Geometric permutations of non-overlapping unit balls revisited

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Geometric Permutations of Non-Overlapping Unit Balls Revisited☆

Jae-Soon Ha[1], Otfried Cheong[1], Xavier Goaoc[2], Jungwoo Yang[3]

Abstract

Given four congruent balls A, B, C, D in $\mathbb{R}^\delta$ that have disjoint interior and admit a line that intersects them in the order $ABCD$, we show that the distance between the centers of consecutive balls is smaller than the distance between the centers of A and D. This allows us to give a new short proof that n interior-disjoint congruent balls admit at most three geometric permutations, two if $n \geqslant 7$. We also make a conjecture that would imply that $n \geqslant 4$ such balls admit at most two geometric permutations, and show that if the conjecture is false, then there is a counter-example of a highly degenerate nature (in the algebraic sense).

Keywords:

1. Introduction

A *line transversal* to a family $\mathfrak{F}$ of pairwise disjoint convex sets in $\mathbb{R}^\delta$ is a line that intersects every element of that family. The study of line transversals, their properties, and conditions for their existence started in the 1950s with the classic work of Grünbaum, Hadwiger, and Danzer; background about the sizable literature on *geometric transversal theory* can be found in the classic survey of Danzer et al. [1], or the more recent ones by Goodman et al. [2], Eckhoff [3], Wenger [4], or Holmsen [5].

An oriented line transversal ℓ to a family $\mathfrak{F}$ induces a linear order on $\mathfrak{F}$:

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Fig. 1(a) shows three oriented transversals to a family of three congruent disks inducing the orders $A \prec C \prec B$, $A \prec B \prec C$, and $B \prec A \prec C$. For

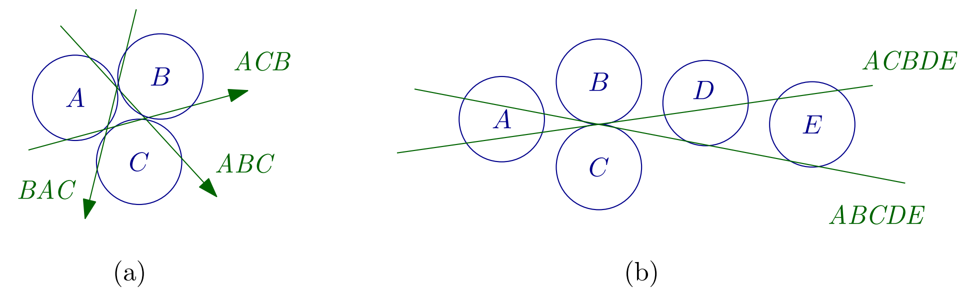


Figure 1: Orders and geometric permutations

conciseness, we usually represent the order by the string listing the elements, the three lines in Fig. 1(a) induce the orders ACB, ABC, and BAC. Natural questions in geometric transversal theory are: Given a family of disjoint convex objects, how many different orders can be realized by line transversals? How much can these orders differ? What becomes of these questions if the objects have a more restricted shape, for instance if they are balls or axis-aligned boxes?

If an order can be realized by an oriented line, so can its reverse and the two orders are therefore equivalent as far as line transversals are concerned. The equivalence classes, that is, pairs of an order and its reverse, are called *geometric permutations*. Fig. 1(b) shows a set of five congruent disks with the two geometric permutations $ABCDE$ and $ACBDE$, which could equally well be written as $EDCBA$ and $EDBCA$. In Fig. 1(b) the disks B and C touch each other. We allow this, *but a line transversal is not allowed to be tangent to these disks in this common point*. Put differently, we can remove the common points of contact from the objects to obtain a family of disjoint convex objects with the same set of line transversals. It is convenient to allow such families, as configurations are often easier to describe when objects touch. We will call a family of compact convex objects in $\mathbb{R}^\delta$ that may touch, but whose interior is disjoint, a *non-overlapping* family.

The study of geometric permutations started in the 1980s with the work by Katchalski et al. [6, 7]. In the plane, n convex objects admit at most $2n-2$ geometric permutations and this bound is tight [8]. One of the intriguing open questions is the corresponding bound for three and higher dimensions: n convex objects in $\mathbb{R}^\delta$ can have $\Omega(n^{\delta-1})$ geometric permutations [9], but the best known upper bound is only $O(n^{2\delta-3}\log n)$ [10]. For balls or similar *fat* objects, the lower bound of $\Omega(n^{\delta-1})$ is known to be tight [9, 11]. Disjoint *congruent* balls, however, have only a constant number of geometric permutations: In two dimensions, $n \geqslant 4$ congruent disks have at most two geometric permutations [9, 12]. In dimension $\delta \geqslant 3$, Cheong et al. [13] proved that n non-overlapping congruent balls have at most three geometric permutations, and at most two geometric permutations when $n \geqslant 9$.

In this paper we revisit the problem of bounding the number of geometric permutations of n non-overlapping congruent balls in $\mathbb{R}^\delta$. Since we can arbitrarily choose the radius of the balls, we will refer to them as *unit balls*. The earlier work of Cheong et al. [13] does not entirely settle the question, as no construction of $n > 3$ non-overlapping unit balls is known that admits more than two geometric permutations. Furthermore, the proof by Cheong et al. is quite technical and relies on delicate geometric lemmas and tedious case analysis. In the first part of this paper, we give a shorter and greatly simplified proof that $n \geqslant 3$ non-overlapping unit balls have at most three geometric permutations. Unlike the previous proof [13], it could be presented in its entirety in an undergraduate course on transversal theory. Our main theorem is the following:

Theorem 1. *Let $\mathfrak{F}$ be a family of n non-overlapping unit balls in $\mathbb{R}^\delta$. The number of geometric permutations of $\mathfrak{F}$ is at most three if $n \leqslant 6$, and at most two if $n \geqslant 7$.*

Theorem 1 slightly improves the previous bound of Cheong et al. [13] by settling the question for $n = 7$ and 8 (so that only the cases $4 \leqslant n \leqslant 6$ remain open). Our proof relies on the following lemma:

Distance Lemma. *If four non-overlapping unit balls A, B, C and D in $\mathbb{R}^\delta$ have a line transversal with the order $ABCD$ then $|ad| > \max\{|ab|, |bc|, |cd|\}$.*

(Here and throughout the paper we will use lower-case letters to denote the centers of balls written with upper-case letters, so a, b, c, and d are the centers of A, B, C, and D.) The lemma is not as obvious as it might appear and is false for three balls: Fig. 1(a) shows that $|ac| < |ab|$ is possible for three unit balls with a transversal with the order ABC.

We prove the Distance Lemma, in Section 3, by first modifying the given configuration into a canonical situation: We shrink the balls, keeping them congruent, until we reach the smallest radius for which they still have a transversal with the given order. This idea has probably been used first by Klee [14] and then by Hadwiger [15]. The resulting canonical configuration $\mathfrak{F}$ has the property that the line transversal ℓ is *pinned* (Lemma 7): This means that any arbitrarily small perturbation of ℓ is no longer a transversal of $\mathfrak{F}$. In other words, ℓ is an isolated point in the space of transversals of $\mathfrak{F}$. The same method for deforming a family of unit balls such that the line transversal becomes pinned has been used by Cheong et al. [16]. The correctness of the method is there deduced from algebraic results by Megyesi and Sottile [17] and by Borcea et al. [18]. This argument requires strict disjointness of the balls, and doesn't meet our goal of a proof presentable to undergraduates. We instead observe that the fact we need is already implicit in a proof by Holmsen et al. [19]. In Appendix Appendix A we examine their proof to prove the correctness of the pinning method for non-overlapping unit balls.

Before proving the Distance Lemma, we show, in Section 2, that it readily simplifies various steps of the proof of Cheong et al. [13], resulting in an elementary proof that the number of geometric permutations of n non-overlapping unit balls is at most three. On the one hand, the Distance Lemma simplifies technical derivations. For example, the fact that the geometric permutations $ABCD$ and $BADC$ are *incompatible* for non-overlapping unit balls, that is, they cannot be realized at the same time by a family of four balls, was given a delicate, five pages long, proof [13, Section 4]; it follows immediately from the Distance Lemma, since $ABCD$ implies that $|ad| > |bc|$ and $BADC$ implies that $|bc| > |ad|$, a contradiction. On the other hand, using the Distance Lemma we can replace rather pedestrian arguments by more conceptual analyses, for instance the mechanical reduction from n to 4 balls [13, Section 2] is done more concisely in Lemma 2.

We conjecture that the geometric permutations $ABCD$ and $ACDB$ are incompatible. If proven, this would show that $n \geqslant 4$ non-overlapping unit balls have at most two geometric permutations, thereby completely closing this question. In the second part of this paper, we analyze the geometry of certain pinning configurations and show that if our conjecture is false then it must admit counter-examples of a highly contrived nature.

2. At most three geometric permutations

We first use the Distance Lemma to reduce the problem from n balls to three or four balls (the same result was obtained by Cheong et al. [13] via a tedious case-analysis):

Lemma 2. *If $n \geqslant 4$ non-overlapping unit balls in $\mathbb{R}^\delta$ have at least $k \in \{3,4\}$ geometric permutations, then k of the balls have k distinct geometric permutations.*

Proof. Let $\mathfrak{F}$ be a family of $n \geqslant 4$ non-overlapping unit balls in $\mathbb{R}^\delta$. We call an element *extreme* in a geometric permutation if it appears first or last in its order. We make two observations:

(i) Any two geometric permutations of $\mathfrak{F}$ have an extreme element in common. Indeed, if two geometric permutations σ_1 and σ_2 of $\mathfrak{F}$ have disjoint sets of extreme elements $\{A_1, B_1\}$ and $\{A_2, B_2\}$ then applying the distance lemma to σ_1 yields $|a_1b_1| > |a_2b_2|$ and applying it to σ_2 yields $|a_2b_2| > |a_1b_1|$, a contradiction.

(ii) If two geometric permutations of $\mathfrak{F}$ share an extreme element A then they differ on $\mathfrak{F} \setminus \{A\}$. Indeed, assume that the first geometric permutations writes $AB \ldots XY$. If the second, which is distinct from the first, coincides with it on $\mathfrak{F} \setminus \{A\}$ then it must be $AYX \ldots B$. The distance lemma then implies both that $|ay| > |ab|$ and that $|ab| > |ay|$, a contradiction.

Assume that $\mathfrak{F}$ has three geometric permutations and let $\mathfrak{G}$ be a minimal subfamily of $\mathfrak{F}$ on which their restrictions τ_1, τ_2, and τ_3 are pairwise distinct. By observation (i), any two of these restrictions have a common extreme element. There cannot be an extreme element common to all three τ_i as observation (ii) would contradict the minimality of $\mathfrak{G}$. Hence, there exist three distinct elements $A, B, C \in \mathfrak{G}$ such that A is extreme in τ_1 and τ_2, B is extreme in τ_2 and τ_3 and C is extreme in τ_1 and τ_3. Then the restrictions of τ_1, τ_2 and τ_3 to $\{A, B, C\}$ are ABC, ACB and BAC, implying $\mathfrak{G} = \{A, B, C\}$.

Assume now that $\mathfrak{F}$ has four geometric permutations and, again, let $\mathfrak{G}$ be a minimal subfamily of $\mathfrak{F}$ on which their restrictions τ_1, τ_2, τ_3, and τ_4 are pairwise distinct. For $H \subseteq \mathfrak{G}$ let $\tau_{i|H}$ denote the restriction of τ_i to H. As we just argued $\mathfrak{G}$ contains a triple $T = \{A, B, C\}$ such that, without loss of generality, A is extreme in τ_1 and τ_2, B is extreme in τ_2 and τ_3 and C is extreme in τ_1 and τ_3; we further have

$$\tau_{1|T} = ABC, \quad \tau_{2|T} = ACB, \quad \tau_{3|T} = BAC.$$

By observation (i) the extreme elements of τ_4 are among $\{A, B, C\}$, say A and C. Since τ_1 and τ_4 have the same extreme elements but are different there must exist a pair $\{D, E\} \subset \mathfrak{G}$ such that the restrictions of τ_1 and τ_4 to $\{A, C, D, E\}$ are different. Assume that $B \notin \{D, E\}$ so that $Q = \{A, B, C, D, E\}$ has size five. We write $\tau_{1|Q} = AX_1X_2X_3C$, $\tau_{2|Q} = AY_1Y_2Y_3B$ and $\tau_{3|Q} = BZ_1Z_2Z_3C$. If $X_3 = B$ then $|bc| < |ac|$ and we must set $Z_1 = A$. This implies that $|ab| < |bc|$ and we must set $Y_1 = C$. This implies that $|ac| < |ab|$ which contradicts the two previous inequalities. It must then be that $X_3 \neq B$. This implies that $|ab| < |ac|$ which forces $Y_3 = C$. This implies $|bc| < |ab|$, which forces $Z_3 = A$. This implies that $|ac| < |bc|$, again a contradiction with the two previous inequalities. As a consequence, $B \in \{D, E\}$ and $\tau_1, \ldots \tau_4$ are already distinct on the quadruple $\{A, C, D, E\}$. $\square$

We can now easily prove that there cannot be more than three geometric permutations.

Theorem 3. *A family of non-overlapping unit balls in $\mathbb{R}^\delta$ has at most three geometric permutations.*

Proof. By Lemma 2 it suffices to prove the statement for families of size four. Let A, B, C, D be four non-overlapping unit balls in $\mathbb{R}^\delta$ and assume that there is a line transversal in the order $ABCD$. the Distance Lemma implies that $|ad| > \max\{|ab|, |bc|, |cd|\}$ and no line can meet these balls in the order $ADCB$ (which implies $|ab| > |ad|$), $BADC$ (which implies $|bc| > |ad|$), $BDAC$ (implying $|bc| > |ad|$), or $CBAD$ (as this entails $|cd| > |ad|$). Of the twelve geometric permutations of four elements, this leaves the seven shown in Fig. 2 as candidates for the remaining geometric permutations of $\mathfrak{F}$. It is

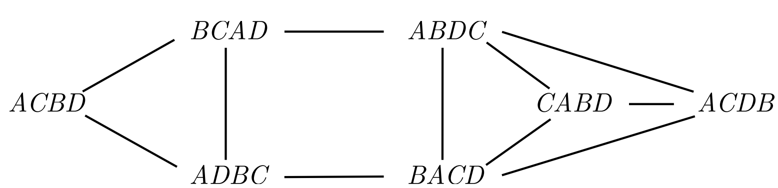


Figure 2: Proof of Theorem 3.

easy to verify that the geometric permutations connected by edges in Fig. 2 are also incompatible by the distance lemma. The resulting graph has no independent set of size larger than two, and so $\mathfrak{F} = \{A, B, C, D\}$ has at most three geometric permutations. $\square$

To prove the stronger statement of Theorem 1, we need two lemmas proven by Cheong et al. [13] (their proofs are short and self-contained). For a directed line ℓ, we write $\vec{\ell}$ for its direction vector, for points $p, q \in \mathbb{R}^\delta$, we will write $\overrightarrow{pq}$ for the vector $q - p$ from p to q.

Lemma 4 ([13, Lemma 7])**.** *Let A, B and C be three non-overlapping unit spheres in $\mathbb{R}^\delta$, and let ℓ be a directed line stabbing them in the order ABC. Then $\angle(\vec{\ell}, \overrightarrow{ac}) < \pi/4$.*

Lemma 5 ([13, Lemma 6])**.** *Let $\mathcal{C}$ be a cylinder of radius one and length less than $s\sqrt{2}$ in $\mathbb{R}^\delta$, for some $s \in \mathbb{N}$. Then $\mathcal{C}$ contains at most $2s$ points with pairwise distance at least two.*

We analyze the intersection of two cylinders more carefully in the following lemma:

Lemma 6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be cylinders of radius one and axes σ_1 and σ_2 in $\mathbb{R}^\delta$. If $\pi/4 < \angle(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}) \leqslant \pi/2$, then the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$ contains at most six points with pairwise distance at least two.*

Proof. We choose a coordinate system where σ_1 is the x_1-axis, and σ_2 is the line $(t\cos\theta, t\sin\theta, h, 0, \ldots, 0)$, where $\theta = \angle(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}) > \pi/4$ and $h \geqslant 0$ is the distance between σ_1 and σ_2. The left side of Fig. 3 shows the projection on the x_1x_2-plane. Consider the points $u = (1/\sin\theta, 0, 0, \ldots, 0)$ and $v = (1/\sin\theta + \cot\theta, 1, 0, \ldots, 0)$ marked in the figure. Since $\theta > \pi/4$, the distance between u and $-u$ is less than $2\sqrt{2}$, and so by Lemma 5 the section of $\mathcal{C}_1$ between $-u$ and u contains at most four points of pairwise distance at least two. All remaining points in $\mathcal{C}_1 \cap \mathcal{C}_2$ must project into the two symmetric shaded regions. We will now show that the parts of the intersection of the two cylinders that project into each of these shaded regions have diameter less than two and can therefore contain only one point each, proving the lemma.

Let $p \in \mathcal{C}_1 \cap \mathcal{C}_2$ be a point with $u_1 < p_1 \leqslant v_1$ (we use indices for the coordinates in $\mathbb{R}^\delta$). We will show that $|pr| < 1$, where $r = \big((u_1 + v_1)/2, \frac{1}{2}, \frac{h}{2}, 0, \ldots, 0\big)$ (note that r does not lie in the x_1x_2-plane). Since

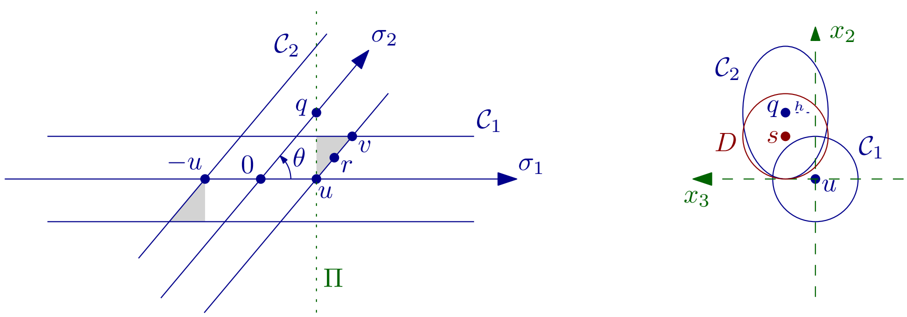


Figure 3: Left: Projection of $\mathcal{C}_1$ and $\mathcal{C}_2$ on the x_1x_2-plane. Right: The intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ with Π.

$\theta > \pi/4$, we have $v_1 - u_1 = \cot\theta < 1$, and so $|p_1 - r_1| < 1/2$. Let Π denote the hyperplane $x_1 = u_1$, and let p^* and r^* be the orthogonal projection of p and r into Π. We observe that $p^* \in \mathcal{C}_1 \cap \mathcal{C}_2$. The intersection $\mathcal{C}_1 \cap \Pi$ is the unit-radius ball around the origin in Π. The intersection $\mathcal{C}_2 \cap \Pi$ is an ellipsoid with center $q = (1/\sin\theta, 1/\cos\theta, h, 0, \dots, 0)$, see right hand side of Fig. 3. $\mathcal{C}_2 \cap \Pi$ contains exactly the points $x = (u_1, x_2, x_3, \dots, x_\delta) \in \Pi$ with

$$(\cos\theta)^2(x_2 - \frac{1}{\cos\theta})^2 + (x_3 - h)^2 + \sum_{i=4}^{\delta} x_i^2 \leqslant 1.$$

Consider the ball D with center $s = (u_1, 1, h, 0, \dots, 0)$ and radius one. For a point $x \in \Pi$ with $0 \leqslant x_2 \leqslant 1$, we have

$$1 - x_2 \leqslant 1 - (\cos\theta)x_2 = (\cos\theta)(\frac{1}{\cos\theta} - x_2),$$

and so $x \in \mathcal{C}_2$ implies $x \in D$. It follows that $p^* \in \mathcal{C}_1 \cap D$. Since $|us| \geqslant 1$ and $r^* = (u+s)/2$ is the midpoint of the two centers we have $|p^*r^*| \leqslant \sqrt{3}/2$. It follows that $|pr|^2 = |p_1 - r_1|^2 + |p^*r^*|^2 < 1/4 + 3/4 = 1$. $\square$

Proof of Theorem 1. We proved the bound for $n \leqslant 6$ in Theorem 3, so it remains to consider families $\mathfrak{F}$ of $n \geqslant 7$ balls. We show that the geometric permutations $XYZU$ and $XUYZ$ are incompatible for $\mathfrak{F}$. Assume for a contradiction that ℓ is an oriented transversal inducing the order $XYZU$, and ℓ' is an oriented transversal inducing the order $XUYZ$. By Lemma 4, we have $\angle(\vec{\ell}, \overrightarrow{\ell'}) \leqslant \angle(\vec{\ell}, \overrightarrow{xz}) + \angle(\overrightarrow{\ell'}, \overrightarrow{xz}) < \pi/2$. Since ℓ' meets U before Y, we have $\angle(\overrightarrow{\ell'}, \overrightarrow{yu}) > \pi/2$, and by Lemma 4 again we have $\angle(\vec{\ell}, \overrightarrow{yu}) < \pi/4$, implying $\angle(\vec{\ell}, \overrightarrow{\ell'}) > \pi/4$. Consider now the cylinders $\mathcal{C}$ and $\mathcal{C}'$ of radius one with axes ℓ

and ℓ'. Since ℓ and ℓ' are transversals for $\mathfrak{F}$, the centers of all balls in $\mathfrak{F}$ are contained in $\mathcal{C} \cap \mathcal{C}'$. By Lemma 6, this implies $n \leqslant 6$, a contradiction.

We now assume that $\mathfrak{F}$ has three geometric permutations. By Lemma 2 there is a subset $\mathfrak{G} = \{A, B, C, D\}$ of four balls such that $\mathfrak{G}$ already has three geometric permutations. We can assume $ABCD$ is one of them. The incompatible pair $(XYZU, XUYZ)$ implies that $ACDB$, $ADBC$, $CABD$, and $BCAD$ cannot exist. Of the geometric permutations shown in Fig. 2, this only leaves $ACBD$, $ABDC$, and $BACD$. Since we already know $ABDC$ and $BACD$ to be incompatible by the distance lemma (see Fig. 2), the last pair must therefore include $ACBD$. But this permutation is incompatible with the other two because they form pairs of the form $(XYZU, XUYZ)$. $\square$

3. Proof of the Distance Lemma

We say that a family $\mathfrak{F}$ *pins* a line ℓ or that ℓ is pinned by $\mathfrak{F}$ if ℓ is a line transversal to $\mathfrak{F}$ and any arbitrarily small perturbation of ℓ is not a line transversal to $\mathfrak{F}$. [4] It is often convenient to deform a family of balls and lines into a configuration where the lines are pinned. The following lemma describes such a deformation.

Lemma 7. *Let $\mathfrak{F}(t) = \{B_1(t), B_2(t), \ldots, B_n(t)\}$ be a parameterized family of non-overlapping balls of radius $t \in [0,1]$ in $\mathbb{R}^3$, with the property that $B_i(s) \subset B_i(t)$ for any $1 \leqslant i \leqslant n$ and $0 \leqslant s < t \leqslant 1$. If $\mathfrak{F}(1)$ has a line transversal in the order $B_1(1)B_2(1)\ldots B_n(1)$ then there exists $t^* \in [0,1]$ such that $\mathfrak{F}(t^*)$ has a pinned line transversal in the order $B_1(t^*)B_2(t^*)\ldots B_n(t^*)$.*

The proof of Lemma 7 is already implicit in Holmsen et al. [19]. For completeness, we revisit their proof in Appendix Appendix A and make the necessary adjustments.

The following lemma allows us to reduce the dimension in which we have to prove our statements.

Lemma 8. *Let $\mathfrak{F}$ be a family of non-overlapping unit balls in $\mathbb{R}^\delta$ with a line transversal ℓ, and let $\mathfrak{S}$ be an affine subspace containing the centers of all*

[4] Equivalently, a line is pinned by $\mathfrak{F}$ if it is an isolated point in the space of line transversals to $\mathfrak{F}$ endowed with the natural topology on the space of lines, for instance as given by the Grassmann-Plücker coordinates.

balls in $\mathfrak{F}$. The orthogonal projection ℓ' of ℓ into the subspace $\mathfrak{S}$ is a line transversal to $\mathfrak{F}$ realizing the same geometric permutation.

Proof. If p is a point in a ball $B \in \mathfrak{F}$ and p' is the projection of p into $\mathfrak{S}$, then $|bp'| \leqslant |bp|$, and so $p' \in B$. The lemma follows. $\square$

We will use the following folklore characterization of triples of balls pinning a line (we include a proof for completeness).

Lemma 9. *A set $\{A, B, C\}$ of three non-overlapping unit balls in $\mathbb{R}^3$ pins a line ℓ if and only if they are tangent to ℓ, their centers a, b, c are coplanar with ℓ, and in that plane ℓ separates the center of the middle ball (in the order of tangency) from the other two centers.*

Proof. If $\{A, B, C\}$ satisfies the condition then the ball B is separated from A and C by the plane Π perpendicular in ℓ to the plane of centers. Any line transversal to $\{A, B, C\}$ in the same order as ℓ must be contained in Π, and thus ℓ is pinned.

Conversely, assume that $\{A, B, C\}$ pins ℓ. If ℓ is not contained in the plane of centers, then rotating ℓ toward its orthogonal projection into that plane decreases the distances to all centers as in Lemma 8. Since any intermediate line in this rotation is therefore a line transversal, ℓ is not pinned. The line ℓ is thus contained in the plane of centers, and the necessity of the separation condition is easily checked. $\square$

Proof of distance lemma. We first argue that the statement follows from the case $\delta = 3$. Indeed, let $\mathfrak{S}$ denote a 3-dimensional space containing the four balls' centers (if they are not coplanar, then $\mathfrak{S}$ is uniquely defined). The space $\mathfrak{S}$ intersects the four non-overlapping δ-dimensional unit balls in four non-overlapping 3-dimensional unit balls with the same centers. Let ℓ' denote the orthogonal projection of ℓ into $\mathfrak{S}$. By Lemma 8, ℓ' is a line transversal to the four 3-dimensional balls with the same geometric permutation as ℓ. It therefore suffices to prove the claim for the 3-dimensional balls and ℓ'.

We now assume that we are in the case $\delta = 3$. We shrink the balls uniformly around their center. By Lemma 7 we will reach a configuration with transversal ℓ that is pinned by four non-overlapping unit balls $\{A, B, C, D\}$. If the four centers are collinear then the statement is clear, so we assume otherwise.

We will use the following notation (refer to Fig. 4). Let $\ell^\perp$ denote a plane orthogonal to ℓ. For $x \in \{a, b, c, d\}$, let x' denote the orthogonal projection

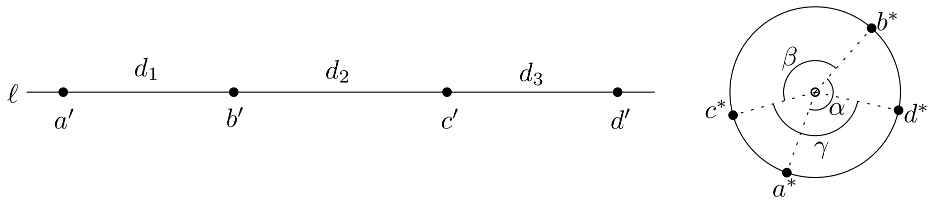


Figure 4: Notation for the proof of the Distance Lemma (here all four balls are shown tangent to the pinned line).

of x onto ℓ (that is, the point closest to x on ℓ), and let x^* denote the orthogonal projection of x onto $\ell^\perp$. We set $d_1 = |a'b'|$, $d_2 = |b'c'|$, $d_3 = |c'd'|$, and $\Delta = 2(d_1d_2 + d_1d_3 + d_2d_3)$. We have:

$$|ad|^2 = (d_1 + d_2 + d_3)^2 + |a^*d^*|^2, \quad |ab|^2 = d_1^2 + |a^*b^*|^2,$$

$$|bc|^2 = d_2^2 + |b^*c^*|^2, \quad |cd|^2 = d_3^2 + |c^*d^*|^2.$$

Since ℓ is a line transversal with order $ABCD$, we have $d_1, d_2, d_3 > 0$ and therefore $\Delta > 0$.

Our goal is to prove that

$$|ad|^2 - |ab|^2 = \Delta + d_2^2 + d_3^2 + |a^*d^*|^2 - |a^*b^*|^2 > 0 \tag{1}$$
$$|ad|^2 - |bc|^2 = \Delta + d_1^2 + d_3^2 + |a^*d^*|^2 - |b^*c^*|^2 > 0 \tag{2}$$
$$|ad|^2 - |cd|^2 = \Delta + d_1^2 + d_2^2 + |a^*d^*|^2 - |c^*d^*|^2 > 0 \tag{3}$$

Assume first that three of the balls already pin ℓ. There are essentially two cases:

- If the three balls are $\{A, B, C\}$ then, by Lemma 9, $a^* = c^*$ and $|b^*c^*|^2 = |a^*b^*|^2 = 4$. The fact that $a^* = c^*$ immediately implies Inequality (3). Since C and D are non-overlapping we have

  $$d_3^2 + |a^*d^*|^2 = d_3^2 + |c^*d^*|^2 = |cd|^2 \geqslant 4,$$

  which implies Inequalities (1) and (2). The case where the three balls are $\{B, C, D\}$ is symmetric.

- If the three balls are $\{A, B, D\}$ then, by Lemma 9, $a^* = d^*$ and $|a^*b^*|^2 = |b^*d^*|^2 = 4$. Since B, C and D are non-overlapping, we have

  $$d_2^2 = |bc|^2 - |b^*c^*|^2 \geqslant 4 - |b^*c^*|^2$$

$$\text{and} \quad d_3^2 = |cd|^2 - |c^*d^*|^2 \geqslant 4 - |c^*d^*|^2 = 4 - |c^*a^*|^2.$$

Using $|a^*d^*|^2 = 0$ and $|a^*b^*|^2 = 4$, we can bound

$$\begin{aligned}
\Delta + d_2^2 + d_3^2 + |a^*d^*|^2 - |a^*b^*|^2 &\geqslant \Delta + 4 - |b^*c^*|^2 + 4 - |c^*a^*|^2 - 4 \\
&= \Delta + 4 - |b^*c^*|^2 - |c^*a^*|^2 \\
\Delta + d_1^2 + d_3^2 + |a^*d^*|^2 - |b^*c^*|^2 &\geqslant \Delta + d_1^2 + 4 - |c^*a^*|^2 - |b^*c^*|^2 \\
\Delta + d_1^2 + d_2^2 + |a^*d^*|^2 - |c^*d^*|^2 &\geqslant \Delta + d_1^2 + 4 - |b^*c^*|^2 - |c^*a^*|^2
\end{aligned}$$

Since c^* lies in the disk of diameter a^*b^*, the triangle $a^*b^*c^*$ is right or obtuse, and

$$|a^*c^*|^2 + |c^*b^*|^2 \leqslant |a^*b^*|^2 = 4.$$

This implies Inequalities (1)–(3). The case where the three balls are $\{A, C, D\}$ is symmetric.

It remains to handle the case where no three balls in $\{A, B, C, D\}$ pin ℓ. This implies that ℓ is tangent to all four balls, and the points a^*, b^*, c^*, d^* lie on the unit circle around $\ell^* = \ell \cap \ell^\perp$ in $\ell^\perp$. We let α be the angle made by a^* and b^* at ℓ^*, β the angle made by b^* and c^* at ℓ^*, and γ the angle made by c^* and d^* at ℓ^*, see Fig. 4. All the angles are measured counterclockwise (even if they are larger than π) so that the angle made by a^* and d^* is the same as $\alpha + \beta + \gamma$ modulo 2π.

We define a function g by $g(\phi) = \sqrt{2 + 2\cos\phi}$ for $\phi \in \mathbb{R}$. We claim that for any three angles x, y, z we have

$$g(x + y + z) \leqslant g(x) + g(y) + g(z), \tag{4}$$

and that the inequality is strict unless two of x, y, z are equal to π modulo 2π. Indeed, let $f(\phi) = \sqrt{2 - 2\cos\phi} = g(\pi - \phi)$ and observe that $f(\phi)$ is the distance between two points on the unit circle that make an angle of ϕ at the center of the unit circle. The triangle inequality immediately implies that for any two angles ϕ and θ we have $f(\phi + \theta) \leqslant f(\phi) + f(\theta)$, where equality holds only if ϕ or θ is equal to 0 modulo 2π. Thus, for any three angles x, y, z we have

$$\begin{aligned}
f\big(\pi - (x + y + z)\big) &= f\big(3\pi - (x + y + z)\big) \\
&= f\big((\pi - x) + (\pi - y) + (\pi - z)\big) \\
&\leqslant f(\pi - x) + f(\pi - y) + f(\pi - z),
\end{aligned}$$

and Inequality (4) follows, with equality only if two of x, y, z are equal to π modulo 2π.

Consider the angles α, β, and γ. If $\alpha = \beta = \pi$, then by Lemma 9, $\{A, B, C\}$ already pin ℓ, a contradiction. If $\beta = \gamma = \pi$, then $\{B, C, D\}$ already pin ℓ, again a contradiction. We thus have

$$g(\alpha + \beta + \gamma) \leqslant g(\alpha) + g(\beta) + g(\gamma), \tag{5}$$

and the inequality is strict unless $\alpha = \gamma = \pi$. In this case[5] $|a^* b^*| = |c^* d^*| = 2$, and therefore $|ab| > 2$ and $|cd| > 2$.

We observe that

$$\begin{aligned} |ab|^2 &= d_1^2 + 2 - 2\cos\alpha = d_1^2 + 4 - g(\alpha)^2, \\ |bc|^2 &= d_2^2 + 2 - 2\cos\beta = d_2^2 + 4 - g(\beta)^2, \\ |cd|^2 &= d_3^2 + 2 - 2\cos\gamma = d_3^2 + 4 - g(\gamma)^2. \end{aligned}$$

Since the balls are non-overlapping, this implies $d_1 \geqslant g(\alpha)$, $d_2 \geqslant g(\beta)$, and $d_3 \geqslant g(\gamma)$. In particular, when Inequality (5) is not strict, then $d_1 > g(\alpha)$ and $d_3 > g(\gamma)$.

Recall that $\Delta = 2(d_1 d_2 + d_1 d_3 + d_2 d_3)$ and set $\Delta' = 2\big(g(\alpha)g(\beta) + g(\alpha)g(\gamma) + g(\beta)g(\gamma)\big) \leqslant \Delta$. We can write

$$\begin{aligned} |ad|^2 &= (d_1 + d_2 + d_3)^2 + 4 - g(\alpha + \beta + \gamma)^2 \\ &= d_1^2 + d_2^2 + d_3^2 + \Delta + 4 - g(\alpha + \beta + \gamma)^2 \\ &\geqslant d_1^2 + d_2^2 + d_3^2 + \Delta' + 4 - g(\alpha + \beta + \gamma)^2. \end{aligned}$$

We can now complete the proof.

$$\begin{aligned} |ad|^2 &\geqslant d_1^2 + d_2^2 + d_3^2 + \Delta' + 4 - g(\alpha + \beta + \gamma)^2 \\ &\geqslant d_1^2 + g(\beta)^2 + g(\gamma)^2 + \Delta' + 4 - g(\alpha + \beta + \gamma)^2 \qquad (6) \\ &= \big(g(\alpha) + g(\beta) + g(\gamma)\big)^2 - g(\alpha + \beta + \gamma)^2 + d_1^2 + 4 - g(\alpha)^2 \\ &\geqslant d_1^2 + 4 - g(\alpha)^2 = |ab|^2 \qquad (7) \end{aligned}$$

[5]Theorem 14 actually implies that in this case ℓ is not pinned at all, the argument here keeps the proof self-contained.

If Inequality (7) is not strict, then Inequality (6) is strict, and so $|ad| > |ab|$.

$$\begin{aligned}
|ad|^2 &\geqslant d_1^2 + d_2^2 + d_3^2 + \Delta' + 4 - g(\alpha+\beta+\gamma)^2 \\
&\geqslant g(\alpha)^2 + d_2^2 + g(\gamma)^2 + \Delta' + 4 - g(\alpha+\beta+\gamma)^2 \qquad (8)\\
&= \big(g(\alpha)+g(\beta)+g(\gamma)\big)^2 - g(\alpha+\beta+\gamma)^2 + d_2^2 + 4 - g(\beta)^2 \\
&\geqslant d_2^2 + 4 - g(\beta)^2 = |bc|^2 \qquad (9)
\end{aligned}$$

Again, if Inequality (9) is not strict, then Inequality (8) is, and we have $|ad| > |bc|$. By symmetry, we also obtain $|ad| > |cd|$. □

We note that the Distance Lemma does not generalize to arbitrary translates of a centrally symmetric convex set (it already fails for parallel unit segments). This does not come as a surprise given how the proof hinges on Pythagoras' theorem.

4. Conjectures on four unit balls and two lines

We conjecture that the geometric permutations $ABCD$ and $ACDB$ are incompatible for non-overlapping unit balls:

Conjecture 1. *There is no set of four non-overlapping unit balls in $\mathbb{R}^3$ admitting the geometric permutations $ABCD$ and $ACDB$.*

In the plane, $ABCD$ and $ACDB$ can be realized by translates of some convex figure (cf. Asinowski et al. [12], more precisely the discussion on type 6) but are incompatible for disjoint unit disks [12, Lemma 1].

As we have seen in the proof of Theorem 1 in Section 2, Conjecture 1 would imply that a family of at least four non-overlapping unit balls in $\mathbb{R}^\delta$ has at most two geometric permutations, settling our question entirely. In this section, we study what a counter-example to our conjecture would look like.

We first employ the shrinking technique to obtain a configuration where both line transversals are pinned.

Lemma 10. *If Conjecture 1 is false then there exist four non-overlapping unit balls in $\mathbb{R}^3$ that pin two lines realizing the geometric permutations $ABCD$ and $ACDB$.*

Proof. Consider four non-overlapping unit balls $\mathfrak{F} = \{A, B, C, D\}$ in $\mathbb{R}^3$ that admit line transversals with orders $ABCD$ and $ACDB$. We uniformly shrink the four balls about their centers. By Lemma 7, we will reach a radius $t_1 > 0$ where the transversal σ_1 for one of the two orders is pinned, while a transversal for the other order still exists. For each ball $X \in \mathfrak{F}$, we pick a point $x_0 \in X \cap \sigma_1$. We continue shrinking the balls, but now we shrink X with homothety center x_0. By Lemma 7, we will reach a radius $t_2 > 0$ where the transversal σ_2 for the second order is pinned. Since the points x_0 lie in the shrunken balls, σ_1 is still a transversal, and since the balls of radius t_2 lie inside the balls of radius t_1, σ_1 is still pinned. By scaling the balls back to unit radius, we obtain the configuration announced by the lemma. $\square$

4.1. Configurations with two pinned transversals

In this section we restrict the geometry of configurations as in Lemma 10. We start with some geometric preliminaries. Throughout this section we will be dealing with families of at most four balls.

Lemma 11. *If three non-overlapping unit balls X, Y and Z in $\mathbb{R}^3$ admit a line transversal with order XYZ, then the angles $\angle(yxz)$ and $\angle(xzy)$ are acute.*

Proof. Using Lemma 7, we shrink the balls uniformly around their centers until a line transversal ℓ with order XYZ is pinned. By Lemma 9, ℓ is parallel to xz and y lies inbetween x and y in the projection on ℓ, implying the claim. $\square$

Lemma 12. *If three non-overlapping unit balls X, Y and Z in $\mathbb{R}^3$ admit two line transversals with the orders XYZ and XZY, then the triangle $\triangle xyz$ is acute and $|yz| < 2\sqrt{2}$.*

Proof. Lemma 11 implies that $\triangle xyz$ is acute. To prove the last statement, we shrink the balls uniformly around their centers. By Lemma 7, there exists $t > 0$ such that without loss of generality the following holds: X, Y, Z are now balls with radius $t \leqslant 1$ pinning a line transversal ℓ with order XYZ, and the balls have a second line transversal ℓ' with order XZY. By Lemma 9, ℓ lies in the plane containing x, y, z, and separates y from x and z. Let x', y', z' be the projection of x, y, z onto ℓ, they appear in this order along ℓ. If $|yz| \geqslant 2\sqrt{2}$, then $|y'z'| \geqslant 2$, and there is a plane orthogonal to ℓ that separates Z on one side from X and Y on the other side. But that contradicts the existence of ℓ'. $\square$

We can now state our restriction on a possible counter-example of Conjecture 1.

Theorem 13. *If four non-overlapping unit balls $\{A, B, C, D\}$ in $\mathbb{R}^3$ pin two lines with the geometric permutations $ABCD$ and $ACDB$, then these lines are not pinned by a proper subset of the balls.*

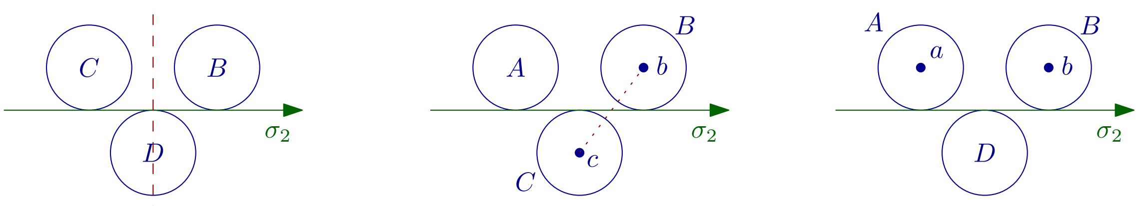
Proof. Consider four non-overlapping unit balls $\{A, B, C, D\}$ in $\mathbb{R}^3$ that pin two lines σ_1 and σ_2 realizing, respectively, the geometric permutations $ABCD$ and $ACDB$. Let us first remark that

$$\pi/4 < \angle(\overrightarrow{ab}, \overrightarrow{\sigma_1}) < \pi/2 \tag{10}$$

Indeed, since σ_1 meets A before B, we have $\angle(\overrightarrow{ab}, \overrightarrow{\sigma_1}) < \pi/2$. Moreover, if $\angle(\overrightarrow{ab}, \overrightarrow{\sigma_1}) \leqslant \pi/4$, since Lemma 4 yields that $\angle(\overrightarrow{\sigma_1}, \overrightarrow{bd}) < \pi/4$, we would have $\angle(\overrightarrow{ab}, \overrightarrow{bd}) \leqslant \angle(\overrightarrow{ab}, \overrightarrow{\sigma_1}) + \angle(\overrightarrow{\sigma_1}, \overrightarrow{bd}) < \pi/2$, a contradiction with $\triangle abd$ being acute by Lemma 12. Let us also remark that

$$|ab| < 2\sqrt{2} \tag{11}$$

as otherwise any line meeting A before B would make an angle less than $\pi/4$ with $\overrightarrow{ab}$, contradicting Equation (10).



σ_2 pinned by C, D, B. σ_2 pinned by A, C, B. σ_2 pinned by A, D, B.

Figure 5: Proof of Theorem 13

Let us assume for a contradiction that σ_2 is pinned by three of the four balls. We first remark that σ_2 can only be pinned by A, D, B (refer to Fig. 5):

- If σ_2 is pinned by C, D, and B, then by Lemma 9, σ_2 lies in the plane of bcd and is parallel to cb. Since σ_2 meets A before C, the bisecting plane of C and B separates B from A and C, a contradiction to the existence of σ_1.
- If σ_2 is pinned by A, C, D, then it is the only line meeting the three balls in this order. But then $\sigma_2 = \sigma_1$, a contradiction.
- If σ_2 is pinned by A, C, B, then by Lemma 9 it lies in the plane of acb and is parallel to ab. Lemma 12 implies that $|bc| < 2\sqrt{2}$ and so $\angle(\overrightarrow{cb}, \overrightarrow{\sigma_2}) > \pi/4$. Lemma 4 yields, however, that $\angle(\overrightarrow{cb}, \overrightarrow{\sigma_2}) < \pi/4$, a contradiction.

So assume that σ_2 is pinned by A, D, and B. By Lemma 9, σ_2 lies in the plane of adb and is parallel to $\overrightarrow{ab}$. Thus, $\angle(\overrightarrow{ab}, \overrightarrow{cb}) = \angle(\overrightarrow{\sigma_2}, \overrightarrow{cb}) < \pi/4$ by Lemma 4. It follows that c is contained in $\mathfrak{C}(b, \overrightarrow{ba}, \pi/4)$, where $\mathfrak{C}(u, \vec{v}, \alpha)$ denotes the cone of all points p such that $\angle(\overrightarrow{up}, \vec{v}) \leqslant \alpha$. Since σ_1 intersects C after A and B, the center c must lie in the cone with apex a spanned by the ball of radius 2 and center b. With $\gamma = \arcsin(2/ab)$, this cone is $\mathfrak{C}(a, \overrightarrow{ab}, \gamma)$. By Equation (11), $2 \leqslant |ab| < 2\sqrt{2}$, and so $\pi/4 < \gamma \leqslant \pi/2$. Let p be a point in the plane containing a, b and c and such that $\angle(bap) = \gamma$ and $\angle(abp) = \pi/4$ (see Fig. 6). Notice that c lies in the triangle $\triangle abp$. We claim that any point in this triangle is at distance less than 2 from a or b. In particular, there is no way to place c so as to make A, B, C non-overlapping unit balls and σ_2 cannot be pinned by A, D, and B, or, more generally, by three of the balls.

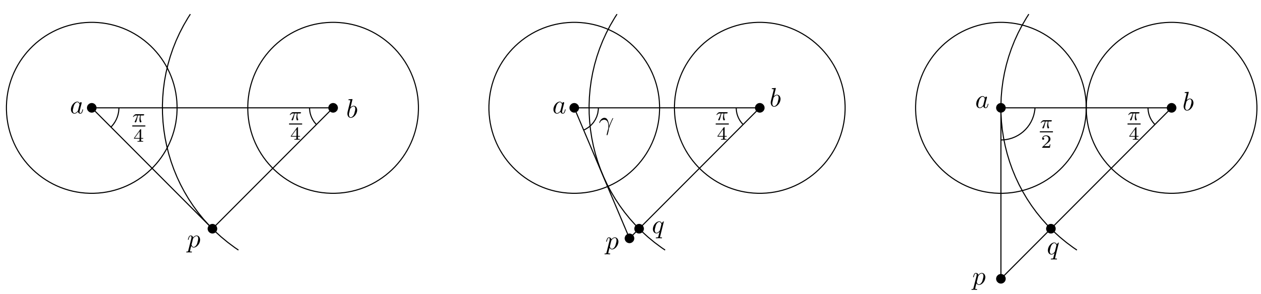


Figure 6: $\triangle abp$ for three values of γ.

It remains to prove the claim on $\triangle abp$. Since $\gamma > \pi/4$ we have $|bp| > 2$. Let q be the point on the segment bp with $|bq| = 2$. Since the segment ap touches the circle of radius 2 around b, we have $\angle(aqp) > \pi/2$, and so $|aq| <$

$|ap|$. We claim that $|ap| < 2$. Indeed, the law of sines gives $|ab|/\sin(3\pi/4 - \gamma) = |ap|/\sin(\pi/4)$, and so

$$|ap| = \frac{\sin(\pi/4)}{\sin(\pi/4+\gamma)}|ab| = \frac{\frac{1}{2}\sqrt{2}}{\sin(\pi/4+\gamma)}\frac{2}{\sin\gamma}$$
$$= \frac{\sqrt{2}}{\sin(\pi/4+\gamma)\sin\gamma} = \frac{2}{\sin\gamma(\sin\gamma+\cos\gamma)}.$$

Define $f(x) = \sin x(\sin x + \cos x) = \sin^2 x + \frac{1}{2}\sin 2x$. Since $f'(x) = \sin 2x + \cos 2x$, the function f is (strictly) increasing from $x = 0$ to $x = 3\pi/8$, and (strictly) decreasing from $x = 3\pi/8$ to $x = 7\pi/8$. Since $f(\pi/4) = f(\pi/2) = 1$, it follows that $f(x) > 1$ for $\pi/4 < x < \pi/2$; this proves our claim that $|ap| < 2$. Now, if a point $u \in \triangle apb$ lies to the left of the vertical line through q, then it is at distance less than 2 from a, if u lies to the right of q then it has distance less than 2 from b.

It follows that σ_2 cannot be pinned by any three of the balls, and is tangent to all four. We now assume, for a contradiction, that σ_1 is pinned by three of the balls. Again, we easily dismiss three of the cases:

- If σ_1 is pinned by B, C, and D, then by Lemma 9 σ_1 lies in the plane of bcd and is parallel to bd. Since σ_1 meets A before B, the bisecting plane of B and D separates D from A and B, a contradiction to the existence of σ_2.
- If σ_1 is pinned by A, C, D, then $\sigma_2 = \sigma_1$, a contradiction.
- If σ_1 is pinned by A, B, D, then it lies in the plane of abd and is parallel to ad. Since $|bd| < 2\sqrt{2}$ by Lemma 12, we have $\angle(\overrightarrow{bd}, \overrightarrow{\sigma_1}) > \pi/4$, contradicting Lemma 4.

So σ_1 must be pinned by A, B, and C and lie in the plane spanned by abc, see Fig. 7. Since σ_1 is a transversal, the center d of ball D must lie inside the cylinder $\mathcal{C}$ of radius one with axis σ_1. Since σ_2 meets A, D, and B in this order, d must also lie in the cylinder of radius two with axis ab. Since $\angle(abd) < \pi/2$ by Lemma 12, d lies above the plane orthogonal to ab through b, and since σ_1 meets D after C, d lies to the right of the plane orthogonal to σ_1 through c. In the projection on the abc-plane, this restricts d to the shaded area in Fig. 7. Let p be the rightmost point of this feasible region for d, that is, the point in the abc-plane such that $|bp| = 2$ and $\angle(abp) = \pi/2$. For any point u, let $\sigma_1^\perp(u)$ be the plane orthogonal to σ_1 passing through u. The center d lies in the cylinder $\mathcal{C}$ between the planes $\sigma_1^\perp(c)$

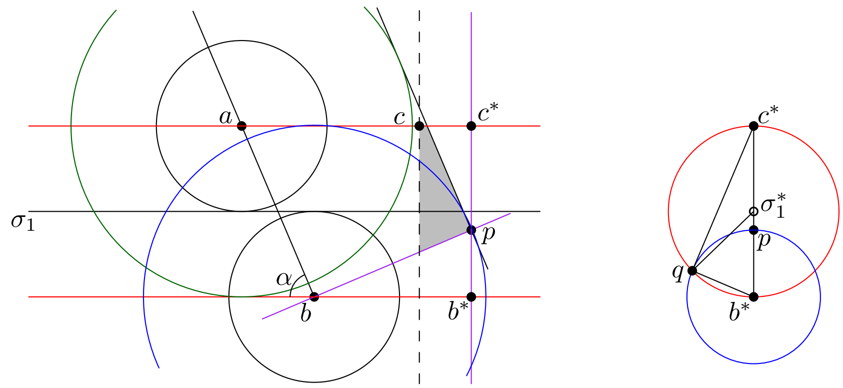


Figure 7: When σ_1 is pinned by A, B, C

and $\sigma_1^\perp(p)$.

We will now show that any point d in this cylinder has distance less than two from b or c, and so D cannot be non-overlapping with B and C, a contradiction. Clearly it suffices to show this for the disk S of radius one around σ_1 in the plane $\sigma_1^\perp(p)$. Let q be a point on the boundary of S with $|bq| = 2$ (see right side of Fig. 7). It suffices to show that $|cq| < 2$. Let $\alpha = \angle(\overrightarrow{ab}, \overrightarrow{\sigma_1})$. By Equation (10), $\pi/4 < \alpha < \pi/2$. Since $|qb^*| = |pb^*| = 2\sin(\pi/2 - \alpha) = 2\cos\alpha$ and $\angle(b^* q c^*) = \pi/2$, $|qc^*|^2 = 4 - 4\cos^2\alpha = 4\sin^2\alpha$. We have $|cc^*| = |ac^*| - |ac| = |ab|\cos\alpha + |bb^*| - |ac|$, and with $|ab| = 2/\sin\alpha$, $|bb^*| = 2\cos(\pi/2 - \alpha) = 2\sin\alpha$, and $|ac| \geqslant 2$ this implies $|cc^*| \leqslant 2\cot\alpha + 2\sin\alpha - 2$. Thus $|cq|^2 = |qc^*|^2 + |cc^*|^2 \leqslant 4\sin^2\alpha + (2\cot\alpha + 2\sin\alpha - 2)^2$.

We have

$$\begin{aligned}
&\sin^2 x + \big(\cot x + \sin x - 1\big)^2 - 1 && (12)\\
&\quad = \big(\cot x + \sin x - 1\big)^2 - \cos^2 x \\
&\quad = \big(\cot x + \sin x + \cos x - 1\big)\big(\cot x + \sin x - \cos x - 1\big) \\
&\quad = \big(\cot x + \sin x + \cos x - 1\big)(\cot x - 1)(1 - \sin x). && (13)
\end{aligned}$$

On the interval $\pi/4 < x < \pi/2$, we have $\sqrt{2}/2 < \sin x < 1$, $\sqrt{2}/2 > \cos x > 0$, and $1 > \cot x > 0$, implying $\cot x - 1 < 0$ and $1 - \sin x > 0$. Furthermore

$$\sin x + \cos x = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) > 1 \qquad \text{for } \pi/4 < x < \pi/2,$$

and so the first term in Equation (13) is positive. It follows that $\sin^2 x + (\cot x + \sin x - 1)^2 < 1$. This implies $|cq|^2 < 4$, and we arrived at the contradiction for this final case. $\square$

4.2. Minimal pinnings by four balls

By Lemma 10 and Theorem 13, if Conjecture 1 is false then there exist four non-overlapping unit balls with two transversals with the orders $ABCD$ and $ACDB$ that are both pinned by the four balls but no three of them. We now analyze the geometry of such *minimal pinnings* by four balls (Theorem 14) and derive a statement equivalent to Conjecture 1 but more restrictive (Conjecture 2).

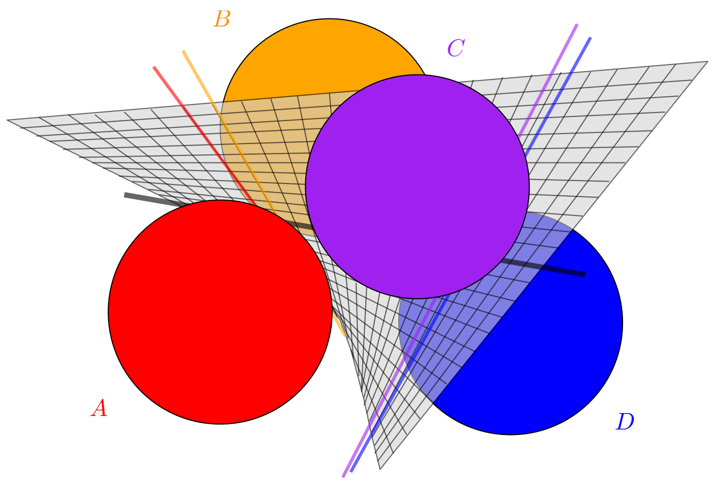


Figure 8: An alternating hyperboloidal configuration

If X is a ball tangent to a line ℓ, the *ridge* $r_\ell(X)$ of X with respect to ℓ is the line tangent to X and perpendicular to ℓ in $X \cap \ell$. We say that four or more lines are in *hyperboloidal configuration* if they are all contained in the same family of rulings of a hyperbolic paraboloid or a hyperboloid of one sheet (see [20] for a classical discussion of such configurations). An *alternating hyperboloidal configuration* is a pair $(\mathfrak{F}, \ell)$ where $\mathfrak{F}$ is a family of four unit balls balls and ℓ is a line tangent to every member of $\mathfrak{F}$ satisfying the two following conditions:

(i) the ridges $\{r_\ell(X) \mid X \in \mathfrak{F}\}$ are in hyperboloidal configuration, witnessed by a hyperbolic paraboloid $\mathcal{H}$,

(ii) the normals to $\mathcal{H}$ at its tangency point with the balls of $\mathfrak{F}$, directed towards the center of that ball and ordered along ℓ, point to alternating sides of $\mathcal{H}$.

(See Fig. 8 for an example.) Since in an alternating hyperboloidal configuration the ridges are all perpendicular to ℓ, and therefore parallel to a common plane, the quadric they span can only be a hyperbolic paraboloid. Condition (i) then forces the line ℓ to intersect $\mathcal{H}$ in at least four points and therefore $\ell \subset \mathcal{H}$. This in turn implies that every ball $X \in \mathfrak{F}$ is tangent to $\mathcal{H}$ in $X \cap \ell$ as the tangent plane to both X and $\mathcal{H}$ in $X \cap \ell$ contains the two lines (but X does not need to intersect $\mathcal{H}$ in a single point). That and the fact that a hyperbolic paraboloid separates $\mathbb{R}^3$ into two connected components makes condition (ii) well-defined.

Theorem 14. *If a family $\mathfrak{F}$ of 4 non-overlapping unit balls in $\mathbb{R}^3$ minimally pins a line ℓ then $(\mathfrak{F}, \ell)$ is an alternating hyperboloidal configuration.*

Proof. Let σ be a line minimally pinned by $ABCD$ in that order. We assume that σ coincides with the z-axis. The center of ball X is denoted by x and its contact point with σ is denoted by x'. We parameterize the space of lines by $\mathbb{R}^4$ using the coordinates of the intersections with the planes $z = 0$ and $z = 1$: the parameters (u_1, u_2, u_3, u_4) corresponds to the line through $(u_1, u_2, 0)$ and $(u_3, u_4, 1)$. The lines lying in a plane with constant z are not represented, but this will not be an issue. The point $(0, 0, 0, 0)$ corresponds to σ.

To every ball $X \in \{A, B, C, D\}$ we associate the *screen* $S_\sigma(X)$ that is the intersection of the closed halfspace bounded by the tangent plane to X in x' that contains X, and the plane perpendicular to σ in x'. The screen $S_\sigma(X)$ is a halfplane that lies in a plane perpendicular to σ and is bounded, in that plane, by the ridge $r_\sigma(X)$. Now, the line transversals to $S_\sigma(X)$ form, in our $\mathbb{R}^4$, a halfspace $H(X)$ bounded by a hyperplane through the origin [21, pp. 4–5]. We let $n(X)$ denote the outer normal of $H(X)$ and observe that the boundary of $H(X)$ is the set of lines intersecting the ridge $r_\sigma(X)$.

Now let $I = H(A) \cap H(B) \cap H(C) \cap H(D)$. A necessary condition for the balls $\{A, B, C, D\}$ to pin σ is that I has empty interior [22, Lemma 9]. This implies that the family of normals $\{n(A), n(B), n(C), n(D)\}$ is linearly dependent (since, in $\mathbb{R}^4$, four halfspaces with linearly independent normals intersect with non-empty interior). It could be that two, three or all four vectors are *minimally* linearly dependent. Geometric interpretations of these situations were given in [21, Lemma 15]:

- If two normals are dependent then the two corresponding ridges are equal. This cannot happen for non-overlapping balls.
- If three normals are dependent, then it must be that the three corresponding ridges are either coplanar with or concurrent on σ. Con-

currency is again ruled out for non-overlapping balls, and we rule out coplanarity in the next paragraph.

- If no three normals are dependent then the four ridges are in hyperboloidal configuration (the other case with concurrent ridges can again not occur in our situation).

Let us observe that the case where three normals are linearly dependent cannot correspond to a minimal pinning of σ by the four balls. As mentioned, the three corresponding ridges must lie, together with σ, in some plane Π. Let us denote them r_1, r_2 and r_3 in the order in which σ meets them, and let X_i be the ball corresponding to r_i. Since a triple of balls does not suffice to pin σ, Π does not separate X_2 from X_1 and X_3; by symmetry we can assume that either all three balls are on the same side of Π or Π separates X_1 and X_2 from X_3. Since I has empty interior, the fourth ridge also lies in the plane Π. Then, either three of the balls pin σ or all four do not, a contradiction.

We must therefore be in the situation where the four ridges are in hyperboloidal configuration. Observe that since any three normals are linearly independent, the intersection I must be exactly a line in $\mathbb{R}^4$. That line $I \subset \mathbb{R}^4$ corresponds to the set of lines intersecting all four ridges. Let Q denote the quadric formed by the union (in $\mathbb{R}^3$) of these line transversals to the four ridges. Let us orient Q, that is choose an outward normal (defined continuously over all of Q). There are two connected components in $\mathbb{R}^3 \setminus Q$, which we call *sides* of Q. As we move a point p along σ, the outward normal of Q in p keeps pointing into the same side but rotates continuously around σ; as p ranges over all of σ, that normal turns by a total angle of π. For each ball $X \in \{A, B, C, D\}$, $S(X) \setminus r(X)$ is contained either in the positive side or in the negative side of Q.

Now consider the orthogonal projection of the four screens on the plane $z = 0$. The circular order in which the projections of the ridges appear matches the order in which σ meets the screens; indeed, the projection of ridge $r(X)$ is simply the trace in $z = 0$ of the plane tangent to Q in x', and we observed that the tangent plane turns continuously, and by a total angle of π, as the contact point ranges over all of σ. Moreover, (the relative interiors of) any two consecutive screens are contained in opposite sides of Q as otherwise we can perturb the plane $z = 0$ so that the projections of the four screens intersect with non-empty interior, and I cannot have empty interior. Altogether, this proves that a quadruple of balls minimally pinning a line must form an alternating hyperboloidal configuration. $\square$

Theorem 14, Lemma 10, and Theorem 13 imply that Conjecture 1 can be reformulated in the following form:

Conjecture 2. *There is no set $\mathfrak{F}$ of four non-overlapping unit balls in $\mathbb{R}^3$ with two line transversals σ_1, σ_2 that realize the geometric permutations $ABCD$ and $ACDB$ and such that $(\mathfrak{F}, \sigma_1)$ and $(\mathfrak{F}, \sigma_2)$ are alternating hyperboloidal configurations.*

5. Concluding remarks

Conjectures 1 and 2 can be expressed by asking whether a system of low-degree polynomial equations and inequalities in a small number of variables has a solution (see Appendix Appendix B). In principle, such systems can be solved by computer algebra software based on Gröbner basis computation such as the RAGLIB library. Inequalities are harder to handle than equalities by these solvers, so this is one reason why Theorem 14, Lemma 10, and Theorem 13 are interesting: Conjecture 2 replaces most of the inequalities in Conjecture 1 by equalities.

Our attempts using algebraic solver software were inconclusive. These questions may constitute interesting challenges for the computer algebra community.

We speculate that Theorem 14 can be turned into an equivalence, at least when the balls are disjoint. One approach could be to remark that if $(\mathfrak{F}, \sigma)$ is an alternating hyperboloidal configuration where σ realizes the geometric permutation $ABCD$, then the direction $\vec{u}$ of σ is on the boundary of the so-called *cone of directions* of transversals to the triples ABC and BCD (using ideas such as eg. [18, Proposition 3]). These cones are strictly convex in $\vec{u}$, are mutually tangent in $\vec{u}$ (their common supporting great circle is the set of directions of transversals to the four ridges) and the alternating property of the configuration $(\mathfrak{F}, \sigma)$ ensures that this tangency is *external*. Spelling out this outline requires non-trivial technical developments, all the more if one cares for the setting of non-overlapping balls, and is not needed for our main result of Section 4; we thus leave it to the interested reader to check the validity of this approach.

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Appendix A. Correctness of the pinning method

We start with a lemma in two dimensions:

Lemma 15. *Let $\mathfrak{F}$ be a family of non-overlapping, but not necessarily congruent disks in the plane. If $\mathfrak{F}$ admits two distinct line transversals with the same order, then there is a transversal with the same order that intersects the interior of every disk.*

Proof. Consider one transversal ℓ of the two. The disks project along ℓ onto its orthogonal complement $\ell^{\perp}$ as intervals. Since ℓ is a transversal, the intersection of the intervals is not empty. If the intersection is an interval, we are done. If it is a point, then ℓ is tangent to some of the disks. The tangent disks cannot alternate, as then ℓ would be pinned, and no second transversal could exist. It follows that the tangencies on the left strictly follow the tangencies on the right, or vice versa, and we can slightly rotate ℓ to obtain the desired transversal. $\square$

We now closely follow Holmsen et al. [19]. Let $H(z)$ be the plane parallel to the xy-plane at height z. For two non-overlapping unit balls A, B in $\mathbb{R}^3$ and any $z \in \mathbb{R}$, let $\mathfrak{K}(AB, z)$ be the set of angles θ such that there is a directed transversal meeting A before B, lying in $H(z)$, and making angle θ with the positive x-axis.

Lemma 16. *Given two non-overlapping unit balls A and B in $\mathbb{R}^3$ with $\theta_1 \in \mathfrak{K}(AB, z_1)$ and $\theta_2 \in \mathfrak{K}(AB, z_2)$. Then there is an $\varepsilon > 0$ such that the interval $[\theta_0 - \varepsilon, \theta_0 + \varepsilon] \subset \mathfrak{K}(AB, z_0)$, where $z_0 = (z_1 + z_2)/2$ and θ_0 bisects the smaller angle between θ_1 and θ_2.*

Proof. Since the statement of the lemma is invariant under coordinate transformations that keep the normal vector of $H(z)$ fixed, we can assume that A has center $(0, 0, 0)$ and B has center $(d, 0, b)$, with $d > 0$ and $0 \leqslant b \leqslant 2$. For $b - 1 \leqslant z \leqslant 1$ the intersections $H(z) \cap A$ and $H(z) \cap B$ are two disks of radius $R(z)$ and $R(z - b)$, respectively, where

$$R(z) = \sqrt{1 - z^2}.$$

A directed line in $H(z)$ meeting A before B makes an angle $\alpha \in (-\pi/2, \pi/2)$ with the positive x-axis. Such transversals exist for $b - 1 \leqslant z \leqslant 1$, and so we can assume $b-1 \leqslant z_1 \leqslant z_0 \leqslant z_2 \leqslant 1$. For a fixed z with $b-1 \leqslant z \leqslant 1$, transversals with orientation α exist for $-G(z) \leqslant \alpha \leqslant G(z)$, where

$$G(z) = \arcsin(f(z)) \qquad \text{and} \qquad f(z) = \frac{R(z) + R(z-b)}{d},$$

with one exception: If A and B touch, they do so in $H(b/2)$. In that case $f(b/2) = 1$ and $G(b/2) = \pi/2$, but transversals exist only for $-\pi/2 < \alpha < \pi/2$.

To prove the lemma, it suffices to show that $G(z_0) > (G(z_1) + G(z_2))/2$. We will show that in fact the function $G(z)$ is strictly convex by showing that $G''(z) < 0$.

The function $G(z)$ is symmetric about $z = b/2$. When A and B touch, then $G(z)$ is not differentiable in $z = b/2$, but in that case $G(b/2) = \pi/2$ and this is clearly the maximum. It therefore suffices to show $G''(z) < 0$ for $b/2 < z < 1$. In this range, we have $f(z) < 1$, and

$$f'(z) = \frac{R'(z) + R'(z-b)}{d}, \qquad f''(z) = \frac{R''(z) + R''(z-b)}{d},$$

$$f'''(z) = \frac{R'''(z) + R'''(z-b)}{d},$$

where

$$R'(z) = \frac{-z}{(1-z^2)^{1/2}}, \qquad R''(z) = \frac{-1}{(1-z^2)^{3/2}},$$

$$R'''(z) = \frac{-3z}{(1-z^2)^{5/2}}, \qquad R''''(z) = \frac{-12z^2 - 3}{(1-z^2)^{7/2}}.$$

We note that $R''(z) < 0$ and $R''''(z) < 0$ for all $-1 < z < 1$, which implies that $R'(z)$ and $R'''(z)$ are strictly decreasing in this range. Therefore $f'(z)$ and $f'''(z)$ are strictly decreasing in the range $b/2 < z < 1$. Since $f'(b/2) = f'''(b/2) = 0$, this means that $f'(z) < 0$ and $f'''(z) < 0$ for $b/2 < z < 1$. We have next

$$G'(z) = \frac{f'(z)}{(1-(f(z))^2)^{1/2}} \qquad \text{and} \qquad G''(z) = \frac{g(z)}{(1-(f(z))^2)^{3/2}},$$

where

$$g(z) = f''(z)(1-(f(z))^2) + f(z)(f'(z))^2.$$

The sign of $G''(z)$ is determined by $g(z)$, which is well defined and differentiable at $z = b/2$ even when A and B touch. Since $f'(b/2) = 0$ and

$f''(b/2) < 0$ we have $g(b/2) \leqslant 0$, with equality only if A and B touch. We have

$$g'(z) = f'''(z)(1 - (f(z))^2) + (f'(z))^3 < 0 \qquad \text{for } b/2 < z < 1,$$

since $f'''(z) < 0$ and $f'(z) < 0$. It follows that $g(z)$ is strictly decreasing in the range $b/2 < z < 1$, which implies $g(z) < g(b/2) \leqslant 0$. Consequently $G''(z) < 0$ for $b/2 < z < 1$, completing the proof. □

We now have all the necessary tools.

Proof of Lemma 7. The lemma is true with $t^* = 0$ if the centers are all collinear, so assume this is not the case. We set $t^* > 0$ to be the infimum over all radii t where $\mathfrak{F}(t)$ has a transversal with the given order. It follows from the compactness of the balls that $\mathfrak{F}(t^*)$ has a transversal ℓ_1 with the correct order. Assume for a contradiction, that $\mathfrak{F}(t^*)$ has a second line transversal $\ell_2 \neq \ell_1$ with the same order. We will argue that then there is another transversal ℓ with the same order that intersects the interior of every ball in $\mathfrak{F}(t^*)$. This implies that there is an $\varepsilon > 0$ such that ℓ is a transversal for the family $\mathfrak{F}(t^* - \varepsilon)$ as well, a contradiction.

If ℓ_1 and ℓ_2 are parallel, then the entire strip bounded by the two lines intersects all balls, and we can choose ℓ to be any line inbetween. Assume next that ℓ_1 and ℓ_2 are not parallel, and choose a coordinate system where they are parallel to the xy-plane. Let ℓ_i, for $i \in \{1, 2\}$, lie in the plane $H(z_i)$ and make angle θ_i with the positive x-axis. Let $z_0 = (z_1 + z_2)/2$ and let θ_0 be the angle bisecting θ_1 and θ_2 as in Lemma 16.

For every pair $1 \leqslant i < j \leqslant j$, Lemma 16 guarantees the existence of an $\varepsilon_{ij} > 0$ such that the interval $[\theta_0 - \varepsilon_{ij}, \theta_0 + \varepsilon_{ij}] \subset \mathfrak{K}(B_i(t^*)B_j(t^*), z_0)$. Setting $\varepsilon = \min_{i<j} \varepsilon_{ij}$, we have

$$[\theta_0 - \varepsilon, \theta_0 + \varepsilon] \subset \bigcap_{i<j} \mathfrak{K}(B_i(t^*)B_j(t^*), z_0).$$

Consider now the family of disks in $H(z_0)$ obtained as the intersection of each ball $B_i(t^*)$ with $H(z_0)$. For any angle $\theta \in [\theta_0 - \varepsilon, \theta_0 + \varepsilon]$, consider the projection of the disks on the orthogonal complement $\ell^\perp$ of a line with orientation θ. Each disk projects on an interval. The intersection of the projections of $B_i(t^*)$ and $B_j(t^*)$ is non-empty, since $\theta \in \mathfrak{K}(B_i(t^*)B_j(t^*), z_0)$. By Helly's theorem in one dimension, this implies that the intersection of all intervals is not empty. Therefore there exists a line transversal to the disks with orientation θ, and by construction it meets the disks in order. Since this holds for any angle θ in this interval, we have a transversal intersecting the interior of each disk by Lemma 15. □

Appendix B. Semi-algebraic reformulation of Conjectures 1 and 2

Counter-examples to Conjectures 1 and 2 can be expressed as solutions of sets of polynomial equalities and inequalities, therefore reducing these conjectures to the question of the emptiness of a semi-algebraic set. Various algorithms are known to answer this question and several implementations are available [23], so in principle settling our conjecture is only a matter of computational resources. The resources needed to solve a given problem can be greatly influenced by the modeling of the problem. We therefore believe that although our attempts in this direction failed, there is value in summarizing our efforts.

Tangency condition.. Our first formulation yields a semi-algebraic set defined in $\mathbb{R}^{10}$ by four equalities (two of of degree 6, two of degree 10) and twelve quadratic inequalities. It describes the configurations of four non-overlapping unit balls $\{A, B, C, D\}$ and two common tangents to these balls in the geometric permutations $ABCD$ and $ACDB$. It follows from Lemma 10 and Theorem 13 that the existence of such a configuration is equivalent to falsifying Conjectures 1 and 2.

Up to translation and symmetry we can assume that a is at the origin, b is on the x-axis and c is on the xy-plane. The four points can thus be described using six variables $x_b, x_c, y_c, x_d, y_d, z_d$. We next argue that parameterizing the directions of the two lines, rather than the lines themselves, is sufficient. Indeed, the geometric permutation realized by a line tangent or transversal to the four balls can be read from its direction vector $\vec{v}$ alone: that line meets X before Y if and only if $\vec{v} \cdot \overrightarrow{xy} > 0$. Also, we can assume that the centers of the balls are not coplanar, since in this case Conjecture 1 is known to hold, so Equation (6) of Borcea et al. [24] allows to retrieve the full description of the line from its direction and the coordinates of the centers of the balls.

So let $\overrightarrow{v_1}, \overrightarrow{v_2}$ denote the direction vectors of two common tangents in order, respectively, $ABCD$ and $ACDB$. Since no line parallel to the yz-plane can be a common tangent to the balls A and B, up to scaling we can write the two vectors $\overrightarrow{v_1} = (1, q, r)$ and $\overrightarrow{v_2} = (1, s, t)$. The condition that $\overrightarrow{v_i}$ is a direction of a common tangent to the four balls is equivalent to Equations (7) and (8) of Borcea et al. [24]; we thus have for each $\overrightarrow{v_i}$ two equalities, one of degree 6 with 27 terms, the other of degree 10 with 195 terms. We then require that the balls be disjoint by adding six quadratic inequalities that require that the squared distance between any two centers is at least 4. We finally ensure that the two lines meet the balls in the right order by six bilinear inequalities that constrain the signs of $\overrightarrow{v_1} \cdot \overrightarrow{ab}$, $\overrightarrow{v_1} \cdot \overrightarrow{bc}$, $\overrightarrow{v_1} \cdot \overrightarrow{cd}$ and likewise for $\overrightarrow{v_2}$.

Pinning conditions.. Our second formulation builds on Conjecture 2 and yields a semi-algebraic set defined, essentially, in $\mathbb{R}^{10}$ by six equalities of degree 4, six linear inequalities and six inequalities of degree 4.

We first describe an alternating hyperboloidal configuration $(\mathfrak{F}, \sigma)$ using 5 variables $(h, t_a, t_b, t_c, t_d) \in \mathbb{R}^5$, six degree-four inequalities and three linear inequalities. Specifically, we equip $\mathbb{R}^3$ with an orthonormal frame such that σ is the x-axis and its minimal pinning by $\mathfrak{F}$ is witnessed by the hyperbolic paraboloid with equation $xy = -hz$; the centers of the balls are thus on the quadric with equation $xy = hz$. The position of ball W is given by the variable t_w that represents the x-coordinate of the tangency point of W and σ. Assuming the balls have unit radius, the position of the center is then:

$$w = \left(\frac{2ht_w}{1 - t_w^2}, \frac{1 - t_w^2}{1 + t_w^2}, \frac{2t_w}{1 + t_w^2} \right)$$

It remains to require that the balls be non-overlapping (six inequalities of degree four bounding from below the squared distances between centers) and to specify the order in which the balls touch the line σ (three linear inequalities ordering the t_w's).

Our semi-algebraic set then describes the existence of two configurations of the previous type that realize the geometric permutations $ABCD$ and $ACDB$ and where the tetrahedra of centers are isometric.[6] Indeed, if two such configurations exist then a rigid motion that maps each ball from the first configuration to the matching ball in the second configuration will send the

[6]Two tetrahedra with equal edge-lengths are either isometric or one is isometric to a reflection of the other. A point satisfying all these constraints may thus not correspond to a pair of isometric minimal-pinning configurations. Yet, mirroring one of the configurations would give a pair of isometric minimal pinnings so as far as we only care about existence, this system is fine as is.

x-axis of the first configuration to a transversal realizing $ABCD$ in the second configuration. Conversely, if a counter-example to Conjectures 2 exists then it gives rise to a pair of configuration as described above.

We build our system by picking two independent sets of variables (h, t_a, t_b, t_c, t_d) and $(h', t'_a, t'_b, t'_c, t'_d)$, collecting the linear inequalities enforcing the orders $ABCD$ on one configuration and $ACDB$ on the other, collecting the six degree-four inqualities enforcing that the first configuration is non-overlapping (we drop their counterparts in the second configuration as they are redundant if the configurations are isometric) and adding the condition $|uv|^2 = |u'v'|^2$ for each of the six pairs in $\{a, b, c, d\}$.

We expect the variety defined by these equations (ie. dropping the inequations) to have dimension 4: the lower bound comes from the system (10 variables minus 6 equations, hoping for transverse intersection) and the upper bound comes from the geometry (dimension 5 or more would imply that in the space of configurations of four balls with two tangents, the set of configurations where both tangents are pinned has codimension 1 whereas we expect that when deforming such a configuration, both pinnings need not happen simultaneously). The system contains a 5-dimensional degenerate component that corresponds to $hh' = 0$, as the parameterization then degenerates. We therefore add one variable u and the equality $u * h * h' = 1$ so that the existence of a solution in u forces the other term to be non-zero.[7]

Discussion.. The first system seems a challenge for symbolic methods such as critical point techniques. Indeed, such methods typically first operate on the underlying variety, obtained by dropping from the system all inequalities. This variety therefore corresponds to configurations of four, possibly intersecting, unit balls with two common tangents, in any order. That variety appears to be unmixed, that is, it contains components of different dimensions, and to contain a singular locus of dimension at least 2 (those configurations where a tangent is pinned). Already an equidimensional decomposition of this system seems out of reach at the moment. The second system seems better suited but also seems out of reach at the moment.

[7]This trick of enforcing an inequality $I \neq 0$ by adding a variable u and an equality $uI = 1$ is known as "saturation".