Convergence in distribution norms in the CLT for non identical distributed random variables

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Abstract

We study the convergence in distribution norms in the Central Limit Theorem for non identical distributed random variables that is

$$\varepsilon_n(f) := \mathbb{E}\Big(f\Big(\sum_{i=1}^n Z_i\Big)\Big) - \mathbb{E}\big(f(G)\big) \to 0$$

where $S_n = \sum_{i=1}^n Z_i$ with Z_i centred independent random variables (with a suitable re-normalization for S_n) and G is standard normal. We also consider local developments (Edgeworth expansion). This kind of results is well understood in the case of smooth test functions f. If one deals with measurable and bounded test functions (convergence in total variation distance), a well known theorem due to Prohorov shows that some regularity condition for the law of the random variables X_n, $n \in \mathbb{N}$, on hand is needed. Essentially, one needs that the law of X_n is locally lower bounded by the Lebesgue measure (Doeblin's condition). This topic is also widely discussed in the literature (see Battacharaya and Rao [12]). Our main contribution is to discuss convergence in distribution norms, that is to replace the test function f by some derivative $\partial^\alpha f$ and to obtain upper bounds for $\varepsilon_n(\partial^\alpha f)$ in terms of the original function f.

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1 Introduction

We consider a sequence of centred independent random variables $Z_k \in \mathbb{R}^d, k \in \mathbb{N}$ with covariance matrixes $\sigma_k^{i,j} = \mathbb{E}(Z_k^i Z_k^j)$ and we look to

$$S_n(Z) = \sum_{k=1}^{n} Z_k. \tag{1.1}$$

Our aim is to obtain a Central Limit Theorem as well as Edgeworth developments in this framework. The basic hypotheses are the following. We assume the normalization condition

$$\sum_{k=1}^{n} \sigma_k = I_d \tag{1.2}$$

where $I_d \in \mathcal{M}_{d\times d}$ is the identity matrix. Moreover we assume that for each $p \in \mathbb{N}$ there exists a constant $C_p \geq 1$ such that

$$\max_k \mathbb{E}(|Z_k|^p) \leq \frac{C_p(Z)}{n^{p/2}}. \tag{1.3}$$

Let $\|f\|_{k,\infty}$ denote the norm in $W^{k,\infty}$, that is the uniform norm of f and of all its derivatives of order less or equal to k. First, we want to prove that

$$\left|\mathbb{E}(f(S_n(Z)) - \int_{\mathbb{R}^d} f(x)\gamma_d(x)dx\right| \leq \frac{C_0}{n^{\frac{1}{2}}} \|f\|_{3,\infty} \tag{1.4}$$

where $\gamma_d(x) = (2\pi)^{-d/2}\exp(-\frac{1}{2}|x|^2)$ is the density of the standard normal law. This corresponds to the Central Limit Theorem (hereafter CLT). Moreover we look for some functions (polynomials) $\psi_k : \mathbb{R}^d \to \mathbb{R}$ such that for $N \in \mathbb{N}$ and for every $f \in C_b^{(N+1)(N+3)}(\mathbb{R}^d)$

$$\left|\mathbb{E}(f(S_n(Z)) - \int_{\mathbb{R}^d} f(x)\Big(\sum_{k=0}^{N} \frac{1}{n^{k/2}}\psi_k(x)\Big)\gamma_d(x)dx\right| \leq \frac{C_N}{n^{\frac{1}{2}(N+1)}} \|f\|_{(N+1)(N+3),\infty}. \tag{1.5}$$

This is the Edgeworth development of order N. In the case of smooth test functions f (as it is the case in (1.5)), this topic has been widely discussed and well understood: such development has been obtained by Sirazhdinov and Mamatov [21] in the case of identically distributed random variables and then by Götze and Hipp [16] in the non identically distributed case. A complete presentation of this topic may be found in the book of Battacharaya and Rao [12]. It it worth to mention that the classical approach used in the above papers is based on Fourier analysis. In particular, the coefficients ψ_k in the above development are given as inverse Fourier transform of some suitable functions, so the expression of ψ_k is not completely transparent and its explicit computation requires some effort.

In our paper we use a different approach based on the Lindemberg method for Markov semigroups (this is inspired from works concerning the parametrix method for Markov semigroups in [9]). This alternative approach is convenient for the proof of our main result concerning "distribution norms" (see below). But, even in the case of smooth test functions, this allows to obtain slightly more clear and precise results: we prove that ψ_k are linear combination of Hermite polynomials of order less or equal to k, whose coefficients are explicit and computed starting with the moments of Z_i and G_i, G_i denoting a Gaussian random variable with the same covariance matrix as Z_i. So the computation of these coefficients is easier. Moreover, our estimates hold for each fixed n (in contrast with the ones in the above papers, which are just asymptotic).

A second problem is to obtain the estimate (1.5) for test functions f which are not regular, in particular to replace $\|f\|_{(N+1)(N+3),\infty}$ by $\|f\|_{\infty}$. This amounts to estimate the error in total variation distance. In the case of identically distributed random variables, and for $N=0$ (so at the level of the standard CLT), this problem has been widely studied. First of all, one may prove the convergence in Kolmogorov distance, that is for $f=1_D$ where D is a rectangle. Many refinements of this type of result has been obtained by Battacharaya and Rao and they are presented in [12]. But it turns out that one may not prove such a result for a general measurable set D without assuming more regularity on the law of Z_k, $k\in\mathbb{N}$. Indeed, in his seminal paper [20] Prohorov proved that the convergence in total variation distance is equivalent to the fact that there exists m such that the law of $Z_1+\cdots+Z_m$ has an absolutely continuous component. In [3] Bally and Caramellino obtained (1.5) in total variation distance, for identically distributed random variables, under the hypothesis that the law of Z_k is locally lower bounded by the Lebesgue measure. We assume this type of hypothesis in this paper also. More precisely we assume that there exists $r,\varepsilon>0$ and there exists $z_k\in\mathbb{R}^d$ such that for every measurable set $A\subset B_r(z_k)$

$$\mathbb{P}(Z_k\in A)\geq\varepsilon\lambda(A) \tag{1.6}$$

where λ is the Lebesgue measure. This condition is known in the literature as Doeblin's condition. Under this hypothesis we are able to obtain (1.5) in total variation distance. It is clear that (1.6) is more restrictive than Prohorov's condition. However we prove that in the framework of the CLT for identically distributed random variables, if we have Prohorov's condition we may produce doubling condition as well, just working with the packages $Y_k=\sum_{i=2km+1}^{2(k+1)m}Z_i$. This allows us to prove Corollary 3.12 which is a stronger version of Prohorov's theorem.

Let us finally mention another line of research which has been strongly developed in the last years: it consists in estimating the convergence in the CLT in entropy distance. This starts with the papers of Barron [11] and Johnson and Barron [14]. In these papers the case of identically distributed random variables is considered, but recently, in [13] Bobkov, Chistyakov and Götze obtained the estimate in entropy distance for the case of random variables which are no more identically distributed as well. We recall that the convergence in entropy distance implies the convergence in total variation distance, so such results are stronger. However, in order to work in entropy distance one has to assume that the law of Z_k is absolutely continuous with respect to the Lebesgue measure and have finite entropy and this is more limiting than (1.6). So the hypothesis and the results are slightly different.

A third problem is to obtain the CLT and the Edgeworth development with the test function f replaced by a derivative $\partial_\gamma f$. If the law of $S_n(Z)$ is absolutely continuous with respect to the Lebesgue measure, this means that we prove the convergence of the density and of its derivatives as well (which corresponds to the convergence in distribution norms). Unfortunately we fail to obtain such a result in the general framework: this is moral because we do not assume that the laws of Z_k, $k=1,...,n$ are absolutely continuous, and then the law of $S_n(Z)$ may have atoms. However we obtain a similar result, but we have to keep a "small error". Let us give a precise statement of our result. For a function $f\in C_p^m(\mathbb{R}^d)$ (m times differentiable with polynomial growth) we define $L_m(f)$ and $l_m(f)$ to be two constants such that

$$\sum_{0\leq|\alpha|\leq m}|\partial_\alpha f(x)|\leq L_m(f)(1+|x|)^{l_m(f)}. \tag{1.7}$$

Our main result is the following: for a fixed $m\in\mathbb{N}$, there exist some constants $C_N\geq 1\geq c_N>0$ (depending on r,ε from (1.6) and on C_p from (1.3)) such that for every multi-index γ with $|\gamma|=m$ and for every $f\in C_p^m(\mathbb{R}^d)$

$$\begin{aligned}&\Big|\mathbb{E}\Big(\partial_\gamma f(S_n(Z))-\int_{\mathbb{R}^d}\partial_\gamma f(x)\Big(\sum_{q=0}^N\frac{1}{n^{q/2}}\psi_q(x)\Big)\gamma_d(x)dx\Big)\Big|\\ &\leq C_N\Big(L_m(f)e^{-c_N\times n}+\frac{1}{n^{\frac12(N+1)}}L_0(f)\Big).\end{aligned} \tag{1.8}$$

If the random variables $Z_k,k\in\mathbb{N}$ are identically distributed we succeed to obtain exactly the same result under the Prohorov's condition (see Corollary 3.12). So this is a strictly stronger version of Prohorov's theorem (for $m=0$ we get the convergence in total variation). Moreover, such result is used in [6] in order to give invariance principles concerning the variance of the number of zeros of trigonometric polynomials.

However we fail to get convergence in distribution norms because $L_m(f)e^{-c_N\times n}$ appears in the upper bound of the error and $L_m(f)$ depends on the derivatives of f. But we are close to such a result: notice first that if $f_n=f*\phi_{\delta_n}$ is a regularization by convolution with $\delta_n=\exp(-\frac{c_N}{2m}\times n)$ then (1.8) gives

$$\Big|\mathbb{E}\Big(\partial_\gamma f_n(S_n(Z))-\int_{\mathbb{R}^d}\partial_\gamma f_n(x)\Big(\sum_{q=0}^N\frac{1}{n^{q/2}}\psi_q(x)\Big)\gamma_d(x)dx\Big)\Big|\leq\frac{C_N}{n^{\frac12(N+1)}}L_0(f). \tag{1.9}$$

Another way to eliminate $L_m(f)e^{-c_N\times n}$ is to assume that the law of $Z_i,\ i=1,...,m$ are absolutely continuous with the derivative of the density belonging to L^1. This is done in Proposition 4.2: we prove that for every $k\in\mathbb{N}$ and every multi-index α

$$\sup_x(1+|x|^2)^k\,|\partial_\alpha p_{S_n}(x)-\partial_\alpha\gamma(x)|\leq\frac{C}{\sqrt{n}}$$

so, under these stronger conditions, we succeed to obtain convergence in distribution norms.

But the most interesting consequence of our result is given in Theorem 4.1: there we give an invariance principle for the occupation time of a random walk. More precisely we take $\varepsilon_n=n^{-\frac12(1-\rho)}$ with $\rho\in(0,1)$ and we prove that, for every $\rho'<\rho$

$$\Big|\sum_{k=1}^n\mathbb{E}\Big(\frac{1}{\varepsilon_n}1_{(-\varepsilon_n,\varepsilon_n)}(\sum_{i=1}^k Z_i)\Big)-\mathbb{E}\Big(\int_0^1\frac{1}{\varepsilon_n}1_{(-\varepsilon_n,\varepsilon_n)}(W_s)ds\Big)\Big|\leq\frac{C}{n^{\frac12(1+\rho')}}$$

with W_s a Brownian motion (so $\int_0^1\frac{1}{\varepsilon_n}1_{(-\varepsilon_n,\varepsilon_n)}(W_s)ds$ converges to the local time of W). Here the test function is $f_n=\frac{1}{\varepsilon_n}1_{(-\varepsilon_n,\varepsilon_n)}$ and this converges to the Dirac function. This example shows that (1.8) is an appropriate estimate in order to deal with some singular problems.

The paper is organized as follows. In Section 2 we prove the result for smooth test functions (that is (1.5)) and in Section 3 we treat the case of measurable test functions. In order to do it we use some integration by parts technology which has already been used in [3] and which is presented in Section 3.1. We mention that a similar approach has been used by Nourdin and Poly [18], by using the Γ-calculus settled in [10]. The main result in Section 3 is Theorem 3.8. In Section 4 we treat the two applications mentioned above. Finally we leave for Appendix A the explicit calculus of the coefficients ψ_q from (1.5) for $q = 1, 2, 3$ and in Appendix B we prove a technical result which is used in our development.

Although many ideas in our paper come from previous works (mainly from Malliavin calculus), at the end we finish with an approach which is fairly simple and elementary - so we try to give here a presentation which is essentially self contained (even if some cumbersome and straightforward computations are just sketched).

2 Smooth test functions

2.1 Notation and main result

We fix $n \in \mathbb{N}$ and we consider n centred and independent random variables $Z = (Z_k)_{1\leq k\leq n}$ with $Z_k = (Z_k^1, ..., Z_k^d) \in \mathbb{R}^d$. We denote by σ_k the covariance matrix of Z_k that is

$$\sigma_k^{i,j} = \mathbb{E}(Z_k^i Z_k^j), \quad 1 \leq k \leq n.$$

We look to

$$S_n(Z) = \sum_{k=1}^n Z_k. \tag{2.1}$$

Our aim is to compare the law of $S_n(Z)$ with the law of $S_n(G)$ where $G = (G_k)_{1\leq k\leq n}$ denotes n centred and independent Gaussian random variables with the same covariance matrices:

$$\mathbb{E}(G_k^i G_k^j) = \sigma_k^{i,j}.$$

This is a CLT result (but we stress that it is not asymptotic). And we will obtain an Edgeworth development as well.

We assume that Z_k has finite moments of any order and more precisely,

$$\max_{1\leq k\leq n} \mathbb{E}(|Z_k|^i) \leq \frac{C_i(Z)}{n^{i/2}}. \tag{2.2}$$

In particular, for $i = 2$ the inequality (2.2) gives

$$\max_{1\leq k\leq n} \sup_{i,j} |\sigma_k^{i,j}| \leq \frac{C_2(Z)}{n}. \tag{2.3}$$

Since the covariance matrix of G_k is equal to that of Z_k, the inequality (2.2) holds for the G_k's as well, so we can resume by writing

$$\sup_{1\leq k\leq n} \mathbb{E}(|Z_k|^i) \vee \mathbb{E}(|G_k|^i) \leq \frac{C_i(Z)}{n^{i/2}}. \tag{2.4}$$

Without loss of generality, (from Hölder) we can assume that $1 \leq C_i(Z) \leq C_{i+1}(Z)$ and more in general

$$1 \leq C_p(Z) \leq C_{pq}^{1/q}(Z), \quad p, q \geq 1.$$

Remark 2.1. *Although it is not explicitly written, we are assuming that we fix n and that the laws of Z_k and G_k, as well as σ_k, are all depending on n. In our applications, we take a sequence $Y = \{Y_k\}_k$*

of i.i.d. centred r.v's taking values in $\mathbb{R}^m$ *and we consider* $Z_k = \frac{1}{\sqrt{n}} C_k Y_k$*, where* C_k *denotes a* $d\times m$ *matrix. Therefore, we actually study*

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n C_k Y_k$$

Notice that

$$\max_{1\le k\le n} \mathbb{E}(|Z_k|^i) \le \frac{c_i(Y)}{n^{i/2}} \times \max_{1\le k\le n} \|C_k\|^i,$$

in which $c_i(Y)$ *denotes a constant depending only on (the law of) the* Y_k*'s, so that (2.4) actually holds. We will specialize the results to this case. But in order to relax the notation and the proofs, it is much more useful to consider a general* Z_k *instead of* $\frac{1}{\sqrt{n}} C_k Y_k$*.*

In order to give the expression of the terms which appear in the Edgeworth development we need to introduce some notation.
We say that α is a multiindex if $\alpha \in \{1,\ldots,d\}^k$ for some $k\ge 1$, and we set $|\alpha| = k$ its length. We allow the case $k=0$, giving the void multiindex $\alpha = \emptyset$.
Let α be a multiindex and set $k=|\alpha|$. For for $x\in\mathbb{R}^d$ and $f:\mathbb{R}^d\to\mathbb{R}$, we denote $x^\alpha = x_{\alpha_1}\cdots x_{\alpha_k}$ and $\partial_\alpha f(x) = \partial_{x_{\alpha_1}}\cdots\partial_{x_{\alpha_k}} f(x)$, the case $k=0$ giving $x^\emptyset = 1$ and $\partial_\emptyset f = f$. In the following, we denote with $C^k(\mathbb{R}^d)$ the set of the functions f such that $\partial_\alpha f$ exists and is continuous for any α with $|\alpha|\le k$. The set $C^k_p(\mathbb{R}^d)$, resp. $C^k_b(\mathbb{R}^d)$, is the subset of $C^k(\mathbb{R}^d)$ such that $\partial_\alpha f$ has polynomial growth, resp. is bounded, for any α with $|\alpha|\le k$. $C^\infty(\mathbb{R}^d)$, resp. $C^\infty_p(\mathbb{R}^d)$ and $C^\infty_b(\mathbb{R}^d)$, denotes the intersection of $C^k(\mathbb{R}^d)$, resp. of $C^k_p(\mathbb{R}^d)$ and of $C^k_b(\mathbb{R}^d)$, for every k.
For $f\in C^k_b(\mathbb{R}^d)$ we denote

$$\|f\|_{k,\infty} = \|f\|_\infty + \sum_{1\le|\alpha|\le k} \|\partial_\alpha f\|_\infty$$

and for $f\in C^k_p(\mathbb{R}^d)$ we define $L_k(f)$ and $l_k(f)$ to be some constants such that

$$\sum_{0\le|\alpha|\le k} |\partial_\alpha f(x)| \le L_k(f)(1+|x|)^{l_k(f)}. \tag{2.5}$$

Moreover, for a non negative definite matrix $\sigma\in\mathcal{M}_{d\times d}$ we denote by L_σ the Laplace operator associated to σ, i.e.

$$L_\sigma = \sum_{i,j=1}^d \sigma^{i,j}\partial_{z_i}\partial_{z_j}. \tag{2.6}$$

For $r\ge 1$ and $l\ge 0$ we set

$$\Delta_\alpha(r) = \mathbb{E}(Z_r^\alpha) - \mathbb{E}(G_r^\alpha) \quad\text{and}\quad D_r^{(l)} = \sum_{|\alpha|=l} \Delta_\alpha(r)\partial_\alpha. \tag{2.7}$$

Notice that $D_r^{(l)}\equiv 0$ for $l=0,1,2$ and, by (2.4), for $l\ge 3$ and $|\alpha|=l$ then

$$|\Delta_\alpha(r)| \le \frac{2C_l(Z)}{n^{l/2}}, \quad r=1,\ldots,n. \tag{2.8}$$

We construct now the coefficients of our development. Let N be fixed: this is the order of the development that we will obtain. Given $1\le m\le k\le N$ we define

$$\begin{aligned}
\Lambda_m &= \{((l_1,l'_1),\ldots,(l_m,l'_m)) : N+2\ge l_i\ge 3, \overline{N} := [N/2]\ge l'_i\ge 0, i=1,\ldots,m\},\\
\Lambda_{m,k} &= \{((l_1,l'_1),\ldots,(l_m,l'_m))\in\Lambda_m : \sum_{i=1}^m l_i + 2\sum_{i=1}^m l'_i = k+2m\}.
\end{aligned} \tag{2.9}$$

Then, for $1 \leq k \leq N$, we define the differential operator

$$\Gamma_k = \sum_{m=1}^{k} \sum_{((l_1,l_1'),\ldots,(l_m,l_m'))\in\Lambda_{m,k}} \sum_{1\leq r_1<\ldots<r_m\leq n} \prod_{i=1}^{m} \frac{1}{l_i!} D_{r_i}^{(l_i)} \prod_{j=1}^{m} \frac{(-1)^{l_j'}}{2^{l_j'} l_j'!} L_{\sigma_{r_j}}^{l_j'}. \tag{2.10}$$

By using (2.2) and (2.8), one easily gets the following estimates:

$$|\Gamma_k f(x)| \leq C \times \frac{C_{3k}}{n^{k/2}} \sup_{|\alpha|\leq 3k} |\partial_\alpha f(x)|, \quad f \in C_b^{3k}(\mathbb{R}^d), \tag{2.11}$$

$$|\Gamma_k f(x)| \leq C \times \frac{C_{3k}}{n^{k/2}} L_{3k}(f)(1+|x|)^{l_{3k}(f)}, \quad f \in C_p^{3k}(\mathbb{R}^d), \tag{2.12}$$

where $L_{3k}(f)$ and $l_{3k}(f)$ are given in (2.5) and C, C_{3k} are positive constants.
We introduce now the Hermite polynomials. We refer to Nualart [19] for definitions and properties, here we just give the shortest way to introduce them by means of the integration by parts formula. Given a multi-index α, the Hermite polynomial H_α on $\mathbb{R}^d$ is defined by

$$\mathbb{E}(\partial_\alpha f(W)) = \mathbb{E}(f(W)H_\alpha(W)) \quad \forall f \in C_p^\infty(\mathbb{R}^d) \tag{2.13}$$

where W is a standard normal random variable in $\mathbb{R}^d$. Moreover for a differential operator $\Gamma = \sum_{|\alpha|\leq k} a(\alpha)\partial_\alpha$, with $a(\alpha) \in \mathbb{R}$, we denote $H_\Gamma = \sum_{|\alpha|\leq k} a(\alpha)H_\alpha$ so that

$$\mathbb{E}(\Gamma f(W)) = \mathbb{E}(f(W)H_\Gamma(W)). \tag{2.14}$$

Finally we define

$$\Phi_N(x) = 1 + \sum_{k=1}^{N} H_{\Gamma_k}(x) \text{ with } \Gamma_k \text{ defined in (2.10).} \tag{2.15}$$

The main result in this section is the following (recall the constants $L_k(f)$ and $l_k(f)$, $f \in C_p^k(\mathbb{R}^d)$, defined in (2.5)):

Theorem 2.2. *Let $N \in \mathbb{N}$ be given. Then for every $f \in C_p^{2N(\overline{N}+N+3)}(\mathbb{R}^d)$*

$$\begin{aligned} &|\mathbb{E}(f(S_n(Z)) - \mathbb{E}(f(W)\Phi_N(W))| \\ &\leq \mathcal{H}_N C_{2(N+3)}^{2N(N+2\overline{N})}(Z)(1+C_{2l_{\widehat{N}}(f)}(Z))^{2N+3} 2^{(N+2)(l_{\widehat{N}}(f)+1)} L_{\widehat{N}}(f) \times \frac{1}{n^{\frac{N+1}{2}}} \end{aligned} \tag{2.16}$$

in which $\overline{N} = [N/2]$, $\widehat{N} = N(2\overline{N}+N+5)$, $\mathcal{H}_N$ is a positive constant depending on N and W denotes a standard normal random variable in $\mathbb{R}^d$.
As a consequence, taking $f(x) = x^\beta$ with $|\beta| = k$, one gets

$$|\mathbb{E}(S_n(Z)^\beta) - \mathbb{E}(W^\beta \Phi_N(W))| \leq \mathcal{H}_N C_{2(N+3)}^{2N(N+2\overline{N})}(Z)(1+C_{2k}(Z))^{2N+3} 2^{(N+2)(k+1)} \times \frac{1}{n^{\frac{N+1}{2}}}. \tag{2.17}$$

2.2 Basic decomposition and proof of the main result

Let $N \in \{0, 1, \ldots\}$. We define

$$T_{N,r}^0 f(x) = \sum_{l=1}^{N+2} \frac{1}{l!} D_r^{(l)} f(x). \tag{2.18}$$

Since $D_r^{(l)} \equiv 0$ for $l = 0, 1, 2$, the above sum actually begins with $l = 3$ and of course this is the basic fact. Then, with the convention $\sum_{l=3}^{2} = 0$, we have

$$T_{N,r}^{0} f(x) = \sum_{l=3}^{N+2} \frac{1}{l!} D_r^{(l)} f(x).$$

We also define

$$\begin{aligned} T_{N,r}^{1,Z} f(x) &= \frac{1}{(N+2)!} \sum_{|\alpha|=N+3} \int_0^1 (1-\lambda)^{N+2} \mathbb{E}(\partial_\alpha f(x+\lambda Z_r) Z_r^{\alpha}) d\lambda \quad \text{and} \\ T_{N,r}^{1} f(x) &= T_{N,r}^{1,Z} f(x) - T_{N,r}^{1,G} f(x). \end{aligned} \tag{2.19}$$

For a matrix $\sigma \in \mathcal{M}_{d\times d}$ we recall the Laplace operator L_σ associated to σ in (see (2.6)) and we define

$$h_{N,\sigma}^{0} f(x) = f(x) + \sum_{l=1}^{\overline{N}} \frac{(-1)^l}{2^l l!} L_\sigma^l f(x), \quad \text{with} \quad \overline{N} = [N/2], \tag{2.20}$$

$$h_{N,\sigma}^{1} f(x) = \frac{(-1)^{\overline{N}+1}}{2^{\overline{N}+1} \overline{N}!} \int_0^1 s^{\overline{N}} \mathbb{E}(L_\sigma^{\overline{N}+1} f(x + \sigma^{1/2} \sqrt{s}\, W)) ds. \tag{2.21}$$

In (2.21), W stands for a standard Gaussian random variable. Then we define

$$U_{N,r}^{0} f(x) = \mathbb{E}(h_{N,\sigma_r}^{0} f(x + G_r)) \quad \text{and} \quad U_{N,r}^{1} f(x) = h_{N,\sigma_r}^{1} f(x). \tag{2.22}$$

We now put our problem in a semigroup framework. For a sequence X_k, $k \geq 1$, of independent r.v.'s, for $1 \leq k \leq p$ we define

$$P_{k,k}^{X} f(x) = f \text{ and for } p > k \geq 1 \text{ then } P_{k,p}^{X} f(x) = \mathbb{E}\Big(f\Big(x + \sum_{i=k}^{p-1} X_i\Big)\Big). \tag{2.23}$$

We use $P_{k,p}^{Z}$ and $P_{k,p}^{G}$. By using independence, we have the semigroup and the commutative property:

$$P_{k,p}^{X} = P_{r,p}^{X} P_{k,r}^{X} = P_{k,r}^{X} P_{r,p}^{X} \quad k \leq r \leq p. \tag{2.24}$$

Moreover, for $m = 1, \ldots, N$ we denote

$$\begin{aligned} Q_{N,r_1,\ldots,r_m}^{(m)} &= \sum_{\substack{\sum_{i=1}^m q_i + \sum_{i=1}^m q_i' > 0 \\ q_i, q_i' \in \{0,1\}}} \prod_{i=1}^{m} U_{N,r_i}^{q_i'} \prod_{j=1}^{m} T_{N,r_j}^{q_j} \quad \text{and} \\ R_{N,k,n}^{(m)} &= \sum_{k \leq r_1 < \cdots < r_m \leq n} P_{r_m+1,n}^{G} P_{r_{m-1}+1,r_m}^{G} \cdots P_{r_1+1,r_2}^{G} P_{k,r_1}^{G} Q_{N,r_1,\ldots,r_m}^{(m)}. \end{aligned} \tag{2.25}$$

Notice that in the first sum above the conditions $q_i, q_i' \in \{0,1\}$ and $q_1 + \cdots + q_m + q_1' + \cdots + q_m' > 0$ say that at least one of $q_i, q_i', i = 1, ..., m$ is equal to one. We notice that the operators T_{N,r_i}^{1} and $U_{N,\sigma_{r_i}}^{1}$ represent "remainders" and they are supposed to give small quantities of order $n^{-\frac{1}{2}(N+1)}$. So the fact that at least one q_i or q_i' is non null means that the product $(\prod_{i=1}^m U_{N,r_i}^{q_i'})(\prod_{i=1}^m T_{N,r_i}^{q_i})$ has at least one term which is a remainder (so is small), and consequently $R_{N,k,n}^{(m)}$ is a remainder also.

Finally we define

$$
\begin{aligned}
Q^{(N+1)}_{N,r_1,\ldots,r_{N+1}} &= \prod_{i=1}^{N+1}(T^0_{N,r_i}+T^1_{N,r_i}) \quad \text{and} \\
R^{(N+1)}_{N,k,n} &= \sum_{k\leq r_1<\cdots<r_{N+1}\leq n} P^Z_{r_{N+1}+1,n}P^G_{r_N+1,r_{N+1}}\cdots P^G_{r_1+1,r_2}P^G_{k,r_1}Q^{(N+1)}_{N,r_1,\ldots,r_{N+1}}
\end{aligned}
\tag{2.26}
$$

We are now able to give our first result:

Proposition 2.3. *Let $N\geq 1$ and let $T^0_{N,r}$, h^0_{N,σ_r}, $R^{(m)}_{N,k,n}$, $m=1,\ldots,N+1$, be given through (2.18), (2.20), (2.25), (2.26). Then for every $1\leq k\leq n+1$ and $f\in C_p^{N(2\overline{N}+N+3)}(\mathbb{R}^d)$ one has*

$$
P^Z_{k,n+1}f = P^G_{k,n+1}f + \sum_{m=1}^n \sum_{k\leq r_1<\cdots<r_m\leq n} P^G_{k,n+1}\Big(\prod_{i=1}^m T^0_{N,r_i}\Big)\Big(\prod_{j=1}^m h^0_{N,\sigma_{r_j}}\Big)f + \sum_{m=1}^{N+1} R^{(m)}_{N,k,n}f. \tag{2.27}
$$

Proof. **Step 1 (Lindeberg method)** We use the Lindeberg method in terms of semigroups: for $1\leq k\leq n+1$

$$
P^Z_{k,n+1}-P^G_{k,n+1} = \sum_{r=k}^n P^Z_{r+1,n+1}(P^Z_{r,r+1}-P^G_{r,r+1})P^G_{k,r}.
$$

Then we define

$$
A_{k,p} = \mathbf{1}_{1\leq k\leq p-1\leq n}\,(P^Z_{p-1,p}-P^G_{p-1,p})P^G_{k,p-1} \tag{2.28}
$$

and the above relation reads

$$
P^Z_{k,n+1} = P^G_{k,n+1} + \sum_{r=k}^n P^Z_{r+1,n+1}A_{k,r+1}. \tag{2.29}
$$

We will write (2.29) as a discrete time Volterra type equation (this is inspired from the approach to the parametrix method given in [9]: see equation (3.1) there). For a family of operators $F_{k,p}$, $k\leq p$ we define AF by

$$
(AF)_{k,p} = \sum_{r=k}^{p-1} F_{r+1,p}A_{k,r+1}
$$

and we write (2.29) in functional form:

$$
P^Z = P^G + AP^Z. \tag{2.30}
$$

By iteration,

$$
P^Z = P^G + AP^G + \cdots + A^N P^G + A^{N+1}P^Z. \tag{2.31}
$$

By the commutative property in (2.24), straightforward computations give

$$
\begin{aligned}
(A^mP^G)_{k,p} = \mathbf{1}_{k\leq p-m}\sum_{k\leq r_1<\cdots<r_m\leq p-2} & P^G_{r_m+1,p-1}P^G_{r_{m-1}+1,r_m}\cdots P^G_{r_1+1,r_2}P^G_{k,r_1}(P^Z_{p-1,p}-P^G_{p-1,p})\times \\
&\times(P^Z_{r_m,r_m+1}-P^G_{r_m,r_m+1})(P^Z_{r_{m-1},r_{m-1}+1}-P^G_{r_{m-1},r_{m-1}+1})\cdots(P^Z_{r_1,r_1+1}-P^G_{r_1,r_1+1}).
\end{aligned}
\tag{2.32}
$$

Step 2 (Taylor formula) The drawback of (2.31) is that A depends on P^Z also, see (2.28). So, we use now the Taylor's formula in order to eliminate this dependance. We use (2.4) and we consider a

Taylor approximation at the level of an error of order $n^{-\frac{N+2}{2}}$. We use the following expression for the Taylor's formula: for $f \in C_p^\infty(\mathbb{R}^d)$,

$$f(x+y) = f(x) + \sum_{p=1}^{N+2} \frac{1}{p!} \sum_{|\alpha|=p} \partial_\alpha f(x) y^\alpha + \frac{1}{(N+2)!} \sum_{|\alpha|=N+3} y^\alpha \int_0^1 (1-\lambda)^{N+2} \partial_\alpha f(x+\lambda y) d\lambda$$

Then we have, with $D_r^{(l)}$ defined in (2.7),

$$\begin{aligned}
&(P^Z_{r,r+1} - P^G_{r,r+1}) f(x) = \mathbb{E}(f(x+Z_r)) - \mathbb{E}(f(x+G_r)) \\
&= \sum_{l=1}^{N+2} \frac{1}{l!} D_r^{(l)} f(x) + \frac{1}{(N+2)!} \sum_{|\alpha|=N+3} \int_0^1 (1-\lambda)^{N+2} \big[\mathbb{E}(\partial_\alpha f(x+\lambda Z_r) Z_r^\alpha) - \mathbb{E}(\partial_\alpha f(x+\lambda G_r) G_r^\alpha)\big] d\lambda \\
&= T^0_{N,r} f(x) + T^1_{N,r} f(x).
\end{aligned}$$

By using the independence property, one can apply commutativity and by using (2.32) we have

$$(A^m F)_{k,r+1} = \mathbf{1}_{k \le r+1-m} \sum_{k \le r_1 < \cdots < r_m \le r} F_{r_m+1,r+1} P^G_{r_{m-1}+1,r_m} \cdots P^G_{r_1+1,r_2} P^G_{k,r_1} \prod_{j=1}^m (T^0_{N,r_j} + T^1_{N,r_j}). \quad (2.33)$$

Notice that the operator in (2.33) acts on $f \in C^{m(N+3)}$. In particular,

$$(A^m P^G)_{k,n+1} = \mathbf{1}_{k \le n+1-m} \sum_{k \le r_1 < \cdots < r_m \le n} P^G_{r_m+1,n+1} P^G_{r_{m-1}+1,r_m} \cdots P^G_{r_1+1,r_2} P^G_{k,r_1} \prod_{j=1}^m (T^0_{N,r_j} + T^1_{N,r_j}) \quad (2.34)$$

Step 3 (Backward Taylor formula) Since

$$P^G_{r_m+1,n+1} P^G_{r_{m-1}+1,r_m} \cdots P^G_{r_1+1,r_2} P^G_{k,r_1} f(x) = \mathbb{E}\Big(f\Big(x + \sum_{i=k}^n G_k - \sum_{j=1}^m G_{r_j}\Big)\Big),$$

the chain $P^G_{r_m+1,n} \cdots P^G_{r_1+1,r_2} P^G_{k,r_1}$ contains all the steps, except for the steps corresponding to $r_i, i = 1, ..., m$ (remark that for each i, $P^G_{r_i,r_i+1}$ is replaced with $T^0_{N,r_i} + T^1_{N,r_i}$). In order to "insert" such steps we use the backward Taylor formula (B.3) up to order $\overline{N} = [N/2]$ (see next Appendix B). So, we take h^0_{N,σ_r} and h^1_{N,σ_r} as in (2.20) and (2.21) respectively and we have

$$\begin{aligned}
P^G_{r_1+1,r_2} P^G_{k,r_1} f(x) &= \mathbb{E}\Big(f\Big(x + \sum_{i=k}^{r_2-1} G_i - G_{r_1}\Big)\Big) \\
&= \mathbb{E}\Big(h^0_{N,\sigma_{r_1}} f\Big(x + \sum_{i=k}^{r_2-1} G_i\Big)\Big) + \mathbb{E}\Big(h^1_{N,\sigma_{r_1}} f\Big(x + \sum_{i=k}^{r_2-1} G_i - G_{r_1}\Big)\Big) \\
&= P^G_{r_1+1,r_2} P^G_{k,r_1} (P^G_{r_1,r_1+1} h^0_{N,\sigma_{r_1}} + h^1_{N,\sigma_{r_1}}) f(x) \\
&= P^G_{r_1+1,r_2} P^G_{k,r_1} (U^0_{N,r_1} + U^1_{N,r_1}) f(x),
\end{aligned}$$

U^0_{N,r_1} and U^1_{N,r_1} being given in (2.22). We use this formula in (2.34) for every $i = 1, 2, ..., m$ and we get

$$(A^m P^G)_{k,n+1} = \sum_{k \le r_1 < \cdots < r_m \le n} P^G_{r_m+1,n+1} \cdots P^G_{r_1+1,r_2} P^G_{k,r_1} \Big(\prod_{i=1}^m (U^0_{N,r_i} + U^1_{N,r_i})\Big) \Big(\prod_{j=1}^m (T^0_{N,r_j} + T^1_{N,r_j})\Big). \quad (2.35)$$

Notice that the above operator acts on $C_p^{m(2\overline{N}+N+5)}(\mathbb{R}^d)$. Our aim now is to isolate the principal term, that is the sum of the terms where only U^0_{N,r_i} and T^0_{N,r_i} appear. So we write

$$\begin{aligned}(A^mP^G)_{k,n+1} &= \sum_{k\leq r_1<\cdots<r_m\leq n} P^G_{r_m+1,n+1}\cdots P^G_{r_1+1,r_2}P^G_{k,r_1}\Big(\prod_{i=1}^m U^0_{N,r_i}\Big)\Big(\prod_{j=1}^m T^0_{N,r_j}\Big)\\ &\quad+\sum_{k\leq r_1<\cdots<r_m\leq n} P^G_{r_m+1,n+1}\cdots P^G_{r_1+1,r_2}P^G_{k,r_1}Q^{(m)}_{N,r_1,\ldots,r_m}\end{aligned}$$

with $Q^{(m)}_{N,r_1,\ldots,r_m}$ defined in (2.25). The second term is just $R^{(m)}_{N,k,n}$ in (2.25). In order to compute the first one we notice that for every $r'<r<r''$ we have

$$P^G_{r+1,r''}P^G_{r',r}P^G_{r,r+1}=P^G_{r',r''}$$

so that

$$P^G_{r_m+1,n+1}\cdots P^G_{r_1+1,r_2}P^G_{k,r_1}(\prod_{i=1}^m U^0_{N,r_i})=P^G_{k,n+1}(\prod_{i=1}^m h^0_{N,\sigma_{r_i}}).$$

Then, for $m=1,...,N$

$$(A^mP^G)_{k,n+1}=\sum_{k\leq r_1<\cdots<r_m\leq n} P^G_{k,n+1}\Big(\prod_{i=1}^m h^0_{N,\sigma_{r_i}}\Big)\Big(\prod_{i=1}^m T^0_{N,r_i}\Big)+R^{(m)}_{N,k,n}.$$

We treat now $A^{N+1}P^Z$. Using (2.33) we get

$$(A^{N+1}P^Z)_{k,n+1}=\sum_{k\leq r_1<\ldots<r_{N+1}\leq n} P^Z_{r_{N+1}+1,n+1}P^G_{r_N+1,r_{N+1}}\ldots P^G_{r_1+1,r_2}P^G_{k,r_1}\prod_{i=1}^N(T^0_{N,r_i}+T^1_{N,r_i})=R^{(N+1)}_{N,k,n},$$

which acts on $C_p^{N(N+3)}$. $\square$

We give now some useful representations of the remainders.

Lemma 2.4. *Let $m\in\{1,...,N+1\}$ and $r_1<\cdots<r_m\leq n$ be fixed. Set $N_m:=m(2\overline{N}+N+5)$ for $m\leq N$ and $N_m=(N+1)(N+3)$ otherwise. Then, the operators $Q^{(m)}_{N,r_1,\ldots,r_m}$ defined in (2.25) for $m=1,\ldots,N$ and in (2.26) for $m=N+1$, can be written as*

$$Q^{(m)}_{N,r_1,\ldots,r_m}f(x)=\sum_{3\leq|\alpha|\leq N_m} a_n^{r_1,\ldots,r_m}(\alpha)\theta^\alpha_{r_1,\ldots,r_m}\partial_\alpha f(x) \tag{2.36}$$

where $a_n(\alpha)\in\mathbb{R}$ are suitable coefficients with the property

$$|a_n^{r_1,\ldots,r_m}(\alpha)|\leq\frac{(CC_2^{\overline{N}+1}(Z))^m}{n^{\frac{N+3m}{2}}}, \tag{2.37}$$

and $\theta^\alpha_{r_1,\ldots,r_m}:C_p^\infty(\mathbb{R}^d)\to C_p^\infty(\mathbb{R}^d)$ is an operator which verifies

$$|\theta^\alpha_{r_1,\ldots,r_m}\partial_\alpha f(x)|\leq\big(2^{l_{N_m}(f)+1}C^{1/2}_{2(N+3)}(Z)(1+C_{2l_{N_m}(f)}(Z))^2\big)^mL_{N_m}(f)(1+|x|)^{l_{N_m}(f)} \tag{2.38}$$

$C>0$ being a suitable constant. Moreover, $\theta^\alpha_{r_1,\ldots,r_m}$ can be represented as

$$\theta^\alpha_{r_1,\ldots,r_m}f(x)=\int_{(\mathbb{R}^d)^{2m}}f(x+y_1+\cdots+y_{2m})\mu^\alpha_{r_1,\ldots,r_m}(dy_1,\ldots,dy_{2m}) \tag{2.39}$$

where $\mu^\alpha_{r_1,\ldots,r_m}$ is a finite signed measure.

Proof. In a first step we construct the measures $\mu^\alpha_{r_1,\ldots,r_m}$ and the operators $\theta^\alpha_{r_1,\ldots,r_m}$ and in a second step we prove that the corresponding coefficients $a_n^{r_1,\ldots,r_m}(\alpha)$ verify (2.37). We start by representing $T^0_{N,r}$ defined in (2.18). Set

$$\nu_r^{0,\alpha}(dy) = n^{\frac{l}{2}}\Delta_\alpha(r)\delta_0(dy), \quad |\alpha| = l \geq 3.$$

Notice that if $|\alpha| = l \geq 3$ then $n^{\frac{l}{2}}\,|\Delta_\alpha(r)| \leq 2C_l(Z)$. So, we have

$$\begin{aligned} &T^0_{N,r}f(x) = \sum_{l=3}^{N+2}\frac{1}{n^{\frac{l}{2}}}\sum_{|\alpha|=l}\frac{1}{l!}\int_{\mathbb{R}^d}\partial_\alpha f(x+y)\nu_r^{0,\alpha}(dy) \quad \text{with} \\ &\int_{\mathbb{R}^d}(1+|y|)^\gamma|\nu_r^{0,\alpha}|(dy) \leq 2C_l(Z), \quad |\alpha| = l \leq N+2. \end{aligned} \tag{2.40}$$

Hereafter γ denotes a non negative power. Concerning $T^1_{N,r}$ in (2.19), for $|\alpha| = N+3$ set $\nu^{1,\alpha}(dy) = n^{\frac{N+3}{2}}\int_0^1(1-\lambda)^{N+2}(\frac{y}{\lambda})^\alpha\big[\mu_{\lambda Z_r}(dy) - \mu_{\lambda G_r}(dy)\big]d\lambda$, that is

$$\nu^{1,\alpha}(A) = n^{\frac{N+3}{2}}\int_0^1(1-\lambda)^{N+2}\big[\mathbb{E}(Z_r^\alpha\mathbf{1}_{\lambda Z_r\in A}) - \mathbb{E}(G_r^\alpha\mathbf{1}_{\lambda G_r\in A})\big]d\lambda, \quad |\alpha| = N+3,$$

for every Borel set A. Then we have

$$\begin{aligned} &T^1_{N,r}f(x) = \frac{1}{n^{\frac{1}{2}(N+3)}}\sum_{|\alpha|=N+3}\frac{1}{(N+2)!}\int_{\mathbb{R}^d}\partial_\alpha f(x+y)\nu_r^{1,\alpha}(dy) \quad \text{with} \\ &\int_{\mathbb{R}^d}(1+|y|)^\gamma|\nu_r^{1,\alpha}|(dy) \leq \frac{2^{\gamma+1}}{N+3}C^{1/2}_{2(N+3)}(Z)(1+C_{2\gamma}(Z))^{1/2}, \quad |\alpha| = N+3. \end{aligned} \tag{2.41}$$

We represent now the operator $U^0_{N,r}f(x) = \mathbb{E}(h^0_{N,\sigma_r}f(x+G_r))$ with $h^0_{N,\sigma_r}f$ defined in (2.20). Notice that

$$h^0_{N,\sigma_r} = \sum_{l=0}^{\overline{N}}\sum_{|\alpha|=2l}c_n^{\sigma_r}(\alpha)\partial_\alpha \quad \text{with} \quad c_n^{\sigma_r}(\alpha) = \frac{(-1)^l}{2^l l!}\prod_{k=1}^{l}\sigma_r^{\alpha_{2k-1},\alpha_{2k}}, \quad |\alpha| = l.$$

So, by denoting $\rho^0_{\sigma_r}$ the law of G_r, we have

$$\begin{aligned} &U^0_{N,r}f(x) = \mathbb{E}(h^0_{N,\sigma_r}f(x+G_r)) = \sum_{l=0}^{\overline{N}}\sum_{|\alpha|=2l}c_n^{\sigma_r}(\alpha)\int_{\mathbb{R}^d}\partial_\alpha f(x+y)\rho^0_{\sigma_r}(dy) \quad \text{with} \\ &|c_n^{\sigma_r}(\alpha)| \leq \frac{C_2(Z)^l}{2^l l! n^{\frac{l}{2}}} \quad \text{and} \quad \int_{\mathbb{R}^d}(1+|y|)^\gamma|\rho^0_{\sigma_r}|(dy) \leq 2^\gamma(1+C_\gamma(Z)). \end{aligned} \tag{2.42}$$

We now obtain a similar representation for $h^1_{N,\sigma}f(x)$ defined in (2.21). Set

$$\rho^1_\sigma(dy) = \Big(\int_0^1 s^{\overline{N}}\phi_{\sigma^{1/2}\sqrt{s}W}(y)ds\Big)dy,$$

in which $\phi_{\sigma^{1/2}\sqrt{s}W}$ denotes the density of a centred Gaussian r.v. with covariance matrix $s\sigma$. Then we write

$$h^1_{N,\sigma}f(x) = \sum_{|\alpha|=2(\overline{N}+1)}b_n^\sigma(\alpha)\int_{\mathbb{R}^d}\partial_\alpha f(x+y)\rho^1_\sigma(dy) \quad \text{with} \quad b_n^\sigma(\alpha) = \frac{(-1)^{\overline{N}+1}}{2^{\overline{N}+1}\overline{N}!}\prod_{k=1}^{\overline{N}+1}\sigma^{\alpha_{2k-1},\alpha_{2k}}.$$

Since $\overline{N}+1 \geq (N+1)/2$, we have

$$\begin{aligned}
&U^1_{N,r}f(x) = h^1_{N,\sigma_r}f(x) = \sum_{|\alpha|=2(\overline{N}+1)} b^{\sigma_r}_n(\alpha)\int \partial_\alpha f(x+y)\rho^1_{\sigma_r}(dy) \quad \text{with} \\
&|b^{\sigma}_n(\alpha)| \leq \frac{1}{2^{\overline{N}+1}\overline{N}!}C_2(Z)^{\overline{N}+1}n^{-\frac{N+1}{2}} \quad \text{and} \quad \int_{\mathbb{R}^d}(1+|y|)^\gamma |\rho^1_{\sigma_r}|(dy) \leq \frac{2^\gamma}{\overline{N}+1}(1+C_\gamma(Z)).
\end{aligned} \tag{2.43}$$

Using (2.40), (2.41), (2.42) and (2.43) we obtain (2.36) with the measure $\mu^\alpha_{r_1,\ldots,r_m}$ from (2.39) constructed in the following way:

$$\begin{aligned}
&\int_{\mathbb{R}^{d\times 2m}} f(y_1,\ldots,y_m,\bar{y}_1,\ldots,\bar{y}_m)\mu^\alpha_{r_1,\ldots,r_m}(dy_1,\ldots,dy_m,d\bar{y}_1,\ldots,d\bar{y}_m) \\
&= \int_{\mathbb{R}^{d\times 2m}} f(y_1,\ldots,y_m,\bar{y}_1,\ldots,\bar{y}_m)\eta_1(dy_1)\cdots\eta_m(dy_m)\bar{\eta}_1(d\bar{y}_1)\cdots\bar{\eta}_m(d\bar{y}_m)
\end{aligned}$$

where η_i is one of the measures $\nu^{q,\beta}_{r_i}$, $q=0,1$, and $\bar{\eta}_i$ is one of the measures $\rho^q_{\sigma_{r_i}}$, $q=0,1$.
Let us check that the coefficients $a^{r_1,\ldots,r_m}_n(\alpha)$ which will appear in (2.36) verify the bounds in (2.37). Take first $m \in \{1,...,N\}$. Then $Q^{(m)}_{r_1,\ldots,r_m}$ is the sum of $(\prod_{i=1}^m U^{q'_i}_{N,r_i})(\prod_{j=1}^m T^{q_j}_{N,r_j})$ where $q_i, q'_i \in \{0,1\}$ and at least one of them is equal to one. And $a^{r_1,\ldots,r_m}_n(\alpha)$ is the product of coefficients which appear in the representation of $U^{q'_i}_{N,r_i}$ and $T^{q_j}_{N,r_j}$. Recall that the coefficients of T^0_{N,r_j} are all bounded by $Cn^{-3/2}$ and the coefficients of T^1_{N,r_j} are bounded by $Cn^{-\frac{1}{2}(N+3)}$. Moreover the coefficients of U^0_{N,r_i} are bounded by $CC_2^{\overline{N}}(Z)$ and the coefficients of U^1_{N,r_i} are bounded by $CC_2^{\overline{N}+1}(Z)n^{-(\overline{N}+1)}$. Therefore, $(\prod_{i=1}^m U^{q'_i}_{N,r_i})(\prod_{j=1}^m T^{q_j}_{N,r_j})$ is upper bounded by

$$\begin{aligned}
&\Big(\frac{C}{n^{\frac{1}{2}(N+3)}}\Big)^{\sum_{i=1}^m q_i} \times \Big(\frac{C}{n^{3/2}}\Big)^{\sum_{i=1}^m (1-q_i)} \times \Big(\frac{CC_2^{\overline{N}+1}(Z)}{n^{\overline{N}+1}}\Big)^{\sum_{i=1}^m q'_i} \times \big(CC_2^{\overline{N}}(Z)\big)^{\sum_{i=1}^m (1-q'_i)} \\
&\leq \Big(\frac{1}{n^{\frac{1}{2}N}}\Big)^{\sum_{i=1}^m q_i} \times \frac{C^m}{n^{\frac{3m}{2}}} \times \Big(\frac{1}{n^{\overline{N}+1}}\Big)^{\sum_{i=1}^m q'_i} \times (CC_2^{\overline{N}+1}(Z))^m \\
&\leq \frac{(CC_2^{\overline{N}+1}(Z))^m}{n^{\frac{N}{2}\sum_{i=1}^m q_i + (\overline{N}+1)\sum_{i=1}^m q'_i + \frac{3m}{2}}} \leq \frac{(CC_2^{\overline{N}+1}(Z))^m}{n^{\frac{N}{2}(\sum_{i=1}^m q_i + \sum_{i=1}^m q'_i)+\frac{3m}{2}}} \leq \frac{(CC_2^{\overline{N}+1}(Z))^m}{n^{\frac{N+3m}{2}}}.
\end{aligned}$$

We finally prove (2.38). We have

$$\begin{aligned}
&|\theta^\alpha_{r_1,\ldots,r_m}\partial_\alpha f(x)| \leq \int_{\mathbb{R}^{d\times 2m}} |\partial_\alpha f|\Big(x+\sum_{i=1}^m y_i + \sum_{j=1}^m \bar{y}_j\Big)|\eta|_1(dy_1)\cdots|\eta_m|(dy_m)|\bar{\eta}|_1(d\bar{y}_1)\cdots|\bar{\eta}_m|(d\bar{y}_m) \\
&\leq L_{N_m}(f)(1+|x|)^{l_{N_m}(f)}\Big(\prod_{i=1}^m\int_{\mathbb{R}^d}(1+|y|)^{l_{N_m}(f)}|\eta_i|(dy)\Big)\Big(\prod_{i=1}^m\int_{\mathbb{R}^d}(1+|y|)^{l_{N_m}(f)}|\bar{\eta}_i|(dy)\Big) \\
&\leq L_{N_m}(f)(1+|x|)^{l_{N_m}(f)}\Big((2C_{N+2}(Z)) \vee (2^{l_{N_m}(f)+1}C^{1/2}_{2(N+3)}(Z)(1+C_{2l_{N_m}(f)}(Z)\Big)^m \\
&\quad \times \Big(2^{l_{N_m}(f)}(1+C_{l_{N_m}(f)}(Z))\Big)^m \\
&\leq \big(2^{l_{N_m}(f)+1}C^{1/2}_{2(N+3)}(Z)(1+C_{2l_{N_m}(f)}(Z))^2\big)^m L_{N_m}(f)(1+|x|)^{l_{N_m}(f)}
\end{aligned}$$

because $C_{N+2}(Z) \leq C_{2(N+3)}(Z)^{\frac{N+2}{2(N+3)}} \leq C_{2(N+3)}(Z)^{\frac{1}{2}}$. So the proof concerning $Q^{(m)}_{N,r_1,\ldots,r_m}$, $m=1,...,N$, is completed. The proof for $Q^{(N+1)}_{N,r_1,\ldots,r_{N+1}}$ is clearly the same. $\square$
We give now the representation of the "principal term":

Lemma 2.5. *Let the set-up of Proposition 2.3 hold. Then,*

$$\sum_{m=1}^{N}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}T^0_{N,r_i}\Big)\Big(\prod_{j=1}^{m}h^0_{N,\sigma_{r_j}}\Big)=\sum_{m=1}^{N}\Gamma_k+Q^0_{N,n} \tag{2.44}$$

with Γ_k *defined in (2.10) and*

$$Q^0_{N,n}=\sum_{N+1\le|\alpha|\le N(N+2\overline{N})}c_n(\alpha)\partial_\alpha \quad \textit{with} \quad |c_n(\alpha)|\le\frac{(CC_{N+1}(Z)C_2(Z))^{N(N+2\overline{N})}}{n^{\frac{N+1}{2}}} \tag{2.45}$$

Proof. Let Λ_m and $\Lambda_{m,k}$ be the sets in (2.9). Notice that, for fixed m, the $\Lambda_{m,k}$'s are disjoint as k varies. Suppose that $m\in\{1,...,N\}$. Then $\Lambda_{m,k}=\emptyset$ if $k\notin\{m,\ldots,N(N+2\overline{N})\}$ so that $\Lambda_m=\cup_{k=m}^{2N(N+2)}\Lambda_{m,k}$ and consequently

$$\cup_{m=1}^{N}\Lambda_m=\cup_{m=1}^{N}\cup_{k=m}^{N(N+2\overline{N})}\Lambda_{m,k}=\cup_{k=1}^{N(N+2\overline{N})}\cup_{m=1}^{k}\Lambda_{m,k}.$$

It follows that

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}T^0_{N,r_i}\Big)\Big(\prod_{j=1}^{m}h^0_{N,\sigma_{r_j}}\Big)\\
=\;&\sum_{m=1}^{N}\sum_{l_1,..,l_m=3}^{N+2}\sum_{l'_1,..,l'_m=0}^{\overline{N}}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}\frac{1}{l_i!}D^{(l_i)}_{r_i}\Big)\Big(\prod_{j=1}^{m}\frac{(-1)^{l'_j}}{2^{l'_j}l'_j!}L^{l'_j}_{\sigma_{r_j}}\Big)\\
=\;&\sum_{k=1}^{N(N+2\overline{N})}\sum_{m=1}^{k}\sum_{(l_1,l'_1),\ldots,(l_m,l'_m))\in\Lambda_{m,k}}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}\frac{1}{l_i!}D^{(l_i)}_{r_i}\Big)\Big(\prod_{j=1}^{m}\frac{(-1)^{l'_j}}{2^{l'_j}l'_j!}L^{l'_j}_{\sigma_{r_j}}\Big)\\
=\;&\sum_{k=1}^{N}\Gamma_k+Q^0_{N,n}
\end{aligned}$$

with

$$Q^0_{N,n}=\sum_{k=N+1}^{N(N+2\overline{N})}\sum_{m=1}^{k}\sum_{(l_1,l'_1),\ldots,(l_m,l'_m))\in\Lambda_{m,k}}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}\frac{1}{l_i!}D^{l_i}_{r_i}\Big)\Big(\prod_{j=1}^{m}\frac{(-1)^{l'_j}}{2^{l'_j}l'_j!}L^{l'_j}_{\sigma_{r_j}}\Big),$$

which is a differential operator of the form (2.45). Moreover, the coefficients $c_n(\alpha)$ can be bounded as follows:

$$\begin{aligned}
|c_n(\alpha)|&\le n^m\times\prod_{i=1}^{m}\Big(\frac{1}{l_i!}\frac{2C_{N+1}(Z)}{n^{\frac{l_i}{2}}}\Big)\times\prod_{i=1}^{m}\Big(\frac{1}{2^{l'_i}l'_i!}\frac{C_2^{l'_i}(Z)}{n^{l'_i}}\Big)\le n^m\times\frac{(CC_{N+1}(Z)C_2(Z))^m}{n^{\frac{\sum_{i=1}^m l_i+2\sum_{i=1}^m l'_i}{2}}}\\
&\le n^m\times\frac{(CC_{N+1}(Z)C_2(Z))^m}{n^{\frac{k}{2}+m}}\le\frac{(CC_{N+1}(Z)C_2(Z))^{N(N+2\overline{N})}}{n^{\frac{N+1}{2}}}
\end{aligned}$$

and the estimate in (2.45) holds as well. □

We are now ready for the

Proof of Theorem 2.2 We denote $P_n^X = P_{1,n+1}^X$, with $X = Z$ or $X = G$, so that

$$\mathbb{E}(f(S_n(Z)) - \mathbb{E}(f(W)\Phi_N(W)) = P_n^Z f(0) - P_n^G\Big(\mathrm{Id} + \sum_{k=1}^N \Gamma_k\Big) f(0).$$

We have proved that

$$P_n^Z f(x) = P_n^G\Big(\mathrm{Id} + \sum_{k=1}^N \Gamma_k\Big) f(x) + I_1 f(x) + I_2 f(x) + I_3 f(x) \tag{2.46}$$

with

$$\begin{aligned} I_1 f(x) &= P_n^G Q_{N,n}^0 f(x), \\ I_2 f(x) &= \sum_{1\leq r_1<\ldots<r_{N+1}\leq n} P^Z_{r_{N+1}+1,n} P^G_{r_N+1,r_{N+1}} \ldots P^G_{r_1+1,r_2} P^G_{k,r_1} Q^{(N+1)}_{N,r_1,\ldots,r_{N+1}} f(x) \\ I_3 f(x) &= \sum_{m=1}^N \sum_{1\leq r_1<\ldots<r_m\leq n} P^G_{r_m+1,n} P^G_{r_{m-1}+1,r_m} \ldots P^G_{r_1+1,r_2} P^G_{k,r_1} Q^{(m)}_{N,r_1,\ldots,r_m} f(x), \end{aligned} \tag{2.47}$$

so it is sufficient to study the remaining terms I_1, I_2 and I_3 above.
Consider first $m \in \{1, \ldots, N\}$. We use Lemma 2.4 (recall N_m given therein) and in particular (2.36):

$$\begin{aligned} & P^G_{r_m+1,n} P^G_{r_{m-1}+1,r_m} \ldots P^G_{r_1+1,r_2} P^G_{1,r_1} Q^{(m)}_{N,r_1,\ldots,r_m} f \\ = & \sum_{3\leq|\alpha|\leq N_m} a_n^{r_1,\ldots,r_m}(\alpha) P^G_{r_m+1,n} P^G_{r_{m-1}+1,r_m} \ldots P^G_{r_1+1,r_2} P^G_{1,r_1} \theta^\alpha_{r_1,\ldots,r_m} \partial_\alpha f. \end{aligned}$$

Notice that if $|g(x)| \leq L(1+|x|)^l$ then

$$\begin{aligned} &|P^G_{r_m+1,n} P^G_{r_{m-1}+1,r_m} \ldots P^G_{r_1+1,r_2} P^G_{1,r_1} g(x)| \leq \mathbb{E}\Big(L\Big(1 + \Big|x + \sum_{k=1}^n G_k 1_{k\notin\{r_1,\ldots,r_m\}}\Big|\Big)^l\Big) \\ &\leq L(1+|x|)^l \mathbb{E}\Big(\Big(1 + \Big|\sum_{k=1}^n G_k 1_{k\notin\{r_1,\ldots,r_m\}}\Big|\Big)^l\Big) \leq L(1+|x|)^l \Big(1 + \Big\|\sum_{k=1}^n G_k 1_{k\notin\{r_1,\ldots,r_m\}}\Big\|_l\Big)^l. \end{aligned}$$

Since the $G_k 1_{k\notin\{r_1,\ldots,r_m\}}$'s are centred and independent, we can use the Burkholder inequality (see next (3.26), which gives

$$\Big\|\sum_{k=1}^n G_k 1_{k\notin\{r_1,\ldots,r_m\}}\Big\|_l \leq \Big(\sum_{k=1}^n \|G_k 1_{k\notin\{r_1,\ldots,r_m\}}\|_l^2\Big)^{1/2} \leq \Big(n \sum_{k=1}^n \frac{C_l^{2/l}(Z)}{n}\Big)^{1/2} \leq C_l^{1/l}(Z)$$

and by inserting, we get

$$|P^G_{r_m+1,n} P^G_{r_{m-1}+1,r_m} \ldots P^G_{r_1+1,r_2} P^G_{1,r_1} g(x)| \leq L(1+|x|)^l (1 + C_l^{1/l}(Z))^l \leq 2^l (1 + C_l(Z)) L(1+|x|)^l.$$

We use now this inequality with $g = \theta^\alpha_{r_1,\ldots,r_m} \partial_\alpha f$: by applying (2.38) we get

$$|P^G_{r_m+1,n} P^G_{r_{m-1}+1,r_m} \ldots P^G_{r_1+1,r_2} P^G_{1,r_1} Q^{(m)}_{N,r_1,\ldots,r_m} f(x)| \leq \mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}$$

with

$$\mathcal{K}_{N,m}(f) = 2^{l_{N_m}(f)} (1 + C_{l_{N_m}(f)}(Z)) \big(2^{l_{N_m}(f)+1} C^{1/2}_{2(N+3)}(Z)(1 + C_{2l_{N_m}(f)}(Z))^2\big)^m L_{N_m}(f).$$

Moreover, using (2.37)

$$\begin{aligned}
& |P^G_{r_m+1,n}P^G_{r_{m-1}+1,r_m}...P^G_{r_1+1,r_2}P^G_{1,r_1}Q^{(m)}_{N,r_1,\ldots,r_m}f(x)| \\
\leq\ & \mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}\sum_{0\leq|\alpha|\leq N+1}|a_n^{r_1,\ldots,r_m}(\alpha)| \\
\leq\ & \mathcal{H}_N\mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}(CC_2^{\overline{N}+1}(Z))^m\times\frac{1}{n^{\frac{1}{2}(N+3m)}},
\end{aligned}$$

$\mathcal{H}_N$ denoting a constant depending on N only. Since the set $\{1\leq r_1<\ldots<r_m\leq n\}$ has less than n^m elements, we get

$$\begin{aligned}
|I_3f(x)| &\leq N\times n^m\times\mathcal{H}_N\mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}(CC_2^{\overline{N}+1}(Z))^m\times\frac{1}{n^{\frac{1}{2}(N+3m)}} \\
&\leq N\mathcal{H}_N\mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}(CC_2^{\overline{N}+1}(Z))^m\times\frac{1}{n^{\frac{1}{2}(N+1)}}
\end{aligned}$$

The estimate for $I_2(f)$ is analogous. Concerning I_1f, we use (2.45) in order to obtain

$$\begin{aligned}
|I_1f(x)| &\leq \sum_{N+1\leq|\alpha|\leq N(N+2\overline{N})}|c_n(\alpha)|\,|P_n^G\partial_\alpha f(x)| \\
&\leq \sum_{N+1\leq|\alpha|\leq N(N+2\overline{N})}|c_n(\alpha)|\,L_{N(N+2\overline{N})}(f)(1+|x|)^{l_{N(N+2\overline{N})}(f)}\mathbb{E}\Big(\Big(1+\Big|\sum_{k=1}^n G_k\Big|\Big)^{l_{N(N+2\overline{N})}(f)}\Big) \\
&\leq \frac{(CC_{N+1}(Z)C_2(Z))^{N(N+2\overline{N})}}{n^{\frac{N+1}{2}}}L_{N(N+2\overline{N})}(f)(1+|x|)^{l_{N(N+2\overline{N})}(f)}\times 2^{l_{N(N+2\overline{N})}(f)}(1+C_{2l_{N(N+2\overline{N})}(f)}(Z)),
\end{aligned}$$

in which we have again used the Burkholder inequality (3.26). By using $C_p(Z)\leq C_{pq}^{1/q}(Z)\leq C_{pq}(Z)$, $q\geq 1$, we get

$$\sum_{i=1}^3|I_if(x)|\leq\mathcal{H}_NC_{2(N+3)}^{2N(N+2\overline{N})}(Z)(1+C_{2l_{\widehat{N}}(f)}(Z))^{2N+3}2^{(N+2)(l_{\widehat{N}}(f)+1)}L_{\widehat{N}}(f)(1+|x|)^{l_{\widehat{N}}(f)}\times\frac{1}{n^{\frac{N+1}{2}}}$$

with $\widehat{N}=N(2\overline{N}+N+5)$, and statement (2.16) follows. Concerning (2.17), it suffices to notice that for $f(x)=x^\beta$ with $|\beta|=k$ then $L_{\widehat{N}}(f)=1$ and $l_{\widehat{N}}(f)=k$. $\square$

3 General test functions

3.1 Differential calculus based on a splitting method

In this section we use the variational calculus settled in [2, 1, 7, 8] in order to treat general test functions. Let us give the definitions and the notation.

We say that the law of the random variable $Y\in\mathbb{R}^d$ is locally lower bounded by the Lebesgue measure if there exists $y_Y\in\mathbb{R}^d$ and $\varepsilon,r>0$ such that for every non negative and measurable function $f:\mathbb{R}^d\to\mathbb{R}_+$

$$\mathbb{E}(f(Y))\geq\varepsilon\int f(y-y_Y)1_{B(0,r)}(y-y_Y)dy. \tag{3.1}$$

We denote by $\mathcal{L}(r,\varepsilon)$ the class of the random variables which verify (3.1). Given $r>0$ we consider the functions $a_r,\psi_r:\mathbb{R}\to\mathbb{R}_+$ defined by

$$a_r(t)=1-\frac{1}{1-(\frac{t}{r}-1)^2}\qquad \psi_r(t)=1_{\{|t|\leq r\}}+1_{\{r<|t|\leq 2r\}}e^{a_r(|t|)}. \tag{3.2}$$

If $Y \in \mathcal{L}(2r,\varepsilon)$ then

$$\mathbb{E}(f(Y)) \geq \varepsilon \int f(y-y_Y)\psi_r(|y-y_Y|^2)dy.$$

The advantage of $\psi_r(|y-y_Y|^2)$ is that it is a smooth function (which replaces the indicator function of the ball) and (it is easy to check) that for each $l \in \mathbb{N}, p \geq 1$ there exists a universal constant $C_{l,p} \geq 1$ such that

$$\psi_r(t)|a_r^{(l)}(|t|)|^p \leq \frac{C_{l,p}}{r^{lp}} \tag{3.3}$$

where $a_r^{(l)}$ denotes the derivative of order l of a_r. Moreover one can check (see [3]) that if $Y \in \mathcal{L}(2r,\varepsilon)$ then it admits the following decomposition (the equality is understood as identity of laws):

$$Y = \chi U + (1-\chi)V \tag{3.4}$$

where χ, U, V are independent random variables with the following laws:

$$\begin{aligned} \mathbb{P}(\chi=1) = \varepsilon m(r) \quad &\text{and} \quad \mathbb{P}(\chi=0) = 1-\varepsilon m(r), \\ \mathbb{P}(U \in dy) &= \frac{1}{m(r)}\psi_r(|y-y_Y)|^2)dy \\ \mathbb{P}(V \in dy) &= \frac{1}{1-\varepsilon m(r)}(\mathbb{P}(Z \in dy) - \varepsilon\psi_r(|y-y_Y|^2)dy) \end{aligned} \tag{3.5}$$

with

$$m(r) = \int \psi_r(|y-y_Y|^2)dy. \tag{3.6}$$

We are now able to present our calculus. We fix $r,\varepsilon > 0$ and we consider a sequence of independent random variables $Y_k \in \mathcal{L}(2r,\varepsilon), k \in \mathbb{N}$. Then, using the procedure described above we write

$$Y_k = \chi_k U_k + (1-\chi_k)V_k, \tag{3.7}$$

the law of χ_k, U_k and V_k being given in (3.5). We assume that χ_k, U_k, V_k, $k \in \mathbb{N}$, are independent. We define $\mathcal{G} = \sigma(\chi_k, V_k, k \in \mathbb{N})$. A random variable $F = f(\omega, U_1, ..., U_n)$ is called a simple functional if f is $\mathcal{G} \times \mathcal{B}(\mathbb{R}^{d\times n})$ measurable and for each $\omega, f(\omega,\cdot) \in C_b^\infty(\mathbb{R}^{d\times n})$. We denote $\mathcal{S}$ the space of the simple functionals. Moreover we define the differential operator $D : \mathcal{S} \to l_2 := l_2(\mathbb{R}^d)$ by $D_{(k,i)}F = \chi_k \partial_{u_k^i} f(\omega, U_1, ..., U_n)$. Then the Malliavin covariance matrix of $F \in (F^1, ..., F^m) \in \mathcal{S}^m$ is defined as

$$\sigma_F^{i,j} = \left\langle DF^i, DF^j \right\rangle_{l_2} = \sum_{k=1}^\infty \sum_{p=1}^d D_{(k,p)}F^i \times D_{(k,p)}F^j, \quad i,j = 1,...,m. \tag{3.8}$$

If σ_F is invertible we denote $\gamma_F = \sigma_F^{-1}$.
Moreover, we define the iterated derivatives $D^m : \mathcal{S} \to l_2^{\otimes m}$ by $D^{(m)}_{(k_1,i_1),....,(k_m,i_m)} = D_{(k_1,i_1)}....D_{(k_m,i_m)}$ and on $\mathcal{S}$ we consider the norms

$$|F|_q^2 = |F|^2 + \sum_{m=1}^q |D^m F|^2_{l_2^{\otimes m}} = |F|^2 + \sum_{m=1}^q \sum_{k_1,...,k_m=1}^\infty \sum_{i_1,...,i_m=1}^d \left|D_{(k_1,i_1)}....D_{(k_m,i_m)}F\right|^2$$

and

$$\|F\|_{q,p} = (\mathbb{E}(|F|_q^p))^{1/p}. \tag{3.9}$$

We introduce now the Ornstein-Uhlenbeck operator L. We denote $\theta_{k,i} = \partial_i \ln p_{U_k}(U_k) = 2(U_k - y_Y)^i 1_{r<|U_k-y_Y|^2<2r} a'_r(|U_k - y_Y|^2)$, p_{U_k} being the density of U_k, and we define

$$LF = -\sum_{k=1}^{\infty}\sum_{i=1}^{d}\left(D_{(k,i)}D_{(k,i)}F + D_{(k,i)}F \times \theta_{k,i}\right). \tag{3.10}$$

Using elementary integration by parts on $\mathbb{R}^d$ one easily proves the following duality formula: for $F, G \in \mathcal{S}$

$$\mathbb{E}(\langle DF, DG\rangle_{l_2}) = \mathbb{E}(FLG) = \mathbb{E}(GLF). \tag{3.11}$$

Finally, for $q \geq 2$, we define

$$|||F|||_{q,p} = \|F\|_{q,p} + \|LF\|_{q-2,p}\,. \tag{3.12}$$

We recall now the basic computational rules and the integration by parts formulae. For $\phi \in C^1(\mathbb{R}^d)$ and $F = (F^1, ..., F^d) \in \mathcal{S}^d$ we have

$$D\phi(F) = \sum_{j=1}^{d} \partial_j \phi(F) DF^j, \tag{3.13}$$

and for $F, G \in \mathcal{S}$

$$L(FG) = FLG + GLF - 2\langle DF, DG\rangle\,. \tag{3.14}$$

The formula (3.13) is just the chain rule in the standard differential calculus and (3.14) is obtained using duality. Let $H \in \mathcal{S}$. We use the duality relation and (3.11) we obtain

$$\mathbb{E}(HFLG) = \mathbb{E}(\langle D(HF), DG\rangle_{l_2}) = \mathbb{E}(H\langle DF, DG\rangle_{l_2}) + \mathbb{E}(F\langle DH, DG\rangle_{l_2}).$$

A similar formula holds with GLF instead of FLG. We sum them and we obtain

$$\begin{aligned}\mathbb{E}(H(FLG + GLF)) &= 2\mathbb{E}(H\langle DF, DG\rangle_{l_2}) + \mathbb{E}(\langle DH, D(FG)\rangle_{l_2}) \\ &= 2\mathbb{E}(H\langle DF, DG\rangle_{l_2}) + \mathbb{E}(HL(FG)).\end{aligned}$$

We give now the integration by parts formula (this is a localized version of the standard integration by parts formula from Malliavin calculus).

Theorem 3.1. *Let $\eta > 0$ be fixed and let $\Psi_\eta \in C^\infty(\mathbb{R})$ be such that $1_{[\eta/2,\infty)} \leq \Psi_\eta \leq 1_{[\eta,\infty)}$ and for every $k \in \mathbb{N}$ one has $\|\Psi_\eta^{(k)}\|_\infty \leq C\eta^{-k}$. Let $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$. For every $\phi \in C_p^\infty(\mathbb{R}^d)$, $\eta > 0$ and $i = 1, ..., d$*

$$\mathbb{E}(\partial_i \phi(F) G\Psi_\eta(\det \sigma_F)) = \mathbb{E}(\phi(F) H_i(F, G\Psi_\eta(\det \sigma_F))) \tag{3.15}$$

with

$$H_i(F, G\Psi_\eta(\det \sigma_F)) = \sum_{j=1}^{d}\left(G\Psi_\eta(\det \sigma_F)\right)\gamma_F^{i,j} LF^j + \left\langle D(G\Psi_\eta(\det \sigma_F)\gamma_F^{i,j}), DF^j\right\rangle_{l_2}. \tag{3.16}$$

Let $m \in \mathbb{N}, m \geq 2$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \{1, ..., d\}^m$. Then

$$\mathbb{E}(\partial_\alpha \phi(F) G\Psi_\eta(\det \sigma_F)) = \mathbb{E}(\phi(F) H_\alpha(F, G\Psi_\eta(\det \sigma_F))) \tag{3.17}$$

with $H_\alpha(F, G\Psi_\eta(\det \sigma_F))$ defined by reccurence

$$H_{(\alpha_1,...,\alpha_m)}(F, G\Psi_\eta(\det \sigma_F)) := H_{\alpha_m}(F, H_{(\alpha_1,...,\alpha_{m-1})}(F, G\Psi_\eta(\det \sigma_F))).$$

Proof. We give here only a sketch of the proof, a detailed one can be found e.g. in [4] and [7]. Using the chain rule $D\phi(F) = \nabla\phi(F)DF$ so that

$$\langle D\phi(F), DF\rangle_{l_2} = \nabla\phi(F)\langle DF, DF\rangle_{l_2} = \nabla\phi(F)\sigma_F.$$

It follows that, on the set $\det\sigma_F > 0$,we have $\nabla\phi(F) = \gamma_F\langle D\phi(F), DF\rangle_{l_2}$. Then, by using (3.15) we get

$$\begin{aligned}\mathbb{E}(G\Psi_\eta(\det\sigma_F)\nabla\phi(F)) &= \mathbb{E}(G\Psi_\eta(\det\sigma_F)\gamma_F\langle D\phi(F), DF\rangle_{l_2}) \\ &= \mathbb{E}(\phi(F)G\Psi_\eta(\det\sigma_F)\gamma_F LF) - \mathbb{E}(\phi(F)\langle(G\Psi_\eta(\det\sigma_F)\gamma_F), DF\rangle_{l_2}.\end{aligned}$$

and (3.15)-(3.16) hold. By iteration one obtains the higher order integration by parts formulae. □

We give now useful estimates for the weights which appear in (3.17):

Lemma 3.2. *Let $m, q \in \mathbb{N}$, $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$. There exists a universal constant $C \geq 1$ (depending on d, m, q only) such that for every multi index α with $|\alpha| = q$ one has*

$$|H_\alpha(F, G\Psi_\eta(\det\sigma_F))|_m \leq C\eta^{-q(m+q+1)}(1 + |||F|||_{m+q+1}^{2dq(m+q+3)} + |LF|_{m+q-1}^{2q})\,|G|_{m+q}. \tag{3.18}$$

In particular we have

$$\|H_\alpha(F, G\Psi_\eta(\det\sigma_F))\|_p \leq C\eta^{-(q+1)^2}(1 \vee |||F|||_{q+1,4dq(q+3)p}^{2dq(q+3)})\,\|G\|_{q,2p} \tag{3.19}$$

Proof. A rather long but straightforward computation (see [7] or [4] Theorem 3.4, more precise details are given in [5]) gives

$$\begin{aligned}&|H_\alpha(F, G\Psi_\eta(\det\sigma_F))|_m \\ \leq\ & C(1 \vee (\det\sigma_F)^{-1})^{q(m+q+1)}(1 + |F|_{m+q+1}^{2dq(m+q+2)} + |LF|_{m+q-1}^{2q})\,|G\Psi_\eta(\det\sigma_F)|_{m+q}.\end{aligned}$$

Notice that

$$\begin{aligned}|G\Psi_\eta(\det\sigma_F)|_{m+q} &\leq |G|_{m+q}\,|\Psi_\eta(\det\sigma_F)|_{m+q} \\ &\leq \frac{C}{\eta^{m+q}}\,|G|_{m+q}\,|\det\sigma_F|_{m+q} \leq \frac{C}{\eta^{m+q}}\,|G|_{m+q}\,|F|_{m+q}^d.\end{aligned}$$

Moreover, on the set $\Psi_\eta(\det\sigma_F) \neq 0$ we have $\det\sigma_F \geq \eta/2$. So

$$|H_\alpha(F, G\Psi_\eta(\det\sigma_F))|_m \leq C\eta^{-(q+1)(m+q+1)}(1 + |F|_{m+q+1}^{2dq(m+q+3)} + |LF|_{m+q-1}^{2q})\,|G|_{m+q}$$

so (3.18) is proved. Taking now $m = 0$ and using Schwartz inequality we obtain (3.19). □

We go now on and we give the regularization lemma. We recall that a super kernel $\phi : \mathbb{R}^d \to \mathbb{R}$ is a function which belongs to the Schwartz space $\mathbf{S}$ (infinitely differentiable functions which decrease in a polynomial way to infinity), $\int \phi(x)dx = 1$, and such that for every multi indexes α and β, one has

$$\int y^\alpha\phi(y)dy = 0, \quad |\alpha| \geq 1, \tag{3.20}$$

$$\int |y|^m\,|\partial_\beta\phi(y)|\,dy < \infty. \tag{3.21}$$

As usual, for $|\alpha| = m$ then $y^\alpha = \prod_{i=1}^m y_{\alpha_i}$. Since super kernels play a crucial role in our approach we give here the construction of such an object (we follow [17] Section 3, Remark 1). We do it in dimension

$d = 1$ and then we take tensor products. So, if $d = 1$ we take $\psi \in \mathbf{S}$ which is symmetric and equal to one in a neighborhood of zero and we define $\phi = \mathcal{F}^{-1}\psi$, the inverse of the Fourier transform of ψ. Since $\mathcal{F}^{-1}$ sends $\mathbf{S}$ into $\mathbf{S}$ the property (3.21) is verified. And we also have $0 = \psi^{(m)}(0) = i^{-m}\int x^m\phi(x)dx$ so (3.20) holds as well. We finally normalize in order to obtain $\int \phi = 1$.
We fix a super kernel ϕ. For $\delta \in (0,1)$ and for a function f we define

$$\phi_\delta(y) = \frac{1}{\delta^d}\phi\Big(\frac{y}{\delta}\Big) \quad \text{and} \quad f_\delta = f * \phi_\delta,$$

the symbol $*$ denoting convolution. For $f \in C_p^k(\mathbb{R}^d)$, we recall the constants $L_k(f)$ and $l_k(f)$ in (2.5).

Lemma 3.3. *Let $F \in \mathcal{S}^d$ and $q, m \in \mathbb{N}$. There exists a constant $C \geq 1$, depending on d, m and q only, such that for every $f \in C_p^{q+m}(\mathbb{R}^d)$, every multi index γ with $|\gamma| = m$ and every $\eta, \delta > 0$*

$$\begin{aligned} &|\mathbb{E}(\Psi_\eta(\det\sigma_F))\partial_\gamma f(x+F)) - \mathbb{E}(\Psi_\eta(\det\sigma_F))\partial_\gamma f_\delta(x+F))| \\ &\leq C c_{l_0(f)+q} L_0(f) \left\|F\right\|_{2l_0(f)}^{l_0(f)} \mathcal{C}_{q+m}(F)\frac{\delta^q}{\eta^{(q+m+1)^2}}(1+|x|)^{l_0(f)} \end{aligned} \tag{3.22}$$

with

$$c_p = \int |\phi(z)|\,(1+|z|)^p dz \quad \textit{and} \quad \mathcal{C}_p(F) = 1 \vee |||F|||_{p+1,4dp(p+3)}^{2dp(p+3)}. \tag{3.23}$$

As a consequence, we have

$$\begin{aligned} &|\mathbb{E}(\partial_\gamma f(x+F))) - \mathbb{E}(\partial_\gamma f_\delta(x+F))| \\ &\leq C\left\|F\right\|_{2l_0(f)}^{l_0(f)}\Big(c_{l_m(f)}L_m(f)\mathbb{P}^{1/2}(\det\sigma_F \leq \eta) + \frac{c_{l_0(f)+q}\delta^q}{\eta^{(q+m+1)^2}}L_0(f)\mathcal{C}_{q+m}(F)\Big)(1+|x|)^{l_m(f)}. \end{aligned} \tag{3.24}$$

Proof A. Using Taylor expansion of order q

$$\begin{aligned} \partial_\gamma f(x) - \partial_\gamma f_\delta(x) &= \int(\partial_\gamma f(x) - \partial_\gamma f(y))\phi_\delta(x-y)dy \\ &= \int I_{\gamma,q}(x,y)\phi_\delta(x-y)dy + \int R_{\gamma,q}(x,y)\phi_\delta(x-y)dy \end{aligned}$$

with

$$\begin{aligned} I_{\gamma,q}(x,y) &= \sum_{i=1}^{q-1}\frac{1}{i!}\sum_{|\alpha|=i}\partial_\alpha\partial_\gamma f(x)(x-y)^\alpha, \\ R_{\gamma,q}(x,y) &= \frac{1}{q!}\sum_{|\alpha|=q}\int_0^1\partial_\alpha\partial_\gamma f(x+\lambda(y-x))(x-y)^\alpha(1-\lambda)^q d\lambda. \end{aligned}$$

Using (3.20) we obtain $\int I(x,y)\phi_\delta(x-y)dy = 0$ and by a change of variable we get

$$\int R_{\gamma,q}(x,y)\phi_\delta(x-y)dy = \frac{1}{q!}\sum_{|\alpha|=q}\int_0^1\int dz\phi_\delta(z)\partial_\alpha\partial_\gamma f(x+\lambda z)z^\alpha(1-\lambda)^q d\lambda.$$

So, we have

$$\begin{aligned} &\mathbb{E}(\Psi_\eta(\det\sigma_F)\partial_\gamma f(x+F)) - \mathbb{E}(\Psi_\eta(\det\sigma_F)\partial_\gamma f_\delta(x+F)) \\ &= \mathbb{E}(\int \Psi_\eta(\det\sigma_F)R_{\gamma,q}(x+F,y)\phi_\delta(x+F-y)dy) \\ &= \frac{1}{q!}\sum_{|\alpha|=q}\int_0^1\int dz\phi_\delta(z)\mathbb{E}(\Psi_\eta(\det\sigma_F)\partial_\alpha\partial_\gamma f(x+F+\lambda z))z^\alpha(1-\lambda)^q d\lambda. \end{aligned}$$

Using integration by parts formula (3.17) (with $G=1$)

$$\begin{aligned}
&|\mathbb{E}(\Psi_\eta(\det\sigma_F)\partial_\alpha\partial_\gamma f(x+F+\lambda z))|\\
&=\left|\mathbb{E}(f(F+\lambda z)H_{(\gamma,\alpha)}(F,\Psi_\eta(\det\sigma_F))\right|\\
&\le L_0(f)\mathbb{E}((1+|x|+|z|+|F|)^{l_0(f)}\left|H_{(\gamma,\alpha)}(F,\Psi_\eta(\det\sigma_F))\right|)\\
&\le C(1+|x|)^{l_0(f)}(1+|z|)^{l_0(f)}L_0(f)\left\|F\right\|_{2l_0(f)}^{l_0(f)}(\mathbb{E}(\left|H_{(\gamma,\alpha)}(F,\Psi_\eta(\det\sigma_F))\right|^2)^{1/2}.
\end{aligned}$$

and the upper bound from (3.19) (with $p=2$) we get

$$(\mathbb{E}(\left|H_{(\alpha,\gamma)}(F,\Psi_\eta(\det\sigma_F))\right|^2)^{1/2}\le\frac{C}{\eta^{(q+m+1)^2}}(1\vee|||F|||_{q+m+1,4d(q+m)(q+m+3)}^{2d(q+m)(q+m+3)})$$

And since $\int dz\left|\phi_\delta(z)z^\alpha\right|(1+|z|)^{l_0(f)}\le\delta^q\int\left|\phi(z)z^\alpha\right|(1+|z|)^{l_0(f)}dz$ we conclude that

$$\begin{aligned}
&|\mathbb{E}(\Psi_\eta(\det\sigma_F))\partial_\gamma f(F))-\mathbb{E}(\Psi_\eta(\det\sigma_F))\partial_\gamma f_\delta(F))|\\
&\le C(1+|x|)^{l_0(f)}c_{l_0(f)}L_0(f)\left\|F\right\|_{2l_0(f)}^{l_0(f)}\mathcal{C}_{q+m}(F)\frac{C\delta^q}{\eta^{(q+m+1)^2}},\quad c_{l_0(f)}=\int\left|\phi(z)\right|(1+|z|)^{l_0(f)+q}dz
\end{aligned}$$

and (3.22) holds. Concerning (3.24), we write

$$\begin{aligned}
&|\mathbb{E}((1-\Psi_\eta(\det\sigma_F)))\partial_\gamma f(x+F))-\mathbb{E}((1-\Psi_\eta(\det\sigma_F)))\partial_\gamma f_\delta(x+F))|\\
\le\ &2(L_0(\partial_\gamma f_\delta)\vee L_0(\partial_\gamma f))\mathbb{E}((1-\Psi_\eta(\det\sigma_F)))(1+|x|+|F|)^{l_0(\partial_\gamma f_\delta)\vee l_0(\partial_\gamma f)})\\
\le\ &2(L_0(\partial_\gamma f_\delta)\vee L_0(\partial_\gamma f))(1+|x|)^{l_0(\partial_\gamma f_\delta)\vee l_0(\partial_\gamma f)})\left\|F\right\|_{2l_0(f_\delta)\vee l_0(f)}^{l_0(f_\delta)\vee l_0(f))}\mathbb{P}^{1/2}(\det\sigma_F\le\eta).
\end{aligned}$$

So the proof of (3.24) will be completed as soon as we check that $l_0(\partial_\gamma f_\delta)\le l_m(f)$ and $L_0(\partial_\gamma f_\delta)\le L_m(f)\int(1+|y|)^{l_m(f)}\left|\phi(y)\right|dy$:

$$\begin{aligned}
|\partial_\gamma f_\delta(x)|=\left|\int\partial_\gamma f(x-y)\phi_\delta(y)dy\right|&\le L_m(f)\int(1+|x-y|)^{l_m(f)}\left|\phi_\delta(y)\right|dy\\
&\le L_m(f)(1+|x|)^{l_m(f)}\int(1+|y|)^{l_m(f)}\left|\phi(y)\right|dy.
\end{aligned}$$

□

3.2 CLT and Edgeworth's development

In this section we take $F=S_n(Z)=\sum_{k=1}^n Z_k$ defined in (2.1). It is convenient for us to write $\sigma_k=C_kC_k^*$ with $C_k\in\mathcal{M}_{d\times d}$ symmetric and $Z_k=C_kY_k$. So $S_n(Z)=\sum_{k=1}^n C_kY_k$. We assume that $Y_k\in\mathcal{L}(2r,\varepsilon)$ so we have the decomposition (3.7). Consequently

$$F=S_n(Z)=\sum_{k=1}^n C_kY_k=\sum_{k=1}^n C_k(\chi_kU_k+(1-\chi_k)V_k).$$

We will use Lemma 3.3, so we estimate the quantities which appear in the right hand side of (3.22).

Lemma 3.4. *For every $k\in\mathbb{N}$ and $p\ge1$ there exists a constant C depending on k,p only, such that*

$$\sup_n|||S_n(Z)|||_{k,p}\le C\times\frac{C_p(Z)}{r^k}.\tag{3.25}$$

Proof. We will use the following easy consequence of Burkholder's inequality for discrete martingales: if $M_n = \sum_{k=1}^n \Delta_k$ with $\Delta_k, k = 1, ..., n$ independent centred random variables, then

$$\|M_n\|_p \le C\mathbb{E}\Big(\Big(\sum_{k=1}^n |\Delta_k|^2\Big)^{p/2}\Big)^{1/p} = C\Big\|\sum_{k=1}^n |\Delta_k|^2\Big\|_{p/2}^{1/2} \le C\Big(\sum_{k=1}^n \|\Delta_k\|_p^2\Big)^{1/2}. \tag{3.26}$$

Using this inequality and (2.4) we obtain $\|S_n(Z)\|_p \le C \times C_p(Z)$. We look now to the Sobolev norms. It is easy to see that, $S_n(Z)^i$ denoting the ith component of $S_n(Z)$,

$$D_{(k,j)}S_n(Z)^i = \chi_k C_k^{i,j} \quad \text{and} \quad D^{(l)}S_n(Z) = 0 \text{ for } l \ge 2.$$

Since $\sum_{k=1}^n |\sigma_k| \le C_2(Z)$ it follows that

$$\|S_n(Z)\|_{k,p} \le 2C_p(Z) \quad \forall k \in \mathbb{N}, p \ge 2.$$

Moreover

$$LS_n(Z) = -\sum_{k=1}^n LZ_k = -\sum_{k=1}^n \chi_k C_k A_r(U_k), \quad A_r(U_k) = 1_{r<|U_k-y_Y|^2<2r} \times 2a_r'(|U_k - y_Y|^2)(U_k - y_Y).$$

We prove that

$$\|LS_n(Z)\|_{k,p} \le \frac{C}{r^k} \times C_p(Z), \tag{3.27}$$

C depending on k, p but being independent of n.
Let $k = 0$. The duality relation gives $\mathbb{E}(LZ_k) = \mathbb{E}(\langle D1, DZ_k\rangle_{l_2}) = 0$. Since the LZ_k's are independent, we can apply (3.26) first and (2.4), so that

$$\begin{aligned}\|LS_n(Z)\|_p &\le C\Big(\sum_{k=1}^n \|C_k A_r(U_k)\|_p^2\Big)^{1/2} \\ &\le \Big(\sum_{k=1}^n \frac{C_2(Z)}{n}\|A_r(U_k)\|_p^2\Big)^{1/2}\end{aligned}$$

By (3.3) $\mathbb{E}(|A_r(U_k)|^p) \le Cr^{-p}$ so $\|LS_n(Z)\|_p \le Cr^{-1} \times C_2(Z)$.
We take now $k = 1$. We have

$$D_{(q,j)}LS_n(Z)^i = D_{(q,j)}\big(\chi_k C_q A_r(U_q)\big) = \chi_k C_q D_{(q,j)}A_r(U_q)$$

so that, using again (2.4),

$$|DLS_n(Z)|_{l_2}^2 = \sum_{q=1}^n\sum_{j=1}^d \big|\chi_k C_q D_{(q,j)}A_r(U_q)\big|^2 \le C \times \frac{C_2(Z)}{n}\sum_{q=1}^n\sum_{j=1}^d \big|D_{(q,j)}A_r(U_q)\big|^2.$$

We notice that $D_{(q,j)}A_r(U_q)$ is not null for $r < |U_q - y_Y|^2 < 2r$ and contains the derivatives of a_r up to order 2, possibly multiplied by polynomials in the components of $U_q - y_Y$ of order up to 2. Since $|U_q - y_Y|^2 \le 2r$, by using (3.3) one obtains $\mathbb{E}(|DLF|_{l_2}^p) \le Cr^{-2p} \times C_2^{1/2}(Z)$, so (3.27) holds for $k = 1$ also. And for higher order derivatives the proof is similar. $\square$

We give now estimates of the Malliavin covariance matrix. We have

$$\sigma_{S_n(Z)} = \sum_{k=1}^n \chi_k \sigma_k.$$

We denote

$$\overline{\sigma}_n = \sum_{i=1}^{n} \sigma_i, \quad \underline{\lambda}_n = \inf_{|\xi|=1} \langle \overline{\sigma}_n \xi, \xi \rangle, \quad \overline{\lambda}_n = \sup_{|\xi|=1} \langle \overline{\sigma}_n \xi, \xi \rangle. \tag{3.28}$$

For reasons which will be clear later on we *do not* consider here the normalization condition $\overline{\sigma}_n = \mathrm{Id}$. We have the following result.

Lemma 3.5. *Let $\eta = (\frac{\underline{\lambda}_n m(r)}{2(1+2\overline{\lambda}_n)})^d$. Then*

$$\mathbb{P}(\det \sigma_{S_n(Z)} \leq \eta) \leq \frac{e^3 \overline{c}_d}{9} \Big(\frac{2(1+2\overline{\lambda}_n)}{\underline{\lambda}_n m(r)} \Big)^d \exp\Big(- \frac{\underline{\lambda}_n^2 m^2(r)}{16 \overline{\lambda}_n} \times n \Big), \tag{3.29}$$

$\overline{c}_d$ denoting a positive constant depending on the dimension d only and $\overline{\lambda}_n$ and $\underline{\lambda}_n$ being given in (3.28).

Proof. Since $\sigma_k = C_k C_k^*$ we have

$$\left\langle \sigma_{S_n(Z)} \xi, \xi \right\rangle = \sum_{k=1}^{n} \chi_k \left\langle \sigma_k \xi, \xi \right\rangle = \sum_{k=1}^{n} \chi_k \left| C_k \xi \right|^2.$$

Take $\xi_1, ..., \xi_N \in S_{d-1} =: \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ such that the balls of centers ξ_i and radius $\eta^{1/d}$ cover S_{n-1}. One needs $N \leq \bar{c}_d \eta^{-1}$ points, where $\bar{c}_d$ is a constant depending on the dimension. It is easy to check that $\xi \mapsto \left\langle \sigma_{S_n(Z)} \xi, \xi \right\rangle$ is Lipschitz continuous with Lipschitz constant $2\overline{\lambda}_n$ so that $\inf_{|\xi|=1} \langle \sigma_{S_n(Z)} \xi, \xi \rangle \geq \inf_{i=1,\dots,N} \langle \sigma_{S_n(Z)} \xi_i, \xi_i \rangle - 2\overline{\lambda}_n \eta^{1/d}$. Consequently,

$$\begin{aligned}
\mathbb{P}(\det \sigma_{S_n(Z)} \leq \eta) &\leq \mathbb{P}(\inf_{|\xi|=1} \left\langle \sigma_{S_n(Z)} \xi, \xi \right\rangle \leq \eta^{1/d}) \leq \sum_{i=1}^{N} \mathbb{P}(\left\langle \sigma_{S_n(Z)} \xi_i, \xi_i \right\rangle \leq \eta^{1/d} + 2\overline{\lambda}_n \eta^{1/d}) \\
&\leq \frac{\bar{c}_d}{\eta} \max_{i=1,\dots,N} \mathbb{P}\big(\langle \sigma_{S_n(Z)} \xi_i, \xi_i \rangle \leq \eta^{1/d} (1 + 2\overline{\lambda}_n) \big).
\end{aligned}$$

So, it remains to prove that for every $\xi \in S_{d-1}$ and for the choice $\eta = (\frac{\underline{\lambda}_n m(r)}{2(1+2\overline{\lambda}_n)})^d$,

$$\mathbb{P}\big(\langle \sigma_{S_n(Z)} \xi, \xi \rangle \leq (1 + 2\overline{\lambda}_n) \eta^{1/d} \big) \leq \frac{2e^3}{9} \exp\Big(- \frac{\underline{\lambda}_n^2 m^2(r)}{16 \overline{\lambda}_n} \times n \Big).$$

We recall that $\mathbb{E}(\chi_k) = m(r)$ and we write

$$\begin{aligned}
\mathbb{P}\big(\langle \sigma_{S_n(Z)} \xi, \xi \rangle \leq (1 + 2\overline{\lambda}_n) \eta^{1/d} \big) &= \mathbb{P}\Big(\sum_{k=1}^{n} (\chi_k - m(r)) \left| C_k \xi \right|^2 \leq (1 + 2\overline{\lambda}_n) \eta^{1/d} - m(r) \sum_{k=1}^{n} \left| C_k \xi \right|^2 \Big) \\
&\leq \mathbb{P}\Big(- \sum_{k=1}^{n} (\chi_k - m(r)) \left| C_k \xi \right|^2 \geq \underline{\lambda}_n m(r) - (1 + 2\overline{\lambda}_n) \eta^{1/d} \Big)
\end{aligned}$$

the last equality being true because

$$\sum_{k=1}^{n} \left| C_k \xi \right|^2 = \sum_{k=1}^{n} \langle \sigma_k \xi, \xi \rangle \geq \underline{\lambda}_n \left| \xi \right|^2 = \underline{\lambda}_n.$$

So, we take $\eta = (\frac{\underline{\lambda}_n m(r)}{2(1+2\overline{\lambda}_n)})^d$ and we get

$$\mathbb{P}\big(\langle \sigma_{S_n(Z)} \xi, \xi \rangle \leq (1 + 2\overline{\lambda}_n) \eta^{1/d} \big) \leq \mathbb{P}\Big(- \sum_{k=1}^{n} (\chi_k - m(r)) \left| C_k \xi \right|^2 \geq \frac{\underline{\lambda}_n m(r)}{2} \Big)$$

We now use the following Hoeffding's inequality (in the slightly more general form given in [15] Corollary 1.4): if the differences X_k of a martingale M_n are such that $\mathbb{P}(|X_k| \leq b_k) = 1$ then $\mathbb{P}(M_n \geq x) \leq (2e^3/9)\exp(-|x|^2 \times n/(2(b_1^2 + \cdots + b_n^2)))$. Here, we choose $X_k = -(\chi_k - m(r))\,|C_k\xi|^2$. These are independent random variables and $|X_k| \leq 2\,|C_k\xi|^2$. Then

$$\begin{aligned}\mathbb{P}\Big(-\sum_{k=1}^{n}(\chi_k - m(r))\,|C_k\xi|^2 \geq \frac{\underline{\lambda}_n m(r)}{2}\Big) &\leq \frac{2e^3}{9}\exp\Big(-\frac{\underline{\lambda}_n^2 m^2(r)}{4} \times \frac{n}{4\sum_{k=1}^n |C_k\xi|^2}\Big)\\ &\leq \frac{2e^3}{9}\exp\Big(-\frac{\underline{\lambda}_n^2 m^2(r)}{16\overline{\lambda}_n} \times n\Big).\end{aligned}$$

□

We are now able to give the regularization lemma in our specific framework.

Lemma 3.6. *Let $q, m \in \mathbb{N}$. There exists some constant $C \geq 1$, depending just on q, m, such that for every $\delta > 0$, every multi index γ with $|\gamma| = m$ and every $f \in C_p^m(\mathbb{R}^d)$ one has*

$$\begin{aligned}&|\mathbb{E}(\partial_\gamma f(x + S_n(Z))) - \mathbb{E}(\partial_\gamma f_\delta(x + S_n(Z)))|\\ &\leq CC_{2l_0(f)}^{1/2}(Z)Q_{q,m}(Z)\Big(L_m(f)\exp(-\frac{\underline{\lambda}_n^2 m^2(r)}{32\overline{\lambda}_n} \times n) + \delta^q L_0(f)\Big)(1+|x|)^{l_m(f)}\end{aligned} \tag{3.30}$$

with

$$Q_{q,m}(Z) = \frac{C_{4d(q+m)(q+m+3)}^{2d(q+m)(q+m+3)}(Z)}{r^{q+m+1}}\Big(\frac{1+2\overline{\lambda}_n}{\underline{\lambda}_n m(r)}\Big)^{d(q+m+1)^2} c_{l_m(f)\vee(l_0(f)+q)}, \tag{3.31}$$

c_p being given in (3.23).

Proof. We will use Lemma 3.3. Notice first that, by (3.25), the constant $C_{q+m}(S_n(Z))$ defined in (3.23) is upper bounded by

$$CC_{4d(q+m)(q+m+3)}^{2d(q+m)(q+m+3)}(Z)r^{-(q+m+1)},$$

C depending on d and $q+m$. And by using the Burkholder inequality (3.26), one has $\|S_n(Z)\|_{2l_0(f)}^{l_0(f)} \leq C_{2l_0(f)}^{1/2}(Z)$. So (3.24) gives

$$\begin{aligned}&|\mathbb{E}(\partial_\gamma f(x + S_n(Z))) - \mathbb{E}(\partial_\gamma f_\delta(x + S_n(Z)))|\\ \leq\ &\frac{CC_{2l_0(f)}^{1/2}(Z)}{r^{q+m+1}}C_{4d(q+m)(q+m+3)}^{2d(q+m)(q+m+3)}(Z)c_{l_m(f)\vee(l_0(f)+q)}\Big(L_m(f)\mathbb{P}^{1/2}(\det\sigma_F \leq \eta) + \frac{\delta^q}{\eta^{(q+m+1)^2}}L_0(f)\Big) \times\\ &\times(1+|x|)^{l_m(f)},\end{aligned}$$

$c_p(f)$ being given in (3.23). We take now $\eta = (\frac{\underline{\lambda}_n m(r)}{2(1+\overline{\lambda}_n)})^d$ and we use (3.29) in order to obtain

$$\begin{aligned}&|\mathbb{E}(\partial_\gamma f(x + S_n(Z))) - \mathbb{E}(\partial_\gamma f_\delta(x + S_n(Z)))| \leq \frac{CC_{2l_0(f)}^{1/2}(Z)}{r^{q+m+1}}C_{4d(q+m)(q+m+3)}^{2d(q+m)(q+m+3)}(Z)c_{l_m(f)\vee(l_0(f)+q)}\times\\ &\times\Big(\frac{1+2\overline{\lambda}_n}{\underline{\lambda}_n m(r)}\Big)^{d(q+m+1)^2}\Big(L_m(f)\exp(-\frac{\underline{\lambda}_n^2 m^2(r)}{32\overline{\lambda}_n} \times n) + \delta^q L_0(f)\Big)(1+|x|)^{l_m(f)}\end{aligned}$$

□

We are now able to characterize the regularity of the semigroup P_n^Z :

Proposition 3.7. *Let $f \in C_p^m(\mathbb{R}^d)$. If $|\gamma| = m$ then*

$$
\begin{aligned}
|\mathbb{E}(\partial_\gamma f(x+S_n(Z)))| \leq \Big(& L_m(f)2^{l_m(f)} \times \frac{C(1+C_{2l_m(f)}(Z))(1+2\overline{\lambda}_n)^{d/2}}{(\underline{\lambda}_n m(r))^{d/2}} \exp(-\frac{\underline{\lambda}_n^2 m^2(r)}{32\overline{\lambda}_n} \times n) \\
& + L_0(f)2^{l_0(f)} \times \frac{C(1+C_{2l_0(f)}(Z))C_{8dm(m+3)}^{2dm(m+3)}(Z)(1+2\overline{\lambda}_n)^{d(m+1)^2}}{r^{m+1}(\underline{\lambda}_n m(r))^{d(m+1)^2}} \Big)(1+|x|)^{2l_m(f)}
\end{aligned}
\tag{3.32}
$$

Proof. We take $\eta = (\frac{\underline{\lambda}_n m(r)}{2(1+\overline{\lambda}_n)})^d$ and the truncation function Ψ_η and we write

$$
\mathbb{E}(\partial_\gamma f(x+S_n(Z))) = I + J
$$

with

$$
I = \mathbb{E}(\partial_\gamma f(x+S_n(Z))(1-\Psi_\eta(\det \sigma_{S_n}))), \quad J = \mathbb{E}(\partial_\gamma f(x+S_n(Z))\Psi_\eta(\det \sigma_{S_n})).
$$

We estimate first

$$
\begin{aligned}
|I| & \leq L_m(f)\mathbb{E}((1+|x|+|S_n(Z)|)^{l_m(f)}(1-\Psi_\eta(\det \sigma_{S_n}))) \\
& \leq L_m(f)(\mathbb{E}((1+|x|+|S_n(Z)|^{2l_m(f)})^{1/2}\mathbb{P}^{1/2}(\det \sigma_{S_n} \leq \eta) \\
& \leq CL_m(f)2^{l_m(f)}(1+|x|)^{2l_m(f)}(1+C_{2l_m(f)}(Z))\Big(\frac{2(1+2\overline{\lambda}_n)}{\underline{\lambda}_n m(r)}\Big)^{d/2} \exp\Big(-\frac{\underline{\lambda}_n^2 m^2(r)}{32\overline{\lambda}_n} \times n\Big),
\end{aligned}
$$

In order to estimate J we use integration by parts and we obtain

$$
\begin{aligned}
J & = \mathbb{E}(f(x+S_n(Z))H_\gamma(S_n(Z),\Psi_\eta(\det \sigma_{S_n}))) \\
& \leq L_0(f)\mathbb{E}((1+|x|+|S_n|)^{l_0(f)}|H_\gamma(S_n(Z),\Psi_\eta(\det \sigma_{S_n}))|) \\
& \leq CL_0(f)2^{l_0(f)}(1+|x|)^{l_0(f)}(1+C_{2l_0(f)}(Z))(\mathbb{E}(|H_\gamma(S_n(Z),\Psi_\eta(\det \sigma_{S_n}))|^2))^{1/2}.
\end{aligned}
$$

Then using (3.19) and (3.25)

$$
\begin{aligned}
\big\|H_\gamma(S_n(Z),\Psi_\eta(\det \sigma_{S_n(Z)}))\big\|_2 & \leq C\eta^{-(m+1)^2}(1 \vee \||S_n(Z)\||_{m+1,8dm(m+3)}^{2dm(m+3)}) \\
& \leq Cr^{-(m+1)}C_{8dm(m+3)}^{2dm(m+3)}(Z)\Big(\frac{\underline{\lambda}_n m(r)}{2(1+\overline{\lambda}_n)}\Big)^{-d(m+1)^2}.
\end{aligned}
$$

□

We are now able to give the main result.

Theorem 3.8. *We look to $S_n(Z) = \sum_{k=1}^n Z_k = \sum_{k=1}^n C_k Y_k$ and we assume that $Y_k \in \mathcal{L}(2r,\varepsilon)$ for some $\varepsilon > 0, r > 0$. We also assume that (1.2) and (1.3) hold (for every $p \in \mathbb{N}$). Let $N, q \in \mathbb{N}$ be fixed. We assume that n is sufficiently large in order to have*

$$
n^{\frac{1}{2}(N+1)}e^{-\frac{m^2(r)}{128}\times n} \leq 1 \quad \textit{and} \quad n \geq 4(N+1)C_2(Z).
$$

There exists $C \geq 1$, depending on N and q only, such that for every multi index γ with $|\gamma| = q$ and every $f \in C_p^q(\mathbb{R}^d)$

$$
|\mathbb{E}(\partial_\gamma f(S_n(Z))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \leq C \times C_*(Z)(\frac{1}{n^{\frac{1}{2}(N+1)}}L_0(f) + L_q(f)e^{-\frac{m^2(r)}{32}\times n}) \tag{3.33}
$$

where

$$
\begin{aligned}
C_*(Z) = \quad & (C_2^{\overline{N}+1}(Z))^{N+1}2^{(N+1)(l_0(f)+1)+2}(1+C_{2l_0(f)}(Z))^{2N+3} \\
& \times \frac{C_{p_2}^{p_3}(Z)}{r^{p_1+1}m(r)^{d(p_1+1)^2}}\, c_{l_0(f)+q+(N+1)(N+3)}
\end{aligned} \tag{3.34}
$$

$$
\textit{with} \quad p_1 = 2q+(N+1)(N+3), \quad p_2 = 8dp_1(p_1+3), \quad p_3 = 2dp_1(p_1+3)+\frac{N+1}{2} \tag{3.35}
$$

c_p *being given in (3.23).*

Proof. Step 1. We assume first that $f \in C_p^{q+(N+1)(N+3)}(\mathbb{R}^d)$ and we prove that

$$
\begin{aligned}
& |\mathbb{E}(\partial_\gamma f(S_n(Z))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \\
& \leq \frac{\widehat{C}_1}{n^{\frac{1}{2}(N+1)}}(L_{q+(N+1)(N+3)}(f)2^{l_{q+(N+1)(N+3)}(f)}e^{-\frac{m^2(r)}{64}\times n} + L_0(f)2^{l_0(f)}),
\end{aligned} \tag{3.36}
$$

where

$$
\begin{aligned}
\widehat{C}_1 &= (CC_2^{\overline{N}+1}(Z))^{N+1}C_{2(N+3)}^{(N+1)/2}(Z)B_{q+(N+1)(N+3)} \quad \text{with} \\
B_p &= (2^{l_p(f)+1}(1+C_{2l_p(f)})^2)^{N+1}2^{l_p(f)}\frac{C}{r^{p+1}m(r)^{d(p+1)^2}} \times (1+C_{2l_p(f)}(Z)) \times C_{8dp(p+3)}^{2dp(p+3)}(Z).
\end{aligned}
$$

Notice that (3.36) is analogous to (3.33) but here $L_q(f)$ and $l_q(f)$ are replaced by $L_{q+(N+1)(N+3)}(f)$ and $l_{q+(N+1)(N+3)}(f)$.
We recall that in (2.46) and (2.47) we have proved that

$$
P_n^Z\partial_\gamma f(x) = P_n^G(Id+\sum_{k=1}^N \Gamma_k)\partial_\gamma f(x) + I_1(\partial_\gamma f)(x) + I_2(\partial_\gamma f)(x) + I_3(\partial_\gamma f)(x)
$$

with

$$
\begin{aligned}
I_1(f)(x) &= P_n^G Q_{N,n}^0 f(x), \\
I_2(f)(x) &= \sum_{1\leq r_1<\ldots<r_{N+1}\leq n} P_{r_{N+1}+1,n+1}^Z P_{r_N+1,r_{N+1}}^G \ldots P_{r_1+1,r_2}^G P_{1,r_1}^G Q_{N,r_1,\ldots,r_{N+1}}^{(N+1)} f(x) \\
I_3(f)(x) &= \sum_{m=1}^N \sum_{1\leq r_1<\ldots<r_m\leq n} P_{r_m+1,n+1}^G P_{r_{m-1}+1,r_m}^G \ldots P_{r_1+1,r_2}^G P_{1,r_1}^G Q_{N,r_1,\ldots,r_m}^{(m)} f(x)
\end{aligned} \tag{3.37}
$$

and (see (2.36) and (2.45))

$$
\begin{aligned}
Q_{N,r_1,\ldots,r_m}^{(m)} f(x) &= \sum_{3\leq|\alpha|\leq N_m} a_n^{r_1,\ldots,r_m}(\alpha)\theta_{r_1,\ldots,r_m}^\alpha \partial_\alpha f(x), \\
Q_{N,n}^0 f(x) &= \sum_{N+1\leq|\alpha|\leq N(N+2\overline{N})} c_n(\alpha)\partial_\alpha f(x),
\end{aligned} \tag{3.38}
$$

N_m being given in Lemma 2.4. The coefficients which appear above verify

$$
|a_n^{r_1,\ldots,r_m}(\alpha)| \leq \frac{(CC_2^{\overline{N}+1}(Z))^m}{n^{\frac{N+3m}{2}}} \quad \text{and} \quad |c_n(\alpha)| \leq \frac{(CC_{N+1}(Z)C_2(Z))^{N(N+2\overline{N})}}{n^{\frac{N+1}{2}}} \tag{3.39}
$$

We first estimate $I_2(f)$. Let us prove that for every $r_1 < ... < r_{N+1}$

$$\begin{aligned}&\left|P^Z_{r_{N+1}+1,n}P^G_{r_N+1,r_{N+1}}...P^G_{r_1+1,r_2}P^G_{k,r_1}Q^{(N+1)}_{r_1,...,r_{N+1}}\partial_\gamma f(x)\right| \\ &\leq \frac{\widehat{C}_1}{n^{\frac{1}{2}(4N+3)}}(L_{q+(N+1)(N+3)}(f)2^{l_{q+(N+1)(N+3)}(f)}e^{-\frac{m^2(r)}{64}\times n}+L_0(f)2^{l_0(f)})(1+|x|)^{l_{q+(N+1)(N+3)}(f)}\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}&\widehat{C}_1 = (CC_2^{\overline{N}+1}(Z))^{N+1}C_{2(N+3)}^{(N+1)/2}(Z)B_{q+(N+1)(N+3)} \quad \text{with} \\ &B_p = (2^{l_p(f)+1}(1+C_{2l_p(f)})^2)^{N+1}2^{l_p(f)}\frac{C}{r^{p+1}m(r)^{d(p+1)^2}}\times(1+C_{2l_p(f)}(Z))\times C^{2dp(p+3)}_{8dp(p+3)}(Z).\end{aligned}$$

Recall that $\sigma_{r_i} \leq \frac{1}{n}C_2(Z)$. We take $n \geq 4(N+1)C_2(Z)$ so that

$$\sum_{i=1}^{N+1}\sigma_{r_i} \leq \frac{1}{4}I. \tag{3.41}$$

Recall that $\sum_{i=1}^n \sigma_i = I$. So we distinguish now two cases:

$$\textbf{Case 1:} \qquad \sum_{i=r_{N+1}+1}^{n}\sigma_i \geq \frac{1}{2}I, \tag{3.42}$$

$$\textbf{Case 2:} \qquad \sum_{i=1}^{r_{N+1}}\sigma_i \geq \frac{1}{2}I. \tag{3.43}$$

We treat Case 1. Notice that all the operators coming on in (3.37) commute so, using also (3.38) we obtain

$$\begin{aligned}&P^Z_{r_{N+1}+1,n+1}P^G_{r_N+1,r_{N+1}}...P^G_{r_1+1,r_2}P^G_{k,r_1}Q^{(N+1)}_{N,r_1,...,r_{N+1}}\partial_\gamma f(x) \\ =& \sum_{3\leq|\alpha|\leq(N+1)(N+3)} a_n^{r_1,...,r_{N+1}}(\alpha)\theta^\alpha_{r_1,...,r_{N+1}}P^G_{r_{N-1}+1,r_N}...P^G_{r_1+1,r_2}P^G_{1,r_1}P^Z_{r_N+1,n}\partial_\gamma\partial_\alpha f(x).\end{aligned}$$

Using (3.42) and (3.32) with $m = |\gamma| + |\alpha| \leq q + (N+1)(N+3)$ we get

$$\begin{aligned}&|P^Z_{r_{N+1}+1,n+1}\partial_\gamma\partial_\alpha f(x)| \\ &\leq A_{q+(N+1)(N+3)}(L_{q+(N+1)(N+3)}(f)2^{l_{q+(N+1)(N+3)}(f)}e^{-\frac{m^2(r)}{64}\times n}+L_0 2^{l_0(f)}(f))(1+|x|)^{l_{q+(N+1)(N+3)}(f)}\end{aligned}$$

with

$$A_p = \frac{C}{r^{p+1}m(r)^{d(p+1)^2}}\times(1+C_{2l_p(f)}(Z))\times C^{2dp(p+3)}_{8dp(p+3)}(Z).$$

Therefore, we can write

$$\begin{aligned}&L_0(P^Z_{r_{N+1}+1,n}\partial_\gamma\partial_\alpha f) = A_{q+(N+1)(N+3)}(L_{q+(N+1)(N+3)}(f)2^{l_{q+(N+1)(N+3)}(f)}e^{-\frac{m^2(r)}{64}\times n}+L_0(f)2^{l_0(f)}) \\ &l_0(P^Z_{r_{N+1}+1,n}\partial_\gamma\partial_\alpha f) = l_{q+(N+1)(N+3)}(f).\end{aligned}$$

Now, in the proof of Theorem 2.2 we have proven that

$$|P^G_{r_{N-1}+1,r_N}...P^G_{r_1+1,r_2}P^G_{1,r_1}g(x)| \leq L_0(g)(1+C_{2l_0(g)})2^{l_0(g)}(1+|x|)^{l_0(g)}$$

and following the proof of Lemma 2.4 we have

$$|\theta^{\alpha}_{r_1,\dots,r_m} g(x)| \leq \big(2^{l_0(g)+1} C^{1/2}_{2(N+3)}(Z)(1+C_{2l_0(g)}(Z))^2\big)^m L_0(g)(1+|x|)^{l_0(g)}.$$

So, taking all estimates, we obtain

$$\begin{aligned} &|\theta^{\alpha}_{r_1,\dots,r_{N+1}} P^G_{r_{N-1}+1,r_N} \dots P^G_{r_1+1,r_2} P^G_{1,r_1} P^Z_{r_N+1,n+1} \partial_\gamma \partial_\alpha f(x)| \leq B_{q+(N+1)(N+3)} C^{(N+1)/2}_{2(N+3)}(Z) \\ &\quad \times (L_{q+(N+1)(N+3)}(f) 2^{l_{q+(N+1)(N+3)}(f)} e^{-\frac{m^2(r)}{64}\times n} + L_0(f)2^{l_0(f)}) \times (1+|x|)^{l_{q+(N+1)(N+3)}(f)} \end{aligned}$$

with

$$B_p = (2^{l_p(f)+1}(1+C_{2l_p(f)})^2)^{N+1} A_p 2^{l_p(f)}.$$

So,

$$\begin{aligned} &|P^Z_{r_{N+1}+1,n+1} P^G_{r_N+1,r_{N+1}} \dots P^G_{r_1+1,r_2} P^G_{1,r_1} Q^{(N+1)}_{r_1,\dots,r_{N+1}} \partial_\gamma f(x)| \\ &\leq \sum_{3\leq|\alpha|\leq(N+1)(N+3)} \left|a^{r_1,\dots,r_{N+1}}_n(\alpha)\right| B_{q+(N+1)(N+3)} C^{(N+1)/2}_{2(N+3)}(Z) \times \\ &\times (L_{q+(N+1)(N+3)}(f) 2^{l_{q+(N+1)(N+3)}(f)} e^{-\frac{m^2(r)}{64}\times n} + L_0(f)2^{l_0(f)})(1+|x|)^{l_{q+(N+1)(N+3)}(f)} \\ &\leq \frac{(C\overline{C}^{N+1}_2(Z))^{N+1}}{n^{\frac{4N+3}{2}}} B_{q+(N+1)(N+3)} C^{(N+1)/2}_{2(N+3)}(Z) \\ &\times (L_{q+(N+1)(N+3)}(f) 2^{l_{q+(N+1)(N+3)}(f)} e^{-\frac{m^2(r)}{64}\times n} + L_0(f)2^{l_0(f)})(1+|x|)^{l_{q+(N+1)(N+3)}(f)}, \end{aligned}$$

and (3.40) is proved in Case 1.
We deal now with Case 2. We write

$$\begin{aligned} &P^Z_{r_{N+1}+1,n+1} P^G_{r_N+1,r_{N+1}} \dots P^G_{r_1+1,r_2} P^G_{1,r_1} Q^{(N+1)}_{N,r_1,\dots,r_{N+1}} \partial_\gamma f(x) \\ = &\sum_{3\leq|\alpha|\leq(N+1)^2} a^{r_1,\dots,r_{N+1}}_n(\alpha) \theta^{\alpha}_{r_1,\dots,r_{N+1}} P^Z_{r_{N+1}+1,n+1} P^G_{r_N+1,r_{N+1}} \cdots P^G_{r_1+1,r_2} P^G_{1,r_1} \partial_\gamma \partial_\alpha f(x). \end{aligned}$$

Notice that

$$P^G_{r_N+1,r_{N+1}} \cdots P^G_{r_1+1,r_2} P^G_{1,r_1} \partial_\gamma \partial_\alpha f(x) = \mathbb{E}(\partial_\gamma \partial_\alpha f(x+G))$$

where G is a centred Gaussian random variable of variance $\sum_{i=1}^{r_{N+1}} \sigma_i - \sum_{i=1}^{m} \sigma_{r_i} \geq \frac{1}{4} I$. So standard integration by parts yields

$$|P^G_{r_N+1,r_{N+1}} \dots P^G_{r_1+1,r_2} P^G_{1,r_1} \partial_\gamma \partial_\alpha f(x)| \leq C L_0(f)(1+|x|)^{l_0(f)}.$$

Now the proof follows as in the previous case. So (3.40) is proved. And, summing over $r_1 < r_2 < \cdots < r_{N+1} \leq n$ we get

$$\begin{aligned} &|I_2(\partial_\gamma f)(x)| \\ &\leq n^{N+1} \times \frac{\widehat{C}_1}{n^{\frac{1}{2}(4N+3)}} (L_{q+(N+1)(N+3)}(f) 2^{l_{q+(N+1)(N+3)}(f)} e^{-\frac{m^2(r)}{64}\times n} + L_0(f)2^{l_0(f)})(1+|x|)^{l_{q+(N+1)(N+3)}(f)} \\ &\leq \frac{\widehat{C}_1}{n^{\frac{1}{2}(N+1)}} (L_{q+(N+1)(N+3)}(f) 2^{l_{q+(N+1)(N+3)}(f)} e^{-\frac{m^2(r)}{64}\times n} + L_0(f)2^{l_0(f)})(1+|x|)^{l_{q+(N+1)(N+3)}(f)}. \end{aligned}$$

Exactly as in Case 2 presented above (using standard integration by parts with respect to the law of Gaussian random variables) we obtain

$$|I_1(\partial_\gamma f)(x)| + |I_3(\partial_\gamma f)(x)| \leq \frac{\widehat{C}_1}{n^{\frac{1}{2}(N+1)}} L_0(f)(1+|x|)^{l_0(f)}.$$

So, (3.36) is proved.

Step 2. We now come back and we replace $L_{q+(N+1)(N+3)}(f)$ by $L_q(f)$ in (3.36). We will use the regularization lemma. So we fix $\delta > 0$ (to be chosen in a moment) and we write

$$|\mathbb{E}(\partial_\gamma f(S_n(Z))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \leq A_\delta(f) + A'_\delta(f) + A''_\delta(f)$$

with

$$\begin{aligned}
A_\delta(f) &= |\mathbb{E}(\partial_\gamma f_\delta(S_n(Z))) - \mathbb{E}(\partial_\gamma f_\delta(W)\Phi_N(W))| \\
A'_\delta(f) &= |\mathbb{E}(\partial_\gamma f(S_n(Z))) - \mathbb{E}(\partial_\gamma f_\delta(S_n(Z)))|, \\
A''_\delta(f) &= |\mathbb{E}(\partial_\gamma f(W)\Phi_N(W)) - \mathbb{E}(\partial_\gamma f_\delta(W)\Phi_N(W))|.
\end{aligned}$$

We will use (3.36) for f_δ. Notice that $L_m(f_\delta) \leq \hat{c}_m L_0(f)\delta^{-m}$, with $\hat{c}_m = \max_{0\leq|\alpha|\leq m} \int |\partial_\alpha \phi(x)|dx$, and $l_m(f) = l_0(f)$. So,

$$A_\delta(f) \leq \frac{C_1}{n^{\frac{1}{2}(N+1)}} L_0(f)\Big(\frac{1}{\delta^{q+(N+1)(N+3)}} e^{-\frac{m^2(r)n}{64}} + 1\Big),$$

where

$$\begin{aligned}
C_1 &= (CC_2^{\overline{N}+1}(Z))^{N+1} C_{2(N+3)}^{(N+1)/2}(Z) 2^{(N+1)(l_0(f)+1)+2}(1 + C_{2l_0(f)}(Z))^{2(N+1)+1} B_{q+(N+1)(N+3)} \\
\text{with} \quad B_p &= \frac{C}{r^{p+1}m(r)^{d(p+1)^2}} \times C_{8dp(p+3)}^{2dp(p+3)}(Z).
\end{aligned}$$

We use now (3.30) with $x = 0$ and with some h to be chosen in a moment. We then obtain

$$A'_\delta(f) \leq CC_{2l_0(f)}^{1/2}(Z)Q_{h,q}(Z)\Big(L_q(f)\exp\Big(-\frac{m^2(r)}{32} \times n\Big) + L_0(f)\delta^h\Big)$$

with $Q_{h,q}(Z)$ defined in (3.31) (In order to identify the notation from (3.31) we recall that $q = |\gamma|$ was denoted by m in (3.31) and h, which we may choose as we want, was denoted by q in (3.31)). And we also have $A''_\delta(f) \leq CL_0(f)\delta^h$ (the proof is identical to the one of (3.24) but one employs usual integration by parts with respect to the Gaussian law). We put all this together and we obtain

$$\begin{aligned}
& |\mathbb{E}(\partial_\gamma f(S_n(Z))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \\
\leq \; & \frac{C_1}{n^{\frac{1}{2}(N+1)}} L_0(f)\Big(\frac{1}{\delta^{q+(N+1)(N+3)}} e^{-\frac{m^2(r)}{64}\times n} + 1\Big) \\
& + CC_{2l_0(f)}^{1/2}(Z)Q_{h,q}(Z)\Big(L_q(f)e^{-\frac{m^2(r)}{32}\times n} + L_0(f)\delta^h\Big)
\end{aligned}$$

We take now δ such that

$$\delta^h = \frac{1}{\delta^{q+(N+1)(N+3)}} e^{-\frac{m^2(r)n}{64}}$$

so that

$$\delta^h = e^{-\frac{m^2(r)n}{64} \times \frac{h}{h+q+(N+1)(N+3)}} = e^{-\frac{m^2(r)n}{128}}$$

the last equality being obtained if we take $h=q+(N+1)(N+3)$. With this choice of h and δ we get

$$\begin{aligned}&|\mathbb{E}(\partial_\gamma f(S_n(Z)))-\mathbb{E}(\partial_\gamma f(W)\Phi_N(W))|\\ &\leq (2C_1+CC^{1/2}_{2l_0(f)}(Z)Q_{q+(N+1)(N+3),q}(Z))L_0(f)\Big(e^{-\frac{m^2(r)n}{128}}+\frac{1}{n^{\frac{1}{2}(N+1)}}\Big)\\ &+CC^{1/2}_{2l_0(f)}(Z)Q_{q+(N+1)(N+3),q}(Z)L_q(f)e^{-\frac{m^2(r)}{32}\times n}.\end{aligned}$$

We take now n sufficiently large in order to have

$$n^{\frac{1}{2}(N+1)}e^{-\frac{m^2(r)n}{128}}\leq 1.$$

The statement now follows by observing that, with $C_*(Z)$ given in (3.34),

$$\begin{aligned}2C_1+CC^{1/2}_{2l_0(f)}(Z)Q_{q+(N+1)(N+3),q}(Z)&\leq C\times C_*(Z),\\ C^{1/2}_{2l_0(f)}(Z)Q_{q+(N+1)(N+3),q}(Z)&\leq C\times C_*(Z).\end{aligned}$$

□

The result in Theorem 3.8 holds under the following slightly weaker condition (which will be used in the proof of Corollary 3.12 below).

Proposition 3.9. *Assume that for some $m<n$ one has $Y_k\in\mathcal{L}(2r,\varepsilon)$ for $k\leq n-m$ and $\sum_{k=1}^{n-m}\sigma_k\geq\frac{1}{2}I$. Then (3.33) holds true.*

Proof. The idea is that, since $\sum_{k=1}^{n-m}\sigma_k\geq\frac{1}{2}I$, the random variables $Y_k, k\leq n-m$ contain sufficient noise in order to give the regularization effect.
We show the main changes in the estimate of $I_2(f)$ (for $I_1(f)$, $I_3(f)$ the proof is analogues). We split $P^Z_{r_{N+1}+1,n}=P^Z_{r_{N+1}+1,n-m}P^Z_{n-m,n}$ and we need to have sufficient noise in order that $P^Z_{r_{N+1}+1,n-m}$ gives the regularization effect. Then, the two cases described in (3.42) and (3.43) are replaced now by $\sum_{i=r_{N+1}+1}^{n-m}\sigma_i\geq\frac{1}{4}I$ and $\sum_{i=1}^{r_{N+1}+1}\sigma_i\geq\frac{1}{4}I$ respectively. And the condition (3.41) becomes $\sum_{i=1}^{N+1}\sigma_{r_i}\leq\frac{1}{8}I$. Then the proof follows exactly the same line. □

The result in Theorem 3.8 holds without assuming the normalization condition (1.2). In fact, we can state the following

Proposition 3.10. *Let σ_n denote the covariance matrix of $S_n(Z)$ and assume it is invertible. Then (3.33) reads: for a multi-index α with $|\alpha|=q$*

$$\Big|\mathbb{E}(\partial_\alpha f(S_n(Z)))-\mathbb{E}(\partial_\alpha f(\sigma_n^{1/2}W)\Phi_N^{\sigma_n}(W))\Big|\leq C\times C_*^{\sigma_n}(Z)(\frac{1}{n^{\frac{1}{2}(N+1)}}L_0(f)+L_q(f)e^{-\frac{m^2(\overline{r})}{16}\times n})\quad(3.44)$$

where W is a standard Gaussian random variable and $\Phi_N^{\sigma_n}$ is defined in (2.10) using $\overline{Z}_k=\sigma_n^{-1/2}Z_k$ instead of Z_k (this means that the coefficients $\Delta_\alpha(k)=\mathbb{E}(Z_k^\alpha)-\mathbb{E}(G_k^\alpha)$ are replaced by $\overline{\Delta}_\alpha(k)=\mathbb{E}(\overline{Z}_k^\alpha)-\mathbb{E}(G_k^\alpha)$). And $C_*^{\sigma_n}(Z)=(\overline{\lambda}_n^{dq}/\underline{\lambda}_n^{-q})C_*(Z)$ with $C_*(Z)$ given in (3.34) and $\underline{\lambda}_n$ respectively $\overline{\lambda}_n$ the lower respectively the larger eigenvalue of σ_n. Finally $\overline{r}=r(\underline{\lambda}_n/\overline{\lambda}_n)^d$.

Proof. For a matrix $\sigma\in\mathcal{M}_{d\times d}$ and for two multi-indexes $\alpha=(\alpha_1,...,\alpha_q),\beta=(\beta_1,...,\beta_q)$ we denote $\sigma^{\alpha,\beta}=\prod_{i=1}^q\sigma^{\alpha_i,\beta_i}$. Suppose that σ is invertible and let $\gamma=\sigma^{-1}$. For a function $f=\mathbb{R}^d\to\mathbb{R}$ let $f_a(x)=f(ax)$. A simple computation shows that

$$(\partial_\alpha f)(\sigma x)=\sum_{|\beta|=|\alpha|}\gamma^{\alpha,\beta}\partial_\beta f_\sigma(x).$$

We denote now $\overline{Z}_k = \sigma_n^{-1/2} Z_k$ and we note that $S_n(\overline{Z}) = \sigma_n^{-1/2} S_n(Z)$ verifies the nomalization condition (1.2). So using (3.33) for $S_n(\overline{Z})$ we obtain

$$\begin{aligned}\mathbb{E}(\partial_\alpha f(S_n(Z))) &= \mathbb{E}(\partial_\alpha f(\sigma_n^{1/2} S_n(\overline{Z}))) = \sum_{|\beta|=q} (\sigma_n^{-1/2})^{\alpha,\beta} \mathbb{E}(\partial_\beta f_{\sigma_n^{1/2}}(S_n(\overline{Z})) \\ &= \sum_{|\beta|=q} (\sigma_n^{-1/2})^{\alpha,\beta} (\mathbb{E}(\partial_\beta f_{\sigma_n^{1/2}}(W)\Phi_N^{\sigma_n}(W)) + R_N^\beta(n)) \\ &= \mathbb{E}(\partial_\alpha f(\sigma_n^{1/2} W)\Phi_N^{\sigma_n}(W)) + \sum_{|\beta|=q} (\sigma_n^{-1/2})^{\alpha,\beta} R_N^\beta(n).\end{aligned}$$

The estimate of $R_N(n)$ follows from the fact that $L_q(f_{\sigma_n^{1/2}}) \leq \overline{\lambda}_n^q L_q(f)$ and $\sum_{|\beta|=q} (\sigma_n^{-1/2})^{\alpha,\beta} \leq C \underline{\lambda}_n^{-q} \overline{\lambda}_n^{dq}$. □

Another immediate consequence of Theorem 3.8 is given by the following estimate for an "approximative density" of the law of $S_n(Z)$:

Corollary 3.11. *Suppose that $n^{(N+1)(\frac{1}{2}+\frac{1}{2d})} e^{-\frac{m^2(r)}{32d}\times n} \leq 1$. Let δ_n be such that*

$$n^{(N+1)/2d} e^{-\frac{m^2(r)}{32d}\times n} \leq \delta_n \leq \frac{1}{n^{\frac{1}{2}(N+1)}}.$$

Then

$$\left| \mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|S_n(Z)-a| \leq \delta_n\}}\Big) - \gamma_d(a)\Phi_N(a) \right| \leq \frac{C}{n^{\frac{1}{2}(N+1)}}. \tag{3.45}$$

Here γ_d is the density of the standard normal law in $\mathbb{R}^d$.

Proof. Let $h(x) = \int_{-\infty}^{x_1} dx_1 ... \int_{-\infty}^{x_{d-1}} \frac{1}{\delta_n^d} 1_{\{|x-a| \leq \delta_n\}} dx_d$ so that $\frac{1}{\delta_n^d} 1_{\{|x-a| \leq \delta_n\}} = \partial_{x_1} ... \partial_{x_d} h(x)$. Using Theorem 3.8

$$\begin{aligned}\mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|S_n(Z)-a| \leq \delta_n\}}\Big) &= \mathbb{E}(\partial_{x_1} ... \partial_{x_d} h(S_n(Z))) = \mathbb{E}(\partial_{x_1} ... \partial_{x_d} h(W)\Phi_N(W)) + R_N(n) \\ &= \mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|W-a| \leq \delta_n\}} \Phi_N(W)\Big) + R_N(n)\end{aligned}$$

with

$$|R_N(n)| \leq C\Big(\frac{1}{n^{\frac{1}{2}(N+1)}} + \frac{1}{\delta_n^d} e^{-\frac{m^2(r)}{16}\times n}\Big) \leq \frac{C}{n^{\frac{1}{2}(N+1)}}$$

the last inequality being true by our choice of δ_n. Moreover

$$\begin{aligned}\mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|W-a| \leq \delta_n\}}\Big)\Phi_N(W)) &= \int_{\mathbb{R}^d} \frac{1}{\delta_n^d} 1_{\{|y-a| \leq \delta_n\}})\Phi_N(y)\gamma_d(y)dy \\ &= \Phi_N(a)\gamma_d(a) + R'(n)\end{aligned}$$

with $|R'(n)| \leq \frac{C}{n^{\frac{1}{2}(N+1)}}$ the last inequality being again a consequence of the choice of δ_n. □

We now prove a stronger version of Prohorov's theorem. We consider a sequence of identical distributed, centred random variables $X_k \in \mathbb{R}^d$ which have finite moments of any order and we look to

$$S_n(X) = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k.$$

Following Porhorov we assume that there exist $m \in \mathbb{N}$ such that

$$\mathbb{P}(X_1 + ... + X_m \in dx) = \mu(dx) + \psi(x)dx \tag{3.46}$$

for some measurable non negative function ψ.

Corollary 3.12. *We assume that (3.46) holds. We fix $q, N \in \mathbb{N}$. There exists two constants $0 < c_* \leq 1 \leq C_*$ (depending on N and q) such that the following holds: if*

$$n^{\frac{1}{2}(N+1)} e^{-c_* n} \leq 1$$

then, for every multi-index γ with $|\gamma| \leq q$ and foe every $f \in C_p^q(\mathbb{R}^d)$ one has

$$|\mathbb{E}(\partial_\gamma f(S_n(X))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \leq C_* \Big(\frac{1}{n^{\frac{1}{2}(N+1)}} L_0(f) + L_q(f) e^{-c_* \times n} \Big) \tag{3.47}$$

Proof. We denote

$$Y_k = \sum_{i=2km+1}^{2(k+1)m} X_i \quad \text{and} \quad Z_k = \frac{1}{\sqrt{n}} Y_k.$$

Notice that we may take ψ in (3.46) to be bounded with compact support. Then $\psi * \psi$ is continuous and so we may find some $r > 0, \varepsilon > 0$ and $y \in \mathbb{R}^d$ such that $\psi * \psi \geq \varepsilon 1_{B_r(y)}$. It follows that $Y_k \in \mathcal{L}(2r, \varepsilon)$ and we may use the previous theorem in order to obtain (3.47) for $n = 2m \times n'$ with $n' \in \mathbb{N}$. But this is not satisfactory because we claim that (3.47) holds for every $n \in \mathbb{N}$. This does not follow directly but needs to come back to the proof of Theorem 3.8 and to adapt it in the following way. Suppose that $2mn' \leq n < 2m(n'+1)$. Then

$$S_n(X) = S_{2mn'}(X) + \frac{1}{\sqrt{n}} \sum_{k=2mn'+1}^{n} X_k = \frac{1}{\sqrt{n}} \sum_{k=1}^{n'} Y_k + \frac{1}{\sqrt{n}} \sum_{k=2mn'+1}^{n} X_k.$$

Since $X_k, 2mn'+1 \leq k \leq n$ have no regularity property, we may not use them in the regularization arguments employed in the proof of Theorem 3.8. But $Y_k, 1 \leq k \leq n'$ contain sufficient noise in order to achieve the proof (see Remark 3.9). $\square$

4 Examples

4.1 An invariance principle related to the local time

In this section we consider a sequence of independent identically distributed, centred random variables Y_k, $k \in \mathbb{N}$, with finite moments of any order and we denote

$$S_n(k, Y) = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} Y_i.$$

Our aim is to study the asymptotic behaviour of the expectation of

$$L_n(Y) = \frac{1}{n} \sum_{k=1}^{n} \psi_{\varepsilon_n}(S_n(k, Y)) \quad \text{with} \quad \psi_{\varepsilon_n}(x) = \frac{1}{2\varepsilon_n} 1_{\{|x| \leq \varepsilon_n\}}.$$

So $L_n(Y)$ appears as the occupation time of the random walk $S_n(k, Y), k = 1, ..., n$, and consequently, as $\varepsilon_n \to 0$, one expects that it has to be close to the local time in zero at time 1, denoted by l_1, of the Brownian motion. In fact, we prove now that $\mathbb{E}(L_n(Y)) \to \mathbb{E}(l_1)$ as $n \to \infty$.

Theorem 4.1. *Let* $\varepsilon_n = n^{-\frac{1}{2}(1-\rho)}$ *with* $\rho \in (0,1)$. *We consider a centred random variable* $Y \in \mathcal{L}(r,\varepsilon)$ *which has finite moments of any order and we take a sequence* $Y_i, i \in \mathbb{N}$ *of independent copies of* Y. *We define*

$$N(Y) = \max\{2k : \mathbb{E}(Y^{2k}) = \mathbb{E}(G^{2k})\} - 1 \geq 1$$

and we denote $p_{N(Y)} = 8(1 + (N(Y)+1)(N(Y)+3))(4 + (N(Y)+1)(N(Y)+3))$. *For every* $\eta < 1$ *there exists a constant* C *depending on* $r, \varepsilon, \rho, \eta$ *and on* $\|Y\|_{p_{N(Y)}}$ *such that*

$$\big|\mathbb{E}(L_n(Y)) - \mathbb{E}(L_n(G))\big| \leq \frac{C}{n^{\frac{1}{2}+\frac{\eta\rho N(Y)}{2}}}. \tag{4.1}$$

The above inequality holds for n *which is sufficiently large in order to have*

$$n^{\frac{1}{2}} \exp\Big(-\frac{m^2(r)}{32} \times n^{\rho\eta}\Big) \leq \frac{1}{n^{\frac{1}{2}(N(Y)+1)\eta\rho}} \tag{4.2}$$

As a consequence, we have

$$\lim_{n\to\infty} \mathbb{E}(L_n(Y)) = \mathbb{E}(l_1), \tag{4.3}$$

l_1 *denoting the local time in the point* 0 *at time 1 of a Brownian motion.*

Proof. All over this proof we denote by C a constant which depends on $r, \varepsilon, \rho, \eta$ and on $\|Y\|_{p_{N(Y)}}$ (as in the statement of the lemma) and which may change from a line to another.

Step 1. We take $k_n = n^{\eta\rho}$. Suppose first that $k \leq k_n$. We write

$$\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y))) = \frac{1}{\varepsilon_n}\Big(1 - \mathbb{P}(|S_n(k,Y)| \geq \varepsilon_n)\Big)$$

so that

$$|\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y))) - \mathbb{E}(\psi_{\varepsilon_n}(S_n(k,G)))| \leq \frac{1}{\varepsilon_n}\big(\mathbb{P}(|S_n(k,Y)| \geq \varepsilon_n) + \mathbb{P}(|S_n(k,G)| \geq \varepsilon_n)\big).$$

Using Chebyshev's inequality and Burkholder's inequality we obtain for every $p \geq 2$

$$\begin{aligned}\mathbb{P}(|S_n(k,Y)| \geq \varepsilon_n) &= \mathbb{P}\Big(\Big|\sum_{i=1}^{k} Y_i\Big| \geq \varepsilon_n\sqrt{n}\Big) \leq \frac{1}{(\varepsilon_n\sqrt{n})^p}\mathbb{E}\Big(\Big|\sum_{i=1}^{k} Y_i\Big|^p\Big) \\ &\leq \frac{C}{(\varepsilon_n\sqrt{n})^p}\Big(\sum_{i=1}^{k}\|Y_i\|_p^2\Big)^{p/2} \leq \frac{Ck^{p/2}}{(\varepsilon_n\sqrt{n})^p} = \frac{C}{\varepsilon_n^p} \times \Big(\frac{k}{n}\Big)^{p/2}.\end{aligned}$$

And the same estimate holds with Y_i replaced by G_i. We conclude that

$$\begin{aligned}&\Big|\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,Y))\Big) - \mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,G))\Big)\Big| \leq \frac{C}{\varepsilon_n^{p+1}} \times \frac{1}{n}\sum_{k=1}^{k_n}\Big(\frac{k}{n}\Big)^{p/2} \\ &\leq \frac{C}{\varepsilon_n^{p+1}} \times \int_0^{k_n/n} x^{p/2}dx = \frac{C}{\varepsilon_n^{p+1}} \times \Big(\frac{k_n}{n}\Big)^{\frac{p}{2}+1} \\ &= \frac{C}{n^{\frac{p\rho}{2}(1-\eta)+\frac{1}{2}-(\eta-\frac{1}{2})\rho}} \leq \frac{C}{n^{\frac{p\rho}{2}(1-\eta)}}.\end{aligned}$$

We take $p=\frac{1+\rho\eta N(Y)}{\rho(1-\eta)}$ and we obtain

$$\Big|\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,Y))\Big)-\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,G))\Big)\Big|\leq\frac{C}{n^{\frac{1}{2}+\frac{N(Y)}{2}\eta\rho}}.$$

Step 2. We fix now $k\geq k_n$ and we apply our Edgeworth development (see (3.33)) to

$$\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_i=\sum_{i=1}^{k}Z_i,\quad\text{where}\quad Z_i=\frac{1}{k^{1/2}}Y_i.$$

In particular the constants $C_p(Z)$ defined in (2.4) are given by $C_p(Z)=k^{p/2}\max_i\mathbb{E}(|Z_i|^p)=\|Y\|_p^p$. We denote

$$h_{\alpha,n}(x)=\int_{-\infty}^{\alpha x}\psi_{\varepsilon_n}(y)dy=h_{1,n}(\alpha x).\tag{4.4}$$

This gives $\psi_{\varepsilon_n}(x)=h'_{1,n}(x)$ and $h'_{\alpha,n}(x)=\alpha h'_{1,n}(\alpha x)$. Moreover, $\|h_{\alpha,n}\|_\infty\leq 1$ and $\|h'_{\alpha,n}\|_\infty\leq|\alpha|/\varepsilon_n$, so that

$$L_0(h_{\alpha,n})=1\quad\text{and}\quad L_1(h_{\alpha,n})=|\alpha|\times\frac{1}{\varepsilon_n}.$$

We now write

$$\begin{aligned}\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y)))&=\mathbb{E}(h'_{1,n}(S_n(k,Y)))=\mathbb{E}\Big(h'_{1,n}\Big(\sqrt{\frac{k}{n}}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_i\Big)\Big)\\&=\sqrt{\frac{n}{k}}\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}\Big(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_i\Big)\Big).\end{aligned}$$

We will use (3.33) with $f=h_{\sqrt{\frac{k}{n}},n}$ and ∂_γ will be the first order derivative. Then, by (3.33) with $N=N(Y)$

$$\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y))=\sqrt{\frac{n}{k}}\Big(\mathbb{E}\big(h'_{\sqrt{\frac{k}{n}},n}(W_1)\Phi_{N(Y)}(W_1)\big)+R_{N(Y)}(k)\Big)$$

with

$$\begin{aligned}\big|R_{N(Y)}(k)\big|&\leq\frac{C}{k^{(N(Y)+1)/2}}L_0(h_{\sqrt{\frac{k}{n}},n})+CL_1(h'_{\sqrt{\frac{k}{n}},n})\exp\Big(-\frac{m^2(r)}{32}\times k\Big)\\&\leq\frac{C}{k^{(N(Y)+1)/2}}+C\sqrt{\frac{k}{n}}\times\frac{1}{\varepsilon_n}\exp\Big(-\frac{m^2(r)}{32}\times k\Big).\end{aligned}$$

Here C is the constant from (3.33) defined in (3.34). Notice that by (4.2), for $k\geq k_n=n^{\eta\rho}$ one has

$$\begin{aligned}\sqrt{\frac{k}{n}}\times\frac{1}{\varepsilon_n}\exp\Big(-\frac{m^2(r)}{32}\times k\Big)&\leq n^{\frac{1}{2}}\exp\Big(-\frac{m^2(r)}{32}\times n^{\rho\eta}\Big)\\&\leq\frac{1}{n^{\frac{1}{2}(N(Y)+1)\eta\rho}}=\frac{1}{k_n^{(N(Y)+1)/2}}\leq\frac{C}{k^{(N(Y)+1)/2}},\end{aligned}$$

so that $\left|R_{N(Y)}(k)\right| \leq Ck^{-(N(Y)+1)/2}$. Then

$$\begin{aligned}
\Big|\sum_{k=k_n}^{n}\sqrt{\frac{n}{k}}R_{N(Y)}(k)\frac{1}{n}\Big| &\leq \frac{C}{n^{(N(Y)+1)/2}}\sum_{k=k_n}^{n}\frac{1}{(k/n)^{1+\frac{N(Y)}{2}}}\times\frac{1}{n} \\
&\leq \frac{C}{n^{(N(Y)+1)/2}}\int_{k_n/n}^{1}\frac{ds}{s^{1+\frac{N(Y)}{2}}} \\
&= \frac{C}{n^{(N(Y)+1)/2}}(n/k_n)^{\frac{N(Y)}{2}} = \frac{C}{n^{\frac{1}{2}+\frac{N(Y)\rho\eta}{2}}}.
\end{aligned}$$

We recall now that (see (2.15))

$$\Phi_{N(Y)}(x) = 1 + \sum_{l=1}^{N(Y)} H_{\Gamma_l}(x)$$

with $H_{\Gamma_l}(x)$ linear combinations of Hermite polynomials (see (2.10) and (2.14)). Notice that if l is odd then Γ_l is a linear combination of differential operators of odd order (see the definition of $\Lambda_{m,l}$ in (2.9)). So H_{Γ_l} is an odd function (as a linear combination of Hermite polynomials of odd order) so that $\psi_{\varepsilon_n} \times H_{\Gamma_l}$ is also an odd function. Since W_1 and $-W_1$ have the same law, it follows that

$$\begin{aligned}
\mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)H_{\Gamma_l}(W_1)\Big) &= \mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times(-W_1)\Big)H_{\Gamma_l}(-W_1)\Big) \\
&= -\mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)H_{\Gamma_l}(W_1)\Big)
\end{aligned}$$

and consequently

$$\sqrt{\frac{n}{k}}\times\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}(W_1)H_{\Gamma_l}(W_1)\Big) = \mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)H_{\Gamma_l}(W_1)\Big) = 0.$$

Moreover, by the definition of $N(Y)$, for $2l \leq N(Y)$ we have $\mathbb{E}(Y^{2l}) = \mathbb{E}(G^{2l})$ so that $H_{\Gamma_{2l}} = 0$. We conclude that

$$\sqrt{\frac{n}{k}}\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}(W_1)\Phi_{N(Y)}(W_1)\Big) = \sqrt{\frac{n}{k}}\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}(W_1)\Big) = \mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)\Big) = \mathbb{E}(\psi_{\varepsilon_n}(S_n(k,G))).$$

We put now together the results from the first and the second step and we obtain (4.1).

Step 3. We prove (4.3). Recall first the representation formula

$$\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) = \mathbb{E}\Big(\int\psi_{\varepsilon_n}(a)l_1^a da\Big),$$

where l_1^a denotes the local time in $a \in \mathbb{R}$ at time 1, so that $l_1 = l_1^0$. Since $a \mapsto l_1^a$ is Hölder continuous of order $\frac{\rho'}{2}$ for every $\rho' < 1$, we obtain

$$\Big|\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) - \mathbb{E}(l_1^0)\Big| \leq \varepsilon_n^{\rho'/2} = \frac{1}{n^{\frac{\rho'(1-\rho)}{4}}}. \tag{4.5}$$

We prove now that, for every $\rho' < 1$ and n large enough,

$$\Big|\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) - \mathbb{E}(L_n(G))\Big| \leq \frac{C}{n^{\frac{1+\eta\rho}{2}}}. \tag{4.6}$$

To begin we notice that $S_n(k,G)$ has the same law as $W_{k/n}$, so that we write

$$\mathbb{E}\Big(\int_0^1 \psi_{\varepsilon_n}(W_s)ds\Big) - \mathbb{E}(L_n(G)) = \mathbb{E}\Big(\sum_{k=1}^n \delta_k\Big)$$

with

$$\delta_k = \int_{k/n}^{(k+1)/n} (\psi_{\varepsilon_n}(W_s) - \psi_{\varepsilon_n}(W_{k/n}))ds.$$

As above, we take $k_n = n^{\rho\eta}$ and for $k \le k_n$, we have

$$\mathbb{E}(\delta_k) = -\frac{1}{2\varepsilon_n}\int_{k/n}^{(k+1)/n} \big(\mathbb{P}(|W_s| \ge \varepsilon_n) - \mathbb{P}(|W_{k/n}| \ge \varepsilon_n)\big)ds.$$

Since $\mathbb{P}(|W_s| \ge \varepsilon_n) \le C\exp(-\frac{\varepsilon^2}{2s})$, this immediately gives

$$|\mathbb{E}(\delta_k)| \le \frac{C}{n\varepsilon_n}\exp\Big(-\frac{1}{2}\varepsilon_n^2 \times \frac{n}{k+1}\Big) \le \frac{C}{n\varepsilon_n}\exp\Big(-\frac{1}{2}\varepsilon_n^2 \times \frac{n}{k_n+1}\Big) = \frac{C}{n\varepsilon_n}\exp\Big(-\frac{1}{2}n^{\rho(1-\eta)}\Big)$$

so that

$$\sum_{k=1}^{k_n} |\mathbb{E}(\delta_k)| \le \frac{C}{\varepsilon_n}\exp(-\frac{1}{8}n^{\rho(1-\eta)}) \le \frac{C}{n^{\frac{1+\eta\rho}{2}}},$$

for n large enough.
We consider now the case $k \ge k_n$. Using a formal computation, by applying the standard Gaussian integration by parts formula, we write

$$\begin{aligned}\mathbb{E}(\psi_{\varepsilon_n}(W_s) - \psi_{\varepsilon_n}(W_{k/n})) &= \frac{1}{2}\int_{k/n}^s \mathbb{E}(\psi''_{\varepsilon_n}(W_v))dv = \frac{1}{2}\int_{k/n}^s \mathbb{E}(\psi''_{\varepsilon_n}(\sqrt{v}W_1))dv \\ &= \int_{k/n}^s \mathbb{E}(h'''_{1,n}(\sqrt{v}W_1)H_3(W_1))dv = \int_{k/n}^s \frac{1}{2v^{3/2}}\mathbb{E}(h_{1,n}(\sqrt{v}W_1)H_3(W_1))dv,\end{aligned}$$

in which we have used (4.4) and where H_3 denotes the third Hermite polynomial. The above computation is formal because ψ_{ε_n} is not differentiable. But, since the first and the last term in the chain of equalities depends on ψ_{ε_n} only (and not on the derivatives) we may use regularization by convolution in order to do it rigorously. Notice also that the first equality is obtained using Ito's formula and the last one is obtained using integration by parts. It follows that

$$|\mathbb{E}(\delta_k)| \le \int_{k/n}^{(k+1)/n} ds \int_{k/n}^s \frac{1}{2v^{3/2}}\mathbb{E}(h_{1,\varepsilon_n}(\sqrt{v}W_1)\,|H_3(W_1)|)dv \le \frac{C}{n}\int_{k/n}^{(k+1)/n} \frac{1}{v^{3/2}}dv$$

and consequently

$$\sum_{k=k_n}^{n} |\mathbb{E}(\delta_k)| \le \frac{C}{n}\int_{k_n/n}^1 \frac{1}{v^{3/2}}dv \le \frac{C}{n^{\frac{1+\eta\rho}{2}}}.$$

So (4.6) is proved, and this together with (4.5) and (4.1), give (4.3). □

4.2 Convergence in distribution norms

In this section we prove that, under some supplementary regularity assumptions on the laws of Z_k, $k \in \mathbb{N}$, Theorem 3.8 implies that the density of the law of $S_n(Z)$ converges in distribution norms to the Gaussian density. We write

$$Z_k = \frac{C_k}{\sqrt{n}} Y_k$$

and we denote $\sigma_k = C_k C_k^*$. We assume that

$$0 < \underline{\sigma} \le \sigma_k \le \overline{\sigma} < \infty, \quad \text{and} \quad \sup_k \|Y_k\|_p^p < \infty. \tag{4.7}$$

In particular each σ_k is invertible. We denote $\gamma_k = \sigma_k^{-1}$. Notice that the normalization condition is

$$\frac{1}{n}\sum_{k=1}^n \mathbb{E}(Z_k^i Z_k^j) = 1_{i=j}.$$

For a function $f \in C^1(\mathbb{R}^d)$ and for $k \in \mathbb{N}$ we denote

$$m_{1,k}(f) = \int_{\mathbb{R}^d} (1+|x|)^k \, |\nabla f(x)| \, dx.$$

Proposition 4.2. ***A***. *We fix $q \in \mathbb{N}$ and we also fix a polynomial P. Suppose that $Y_i \in \mathcal{L}(r,\varepsilon), i \in \mathbb{N}$ and (4.7) holds. Moreover we suppose that*

$$\mathbb{P}(Y_i \in dy) = p_{Y_i}(y)dy \quad \textit{with} \quad p_{Y_i} \in C^1(\mathbb{R}^d) \quad \textit{for every for } i = 1, ..., q. \tag{4.8}$$

There exist some constants $c \in (0,1)$ (depending on r and on ε) and $C_q(P) \ge 1$ (depending on $q, \underline{\sigma}, \overline{\sigma}$ and on P) such that, if $n^{(q+1)/2} e^{-cn} \le 1$, then, for every $f \in C_p^q(\mathbb{R}^d)$, and every multi-index α with $|\alpha| \le q$

$$|\mathbb{E}(P(S_n(Z))\partial_\alpha f(S_n(Z)) - \mathbb{E}(P(S_n(G))\partial_\alpha f(S_n(G))| \le \frac{C_q(P)}{\sqrt{n}} \prod_{i=1}^q m_{1,l_0(f)+l_0(P)}(p_{Y_i}) \times L_0(f). \tag{4.9}$$

B. *Moreover, if p_{S_n} is the density of the law of $S_n(Z)$ then, if $n^{(d+q+1)/2} e^{-cn} \le 1$, we have*

$$\sup_{x \in \mathbb{R}^d} |P(x)(\partial_\alpha p_{S_n}(x) - \partial_\alpha \gamma(x))| \le \frac{C_{q+d}(P)}{\sqrt{n}} \prod_{i=1}^{q+d} m_{1,l_0(f)+l_0(P)}(p_{Y_i}) \tag{4.10}$$

where γ is the density of the standard normal law in $\mathbb{R}^d$.

Proof A. We proceed by recurence on the degree k of the polynomial P. First we assume that $k = 0$ (so that P is a constant) and we prove (4.9) for every $q \in \mathbb{N}$. We write

$$S_n(Z) = \frac{1}{\sqrt{n}}\sum_{i=1}^n C_i Y_i = \frac{1}{\sqrt{n}}\sum_{i=1}^q C_i Y_i + S_n^{(q)}(Z).$$

with

$$S_n^{(q)}(Z) = \frac{1}{\sqrt{n}}\sum_{i=q+1}^n C_i Y_i.$$

Then we define

$$g(x)=\mathbb{E}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{q}C_iY_i+x\Big)\Big)$$

and we have

$$\mathbb{E}(\partial_\alpha f(S_n(Z))=\mathbb{E}(\partial_\alpha g(S_n^{(q)}(Z))).$$

Now using (3.44) with $N=0$ for $S_n^{(q)}(Z)$ we get

$$\mathbb{E}(\partial_\alpha g(S_n^{(q)}(Z)))=\mathbb{E}(\partial_\alpha g(S_n^{(q)}(G)))+R_n=\mathbb{E}\Big(\partial_\alpha f\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{q}C_iY_i+S_n^{(q)}(G)\Big)\Big)+R_n \tag{4.11}$$

with

$$|R_n|\le C\Big(\frac{1}{\sqrt{n}}L_0(g)+e^{-cn}L_q(g)\Big). \tag{4.12}$$

Let us estimate $L_q(g)$. We recall that $\gamma_i=\sigma_i^{-1}$. For $\alpha=(\alpha_1,...,\alpha_q)$ we have

$$(\partial_\alpha f)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}C_iy_i+x\Big)=\sum_{\beta_1,...,\beta_q=1}^{d}n^{q/2}\Big(\prod_{i=1}^{q}(\gamma_iC_i)^{\alpha_i,\beta_i}\Big)\times\partial_{y_1^{\beta_1}}....\partial_{y_q^{\beta_q}}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}C_iy_i+x\Big)\Big), \tag{4.13}$$

in which we have assumed that the Y_i's take values in $\mathbb{R}^m$. So

$$\begin{aligned}
\partial_\alpha g(x)&=\mathbb{E}\Big((\partial_\alpha f)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{q}C_iY_i+x\Big)\Big)\\
&=n^{q/2}\sum_{\beta_1,...,\beta_q=1}^{m}\Big(\prod_{i=1}^{q}(\gamma_iC_i)^{\alpha_i,\beta_i}\Big)\int_{\mathbb{R}^{qm}}\partial_{y_1^{\beta_1}}....\partial_{y_q^{\beta_q}}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}C_iy_i+x\Big)\Big)\prod_{i=1}^{q}p_{Y_i}(y_i)dy_1...dy_q\\
&=(-1)^q n^{q/2}\sum_{\beta_1,...,\beta_q=1}^{m}\Big(\prod_{i=1}^{q}(\gamma_iC_i)^{\alpha_i,\beta_i}\Big)\int_{\mathbb{R}^{qm}}f\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}C_iy_i+x\Big)\prod_{i=1}^{q}\partial_{y_i^{\beta_i}}p_{Y_i}(y_i)dy_1...dy_q.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\partial_\alpha g(x)| &\le Cn^{q/2}L_0(f)\int_{\mathbb{R}^q}(1+|x|+\sum_{i=1}^{q}|y_i|)^{l_0(f)}\prod_{i=1}^{q}|\nabla p_{Y_i}(y_i)|\,dy_1...dy_q\\
&\le Cn^{q/2}L_0(f)(1+|x|)^{l_0(f)}\prod_{i=1}^{q}m_{1,l_0(f)}(p_{Y_i}).
\end{aligned}$$

We conclude that $l_q(g)=l_0(f)$ and $L_q(g)\le Cn^{q/2}L_0(f)\prod_{i=1}^{q}m_{1,l_0(f)}(p_{Y_i})$. The same is true for $q=0$ and so (4.12) gives

$$|R_n|\le CL_0(f)\prod_{i=1}^{q}m_{1,l_0(f)}(p_{Y_i})\Big(\frac{1}{\sqrt{n}}+n^{q/2}e^{-cn}\Big)\le CL_0(f)\prod_{i=1}^{q}m_{1,l_0(f)}(p_{Y_i})\times\frac{1}{\sqrt{n}}$$

the last inequality being true if $n^{q/2}e^{-cn}\le n^{-1/2}$.
So (4.11) says that we succeed to replace $Y_i, q+1\le i\le n$ by $G_i, q+1\le i\le n$ and the price to be paid is $CL_0(f)\prod_{i=1}^{q}m_{1,l_0(f)}(p_{Y_i})\times\frac{1}{\sqrt{n}}$. Now we can do the same thing and replace $Y_i, 1\le i\le q$ by $G_i, 1\le i\le q$ and the price will be the same (here we use $C_iG_i, i=q+1,...,2q$ instead of $C_iY_i, i=1,...,q$). So (4.9) is proved for polynomials P of degree $k=0$.

We assume now that (4.9) holds for every polynomials of degree less or equal to $k-1$ and we prove it for a polynomial P of order k. We have

$$\partial_\alpha(P \times f) = \sum_{(\beta,\gamma)=\alpha} \partial_\beta P \times \partial_\gamma f$$

so that

$$P \times \partial_\alpha f = \partial_\alpha(P \times f) - \sum_{\substack{(\beta,\gamma)=\alpha \\ |\beta|\geq 1}} \partial_\beta P_k \times \partial_\gamma f.$$

Since $|\beta| \geq 1$ the polynomial $\partial_\beta P$ has degree at most $k-1$. Then the recurrence hypothesis ensures that (4.9) holds for $\partial_\beta P \times \partial_\gamma f$. Moreover, using again (4.9) for $g = P \times f$ we obtain (4.9) in which $L_0(g) \leq L_0(P)L_0(f)$ and $l_0(g) \leq l_0(P) + l_0(f)$ appear. So **A.** is proved.

Let us prove **B**. We denote $f_x(y) = \prod_{i=1}^d 1_{(x,\infty)}(y)$ and, for a multi-index $\alpha = (\alpha_1, ..., \alpha_q)$ we denote $\overline{\alpha} = (\alpha_1, ..., \alpha_q, 1, ..., d)$. Then, using a formal computation (which may de done rigourously by means of a regularization procedure) we obtain

$$\begin{aligned}
P(x)\partial_\alpha p_{S_n}(x) &= \int \delta_0(y-x)P(y)\partial_\alpha p_{S_n}(y)dy \\
&= (-1)^q \sum_{(\beta,\gamma)=\alpha} \int \partial_\beta \delta_0(y-x)\partial_\gamma P(y)p_{S_n}(y)dy \\
&= (-1)^q \sum_{(\beta,\gamma)=\alpha} \int \partial_{\overline{\beta}} f_x(y)\partial_\gamma P(y)p_{S_n}(y)dy \\
&= (-1)^q \sum_{(\beta,\gamma)=\alpha} \mathbb{E}(\partial_{\overline{\beta}} f_x(S_n(Z))\partial_\gamma P(S_n(Z))).
\end{aligned}$$

A similar computation holds with $S_n(Z)$ replaced by $S_n(G)$. So we have

$$\begin{aligned}
& |P(x)(\partial_\alpha p_{S_n}(x) - \partial_\alpha \gamma(x)| \\
\leq & \sum_{(\beta,\gamma)=\alpha} \left| \mathbb{E}(\partial_{\overline{\beta}} f_x(S_n(Z))\partial_\gamma P(S_n(Z))) - \mathbb{E}(\partial_{\overline{\beta}} f_x(S_n(G))\partial_\gamma P(S_n(G))) \right| \\
\leq & \frac{C_{q+d}(P)}{\sqrt{n}} \prod_{i=1}^{q+d} m_{1,l_0(f)+l_0(P)}(p_{Y_i})
\end{aligned}$$

the last inequality being a consequence of (4.9). $\square$

Remark 4.3. *We would like to obtain Edgeworth expansions as well – but there is a difficulty: when we use the expansion for $S_n^{(q)}(Z)$ we are in the situation when the covariance matrix of $S_n^{(q)}(Z)$ is not the identity matrix. So the coefficients of the expansion are computed using a correction (see the definition of $\overline{\Delta}_k$ in the Remark 3.10). And this correction produces an error of order $n^{-1/2}$. This means that we are not able to go beyond this level (at least without supplementary technical effort).*

A Computation of the first three coefficients

We explicitly write the expression of Γ_k for $k = 1, 2, 3$ (for larger values of k the term Γ_k is difficult to explicitly compute). Recall formulas (2.10) for Γ_k and formula (2.9) for the set $\Lambda_{m,k}$ appearing in (2.10).

Case $k=1$. Then $m=1$ and $l=3, l'=0$. So the first order terms are given by

$$\Gamma_1 = \sum_{r=1}^n \frac{1}{6} D_r^{(3)} = \frac{1}{6} \sum_{|\alpha|=3} \sum_{r=1}^n \Delta_\alpha(r) \partial_\alpha.$$

Case $k=2$. Then $m=1$ or $m=2$. Suppose first that $m=1$. Then we need that $l+2l' = k+2m = 4$. This means that we have $l=4, l'=0$. The corresponding term is

$$\frac{1}{24} \sum_{r=1}^n D_r^{(4)} = \frac{1}{24} \sum_{|\alpha|=4} \sum_{r=1}^n \Delta_\alpha(r)\, \partial_\alpha.$$

Suppose now that $m=2$. Then we need that $l_1+l_2+2(l_1'+l_2') = k+2m = 6$. The only possibility is $l_1=l_2=3, l_1'=l_2'=0$ and the corresponding term is

$$\frac{1}{36} \sum_{0\le r_1<r_2\le n} D_{r_1}^{(3)} D_{r_2}^{(3)} = \frac{1}{36} \sum_{|\alpha|=3} \sum_{|\beta|=3} \sum_{0\le r_1<r_2\le n} \Delta_\alpha(r_1)\Delta_\beta(r_2)\partial_\alpha\partial_\beta.$$

We conclude that

$$\Gamma_2 = \frac{1}{24} \sum_{|\alpha|=4} \sum_{r=1}^n \Delta_\alpha(r)\, \partial_\alpha + \frac{1}{36} \sum_{|\alpha|=3} \sum_{|\beta|=3} \sum_{0\le r_1<r_2\le n} \Delta_\alpha(r_1)\Delta_\beta(r_2)\partial_\alpha\partial_\beta.$$

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$$\Gamma_2 = \frac{1}{24} \sum_{|\alpha|=4} n \times \frac{1}{n^2} \Delta_\alpha \partial_\alpha + \frac{1}{36} \sum_{|\alpha|=3} \sum_{|\beta|=3} \frac{n^2-n}{2} \times \frac{1}{n^3} \Delta_\alpha\Delta_\beta\partial_\alpha\partial_\beta.$$

Case $k=3$. $m=1$. We need that $l+2l' = k+2m = 5$. So $l=3, l'=1$ or $l=5, l'=0$. The term coresponding to $l=3, l'=1$ is

$$\begin{aligned} -\frac{1}{12} \sum_{r=1}^n D_r^{(3)} L^1_{\sigma_r} &= -\frac{1}{12} \sum_{r=1}^n \sum_{|\alpha|=3} \Delta_\alpha(r)\partial_\alpha \sum_{i,j=1}^d \sigma_r^{i,j} \partial_i\partial_j \\ &= -\frac{1}{12} \sum_{|\alpha|=3} \sum_{i,j=1}^d \sum_{r=1}^n \Delta_\alpha(r) \sigma_r^{i,j} \partial_\alpha \partial_i \partial_j. \end{aligned}$$

and the term corresponding to $l=5, l'=0$ is

$$\sum_{r=1}^n \frac{1}{5!} D_r^{(5)} = \frac{1}{5!} \sum_{|\alpha|=5} \sum_{r=1}^n \Delta_\alpha(r)\partial_\alpha$$

$m=2$. We need $l_1+l_2+2(l_1'+l_2') = k+2m = 3+4 = 7$. The only possibility is $l_1=3$, $l_2=4$, $l_1'=l_2'=0$ and $l_1=4$, $l_2=3$, $l_1'=l_2'=0$. The corresponding term is

$$2 \sum_{0\le r_1<r_2\le n} \frac{1}{3!} D_{r_1}^{(3)} \frac{1}{4!} D_{r_2}^{(4)} = \frac{2}{3!4!} \sum_{|\alpha|=3} \sum_{|\beta|=4} \sum_{0\le r_1<r_2\le n} \Delta_\alpha(r_1)\Delta_\beta(r_2)\partial_\alpha\partial_\beta.$$

$m=3$. We need $l_1+l_2+l_3+2(l_1'+l_2'+l_3') = k+2m = 3+6 = 9$. The only possibility is $l_1=l_2=l_3=3, l_1'=l_2'=l_3'$ and the corresponding term is

$$\frac{1}{6^3} \sum_{0\le r_1<r_2<r_3\le n} D_{r_1}^{(3)} D_{r_2}^{(3)} D_{r_3}^{(3)} = \frac{1}{6^3} \sum_{|\alpha|=3} \sum_{|\beta|=3} \sum_{|\gamma|=3} \sum_{0\le r_1<r_2<r_3\le n} \Delta_\alpha(r_1)\Delta_\beta(r_2)\Delta_\gamma(r_3)\partial_\alpha\partial_\beta\partial_\gamma.$$

We conclude that

$$\begin{aligned}\Gamma_3 =& \frac{1}{5!}\sum_{|\alpha|=5}\sum_{r=1}^{n}\Delta_\alpha(r)\partial_\alpha - \frac{1}{12}\sum_{|\alpha|=3}\sum_{i,j=1}^{d}\sum_{r=1}^{n}\Delta_\alpha(r)\sigma_r^{i,j}\partial_\alpha\partial_i\partial_j \\ &+ \frac{2}{3!4!}\sum_{|\alpha|=3}\sum_{|\beta|=4}\sum_{0\le r_1<r_2\le n}\Delta_\alpha(r_1)\Delta_\beta(r_2)\partial_\alpha\partial_\beta \\ &+ \frac{1}{6^3}\sum_{|\alpha|=3}\sum_{|\beta|=3}\sum_{|\gamma|=3}\sum_{0\le r_1<r_2<r_3\le n}\Delta_\alpha(r_1)\Delta_\beta(r_2)\Delta_\gamma(r_3)\partial_\alpha\partial_\beta\partial_\gamma\end{aligned}$$

B A Backward Taylor formula

We consider a non negative definite square matrix $\sigma \in \mathcal{M}_{d\times d}$ and we write it as $\sigma = C \times C^*$, with $C \in \mathcal{M}_{d\times d}$ (so $C = \sigma^{1/2}$). And we denote by L_σ the Laplace operator associated to σ :

$$L_\sigma f = \sum_{i,j=1}^{d}\sigma^{i,j}\partial_i\partial_j f.$$

We also consider a d-dimensional Brownian motion $W = (W^1, ..., W^d)$.

Lemma B.1. *Set $L_\sigma = \sum_{i,j=1}^{d}\sigma^{i,j}\partial_i\partial_j$ and let C denote a matrix such that $CC^T = \sigma$. Then for every $k \in \mathbb{N}$, $k \geq 0$, and $g \in C_b^{2k+2}(\mathbb{R}^d)$ one has*

$$g(0) = \mathbb{E}(CW_1)) + \sum_{\ell=1}^{k}\frac{(-1)^\ell}{2^\ell \ell!}\mathbb{E}(L_\sigma^l g(CW_1)) + \frac{(-1)^{k+1}}{2^{k+1}k!}\int_0^1 s^k\mathbb{E}(L_\sigma^{k+1}g(CW_s))ds, \tag{B.1}$$

in which W denotes a standard Brownian motion in $\mathbb{R}^d$.

Proof. Set $X_t = CW_t$. By Itô's formula, one has $\mathbb{E}(g(X_1)) = \mathbb{E}(g(X_t)) + \frac{1}{2}\int_t^1 \mathbb{E}(L_\sigma g(X_s))ds$, so we can write

$$\mathbb{E}(g(X_t)) = \mathbb{E}(g(X_1)) - \frac{1}{2}\int_t^1 \mathbb{E}(L_\sigma g(X_s))ds. \tag{B.2}$$

Taking $t = 0$, this gives

$$g(0) = \mathbb{E}(g(X_1)) - \frac{1}{2}\int_0^1 \mathbb{E}(L_\sigma g(X_s))ds.$$

We now iterate the above equality. First, we have

$$g(0) = \mathbb{E}(g(X_1)) - \frac{1}{2}\mathbb{E}(L_\sigma g(X_1)) - \frac{1}{2}\int_0^1 \left[\mathbb{E}(L_\sigma g(X_s)) - \mathbb{E}(L_\sigma g(X_1))\right]ds$$

and by using (B.2) we get

$$\begin{aligned}g(0) &= \mathbb{E}(g(X_1)) - \frac{1}{2}\mathbb{E}(L_\sigma g(X_1)) + \frac{1}{4}\int_0^1 ds\int_s^1 \mathbb{E}(L_\sigma^2 g(W_u))du \\ &= \mathbb{E}(g(X_1)) - \frac{1}{2}\mathbb{E}(L_\sigma g(X_1)) + \frac{1}{4}\int_0^1 u\mathbb{E}(L_\sigma^2 g(X_u))du.\end{aligned}$$

By proceeding in the iteration, statement (B.1) follows. □

We consider now a sequence of independent centred Gaussian random variables G_k with covariance matrix σ_k and we denote $S_p = \sum_{k=1}^{p} G_k$. Moreover, for a matrix $\sigma \in \mathcal{M}_{d\times d}$ we define the operators

$$h^0_{N,\sigma}\phi(x) = \phi(x) + \sum_{l=1}^{N} \frac{(-1)^l}{2^l l!} L^l_\sigma \phi(x) \qquad h^1_{N,\sigma}\phi(x) = \frac{(-1)^{N+1}}{2^{N+1}N!} \int_0^1 s^N \mathbb{E}(L^{N+1}_\sigma \phi(x + C_{p+1}W_s))ds.$$

where W is a $d-$dimensional Brownian motion independent of S_p.

Lemma B.2. *For every $\phi \in C^{2N+2}(\mathbb{R}^d)$ one has*

$$\mathbb{E}(\phi(S_p)) = \mathbb{E}(h^0_{N,\sigma_{p+1}}\phi(S_{p+1})) + \mathbb{E}(h^1_{N,\sigma_{p+1}}\phi(S_p)) \tag{B.3}$$

Proof. We notice that G_{p+1} has the same law as $C_{p+1}W_1$, and moreover, we denote $\psi_\omega(x) = \phi(S_p(\omega)+x)$. Then, using the independence property and (B.1) we obtain

$$\begin{aligned}\mathbb{E}(\psi_\omega(0)) &= \mathbb{E}(\psi_\omega(C_{p+1}W_1)) + \sum_{l=1}^{N} \frac{(-1)^l}{2^l l!} \mathbb{E}(L^l_{\sigma_{p+1}} \psi_\omega(C_{p+1}W_1)) \\ &\quad + \frac{(-1)^{N+1}}{2^{N+1}N!} \int_0^1 s^N \mathbb{E}(L^{N+1}_{\sigma_{p+1}} \psi_\omega(C_{p+1}W_s)))ds.\end{aligned}$$

Since $\mathbb{E}(L^l_{\sigma_{p+1}}\psi_\omega(C_{p+1}W_1)) = E(L^l_{\sigma_{p+1}}\phi(S_{p+1}))$ and $\mathbb{E}(L^N_{\sigma_{p+1}}\psi_\omega(C_{p+1}W_1)) = \mathbb{E}(L^{N+1}_{\sigma_{p+1}}\phi(S_p+C_{p+1}W_s))$ the above formula is exactly (B.3). □

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