Persistent homoclinic tangencies and infinitely many sinks for residual sets of automorphisms of low degree in $\mathbb{C}^3$

Contents

1 Introduction

1.1 Background

Hyperbolic systems such as the horseshoe introduced by Smale were originally supposed to be dense in the set of parameters in the 1960's. This was quickly discovered to be false in general for diffeomorphisms of manifolds of dimension greater than 2 (see [1]). The discovery in the seventies of the so-called Newhouse phenomenon, i.e. the existence of residual sets of C^2 diffeomorphisms of compact surfaces with infinitely many sinks (periodic attractors) in [15] showed it was false in dimension 2 too. In a subsequent work, Newhouse showed (see [16]) that such sets appear in fact close to any diffeomorphism with an homoclinic tangency. The technical core of the proof is the reduction to a line of tangency between the stable and unstable foliations where two Cantor sets must have persistent intersection. This gives persistent homoclinic tangencies between the stable and unstable foliations, ultimately leading to infinitely many sinks. Indeed, it is a well known fact that a sink is created in the unfolding of a generic homoclinic tangency.

Palis and Viana showed in [17] an analogous result for real diffeomorphisms in higher dimensions. More precisely, they proved that near any smooth diffeomorphism of $\mathbb{R}^3$ exhibiting a homoclinic tangency associated to a sectionally dissipative saddle, there is a residual subset of an open set of diffeomorphisms such that each of its elements displays infinitely many coexisting sinks.

In the complex setting, this reduction is not possible anymore and to get persistent homoclinic tangencies and then residual sets of diffeomorphisms displaying infinitely many sinks, we have to intersect two Cantor sets in the plane. This was done by Buzzard who proved in [7] that there exists $d > 0$ such that there exists an automorphism $G \in \mathrm{Aut}_d(\mathbb{C}^2)$ and a neighborhood $N \subset \mathrm{Aut}_d(\mathbb{C}^2)$ of G such that N has persistent homoclinic tangencies. Then, there is a residual subset of $\mathrm{Aut}_d(\mathbb{C}^2)$ of automorphisms with infinitely many sinks. In fact, after extending the stable and unstable foliations of a basic set, there is still a complex disk of tangency where intersections with the two foliations are two Cantor sets in two (real) dimensions. Buzzard gives an elegant criterion (see [6]) which generates the intersection of two such Cantor sets, hence leading to persistent homoclinic tangencies. More precisely, he gets persistent intersections between a "spiralic" Cantor set and a second Cantor set with high topological dimension. In his article, Buzzard uses a Runge approximation argument to get a polynomial automorphism, which implies that the degree d remains unknown and is supposedly very high.

In the article [4], Bonatti and Diaz introduce a type of horseshoe they called blender horseshoe. The important property of such hyperbolic sets lies in the fractal configuration of one of their stable/unstable manifold which implies persistent intersection between any well oriented graph and this foliation. In some sense, the foliation behaves just as it had greater Hausdorff dimension than every individual manifold of the foliation. They find how to get robust homoclinic tangencies for some C^r-diffeomorphism of $\mathbb{R}^3$ with an homoclinic tangency by some geometric intersection procedure using the properties which define the blender in [5] . In the article [9] , one can find real polynomial maps of degree 2 with a blender. Other studies using blenders include [8], [2] and [3].

1.2 Results and outline

In this article, we adapt a complex blender. More precisely, we solve here the degree problem in dimension 3 by introducing a kind of 3-dimensional central-stable complex blender. Here is our main result :

Main Theorem. *There exists* $f \in Aut_5(\mathbb{C}^3)$ *such that for every* $g \in Aut(\mathbb{C}^3)$ *sufficiently close to* f, g *admits a tangency between the stable and unstable laminations of some hyperbolic set.*

Notice that in the previous result, g is not assumed to be polynomial.

Corollary 1. *For each* $d \geq 5$, *there exists an open set* $\mathcal{V} \subset Aut_d(\mathbb{C}^3)$ *in which the automorphisms having a homoclinic tangency are dense.*

Corollary 2. *For each* $d \geq 5$, *there exists an open set* $\mathcal{V} \subset Aut_d(\mathbb{C}^3)$ *in which the automorphisms having infinitely many sinks are dense.*

Let us remark that there are classes of interesting polynomial automorphisms of $\mathbb{C}^3$ called regular and semi-regular automorphisms which have received much attention due to their interesting dynamical properties (e.g [13],[18]). It is possible to choose the automorphisms to be regular or semi-regular in the above results because the condition of being regular or semi-regular is dense for the Zariski topology.

Let us present quickly our method to prove this result. We consider an automorphism of $\mathbb{C}^3$ which is a perturbation of the following map f_1 :

$$f_1 : (z_1, z_2, z_3) \to (p(z_1) + b.z_2, z_1, \lambda.z_1 + \mu.z_3) \tag{1}$$

where p is a polynomial and the coefficients b, λ, μ are complex numbers. Then, we prove that f_1 has a hyperbolic set with the property that the stable lamination of dimension 2 can be fixed with enough freedom while the 1-dimensional unstable lamination presents one direction where the lamination is very thick. Moreover, this hyperbolic set has a fixed point which is sectionally dissipative. Perturbating the automorphism f_1 creates an initial heteroclinic tangency. Then, we show that the two laminations are such that there are robust heteroclinc tangencies. Then we get robust homoclinic tangencies. By a classical Baire category argument, this gives a subset of the set of automorphisms of limited degree 5 in which automorphisms with infinitely many sinks are dense.

In section 2, we precise the automorphism. We give first a complex polynomial p, then a 2-dimensional Henon automorphism h using p and finally the 3-dimensional automorphism f_1 by adding to h a third linear component.
In section 3 and 4, we describe the basic set : we prove propositions which precise the transversal shape of the unstable foliation.
In section 5, then we perturb f_1 to get a heteroclinic tangency with good orientation of the two foliations. This is done by increasing the degree of p and adding a small term in the first component of f_1.
In section 6, we prove that these heteroclinic tangencies are robust and we conclude the proof of the main result in section 7.
For the convenience of the reader we recall how to create sinks from homoclinic

tangencies in the sectionally dissipative case.

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2 Preliminaries

2.1 One complex dimension

We denote by $\mathbb{D} \subset \mathbb{C}$ the unit disk.

Lemma 2.1.1. *For $r = \frac{1}{10}$ and $r' = \frac{1}{100}$, there exists a polynomial p of degree 4 such that :*
(i) $p^{-1}(\mathbb{D}) \subset\subset \mathbb{D}$ and $p^{-1}(\mathbb{D})$ admits 3 components D_1, D_2, D_3
(ii) For D_1 and D_2 we have : $D_1 \subset B(1-r, r')$, $D_2 \subset B(r-1, r')$, $p_{|D_i}$ is a biholomorphism and $|p'| > 100$ on D_i for $i \in \{1, 2\}$
(iii) p admits a fixed point $\alpha_p \in D_3$ such $1 < |p'(\alpha_p)| < 1.1$

Proof. Let us start with a real polynomial $z^2 + c$ where $c = 0.249$. We modify it by adding a term of degree 4 : $p(z) = z^2 + 0.249 - \frac{d}{R^2}.z^4$ on a disk of radius R. Then, it suffices to rescale to get the result on $\mathbb{D}$. We want to choose R such that the following inequalities hold :

$$p((1 - r + 2r')R) < -2R \tag{2}$$

$$p((1 - r - 2r')R) > 2R \tag{3}$$

that is :

$$\begin{cases} ((1-r+2r')R)^2 - \frac{d}{R^2}(1-r+2r')^4)R^4 + 0.249 < -2R \\ ((1-r-2r')R)^2 - \frac{d}{R^2}(1-r-2r')^4)R^4 + 0.249 > 2R \end{cases}$$

is equivalent to :

$$\begin{cases} ((1-r+2r')^2 - d(1-r+2r')^4)R^2 + c < -2R \\ ((1-r-2r')^2 - d(1-r-2r')^4)R^2 + c > 2R \end{cases}$$

Once for all, we take $d = \frac{1}{(1-r)^2}$. Then, if we take R sufficiently large, the two inequalities are verified.

The critical points of p are $\{0, c_p = +\frac{R}{\sqrt{2d}}, -\frac{R}{\sqrt{2d}}\}$. Since $p(0) = c \in B(0, R)$ and :

$$p(\frac{R}{\sqrt{2d}}) = \frac{R^2}{2d} + c - d.\frac{R^4}{4d^2R^2} = \frac{R^2}{4d} + c \simeq \frac{R^2}{4d} > R$$

by the Riemann-Hürwitz formula we see that $p^{-1}(B(0, R))$ admits 3 components, two of them are univalent, we call them D_1 and D_2, and one is not, we call it D_3. In $\mathbb{R}$, $p^{-1}((-R, R))$ admits 3 components I_1, I_2, I_3 with I_1 near $-R$, I_2 near $+R$ and $0 \in I_3$ (see Figure 1). Moreover, the 3 components D_1, D_2, D_3 of $p^{-1}(D(0, R))$ in $\mathbb{C}$ are such that $D_i \cap \mathbb{R} = I_i$. Indeed, p is real, the sets D_i are symmetric w.r.t. to $\mathbb{R}$ and simply connected by the maximum principle so the intersections $D_i \cap \mathbb{R}$ must be intervals. $p_{|D_i}$ is a biholomorphism for $i \in \{1, 2\}$ and we have : $p^{-1}(\mathbb{D}) \subset\subset \mathbb{D}$.

Then, let us consider the univalent map $q = p^{-1}_{|D'_1} : B(0, 2R) \to D'_1$ where D'_i is the component of $p^{-1}(D(0, 2R))$ such that : $D_i \subset D'_i$. Since $p((1-r+2r')R) < -2R$ and $p((1-r-2r')R) > 2R$, there is a point $z_q \in B(0, R)$ such that : $|p'(q(z_q)| > \frac{2R}{4r'} = \frac{R}{2r'}$, that is $|q'(z_q)| < \frac{2r'}{R}$. Now, by the Koebe Theorem we know that for an univalent map $g : \mathbb{D} \to \mathbb{C}$ we have the following inequalities :

$$\forall z \in \mathbb{D}, \frac{1-|z|}{(1+|z|)^3} < \frac{|g'(z)|}{|g'(0)|} < \frac{1+|z|}{(1-|z|)^3}$$

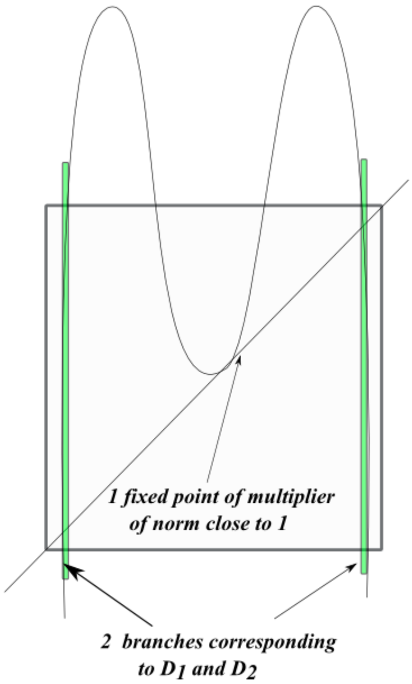

Figure 1 : graph of p on the real line

Applying this result to q and the ball $B(0, R) \subset B(0, 2R)$ we get that $\forall z \in B(0, R), 12.|q'(0)| > |q'(z)| > \frac{4}{27}.|q'(0)|$ then : $\forall z \in B(0, R), |q'(z)| < 144.|q'(z_q)| < \frac{144.2r'}{R}$ then : $\forall z \in D_1, |p'(z)| > \frac{R}{288r'} > \frac{1}{r'} = 100$ and $\forall z \in B(0, R), |q'(z)| < r'$ if $R > 288$. This inequality implies that $D_1 \subset B(1-r, r')$. Then, $|p'| > 100$ on D_1. The corresponding result holds for D_2.

The multiplier of the repelling fixed point $z^2 + 0.249$ is smaller than 1.1 in modulus, so increasing the value of R if necessary we get the same estimate for p and we are done.

□

Remark 2.1.2. *Let us further introduce two sets D''_1 and D''_2 such that : $D_1 \subset D''_1 \subset D'_1$ and $D_2 \subset D''_2 \subset D'_2$, D''_1 and D''_2 are 2 components of $p^{-1}(D(0, \frac{3R}{2}))$. The Koebe Theorem gives us that $|p'| > 100$ on D''_i.*

2.2 Two complex dimensions

We now perturb the polynomial p into a complex Henon map with small Jacobian $-b$.

$$h : (z_1, z_2) \mapsto (p(z_1) + bz_2, z_1)$$

In this subsection, we will denote by C^u the two-dimensional cone centered at e_1 of opening $\frac{1}{10}$ and C^{ss} the two-dimensional cone centered at e_2 of opening $\frac{1}{10000}$. In the next subsection, we will introduce analogous three dimensional cones. To simplify the notations, we will denote them too by C^u and C^{ss}.

Definition 2.2.1. *Given an automorphism $F : \mathbb{C}^2 \to \mathbb{C}^2$, we say that $\bigcap_{-\infty}^{\infty} F^n(\mathbb{D}^2)$ is a horseshoe for F if :*
- $F(\mathbb{D}^2) \cap \mathbb{D}^2$ is an union of two bidisks $\mathcal{D}^1$ and $\mathcal{D}^2$ such that $\mathcal{D}^i \cap \partial(\mathbb{D}^2)$ is included in $\partial(\mathbb{D}) \times \mathbb{D}$ for $i = 1, 2$
- on $F^{-1}(\mathbb{D}^2) \cap F(\mathbb{D}^2)$, the cone field C^u is F-invariant and the cone field C^{ss} is F^{-1}-invariant
- there exists $C_F > 1$ such that on $F^{-1}(\mathbb{D}^2) \cap F(\mathbb{D}^2)$:

$$\forall u \in C^u, ||DF(u)|| > C_F||u|| \text{ and } \forall v \in C^{ss}, ||D(F^{-1})v|| > C_F||v||$$

In the following, we will have to take the constant b such that $0 < |b| < b_i$ where b_i is a bound which will be reduced a finite number of times so that it will ensure some properties on f_1. The following result is classical. For instance, it follows from the work of Hubbard-Oberste-Vorth, see [14] or [11], we just give a justification for the constants of the cones.

Proposition 2.2.2. *There exists a positive number b_0 such that if $|b| < b_0$, then $\bigcap_{-\infty<n<+\infty} h^n(\mathbb{D}^2)$ is a horseshoe for $h_{|\mathbb{D}^2}$. Besides, the fixed point α_h continuation of the fixed point α_p of p is a saddle point of expanding eigenvalue $1 < |\xi_h| < 1.1$.*

Proof. We have that $|p'| > 100$ on $D_1'' \cup D_2''$. This implies that there exists a positive number b_0 such that if $|b| < b_0$, then C^u is h-invariant and C^{ss} is h^{-1}-invariant with $C_h = 50$. □

Definition 2.2.3. *We denote $h[1]$ by the restriction of h to $D_1'' \times \mathbb{D}$ and by $h[2]$ the restriction of h to $D_2'' \times \mathbb{D}$ (where D_1'' and D_2'' were defined in the precedent Remark). Then we inductively define $h[I]$ for an arbitrary sequence of digits by $h[Ij] = h[j] \circ h[I]$ on $(h[I])^{-1}(D_j'' \times \mathbb{D})$. We define : $H_1 = (h[1])(D_1'' \times \mathbb{D}) \cap \mathbb{D}^2$, $H_2 = (h[2])(D_2'' \times \mathbb{D}) \cap \mathbb{D}^2$ and $H_{Ij} = (h[j])(U_I) \cap \mathbb{D}^2$.*

The following Proposition is a consequence of the general study on complex horseshoes.

Proposition 2.2.4.
(i) For each finite sequence of indices $I = (i_1, ..., i_p)$, $i_j \in \{1, 2\}$, the set H_I is of the form : $H_I = \bigcup_{z_1 \in \mathbb{D}} \{z_1\} \times I_I^{z_1}$ where $I_I^{z_1}$ is an open topological disk included in $\mathbb{D}$.

(ii) If I is an infinite sequence $I \in \{1, 2\}^{\mathbb{N}}$, the intersection $H_I = \bigcap_{p \geq 0} H_{I \leq p}$ is a piece of the intersection of the unstable manifold of one point of the set $\bigcap_{-\infty}^{\infty} h^n(\mathbb{D} \times \mathbb{D})$ with $\mathbb{D}^2 \times Q$ (where $I \leq p$ denotes the finite subsequence of the p first digits of I).

Let us call $t_{z_1,I} = \delta_1(I_I^{z_1}) = \delta_2(\{z_1\} \times I_I^{z_1})$. We call $t_n = \max_{z_1,|I|=n} t_{z_1,I}$. We first prove that the rate of decay of t_n can be made arbitrary small.

Lemma 2.2.5. *For all $k \in]0,1[$, there exists $0 < b_1 < b_0$ such that for all $|b| < b_1$, $n \in \mathbb{N}$ we have : $t_{n+1} < kt_n$. Consequently : $\sum_{n=0}^{+\infty} t_n < \frac{1}{1-k}$*

Proof. Let us prove the lemma by induction. The property is obvious for $I = \emptyset$. Let us suppose the lemma is proven for all the I until a certain rank. We bound here $t_{z_1,I\cup\{i_{p+1}\}}$. The intersection of the set $H_I = h[I](\mathbb{D}^2)$ with $p(z_1) + bz_2 = C^{st}$ is a curve Γ_I such that $\delta_1(\Gamma_I) < k.t_n$ if we take $0 < b_1 < b_0$ sufficiently small. Mapping by $h : (z_1, z_2) \mapsto (p(z_1) + b.z_2, z_1)$, we get : $t_{n+1} < k.t_n$. □

2.3 Three complex dimensions

We consider now the 3 dimensional map f_1 introduced in (1) :

$$f_1 : (z_1, z_2, z_3) \mapsto (p(z_1) + b.z_2, z_1, \lambda.z_1 + \mu.z_3)$$

In the following, we will see that the first direction is expanded by f_1 and corresponds to the direction of the unstable manifolds of a hyperbolic set we are going to describe. The second and third directions are contracted by f_1 and correspond to the direction of the stable manifolds of this hyperbolic set. We fix 3 small angular cones C^u, C^{ss}, C^{cs} centered in the three axis of coordinates with thin opening.

Notation. *C^u is the cone centered at e_1 of opening $\frac{1}{10}$, C^{ss} is the cone centered at e_2 of opening $\frac{1}{10000}$ and C^{cs} is the cone centered at e_3 of opening $\frac{1}{10000}$.*

Definition 2.3.1. *Given an automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$, we say that $\bigcap_{-\infty}^{\infty} F^n(\mathbb{D}^2 \times Q)$ is a horseshoe for F if :*
- $F(\mathbb{D}^2 \times Q) \cap (\mathbb{D}^2 \times Q)$ is an union of two tridisks $\mathcal{D}^1$ and $\mathcal{D}^2$ such that $\mathcal{D}^i \cap \partial(\mathbb{D}^2 \times Q)$ is included in $(\partial(\mathbb{D}) \times \mathbb{D} \times Q)$ for $i = 1, 2$
- on $F^{-1}(\mathbb{D}^2 \times Q) \cap F(\mathbb{D}^2 \times Q)$, the cone field C^u is F-invariant and the cone fields C^{ss}, C^{cs} are F^{-1}-invariant
- there exists $C_F > 1$ such that on $F^{-1}(\mathbb{D}^2 \times Q) \cap F(\mathbb{D}^2 \times Q)$:

$$\forall u \in C^u, ||DF(u)|| > C_F||u|| \text{ and } \forall v \in C^{ss} \cup C^{cs}, ||D(F^{-1})v|| > C_F||v||$$

Definition 2.3.2. *We say that a saddle periodic point q of multipliers $|\lambda_1| \leq |\lambda_2| < 1 < |\lambda_3|$ is sectionally dissipative if the product of any two of its eigenvalues is less than 1 in modulus, that is, $|\lambda_1\lambda_3| < 1$ and $|\lambda_2\lambda_3| < 1$.*

Once for all, we fix now :

$$\mu = \frac{9}{10}i, \ \lambda = \frac{1}{10}\frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}} = \frac{1}{10}(1+i) \quad (4)$$

We denote by pr_i the projection on the i^{th} coordinate in $\mathbb{C}^3$.

Proposition 2.3.3. *Let $f_1 : (z_1, z_2, z_3) \to (p(z_1) + b.z_2, z_1, \lambda.z_1 + \mu.z_3)$ with $|b| < b_1$. Then, $\bigcap_{-\infty<n<+\infty} f_1^n(\mathbb{D}^2 \times Q)$ is a horseshoe. Moreover, f_1 has a fixed point α_{f_1} that is sectionally dissipative.*

Proof. The fact that C^u is f_1-invariant and C^{ss} is f_1^{-1}-invariant comes from the analogous result on h. The fact that C^{cs} is f_1^{-1}-invariant comes from the fact that e_3 is an eigenvector at each point for Df_1^{-1} of associated eigenvalue $\frac{1}{\mu} = \frac{10}{9i}$. The existence of α_{f_1} is obvious. Looking at the differential, we see that α_{f_1} has three eigenvalues $\xi_h, \mu, \frac{-b}{\xi_h}$ so the result follows since $1 < |\xi_h| < 1.1$, $|\mu| = \frac{9}{10}$ and $\frac{9}{10}.\frac{11}{10} < 1$. □

Definition 2.3.4. *We denote $f_1[1]$ the restriction of f_1 on $D_1'' \times \mathbb{D} \times Q$ and $f_1[2]$ the restriction of f_1 on $D_2'' \times \mathbb{D} \times Q$ (where D_1'' and D_2'' were defined in Remark 2.1.2). Then, we inductively define $f_1[I]$ for an arbitrary sequence of digits by $f_1[Ij] = f_1[j] \circ f_1[I]$ on $(f_1[I])^{-1}(D_j'' \times \mathbb{D} \times Q)$. We define $U_1 = (f_1[1])(D_1'' \times \mathbb{D} \times Q) \cap \mathbb{D}^3$, $U_2 = (f_1[2])(D_2'' \times \mathbb{D} \times Q) \cap \mathbb{D}^3$ and $U_{Ij} = (f_1[j])(U_I) \cap (\mathbb{D}^2 \times Q)$.*

In the following, we denote by $\mathbb{D}_{.,z_2,z_3}$ the disk parallel to the z_1 axis. In this subsection we gather a few properties of complex horseshoes which will be useful afterwards. In particular, horseshoes are compact, hyperbolic, transitive and locally invariant sets.

Lemma 2.3.5. *For $|b| < b_1$ and for every sequence of digits I, $f_1[I](\mathbb{D}_{.,z_2,z_3})$ is a graph over the first coordinate z_1 of the form*

$$f_1[I](\mathbb{D}_{.,z_2,z_3}) = \{(z_1, y_I^2(z_1), y_I^3(z_1)) : z_1 \in \mathbb{D}\}$$

with derivatives $|(y_I^2)'| < \frac{1}{50}$, $|(y_I^3)'| < \frac{1}{50}$.

Proof. We show the result by induction. We suppose that the conditions are verified for each step until I and we consider the following index $I' = I \cup \{j\}$. In C_I: $z_2 = y_I^2(z_1)$ with the derivative of y_I^2 in modulus bounded by $\frac{1}{5}$: $|(y_I^2)'(z_1)| < \frac{1}{50}$.

Every curve $\{(z_1, y_I^2(z_1), y_I^3(z_1)) : z_1 \in \mathbb{D}\}$ intersects, if $|b| < b_1$, each hypersurface $p(z_1) + bz_2 = C^{st}$ only one and one time. Then the curve $f_1[I'](\mathbb{D}_{.,z_2,z_3})$ is a graph $\{(z_1, y_{Ij}^2(z_1), y_{Ij}^3(z_1)) : z_1 \in \mathbb{D}\}$. The implicit function Theorem says that the derivate of this graph is $\frac{1}{p'(z_1)+b(y_I^2)'(z_1)}$ inside this domain. Then, $\{z_1 : p(z_1) + by_I^2(z_1) \in \mathbb{D}\} = \overline{D_1} \cup \overline{D_2}$ with $\overline{D_i} \subset D_i''$, we have the bound $|p'| \geq 100$. Moreover, $|(y_I^2)'| < \frac{1}{50}$ and $b < 1$, so this gives : $|(y_{Ij}^2)'| < \frac{1}{50}$

So, $f_1[I](\mathbb{D}_{.,z_2,z_3})$ is a graph $\{(z_1, y_I^2(z_1), y_I^3(z_1)) : z_1 \in \mathbb{D}\}$. It remains to prove the bound on the derivative of $y_I^3(z_1)$. We do this by induction. For $|I| = \emptyset$, this is obvious. Then for $I' = I \cup \{j\}$, we see that $f_1[I'](\mathbb{D}_{.,z_2,z_3})$ is the image of a piece of $f_1[I](\mathbb{D}_{.,z_2,z_3})$ so we can write :

$$y_{I'}^3(z_1) = \mu.y_I^3(\zeta(z_1)) + \lambda.\zeta(z_1)$$

$$(y_{I'}^3)'(z_1) = \mu.(y_I^3)'(\zeta(z_1)).\zeta'(z_1)) + \lambda.\zeta'(z_1)$$

where $\zeta(z_1)$ denotes the first coordinate of the inverse image of the point of $C_{I'}$ whose first coordinate is z_1. The derivative of ζ is bounded by $\frac{1}{70}$ because $|p'| > 100$ on $D_1'' \cup D_2''$ and for small values of b, ζ' is near the derivate of the local inverse branch of p. We infer that $|(y_I^3)'(z_1)| \leq B_n$ where B_n satisfies $B_0 = 0$ and $B_{n+1} < \frac{1}{50}|\mu|B_n + \frac{1}{70}$ where $n = |I|$. It follows that for every $n \geq 0$, $B_n < \frac{1}{50}$ and we are done. □

We finish this section by a lemma saying that horizontal graphs are not very perturbated by adding a term of degree 5, $p_5 z^5$, to p.

Lemma 2.3.6. *For every sequence of digits* I, $f[I](\mathbb{D}_{\cdot,z_2,z_3}) = \{(z_1, y^2_{f,I}(z_1), y^3_{f,I}(z_1)) : z_1 \in \mathbb{D}\}$ *is a horizontal graph such that for all* $z_1 \in \mathbb{D}$, $p_5 \mapsto [(y^2_{f,I})(z_1)](p_5)$ *and* $p_5 \mapsto [(y^3_{f,I})(z_1)](p_5)$ *are 2-lipschitz continuous.*

Proof. The lemma is a consequence of the Schwarz-Pick Lemma applied to the two holomorphic maps $\mathbb{D} \mapsto \mathbb{D}, p_5 \mapsto [(y^1_{f,I})(z_1)](p_5)$ and $\mathbb{D} \mapsto \mathbb{D}, p_5 \mapsto [(y^2_{f,I})(z_1)](p_5)$. □

A consequence of this lemma is that the horizontal graphs $\bigcap_{p\geq 0} U_{I\leq p} = \mathcal{W} \cap (\mathbb{D}^2 \times Q) = \{(z_1, y^{\mathcal{W}}_f(z_1), y^{\mathcal{W}}_{f,2}(z_1)) : z_1 \in \mathbb{D}\}$ verify the same 2-Lipschitz continuous property.

3 Structure of the basic set : unperturbed case

3.1 Generalities and main result in the unperturbed case

In this section, we describe the geometry of a horseshoe in $\mathbb{C}^3$ which is induced by f_1. We describe a subset of the set $K_{f_1} = \bigcap_{-\infty}^{\infty} f_1^n(\mathbb{D} \times \mathbb{D} \times Q)$, more precisely $\bigcap_{-\infty}^{\infty} f_1^n((D_1 \cup D_2) \times \mathbb{D} \times Q)$. Recall that $f_1(z_1, z_2, z_3) = (p(z_1) + b.z_2, z_1, \lambda.z_1 + \mu.z_3)$. The choice of Q instead of $\mathbb{D}$ in the last coordinate is a matter of convenience only, since the projection on the 3^{rd} coordinate is easier to analyse in terms of subsquares (see Figure 3).

Definition-Proposition 3.1.1. *Given a set* $E \subset \mathbb{C}^k$ *and* $i \in \{1,2,3\}$, *we call* $\delta_i(E) = diam(pr_i(E))$. *Given* $z_1, z_2 \in \mathbb{C}$, *we let :* $L_{z_1,z_2} = \{(z_1, z_2)\} \times \mathbb{C}$.

Definition 3.1.2. *Given a set* $E \subset \mathbb{C}^2$ *and* $\delta \in (0,1)$, *we say that* E *is of* δ*-product type w.r.t the* 2^{nd} *coordinate if there is a square* $S = c_S + \ell_S.Q \subset \mathbb{C}$ *such that*

$$pr_1(E) \times (c_S + (1-\delta).\ell_S.Q) \subset E \subset pr_1(E) \times (c_S + (1+\delta).\ell_S.Q)$$

We call admissible square for E *a square* S *with this property.*

Proposition 3.1.3.
(i) For each finite sequence of indices $I = (i_1, ..., i_p)$, $i_j \in \{1,2\}$, *the set* U_I *is a fibration by squares over the set* H_I *(see Figures 2,3) :*

$$U_I = \bigcup_{(z_1,z_2)\in H_I} (z_1, z_2) \times Q_{z_1,z_2,I}$$

More precisely, $U_I = \bigcup_{z_1\in\mathbb{D}} \{z_1\} \times R_{z_1,I}$ *where* $R_{z_1,I}$ *is a non empty open connected subset in* $\mathbb{D} \times Q$ *of the form :*

$$R_{z_1,I} = \bigcup_{z_2 \in pr_1(R_{z_1,I})} \{z_2\} \times \{\beta_{z_1,z_2,I} + l_I.Q\} \quad (5)$$

where $Q_{z_1,z_2,I} = \{\beta_{z_1,z_2,I} + l_I.Q\}$ *is a square whose length side* l_I *is independent of* z_1 *and depends only of the length* $|I|$ *of* I *:* $l_I = |\mu|^{|I|}$.

(ii) If I is an infinite sequence $I \in \{1,2\}^{\mathbb{N}}$, the intersection $U_I = \bigcap_{p\geq 0} U_{I\leq p}$ is a piece of the intersection of the unstable manifold of one point of the set $\bigcap_{-\infty}^{\infty} f_1^n(\mathbb{D}\times\mathbb{D}\times Q)$ with $\mathbb{D}^2 \times Q$ (where $I \leq p$ denotes the finite subsequence of the p first digits of I).

The result is essentially a consequence of Proposition 2.2.4 and Lemma 3.1.5 below.

Let us first note the following obvious consequence of the definition of f_1.

Lemma 3.1.4. *For all z_1, z_2, $f_1(L_{z_1,z_2}) = L_{h(z_1,z_2)}$.*

The following lemma describes the geometry of non empty intersections of the form $L_{z_1,z_2} \cap (\{z_1\} \times R_{z_1,I})$:

Lemma 3.1.5. *For every z_1, z_2, if L_{z_1,z_2} intersects $\{z_1\}\times R_{z_1,I}$, then $pr_3(L_{z_1,z_2} \cap (\{z_1\} \times R_{z_1,I}))$ is a square inside the line L_{z_1,z_2}. We denote $\ell_{z_1,z_2,I}$ the length side of the square $L_{z_1,z_2} \cap U_I = L_{z_1,z_2} \cap (\{z_1\} \times R_{z_1,I})$ when this intersection is non empty. Then, for all z_1, I, $\ell_{z_1,z_2,I}$ is constant in z_1, z_2, we denote it by l_I and its value is $l_I = l_{z_1,z_2,I} = |\mu|^{|I|}$.*

Proof. The third coordinate of f, for a fixed value of z_1, is affine in z_3 of multiplier $\mu = \frac{9}{10} i$. This implies : $\forall I, l_{I\cup\{i_{p+1}\}} = \frac{9}{10} l_I$ and the result follows. □

In the following, the projections of the sets $R_{z_1,I}$ will have a special configuration. This is why we introduce the next definition.

Definition 3.1.6. *Let Q the unit square centered at the origin. For any square S whose axes are parallel to those of Q, we denote by NE, SE, SW, NW its four corners. Given such a square S of length size l_S we denote by $S_{c,s,s'}$ where $c \in \{NE, SE, SW, NW\}$, $s \in [\frac{1}{2}, 1[$, $s' \in]0, 1-s[$ the subsquare of side length $s.l_S$ positioned near the corner c at a distance $s' l_S$ of the two sides.*
A configuration NE-SW is a triple $(S, S_{c,s,s'}, S_{c',s,s'})$ where $S_{c,s,s'}$ and $S_{c',s,s'}$ are two subsquares of S such that $\{c, c'\} = \{NE, SW\}$, a configuration NW-SE is a triple $(S, S_{c,s,s'}, S_{c',s,s'})$ such that $S_{c,s,s'}$ and $S_{c',s,s'}$ are two subsquares of S and $\{c, c'\} = \{NW, SE\}$ with $s' < max(\frac{1}{50}, 1-s)$.

Proposition 3.1.7. *There is a constant $0 < b_2 < b_1$ such that for all $|b| < b_2$, we get the following properties :*

1. For every z_1, I, $R_{z_1,I}$ is of $\frac{1}{1000}$-product type.

2. For each admissible square $S_{z_1,I}$ for $R_{z_1,I}$, its length side satisfies

$$\ell_{z_1,I} \in \left(\frac{899}{1000} l_I, \frac{901}{1000} l_I\right)$$

3. Let $S_{z_1,I}$, $S_{z_1,I\cup\{1\}}$ and $S_{z_1,I\cup\{2\}}$ be admissible squares for $R_{z_1,I}$, $R_{z_1,I\cup\{1\}}$ and $R_{z_1,I\cup\{2\}}$ respectively. There are $c, c' \in \{NE, SE, SW, NW\}$, $e, e' \in (\frac{899}{1000}, \frac{901}{1000})$ and $0 < s_1', s_2' < \frac{1}{100}$ such that $S_{z_1,I\cup\{1\}} = (S_{z_1,I})_{c,e,s_1'}$ and $S_{z_1,I\cup\{2\}} = (S_{z_1,I})_{c',e',s_2'}$ and $(S_{z_1,I}, S_{z_1,I\cup\{1\}}, S_{z_1,I\cup\{2\}})$ forms a configuration NE-SW if $|I|$ is even and *a configuration NW-SE if* $|I|$ *is odd.*

4. *The* $\beta_{z_1,z_2,I}$ *depend holomorphically on* z_1, z_2 *and* $|\frac{\partial \beta_{z_1,z_2,I}}{\partial z_1}| < \frac{1}{50}$.

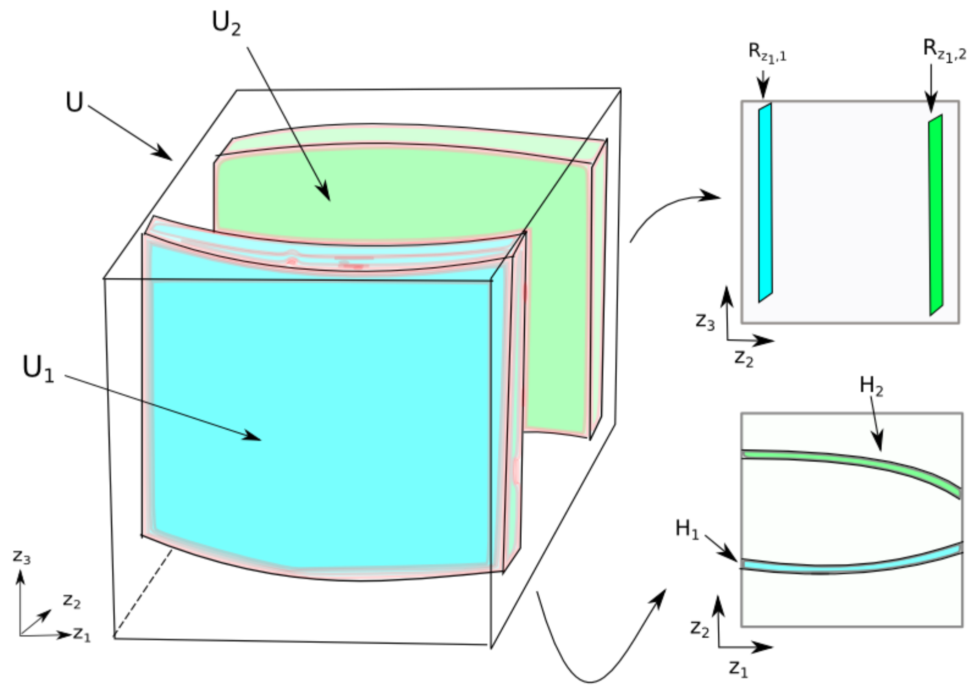

Figure 2 : U, U_1 and U_2

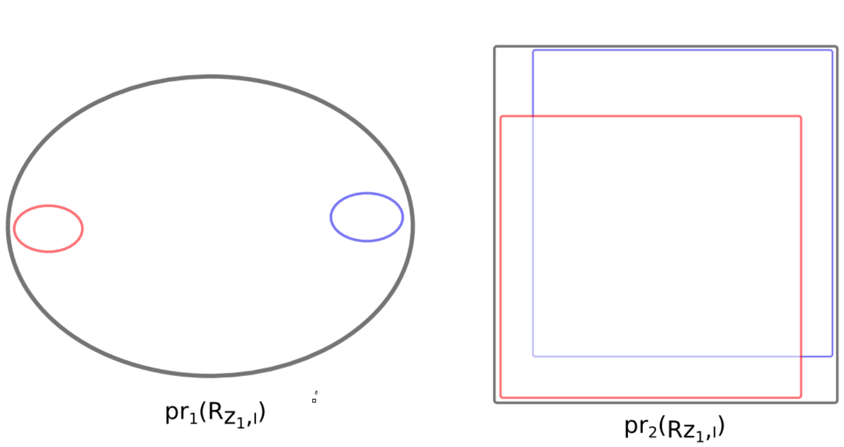

Figure 3 : $R_{z_1,I}$ and its two subsets in projection in the complex case

This proposition simply means that $\mathrm{pr}_3(\{z_1\}\times R_{z_1,I})$ contains a square which contains two subsquares included in, $\mathrm{pr}_3(\{z_1\}\times R_{z_1,I\cap\{1\}})$, $\mathrm{pr}_3(\{z_1\}\times R_{z_1,I\cap\{2\}})$, whose lengths are at least $\frac{899}{1000}$ times the length of the larger one, and positioned in opposite corners.

3.2 Proof of Proposition 3.1.7

Proof. First, let us remark that item 4 is a direct consequence of Lemma 2.3.5. Let us prove items 1 and 2. It is a consequence of the following lemma.

Lemma 3.2.1. *For all I and z_1, there exists a constant r_I which only depends on $|I|$ such that if $|b| < b_1$, for all z_1 and I, $R_{z_1,I}$ is of r_I-product type and for all admissible squares $S_{z_1,I} = c_{z_1,I} + \ell_{z_1,I}.Q \in \mathbb{C}$ for $R_{z_1,I}$ we have :*

$$pr_1(R_{z_1,I}) \times (c_{z_1,I} + (1-r_I).\ell_{z_1,I}.Q) \subset R_{z_1,I} \subset pr_1(R_{z_1,I}) \times (c_{z_1,I} + (1+r_I).\ell_{z_1,I}.Q)$$

Besides, we have :

$$\forall z_1 \in \mathbb{D}, \forall I, r_I \leq \frac{1}{1000}$$

Proof. We have that : $t_{z_1,I} = \delta_2(R_{z_1,I})$. We show Lemma 3.2.1 step by step, the result is the consequence of the convergence of an infinite series. The property is clear for the set U since $U = \bigcup_{z_1 \in \mathbb{D}} (\{z_1\} \times \mathbb{D} \times Q)$. Let us suppose it is true for a finite sequence I and consider a value $z_1 \in \mathbb{D}$, then $z_2 \in I_I^{z_1}$ and two indices $I, I' = I \cup \{i_{p+1}\}$ such that $p(z_1) + bz_2 \in \mathbb{D}$. We study the set $R_{Z_1,I'}$ with the value $Z_1 = p(z_1) + bz_2$. We suppose inductively that for all z_1 such that $f_1^{-1}(R_{Z_1,I'}) \cap R_{z_1,I}$ is not empty, $R_{z_1,I}$ is of r_I-product type and for all admissible square $S_{z_1,I} = c_{z_1,I} + \ell_{z_1,I}.Q \subset \mathbb{C}$ for $R_{z_1,I}$ we have : $\mathrm{pr}_1(R_{z_1,I}) \times (c_{z_1,I} + (1-r_I).\ell_{z_1,I}.Q) \subset R_{z_1,I} \subset \mathrm{pr}_1(R_{z_1,I}) \times (c_{z_1,I} + (1+r_I).\ell_{z_1,I}.Q)$ for a constant r_I. We denote : $R = f_1^{-1}(R_{Z_1,I'})$ and $\delta_{R,I}^{z_1} = \delta_1(R)$. Let us show the following lemma.

Lemma 3.2.2. *If $R_{z_1,I}$ is of r_I-product type, then $R_{Z_1,I'}$ is of $r_{I'}$-product type with : $r_{I'} = r_I + (\frac{1}{50} + \frac{|\lambda|}{2|\mu|})\delta_{R,I}^{z_1}$*

Proof. If $|b| < b_1$, by Property 4, we get that there is a square $S_{Z_1,I'}$ of length $|\mu|.(1 - r_I - \frac{1}{50}.\delta_{R,I}^{z_1}).\ell_{z_1,I}$ such that : $\mathrm{pr}_1(R_{Z_1,I'}) \times S_{Z_1,I'} \subset R_{Z_1,I}$. We consider now the biggest complex disk S' included in $R_{Z_1,I'} \setminus (\mathrm{pr}_1(R_{Z_1,I'}) \times S_{Z_1,I'})$ and Δ the line which contains the preimage of this complex disk by f_1. This line is a line parallel to the z_3 axis, we bound here the diameter $\delta_3(S)$ of the preimage complex disk $S = f_1^{-1}(S')$. Let us call :

$$M = \{(z_1, z_2, z_3) \in R, \mathrm{pr}_2(M \cap \{z_3 = C^{st}\}) = \mathrm{pr}_2(R)\}$$

Then, $\delta_3(S \cap (R/M)) < (r_I + \frac{1}{50}.\delta_{R,I}^{z_1}).\ell_{z_1,I}$ (this is still due to Property 4) so $\delta_3(f_1(S \cap (R/M))) \leq |\mu|.(r_I + \frac{1}{50}.\delta_{R,I}^{z_1}).\ell_{z_1,I}$. Besides, $\delta_1(M) \leq \delta_{R,I}^{z_1}$ so $\delta_3(f_1(S \cap M)) \leq |\lambda|.\delta_{R,I}^{z_1}.\ell_{z_1,I}$. Then the length of S' which is the length of the image of S by f_1 is bounded by : $(|\mu|.(r_I + \frac{1}{50}.\delta_{R,I}^{z_1}) + |\lambda|.\delta_{R,I}^{z_1}).\ell_{z_1,I}$. Then, there is a square $S^+_{Z_1,I'}$ of length $(|\mu|.(1 + r_I + \frac{1}{50}.\delta_{R,I}^{z_1}) + |\lambda|.\delta_{R,I}^{z_1}).\ell_{z_1,I}$ such that : $R_{Z_1,I} \subset \mathrm{pr}_1(R_{Z_1,I'}) \times S^+_{Z_1,I'}$. It implies that $R_{Z_1,I'}$ is of $r_{I'}$-product type with : $\mathrm{pr}_1(R_{Z_1,I'}) \times (c_{Z_1,I'} + (1 - r_{I'}).\ell_{Z_1,I'}.Q) \subset R_{z_1,I'} \subset \mathrm{pr}_1(R_{Z_1,I'}) \times (c_{Z_1,I'} + (1 + r_{I'}).\ell_{Z_1,I'}.Q)$ with $\ell_{Z_1,I'} = \frac{9}{10}\ell_{z_1,I}$ and : $2r_{I'} = 2r_I + \frac{2}{50}\delta_{R,I}^{z_1} + \frac{|\lambda|}{|\mu|}\delta_{R,I}^{z_1}$ so : $r_{I'} = r_I + (\frac{1}{50} + \frac{|\lambda|}{2|\mu|})\delta_{R,I}^{z_1}$. □

The same arguments as in the proof of Lemma 2.2.5 imply that for every $k \in]0, 1[$, there exists $0 < b_2 < b_1$ such that for all $|b| < b_2$, I, i_{p+1}, we have the following : $\delta^{z_1}_{R,I\cup\{i_{p+1}\}} < k\delta^{z_1}_{R,I}$. It thus follows from Lemma 3.2.2 that r_I converges as $|I| \to +\infty$ and

$$r_{I'} \leq (\frac{1}{50} + \frac{|\lambda|}{2|\mu|})(\sum_{n\geq 0} k^n)\delta_0$$

so this can be made smaller than $\frac{1}{1000}$ if δ_0 is small enough.

□

Property 1 of Proposition 3.1.7 is a direct consequence of the previous lemma. Property 2 of Proposition 3.1.7 is a consequence of Property 1 and Lemma 3.1.5. Let us now prove Property 3.

Lemma 3.2.3. *For every z_1, if a complex line Δ parallel to the third axis of coordinates intersects both $\{z_1\} \times R_{z_1,I}$ and $\{z_1\} \times R_{z_1,I\cup\{i_{p+1}\}}$, then the sets $S = pr_3(\Delta \cap (\{z_1\} \times R_{z_1,I}))$ and $S' = pr_3(\Delta \cap (\{z_1\} \times R_{z_1,I\cup\{i_{p+1}\}}))$ are squares such that $S' = S_{c,\frac{9}{10},s'}$ for $c \in \{NE, SE, SW, NW\}$ and $s' \leq \frac{1}{100}$.*

Proof. Let us first study the case where $|I| = 1$.

Lemma 3.2.4. *If a complex line Δ parallel to the third axis of coordinates intersects U_1, then $pr_3(U_1 \cap \Delta)$ is the square $Q_{NE,\frac{9}{10},s}$, and if Δ intersects U_2, then $pr_3(U_2 \cap \Delta)$ is the square $Q_{SW,\frac{9}{10},s}$ with $s < \frac{1}{100}$.*

Proof. This follows from the choice of constants in Section 2. Indeed, the term $\lambda z_1 = \frac{1}{10}\frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}}z_1$ in the expression of $\mathrm{pr}_3 \circ f_1$ is just intended to push the two subsquares in the direction of each of the two NE and SW corners, the two subsquare are of length of side multiplied by the modulus of μ this is $\frac{9}{10}$. □

The demonstration of Lemma 3.2.3 is made by induction on $|I|$. The induction step is simply the fact that the image by $z_3 \mapsto \lambda.z_1 + \mu.z_3$ of a NW (resp. SW,SE,NE) configuration, this is a subsquare positioned in the NW (resp. SW,SE,NE) corner of a greater square, is a SW (resp. SE,NE,NW) configuration with length sides multiplied by $\frac{9}{10}$.

□

Now, for fixed z_1 and I, it is enough to take Δ and Δ' such that Δ intersects $R_{z_1,I\cup\{1\}}$ (resp. Δ' intersects $R_{z_1,I\cup\{2\}}$). We then get two squares which are respectively the square intersection of Δ and $R_{z_1,I\cup\{1\}}$ in Δ, and the square intersection of Δ' and $R_{z_1,I\cup\{2\}}$ in Δ'. These two squares are admissible squares for $R_{z_1,I\cup\{1\}}$ and $R_{z_1,I\cup\{2\}}$. Lemma 3.2.3 and Property 1 give us that every admissible square for $R_{z_1,I}$ with these two subsquares form the configuration given by Item 3.

The third Property has been shown, the proof of Proposition 3.1.7 is now complete.

□

4 Structure of the basic set : perturbed case

4.1 Generalities and main result in the perturbed case

In this section, we analyze the structure of unstable manifolds of the horseshoe induced by an arbitrary perturbation of f_1. Some arguments in Section 3 used the special form of f_1, so we need to adapt the arguments. We can take perturbations of f_1, not necessarily polynomial, in the space of automorphisms of $\mathbb{C}^3$.

Notation. *So far we have considered an automorphism $f_1 = f_1(p, b)$ defined in* (1). *In the following, we will reduce constant b a finite number of times to ensure a certain number of properties. Furthermore, in Section 4, we will introduce decreasing neighborhoods $\nu_i(f_1)$ for $i \in \{0, 1, 2, 3\}$. In Section 5, we perturb f_1 to $f_2 = f_2(\overline{p}, b, \sigma)$ with parameters $\overline{p} = p + p_5 z^5$ and σ such that $f_2 \in \nu_3(f_1)$ and $\nu_4(f_2)$ will denote a non empty neighborhood of f_2 with $\nu_4(f_2) \subset \nu_3(f_1)$. We will subsequently make a number of further reductions until the final result.*

Because of the structural stability of horseshoes, it is well known that Definition 2.3.4 can be applied in a neighborhood $\nu_0(f_1)$ of $f_1 = f_1(p, b)$ in the space of automorphisms of $\mathbb{C}^3$. For such a map $f \in \nu_0(f_1)$, we can define $f[1], f[2]$ and a sequence of sets $U_I = f[I](\mathbb{D}^2 \times Q)$ as before. We have the following properties for the sets U_I :

Proposition 4.1.1. *For every $|b| < b_2$, for every $f \in \nu_0(f_1)$, for every finite sequence of indices $I = (i_1, ..., i_p)$, $i_j \in \{1, 2\}$, the set U_I satisfies the following properties :*

1. $U_I = \bigcup_{z_1 \in \mathbb{D}} \{z_1\} \times R_{z_1,I}$ where $R_{z_1,I}$ is an open connected set with $R_{z_1,I} \subset\subset \mathbb{D} \times Q$.

2. If I is an infinite sequence $I \in \{1, 2\}^{\mathbb{N}}$, the intersection $\bigcap_{p \geq 0} U_{I \leq p}$ is a component of the intersection of the unstable manifold of some point in the set $\bigcap_{n \in \mathbb{Z}} f^n(\mathbb{D}^2 \times Q)$ with $\mathbb{D}^2 \times Q$.

Definition 4.1.2. *A u-curve C is a curve whose tangent vectors are all in C^u (recall the cones C^u, C^{ss}, C^{cs} were defined in Subsection 2.3).*
A s-plane is a plane which admits a basis of two vectors, one of them belonging to C^{ss} and the other one to C^{cs}.
A s-surface is a hypersurface whose tangent planes are all s-planes.

We have the counterpart of Lemma 2.3.5 in this context.

Lemma 4.1.3. *For $|b| < b_2$ and $f \in \nu_0(f_1)$, for every sequence of digits I, $f[I](\mathbb{D}_{.,z_2,z_3})$ is a u-curve.*

Proof. Lemma 2.3.5 defines a cone field which is invariant by f which thus persists under small perturbations. □

The previous Lemma 4.1.3 implies in particular that every piece of unstable manifold $\bigcap_{p \geq 0} U_{I \leq p}$ is a u-curve. The following lemma deals with the horseshoe for $f \in \nu_0(f_1)$.

Lemma 4.1.4. *For each I, the intersection of U_I with a s-surface $\mathcal{S}$ is homeomorphic to $\mathbb{D} \times Q$. Moreover, there is a canonical homeomorphism $\Psi_{\mathcal{S}}$ given by the map $\mathcal{S} \cap U_I \mapsto \mathbb{D} \times Q$, $z \mapsto ((pr_2, pr_3) \circ (f[I])^{-1})(z)$.*

Proof. The property can be proven by induction. For $U_\emptyset = \mathbb{D}^2 \times Q$, the property is obvious. Let us suppose it is true until the rank I. Let $\mathcal{S}$ be a s-surface intersecting $U_{I'} = U_{I \cup \{i'\}}$ with $i' \in \{1, 2\}$. Then $(f[i'])^{-1}(\mathcal{S} \cap (\mathbb{D}^2 \times Q)) \cap (\mathbb{D}^2 \times Q)$ is a s-surface too. Then, its intersection with U_I is by the induction hypothesis homeomorphic to $\mathbb{D} \times Q$ by the map $z \mapsto ((\mathrm{pr}_2, \mathrm{pr}_3) \circ (f[I])^{-1})(z)$. Now, taking its image by f, we have that : $\mathcal{S} \cap U_{I'}$ is homeomorphic to $\mathbb{D} \times Q$ by the map $z \mapsto ((\mathrm{pr}_2, \mathrm{pr}_3) \circ (f[I'])^{-1})(z)$. □

In particular, we have :

Lemma 4.1.5. *For all z_1, I, $R_{z_1,I}$ is homeomorphic to $\mathbb{D} \times Q$.*

Definition 4.1.6. *Given a set $E \subset \mathbb{C}$ and $\delta \in (0, 1)$, we say that E is δ-square shaped if there is a square $S = c_S + \ell_S.Q \subset \mathbb{C}$ (ℓ_S can be complex non real) such that*

$$(c_S + (1 - \delta).\ell_S.Q) \subset E \subset (c_S + (1 + \delta).\ell_S.Q)$$

We call admissible square for E a square S with this property.

Definition 4.1.7. *Let $S, S_{11}, S_{12}, S_{21}, S_{22}$ be squares in $\mathbb{C}$. We say that S is well subdivided by $S_{11}, S_{12}, S_{21}, S_{22}$ if there exist squares S_1, S_2 such that $S_1, S_2 \subset S$, $S_{11}, S_{12} \subset S_1$ and $S_{21}, S_{22} \subset S_2$ and a linear isomorphism ψ of $\mathbb{C}$ such that $(\psi(S), \psi(S_1), \psi(S_2))$ is a configuration NE-SW, and $(\psi(S_1), \psi(S_{11}), \psi(S_{12}))$, $(\psi(S_2), \psi(S_{21}), \psi(S_{22}))$ are configurations NW-SE (see Definition 3.1.6).*
Let $\delta \in (0, \frac{1}{100})$. Given five δ-square shaped sets $(E_i)_{1 \leq i \leq 5}$, we say that E_1 is well subdivided by the other sets if there are 5 admissible squares $(S_i)_{1 \leq i \leq 5}$ respectively for E_i such that S_1 is well subdivided by the other squares.

The following Proposition is an analogous of Proposition 3.1.7.

Proposition 4.1.8. *For every $|b| < b_2$, there exists a neighborhood $\nu_3(f_1) \subset \nu_0(f_1)$ of $f_1 = f_1(p, b)$ such that for every $f \in \nu_3(f_1)$, we have :*

1. For every z_1, I, $\mathrm{pr}_3(\{z_1\} \times R_{z_1,I})$ is $\frac{1}{500}$-square shaped (recall that the sets $R_{z_1,I}$ were defined in Proposition 4.1.1).

2. If $S_{z_1,I}$ and $S_{z_1,I \cup \{i_{p+1}\}}$ are admissible squares for $R_{z_1,I}$ and $R_{z_1,I \cup \{i_{p+1}\}}$ respectively, their length sides satisfy :

$$\ell_{z_1,I \cup \{i_{p+1}\}} \in \left(\frac{895}{1000} \ell_{z_1,I}, \frac{905}{1000} \ell_{z_1,I} \right)$$

3. Let $S_{z_1,I}$, $S_{z_1,I \cup \{11\}}$, $S_{z_1,I \cup \{12\}}$, $S_{z_1,I \cup \{21\}}$ and $S_{z_1,I \cup \{22\}}$ be admissible squares for $R_{z_1,I}$, $R_{z_1,I \cup \{11\}}$, $R_{z_1,I \cup \{12\}}$, $R_{z_1,I \cup \{21\}}$ and $R_{z_1,I \cup \{22\}}$ respectively. Then, $S_{z_1,I}$ is well subdivided by these 4 subsquares.

So $\mathrm{pr}_3(R_{z_1,I})$ contains a square which contains two subsquares included in $\mathrm{pr}_3(R_{z_1,I \cap \{1\}})$, $\mathrm{pr}_3(R_{z_1,I \cap \{2\}})$ whose lengths are the length of the great square times $> \frac{895}{1000}$ and the two squares are near two opposite corners of the great square, with a change of direction at each step.

4.2 Proof of Proposition 4.1.8

If we take $\nu_1(f_1) \subset \nu_0(f_1)$ sufficiently small, the properties are automatically verified for (U, U_1, U_2). Our strategy is to consider the sets $U_{I'}$ as images of pieces of the sets U_I by f which looks like locally more and more precisely to a linear function when we map the sets U_I at a smaller scale. Then, by an infinite series argument we will get a result which looks like a bounded distorsion property.

Notation. *We will call pr_{w_3,w_2} the orthogonal projection on a vector $w_3 \in C^{cs}$ parallel to $w_2 \in C^{ss}$.*

Definition 4.2.1. *We say that a family of sets $\{W_{I'}, I' \in \{I, I\cup\{1\}, I\cup\{2\}, I\cup\{11\}, I\cup\{12\}, I\cup\{21\}, I\cup\{22\}\}$ is a family of u-oriented-subdivided tridisks if we have that :*
- each $W_{I'}$ is an union of u-curves
- $W_{I\cup\{1\}}, W_{I\cup\{2\}} \subset W_I$, $W_{I\cup\{11\}}, W_{I\cup\{12\}} \subset W_{I\cup\{1\}}$ and $W_{I\cup\{21\}}, W_{I\cup\{22\}} \subset W_{I\cup\{2\}}$

In particular, for each finite sequence I, we easily get this new property for the sets U_I : $(U_I, U_{I\cup\{1\}}, U_{I\cup\{2\}}, U_{I\cup\{11\}}, U_{I\cup\{12\}}, U_{I\cup\{21\}}, U_{I\cup\{22\}})$ is a family of u-oriented-subdivided tridisks

Definition 4.2.2. *Let I be a finite sequence and $\eta_1, \eta_2 < \frac{1}{10}$. Let $\{W_{I'}, I' \in \{I, I\cup\{1\}, I\cup\{2\}, I\cup\{11\}, I\cup\{12\}, I\cup\{21\}, I\cup\{22\}\}$ be a family of u-oriented-subdivided tridisks. We say that $\{W_{I'}, I' \in \{I, I\cup\{1\}, I\cup\{2\}, I\cup\{11\}, I\cup\{12\}, I\cup\{21\}, I\cup\{22\}\}$ is of (η_1,η_2)-type if for every s-plane P and for every $w_3 \in \vec{P}\cap C^{cs}, w_2 \in \vec{P}\cap C^{ss}$, denoting by $\Pi_{I'} = pr_{w_3,w_2}(P\cap W_{I'})$, we have :*
- For every I', $\Pi_{I'}$ is η_1-square shaped.
- If S_I and $S_{I\cup\{i_{p+1}\}}$ are admissible squares for Π_I and $\Pi_{I\cup\{i_{p+1}\}}$ respectively, their length sides satisfy :

$$\ell_{I\cup\{i_{p+1}\}} \in \left(\left(\frac{9}{10}-\eta_2\right)\ell_I, \left(\frac{9}{10}+\eta_2\right)\ell_I\right)$$

- Let S_I, $S_{I\cup\{11\}}$, $S_{I\cup\{12\}}$, $S_{I\cup\{21\}}$ and $S_{I\cup\{22\}}$ be admissible squares for Π_I, $\Pi_{I\cup\{11\}}$, $\Pi_{I\cup\{12\}}$, $\Pi_{I\cup\{21\}}$ and $\Pi_{I\cup\{22\}}$. Then, S_I is well subdivided by these 4 subsquares.

Definition 4.2.3. *We say that an invertible linear function h of $\mathbb{C}^3$ is well adapted to the cones (C^u, C^{cs}, C^{ss}) if C^u is invariant by h, C^{ss} and C^{cs} are invariant by h^{-1} and for every $u \in C^u, v \in C^{ss}, w \in C^{cs}$, we have that : $||h(u)|| > 10||u||$, $||h(v)|| < \frac{1}{10}||v||$ and $||h(w)|| \in (\frac{9}{10}||w||; \frac{11}{10}||w||)$.*

We easily get the following :

Proposition 4.2.4. *For each $f \in \nu_1(f_1)$, for each point $P = (P_1, P_2, P_3)$ in $(D_1'' \times \mathbb{D} \times Q) \cup (D_2'' \times \mathbb{D} \times Q)$, the differential Df_P is well adapted to the cones (C^u, C^{ss}, C^{cs}).*

Proof. Df_P is diagonalizable : there is an obvious eigenvalue of f_1 which is μ of eigenvector e_3 such that $|\mu| = \frac{9}{10}$ which persists for $f \in \nu_1(f_1)$ with its eigenvector still in C^{cs}. Besides, the differential of $Df_1(P)$ is equal to :

$$\begin{pmatrix} p'(P_1) & b & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & \mu \end{pmatrix}$$

and gives two other eigenvalues λ_1 and λ_2 such that $|\lambda_1| > 10$, $|\lambda_2| < \frac{1}{10}$ of eigenvectors in C^u and C^{ss} because $|p'| > 50$ on $(D_1'' \cup D_2'')$ and $|b| < b_2$. $\square$

Proposition 4.2.5. *The image of any family of u-oriented subdivided tridisks of (η_1, η_2)-type by any invertible linear map h which is well adapted to the cones (C^u, C^{ss}, C^{cs}) is a tridisk configuration of (η_1, η_2)-type .*

Proof. Let us take a s-plane P' and vectors $w_3' \in \vec{P'} \cap C^{cs}, w_2' \in \vec{P'} \cap C^{ss}$. Let us take $P = h^{-1}(P')$, $w_3 = h^{-1}(w_3')$ and $w_2 = h^{-1}(w_2')$ which are respectively a s-plane and vectors in $\vec{P} \cap C^{cs}$ and $\vec{P} \cap C^{ss}$ since h is well adapted to the cones (C^u, C^{ss}, C^{cs}). Then, denoting by $\Pi_{I'} = \mathrm{pr}_{w_3, w_2}(P \cap W_{I'})$ and $\Pi'_{I'} = \mathrm{pr}_{w_3', w_2'}(P' \cap h(W_{I'}))$ for $I' \in \{I, I\cup\{1\}, I\cup\{2\}, I\cup\{11\}, I\cup\{12\}, I\cup\{21\}, I\cup\{22\}\}$, we have :
- For every I', $\Pi_{I'}$ is η_1-square shaped.
- If S_I and $S_{I\cup\{i_{p+1}\}}$ are admissible squares for Π_I and $\Pi_{I\cup\{i_{p+1}\}}$ respectively, their length sides satisfy :

$$\ell_{I\cup\{i_{p+1}\}} \in \left(\left(\frac{9}{10} - \eta_2\right)\ell_I, \left(\frac{9}{10} + \eta_2\right)\ell_I\right)$$

- Let S_I, $S_{I\cup\{11\}}$, $S_{I\cup\{12\}}$, $S_{I\cup\{21\}}$ and $S_{I\cup\{22\}}$ be admissible squares for Π_I, $\Pi_{I\cup\{11\}}$, $\Pi_{I\cup\{12\}}$, $\Pi_{I\cup\{21\}}$ and $\Pi_{I\cup\{22\}}$. Then, S_I is well subdivided by these 4 subsquares.

Since h is an invertible linear map and $h(w_3) = w_3'$, $h(w_2) = w_2'$, we have by linearity that :
- For every I', $\Pi'_{I'}$ is η_1-square shaped.
- If S'_I and $S'_{I\cup\{i_{p+1}\}}$ are admissible squares for Π'_I and $\Pi'_{I\cup\{i_{p+1}\}}$ respectively, their length sides satisfy :

$$\ell'_{I\cup\{i_{p+1}\}} \in \left(\left(\frac{9}{10} - \eta_2\right)\ell'_I, \left(\frac{9}{10} + \eta_2\right)\ell'_I\right)$$

- Let S'_I, $S_{I\cup\{11\}}$, $S'_{I\cup\{12\}}$, $S_{I\cup\{21\}}$ and $S'_{I\cup\{22\}}$ be admissible squares for Π'_I, $\Pi'_{I\cup\{11\}}$, $\Pi'_{I\cup\{12\}}$, $\Pi'_{I\cup\{21\}}$ and $\Pi'_{I\cup\{22\}}$. Then, S'_I is well subdivided by these 4 subsquares. Finally, $\{W'_{I'}, I' \in \{I, I\cup\{1\}, I\cup\{2\}, I\cup\{11\}, I\cup\{12\}, I\cup\{21\}, I\cup\{22\}\}$ is a configuration of (η_1, η_2)-type. $\square$

Definition 4.2.6. *Let $\mathcal{B}$ be a ball of radius $\rho_{\mathcal{B}}$ in $\mathbb{C}^3$. We say that a map f defined on $\mathcal{B}$ is quasi-linear well adapted to the cones (C^u, C^{ss}, C^{cs}) if there exists a linear map h well adapted to the cones (C^u, C^{ss}, C^{cs}) such that : $inf_{w:||w||=1}||Dh(w)|| > 10.||d^2(f - h_{|\mathcal{B}})||\rho_{\mathcal{B}}$. We call*

$$NL(f) = inf\, ||d^2(f-h)_{|\mathcal{B}}||\rho_{\mathcal{B}}$$

the non-linearity of f, the infimum being taken on all the linear invertible maps h well adapted to the cones (C^u, C^{ss}, C^{cs}).

Proposition 4.2.7. *The image of any family of u-oriented-subdivided tridisks of (η_1, η_2)-type included in a ball $\mathcal{B}$ by any linear map f well adapted to the cones (C^u, C^{cs}, C^{ss}) and defined on $\mathcal{B}$ such that $NL(f) < \frac{1}{100}$ is a family of u-oriented-subdivided tridisks of $(\eta_1 + 10NL(f), \eta_2 + 10NL(f))$-type .*

Proof. Let us take a s-plane P', vectors $w_3' \in \vec{P}' \cap C^{cs}, w_2' \in \vec{P}' \cap C^{ss}$ and O the center of the ball $\mathcal{B}$. Let us call $P = f^{-1}(P')$, $w_3 = Df_O^{-1}(w_3')$ and $w_2 = Df_O^{-1}(w_2')$ which are respectively a s-surface and vectors of C^{cs} and C^{ss} since f is well adapted to the cones (C^u, C^{ss}, C^{cs}). Then, denoting by $\Pi_{I'} = \mathrm{pr}_{w_3,w_2}(P \cap W_{I'})$ and $\Pi'_{I'} = \mathrm{pr}_{w_3',w_2'}(P' \cap f(W_{I'}))$ for $I' \in \{I, I \cup \{1\}, I \cup \{2\}, I \cup \{11\}, I \cup \{12\}, I \cup \{21\}, I \cup \{22\}\}$. The following lemma is obvious :

Lemma 4.2.8. *P is a s-surface.*

Since each $W_{I'}$ is an union of u-curves, simple calculation gives us that :
- For every I', $\Pi_{I'}$ is $\eta_1 + 2NL(f)$-square shaped.
- If S_I and $S_{I\cup\{i_{p+1}\}}$ are admissible squares for Π_I and $\Pi_{I\cup\{i_{p+1}\}}$ respectively, their length sides satisfy :

$$\ell_{I\cup\{i_{p+1}\}} \in ((1 - (\eta_2 + 2\mathrm{NL}(f))\ell_I, (1 + (\eta_2 + 2\mathrm{NL}(f))\ell_I)$$

- Let S_I, $S_{I\cup\{11\}}$, $S_{I\cup\{12\}}$, $S_{I\cup\{21\}}$ and $S_{I\cup\{22\}}$ be admissible squares for Π_I, $\Pi_{I\cup\{11\}}$, $\Pi_{I\cup\{12\}}$, $\Pi_{I\cup\{21\}}$ and $\Pi_{I\cup\{22\}}$. Then, S_I is well subdivided by these 4 subsquares.

Then, mapping by f and since f is quasi-linear well adapted to the cones (C^u, C^{ss}, C^{cs}), we have that
- For every I', $\Pi'_{I'}$ is of $\eta_1 + 10\mathrm{NL}(f)$-square shaped.
- If S'_I and $S'_{I\cup\{i_{p+1}\}}$ are admissible squares for Π'_I and $\Pi'_{I\cup\{i_{p+1}\}}$ respectively, their length sides satisfy :

$$\ell'_{I\cup\{i_{p+1}\}} \in ((1 - (\eta_2 + 10\mathrm{NL}(f))\ell'_I, (1 + (\eta_2 + 10\mathrm{NL}(f))\ell'_I)$$

- Let S'_I, $S_{I\cup\{11\}}$, $S'_{I\cup\{12\}}$, $S_{I\cup\{21\}}$ and $S'_{I\cup\{22\}}$ be admissible squares for Π'_I, $\Pi'_{I\cup\{11\}}$, $\Pi'_{I\cup\{12\}}$, $\Pi'_{I\cup\{21\}}$ and $\Pi'_{I\cup\{22\}}$. Then, S'_I is well subdivided by these 4 subsquares. Finally, $\{W'_{I'}, I' \in \{I, I\cup\{1\}, , I\cup\{2\}, I\cup\{11\}, I\cup\{12\}, I\cup\{21\}, I\cup\{22\}\}$ is a configuration of (η_1, η_2)-type. Indeed, the two first properties are the consequence that the quotients of length are multiplied by a number between

$$\frac{1 - \mathrm{NL}(f)}{1 + \mathrm{NL}(f)} \geq 1 - 2\mathrm{NL}(f)\frac{1}{1 + \mathrm{NL}(f))} \geq 1 - 4\mathrm{NL}(f)$$

and

$$\frac{1 + \mathrm{NL}(f)}{1 - \mathrm{NL}(f)} \leq 1 + 2\mathrm{NL}(f)\frac{1}{1 - \mathrm{NL}(f))} \leq 1 + 4\mathrm{NL}(f)$$

and we have the following inequalities :

$$(1 + 4\mathrm{NL}(f))(1 + \eta_2 + 2\mathrm{NL}(f)) \leq 1 + \eta_2 + 2\mathrm{NL}(f) + 8\mathrm{NL}(f) = 1 + \eta_2 + 10\mathrm{NL}(f)$$

$$(1 - 4\mathrm{NL}(f))(1 - \eta_2 - 2\mathrm{NL}(f)) \geq 1 - \eta_2 - 2\mathrm{NL}(f) - 8\mathrm{NL}(f) = 1 - \eta_2 - 10\mathrm{NL}(f)$$

Finally, $\{W'_{I'}, I' \in \{I, I \cup \{1\}, I \cup \{2\}, I \cup \{11\}, I \cup \{12\}, I \cup \{21\}, I \cup \{22\}\}$ is a configuration of $(\eta_1 + 10\mathrm{NL}(f), \eta_2 + 10\mathrm{NL}(f))$-type. □

Proof of Proposition 4.1.8. Reducing the neighborhood $\nu_2(f_1) \subset \nu_1(f_1)$ if necessary, for every finite sequence I, for every point $P \in U_I$, there exist I' and a sequence of balls $\mathcal{B}_{I',P}, ..., \mathcal{B}_{I,P}$ of radii $\rho_P^{I'}, ..., \rho_P^I$ such that, denoting $U_J \cap \mathcal{B}_{J',P}$ by $U_{J,J',P}$, we have :
- $(U_{j,J,P}, j \in \{J, J \cup \{1\}, J \cup \{2\}, J \cup \{11\}, J \cup \{12\}, J \cup \{21\}, J \cup \{22\}\})$ is a family of u-oriented-subdivided tridisks for $J \in \{I', ..., I\}$
- $(U_{j,J,P}, j \in \{Ji, Ji \cup \{1\}, Ji \cup \{2\}, Ji \cup \{11\}, Ji \cup \{12\}, Ji \cup \{21\}, Ji \cup \{22\}\})$ is included in the image by f of $(U_{j,J,P}, j \in \{J, J \cup \{1\}, J \cup \{2\}, J \cup \{11\}, J \cup \{12\}, J \cup \{21\}, J \cup \{22\}\})$
- $(U_{j,I',P}, j \in \{I', I' \cup \{1\}, I' \cup \{2\}, I' \cup \{11\}, I' \cup \{12\}, I' \cup \{21\}, I' \cup \{22\}\})$ is a family of u-oriented-subdivided tridisks of $(\frac{1}{1000}, \frac{1}{500})$-type.

Moreover, it is possible to have :

$$\frac{1}{1000} > \sum_{J=I'}^{J=I} ||d^2 f|| \rho_P^J > \sum_{J=I'}^{J=I} NL(f_{|\mathcal{B}_{J,P}})$$

Indeed, there exists m such that if $|J'| > |J| + m$, then :

$$||d^2 f|| \rho_P^{J'} < \frac{||d^2 f|| \rho_P^J}{2}$$

Thus the above sum converges and reducing once again the neighborhood $\nu_3(f_1) \subset \nu_2(f_1)$ if necessary, we get it is bounded by $\frac{1}{1000}$. Then, using by iteration the last Proposition, $(U_I, U_{I\cup\{1\}}, U_{I\cup\{2\}}, U_{I\cup\{11\}}, U_{I\cup\{12\}}, U_{I\cup\{21\}}, U_{I\cup\{22\}})$ is a tridisk configuration of $(\frac{5}{1000}, \frac{1}{500})$-type for every I, which is the result we wanted to prove.

□

4.3 Central curves

Definition 4.3.1. *Given a square of length l, we call the middle square of the square the square of length side $\frac{1}{100}.l$ which is centered at the center of the square.*

Definition 4.3.2. *A C^1 graph $C = \{C_{z_1} : z_1 \in \mathbb{D}\}$ over z_1 in $\mathbb{D}^2 \times Q$ is a central curve for U_I if for every $z_1 \in \mathbb{D}$, C_{z_1} is in the middle of an admissible square for $R_{z_1,I}$.*

Proposition 4.3.3. *For all index I, the set U_I admits a central curve which is a u-curve.*

Proof. It is clear that for all I, the set U_I admits a central curve. Because of Proposition 4.1.3, it is possible to pick a family of admissible squares $\mathcal{S}_{z_1,I}$ for $R_{z_1,I}$ so that the maps which respectively send z_1 to the coordinates z_2 and z_3 of the center of $\mathcal{S}_{z_1,I}$ have derivates inferior to $\frac{1}{50}$ in modulus. So, the curve going through all these centers is both a central curve for U_I and a u-curve. □

5 First heteroclinic tangency

5.1 Perturbation

So far we have constructed a mapping f_1 with a blender type horseshoe. In this section, we make a further perturbation of f_1 in such a way that a heteroclinic tangency is obtained with good orientation of the stable and unstable foliations. We consider now the following automorphisms family which we name $f = f_{\overline{p},b,\sigma}$ ($\overline{p}$ includes a new term of degree 5, $\overline{p}(z) = p(z) + p_5 z^5$) :

$$f_{\overline{p},b,\sigma} : (z_1, z_2, z_3) \mapsto (\overline{p}(z_1) + b.z_2 + \sigma.(\lambda.z_1 + \mu.z_3), z_1, \lambda.z_1 + \mu.z_3) \quad (6)$$

which is obviously an automorphism, it is the composition of the previous automorphism with : $(z_1, z_2, z_3) \mapsto (z_1 + \sigma.z_3, z_2, z_3)$ where $\sigma \in \mathbb{C}$, $\sigma \neq 0$ The degree of p is increased by 1 to get a heteroclinic tangency.

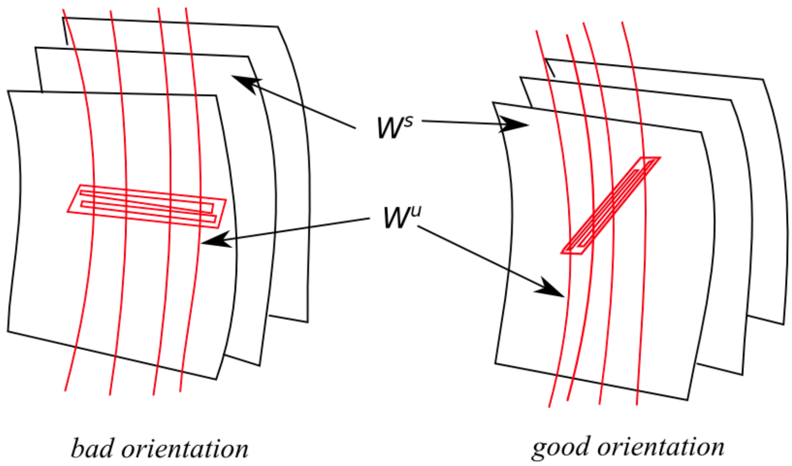

Figure 4 : Orientation

Proposition 5.1.1. *There exists $0 < b_3 < b_2$, $\sigma_0 > 0$ such that for all $|b| < b_3$, $\eta > 0$ and $|\sigma| < \sigma_0$, there exists $p_5 = p_5(b, \sigma)$ with $|p_5| < \eta$ such that $f_2 = f_{\overline{p},b,\sigma}$ is in $\nu_3(f_1)$ and has a tangency between $W^s(\alpha_{f_{\overline{p},b,\sigma}})$ and the unstable manifold of a point of the horseshoe.*

Proof. Consider perturbation of the form : $\overline{p}(z) = p(z) + p_5(z-\alpha_p)(z-c_p)^2 z^2$. Then, it is clear that for all p_5 we still have a critical point $c_{\overline{p}} = c_p$ and a fixed point $\alpha_{\overline{p}} = \alpha_p$. First, we take a value p_5^{pc} such that the point c_p becomes preperiodic and is sent on α_p, where α_p and c_p both are real. It is possible to take p_5 real and n (depending on p_5) such that $|p_5| < \frac{\eta}{2}$ and $(\overline{p}^{pc})^n(c_p) = \alpha_p$, that is with a preperiodic critical point. Indeed, given such a real value $0 < p_5 < \frac{\eta}{2}$, it is possible to get n such that $\overline{p}^n(c_p) < \alpha_p$: for small values of p_5, the Julia set of p is a Cantor set, so $c_p = c_{\overline{p}}$ escapes to infinity. The orbit of c_p stays in the real line and $p_5 > 0$, then there exists n such that $\overline{p}^n(c_p) > \alpha_p$. Since $p^n(c_p) < \alpha_p$, the intermediate value property gives us a new value of p_5^{pc} between 0 and the old one such that for this value $(\overline{p}^{pc})^n(c_p) = \alpha_p$. Besides, it is possible to take

σ_0 such that for all $p_5 \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2})$, b such that $|b| < b_2$, σ such that $|\sigma| < \sigma_0$, we have : $f_{\overline{p},b,\sigma} \in \nu_3(f_1)$.

We denote $\mathcal{N}$ some small neighborhood of c_p and $\mathcal{N}'$ the set $\mathcal{N} \times \mathbb{D} \times Q$. Given a piece of unstable manifold $\mathcal{W}^u \cap (\mathbb{D}^2 \times Q)$ of the horseshoe, reducing σ_0 if necessary, and because c_p is a critical point for $\overline{p}^{pc}$, $f^n(\mathcal{W}^u)$ contains a point Tan_f of tangency with the foliation $\{z_1 = C^{st}\}$ for each $f \in \nu_3(f_1)$. Indeed, the image of a u-curve $C_I^f = \{(z_1, y_{f,I}^2(z_1), y_{f,I}^3(z_1), z_1 \in \mathbb{D}\}$ by $f_{\overline{p},b} : (z_1, z_2, z_3) \mapsto (\overline{p}(z_1) + b.z_2, z_1, \lambda.z_1 + \mu.z_3)$, will be the curve $f_{\overline{p},b}(C_I^f) = \{(\overline{p}(z_1) + by_{f,I}^2(z_1), z_1, \lambda.z_1 + \mu.y_{f,I}^3(z_1)), z_1 \in \mathbb{D}\}$. Then it has a point of quadratic tangency with the foliation $z_1 = C^{st}$ because the derivative of the first coordinate $\overline{p}(z_1) + by_{f,I}^2(z_1)$ vanishes for a value of z_1 and c_p is such that : $p'(c_p) = 0$ but $p''(c_p) \neq 0$. Moreover moving slightly the parameters if necessary, the iterate images of this first tangency are all points of quadratic tangency too. This is still true for f near $f_{\overline{p},b,\sigma}$, we call Tan_f the first coordinate of the point of vertical tangency.

We will conclude by an argument based on the Argument Principle. Indeed, it is possible to ensure that $\{\mathrm{Tan}_{f_{\overline{p},b}} : p_5 \in \partial\mathbb{D}(p_5^{pc}, \frac{\eta}{2})\}$ is a curve in the plane such that $B(\alpha_p, \frac{1}{10000})$ lies in a bounded connected component of its complement. In particular, there is a parameter $p_5 \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2})$ such that $\mathrm{Tan}_{f_{\overline{p},b}}$ belongs to $B(\alpha_p, \frac{1}{10000})$ (in particular, $f^n(\mathcal{W}^u \cap \mathcal{N}')$ is not the union of graphs over z_1) and for $p_5 \in \partial\mathbb{D}(p_5^{pc}, \frac{\eta}{2})$, $f^n(\mathcal{W}^u \cap \mathcal{N}')$ is the union of 2 graphs over z_1. The following lemma is the analogous in dimension 3 of Proposition 8.1 of [10]. The proof is essentially the same and relies on the continuity of the intersection index of properly intersecting analytic sets of complementary dimensions.

Lemma 5.1.2. *Let $(V_{p_5})_{p_5 \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2})}$ be a holomorphic family of curves of degree 2 in $\mathbb{D}^2 \times Q$. We assume that:*
(i) There exists a compact subset $\mathbb{D}(p_5^{pc}, \eta') \subset \mathbb{D}(p_5^{pc}, \frac{\eta}{2})$ such that if $p_5 \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2}) - \mathbb{D}(p_5^{pc}, \eta')$, V_{p_5} is the union of 2 graphs upon the first coordinate.
(ii) There exists $\overline{p}_5 \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2})$ such that $V_{\overline{p}_5}$ is not the union of 2 graphs upon the first coordinate.
Then, if $(W_{p_5})_{p_5 \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2})}$ is any holomorphic family of s-surfaces in $\mathbb{D}^2 \times Q$, there exists $p_{5,t} \in \mathbb{D}(p_5^{pc}, \frac{\eta}{2})$ such that $V_{p_{5,t}}$ and $W_{p_{5,t}}$ admit a point of tangency.

Reducing $0 < b_3 < b_2$ if necessary gives us that the family of surfaces $(W^s(\alpha_f))_f = (W^s(\alpha_f))_{p_5}$ is a family of s-surfaces. We apply the previous lemma, taking the family $f^n(\mathcal{W}^u \cap \mathcal{N}')$ as curves and the family of stable manifolds $(W^s(\alpha_f))_f$ as s-surfaces. We can conclude that there is a parameter $\tilde{p}_5$ such that $f_{\overline{p},b}$ is in $\nu_3(f_1)$ and has a heteroclinic tangency between $W_s(\alpha_{f_{\overline{p},b}})$ and the unstable manifold of a point of the horseshoe.

□

Proposition 5.1.3. *There exists $0 < b_4 < b_3$ such that the following property holds :*

For each b and σ such that $|b| < b_4$ and $0 < |\sigma| < \sigma_0$, for each $f_2 = f_{\overline{p},b,\sigma}$ as in Proposition 5.1.1, there exists $N \geq n$ such that $f_2^N(o_t)$ is a point of tangency between $W^s(\alpha_{f_{\overline{p},b,\sigma}})$ and the unstable manifold of a point of the horseshoe *such that in addition :*

$$D(f_2^N)_{o_t}(e_3) = \kappa_1.e_1 + \kappa_2.e_2 + \kappa_3.e_3 \text{ with } |\kappa_2| < 10^5|\kappa_1| \text{ and } |\kappa_3| < 10^5|\kappa_1| \quad (7)$$

Proof. The point of tangency is the image by f_2^n (for a certain integer n) of a point in $\bigcap_I U_I$. We have : $D(f_2^n)_{o_t}(e_3) = \overline{\kappa_1}.e_1 + \overline{\kappa_2}.e_2 + \overline{\kappa_3}.e_3$ with $\overline{\kappa_1} \neq 0$ because $\sigma \neq 0$. We are going to take a point of its forward orbit which will be a point of heteroclinic tangency with the above inequalities. Reducing $0 < b_4 < b_3$ again if necessary, the manifold $W^s(\alpha_{f_2})$ is a graph over the coordinates z_2, z_3 in the ball $B(0, 2|(\overline{p}^{pc})^n(c_p)|)$ with derivates less than $\frac{1}{(10000.||f_2^n(o_t)||)}$ in modulus. Now, all the iterates by f_2 of $(f_2)^n(o_t)$ are in $B(0, 2|(\overline{p}^{pc})^{n-1}(c_p)|) \cap W^s(\alpha_{f_2})$ and they tend to α_{f_2}. All of them are points of heteroclinic tangency between $W^s(\alpha_{f_2})$ and the unstable manifold of the same point of the horseshoe. Since for all $k \geq 0$,

$$D(f_2)_{f_2^{n+k}(o_t)} = \begin{pmatrix} p'_k & b & \sigma.\mu \\ 1 & 0 & 0 \\ \lambda & 0 & \mu \end{pmatrix}$$

for $|b| < 1$,$|\lambda| < 1$,$|\sigma.\mu| < 1$, $|\mu| < 1$ and $|p'_k| > 1+s$ for a certain $s > 0$ and for all k, by mapping e_3 by $D(f_2^N) = Df_{f_2^N(o_t)} \circ ... \circ Df_{f_2^n(o_t)}$, there is an integer N such that $f_2^N(o_t)$ is a heteroclinic tangency with $D(f_2^N)_{o_t}(e_3) = \kappa_1.e_1 + \kappa_2.e_2 + \kappa_3.e_3$ with $|\kappa_2| < 10^5|\kappa_1|$ and $|\kappa_3| < 10^5|\kappa_1|$. □

5.2 Motion of the point of tangency

In the following, we fix N and the point of tangency $f_2^N(o_t)$ given by the previous Proposition. The following lemma describes the motion of the point of tangency as a function of the parameter p_5.

Lemma 5.2.1. *There exists a neighborhood $\mathcal{U}$ of $f_2^N(o_t)$, $0 < b_5 < b_4$ and $\eta'' > 0$ such that for every $|b| < b_5$, for every p_5 in the neighborhood $\mathbb{D}(\tilde{p}_5, \eta'')$ of the parameter $\tilde{p}_5$ given by Proposition 5.1.1 and for each point $\overline{f}^N((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1))$ of an unstable manifold $\mathcal{W}$ in $\mathcal{U}$, we have that $\frac{d(\overline{f}^N((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1)))}{dp_5} \in C^u$ with $|\frac{d(\overline{f}^N((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1)))}{dp_5}| > 10$.*

Proof. It is easy to get that $|\frac{d(\overline{p}^N(c_p))}{dp_5}| > 20$ in a small neighborhood of the parameter p_5^{pc} given by Proposition 5.1.1 because $p^N(c_p) = \overline{p}^N_{p_5=0}(c_p) > 100$ (it is a real number), $\overline{p}^N_{p_5}(c_p)$ decreases as a real function of p_5 with its negative derivative increasing in modulus and $\overline{p}^N_{p_5^{pc}}(c_p) = 0$ for the parameter given by Proposition 5.1.1 so $\frac{d(\overline{p}^N(c_p))}{dp_5} < -100/p_5^{pc} < -100$. Since $p_5^{pc} < 1$, we have that $|\frac{d(\overline{p}^N(c_p)}{p_5}| > 11$ in a small neighborhood of p_5^{pc}. Taking $0 < b_5 < b_4$ sufficiently small, using Lemma 2.3.6 and applying f^N to the pieces of unstable manifolds in $\mathcal{N}'$ implies both $\frac{d(\overline{f}^N((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1)))}{dp_5} \in C^u$ with $|\frac{d(\overline{f}^N((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1)))}{dp_5}| > 10$ for each point $\overline{f}^N((z_1, y^{\mathcal{W}}_f(z_1), y^{\mathcal{W}}_{f,2}(z_1))$ of an unstable manifold $\mathcal{W}$ near the tangency for $p_5 \in \mathbb{D}(\tilde{p}_5, \eta'')$ ($\tilde{p}_5$ and the small disk around it will be included inside the previous neighborhood of p_5^{pc}). □

6 Persistent homoclinic tangencies

6.1 Persistent heteroclinic tangencies

In this section we finally get persistent heteroclinic tangencies between $W^s(\alpha_f)$ and the unstable foliation of the horseshoe. We do not use the technique of a "disk of tangency" like in [7] but rather study the intersection of the sets $f^N(U_I)$ and $W^s(\alpha_f)$ in $\mathbb{C}^3$. This is possible because by the results of Section 5 the two foliations are conveniently "oriented".

First, it will be convenient to make a local change of coordinates in which $W^s(\alpha_f)$ is a vertical surface and $D(f_2^N)_{o_t}(e_3)$ is colinear to the first axis of coordinates. We take the new coordinates $\phi_f(z_1, z_2, z_3) = (z_1', z_2', z_3')$ in the neighborhood $\mathcal{U}$ of the heteroclinic tangency given by Proposition 5.2.1. Remark that taking $b_6 < b_5$, then in $\mathbb{D}^2 \times Q$, $W^s(\alpha_f)$ is a vertical surface and then a graph $\{Z_1 = w(z_2, z_3) : z_2, z_3\}$ over the coordinates z_2 and z_3. Then, consider the map ϕ_f defined in $\mathcal{U}$ which sends a point $P = (z_{1,P}, z_{2,P}, z_{3,P})$ to the triplet $(z'_{1,P}, z'_{2,P}, z'_{3,P}) = (z'_{1,P}, z_{2,P}, z_{3,P})$ where $z'_{1,P} = Z_{1,P} - w(z_2, z_3)$. Since when $|b| < b_6$ tends to 0, $W^s(\alpha_f)$ tends to a vertical plane directed by e_2 and e_3, then ϕ_f tends to a linear map on $\mathcal{U}$ when b tends to 0. Then, for every $f \in \nu_3(f_1)$:

Notation. *Let us denote V_J the sequence of open sets of the form : $V_J = \{(z_1', z_2', z_3'), z_1' \in W_j\}$, where W_j is the ball of radius $\frac{1}{10^j}$ in $\mathbb{C}$, such that*

$$(\bigcap_J V_J) \cap \mathcal{U} = \phi_f(W^s(\alpha_f) \cap \mathcal{U})$$

Let us denote by τ the tangent vector at the unstable manifold at the point o_t. Increasing N by iterating f_2 a finite number of additional times if necessary, we can suppose that $D(f_2^N)_{o_t}(\tau) \in C^{cs}$. This is due to the fact that the vector $D(f_2^N)_{o_t}(\tau)$ lies in the tangent plane to $W^s(\alpha_f)$ which is a s-plan and to the shape of the matrix

$$D(f_2)_{f_2^{n+k}(o_t)} = \begin{pmatrix} p_k' & b & \sigma.\mu \\ 1 & 0 & 0 \\ \lambda & 0 & \mu \end{pmatrix}$$

Then, reducing $\mathcal{U}$ if necessary, locally, $W^u(o_t)$ is a graph over the third coordinate z_3'. Then there is an integer i such that for all $|I| > i$, we have $U_I' = \bigcup_{z_3' \in \mathbb{D}'} \{z_3'\} \times R'_{z_3',I}$ where $R'_{z_3',I}$ is a non empty open set.

Lemma 6.1.1. *If f_2 is as in Section 5, this is $f_2 = f_2(\overline{p}, b, \sigma)$ with $|b| < b_5$ and $\overline{p}, \sigma$ well chosen, there exists a neighborhood $\nu_4(f_2)$ of f_2 such that $\nu_4(f_2) \subset \nu_3(f_1)$ and for each $f \in \nu_4(f_2)$, for all $|I| > i$, for all $z_3' \in \mathbb{D}'$, $pr_1(R'_{z_3',I})$ is $\frac{1}{400}$-square shaped and well subdivided.*

Proof. Let us consider the inverse image of $e_2' \in C^{ss}$ by $D(f_2^N)_{o_t}$. Reducing the value of σ_0 if necessary, we can have that C^{ss} is invariant by $D(f_2^{-n})_{o_t}$. Since C^{ss} is invariant by each $D(f_2^{-1})_{f_2^{n+k}(o_t)}$ because in $\mathbb{D}^2 \times Q$, f_2 is such that C^{ss} is invariant by f_2^{-1}. For each $e > 0$, it is possible to reduce $\mathcal{U}$ if necessary, such that the inverse image by $D(f_2^N)_{o_t}$ of any plane directed by e_1' and e_2' and going though $\mathcal{U}$ is a e-s-surface (we call here a e-s-surface a s-surface which admits a parametrization e-near the parametrization of a s-plane in the C^2 topology) and f_2^{-N} is e-near $D(f_2^{-N})_{o_t}$ in the C^2 topology, so reducing $\mathcal{U}$ if necessary and taking e sufficiently small, we have that $R'_{z'_3,I}$ is $\frac{1}{400}$-square shaped and $R'_{z'_3,I}$ is well subdivided. This is still true for f in a sufficiently small neighborhood $\nu_4(f_2) \subset \nu_3(f_2)$ of f_2. $\square$

By analogy, we introduce the following definition :

Definition 6.1.2. *A C^1 graph C over z'_3 in $\mathcal{U}$ is a central curve for U'_I if for every $z'_3 \in \mathbb{D}'$, $C_{z'_3}$ is in the middle of an admissible square for $R'_{z'_3,I}$.*

It is easily shown that the image of a central curve for U_I is a central curve for U'_I. Let $\mathcal{C}_I$ be a central curve for the set U'_I. Reducing $0 < b_8 < b_7$ if necessary, we see after calculation that :

Lemma 6.1.3. *For every $|b| < b_8$, for every $f \in \nu_4(f_2)$, every horizontal graph in a small neighborhood of o_t is sent by $\phi_f \circ f^N$ to a degree 2 curve in $\mathcal{U}$, $z'_3 \mapsto (s_1(z'_3), s_2(z'_3), z'_3)$ in $\mathcal{U}$ over z'_3, such that :*
- $z'_3 \mapsto s_1(z'_3)$ is a ramified covering with only a point of ramification of degree 2, in particular, $z'_3 \mapsto (s_1(z'_3), s_2(z'_3), z'_3)$ has only one point of tangency $(z'_{1,t}, z'_{2,t}, z'_{3,t})$ with $z_1 = C^{st}$ inside $\mathcal{U}$ (with $z_1 = z'_{1,t}$).
- for every $z_1 \in pr_1(\mathcal{U})$, if $z_1 = C^{st}$ ($C^{st} \neq z_{1,t}$) intersects C at a point of coordinate z'_3, then the other point of the intersection $\{z_1 = C^{st}\} \cap C$ has its second coordinate inside $z'_{3,t} - B(0, \frac{1}{10})(z'_3 - z'_{3,t})$.

Proof. The unstable manifold which presents the tangency is a u-curve whose a small neighborhood of o_t is sent by f^N to a graph $z'_3 \mapsto (s_1(z'_3), s_2(z'_3), z'_3)$ in $\mathcal{U}$ over z'_3 with a unique point of (quadratic) tangency since $c_{\overline{p}}$ is a simple critical point of p. From now, let us denote by $P : z'_3 \mapsto s_1(z'_{3,t}) + P_2.(z'_3)^2$ the second order Taylor approximation of $s_1(z'_3)$ at $\mathrm{pr}_{e'_3}(f^N(o_t)) = z'_{3,t}$. Reducing $\mathcal{U}$ if necessary, we can suppose there exits ϕ defined on $\mathrm{pr}_{e'_3}(\mathcal{U})$ such that : $s_1(z'_3) = s_1(z'_{3,t}) + P_2.(z'_3)^2 + \chi(z'_3)$ with : $|\chi(z'_3)| < \frac{1}{100}|P_2.(z'_3)^2|$ for each $z'_3 \in \mathrm{pr}_{e'_3}(\mathcal{U})$. This implies the second property. $\square$

In particular, the image of any central curve for U_I by $\phi_f \circ f^N$ is a degree 2 curve in $\mathcal{U}$ and in particular has a unique point of tangency with $z_1 = C^{st}$ of coordinate $z'_{3,I}$.

Definition 6.1.4. *We call $R'_{z'_{3,I},I}$ a zone of tangency for U'_I, we denote it by Z_I.*

We have the following lemma :

Lemma 6.1.5. *For any $j > 0$, if there is an index I such that $Z_I \subset W_{j+1}$, then there is an image by ϕ_f of an unstable manifold which has a point of tangency with $z_1 = C^{st}$ in V_j.*

Proof. Let us suppose that there is an index I such that $Z_I \subset W_{j+1}$. Let us consider the image by ϕ_f of a piece of unstable manifold $\mathcal{W}_I$ included inside U'_I. Since it is the image by $\phi_f \circ f^N$ of a degree 2 curve, it admits inside $\phi_f(\mathcal{U})$ an unique point of tangency with $z_1 = C^{st}$. There is a central curve $\mathcal{C}_I$ for U'_I which is a degree 2 curve and such that $\phi_f(\mathcal{C}_I)$ has its unique point of tangency with $z_1 = C^{st}$ inside V_{j+1}. In particular, for all $z_1 \in W_1 - W_j$, we have that : $\{z_1 = C^{st}\} \cap \phi_f(\mathcal{C}_I)$ is the union of two points and then, $\{z_1 = C^{st}\} \cap \phi_f(\mathcal{W}_I)$ has two points too because $\delta_1(R'_{z'_3,I}) < \frac{1}{10^{j+1}}$ inside $\mathcal{U}$. This implies that the unique point of tangency of $\phi_f(\mathcal{W}_I)$ with $z_1 = C^{st}$ is inside V_j because $\{z_1 = C^{st}\} \cap \phi_f(\mathcal{W}_I)$ is the union of two points for all $z_1 \in W_1 - W_j$ but is a degree 2 curve. □

It is implied by Lemma 6.1.1 that Z_I is square type of admissible square Q_I. Remark that given another central curve for U'_I, using Property 1 of Proposition 4.1.8, we get another zone of tangency which is square type of admissible square Q'_I such that Q'_I is in a neighborhood of Q_I of length size $\frac{1}{400}$ times the length side of Q_I. Let $Z_{I\cup\{1\}}$, $Z_{I\cup\{2\}}$, $Z_{I\cup\{11\}}$,$Z_{I\cup\{12\}}$, $Z_{I\cup\{21\}}$ and $Z_{I\cup\{22\}}$ be zones of tangency for $U'_{I\cup\{1\}}$, $U'_{I\cup\{2\}}$,$U'_{I\cup\{11\}}$,$U'_{I\cup\{12\}}$, $U'_{I\cup\{21\}}$ and $U'_{I\cup\{22\}}$. Property 3 of Proposition 4.1.8 implies that Z_I is well subdivided by $Z_{I\cup\{11\}}$,$Z_{I\cup\{12\}}$, $Z_{I\cup\{21\}}$ and $Z_{I\cup\{22\}}$.

Lemma 6.1.6. *For all $f \in \nu_4(f_2)$, the set $\bigcap_{n>i} \bigcup_{n=|I|} Z_I(f)$ contains an open set.*

Proof. For $I \geq i$, Z_I is $\frac{1}{400}$-square shaped and well divided. We take admissible square $S_{I'}$ for $Z_{I'}$, where $I' \in \{I, I\cup\{1\}, I\cup\{2\}, I\cup\{1,1\}, I\cup\{1,2\}, I\cup\{2,1\}, I\cup\{2,2\}\}$. We denote by $B_{I'}$ the ball of diameter $\frac{95}{100}$ times the length side of $S_{I'}$ centered at the center of $S_{I'}$. Let us denote by $\mathcal{B}_{I'}$ the union of all the balls $B_{I'}$ when $S_{I'}$ describes all the admissible squares for $Z_{I'}$. Then, because Z_I is well divided, we have for any choice of admissible squares (verification is left to the reader) :

$$B_I \subset B_{I\cup\{11\}} \cup B_{I\cup\{12\}} \cup B_{I\cup\{21\}} \cup B_{I\cup\{22\}}$$

This gives us : $\mathcal{B}_I \subset \mathcal{B}_{I\cup\{11\}} \cup \mathcal{B}_{I\cup\{12\}} \cup \mathcal{B}_{I\cup\{21\}} \cup \mathcal{B}_{I\cup\{22\}}$ Then : $\mathcal{B}_I \subset \bigcap_{n>i} \bigcup_{n=|I|} Z_I$. Since $\mathcal{B}_I$ is an open set, the result follows. □

Proposition 6.1.7. *There are persistent heteroclinic tangencies between $W^s(\alpha_f)$ and the unstable set of the horseshoe for every $f \in \nu_4(f_2)$.*

Proof. Because there is initially a heteroclinic tangency, we have that the complex number $\mathrm{pr}_1(W^s(\alpha_f)) = 0$ is in the open set given by the previous Lemma 6.1.6 for the initial map. Then, for all J, it is possible to find I_J such that $Z_{I_J} \subset W_J$ and $\bigcap_{I_J} Z_{I_J} = \{0\}$. Then, we get a sequence (I_J) such that $\bigcap_J U'_{I_J}$ is a piece of unstable manifold of a point of the horseshoe which is tangent to $W^s(\alpha_f)$ (if it was not the case, we would get for some step J an intersection $U'_{I_J} \cap V_J$ with no tangency inside). Since this property holds for each $f \in \nu_4(f_2)$, we get persistent heteroclinic tangencies.

□

6.2 Persistent homoclinic tangencies

Proposition 6.2.1. *There is a dense subset of the open set of automorphisms $\nu_4(f_2)$ such that there is a homoclinic tangency between $W^s(\alpha_f)$ and $W^u(\alpha_f)$.*

Proof. The previous Proposition gives us persistent heteroclinic tangencies. The result is just a simple consequence of the following lemma. □

Lemma 6.2.2. *Let $P_5 > 0$ and $\mathcal{W}_{p_5}$ be a piece of unstable manifold of a point of the horseshoe having a tangency with $W^s_{loc}(f)$ for the value p_5^0. Then, there is a p_5^1 such that $|p_5^1 - p_5^0| < P_5$ and $W^u(\alpha_f)$ has a tangency with $W^s(\alpha_f)$.*

Proof. We pick back again the local coordinates ϕ_f in which $W^s(\alpha_f) = \{z_1 = 0\}$. $\mathcal{W}_{p_5}$ is of the form $(f^N((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1)) : z_1 \in \mathcal{N})_{p_5}$ (remind $\mathcal{N}$ is a small neighborhood of c_p). It is having a tangency with $z_1 = 0$ for the value p_5^0 and the value $z_{1,t}$. Denoting Tan_{p_5} the first coordinate of the tangency point of $\phi_f(\mathcal{W}_{p_5})$ with the vertical foliation $z_1 = C^{st}$ and $\mathrm{Tan} : p_5 \mapsto \mathrm{Tan}_{p_5}$, we have that Tan is holomorphic in a neighborhood of p_5^0. More precisely, for each p_5, there is only one value $z_1(p_5)$ of z_1 such that the map $z_1 \mapsto \mathrm{pr}_1 \circ (f^N)((z_1, y^{\mathcal{W}}_{f,1}(z_1), y^{\mathcal{W}}_{f,2}(z_1))$ has its derivative which vanishes at $z_1(p_5)$ and $\mathrm{Tan}_{p_5} = \mathrm{pr}_1 \circ (f^N)((z_1(p_5), y^{\mathcal{W}}_{f,1}(z_1(p_5)), y^{\mathcal{W}}_{f,2}(z_1(p_5)))$. For all $\eta_1 > 0$, reducing if necessary the values of b and σ, we can bound by η_1 for p_5 in a small neighborhood of 0 the derivate $\frac{d(z_1(p_5))}{dp_5}$. Then, we have that $\frac{d\mathrm{Tan}_{p_5}}{dp_5} = \frac{\partial(f_2^N)}{\partial z_1}.\frac{dz_1(p_5)}{dp_5} + \frac{\partial(f_2^N)}{\partial p_5}$. Taking a sufficiently small value of η_1 and since $|\frac{\partial(f_2^N)}{\partial p_5}|$ is bounded by below (see Lemma 5.2.1), we get that $|\frac{\mathrm{Tan}_{p_5}}{dp_5}|$ is bounded below. It follows that the function Tan is not constant. This implies that there is a small disk $\mathbb{D}_5 \subset B(p_5^0, P_5)$ such that there is a disk $\mathbb{D}_t \subset \mathrm{Tan}(\mathbb{D}_5)$ with $0 \in \mathbb{D}_t$.

There exists a $\eta_2 > 0$ such that the following property holds : if $\mathcal{W}'_{p_5}$ is a family of curves such that for each $p_5 \in \mathbb{D}_5$, $\mathcal{W}'_{p_5}$ is η_2-near $\mathcal{W}_{p_5}$ in the C^2-topology, then for all $p_5 \in \frac{1}{2}\mathbb{D}_5$, $\phi_f(\mathcal{W}'_{p_5})$ has its point of tangency Tan'_{p_5} with $z_1 = C^{st}$ inside $\mathrm{Tan}(\mathbb{D}_5)$ and such that $0 \in \mathbb{D}_t \subset \mathrm{Tan}'(\frac{1}{2}\mathbb{D}_5)$. This is due to the Argument Principle.

But by the inclination lemma, it is possible to find a portion $\mathcal{W}^u_{\alpha_f}$ of $W^u(\alpha_f)$ such that for every $p_5 \in \mathbb{D}_5$, $\mathcal{W}^u_{\alpha_f}$ is η_2-near $\mathcal{W}$ in the C^2-topology. Then, we get for a parameter p_5 with $|p_5| < P_5$ and such that $\phi_f(\mathcal{W}^u_{\alpha_f})$ has a tangency with $z_1 = 0$. □

7 Proof of the main result

Proof. We are now ready to conclude. It is possible to take $b_9 < b_8$ such that for every $|b| < b_9$, for every $f \in \nu_4(f_2)$, the smallest eigenvalue λ^{ss} of any $f \in \nu_4(f_2)$ is inferior to $c_{\max}.\lambda^s$ in modulus (see the Appendix for the definition of $c_{\max}$). We take the automorphism f_2 and its neighborhood $\nu_4(f_2)$. In the previous section, we obtained persistent homoclinic tangencies for the periodic point α_f. The point α_f has the property of being sectionally dissipative. Then, the Proposition in the Appendix which gives the creation of sinks from homoclinic tangencies (the proof is given in Appendix for the convenience of the reader) with a classical Baire category argument already used in [7] allow us to conclude to the existence of a residual set of $\mathrm{Aut}_5(\mathbb{C}^3)$ of automorphisms displaying infinitely many sinks: we just have to take $b = \frac{b_9}{2}$ and the neighborhood $\nu_4(f_2)$ as open set. The proof is complete. □

A From homoclinic tangencies to sinks

It is known since the work of Newhouse how to get a sink from a homoclinic tangency. The following adaptation to the holomorphic case was obtained by Gavosto in [12].

Proposition A.1. *Let F_μ be a family of holomorphic mappings on $\mathbb{C}^2$ with a generic homoclinic tangency at the point Q between $W^s(P_0)$ and $W^u(P_0)$ for the hyperbolic fixed point P_0 for $\mu = 0$. We also assume that the eigenvalues of $Df_1(P_0)$, λ_0^s and λ_0^u satisfy $|\lambda_0^s \lambda_0^u| < 1$. Then for each neighborhood V of Q and every neighborhood S of 0 there exists an attracting periodic fixed point in V for some $\mu \in S$.*

Let us explain Gavosto's method. After reparametrizing, one can suppose that the unstable manifold goes through the quadratic tangency with a positive speed. Then, taking a small bidisk which is a neighborhood B of the tangency, one can find an iterate $F_\mu^{N+n_1+M}(B)$ which intersects B. Inside the intersection $B \cap F_\mu^{N+n_1+M}(B)$, one can find a periodic point P for a map F_μ corresponding to a well chosen parameter μ, of period $N + n_1 + M$ such that there is a basis $((1, \alpha(\mu)), (0, 1))$ in which the matrix of the differential of the periodic point is :

$$DF_\mu^{N+n_1+M}(P) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \simeq 0$, $B \simeq (\lambda_0^u)^{n_1}$, $C \simeq (\lambda_0^s)^{n_1}$ and $D \simeq (2\mathrm{pr}_1(P) - \alpha(\mu))(\lambda_0^u)^{n_1}$. Then, we find two eigenvalues depending from $(\lambda_0^u)^{n_1}(\lambda_0^s)^{n_1}$: $\pm\sqrt{(\lambda_0^u)^{n_1}(\lambda_0^s)^{n_1}}$ which have their modulus smaller than 1 in modulus ($|\lambda_0^s \lambda_0^u| < 1$). In the following, we adapt the proof to get an analogous result in dimension 3 when the third eigenvalue is very small.

Proposition A.2. *There is a constant c_{max} such that for all family F_μ of holomorphic mappings of $\mathbb{C}^3$ with a generic homoclinic tangency at the point Q corresponding to the hyperbolic fixed point P_0 for $\mu = 0$ with the eigenvalues of $Df_1(P_0)$, λ_0^{ss}, λ_0^s and λ_0^u satisfying $|\lambda_0^s \lambda_0^u| < 1$ and $|\lambda_0^{ss}| < c_{max}.\lambda_0^s$, for each neighborhood V of Q and every neighborhood S of 0 there exists an attracting periodic fixed point in V for some $\mu \in S$.*

Proof. This is just a perturbation of the previous proof since the new eigenvalue is very small. Using the same method as in [12] we can pick local coordinates such that in the neighborhood of the tangency point Q, the stable manifold M_μ^s is $z_3 = 0$. Using the inclination lemma, we have that the unstable manifolds M_μ^u are given by graphs over z_1 of the form $\{(z_1, w_2(z_1), w_3(z_1)) : z_1\}$ with $|w_2(z_1)| << |w_3(z_1)|$. We have that : $w_3(z_1) = w_3(z_1, \mu) = w_3(0, \mu) + \frac{\partial w_3}{\partial z_1}(0, \mu) z_1 + \frac{\partial^2 w_3}{\partial z_1^2}(0, \mu) z_1^2 + ...$ Since $\frac{\partial^2 w_3}{\partial z_1^2}(0, \mu) \neq 0$, the implicit function Theorem gives us a $z_{1,\mu}$ such that $\frac{\partial^2 w_3}{\partial z_1^2}(z_{1,\mu}, \mu) = 0$. In the coordinates : $Z_1 = z_1 - z_{1,\mu}$,$Z_2 = z_2$ and $Z_3 = z_3$, the unstable manifolds are : $w_3(Z_1) = w_3(Z_1, \mu) = w_3(0, \mu) + \frac{\partial^2 w_3}{\partial Z_1^2}(0, \mu) Z_1^2 + ...$ Then, in the coordinates : $\tilde{Z}_1 = Z_1$,$\tilde{Z}_2 = Z_2$ (note that : $\frac{\partial w_3}{\partial \mu} \neq 0$) and $\tilde{Z}_3 = \mu \frac{Z_3}{w_3(0,\mu)}$, the unstable manifolds are : $w_3(\tilde{Z}_1) = \mu + \tilde{Z}_1^2 h(\tilde{Z}_1, \mu)$ with $h(0, \mu) \neq 0$. The last changement of coordinates $\overline{Z}_1 = \tilde{Z}_1 \sqrt{h(\tilde{Z}_1, \mu)}$ (where $\sqrt{h(\tilde{Z}_1, \mu)}$ is a complex square root of $h(\tilde{Z}_1, \mu))$, $\overline{Z}_2 = \tilde{Z}_2$ and $\overline{Z}_3 = \tilde{Z}_3$ we finally get that the unstable manifolds are given by $\overline{Z}_3 = \overline{Z}_1^2 + \mu$ and $|\overline{Z}_2| << |\overline{Z}_3|$. From now, we take back the notations z_1, z_2, z_3 for these coordinates. We take a tridisk B around Q : $B = \{(z_1, z_2, z_3) : |z_1| < \delta_1, |z_2| < \delta_2, |z_3| < \delta_3\}$ where $\delta_2 = \delta_3 << \delta_1 << 1$. We pick a small neighborhood V_0 of the periodic point whose stable and unstable manifolds have the tangency and by composing by F_μ^N for an integer N if necessary we have that B is in V_0. Using the inclination lemma, for δ_{z_3}, μ small and n large, $F_\mu^n(B)$ will intersect B.

Let us now denote $\Delta_{z_1,z_2} = \{(z_1, z_2, z_3) : (z_1, z_2, z_3) \in B\}$ for $|z_1| < \delta_1, |z_2| < \delta_3$. For a given z_1 such that $|z_1| < \delta_1$, since it is possible to increase n such that $F_\mu^n(B)$ will intersect B in $\{(z_1, z_2, z_3) : |z_2| < \frac{1}{10}, (z_1, z_2, z_3) \in B\}$, by the Argument principle, there is a $z_2(z_1)$ such that $F^n(\Delta_{z_1,z_2(z_1)})$ intersects $\Delta_{z_1,z_2(z_1)}$: indeed, for a given z_1, by the inclination lemma, $F_\mu^N(\Delta_{z_1,z_2})$ is locally a graph $\{(z_1, d_2(z_1), d_3(z_1)) : z_1\}$ with $|d_2(z_1)| < \delta_2$ and $|d_3(z_1)| < \delta_3$. So, since $|d_2(z_1)| < \delta_2$, when z_2 describes a ball of radius δ_2, there is a z_2 such that $z_2 = d_2(z_1)$ by the Argument Principle. Since $|d_3(z_1)| < \delta_3$, this gives us an intersection between Δ_{z_1,z_2} and $F_\mu^N(\Delta_{z_1,z_2})$. In particular, there is a point $P_{z_1} = (z_1, z_2(z_1), f_\mu(z_1))$ such that denoting: $R_{z_1} = (z_1, z_2(z_1), g_\mu(z_1))$, P_{z_1} is the point of $\Delta_{z_1,z_2(z_1)}$ sent on $\Delta_{z_1,z_2(z_1)}$. This point is arbitrarily close to the stable manifold and R_{z_1} is arbitrarily close to the unstable manifold so that : $g_\mu(z_1) - f_\mu(z_1) = z_1^2 + \mu + o(z_1)$. Then, F_μ^n has two fixed points, given by the two zeros of $g_\mu(z_1) - f_\mu(z_1)$ which are small perturbations of the complex square root of μ and its opposite. Let us now pick one of them $P_f = (z_{1,f}, z_{2,f}, z_{3,f})$. We show it is possible to pick the parameter μ such that P_f is a sink.

We can fix a constant $\eta > 0$ such that $(|\lambda_\mu^s| + \eta)(|\lambda_\mu^u| + \eta) < 1$ and such that in V_0 we have that for every vector of C^u is contracted by at least $|\lambda_\mu^s| + \eta$ and at most $|\lambda_\mu^s| - \eta$, every vector of C^{ss} is contracted by at least $|\lambda_\mu^{ss}| + \eta$ and at most $|\lambda_\mu^{ss}| - \eta$ and every vector of C^{cs} is dilated by at least $|\lambda_\mu^u| - \eta$ and at most $|\lambda_\mu^u| + \eta$. Taking $n = n_1 + M$ where for $k = 1, ..., n_1$, the point $F_\mu^k(Q)$ is in a small neighborhood of the periodic point. Let us now take the basis given by e_1',e_2' the two eigenvectors of $D(F_\mu^{-n_1})_{F_\mu^{n_1}(P_f)}$ respectively in C^u and C^{ss}, and e_3. Then, for all $0 \leq k \leq n_1, F_\mu^k(e_1') \in C^1, F_\mu^k(e_2') \in C^{ss}, F_\mu^k(e_3) \in C^{cs}$ because C^u, C^{ss} are invariant by DF_μ^{-1} and C^{cs} is invariant by DF_μ. Besides, we have that : $DF_\mu^{n_1}(e_1') = \Lambda_1 e_1'$ with : $|\Lambda_1| < (|\lambda_\mu^s| + \eta)^{n_1}$. Similarly, we have that : $DF_\mu^{n_1}(e_2') = \Lambda_2 e_2'$ with : $|\Lambda_2| < (|\lambda_\mu^{ss}| + \eta)^{n_1}$ and $DF_\mu^{n_1}(e_3) = \Lambda_3 e_3 + \epsilon_{3,1} e_1' + \epsilon_{3,2} e_2'$ with : $|\Lambda_3| < (|\lambda_\mu^u| + \eta)^{n_1}$, $|\Lambda_3| > \frac{|\epsilon_{3,1}|}{100}$, $|\Lambda_3| > \frac{|\epsilon_{3,2}|}{100}$. Then, in this basis, the matrix DF_μ^n is of the form :

$$DF^{n_1+M}(P_f) = DF_\mu^M(F_\mu^{n_1}(P_f)).DF_\mu^{n_1}(P_f)$$

Since $DF_\mu^M(F_\mu^{n_1}(P_f))$ is of the form :

$$\begin{pmatrix} A & B & C \\ D & E & \epsilon_1 \\ F & G & \epsilon_2 \end{pmatrix}$$

for some fixed constants $A, B, C, D, E, F, G, |\epsilon_1| < \frac{|C|}{100}, |\epsilon_2| < \frac{|C|}{100}$. The factor

$DF_\mu^{n_1}(P_f)$ is of the form :

$$\begin{pmatrix} \Lambda_1 & 0 & \epsilon_{3,1} \\ 0 & \Lambda_2 & \epsilon_{3,2} \\ 0 & 0 & \Lambda_3 \end{pmatrix}$$

Then $DF_\mu^{n_1+M}(P_f)$ of the form : $\begin{pmatrix} A\Lambda_1 & B\Lambda_2 & C\Lambda_3 \\ D\Lambda_1 & E\Lambda_2 & \epsilon_1\Lambda_3 \\ F\Lambda_1 & G\Lambda_2 & \epsilon_2\Lambda_3 \end{pmatrix}$ with : $|\epsilon_1\Lambda_3| << |C\Lambda_3|$, $|\epsilon_2\Lambda_3| << |C\Lambda_3|$ and if $|\lambda_0^{ss}| < c_{\max}|\lambda_0^s|$ for a certain constant $c_{\max}$, the second column is very small compared to the two others. Taking the determinant of :

$$XI_3 - DF_\mu^{n_1+M}(P_f) = \begin{pmatrix} X - A\Lambda_1 & -B\Lambda_2 & -C\Lambda_3 \\ -D\Lambda_1 & X - E\Lambda_2 & -\epsilon_1\Lambda_3 \\ -F\Lambda_1 & -G\Lambda_2 & X - \epsilon_2\Lambda_3 \end{pmatrix}$$

The characteristic polynomial of $DF_\mu^{n_1+M}(P_f)$, this is the determinant of this matrix, is the sum of X^3, $-CF\Lambda_1\Lambda_3X$ and terms of coefficient at least 100 times smaller than these ones so the characteristic polynomial is a small perturbation of : $X(X^2 - CF(\Lambda_1\Lambda_3))$ and so the eigenvalues are near 0 and the two complex square roots of $CF(\Lambda_1\Lambda_3)$. Since we have that : $|\Lambda_1\Lambda_3| < (((|\lambda_\mu^s| + \eta)(|\lambda_\mu^u| + \eta))^{n_1} < 1$. In particular, if n_1 is sufficiently high, the three eigenvalues are of modulus inferior to 1 and so P_f is a sink. □

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LAMA, UNIVERSITE PARIS-EST MARNE LA VALLEE, 5 BOULEVARD DESCARTES, 77454 CHAMPS SUR MARNE, FRANCE

E-mail address : sebastien.biebler@u-pem.fr