N\Delta t$, i_0, $i_1 \in \mathbb{N}$ such* *that* $i_0 < i_1$*. Let* U_Δ *be the approximate solution* (1.28) *and* $\widetilde{U}_\Delta$ *defined by* (3.1)*. Then*

$$\left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - \widetilde{U}_\Delta|^2 \, dtdx\right)^{\frac{1}{2}} \leq C\sqrt{\Delta x} \tag{3.8}$$

with $|\cdot|$ *defined by* (2.1)*. The constant* C *depends only on* g*,* h_m*,* h_M*,* u_M*,* v_m*,* β*,* T*,* $|x_{i_0-3/2} - x_{i_1+1/2}|$*,* $\|z\|_{L^\infty}$*,* $\|\eta(U_0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ *and* $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$*,* $I^{v^*}_{i_0-1,i_1+1}$ *defined in* (2.2)*.*

Proof. On the one hand we use (1.28), the definition of U_Δ, and we write

$$\begin{aligned}
&U_\Delta - U_i^n \\
=&\frac{1}{\Delta t}\left[\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1} - U_{i+1}^n + U_{i-1}^n}{2\Delta x}(x - x_{i-1/2}) + U_i^{n+1} - U_i^n\right](t - t_n) \\
&+ \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}(x - x_{i-1/2}),
\end{aligned} \tag{3.9}$$

for all $x_{i-1/2} < x < x_{i+1/2}$ and $t_n \leq t < t_{n+1}$.

Using the triangle inequality, we obtain

$$\begin{aligned}
|U_\Delta - U_i^n| \leq \frac{1}{2}|U_{i+1}^{n+1} - U_i^{n+1}| + \frac{1}{2}|U_i^{n+1} - U_{i-1}^{n+1}| \\
+\frac{1}{2}|U_{i+1}^n - U_i^n| + \frac{1}{2}|U_i^n - U_{i-1}^n| + |U_i^{n+1} - U_i^n| \\
+\frac{1}{2}|U_{i+1}^n - U_i^n| + \frac{1}{2}|U_i^n - U_{i-1}^n|.
\end{aligned} \tag{3.10}$$

Thus,

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |U_\Delta - U_i^n|^2 dxdt \leq C_1 \Delta t \Delta x\Big(|U_{i+1}^{n+1} - U_i^{n+1}|^2 + |U_i^{n+1} - U_{i-1}^{n+1}|^2 \\
+|U_{i+1}^n - U_i^n|^2 + |U_i^n - U_{i-1}^n|^2 + |U_i^{n+1} - U_i^n|\Big).
\end{aligned} \tag{3.11}$$

with $C_1 > 0$ an absolute constant.

Next, we set

$$U^1_\Delta(t,x) = U_i^n, \tag{3.12}$$

for $x_{i-1/2} < x < x_{i+1/2}, t^n < t < t^{n+1}$. Now, taking the sum over n and i and making substitutions of indices, we get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dxdt \leq 4C_1\Delta x \sum_{n=0}^{N+1} \sum_{i=i_0-1}^{i_1+1} \Delta t |U_{i+1}^n - U_i^n|^2$$

$$+C_1\Delta x \sum_{n=0}^{N+1} \sum_{i=i_0-1}^{i_1+1} \Delta t |U_i^{n+1} - U_i^n|^2. \quad (3.13)$$

Then we use the discrete gradient estimates (2.3), (2.4) and CFL condition (1.27) in order to get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dxdt \leq C_2\Delta x, \quad (3.14)$$

with C_2 a constant depending on g, h_m, h_M, u_M, v_m, β, T, $|x_{i_0-3/2} - x_{i_1+1/2}|$, $\|z\|_{L^\infty}$, $\|\eta(U_0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$.

On the other hand we use (3.1), the definition of $\widetilde{U}_\Delta$, and we get

$$U_i^n - \widetilde{U}_\Delta$$
$$= \frac{t-t_n}{\Delta x}\Bigg(F^+(U_{i+1/2-}) + F^-(U_{i+1/2+}) - F^+(U_{i-1/2-}) - F^-(U_{i-1/2+})$$
$$- \frac{g}{2}\begin{pmatrix} 0 \\ h_i^2 - h_{i+1/2-}^2 - (h_{i+1}^2 - h_{i+1/2+}^2) \end{pmatrix}\Bigg), \quad (3.15)$$

for all $t_n \leq t < t_{n+1}$, with F^+, F^- defined in (1.25), $U_{i+1/2-}$, $U_{i+1/2+}$ defined in (1.21), $h_{i+1/2+}$, $h_{i+1/2-}$ defined in (1.22).

Then, using that F^+ and F^- are Lipschitz continuous, see (5.112) and (5.113), with the CFL condition (1.27) we obtain that there exists $C_3 > 0$, depending on g, h_M, u_M and v_m such that

$$|U_i^n - \widetilde{U}_\Delta|$$
$$\leq C_3\Bigg(|U_{i+1/2-} - U_{i-1/2-}| + |U_{i+1/2+} - U_{i-1/2+}|$$
$$+ \frac{g}{2}\left|h_i^2 - h_{i+1/2-}^2 - (h_{i+1}^2 - h_{i+1/2+}^2)\right| \Bigg), \quad (3.16)$$

for all $t_n \leq t < t_{n+1}$.

Moreover using (1.21)-(1.23), (2.1), (2.20), (2.19), we get that there exists $C > 0$, depending only on g and h_m such that

$$|U_{i+1/2-} - U_{i-1/2-}| \leq C\left(|h_i - h_{i-1}| + |z_{i+1} - z_i| + |z_i - z_{i-1}|\right), \quad (3.17)$$

$$|U_{i+1/2+} - U_{i-1/2+}| \leq C\left(|h_{i+1} - h_i| + |z_{i+1} - z_i| + |z_i - z_{i-1}|\right). \qquad (3.18)$$

In addition, (5.133), (5.133),we deduce that there exists C, depending on h_M such that

$$\frac{g}{2}\left|h_i^2 - h_{i+1/2-}^2 - (h_{i+1}^2 - h_{i+1/2+}^2)\right| \leq C|z_{i+1} - z_i| + C|z_i - z_{i-1}|. \qquad (3.19)$$

Thus, from (3.16) using the triangle inequality with (3.17), (3.18) and (3.19), there exists C depending on g, h_m, h_M such that

$$\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |U_i^n - \widetilde{U}_\Delta|^2 dtdx$$
$$\leq C\Delta t\Delta x\left(|h_i - h_{i-1}|^2 + |h_{i+1} - h_i|^2 + |z_{i+1} - z_i|^2 + |z_i - z_{i-1}|^2\right). \qquad (3.20)$$

Now, taking the sum over n and i and making substitutions of indices, we get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta^1 - \widetilde{U}_\Delta|^2 dxdt$$
$$\leq C\Delta x\left(\sum_{n=0}^{N+1}\sum_{i=i_0-1}^{i_1+1} \Delta t|U_{i+1}^n - U_i^n|^2 + \sum_{n=0}^{N+1}\sum_{i=i_0-1}^{i_1+1} \Delta t|z_{i+1} - z_i|^2\right), \qquad (3.21)$$

with U_Δ^1 defined in (3.12) and C is a constant depending on g, h_m, h_M. Next, using (2.28) and the gradient estimate (2.3), we get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta^1 - \widetilde{U}_\Delta|^2 dxdt \leq C_2\Delta x, \qquad (3.22)$$

with C_2 a constant depending on g, h_m, h_M, u_M, v_m, β, T, $|x_{i_0-3/2} - x_{i_1+1/2}|$, $\|z\|_{L^\infty}$, $\|\eta(U_0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$.

Finally, noticing that $U_\Delta - \widetilde{U}_\Delta = \left(U_\Delta - U_\Delta^1\right) + \left(U_\Delta^1 - \widetilde{U}_\Delta\right)$, we get

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - \widetilde{U}_\Delta|^2 dtdx$$
$$\leq 2\left(\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dtdx + \int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta^1 - \widetilde{U}_\Delta|^2 dtdx\right). \qquad (3.23)$$

With (3.14), (3.22) we get (3.8), which concludes the proof. $\square$

3.3 Estimate of $\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |F(U_\Delta) - \widetilde{F}_\Delta|^2 \, dtdx$

We will see later on that, in order to prove compactness of the sequence $\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta)$ in H^{-1}_{loc}, we will need an estimate on $F(U_\Delta) - \widetilde{F}_\Delta$. It is the following proposition:

Lemma 3.2. *Let $\beta > 0$ and v^*, h_m, h_M, $u_M>0$, involved respectively in assumption* (1.27) *and* (1.31)-(1.34). *Let $N \in \mathbb{N}$, $T = N\Delta t$, i_0, $i_1 \in \mathbb{N}$ such that $i_0 < i_1$. Let U_Δ be the approximate solution* (1.28) *and $\widetilde{F}_\Delta$ defined by* (3.3)*. Then*

$$\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |F(U_\Delta) - \widetilde{F}_\Delta|^2 \, dtdx \le C\Delta x \tag{3.24}$$

with $|\cdot|$ defined by (2.1)*. The constant C depends only on g, h_m, h_M, u_M, v_m, β, T, $|x_{i_0-3/2} - x_{i_1+3/2}|$, $\|z\|_{L^\infty}$, $\|\eta(U_0)\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$ and $\|h^0\|_{L^1(I^{v^*}_{i_0-1,i_1+1})}$, $I^{v^*}_{i_0-1,i_1+1}$ defined in* (2.2).

Proof. We recall here (3.3)

$$\begin{aligned}\widetilde{F}_\Delta(x) = & \frac{x - x_{i-1/2}}{\Delta x}\left(F^+(U_{i+1/2-}) + F^-(U_{i+1/2+})\right) \\ & + \frac{x_{i+1/2} - x}{\Delta x}\left(F^+(U_{i-1/2-}) + F^-(U_{i-1/2+})\right),\end{aligned} \tag{3.25}$$

for all $x_{i-1/2} < x < x_{i+1/2}$. Moreover, we have

$$F(U_\Delta) = F^+(U_\Delta) + F^-(U_\Delta). \tag{3.26}$$

Thus, using triangle inequality, for all $x_{i-1/2} < x < x_{i+1/2}$, we get

$$\begin{aligned} & |\widetilde{F}_\Delta(x) - F(U_\Delta)| \\ \le & \frac{1}{2}\left|F^+(U_{i+1/2-}) - F^+(U_\Delta)\right| + \frac{1}{2}\left|F^-(U_{i+1/2+}) - F^-(U_\Delta)\right| \\ & + \frac{1}{2}\left|F^+(U_{i-1/2-}) - F^+(U_\Delta)\right| + \frac{1}{2}\left|F^-(U_{i+1/2+}) - F^-(U_\Delta)\right|. \end{aligned} \tag{3.27}$$

Then, using that F^+ and F^- are Lipschitz continuous, see (5.112) and (5.113), with the CFL condition (1.27) we obtain that there exists $C > 0$, depending on g, h_m, h_M, u_M and v_m such that

$$\begin{aligned} & |\widetilde{F}_\Delta(x) - F(U_\Delta)| \\ \le & C\left(\left|U_{i+1/2-} - U_\Delta\right| + \left|U_{i+1/2+} - U_\Delta\right| + \left|U_{i-1/2-} - U_\Delta\right| + \left|U_{i-1/2+} - U_\Delta\right|\right) \end{aligned} \tag{3.28}$$

Moreover using (1.21), (2.19), (2.20), we get

$$
\begin{aligned}
&|\widetilde{F}_\Delta(x) - F(U_\Delta)| \\
&\leq C\left(2\,|U_i - U_\Delta| + |U_{i+1} - U_\Delta| + |U_{i-1} - U_\Delta| + |z_{i+1} - z_i| + |z_i - z_{i-1}|\right)
\end{aligned}
\tag{3.29}
$$

with $C > 0$, depending on g, h_m, h_M, u_M and v_m.

Thus we get

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |\widetilde{F}_\Delta(x) - F(U_\Delta)|^2 dt dx \\
&\leq C \int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |U_i - U_\Delta|^2\, dt dx \\
&+ C\Delta t \Delta x \left(|U_{i+1} - U_i|^2 + |U_{i-1} - U_i|^2 + |z_{i+1} - z_i|^2 + |z_i - z_{i-1}|^2\right),
\end{aligned}
\tag{3.30}
$$

with $C > 0$ an absolute contant.

Now, taking the sum over n and i and making substitutions of indices, we get

$$
\begin{aligned}
\int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |\widetilde{F}_\Delta(x) - F(U_\Delta)|^2 dx dt &\leq \int_0^T \int_{x_{i_0-1/2}}^{x_{i_1+1/2}} |U_\Delta - U_\Delta^1|^2 dx dt \\
+ C\Delta x &\left(\sum_{n=0}^{N+1} \sum_{i=i_0-1}^{i_1+1} \Delta t |U_{i+1}^n - U_i^n|^2 + \sum_{n=0}^{N+1} \sum_{i=i_0-1}^{i_1+1} \Delta t |z_{i+1} - z_i|^2 \right).
\end{aligned}
\tag{3.31}
$$

Finally, using (2.28), the gradient estimate (2.3) and previous estimation (3.14), involving $U_\Delta - U_i^n$, we get (3.24), which concludes the proof. □

4 Proof of theorem 1.1

Using (4.1) we compute

$$
\partial_t U_\Delta + \partial_x F(U_\Delta) = \partial_t (U_\Delta - \widetilde{U}_\Delta) + \partial_x \left(F(U_\Delta) - \widetilde{F}_\Delta\right) + \widetilde{S}_\Delta, \tag{4.1}
$$

with $U_\Delta(t,x)$ defined in (1.28). We multiply (4.1) by $\eta'(U_\Delta)$ and we get, for any entropy-entropy flux (η, G), the following decomposition

$$
\begin{aligned}
\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta) = \; &\eta'(U_\Delta) \cdot \partial_t (U_\Delta - \widetilde{U}_\Delta) \\
&+ \eta'(U_\Delta) \cdot \partial_x \left(F(U_\Delta) - \widetilde{F}_\Delta\right) \\
&+ \; \eta'(U_\Delta) \cdot \widetilde{S}_\Delta \\
= \; &R_1 + M_1 + R_2 + M_2 - \eta'(U_\Delta) \cdot \widetilde{S}_\Delta
\end{aligned}
\tag{4.2}
$$

where

$$\begin{aligned}
R_1 =&\ \partial_t \left(\eta'(U_\Delta)\cdot(U_\Delta - \widetilde{U}_\Delta)\right),\\
M_1 =&\ -\eta''(U_\Delta)\cdot\partial_t U_\Delta \cdot \left(U_\Delta - \widetilde{U}_\Delta\right),\\
R_2 =&\ \partial_x \left(\eta'(U_\Delta)\cdot\left(F(U_\Delta) - \widetilde{F}_\Delta\right)\right),\\
M_2 =&\ -\eta''(U_\Delta)\cdot\partial_x U_\Delta \cdot \left(F(U_\Delta) - \widetilde{F}_\Delta\right).
\end{aligned} \tag{4.3}$$

First we have, using (3.24)

$$\begin{aligned}
&\int_0^T \int_{-R}^R \left|\eta'(U_\Delta)\cdot\left(F(U_\Delta) - \widetilde{F}_\Delta\right)\right|^2 dxdt\\
&\le \|\eta'(U_\Delta)\|_{L^\infty(]0,T[\times]-R,R[)} \int_0^T \int_{-R}^R \left|\left(F(U_\Delta) - \widetilde{F}_\Delta\right)\right|^2 dxdt\\
&\le C_R\sqrt{\Delta x}
\end{aligned} \tag{4.4}$$

thus R_2 goes to zero in $H^-_{loc}1$ as $\Delta x \to 0$. Similarly, using (3.8), R_1 goes to zero in $H^-_{loc}1$ as $\Delta x \to 0$.

Futhermore, using (2.5) and (3.24), we have

$$\begin{aligned}
&\int_0^T \int_{-R}^R |M_2| dxdt\\
\le&\ \|\eta''(U_\Delta)\|_{L^\infty(]0,T[\times]-R,R[)} \left(\iint |\partial_t U_\Delta|^2 dxdt\right)^{1/2} \left(\int_0^T \int_{-R}^R \left|\left(F(U_\Delta) - \widetilde{F}_\Delta\right)\right|^2 dxdt\right)^{1/2}\\
\le&\ \|\eta''(U_\Delta)\|_{L^\infty(]0,T[\times]-R,R[)} \frac{C_1}{\sqrt{\Delta x}} C_3\sqrt{\Delta x}\\
\le&\ C_R
\end{aligned} \tag{4.5}$$

thus M_2 is bounded in $\mathcal{M}_{loc}((0,T)\times\mathbb{R})$. Similarly, using (2.6) and (3.8), M_1 is bounded in $\mathcal{M}_{loc}((0,T)\times\mathbb{R})$.

Using (5.133), if $z_{i+1} - z_i \ge 0$, we have

$$\frac{|S_{i+1/2-}|}{\Delta x} \le C\mathrm{Lip}(z_\Delta)\left(\|U_\Delta\|_\infty + \Delta x \mathrm{Lip}(z)\right), \tag{4.6}$$

and if not, $h_i = h_{i+1/2-}$ and the last inequality holds.

Similarly, using (5.134), if $z_{i+1} - z_i \ge 0$, we have

$$\frac{|S_{i-1/2+}|}{\Delta x} \le C\mathrm{Lip}(z_\Delta)\left(\|U_\Delta\|_\infty + \Delta x \mathrm{Lip}(z)\right), \tag{4.7}$$

and if not, $h_i = h_{i-1/2+}$ and the last inequality holds. Using (4.6), (4.7), we get

$$\|\widetilde{S}_\Delta\|_\infty \leq C\mathrm{Lip}(z_\Delta)\left(\|U_\Delta\|_\infty + \Delta x \mathrm{Lip}(z)\right). \tag{4.8}$$

Moreover, according to (4.2) and (4.3), one has

$$\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta) - R_1 - R_2 = M_1 + M_2 + \eta'(U_\Delta) \cdot \widetilde{S}_\Delta \tag{4.9}$$

thus $M_1 + M_2 + \eta'(U_\Delta) \cdot \widetilde{S}_\Delta$ is bounded in $W^{-1,p}_{loc} \cap \mathcal{M}_{loc}$, $\forall p$, $1 < p < +\infty$, as a consequence it is compact in H^{-1}_{loc}. At this point, we know that $R_1 + R_2$ and $M_1 + M_2 + \eta'(U_\Delta) \cdot \widetilde{S}_\Delta$ are compact in H^{-1}_{loc}, therefore their sum, which is equal to $\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta)$, is compact in H^{-1}_{loc}. Furthermore, $(U_\Delta)_{\Delta>0}$ is bounded since we assume that $(U_i^n)_{i,n}$ is a bounded sequence. We are now able to apply the compensated compactness method and we get that up to a subsequence $U_\Delta \to U$ a.e. and in $L^1_{loc,t,x}$ as $\Delta t \to 0$ and $\Delta x \to 0$, see [23].

Moreover, according to lemma 5.7, $\partial_t U_\Delta$ is bounded in $L^\infty_t(\mathcal{D}'_x)$ and therefore we get

$$d\left((U_\Delta(t_1), U_\Delta(t_2))_{(W^{1,1})'} \leq C\|\partial_t U_\Delta\|_{L^\infty_t(\mathcal{D}'_x)}|t_1 - t_2|, \tag{4.10}$$

and we conclude that $U_\Delta \to U$ in $C_t([0,T], L^\infty_{x,w*}(\mathbb{R}))$, by Arzelà–Ascoli theorem.

Then, knowing that U_Δ converges in L^p_{loc} to U, we can apply lemma 5.8, which concludes the convergence of the approximate source term $\widetilde{S}_\Delta$ to S.

Finally, we pass to the limit in (4.1) using (3.8), (3.24), and (5.131), which enables us to get that the limit U is a solution to the system. Moreover passing to the limit with a test function ϕ in (2.16) we get (1.2).

This ends the proof of theorem 1.1.

5 Appendix

We prove here some technical results used throughout the paper.

Lemma 5.1. *Let $Uk = (h_k, h_k u_k)$ for $k = 1$, 2 with $h_k \geq 0$. Then*

$$\begin{aligned}
&\frac{g^2\pi^2}{6}(2M_1 + M_2)(M_1 - M_2)^2 \\
&= H(M_2) - H(M_1) - \eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix}(M_2 - M_1) \\
&- \mathbb{1}_{(\xi-u_1)^2 > 2gh_1}\ M_2\left(\frac{(\xi-u_1)^2}{2} - gh_1\right),
\end{aligned} \tag{5.1}$$

where $M_k \equiv M_k(\xi) \equiv M(U_k, \xi)$, and $M(U,\xi)$ is defined in (1.8) *and $H(f) \equiv H(f,\xi)$ is defined in* (1.8).

Proof. Using the identity

$$b^3 - a^3 - 3a^2(b-a) = (b+2a)(b-a)^2, \tag{5.2}$$

one has

$$\frac{g^2\pi^2}{6}\left(2M_1 + M_2\right)\left(M_1 - M_2\right)^2 = H(M_2) - H(M_1) - H'(M_1)\left(M_2 - M_1\right), \tag{5.3}$$

where we donote $H'(f,\xi) \equiv \frac{\partial}{\partial f} H(f,\xi)$. Thus it is sufficient to prove

$$\begin{aligned}&\left(\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} - H'(M_2)\right)(M_2 - M_1)\\ &= -\mathbb{1}_{(\xi-u_1)^2>2gh_1}\ M_2\left(\frac{(\xi-u_1)^2}{2} - gh_1\right).\end{aligned} \tag{5.4}$$

On the one hand we compute

$$\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} = \left(gh_1 - \frac{u_1^2}{2} + u_1\xi\right). \tag{5.5}$$

On the other hand we get

$$\begin{aligned}H'(M_1) &= \frac{\xi^2}{2} + \frac{g^2\pi^2}{2}M_1^2\\ &= \frac{\xi^2}{2} + \left(gh_1 - \frac{(\xi-u_1)^2}{2}\right)_+.\end{aligned} \tag{5.6}$$

In consequence, by adding (5.5) and (5.6) we get

$$\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} - H'(M_2) = -\mathbb{1}_{(\xi-u_1)^2>2gh_1}\left(\frac{(\xi-u_1)^2}{2} - gh_1\right). \tag{5.7}$$

and therefore

$$\begin{aligned}&\left(\eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix} - H'(M_2)\right)(M_2 - M_1)\\ &= -\ \mathbb{1}_{(\xi-u_1)^2>2gh_1}\ \left(\frac{(\xi-u_1)^2}{2} - gh_1\right)(M_2 - M_1).\end{aligned} \tag{5.8}$$

Finally we notice that

$$(\xi - u_1)^2 > 2gh_1 \Longleftrightarrow M_1 = 0 \tag{5.9}$$

and we get (5.4), which concludes the proof. ◻

Lemma 5.2. *There exists some constant $\alpha > 0$, depending only on on gravity constant g, on constants h_m, h_M, u_M, which are involved in* (1.30)*, such that*

$$\int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi$$
$$\geq \alpha \left(\eta(U_2) - \eta(U_1) - \eta'(U_1)\,(U_2 - U_1) \right) \tag{5.10}$$

for every U_1, $U_2 \in \mathcal{U}_{h_m,h_M,u_M}$ defined by (1.30) *and where $M_k \equiv M_k(\xi) \equiv M(U_k, \xi)$, with $M(U,\xi)$ defined in* (1.8)*, $H(f) \equiv H(f,\xi)$ is defined in* (1.8) *and $\eta(U)$ defined in* (1.3)*.*

Proof. We set

$$\widehat{\mathcal{U}_m} = \left\{ (h,hu) \in \mathbb{R}^2,\ h \geq h_m \right\} \tag{5.11}$$

and we first deal with the case

$$U_1 = \begin{pmatrix} h_1 \\ h_1 u_1 \end{pmatrix} \text{ and } U_2 = \begin{pmatrix} h_2 \\ h_2 u_2 \end{pmatrix} \in \widehat{\mathcal{U}_m}, \text{ such that } |u_1 - u_2| \leq \sqrt{g h_m}. \tag{5.12}$$

In this case we have

$$\forall t \in [0,1],\ (1-t)\eta'(U_1) + t\eta'(U_2) \in \eta'(\widetilde{\mathcal{U}_m}). \tag{5.13}$$

with

$$\widetilde{\mathcal{U}_m} = \left\{ (h,hu) \in \mathbb{R}^2,\ h \geq \frac{h_m}{2} \right\}. \tag{5.14}$$

Indeed we notice that

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \eta'(\widetilde{\mathcal{U}_m}) \Longleftrightarrow V_1 \geq g\frac{h_m}{2} - \frac{V_2^2}{2}. \tag{5.15}$$

Thus (5.13) is equivalent to

$$\forall t \in [0,1], \forall h_1, h_2 \geq h_m, \forall u_1, u_2 \in \mathbb{R}, \text{ such that } |u_1 - u_2| \leq \sqrt{g h_m}$$
$$(1-t)\left(g h_1 - \frac{u_1^2}{2} \right) + t\left(g h_2 - \frac{u_2^2}{2} \right) \geq g\frac{h_m}{2} - \frac{1}{2}\Big((1-t)u_1 + t u_2 \Big)^2. \tag{5.16}$$

Thus it is sufficient to check that

$$\forall t \in [0,1], \forall u_1, u_2 \in \mathbb{R}, \text{ such that } |u_1 - u_2| \leq \sqrt{g h_m}$$
$$(1-t)\left(g h_m - \frac{u_1^2}{2} \right) + t\left(g h_m - \frac{u_2^2}{2} \right) \geq g\frac{h_m}{2} - \frac{1}{2}\Big((1-t)u_1 + t u_2 \Big)^2. \tag{5.17}$$

This inequality simplifies to

$$\forall t \in [0,1], \forall u_1, u_2 \in \mathbb{R} \quad \frac{gh_m}{2} \geq \frac{t(1-t)}{2} (u_1 - u_2)^2 \tag{5.18}$$

which is true if $|u_1 - u_2| \leq 2\sqrt{gh_m}$.

We want now to use property (5.13) and define a path $v(t) \in \widetilde{\mathcal{U}_m}$, connecting two states U_1, U_2 satisfying (5.12) by

$$\eta'(v(t)) = (1-t)\eta'(U_1) + t\eta'(U_2) \tag{5.19}$$

for $0 \leq t \leq 1$. Such a definition is possible because η is strictly convex and η' is a diffeormorphism. It enables us to set

$$\begin{aligned}\phi(t) = \int_{\mathbb{R}} |\xi| \Big(& H(M(v(t), \xi)) - H(M(U_1, \xi)) \\ & - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M(v(t), \xi) - M(U_1, \xi)) \Big) d\xi \\ & - \alpha \left(\eta(v(t)) - \eta(U_1) - \eta'(U_1) (v(t) - U_1) \right).\end{aligned} \tag{5.20}$$

We notice that $\phi(0) = 0$, and the result of (5.10) is equivalent to $\phi(1) \geq 0$. Thus it is sufficient to prove that ϕ is non-decreasing. Using the fact that

$$\eta'(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} = H'\left(M(U, \xi), \xi\right), \text{ for all } \xi \in \mathbb{R} \text{ such that } M(U, \xi) > 0, \tag{5.21}$$

we can compute

$$\begin{aligned}\phi'(t) = & \int_{\mathbb{R}} |\xi| \left(\eta'(v(t)) - \eta'(U_1) \right) \begin{pmatrix} 1 \\ \xi \end{pmatrix} M'(v(t), \xi) \cdot v'(t) d\xi \\ & - \left(\eta'(v(t)) - \eta'(U_1) \right) \cdot v'(t).\end{aligned} \tag{5.22}$$

Moreover using

$$\eta'(v(t)) - \eta'(U_1) = t\eta''(v(t)) \cdot v'(t) \tag{5.23}$$

we get

$$\begin{aligned}\phi'(t) = t & \int_{\mathbb{R}} |\xi| \eta''(v(t)) \cdot v'(t) \cdot \begin{pmatrix} 1 \\ \xi \end{pmatrix} M'(v(t), \xi) \cdot v'(t) d\xi \\ & - t\, \eta''(v(t)) \cdot v'(t) \cdot v'(t),\end{aligned} \tag{5.24}$$

which can be rewritten as

$$
\begin{aligned}
&\phi'(t)\\
&= -t \int_{\mathbb{R}} |\xi| \left(M'(v(t),\xi)^t \begin{pmatrix} 1 \\ \xi \end{pmatrix}^t \eta''(v(t)) \right) \cdot v'(t) \cdot v'(t) d\xi \\
&\quad - t\, \eta''(v(t)) \cdot v'(t) \cdot v'(t) \\
&= -t \int_{\mathbb{R}} |\xi| M'(v(t),\xi) \otimes \left(\eta''(v(t)) \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right) \cdot v'(t) \cdot v'(t) d\xi \\
&\quad - t\, \eta''(v(t)) \cdot v'(t) \cdot v'(t).
\end{aligned} \tag{5.25}
$$

Thus now it is sufficient for getting (5.10) to prove that

$$
\begin{aligned}
&\forall U \in \widetilde{\mathcal{U}_m}, \forall X \in \mathbb{R}^2 \\
&\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes \left(\eta''(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right) \cdot X \cdot X d\xi \geq \alpha\, \eta''(U) \cdot X \cdot X.
\end{aligned} \tag{5.26}
$$

For all $U \in \widetilde{\mathcal{U}_m}$ and $\xi \in \mathbb{R}$ such that $M(U,\xi) > 0$, we compute

$$
\eta'(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} = H'\left(M(U,\xi),\xi\right) \tag{5.27}
$$

and

$$
\eta''(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} = H''\left(M(U,\xi),\xi\right) M'(U). \tag{5.28}
$$

Moreover one can check that

$$
H''\left(M(U,\xi)\right) = g^2 \pi^2 M(U,\xi) \tag{5.29}
$$

and we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes \left(\eta''(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right) d\xi \\
=&g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi) \cdot M'(U,\xi) \otimes M'(U,\xi) d\xi.
\end{aligned} \tag{5.30}
$$

which is equivalent to

$$
\begin{aligned}
&\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes (\eta''(U) X) \cdot X \cdot X d\xi \\
=&g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi) \left(M'(U,\xi) \cdot X\right)^2 d\xi
\end{aligned} \tag{5.31}
$$

for all $X \in \mathbb{R}^2$.

Now we denote $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and compute $M'(U,\xi) \cdot X$. We recall that

$$M(U,\xi) = \frac{1}{g\pi}\left(2gh - (\xi - u)^2\right)_+^{1/2}, \quad U = (h, hu) \tag{5.32}$$

so we can rewrite

$$M(U,\xi) = \frac{1}{g\pi}\left(2gh - \left(\xi - \frac{hu}{h}\right)^2\right)_+^{1/2} \tag{5.33}$$

and compute partial derivatives

$$\partial_h M(U,\xi) = \frac{1}{2g\pi}\left(2gh - (\xi - u)^2\right)^{-1/2}\left(2g - 2\frac{u}{h}(\xi - u)\right) \tag{5.34}$$

and

$$\partial_{hu} M(U,\xi) = \frac{1}{2g\pi}\left(2gh - (\xi - u)^2\right)^{-1/2}\left(\frac{2}{h}(\xi - u)\right). \tag{5.35}$$

Finally it leads to the formula

$$M'(U,\xi) \cdot X = \frac{M(U,\xi)^{-1}}{g^2\pi^2}\left(gx_1 + \frac{(\xi - u)}{h}(x_2 - ux_1)\right). \tag{5.36}$$

Now we denote

$$x_3 = \frac{1}{h}(x_2 - ux_1) \tag{5.37}$$

in order to write

$$M'(U,\xi) \cdot X = \frac{M(U,\xi)^{-1}}{g^2\pi^2}(gx_1 + (\xi - u)x_3) \tag{5.38}$$

and using (5.38) and (5.31) we get

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes \left(\eta''(U)\begin{pmatrix} 1 \\ \xi \end{pmatrix}\right) \cdot X \cdot X d\xi \\
&= g^2\pi^2 \int_{M(U,\xi)>0} |\xi|\ M(U,\xi)\left(M'(U,\xi) \cdot X\right)^2 d\xi \\
&= \frac{1}{g^2\pi^2}\int_{M(U,\xi)>0} |\xi|\ \frac{1}{M(U,\xi)}\left(gx_1 + (\xi - u)x_3\right)^2 d\xi \\
&\geq \frac{1}{g\pi\sqrt{2gh}}\int_{|\xi - u| \leq \sqrt{2gh}} |\xi|\left(gx_1 + (\xi - u)x_3\right)^2 d\xi := I.
\end{aligned} \tag{5.39}$$

Last estimate In order to get the we used the fact that

$$M(U,\xi) = \frac{1}{g\pi}\left(2gh - (\xi - u)^2\right)_+^{1/2} \leq \frac{\sqrt{2gh}}{g\pi}. \tag{5.40}$$

Using the substitution $v = \xi - u$ and using the convention that

$$\text{if } u = 0 \text{ then } \operatorname{sgn}(u) = 1 \tag{5.41}$$

we obtain

$$I = \frac{1}{g\pi\sqrt{2gh}} \int_{|v|\leq\sqrt{2gh}} |v+u| \left(gx_1 + vx_3\right)^2 dv \tag{5.42}$$

$$\geq \frac{1}{g\pi\sqrt{2gh}} \int_{|v|\leq\sqrt{2gh},\, \operatorname{sgn}(v)=\operatorname{sgn}(u)} (|v| + |u|) \left(gx_1 + v\operatorname{sgn}(u)x_3\right)^2 dv \tag{5.43}$$

$$\geq \frac{1}{g\pi\sqrt{2gh}} \int_0^{\sqrt{2gh}} v \left(gx_1 + v\operatorname{sgn}(u)x_3\right)^2 dv \tag{5.44}$$

$$\geq \frac{1}{2g\pi} \int_{\frac{\sqrt{2gh}}{2}}^{\sqrt{2gh}} \left(gx_1 + v\operatorname{sgn}(u)x_3\right)^2 dv. \tag{5.45}$$

Using the substitution $\xi = \frac{v}{\sqrt{2gh}}$ we obtain

$$\begin{aligned} &\frac{1}{2g\pi} \int_{\frac{\sqrt{2gh}}{2}}^{\sqrt{2gh}} \left(gx_1 + v\operatorname{sgn}(u)x_3\right)^2 dv \\ &= \frac{\sqrt{h}}{\sqrt{2g}\pi} \int_{1/2}^{1} \left(gx_1 + \xi\sqrt{2gh}\operatorname{sgn}(u)x_3\right)^2 d\xi \end{aligned} \tag{5.46}$$

which is a positive definite quadratic form with respect to $y_1 = gx_1$ and $y_3 = \operatorname{sgn}(u)\sqrt{gh}x_3$. Thus we have for some absolute constant $C > 0$

$$\begin{aligned} &\frac{\sqrt{h}}{\sqrt{2g}\pi} \int_{1/2}^{1} \left(gx_1 + \xi\sqrt{2gh}\operatorname{sgn}(u)x_3\right)^2 d\xi \\ &\geq C\frac{\sqrt{h}}{\sqrt{2g}\pi}\left((gx_1)^2 + 2ghx_3^2\right) \\ &\geq C\frac{\sqrt{h}}{\sqrt{2g}\pi}\left((gx_1)^2 + \frac{2g}{h}\left(x_2 - ux_1\right)^2\right) \\ &= C\frac{\sqrt{gh}}{\sqrt{2}\pi}\left(gx_1^2 + \frac{2}{h}\left(x_2 - ux_1\right)^2\right) \end{aligned} \tag{5.47}$$

and by (5.39) (5.46) (5.47), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| M'(U,\xi) \otimes \left(\eta''(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right) \cdot X \cdot X d\xi \\
&\geq C \frac{\sqrt{gh}}{\sqrt{2}\pi} \left(g x_1^2 + \frac{2}{h} \left(x_2 - u x_1 \right)^2 \right).
\end{aligned} \tag{5.48}$$

Besides, we have

$$\eta(h,q) = \left(\frac{1}{2} \frac{q^2}{h} + g \frac{h^2}{2} \right), \tag{5.49}$$

$$\eta'(h,q) = \left(-\frac{1}{2} \frac{q^2}{h^2} + gh, \frac{q}{h} \right), \tag{5.50}$$

$$\eta''(h,q) = \begin{pmatrix} \frac{q^2}{h^3} + g & -\frac{q}{h^2} \\ -\frac{q}{h^2} & \frac{1}{h} \end{pmatrix}, \tag{5.51}$$

$$\eta''(h,hu) = \begin{pmatrix} \frac{u^2}{h} + g & -\frac{u}{h} \\ -\frac{u}{h} & \frac{1}{h} \end{pmatrix}, \tag{5.52}$$

and finally we get

$$\begin{aligned}
&\eta''(U) \cdot X \cdot X \\
&= \left(\left(g + \frac{u^2}{h} \right) x_1^2 + \frac{1}{h} x_2^2 - \frac{2u}{h} x_1 x_2 \right) \\
&= g x_1^2 + \frac{1}{h} \left(x_2 - u x_1 \right)^2.
\end{aligned} \tag{5.53}$$

Thus we find that

$$\int_{\mathbb{R}} |\xi| M'(U) \otimes \left(\eta''(U) \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right) \geq \frac{C}{\sqrt{2}\pi} \sqrt{g h_m} \eta''(U) \tag{5.54}$$

At this point, with the last estimate we obtain that (5.26) holds, and therefore we have the result (5.10) for all $U_1, U_2 \in \widehat{\mathcal{U}_m}$ such that $|u_1 - u_2| \leq \sqrt{g h_m}$, with the constant $\alpha_m = \frac{C}{\sqrt{2}\pi} \sqrt{g h_m}$, where $C > 0$ is an absolute constant.

Thus, it is now sufficient to prove that

$$\begin{aligned}
&\exists \alpha_1 > 0, \quad \forall U_1, U_2 \in \mathcal{U}_{h_m, h_M, u_M}, \\
&|u_1 - u_2| > \sqrt{g h_m} \\
&\Rightarrow \int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \geq \alpha_1.
\end{aligned} \tag{5.55}$$

Indeed, when (5.55) holds, we have

$$\begin{aligned}
&(\eta(U_2) - \eta(U_1) - \eta'(U_1)\,(U_2 - U_1))\\
&= g\frac{(h_2-h_1)^2}{2} + h_2\frac{(u_2-u_1)^2}{2}\\
&\le C(h_M, u_M)\\
&\le \frac{C(h_M,u_M)}{\alpha_1}\int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix}(M_2 - M_1)\right) d\xi
\end{aligned} \tag{5.56}$$

which proves (5.10). Using reductio ad absurdum, we suppose (5.55) does not hold. Thus

$$\begin{aligned}
&\forall n > 0, \quad \exists U_1^n, U_2^n \in \mathfrak{U}_{h_m,h_M,u_M}, \text{ such that}\\
&|u_1^n - u_2^n| > \sqrt{gh_m}\\
&\text{and } \int_{\mathbb{R}} |\xi| \left(H(M_2^n) - H(M_1^n) - \eta'(U_1^n)\begin{pmatrix}1\\ \xi\end{pmatrix}(M_2^n - M_1^n)\right) d\xi \le \frac{1}{n}
\end{aligned} \tag{5.57}$$

where $M_i^n = M(U_i^n, \xi)$.

As $\mathfrak{U}_{h_m,h_M,u_M}$ is a closed and bounded set, we can take 2 subsequences which we also denote U_1^n, U_2^n such that

$$U_1^n \to U_1 \in , \quad U_2^n \to U_2 \in \mathfrak{U}_m \tag{5.58}$$

with

$$|u_1 - u_2| \ge \sqrt{gh_m} \tag{5.59}$$

and by dominated converge theorem

$$\int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix}(M_2 - M_1)\right) d\xi = 0. \tag{5.60}$$

We also know by (5.1) that

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1)\begin{pmatrix}1\\ \xi\end{pmatrix}(M_2 - M_1)\right) d\xi\\
&\ge \int_{\mathbb{R}} |\xi|\,(2M_1 + M_2)\,(M_1 - M_2)^2 d\xi
\end{aligned} \tag{5.61}$$

and therefore we get

$$(2M_1 + M_2)\,(M_1 - M_2)^2 = 0 \quad \text{almost everywhere} \tag{5.62}$$

itself implying that $M_1 = M_2$ a.e. and therefore $U_1 = U_2$, the later being in contradiction with (5.59). □

Lemma 5.3. *Let $g>0$ be the gravity constant. One has*

$$\int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2}-gh_1\right)d\xi$$
$$\leq \frac{4\left(|u_2|+\sqrt{2gh_2}\right)}{g\pi\sqrt{gh_2}}\left(g|h_1-h_2|+(|u_2|+\sqrt{2gh_2})|u_1-u_2|+\frac{1}{2}|u_1^2-u_2^2|\right)^{\frac{5}{2}} \tag{5.63}$$

for every $U_1=(h_1,h_1u_1)$, $h_1>0$ and $U_2=(h_2,h_2u_2)$, $h_2>0$, where $M(U,\xi)$ is defined in (1.8).

Proof. We set

$$K = g|h_1-h_2|+(|u_2|+\sqrt{2gh_2})|u_1-u_2|+\frac{1}{2}|u_1^2-u_2^2|. \tag{5.64}$$

Thus we can rewrite (5.63) as

$$\int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2}-gh_1\right)d\xi$$
$$\leq \frac{4}{g\pi\sqrt{gh_2}}(|u_2|+\sqrt{2gh_2})K^{\frac{5}{2}} \tag{5.65}$$

for every $U_1, U_2 \in \mathcal{U}_{h_m,h_M,u_M}$ defined by (1.30).

We notice that $\forall \xi \in \mathrm{supp}(M_2)$, one has $|\xi| \leq |u_2|+\sqrt{2gh_2}$ and we get

$$\begin{aligned}
&\left|gh_2-\frac{(\xi-u_2)^2}{2}-\left(gh_1-\frac{(\xi-u_1)^2}{2}\right)\right| \\
&=\left|g(h_2-h_1)+\xi(u_2-u_1)-\frac{1}{2}(u_2^2-u_1^2)\right| \\
&\leq K
\end{aligned} \tag{5.66}$$

with K defined by (5.64). Moreover using that $\xi \in \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)$ iff $gh_2-\frac{(\xi-u_2)^2}{2} \geq 0$ and $\frac{(\xi-u_1)^2}{2}-gh_1 \geq 0$, we get

$$\begin{aligned}
0 &\leq gh_2-\frac{(\xi-u_2)^2}{2} \\
&\leq gh_2-\frac{(\xi-u_2)^2}{2}+\frac{(\xi-u_1)^2}{2}-gh_1 \\
&= gh_2-\frac{(\xi-u_2)^2}{2}-\left(gh_1-\frac{(\xi-u_1)^2}{2}\right) \\
&\leq K.
\end{aligned} \tag{5.67}$$

Similarly we obtain

$$\begin{aligned} 0 \leq & \frac{(\xi - u_1)^2}{2} - gh_1 \\ \leq & K. \end{aligned} \tag{5.68}$$

Finally using (1.8), (5.67) and (5.68), we get

$$\begin{aligned} & \int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi) \left(\frac{(\xi - u_1)^2}{2} - gh_1 \right) d\xi \\ = & \frac{\sqrt{2}}{g\pi} \int_{(\xi-u_1)^2>2gh_1} |\xi| \left(gh_2 - \frac{(\xi - u_2)^2}{2} \right)^{1/2} \left(\frac{(\xi - u_1)^2}{2} - gh_1 \right) d\xi \\ \leq & \frac{\sqrt{2}}{g\pi} (|u_2| + \sqrt{2gh_2}) \left| \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2) \right| K^{3/2}. \end{aligned} \tag{5.69}$$

Thus it is now sufficient for getting (5.65) to prove that

$$\left| \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2) \right| \leq \frac{4K}{\sqrt{2gh_2}}. \tag{5.70}$$

Moreover from (5.67) one has for $\xi \in \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)$ that

$$P(\xi) \leq 0 \tag{5.71}$$

where

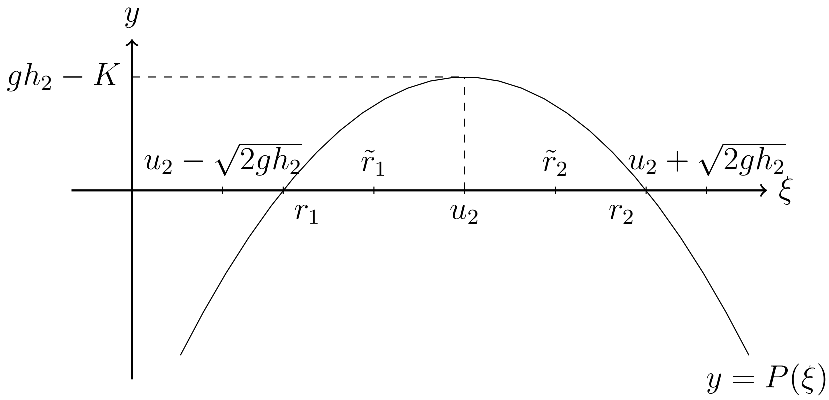
$$P(\xi) = gh_2 - \frac{(\xi - u_2)^2}{2} - K. \tag{5.72}$$

We notice that when $\xi = u_2$, P reaches a maximum equals to $gh_2 - K$, and we distinguish:

- if $K < gh_2$, then the maximum of P is positive and using (5.71) we get that for $\xi \in \mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)$ we have

$$\xi \in \left[u_2 - \sqrt{2gh_2}, r_1 \right] \bigcup \left[r_2, u_2 + \sqrt{2gh_2} \right] \tag{5.73}$$

  with $r_1 < u < r_2 \in \mathbb{R}$ are such that $P(r_1) = P(r_2) = 0$, we have $u_2 - \sqrt{2gh_2} < r_1$ because $P(u_2 - \sqrt{2gh_2}) = -K < 0$ and $r_2 < u_2 + \sqrt{2gh_2}$ because $P(u_2 + \sqrt{2gh_2}) = -K < 0$. This configuration is illustrated in the following picture.



Graph of $\xi \mapsto P(\xi)$ when $K < gh_2$

Thus

$$|\mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)| \leq \left| r_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - r_2 \right|. \tag{5.74}$$

We set

$$\tilde{r}_1 = u_2 - \sqrt{2gh_2} + \frac{2K}{\sqrt{2gh_2}} \tag{5.75}$$

and we notice that

$$\frac{2K}{\sqrt{2gh_2}} < \sqrt{2gh_2} \tag{5.76}$$

because of the assumption $K < gh_2$. Thus we obtain that

$$\tilde{r}_1 < u_2. \tag{5.77}$$

Moreover

$$\begin{aligned} & gh_2 - \frac{(\tilde{r}_1 - u_2)^2}{2} \\ & = gh_2 - \frac{1}{2}\left(-\sqrt{2gh_2} + \frac{2K}{\sqrt{2gh_2}} \right)^2 \\ & = -\frac{K^2}{gh_2} + 2K = K\left(-\frac{K}{gh_2} + 2 \right) \end{aligned} \tag{5.78}$$

and again using the asumption $K < gh_2$ we notice that

$$-\frac{K}{gh_2} + 2 > 1 \tag{5.79}$$

therefore

$$gh_2 - \frac{(\tilde{r}_1 - u_2)^2}{2} > K \tag{5.80}$$

which means that $P(\tilde{r}_1) > 0$. In consequence, using (5.77), we deduce that

$$r_1 < \tilde{r}_1 < u_2. \tag{5.81}$$

Similarly we set

$$\tilde{r}_2 = u_2 + \sqrt{2gh_2} - \frac{2K}{\sqrt{2gh_2}} \tag{5.82}$$

and by the same arguments we obtain that

$$u_2 < \tilde{r}_2 < r_2. \tag{5.83}$$

Putting together (5.81) and (5.83), we get

$$\begin{aligned} &\left| r_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - r_2 \right| \\ \leq &\left| \tilde{r}_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - \tilde{r}_2 \right|. \end{aligned} \tag{5.84}$$

Finally, using (5.74), we get

$$\begin{aligned} |\mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)| &\leq \left| \tilde{r}_1 - \left(u_2 - \sqrt{2gh_2} \right) \right| + \left| u_2 + \sqrt{2gh_2} - \tilde{r}_2 \right| \\ &\leq \frac{4K}{\sqrt{2gh_2}}. \end{aligned} \tag{5.85}$$

- if $K \geq gh_2$ then

$$|\mathrm{supp}(M_1)^c \cap \mathrm{supp}(M_2)| \leq |\mathrm{supp}(M_2)| \leq 2\sqrt{2gh_2} = 2\frac{2gh_2}{\sqrt{2gh_2}} \leq \frac{4K}{\sqrt{2gh_2}} \tag{5.86}$$

which concludes (5.70) and the proof.

□

Lemma 5.4. *There exists some $C > 0$ depending only on g, h_m, h_M, u_M such that*

$$\begin{aligned} &\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} \left(2M_1 + M_2\right) \left(M_1 - M_2\right)^2 d\xi \\ \geq &C \left(g\frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2} \right). \end{aligned} \tag{5.87}$$

for every U_1 , $U_2 \in \mathfrak{U}_{h_m,h_M,u_M}$ defined by (1.30) *and where $M_k \equiv M_k(\xi) \equiv M(U_k, \xi)$, with $M(U, \xi)$ defined in* (1.8).

Proof. Let $U_1, U_2 \in \mathfrak{U}_{h_m,h_M,u_M}$. We use lemma 5.1 and get

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2) (M_1 - M_2)^2 d\xi \\
&= \int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \\
&\quad - \int_{(\xi-u_1)^2 > 2gh_1} |\xi| M(U_2, \xi) \left(gh_1 - \frac{(\xi - u_1)^2}{2} \right) d\xi. \qquad (5.88)
\end{aligned}$$

We first we deal with the case

$$U_1, U_2 \text{ such that } |h_1 - h_2| \le \frac{1}{4\widetilde{C}_1^2} \text{ and } |u_1 - u_2| \le \frac{1}{4\widetilde{C}_2^2} \qquad (5.89)$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ are positive constants depending on g, h_m, h_M, u_M such that $4\widetilde{C}_2^2 \ge \frac{1}{\sqrt{gh_m}}$.

In this case, we are going to estimate the right-hand side of (5.88). On the one hand, in order to estimate the first term in the RHS of (5.88), we apply lemma 5.2 and since $4\widetilde{C}_2^2 \ge \frac{1}{\sqrt{gh_m}}$ we are in the case (5.12) and we get

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi| \left(H(M_2) - H(M_1) - \eta'(U_1) \begin{pmatrix} 1 \\ \xi \end{pmatrix} (M_2 - M_1) \right) d\xi \\
&\ge \alpha_m \left(\eta(U_2) - \eta(U_1) - \eta'(U_1) (U_2 - U_1) \right) \\
&= \alpha_m \left(g \frac{(h_2 - h_1)^2}{2} + h_1 \frac{(u_2 - u_1)^2}{2} \right) \\
&\ge \alpha_m \left(g \frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2} \right) \qquad (5.90)
\end{aligned}$$

with $\alpha_m = \frac{C}{\sqrt{2\pi}} \sqrt{gh_m}$ and $C > 0$ and absolute constant. One may notice that in order to obtain the last inequality we only used the fact that $h_1 \ge h_m$. On the other hand, in order to estimate the second term in the RHS of (5.88), we use lemma 5.3 and obtain

$$\begin{aligned}
&\int_{(\xi-u_1)^2 > 2gh_1} |\xi| M(U_2, \xi) \left(\frac{(\xi - u_1)^2}{2} - gh_1 \right) d\xi \\
&\le \frac{4 \left(|u_2| + \sqrt{2gh_2} \right)}{g\pi\sqrt{gh_2}} \left(g|h_1 - h_2| + (|u_2| + \sqrt{2gh_2})|u_1 - u_2| + \frac{1}{2}|u_1^2 - u_2^2| \right)^{\frac{5}{2}} \\
&\le C_1(h_m,h_M,u_M) \left(g|h_1 - h_2| + C_2(h_m,h_M,u_M)|u_1 - u_2| \right)^{\frac{5}{2}} \qquad (5.91)
\end{aligned}$$

with

$$C_1(h_m,h_M,u_M) = \frac{4\left(|u_M| + \sqrt{2gh_M}\right)}{g\pi\sqrt{gh_m}},$$
$$C_2(h_m,h_M,u_M) = 2|u_M| + \sqrt{2gh_M} \tag{5.92}$$

where $u_M = \dfrac{u_M}{h_m}$. Using Hölder inequality on $\mathbb{R}^2$ we get that for $a,b > 0$, $(a+b)^{5/2} \leq 2^{3/2}(a^{5/2} + b^{5/2})$, we get

$$\int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2} - gh_1\right) d\xi$$
$$\leq 2^{3/2} C_1(h_m,h_M,u_M)\left(g^{\frac{5}{2}}|h_1-h_2|^{\frac{5}{2}} + C_2(h_m,h_M,u_M)^{\frac{5}{2}}|u_1-u_2|^{\frac{5}{2}}\right). \tag{5.93}$$

Thus, putting together the two estimates (5.90) and (5.93) of the RHS of (5.88), we get

$$\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6}(2M_1+M_2)(M_1-M_2)^2 d\xi$$
$$\geq \alpha_m\left(g\frac{(h_2-h_1)^2}{2} + h_m\frac{(u_2-u_1)^2}{2}\right)$$
$$-\, 2^{3/2}C_1(h_m,h_M,u_M)\left(g^{\frac{5}{2}}|h_1-h_2|^{\frac{5}{2}} + C_2(h_m,h_M,u_M)^{\frac{5}{2}}|u_1-u_2|^{\frac{5}{2}}\right)$$
$$= \alpha_m\frac{g(h_2-h_1)^2}{2}\left(1 - \widetilde{C}_1|h_1-h_2|^{\frac{1}{2}}\right)$$
$$+\, \alpha_m\frac{h_m(u_2-u_1)^2}{2}\left(1 - \widetilde{C}_2|u_1-u_2|^{\frac{1}{2}}\right) \tag{5.94}$$

with

$$\alpha_m = \frac{C}{\sqrt{2\pi}}\sqrt{gh_m}, \quad C > 0 \text{ an absolute constant,} \tag{5.95}$$

and

$$\widetilde{C}_1 = \frac{2^{3/2+1}C_1(h_m,h_M,u_M)g^{\frac{5}{2}}}{\alpha_m g},$$
$$\widetilde{C}_2 = \frac{2^{3/2+1}C_1(h_m,h_M,u_M)g^{\frac{5}{2}}C_2(h_m,h_M,u_M)^{\frac{5}{2}}}{\alpha_m g}. \tag{5.96}$$

From (5.94), using that we deal with U_1, U_2 satisfying (5.89), we get

$$\int_{(\xi-u_1)^2>2gh_1} |\xi| M(U_2,\xi)\left(\frac{(\xi-u_1)^2}{2} - gh_1\right) d\xi$$
$$\geq \frac{\alpha_m}{2}\left(g\frac{(h_2-h_1)^2}{2} + h_m\frac{(u_2-u_1)^2}{2}\right). \tag{5.97}$$

At this point we have the result (5.87) for all $U_1, U_2 \in \mathfrak{U}_{h_m,h_M,u_M}$ satisfying (5.89). Thus, it is now sufficient to prove that

$$\begin{aligned}
&\exists \alpha_1 > 0, \quad \forall U_1, U_2 \in \mathfrak{U}_{h_m,h_M,u_M} \text{ such that} \\
&|h_1 - h_2| > \frac{1}{4\widetilde{C}_1^2} \text{ or } |u_1 - u_2| > \frac{1}{4\widetilde{C}_2^2}, \\
&\text{ we have } \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2 d\xi \geq \alpha_1.
\end{aligned} \tag{5.98}$$

Indeed, this last inequality implies that

$$\begin{aligned}
&g\frac{(h_2 - h_1)^2}{2} + h_m \frac{(u_2 - u_1)^2}{2} \\
&\leq C(h_M, u_M) \\
&\leq \frac{C(h_M, u_M)}{\alpha_1} \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2 d\xi
\end{aligned} \tag{5.99}$$

which proves (5.87). Using reductio ad absurdum as in the proof of lemma 5.2, we suppose that (5.98) does not hold. Thus

$$\begin{aligned}
&\forall n > 0, \quad \exists U_1^n, U_2^n \in \mathfrak{U}_{h_m,h_M,u_M}, \text{ such that} \\
&4\widetilde{C}_1^2 |h_1^n - h_2^n| + 4\widetilde{C}_2^2 |u_1^n - u_2^n| > 1 \\
&\text{and } \int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1^n + M_2^n)(M_1^n - M_2^n)^2 d\xi \leq \frac{1}{n}
\end{aligned} \tag{5.100}$$

where $M_i^n = M(U_i^n, \xi)$. As $\mathfrak{U}_{h_m,h_M,u_M}$ is a closed and bounded set, we can take 2 subsequences which we also denote U_1^n, U_2^n such that

$$U_1^n \to U_1 \in \mathfrak{U}_m, \quad U_2^n \to U_2 \in \mathfrak{U}_m \tag{5.101}$$

with

$$4\widetilde{C}_1^2 |h_1 - h_2| + 4\widetilde{C}_2^2 |u_1 - u_2| \geq 1 \tag{5.102}$$

and by dominated converge theorem

$$\int_{\mathbb{R}} |\xi| \frac{g^2\pi^2}{6} (2M_1 + M_2)(M_1 - M_2)^2 d\xi = 0. \tag{5.103}$$

Therefore we get

$$(2M_1 + M_2)(M_1 - M_2)^2 = 0 \quad \text{almost everywhere} \tag{5.104}$$

itself implying that $M_1 = M_2$ a.e. and therefore $U_1 = U_2$, the later being in contradiction with (5.102). $\square$

Lemma 5.5. *Let $U_k = (h_k, h_k u_k)$, $k = 1,2$ with $h_k \geq 0$. Then*

$$\int_{\mathbb{R}} |(M(U_1,\xi) - M(U_2,\xi)|)\, d\xi \\ \leq \frac{2\sqrt{3}}{\sqrt{g}} \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2 \right)^{\frac{1}{2}}, \tag{5.105}$$

with $M(U, \xi)$ defined in (1.8).

Proof. Let us recall some result from [5]

$$\int_{\mathbb{R}} M(U_1,\xi) \left(M(U_1,\xi) - M(U_2,\xi)\right)^2 d\xi \\ \leq \frac{3}{g^2\pi^2} \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2 \right). \tag{5.106}$$

We compute

$$\begin{aligned} &\int_{\mathbb{R}} |(M(U_1,\xi) - M(U_2,\xi)|)\, d\xi \\ &\leq \int_{M_1>0} |M_1 - M_2|\, d\xi + \int_{M_2>0} |M_1 - M_2|\, d\xi \\ &\leq \left(\int_{M_1>0} \frac{1}{M_1} d\xi \right)^{1/2} \left(\int_{M_1>0} M_1 \left(M_1 - M_2\right)^2 d\xi \right)^{1/2} \\ &+ \left(\int_{M_2>0} \frac{1}{M_2} d\xi \right)^{1/2} \left(\int_{M_2>0} M_2 \left(M_1 - M_2\right)^2 d\xi \right)^{1/2} \end{aligned} \tag{5.107}$$

where last estimate is obtained by using Cauchy-Schwarz inequality.

Using the substitution $v = \frac{\xi - u}{\sqrt{2gh}}$ we get

$$\begin{aligned} &\int_{M(U,\xi)>0} \frac{1}{M(U,\xi)} d\xi = \int_{u-\sqrt{2gh}}^{u+\sqrt{2gh}} \frac{g\pi}{\left(2gh - (\xi - u)^2\right)^{1/2}} d\xi \\ &= \int_{-1}^{1} \frac{g\pi\sqrt{2gh}}{\sqrt{2gh}\,(1 - v^2)^{1/2}} dv = g\pi \Big[\arcsin(v) \Big]_{-1}^{1} = g\pi^2. \end{aligned} \tag{5.108}$$

Now from (5.107), using (5.106) and (5.108), we get

$$\int_{\mathbb{R}} |(M(U_1,\xi) - M(U_2,\xi))|\, d\xi \\ \leq 2 \cdot \frac{\sqrt{3}}{\sqrt{g}} \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2 \right)^{\frac{1}{2}} \tag{5.109}$$

i.e. we find (5.105), which concludes the proof. □

Lemma 5.6. *Let $U_k = (h_k, h_k u_k)$, $k = 1{,}2$ with $h_k \geq 0$. Moreover we set here*

$$C = \max_{v \in \left\{ |u_1| + \sqrt{gh_1}, |u_2| + \sqrt{gh_2} \right\}} |v| \left(1 + v^2\right)^{\frac{1}{2}}. \tag{5.110}$$

Then one has

$$|F(U_1) - F(U_2)| \leq \frac{2\sqrt{3}}{\sqrt{g}} C \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}}, \tag{5.111}$$

$$\left|F^+(U_1) - F^+(U_2)\right| \leq \frac{2\sqrt{3}}{\sqrt{g}} C \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}}, \tag{5.112}$$

$$\left|F^-(U_1) - F^-(U_2)\right| \leq \frac{2\sqrt{3}}{\sqrt{g}} C \left(g(h_2 - h_1)^2 + \min(h_1, h_2)(u_2 - u_1)^2\right)^{\frac{1}{2}}. \tag{5.113}$$

Proof. We recall that

$$\begin{aligned} F^+(U) &= \int_{\mathbb{R}} \xi \mathbb{1}_{\xi > 0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi, \\ F^-(U) &= \int_{\mathbb{R}} \xi \mathbb{1}_{\xi < 0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi \\ \text{and} \quad F(U) &= \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U,\xi) d\xi. \end{aligned} \tag{5.114}$$

Thus the result is an immediate consequence of lemma 5.5 and the fact that

$$\forall \xi \in \text{supp} M_1 \cap \text{supp} M_2, \qquad \left| \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right| \leq \underbrace{\max_{\xi \in \left\{ |u_1| + \sqrt{gh_1}, |u_2| + \sqrt{gh_2} \right\}} |\xi| \left(1 + \xi^2\right)^{\frac{1}{2}}}_{= C} \tag{5.115}$$

with C defined by (5.110). □

Lemma 5.7. *Let (U_i^n) defined by* (1.15)-(1.8) *and U_Δ defined by* (1.28). *Let $\phi \in \mathcal{D}$ and we assume* (1.32)-(1.34). *Then, under the CFL condition* (1.27) *there exists some $C > 0$ depending only on $\phi, g, h_m, h_M, u_M, v_m$ such that*

$$\forall t \in [0,T], \quad < \partial_t U_\Delta(t,\cdot), \phi > \leq C. \tag{5.116}$$

Proof. Using (2.47) we get

$$< \partial_t U_\Delta, \phi >= A + B \tag{5.117}$$

with

$$A = \sum_i \frac{1}{\Delta t} \left[\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1} - U_{i+1}^{n} + U_{i-1}^{n}}{2\Delta x} \right] \int_{x_{i-1/2}}^{x_{i+1/2}} (x - x_{i-1/2}) \phi(x) dx \tag{5.118}$$

and

$$B = \sum_i \frac{1}{\Delta t} \left[U_i^{n+1} - U_i^n \right] \int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x) dx. \tag{5.119}$$

First we notice that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (x - x_{i-1/2}) \phi(x) dx = \psi(x_{i+1/2}) \Delta x - \int_{x_{i-1/2}}^{x_{i+1/2}} \psi(x) dx, \tag{5.120}$$

where ψ is an antiderivative of ϕ. Then, using last equality in (5.118), we get

$$A = \sum_i \frac{1}{\Delta t} \left[\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1} - U_{i+1}^{n} + U_{i-1}^{n}}{2\Delta x} \right] \Delta \psi_i \Delta x, \tag{5.121}$$

with $\Delta \psi_i := \psi(x_{i+1/2}) - \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \psi(x) dx$. Moreover, by making substitutions of indices we get

$$A = \sum_i \frac{1}{\Delta t} \left[\frac{U_{i+1}^{n+1} - U_{i+1}^{n}}{2} \right] \left[\Delta \psi_i - \Delta \psi_{i+2} \right]. \tag{5.122}$$

Next using that (U_i^n) is a bounded sequence we get that

$$|U_{i+1}^{n+1} - U_{i+1}^{n}| \leq \frac{\Delta t}{\Delta x} \left(\|F^+(U)\|_\infty + F^-(U)\|_\infty \right). \tag{5.123}$$

Moreover we notice that

$$\left| \frac{\Delta \psi_i - \Delta \psi_{i+2}}{2} \right| \leq \Delta x^2 \mathrm{Lip}(\phi), \tag{5.124}$$

which enables us to get

$$|A| \leq 2C \|(U_i^n)\|_{l^\infty} \mathrm{Lip}(\phi) \left(\Delta x |\mathrm{supp} \phi| \right), \tag{5.125}$$

with $C > 0$ a constant depending only on ϕ. Finally using CFL, it is bounded. Next, from (5.119), we use (1.17) and we get

$$B = \sum_i -\frac{1}{\Delta t}\frac{\Delta t}{\Delta x}\left[F_{i+1/2-} - F_{i-1/2+}\right]\Delta x\phi_i, \tag{5.126}$$

with $\phi_i := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x)dx$. Using (1.18)-(1.20) we get

$$\begin{aligned} B &= \sum_i -\left[F_{i+1/2-} - F_{i-1/2+}\right]\phi_i \\ &= \sum_i -\left[\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right]\phi_i + \sum_i -\left[S_{i+1/2-} - S_{i-1/2+}\right]\phi_i \\ &= \sum_i -\mathcal{F}_{i+1/2}\left[\phi_i - \phi_{i+1}\right] + \sum_i -\left[S_{i+1/2-} - S_{i-1/2+}\right]\phi_i. \end{aligned} \tag{5.127}$$

Moreover

$$|\phi_i - \phi_{i+1}| \leq \Delta x \mathrm{Lip}(\phi) \tag{5.128}$$

and

$$|S_{i+1/2-} - S_{i-1/2+}| \leq C|z_{i+1} - z_i| \leq C|\Delta x|. \tag{5.129}$$

Furthermore, using that (U_i^n) is a bounded sequence we get

$$|B| \leq C\left(\|F^+(U)\|_\infty + \|F^-(U)\|_\infty\right)\mathrm{Lip}(\phi) + C\|\widetilde{S}_\Delta\|_\infty\|\phi\|_\infty, \tag{5.130}$$

with C depending only on ϕ. □

Lemma 5.8. *Let $U_\Delta = (h_\Delta, h_\Delta u_\Delta)$ be the approximate solution of* (1.1) *defined by* (1.28) *and $\widetilde{S}_\Delta$ be the approximate source defined by* (3.7). *We assume that there exists U such that U_Δ tends to U a.e. and in L^p_{loc}, as $\Delta x, \Delta t \to 0$. Then we get that*

$$\forall\phi(t,xt) \in \mathcal{D}(\mathbb{R}^2), \qquad \iint \widetilde{S}_\Delta(t,x)\phi dtdx \underset{\Delta x,\Delta t\to 0}{\longrightarrow} \iint S(t,x)\phi(t,x)dtdx, \tag{5.131}$$

with $S(t,x) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix}$.

Proof. Let $\phi(t,x) \in \mathcal{D}(\mathbb{R}^2)$. We study the following integral

$$\iint \widetilde{S}_\Delta(t,x)\phi dt dx = \sum_{n=0}^{N+1} \sum_{i=i_0-1}^{i_1+1} \Delta t \Delta x \left(S_{i+1/2-} + S_{i-1/2+}\right) \phi_i$$

with $\phi_i^n = \dfrac{1}{\Delta t}\dfrac{1}{\Delta x}\displaystyle\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi(t,x) dt dx$. Next we develop and make a translation over index i and we get

$$\begin{aligned}
&\sum_n \sum_i \Delta t \Delta x \left(S_{i+1/2-} + S_{i-1/2+}\right) \phi_i \\
=&\sum_n \Delta t \Delta x \sum_i S_{i+1/2-}\phi_i + \sum_n \Delta t \Delta x \sum_i S_{i-1/2+}\phi_i, \\
=&\sum_n \Delta t \Delta x \sum_i S_{i+1/2-}\phi_i + \sum_n \Delta t \Delta x \sum_i S_{i+1/2+}\phi_{i+1}.
\end{aligned}$$

Then we notice that $|\phi_{i+1}^n - \phi_i^n| \leq C\Delta x$ with constant $C > 0$ and we obtain that

$$\begin{aligned}
&\left| \iint \widetilde{S}_\Delta(t,x)\phi dt dx - \sum_n \sum_i \Delta t \Delta x \left(S_{i+1/2-} + S_{i+1/2+}\right) \phi_i \right| \\
&\qquad \leq C\Delta x \sum_{n, t_n \in supp\phi} \Delta t \sum_{i, x_{i+1/2} \in supp\phi} S_{i+1/2+}\Delta x. \quad (5.132)
\end{aligned}$$

Since $S_{i+1/2+}$ is bounded the RHS tends to 0. Next, for $\Delta x, |z_{i+1} - z_i|$ small enough, we have on the one hand

$$\begin{aligned}
\Delta x S_{i+1/2-}^{hu} =& g\frac{h_{i+1/2-}^2}{2} - g\frac{h_i^2}{2} \\
=& g\frac{(h_i + z_i - z_{i+1/2})^2}{2} - g\frac{h_i^2}{2}, \text{ (by assumption (1.32))} \\
=& g(z_i - z_{i+1/2})\left(h_i + \frac{z_i - z_{i+1/2}}{2}\right). \quad (5.133)
\end{aligned}$$

On the other hand, as in (5.133), we obtain

$$\Delta x S_{i+1/2+}^{hu} = g\frac{h_{i+1}^2}{2} - g\frac{h_{i+1/2+}^2}{2} = -g(z_{i+1} - z_{i+1/2})\left(h_{i+1} + \frac{z_{i+1} - z_{i+1/2}}{2}\right). \quad (5.134)$$

Moreover noticing that $h_{i+1} = h_i + (h_{i+1} - h_i)$, with (5.133),(5.134) we get

$$\begin{aligned}
&\sum_n \sum_i \Delta t \Delta x \left(S^{hu}_{i+1/2-} + S^{hu}_{i+1/2+} \right) \phi_i \\
&= -\sum_n \sum_i \Delta t \Delta x \, g(z_{i+1} - z_i) h_i \phi_i + \sum_n \sum_i \Delta t \Delta x R^n_i \phi_i
\end{aligned} \tag{5.135}$$

with

$$R^n_i = -g(z_{i+1} - z_{i+1/2})\,(h_{i+1} - h_i) + g\frac{(z_i - z_{i+1/2})^2}{2} \quad -g\frac{(z_{i+1} - z_{i+1/2})^2}{2}. \tag{5.136}$$

First term in the RHS of (5.135) converges to the source term:

$$\begin{aligned}
&-\sum_n \sum_i \Delta t g(z_{i+1} - z_i) h_i \phi_i = -\sum_n \sum_i \Delta t \Delta x \, g \frac{z_{i+1} - z_i}{\Delta x} h_i \phi_i \\
&= \iint -g \frac{\mathrm{d}z_\Delta(x)}{\mathrm{d}x} h_\Delta(t,x) \phi(x) dx \to \iint -g \frac{\mathrm{d}z(x)}{\mathrm{d}x} h(x) \phi(x) dx,
\end{aligned}$$

the convergence holds because we supposed $h_\Delta \to h$, in L^p_{loc} and $\frac{\mathrm{d}z_\Delta(x)}{\mathrm{d}x} \to \frac{\mathrm{d}z(x)}{\mathrm{d}x}$, in L^∞_{loc}. In order to conclude, according to (5.132) we need to prove that the remaining terms $\sum_n \sum_i \Delta t \Delta x R^n_i \phi_i$ tend to 0 as $\Delta t, \Delta x \to 0$.

Using that $0 \le z_{i+1/2} - z_{i+1} \le |z_{i+1} - z_i|$, $0 \le z_{i+1/2} - z_i \le |z_{i+1} - z_i|$ and $|z_{i+1} - z_i| \le C\Delta x$, we get:

$$|R^n_i| \le C\Delta x \, |h_{i+1} - h_i| + C\Delta x^2 \tag{5.137}$$

and

$$\left| \sum_n \sum_i \Delta t R^n_i \phi_i \right| \le C \sum_n \sum_i \Delta t \Delta x \, |h_{i+1} - h_i| + C \sum_n \sum_i \Delta t \Delta x^2 \, |\phi_i| \, . \tag{5.138}$$

On the one hand we have

$$\sum_n \sum_i \Delta t \Delta x \, |h_{i+1} - h_i| \, \phi_i = O(\Delta x^{1/2}), \tag{5.139}$$

using Cauchy Swartz and $\sum_n \sum_i \Delta t (h_{i+1} - h_i)^2 < C$, see (2.3).

On the other hand we have

$$\left| \sum_n \sum_i \Delta t \Delta x^2 \phi_i \right| \le \Delta x \|\phi\|_{L^1} \tag{5.140}$$

Thus

$$\sum_n \sum_i \Delta t R^n_i \phi_i \longrightarrow 0, \tag{5.141}$$

which concludes the proof. □

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