# Random walk in a stratified medium

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**Abstract**

We give a recurrence criterion for a Markov chain in $\mathbb{Z}^{d+1}$ in a medium stratified by parallel affine hyperplanes. The asymptotics of the random walk is governed by some notion of directional flux variance, describing the dispersive power of some associated average flow. The result admits a geometrical interpretation, surprisingly intrinsically non-Euclidean. Some applications and open questions are discussed.

## 1 Introduction

We study the recurrence properties of an inhomogeneous Markov chain $(S_n)_{n\geq 0}$ in $\mathbb{Z}^d\times\mathbb{Z}$, where $d\geq 1$. Starting the random walk at 0, let $S_n=(S_n^1,S_n^2)\in\mathbb{Z}^d\times\mathbb{Z}$. We call "vertical" the quantities relative to the second coordinate. The environment is invariant under $\mathbb{Z}^d$-translations, i.e. the collection of transitions laws is stratified with respect to the affine hyperplanes $(\mathbb{Z}^d\times\{n\})_{n\in\mathbb{Z}}$. We make no hypothesis on the relative dependence between transitions laws in distinct hyperplanes.

A planar random walk of this type was proposed by Campanino and Petritis [3] in 2003, as a simplified probabilistic version of PDE transport models in stratified porous medium considered by Matheron and de Marsily [6]. Following this line of research, we focus on a more general case in $\mathbb{Z}^{d+1}$. For the sequel we fix Euclidean Norms and denote scalar product by a dot.

Let us state the model. For each vertical $n\in\mathbb{Z}$, let reals $p_n,q_n,r_n$ with $p_n+q_n+r_n=1$ and a probability measure $\mu_n$ with support in $\mathbb{Z}^d$. We suppose that for some $\delta>0$ and all $n\in\mathbb{Z}$ :

1) $\min\{p_n,q_n,r_n\}\geq\delta$,

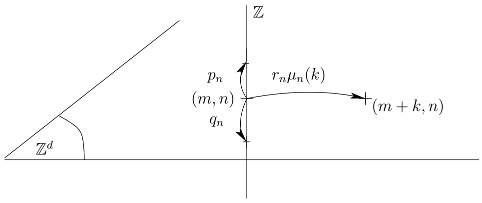
2) $\sum_{k\in\mathbb{Z}^d}\|k\|^{\max(d,3)}\mu_n(k)\leq 1/\delta$,

3) the eigenvalues of the real symmetric matrix $\sum_{k\in\mathbb{Z}^d}kk^T\ \mu_n(k)$ are $\geq\delta$. Equivalently :

$$\sum_{k\in\mathbb{Z}^d}(t.k)^2\mu_n(k)\geq\delta\|t\|^2,\ t\in\mathbb{R}^d.$$

Notice that the last condition implies that the subgroup of $(\mathbb{Z}^d,+)$ generated by $\mathrm{supp}(\mu_n)$ is $d$-dimensional. The transition laws are then defined, for all $(m,n)\in\mathbb{Z}^d\times\mathbb{Z}$ and $k\in\mathbb{Z}^d$, by :

$$\mathbb{P}_{(m,n),(m,n+1)}=p_n,\ \mathbb{P}_{(m,n),(m,n-1)}=q_n,\ \mathbb{P}_{(m,n),(m+k,n)}=r_n\mu_n(k).$$

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The model of Campanino-Petritis corresponds to taking $d = 1$, with $p_n = q_n = p \in (0, 1)$ and $\mu_n = \delta_{\varepsilon_n}$, fixing some sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$ of $\pm 1$. Campanino and Petritis [3] for instance show recurrence when $\varepsilon_n = (-1)^n$ and transience for $\varepsilon_n = 1_{n\geq 0} - 1_{n<0}$ or when the $(\varepsilon_n)$ are typical realizations of i.i.d. random variables with law $(\delta_1+\delta_{-1})/2$. In some neighbourhood of this setting, several variations, extensions and second order questions were subsequently considered by various authors; see the introduction of [2]. In [2], a recurrence criterion was given for the model introduced above when $d = 1$, assuming the local vertical symmetries $p_n = q_n$, $n \in \mathbb{Z}$. In this family of random walks, planar simple random walk, hardly recurrent, is the most recurrent one. This explains the prevalence of transience results on the Campanino-Petritis model. Mention that for the latter, a growth condition larger than $\log n$ on $\varepsilon_1 + \cdots + \varepsilon_n$ is sufficient to ensure transience.

Pushing to some natural limit the method used in [2], we establish in this article a recurrence criterion for the model described above. This furnishes a large class of recurrent random walks in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. The mechanism governing the asymptotic behaviour of the random walk reveals some familiarity with classical Electromagnetism, involving notions such as flux variations. The latter represent the dispersive properties of some average flow associated with the random walk. Variations are measured in a probabilistic sense, via some empirical variances. We also provide a geometrical interpretation of the recurrence criterion. Surprisingly it involves hyperbolic geometry, stereographic projections and some kind of anisotropic pseudosphere.

# 2 Statement of the result

## 2.1 Notations and result

**Definition 2.1**

- *For $n \in \mathbb{Z}$, let $m_n = \sum_{k\in\mathbb{Z}^d} k\mu_n(k)$ be the expectation of $\mu_n$.*
- *For $n \in \mathbb{Z}$, let $p'_n = p_n/(p_n + q_n)$, $q'_n = q_n/(p_n + q_n)$.*
- *For $n \in \mathbb{Z}$, let $a_n = q'_n/p'_n = q_n/p_n$ and $b_n = 1/p'_n = 1 + a_n$.*
- *Set :*

$$\rho_n = \begin{cases} a_1 \cdots a_n, & n \geq 1, \\ 1, & n = 0, \\ (1/a_{n+1}) \cdots (1/a_{-1})(1/a_0) & n \leq -1. \end{cases}$$

- *For $n \geq 0$, let :*

$$v_+(n) = \sum_{0\leq k\leq n} \rho_k \text{ and } v_-(n) = a_0 \sum_{-n-1\leq k\leq -1} \rho_k,$$

*as well as :*

$$w_+(n) = \sum_{0\leq k\leq n} (1/\rho_k) \text{ and } w_-(n) = (1/a_0) \sum_{-n-1\leq k\leq -1} (1/\rho_k).$$

We denote by $\theta$ the "left shift" on indices. Given $f = f((q_i/p_i)_{i\in\mathbb{Z}})$, set $\theta f = f((q_{i+1}/p_{i+1})_{i\in\mathbb{Z}})$. In particular the cocycle relation for $(\rho_n)$ reads as :

$$\forall (n, k) \in \mathbb{Z}^2,\ \rho_{n+k} = \rho_n \theta^n \rho_k.$$

We next need a definition of inverse function for non-decreasing functions defined on the set of non-negative integers $\mathbb{N} = \{0, 1, \cdots\}$ and having values in $\mathbb{R}_+ \cup \{+\infty\}$.

**Definition 2.2**
*Let $f : \mathbb{N} \to \mathbb{R}_+ \cup \{+\infty\}$, non-decreasing. For $x \in \mathbb{R}$, let $f^{-1}(x) = \sup\{n \in \mathbb{N} \mid f(n) \leq x\}$, with $\sup\{\mathbb{N}\} = +\infty$ and $\sup\{\varnothing\} = 0$.*

We next turn to notions related to directional fluxes and their variations.

**Definition 2.3**

- *Let* $S_+^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1,\ x_1 \geq 0\}$ *be a half unit Euclidean sphere of* $\mathbb{R}^d$.
- *For* $u \in S_+^{d-1}$ *and* $k \leq l$ *in* $\mathbb{Z}$, *introduce :*

$$R_k^l(u) = \sum_{s=k}^{l} \frac{r_s}{p_s}\frac{\rho_l}{\rho_s} m_s.u \text{ and } T_k^l(u) = \frac{\rho_{k-1}}{\rho_l}(R_k^l(u))^2 = \rho_{k-1}\rho_l \left(\sum_{s=k}^{l} \frac{r_s}{p_s \rho_s} m_s.u\right)^2.$$

- *For* $m \geq 0$, $n \geq 0$, *let* $\psi(-m,n)$ *be the positive (maybe* $+\infty$*) quantity such that :*

$$\psi^2(-m,n) = nw_+ \circ v_+^{-1}(n) + mw_- \circ v_-^{-1}(m).$$

  *We also set* $\psi(n) = \psi(-n,n)$, $\psi_+(n) = \psi(0,n)$, $\psi_-(n) = \psi(-n,0)$, *for* $n \geq 0$.

- *For* $u \in S_+^{d-1}$, $m \geq 0$, $n \geq 0$, *let* $\varphi_u(-m,n)$ *be the positive (maybe* $+\infty$*) quantity such that :*

$$\varphi_u^2(-m,n) = \psi^2(-m,n) + \sum_{-v_-^{-1}(m) \leq k \leq l \leq v_+^{-1}(n)} T_k^l(u).$$

  *Set for* $n \geq 0$, $\varphi_u(n) = \varphi_u(-n,n)$ *and* $\varphi_{u,+}(n) = \psi^2(-n,n) + \sum_{-v_-^{-1}(n) \leq k \leq l \leq v_+^{-1}(n), kl>0} T_k^l(u)$. *Introduce also :*

$$\varphi_{u,++}(n) = \psi^2(0,n) + \sum_{1 \leq k \leq l \leq v_+^{-1}(n)} T_k^l(u) \text{ and } \varphi_{u,+-}(n) = \psi^2(-n,0) + \sum_{-v_-^{-1}(n) \leq k \leq l \leq -1} T_k^l(u).$$

The aim of the article is to prove the following result.

**Theorem 2.4**
*The random walk is recurrent if and only if :*

$$\sum_{n \geq 1} n^{-d-1} \int_{S_+^{d-1}} \frac{(\varphi_u^{-1}(n))^2}{\varphi_{u,+}^{-1}(n)}\, du = +\infty.$$

## 2.2 Geometrical interpretation; corollaries

Let us detail a geometrical interpretation of the above result. What comes out of the computations is the integral :

$$\int_{u \in S_+^{d-1}, 0<t<1} \frac{(\varphi_u^{-1}(1/t))^2}{\varphi_{u,+}^{-1}(1/t)}\, t^{d-1} du dt.$$

It will be explained later why this quantity has the same order as the one appearing in the statement of the theorem. The term $(\varphi_u^{-1}(1/t))^2/\varphi_{u,+}^{-1}(1/t)$ essentially comes from a stereographic projection. We draw below a picture when $d = 2$ (hence in $\mathbb{R}^3$) showing that the previous integral is the volume of some anisotropic version of Beltrami's pseudosphere. The classical pseudosphere is a model in $\mathbb{R}^3$ of a part of the hyperbolic plane (the whole hyperbolic plane cannot be represented in $\mathbb{R}^3$; theorem of Hilbert, 1901). Here is a way of visualizing this integral ($d = 2$) :

- Draw the vertical line passing at 0, directed by $e_3$, the third vector of the canonical basis of $\mathbb{R}^3$. Fix $u \in S_+^1$. Let $P_0 = u^\perp$ be the vectorial plane orthogonal to $u$. For $0 < t < 1$, let $P_t$ be the affine plane parallel to $P_0$ and passing through $tu$.

- We parametrize points on the left half of $P_0$ in polar coordinates $\rho e^{i\alpha}$, with $0 \leq \alpha \leq \pi$ and $\rho > 0$, as shown, starting from the Northern part of the vertical axis and turning counterclockwise.

- At each $\rho e^{i\alpha}$ we plug in direction $u$ (therefore orthogonally to the plane $P_0$) the length $1/\varphi_u(-\rho\sin(\alpha/2), \rho\cos(\alpha/2))$. When $0 \leq \alpha \leq \pi$ and $\rho > 0$, the $[-\rho\sin(\alpha/2), \rho\cos(\alpha/2)]$ describe all the intervals in the vertical direction containing the point 0. We hence obtain a surface above the left half of $P_0$ in direction $u$, parametrized by $\rho > 0$ and $0 \leq \alpha \leq \pi$.

- For $0 < t < 1$, the plane $P_t$ cuts this surface along some dashed line shown on the picture. The point on this line and in the horizontal plane has coordinates $(t, -\sqrt{2}\varphi_u^{-1}(1/t), 0)$ in the basis $(u, u', e_3)$, where $u' = e_3 \wedge u$. The point on the level line lying on the vertical line passing through $(t, 0, 0)$ has components $(t, 0, \varphi_{u,++}^{-1}(1/t))$.

- A little of geometry, related to some kind of stereographic projection, shows how to obtain $z_t$ equal to $(\varphi_u^{-1}(1/t))^2/\varphi_{u,++}^{-1}(1/t)$ up to multiplicative constants on the picture, at a point of coordinates $(t, 0, z_t)$, still with respect to $(u, u', e_3)$. We use that $\varphi_u^{-1}(x) \leq \varphi_{u,+}^{-1}(x) \leq \varphi_{u,+\pm}^{-1}(x)$, giving that the orthogonal triangle in $P_t$ with vertices $(t, -\sqrt{2}\varphi_u^{-1}(1/t), 0)$, $(t, 0, 0)$ and $(t, 0, \varphi_{u,++}^{-1}(1/t))$ has a vertical edge larger or equal to the horizontal side.

- In the plane generated by $u$ and $e_3$ we have a function $t \longmapsto z_t$, $0 < t < 1$. The integral of this function is the top of the hatched area and has order :

$$\int_{0<t<1} \frac{(\varphi_u^{-1}(1/t))^2}{\varphi_{u,++}^{-1}(1/t)}\, dt.$$

Do next the same work on the Southern side and obtain a similar area (not equal to previous one in general) corresponding to :

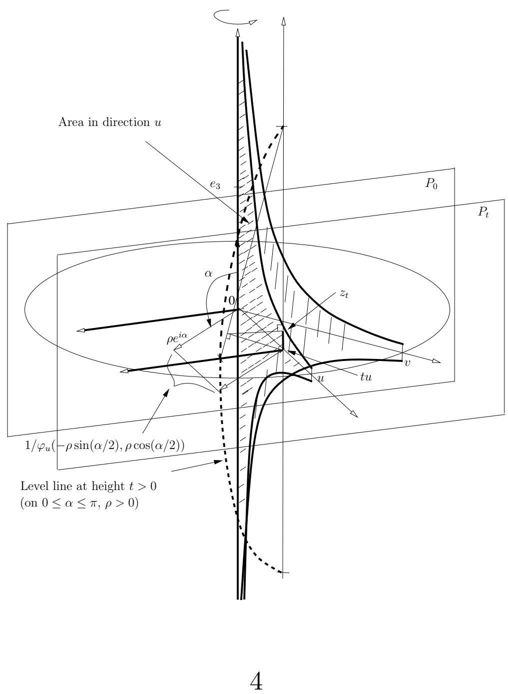
$$\int_{0<t<1} \frac{(\varphi_u^{-1}(1/t))^2}{\varphi_{u,+-}^{-1}(1/t)}\, dt.$$

When summing the two last integrals, one globally obtains the full hatched area, in the plane generated by $e_3$ and $u$ and this has order :

$$\int_{0<t<1} \frac{(\varphi_u^{-1}(1/t))^2}{\varphi_{u,+}^{-1}(1/t)}\, dt.$$

- Rotating the picture with respect to $u \in S^1_+$, one gets a three-dimensional object, looking like some half pseudosphere. The corresponding volume equals, up to constants the volume of the integral we wish to illustrate.

Let us precise that when the random walk goes frankly in some direction $u \in S^1$, then for all $v \in S^1_+$ not orthogonal to $u$, some pinching effect occurs towards the horizontal plane in the sliced picture in direction $v$, making the area (and thus the global volume) smaller.



We now discuss some consequences of the theorem.

**Corollary 2.5**
*For the general model, a sufficient condition for transience is :*

$$\sum_{n\geq 1}\int_{S_+^{d-1}} \frac{1}{(\varphi_u(n))^d}\, du < +\infty.$$

*It is true under the condition* $\sum_{n\geq 1}\psi(n)^{-d} < +\infty$*, depending only on the vertical. The latter is satisfied in the following cases :*

- $d \geq 3$,
- $d = 2$ *and* $w_+ \circ v_+^{-1}(n) + w_- \circ v_-^{-1}(n) \geq (\log n)^{1+\varepsilon}$ *and in particular if* $p_n = q_n$*,* $n \in \mathbb{Z}$*,*
- $d = 1$ *and* $w_+ \circ v_+^{-1}(n) + w_- \circ v_-^{-1}(n) \geq n(\log n)^{2+\varepsilon}$*.*

In the antisymmetric case, an explicit criterion is available.

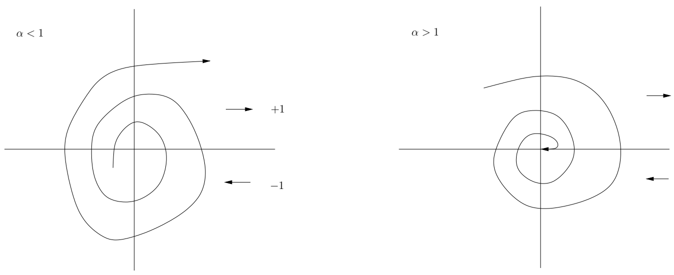
**Proposition 2.6**
*Antisymmetric case. Suppose that* $m_{-n} = -m_n$ *and* $\rho_{-n} = \rho_n$*,* $n \geq 0$*. The random walk is transient if and only if :*

$$\sum_{n\geq 1}\int_{S_+^{d-1}} \frac{1}{(\varphi_{u,++}(n))^d}\, du < +\infty.$$

*In particular, let* $m_n = -m_{-n} = c \neq 0$*,* $n \geq 1$*, with* $m_0 = 0$*, and suppose that* $c_1 n^\alpha \leq \rho_n \leq c_2 n^\alpha$*,* $n \geq 0$*, where* $\alpha \in \mathbb{R}$*. Then :*

- *If* $d = 1$*, the random walk is recurrent if and only if* $\alpha \geq 1$*.*
- *If* $d = 2$*, the random walk is recurrent if and only if* $\alpha \geq 3$*.*

There would be many other cases to consider. The Campanino-Petritis model, i.e. $d = 1$, $p_n = q_n$ and $\mu_n = \delta_{\varepsilon_n}$, in the case when $\varepsilon_n = 1_{n\geq 0} - 1_{n<0}$ corresponds to the first example with $\rho_n = 1$, so $\alpha = 0$, and the random walk is transient. As already indicated in [3], the parameters are largely interior to the transience domain. Remark that in the antisymmetric case when $d = 1$, taking $\mu_n = \delta_1$, $\mu_{-n} = \delta_{-1}$, for $n \geq 1$, and $\rho_n \sim n^\alpha$, $n \geq 1$, since horizontal steps are restricted to $+1$ in the North and to $-1$ in the South, the random walk (recurrent or transient) necessarily makes spirals.




In contrast with the flat case ($p_n = q_n$, $n \in \mathbb{Z}$), one can for this model in some sense “suppress” the vertical dimension for some values of the parameters. Indeed, when $\sum_{n\in\mathbb{Z}}(1/\rho_n) < +\infty$, the vertical component is positive recurrent, hence admits an invariant probability measure. In this sense, the random walk is then “essentially” $d$-dimensional. When $d = 1$, this is a kind of random walk in a half-pipe. This explains the critical values of $d$ appearing in the corollary and in particular the fact that the random walk (in $\mathbb{Z}^{d+1}$) can be recurrent when $d = 2$.

**Proposition 2.7**
*Suppose that* $\sum_{n\in\mathbb{Z}}(1/\rho_n) < +\infty$ *and* $d = 1$*.*

*i) If* $\sum_{n\in\mathbb{Z}}(m_n/\rho_n) \neq 0$*, then the random walk is transient.*

*ii) If* $\sum_{n\in\mathbb{Z}}(m_n/\rho_n) = 0$ *and* $p_{-n} = q_n$*,* $r_{-n} = r_n$*,* $\mu_n = \mu_{-n}$*, for* $n \geq 0$*, then the random walk is recurrent.*

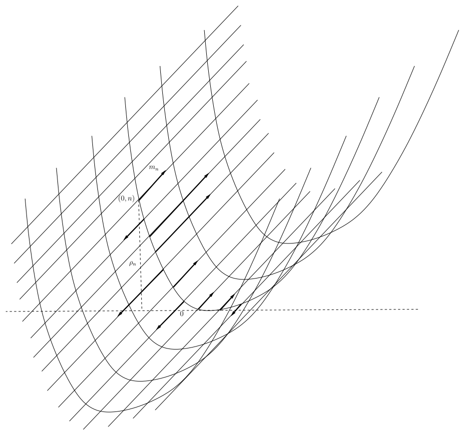
Hence, when $\sum_{n\in\mathbb{Z}}(1/\rho_n) < +\infty$, the finiteness condition for transience is replaced by some non-zero condition. Fixing $d = 1$ and, breaking momentarily the assumptions, suppose that brutally $\rho_1 = \rho_{-1} = +\infty$ (giving $\rho_k = +\infty$, $k \neq 0$). One then recovers that the condition $m_0 \neq 0$ is necessary and sufficient for transience, which is a standard result for one-dimensional i.i.d random walk with integrable step.

Notice in such a model the important role of a single hyperplane, as the latter can modify the asymptotics. This is not true if $\sum_{n\in\mathbb{Z}} 1/\rho_n$, for example if $p_n = q_n$, $n \in \mathbb{Z}$, and $d = 1$, where changing one line did not modify the asymptotics (see the introduction in [2]).

We give an application to a random walk in a half-pipe, with independent level lines.

**Corollary 2.8**
*Let $d = 1$ and $\sum_{n\in\mathbb{Z}}(1/\rho_n) < +\infty$. Suppose that the $(m_n)_{n\in\mathbb{Z}}$ are a typical realization of some independent uniformly bounded random variables, at least one having a density. Then, almost-surely, the associated random walk is transient.*

Indeed, it is clear from the hypotheses that the random variable $\omega \longmapsto \sum_{n\in\mathbb{Z}}(m_n(\omega)/\rho_n)$ admits a density, so equals 0 with zero probability. We next apply the result of the previous proposition.




In this picture in $\mathbb{R}^3$ of the $\mathbb{Z}^2$-half-pipe, we have drawn the points $(k, l) \in \mathbb{Z}^2$ at height $\rho_l$. The quantity $\rho_l$ can be considered as the "level of the sea" at $(k, l)$. The borders of the half-pipe are very steep due to the condition $\sum_{n\in\mathbb{Z}}(1/\rho_n) < +\infty$.

# 3 Preliminaries

## 3.1 Sleszynski-Pringsheim continued fractions

Formally, a general finite continued fraction is written as follows :

$$[(c_1, d_1); (c_2, d_2); \cdots ; (c_n, d_n)] = \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \cfrac{\cdots}{\cdots + \cfrac{c_n}{d_n}}}}.$$

We shall consider finite continued fractions corresponding to the application to some $z_0 \in \mathbb{C}$ in the unit disk of functions of the form $z \longmapsto c/(d + z)$, with complex numbers $c \neq 0$ and $d$ so that $|c| + 1 \leq |d|$, hence preserving the closed unit disk. Such finite continued fractions are usually called finite Sleszynski-Pringsheim (SP) continued fractions.

Infinite SP-continued fractions, written $[(c_1, d_1); (c_2, d_2); \cdots]$, also converge, by the Sleszynski-Pringsheim theorem (see [5]). We will reproduce the arguments proving this result.

For $n \geq 0$, the finite continued fraction $[(c_1, d_1); (c_2, d_2); \cdots ; (c_n, d_n)]$ can be reduced as a fraction $A_n/B_n$, where the $(A_n)$ and $(B_n)$ satisfy the recursive relations :

$$\begin{cases} A_n = d_n A_{n-1} + c_n A_{n-2}, \ n \geq 1, \ A_{-1} = 1, \ A_0 = 0, \\ \\ B_n = d_n B_{n-1} + c_n B_{n-2}, \ n \geq 1, \ B_{-1} = 0, \ B_0 = 1. \end{cases}$$

In our setting it will be directly checked that $B_n$ is never zero for $n \geq 0$. We require the following classical determinant. For $n \geq 1$ :

$$\begin{aligned} A_n B_{n-1} - A_{n-1} B_n &= (-c_n)(A_{n-1} B_{n-2} - A_{n-2} B_{n-1}) = \cdots \\ &= (-1)^n c_1 \cdots c_n (A_0 B_{-1} - A_{-1} B_0) = (-1)^{n+1} c_1 \cdots c_n. \end{aligned}$$

This gives the following representation as a series :

$$[(c_1, d_1); (c_2, d_2); \cdots ; (c_n, d_n)] = \frac{A_n}{B_n} = \sum_{k=1}^{n} \left( \frac{A_k}{B_k} - \frac{A_{k-1}}{B_{k-1}} \right) = \sum_{k=1}^{n} \frac{(-1)^{k+1} c_1 \cdots c_k}{B_k B_{k-1}}. \tag{1}$$

We now focus on a particular class of SP-continued fractions that will appear frequently.

**Lemma 3.1**
*Assume that* $\lim_{n \to +\infty} v_+(n) = +\infty$.

1. *Let* $(\gamma_n)_{n \geq 1}$ *and* $(\gamma'_n)_{n \geq 1}$ *be sequences of complex numbers with* $0 < |\gamma_n| \leq 1$, $|\gamma'_n| \leq 1$. *Then :*

$$[(a_1, b_1/\gamma_1); (-a_2, b_2/\gamma_2); \cdots ; (-a_{n-1}, b_{n-1}/\gamma_{n-1}); (-a_n, b_n/\gamma_n - \gamma'_n)]$$

*is well-defined. It converges to* $[(a_1, b_1/\gamma_1); (-a_2, b_2/\gamma_2); \cdots ; (-a_n, b_n/\gamma_n); \cdots]$, *as* $n \to +\infty$, *an infinite SP-continued fraction. The latter is the limit of* $A_n/B_n$, *as* $n \to +\infty$, *where :*

$$\begin{cases} A_n = \dfrac{b_n}{\gamma_n} A_{n-1} - a_n A_{n-2}, \ n \geq 2, \ A_{-1} = 1, \ A_0 = 0, \ A_1 = a_1, \\ \\ B_n = \dfrac{b_n}{\gamma_n} B_{n-1} - a_n B_{n-2}, \ n \geq 2, \ B_{-1} = 0, \ B_0 = 1, \ B_1 = b_1/\gamma_1. \end{cases} \tag{2}$$

2. *Set* $v_+(-1) = 0$. *The solutions* $(B_n)$ *of* (2) *check :*

$$|B_n| - |B_{n-1}| \geq a_n(|B_{n-1}| - |B_{n-2}|), \ n \geq 1.$$

*As a result,* $|B_n| \geq v_+(n)$, $n \geq -1$. *If the* $0 < \gamma_n \leq 1$ *are real, then* $B_n > B_{n-1} > \cdots > B_{-1} = 0$. *When* $\gamma_n = 1$, $n \geq 1$, *then* $B_n = v_+(n)$, $n \geq -1$, *as well as* $A_n = v_+(n) - 1$, $n \geq 0$.

3. *In* (2), $n \longmapsto |B_n|/v_+(n)$, $n \geq 0$, *is non-decreasing. Also, for* $n \geq 1$ *:*

$$\sum_{k > n} \frac{\rho_k}{|B_k B_{k-1}|} \leq \frac{v_+(n)}{|B_n|^2} \leq \frac{1}{|B_n|}.$$

*Proof of the lemma :*
The solutions of (2) check $|B_n| \geq b_n |B_{n-1}| - a_n |B_{n-2}|$, $n \geq 1$. Hence :

$$|B_n| - |B_{n-1}| \geq a_n(|B_{n-1}| - |B_{n-2}|), \ n \geq 1.$$

When iterating, $|B_n| - |B_{n-1}| \geq \rho_n$. Thus $|B_n| \geq v_+(n)$.

In point 1., the finite continued fraction is well-defined because $a_k \neq 0$, $|b_k/\gamma_k| - a_k \geq b_k - a_k = 1$ and $|\gamma'_n| \leq 1$. We obtain from (1) :

$$[(a_1, b_1/\gamma_1); (-a_2, b_2/\gamma_2); \cdots ; (-a_n, b_n/\gamma_n - \gamma'_n)] = \sum_{k=1}^{n-1} \frac{\rho_k}{B_k B_{k-1}} + \frac{\rho_n}{B_{n-1}\tilde{B}_n}, \tag{3}$$

where $\tilde{B}_n = (b_n/\gamma_n - \gamma'_n)B_{n-1} - a_n B_{n-2}$. We get :

$$|\tilde{B}_n| \geq (b_n - 1)|B_{n-1}| - a_n|B_{n-2}| \geq a_n(|B_{n-1}| - |B_{n-2}|) \geq a_n\rho_{n-1} = \rho_n.$$

In (3) the first term in the right-hand side is absolutely convergent, because, as $|B_k| \geq v_+(k)$ :

$$\sum_{k\geq 1} \frac{\rho_k}{|B_k B_{k-1}|} \leq \sum_{k\geq 1} \frac{v_+(k) - v_+(k-1)}{v_+(k)v_+(k-1)} = \sum_{k\geq 1} \left(\frac{1}{v_+(k-1)} - \frac{1}{v_+(k)}\right) = 1.$$

As $|B_{n-1}| \to +\infty$, we conclude that the right-hand side in (3) converges to $\sum_{k\geq 1} \rho_k/(B_k B_{k-1})$.

When the $\gamma_n$ are real, write $B_n - B_{n-1} = \frac{(1-\gamma_n)b_n}{\gamma_n} B_{n-1} + a_n(B_{n-1} - B_{n-2})$. The condition "$B_n > B_{n-1} \geq 0$" is then transmitted recursively. If $\gamma_n = 1$, then $B_n - B_{n-1} = a_n(B_{n-1} - B_{n-2})$, giving $B_n = v_+(n)$, $n \geq 0$. Similarly, $A_n = v_+(n) - 1$, $n \geq 0$, as :

$$A_n - A_{n-1} = a_n(A_{n-1} - A_{n-2}) = \cdots = a_n \cdots a_2(A_1 - A_0) = \rho_n \text{ and } A_0 = 0.$$

For the last point, we first show that $n \longmapsto |B_n|/v_+(n)$, $n \geq 0$, is non-decreasing. We will require it in the equivalent form $|B_n|/(|B_{n+1}| - |B_n|) \leq v_+(n)/(v_+(n+1) - v_+(n))$. Write :

$$\begin{aligned} v_+(n)|B_{n+1}| - v_+(n+1)|B_n| &\geq v_+(n)\left(b_{n+1}|B_n| - a_{n+1}|B_{n-1}|\right) - v_+(n+1)|B_n| \\ &\geq a_{n+1}\left(v_+(n-1)|B_n| - v_+(n)|B_{n-1}|\right) \\ &\geq \cdots \geq \rho_{n+1}\left(|B_0|v_+(-1) - v_+(0)|B_{-1}|\right) = 0. \end{aligned}$$

Finally, using the previous results, for $n \geq 1$ :

$$\begin{aligned} \sum_{k>n} \frac{\rho_k}{|B_k B_{k-1}|} &= \sum_{k>n} \rho_k \left(\frac{1}{|B_{k-1}|} - \frac{1}{|B_k|}\right) \frac{1}{|B_k| - |B_{k-1}|} \\ &\leq \sum_{k>n} \rho_k \left(\frac{1}{|B_{k-1}|} - \frac{1}{|B_k|}\right) \frac{1}{a_k \cdots a_{n+2}(|B_{n+1}| - |B_n|)} \\ &\leq \frac{\rho_{n+1}}{|B_{n+1}| - |B_n|} \sum_{k>n} \left(\frac{1}{|B_{k-1}|} - \frac{1}{|B_k|}\right) \leq \frac{v_+(n+1) - v_+(n)}{|B_{n+1}| - |B_n|} \frac{1}{|B_n|} \leq \frac{v_+(n)}{|B_n|^2}. \end{aligned}$$

This completes the proof of the lemma. □

## 3.2 Asymptotical behavior of the vertical component

The question of the recurrence/transience of the vertical component of the random walk is classical. Indeed the vertical component restricted to the subsequence of vertical movements is the random walk on $\mathbb{Z}$ with transition probabilities $\mathbb{P}_{n,n-1} = q'_n$ and $\mathbb{P}_{n,n+1} = p'_n$, $n \in \mathbb{Z}$.

**Lemma 3.2**
*The Markov chain on $\mathbb{Z}$ so that $\mathbb{P}_{n,n+1} = p'_n$ and $\mathbb{P}_{n,n-1} = q'_n$, $n \in \mathbb{Z}$, is recurrent if and only if $\lim_{n\to+\infty} v_+(n) = +\infty$ and $\lim_{n\to+\infty} v_-(n) = +\infty$.*

*Proof of the lemma :*
Fix $N > 1$ and let $f(k) = \mathbb{P}_k(\text{exit } [0, N] \text{ on the left side})$, $0 \leq k \leq N$. The Markov property implies that $k \longmapsto f(k)$ is harmonic in the interior of this interval. Precisely, for $1 \leq k \leq N - 1$ :

$$f(k) = p'_k f(k+1) + q'_k f(k-1).$$

Let $g(k) = f(k) - f(k-1)$. We obtain $g(k) = (p_k/q_k)g(k+1)$ and therefore $g(k) = \rho_{k-1}g(1)$, $1 \leq k \leq N$. As a result :

$$-1 = \sum_{k=1}^{N} g(k) = -\mathbb{P}_1(\text{exit } [0,N] \text{ at } N) \sum_{1 \leq k \leq N} \rho_{k-1}.$$

Hence $\mathbb{P}_1(\text{reach } 0) = 1 \Leftrightarrow \lim_{n\to+\infty} v_+(n) = +\infty$. Idem $\mathbb{P}_{-1}(\text{reach } 0) = 1 \Leftrightarrow \lim_{n\to+\infty} v_-(n) = +\infty$. This furnishes the desired result. □

The previous criterion can be reformulated using trees. Let us say that a random variable $X$ has the geometrical law $\mathcal{G}(p)$, $0 < p < 1$, if $\mathbb{P}(X = n) = p^n(1-p)$, $n \geq 0$.

**Lemma 3.3**
*Consider the Galton-Watson tree $(Z_n^+)_{n\geq 1}$ with $Z_1^+ = 1$ and, independently, the law of the number of children at level $n+1$ of an individual at level $n \geq 1$ is $\mathcal{G}(p'_n)$. Then this tree is finite almost-surely if and only if $\lim_{n\to+\infty} v_+(n) = +\infty$.*

*Proof of the lemma :*
As usual, since $\{Z_n^+ = 0\} \subset \{Z_{n+1}^+ = 0\}$, the almost-sure finiteness is equivalent to $\mathbb{P}(Z_n^+ = 0) \to 1$. Fix $0 < s < 1$ and recall that $\mathbb{E}(s^{Z_n^+}) - s \leq \mathbb{P}(Z_n^+ = 0) \leq \mathbb{E}(s^{Z_n^+})$. Taking $n \geq 2$ :

$$\mathbb{E}\left(s^{Z_n^+}\right) = \mathbb{E}\left[\left(\frac{1-p'_{n-1}}{1-sp'_{n-1}}\right)^{Z_{n-1}^+}\right] = \mathbb{E}\left[\left(\frac{a_{n-1}}{b_{n-1}-s}\right)^{Z_{n-1}^+}\right].$$

Iterating (using $a_{n-1}/(b_{n-1}-s)$ in place of $s$), we obtain the following SP-continued fraction :

$$\mathbb{E}\left(s^{Z_n^+}\right) = [(a_1,b_1); (-a_2,b_2); \cdots ; (-a_{n-2},b_{n-2}); (-a_{n-1},b_{n-1}-s)].$$

This corresponds to $\gamma_k = 1$ and $\gamma'_n = s$ in lemma 3.1. From lemma 3.1 and relation (3) :

$$\mathbb{E}\left(s^{Z_n^+}\right) = \frac{v_+(n-2)-1}{v_+(n-2)} + \frac{\rho_{n-1}}{v_+(n-1)\tilde{B}_{n-1}},$$

with $\tilde{B}_{n-1} = (b_{n-1}-s)v_+(n-2) - a_{n-1}v_+(n-3)$, so that $\tilde{B}_{n-1} \geq \rho_{n-1}$ and $\tilde{B}_{n-1} \geq (1-s)v_+(n-2)$. If $v_+(n) \to +\infty$, then $\mathbb{E}(s^{Z_n^+}) \to 1$ uniformly in $0 < s < 1$, giving $\mathbb{P}(Z_n^+ = 0) \to 1$. If $v_+(n) \to_{n\to+\infty} b \in (0,+\infty)$, then $\rho_n \to 0$ and for fixed $0 < s < 1$ we have $\liminf_n \tilde{B}_{n-1} \geq (1-s)b > 0$, so that $\mathbb{E}(s^{Z_n^+})$ tends to $(b-1)/b < 1$, giving $\lim_n \mathbb{P}(Z_n^+ = 0) = (b-1)/b$. □

*Remark.* — There is naturally a symmetric result for the Southern direction of the vertical component. One introduces, with decreasing indices $n \leq -1$, the Galton-Watson tree $(Z_n^-)_{n\leq -1}$ with $Z_{-1}^- = 1$ such that, independently, the law of the number of children at level $n-1$ of an individual at level $n$ is $\mathcal{G}(q'_n)$. The tree is almost-surely finite if and only if $\lim_{k\to+\infty} v_-(k) = +\infty$.

# 4 Reduction to an i.i.d. random walk in $\mathbb{Z}^d$

For the rest of the article we therefore suppose the vertical component recurrent. Equivalently, from the previous section, this means $\lim_{n\to+\infty} v_\pm(n) = +\infty$. Just observe that if for example $v_+$ is bounded by some $v_+(\infty) < \infty$, then $+\infty = \varphi_u(n) = \varphi_{u,+}(n) = \psi(n)$ for $n > v_+(\infty)$, so the reversed functions are bounded quantities and the integral involved in the theorem is finite. The same happens if $v_-$ is bounded.

We can now introduce the random times $0 = \sigma_0 < \tau_0 < \sigma_1 < \tau_1 < \cdots$, where :

$$\tau_k = \min\{n > \sigma_k \mid S_n^2 \neq 0\},\ \sigma_{k+1} = \{n > \tau_k \mid S_n^2 = 0\}.$$

Introduce the $\mathbb{Z}^d$-displacement $D_n = S^1_{\sigma_n} - S^1_{\sigma_{n-1}}$. As the environment is invariant under $\mathbb{Z}^d$-translations, the $(D_n)_{n\geq 1}$ are globally independent and identically distributed. The following lemma is essentially contained in [3].

**Lemma 4.1**
*Let $T_0 = 0$ and $T_n = D_1 + \cdots + D_n$, $n \geq 1$. The random walk $(S_n)_{n\geq 0}$ is recurrent is $\mathbb{Z}^{d+1}$ if and only if $(T_n)_{n\geq 0}$ is recurrent in $\mathbb{Z}^d$.*

*Proof of the lemma :*
If $(T_n)_{n\geq 0}$ is recurrent in $\mathbb{Z}^d$, then $(S_n)$ is recurrent in $\mathbb{Z}^{d+1}$, as $S_{\sigma_n} = (T_n, 0)$. In case of transience of $(T_n)$, using again the invariance of the environment under $\mathbb{Z}^d$-translations, we have :

$$\exists C,\ \forall x \in \mathbb{Z}^d,\ \sum_{n\geq 1} \mathbb{P}\left(T_n = x\right) \leq C.$$

Let $\Gamma \sim \mathcal{G}(r_0)$ and $\xi_k \sim \mu_0$, for $k \geq 1$, so that $((\xi_k)_{k\geq 1}, \Gamma)$ are globally independent and also from the sequence $(T_n)$. Remark that $(S^1_l)_{l\in[\sigma_k,\tau_k)}$ and $(T_k + \sum_{1\leq m\leq l} \xi_m)_{0\leq l\leq \Gamma}$ have the same law. Introduce the real random variable :

$$H = \sum_{1\leq k\leq \Gamma} \|\xi_k\|.$$

Observe now that $S_n$ can be 0 only for $n$ in some $[\sigma_k, \tau_k)$ and that :

$$\mathbb{P}(\exists n \in [\sigma_k, \tau_k),\ S_n = 0) \leq \mathbb{P}(H \geq \|T_k\|).$$

This provides :

$$\begin{aligned}
\sum_{k\geq 1} \mathbb{P}(\exists n \in [\sigma_k, \tau_k),\ S_n = 0) \leq \sum_{k\geq 1} \mathbb{P}(H \geq \|T_k\|) &\leq \sum_{x\in\mathbb{Z}^d} \sum_{k\geq 1} \mathbb{P}(T_k = x)\mathbb{P}\left(H \geq \|x\|\right) \\
&\leq C \sum_{x\in\mathbb{Z}^d} \mathbb{P}(H \geq \|x\|) \leq C'\mathbb{E}(H^d).
\end{aligned}$$

Finally, this gives :

$$\begin{aligned}
\mathbb{E}(H^d) = \sum_{n\geq 0} \mathbb{P}(\Gamma = n)\mathbb{E}\left[\left(\sum_{1\leq k\leq n} \|\xi_k\|\right)^d\right] &\leq (1 - r_0) \sum_{n\geq 0} r_0^n n^{d-1} \mathbb{E}\left(\sum_{1\leq k\leq n} \|\xi_k\|^d\right) \\
&\leq (1 - r_0) \sum_{n\geq 0} r_0^n n^d \mathbb{E}(\|\xi_1\|^d) < \infty.
\end{aligned}$$

By the Borel-Cantelli lemma, $(S_n)$ is transient. This completes the proof of the lemma. □

This reduces the problem of the recurrence of $(S_n)$ to that of $(T_n)$. Set :

$$D = D_1 \text{ and } \chi_D(t) = \mathbb{E}(e^{it.D}),\ t \in \mathbb{R}^d.$$

We shall use the following theorem, the strong form of the Chung-Fuchs recurrence criterion, giving an analytical recurrence criterion for a *i.i.d.* random walk in $\mathbb{Z}^d$. See Spitzer [7]. Recall that $S_+^{d-1}$ denotes the half unit sphere and let $B_d(0, \eta)$ be the ball of center 0 and radius $\eta > 0$ in $\mathbb{R}^d$.

**Theorem 4.2**
*Suppose that the subgroup of $(\mathbb{Z}^d, +)$ generated by the support of the law of $D$ is $\mathbb{Z}^d$. Then the random walk $(T_n)_{n\geq 0}$ is transient if and only if for some $\eta > 0$ :*

$$\int_{B_d(0,\eta)} Re\left(\frac{1}{1 - \chi_D(x)}\right)\ dx < +\infty. \tag{4}$$

Notice that one can restrict the integral to the half unit ball $S_+^{d-1}.]0,\eta[$. Forgetting the multiplicative constant coming from the change of variables in polar coordinates, we next decompose the integral in the form :

$$\int_{S_+^{d-1}\times]0,\eta[} \mathrm{Re}\left(\frac{1}{1-\chi_D(ut)}\right)\, t^{d-1}dudt, \text{ with } (u,t)\in S_+^{d-1}\times]0,\eta[.$$

From our assumptions, the subgroup $G_D$ of $(\mathbb{Z}^d,+)$ generated by the support of the law of $D$ is $d$-dimensional. Observe that $(T_n)$ lives in $G_D$ and recall that $G_D$ admits a basis over $\mathbb{Z}$. Reparametrizing $G_D$ corresponds to making a linear change of variables in (4). The properties of dominated variations shown below in lemma 6.1 imply that we can assume that $G_D=\mathbb{Z}^d$ from the beginning. This is what we do in the sequel.

The only singularity of $1/(1-\chi_D)$ in $\mathbb{R}^d/\mathbb{Z}^d$ is now 0. Fixing $0<\eta<1/2$ small enough, we take $u\in S_+^{d-1}$ and $0<t<\eta$.

## 4.1 Local time and contour of a Galton-Watson tree

For $u\in S_+^{d-1}$ we study the behavior near $0^+$ of $t\longmapsto \chi_D(ut)$. Let us introduce the one-dimensional random walk $(Y_n)_{n\geq 0}$ on $\mathbb{Z}$ such that $Y_0=0$ and $\mathbb{P}_{n,n-1}=q'_n$ and $\mathbb{P}_{n,n+1}=p'_n$, for $n\in\mathbb{Z}$. This is $(S_n^2)_{n\geq 0}$ restricted to the sequence of vertical jumps.

Let $\sigma=\min\{k\geq 1 \mid Y_k=0\}$ be the return time to 0. Grouping in packets the successive $\mathbb{Z}^d$-steps of the random walk, observe that $D$ can be written as :

$$D=\sum_{k=0}^{\sigma-1}\left(\sum_{m=1}^{\Gamma_k}\xi_m^{(k)}\right),$$

where, conditionally on the $(Y_l)_{l\geq 0}$, the $((\xi_m^{(k)})_{m\geq 1,k\geq 0},(\Gamma_k)_{k\geq 0})$ are independent with $\xi_m^{(k)}\sim\mu_{Y_k}$ and $\Gamma_k\sim(\mathcal{G}(r_{Y_k}))$, for all $k\geq 0$. To detail $\chi_D$, define for $n\in\mathbb{Z}$ :

$$\varphi_n(ut)=\mathbb{E}\left(\exp\left(itu.\sum_{m=1}^{\Gamma}\xi_m\right)\right),\ t\in\mathbb{R}, \tag{5}$$

with random variables $\Gamma\sim\mathcal{G}(r_n)$ and $\xi_m\sim\mu_n$, for $m\geq 1$, all being independent. Conditioning on the $(Y_l)_{l\geq 0}$, we obtain the equality :

$$\chi_D(ut)=\mathbb{E}\left(\prod_{k=0}^{\sigma-1}\varphi_{Y_k}(ut)\right)=\varphi_0(ut)\mathbb{E}\left(\prod_{k=1}^{\sigma-1}\varphi_{Y_k}(ut)\right).$$

The only remaining alea is that of the $(Y_l)_{l\geq 0}$. Introduce the conditional expectations :

$$\mathbb{E}^+(.)=\mathbb{E}(.\mid Y_1=1) \text{ and } \mathbb{E}^-(.)=\mathbb{E}(.\mid Y_1=-1).$$

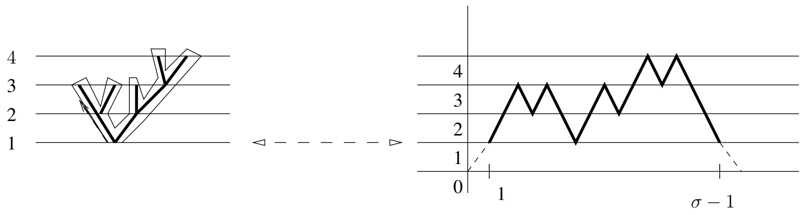
Setting $\chi_D^{\pm}(ut)=\mathbb{E}^{\pm}\left(\prod_{k=1}^{\sigma-1}\varphi_{Y_k}(ut)\right)$, this leads to :

$$\chi_D(ut) \;=\; \varphi_0(ut)(p'_0\chi_D^+(ut)+q'_0\chi_D^-(ut)). \tag{6}$$

We next restrict the analysis to $\chi_D^+$, the case of $\chi_D^-$ being symmetric. Introducing the local times $N_n=\#\{1\leq k\leq\sigma-1,\ Y_k=n\}$, $n\geq 1$, we obtain :

$$\chi_D^+(ut)=\mathbb{E}^+\left(\prod_{n\geq 1}(\varphi_n(ut))^{N_n}\right).$$

The alea now is on the $(N_n)_{n\geq 1}$. To describe these local times, one classically introduces (cf [4] for instance) the Galton-Watson tree $(Z_n^+)_{n\geq 1}$ with $Z_1^+=1$ such that, independently, the law of the number of children at level $n+1$ of an individual at level $n$ is $\mathcal{G}(p'_n)$. This tree is almost-surely finite, from the hypothesis $\lim_{n\to+\infty} v_+(n) = +\infty$.




As shown on the left-hand side of the picture, we make the contour process of the tree, starting from the root of the tree and turning clockwise. We associate to each ascending/descending movement a $+1/-1$ step. This gives the picture on the right-hand side, where we recover a positive excursion of the random walk $(Y_n)$ in the time interval $[1, \sigma-1]$.

Observe that the total number of visits of the random walk at level $n \geq 1$ is $N_n = Z_n^+ + Z_{n+1}^+$. This furnishes :

$$\prod_{n\geq 1} (\varphi_n(ut))^{N_n} = \prod_{n\geq 1} (\varphi_n(ut))^{Z_n^+ + Z_{n+1}^+} = \varphi_1(ut) \prod_{n\geq 1} [\varphi_n(ut)\varphi_{n+1}(ut)]^{Z_{n+1}^+}.$$

Finally :

$$\chi_D^+(ut) = \varphi_1(ut)\mathbb{E}^+ \left( \prod_{n\geq 1} [\varphi_n(ut)\varphi_{n+1}(ut)]^{Z_{n+1}^+} \right).$$

## 4.2 Development of $\chi_D^+$ in SP-continued fraction

We now express $\chi_D^+$ as a SP-continued fraction. For $N \geq 1$ set :

$$\chi_D^{+,N}(ut) = \varphi_1(ut)\mathbb{E}^+ \left( \prod_{n=1}^{N} [\varphi_n(ut)\varphi_{n+1}(ut)]^{Z_{n+1}^+} \right). \tag{7}$$

Let $(R_k^{(n)})_{n\geq 1, k\geq 1}$ be independent random variables such that $R_k^{(n)} \sim \mathcal{G}(p'_n)$. Then $(Z_n^+)_{n\geq 1}$ admits the following classical description :

$$Z_1^+ = 1,\ Z_{n+1}^+ = \sum_{k=1}^{Z_n^+} R_k^{(n)},\ n \geq 1.$$

Recall that the generating function of $\mathcal{G}(p'_n)$ is $s \longmapsto q'_n/(1-p'_n s) = a_n/(b_n - s)$, $0 \leq s \leq 1$. Using conditioning on the first step, this allows to write :

$$\begin{aligned}
\chi_D^{+,N}(ut) &= \varphi_1(ut)\mathbb{E}^+ \left( \prod_{n=1}^{N-1} [\varphi_n(ut)\varphi_{n+1}(ut)]^{Z_{n+1}^+} (\varphi_N(ut)\varphi_{N+1}(ut))^{Z_{N+1}^+} \right) \\
&= \varphi_1(ut)\mathbb{E}^+ \left( \prod_{n=1}^{N-1} [\varphi_n(ut)\varphi_{n+1}(ut)]^{Z_{n+1}^+} \left( \frac{a_N}{b_N - \varphi_N(ut)\varphi_{N+1}(ut)} \right)^{Z_N^+} \right) \\
&= \varphi_1(ut)\mathbb{E}^+ \left( \prod_{n=1}^{N-2} [\varphi_n(ut)\varphi_{n+1}(ut)]^{Z_{n+1}^+} \left( \frac{a_N \varphi_{N-1}(ut)\varphi_N(ut)}{b_N - \varphi_N(ut)\varphi_{N+1}(ut)} \right)^{Z_N^+} \right).
\end{aligned}$$

Replacing $\varphi_N\varphi_{N+1}$ of the first line by the quantity $\dfrac{a_N \varphi_{N-1}\varphi_N}{b_N - \varphi_N\varphi_{N+1}}$, we iterate and obtain :

$$\chi_D^{+,N} = \frac{\varphi_1 a_1}{b_1 - \frac{\varphi_1 \varphi_2 a_2}{b_2 \cdots - \frac{\varphi_{N-1}\varphi_N a_N}{b_N - \varphi_N \varphi_{N+1}}}}.$$

Dividing by $\varphi_1, \cdots, \varphi_N$ at each successive level, using that the $\varphi_n$ are close to 1, hence not 0, uniformly in $n$ and $u \in S_+^{d-1}$ for small $t$, we get :

$$\chi_D^{+,N}(ut) = [(a_1, b_1/\varphi_1(ut)); (-a_2, b_2/\varphi_2(ut)); \cdots ; (-a_N, b_N/\varphi_N(ut) - \varphi_{N+1}(ut))].$$

Now in (7), $\chi_D^{+,N}$ converges pointwise to $\chi_D$ by dominated convergence. Hence, by lemma (3.1) :

$$\chi_D^{+}(ut) = [(a_1, b_1/\varphi_1(ut)); (-a_2, b_2/\varphi_2(ut)); \cdots ; (-a_n, b_n/\varphi_n(ut)) \cdots ].$$

A similar expression is true for $\chi_D^{-}(ut)$. We have in fact shown something slightly stronger :

**Lemma 4.3**
*Let $(\gamma_n)_{n\geq 1}$ be a sequence of complex numbers with $0 < |\gamma_n| \leq 1$. Then :*

$$\mathbb{E}^+ \prod_{k=1}^{\sigma-1} \gamma_{Y_k} = \gamma_1 \mathbb{E}^+ \prod_{n\geq 1} [\gamma_n \gamma_{n+1}]^{Z_{n+1}^+} = [(a_1, b_1/\gamma_1); (-a_2, b_2/\gamma_2); \cdots ; (-a_n, b_n/\gamma_n); \cdots ].$$

## 4.3 Another reduction

Let $E_n(ut) = \sum_{k\in\mathbb{Z}^d} e^{itu.k}\mu_n(k)$, $t \in \mathbb{R}$. From (5), $\varphi_n(ut) = (1-r_n)/(1-r_n E_n(ut))$, giving :

$$\frac{1}{\varphi_n(ut)} = 1 - itu.m_n \frac{r_n}{1-r_n} + O(t^2),$$

with $O$ uniform in $n$ and $u \in S_+^{d-1}$. We shall replace below the $\varphi_n(ut)$ by the $\psi_n(ut)$ in the recursive relation (2) satisfied by the $(B_n)$, where :

**Definition 4.4**
*For $n \in \mathbb{Z}$, $u \in S_+^{d-1}$ and $t \in \mathbb{R}$, set $\eta_n = r_n m_n / p_n$ and* $\dfrac{1}{\psi_n(ut)} = 1 - itu.\eta_n \dfrac{p_n}{1-r_n} = 1 - itu.\eta_n / b_n$.

**Lemma 4.5**
*Let $c = \delta^3/4 > 0$. For small $t > 0$, uniformly in $n$ and $u \in S_+^{d-1}$ :*

$$|\varphi_n(ut)| \leq 1 - ct^2. \tag{8}$$

*Proof of the lemma :*
Let $M_{2,n}(u) = \sum_{k\in\mathbb{Z}^d} (k.u)^2 \mu_n(k)$, $m_n(u) = m_n.u$ and $Var_n(u) = M_{2,n}(u) - m_n(u)^2$. A computation gives :

$$|\varphi_n(ut)| = 1 - \frac{t^2}{2} \frac{r_n}{(1-r_n)^2} (M_{2,n}(u) - r_n Var_n(u)) + O(t^3),$$

with $O$ uniform in $n$ and $u \in S^{d-1}$, due to the uniformly bounded third moment of $\mu_n$. Using the hypotheses, we have $\delta^2 \leq \delta M_{2,n}(u) \leq M_{2,n}(u) - r_n Var_n(u)$. Hence :

$$|\varphi_n(ut)| \leq 1 - \frac{t^2\delta^3}{2} + O(t^3) \leq 1 - \frac{t^2\delta^3}{4},$$

for $t$ small enough, uniformly in $n$ and $u \in S_+^{d-1}$. $\square$

**Lemma 4.6**
*Let $R^+(t) = 1 - \mathbb{E}^+((1-t^2)^{\sigma-1})$ and $f^+(ut) = \mathbb{E}^+(\prod_{k=1}^{\sigma-1} \psi_{Y_k}(ut))$.*

1. *For all $C \geq 1$, for $x > 0$ large enough : $\psi_+^{-1}(Cx) \leq 2C^2\psi_+^{-1}(x)$.*
2. *There exists $\alpha \geq 1$ so that for small $t > 0$ :*

$$\frac{1}{\alpha} \leq R^+(t)\psi_+^{-1}(1/t) \leq \alpha.$$

3. *There exist constants $C_1 > 0, C_2 > 0$ so that for small $t > 0$, uniformly in $u \in S_+^{d-1}$ :*

$$1 - |\chi_D^+(ut)| \geq C_1 R^+(t) \text{ and } \left|\chi_D^+(ut) - f^+(ut)\right| \leq C_2 R^+(t).$$

*Proof of the lemma :*
1. Recall that $\psi_+^2(n) = nw_+ \circ v_+^{-1}(n)$, so $n \longmapsto \psi_+^2(n)/n$ is non-decreasing. Let $C \geq 1$ and $x > 0$. Set $n = \psi_+^{-1}(x)$ and suppose that $n \geq 1$. By definition, $\psi_+(n) \leq x < \psi_+(n+1)$. Similarly, let $n + p = \psi_+^{-1}(Cx)$. Then :

$$\frac{\psi_+^{-1}(Cx)}{\psi_+^{-1}(x)} = \frac{n+p}{n} \leq 2\frac{n+p}{n+1} \leq 2\frac{\psi_+^2(n+p)}{\psi_+^2(n+1)} \leq 2\frac{C^2x^2}{x^2} = 2C^2.$$

2. As a preliminary point, for $n \geq 1$, let $\Theta_+(n) > 0$ be such that $\Theta_+^2(n) = \sum_{1\leq k\leq l\leq n}(\rho_l/\rho_k)$. Fix constants $c > 0, c' > 0$ so that $cw_+(k) \leq \sum_{1\leq u\leq k}(1/\rho_u)$ and $v_+(k+1) \leq c'v_+(k)$, for all $k \geq 1$. We claim that there exists $C > 0$ so that for all $x > 0$ large enough :

$$(1/C)v_+ \circ \Theta_+^{-1}(x) \leq \psi_+^{-1}(x) \leq Cv_+ \circ \Theta_+^{-1}(x).$$

The second inequality follows from the remark that $\Theta_+ \circ v_+^{-1} \leq \psi_+$, giving $v_+^{-1} \circ \psi_+^{-1} \leq \Theta_+^{-1}$, and the fact that $v_+(v_+^{-1}(x)) \geq c''x$, for some constant $c'' > 0$. For the first one, let $x > 0$ and $n = \Theta_+^{-1}(x)$. For any $1 \leq m \leq n$ :

$$x^2 \geq (v_+(n) - v_+(m)) \sum_{1\leq k\leq m}(1/\rho_k) \geq c(v_+(n) - v_+(m))w_+(m).$$

Choose $m \leq n$ so that $v_+(m) \leq v_+(n)/2 < v_+(m+1)$. Hence, as $m = v_+^{-1}(v_+(m))$ :

$$x^2 \geq (c/2)v_+(m)w_+(m) \geq (c/2)\psi_+^2(v_+(m)).$$

We obtain, using at the end the first point of the lemma :

$$v_+(n)/(2c') \leq v_+(m) \leq \psi_+^{-1}(x/\sqrt{c/2}) \leq (4/c)\psi_+^{-1}(x).$$

This completes the proof of the claim.

Let us now turn to the evaluation of $R^+(t)$. Using lemmas 4.3 and 3.1 we have :

$$\mathbb{E}^+((1-t^2)^{\sigma-1}) = \lim_{n\to+\infty}\frac{\alpha_n(t)}{\beta_n(t)} = \sum_{n\geq 1}\frac{\rho_n}{\beta_n\beta_{n-1}},$$

where $\beta_{-1} = 0$, $\beta_0 = 1$ and $\beta_n = (b_n/(1-t^2))\beta_{n-1} - a_n\beta_{n-2}$. We omit the dependence in $t$. The $(\alpha_n)$ satisfy the same recursive relation with this time $\alpha_{-1} = 1$ and $\alpha_0 = 0$. First, as there is a constant $C > 0$ so that for all $n \geq 1$, $\Theta_+(n) \leq \Theta_+(n+1) \leq C\Theta_+(n)$, we deduce that for any constant $c > 0$ (chosen later), there exists a constant $c' > 0$ so that for small enough $t > 0$ there is an integer $N(t)$ so that :

$$\frac{c'}{t^2} \leq \Theta_+^2(N(t)) \leq \frac{c}{t^2}.$$

Next, using lemma 3.1 :

$$\left|R^+(t)-\left(\frac{\beta_{N(t)}(t)-\alpha_{N(t)}(t)}{\beta_{N(t)}(t)}\right)\right| \le \sum_{n>N(t)} \frac{\rho_n}{\beta_n(t)\beta_{n-1}(t)} \le \frac{1}{\beta_{N(t)}}. \tag{9}$$

We shall show that there exists a constant $\varepsilon > 0$ so that $1+\varepsilon \le \beta_{N(t)}(t) - \alpha_{N(t)}(t) \le 1/\varepsilon$ and next that $v_+(N(t)) \le \beta_{N(t)}(t) \le v_+(N(t))/\varepsilon$. These two properties imply that $R^+(t)$ has exact order $1/v_+(N(t))$ and so $1/\psi_+^{-1}(1/t)$, by the claim and the first point.

We have $b_n/(1-t^2) = b_n + t^2 c_n(t)$, with $(1/\alpha) \le c_n(t) \le \alpha$, for some constant $\alpha > 0$. Next :

$$\begin{pmatrix}\beta_n\\ \beta_{n-1}\end{pmatrix} = \begin{pmatrix} b_n+t^2c_n(t) & -a_n\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} b_1+t^2c_1(t) & -a_1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}.$$

Setting $C_n = \begin{pmatrix} b_n & -a_n\\ 1 & 0\end{pmatrix}$, $B = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and since $\beta_n(0) = v_+(n)$, we obtain :

$$\begin{aligned}
\beta_n &= v_+(n) + \sum_{r=1}^{n} t^{2r} \sum_{1\le k_1<\cdots<k_r\le n} c_{k_1}(t)\cdots c_{k_r}(t)\langle e_1, C_n\cdots C_{k_r+1}B\cdots BC_{k_1-1}\cdots C_1e_1\rangle\\
&= v_+(n) + \sum_{r=1}^{n} t^{2r} \sum_{1\le k_1<\cdots<k_r\le n} (c_{k_1}\cdots c_{k_r})(t)v_+(k_1-1)\theta^{k_1}v_+(k_2-k_1-1)\cdots\theta^{k_r}v_+(n-k_r).
\end{aligned}$$

Idem, since $\alpha_n = a_1\theta\beta_{n-1}$ :

$$\alpha_n = v_+(n) - 1 + \sum_{r=1}^{n} t^{2r} \sum_{2\le k_1<\cdots<k_r\le n} (c_{k_1}\cdots c_{k_r})(t)(v_+(k_1-1)-1)\theta^{k_1}v_+(k_2-k_1-1)\cdots\theta^{k_r}v_+(n-k_r).$$

This furnishes :

$$\beta_n - \alpha_n = 1 + \sum_{r=1}^{n} t^{2r} \sum_{1\le k_1<\cdots<k_r\le n} (c_{k_1}\cdots c_{k_r})(t)\theta^{k_1}v_+(k_2-k_1-1)\cdots\theta^{k_r}v_+(n-k_r).$$

As a result $\beta_n \le v_+(n)(1+\sum_{1\le r\le n}\alpha^r t^{2r}(\Theta_+^2(n))^r)$ and $\beta_n - \alpha_n \ge 1 + t^2\Theta_+^2(n)/\alpha$. We simply choose $0 < c \le \alpha/2$ to get the desired result.

3. We have $\chi_D^+(ut) = \mathbb{E}^+(\prod_{k=1}^{\sigma-1}\varphi_{Y_k}(ut))$. By (8), $|\chi_D^+(ut)| \le \mathbb{E}^+((1-ct^2)^{\sigma-1})$. This gives the first inequality, as the first point of the lemma says that $R^+(\sqrt{c}\,t) \le CR^+(t)$, for some constant $C$ depending on $c$. Concerning the second inequality :

$$\begin{aligned}
\left|\chi_D^+(ut) - f^+(ut)\right| &= \left|\mathbb{E}^+\left(\prod_{k=1}^{\sigma-1}\varphi_{Y_k}(ut)\right) - \mathbb{E}^+\left(\prod_{k=1}^{\sigma-1}\psi_{Y_k}(ut)\right)\right|\\
&= \left|\mathbb{E}^+\left(\sum_{k=1}^{\sigma-1}\left(\prod_{l=1}^{k-1}\varphi_{Y_l}(ut)(\varphi_{Y_k}(ut)-\psi_{Y_k}(ut))\prod_{l=k+1}^{\sigma-1}\psi_{Y_l}(ut)\right)\right)\right|.\\
&\le \mathbb{E}^+\left(\sum_{k=1}^{\sigma-1}\left(\prod_{l=1}^{k-1}|\varphi_{Y_l}(ut)||\varphi_{Y_k}(ut)-\psi_{Y_k}(ut)|\prod_{l=k+1}^{\sigma-1}|\psi_{Y_l}(ut)|\right)\right).
\end{aligned}$$

Using now that for some $C > 0$ and small enough $t > 0$, uniformly in $n$ and $u \in S_+^{d-1}$, $|\varphi_n(ut) - \psi_n(ut)| \le Ct^2$, as well as $|\varphi_n(ut)| \le 1 - ct^2$ and $|\psi_n(ut)| \le 1$, we get for small $t > 0$ :

$$\left|\chi_D^+(ut) - f^+(ut)\right| \le Ct^2\mathbb{E}^+\left(\sum_{k=1}^{\sigma-1}(1-ct^2)^{k-1}\right) = Ct^2\mathbb{E}^+\left(\frac{1-(1-ct^2)^{\sigma-1}}{ct^2}\right) = \frac{C}{c}R^+(\sqrt{c}\,t).$$

The conclusion now comes from the first point of the lemma.

□

## 5 Precise analysis of some convergents

As a summary, from the previous section, uniformly in $u \in S_+^{d-1}$ :

$$\chi_D^+(ut) = f^+(ut) + O(R^+(t)),$$

with $f^+(ut) = \lim_{n\to+\infty} A_n(ut)/B_n(ut)$, where now :

$$\begin{pmatrix} B_n(ut) \\ B_{n-1}(ut) \end{pmatrix} = \begin{pmatrix} b_n - itu.\eta_n & -a_n \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 - itu.\eta_1 & -a_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

together with $A_n(ut) = a_1\theta B_{n-1}(ut)$.

Recall the definitions $R_k^l(u) = \sum_{k\le r\le l} \eta_r.u(\rho_l/\rho_r)$ and $T_k^l(u) = (R_k^l(u))^2\rho_{k-1}/\rho_l,\ k \le l$. For fixed $u \in S_+^{d-1}$, notice that these quantities depend only on the data in $[k,l]$.

**Definition 5.1**
*We fix $u \in S_+^{d-1}$. Omitting the dependence with respect to $u$, set :*

$$\Delta_r^n = \sum_{1\le k_1<\cdots<k_r\le n} R_1^{k_1} R_{k_1+1}^{k_2} \cdots R_{k_{r-1}+1}^{k_r},$$

*with $\Delta_0^n = 1$ and $\Delta_r^n = 0$ if $r > n$ or $r < 0$.*

Proceeding as in the previous section, setting $\eta'_k = u.\eta_k$, we develop :

$$\begin{aligned} B_n(ut) &= v_+(n) + \sum_{r=1}^n (-it)^r \sum_{1\le k_1<\cdots<k_r\le n} \eta'_{k_1}\cdots\eta'_{k_r} v_+(k_1-1)\theta^{k_1} v_+(k_2-k_1-1)\cdots\theta^{k_r} v_+(n-k_r), \\ A_n(ut) &= v_+(n) - 1 + \sum_{r=1}^n (-it)^r \sum_{2\le k_1<\cdots<k_r\le n} \eta'_{k_1}\cdots\eta'_{k_r} (v_+(k_1-1)-1)\theta^{k_1} v_+(k_2-k_1-1)\cdots\theta^{k_r} v_+(n-k_r). \end{aligned}$$

We therefore obtain :

$$B_n(ut) - A_n(ut) = 1 + \sum_{r=1}^n (-it)^r \sum_{1\le k_1<\cdots<k_r\le n} \eta'_{k_1}\cdots\eta'_{k_r}\theta^{k_1} v_+(k_2-k_1-1)\cdots\theta^{k_r} v_+(n-k_r).$$

In the last sum, fix $k_2, \cdots, k_r$ and write :

$$\sum_{1\le k_1<k_2} \eta'_{k_1}\theta^{k_1} v_+(k_2-k_1-1) = \sum_{1\le k_1<k_2} \eta'_{k_1} \sum_{k_1\le l<k_2} \frac{\rho_l}{\rho_{k_1}} = \sum_{1\le l<k_2}\sum_{1\le k_1\le l} \eta'_{k_1}\frac{\rho_l}{\rho_{k_1}} = \sum_{1\le l<k_2} R_1^l.$$

Successively iterate this manipulation for $k_2, \cdots, k_r$ in the formula for $B_n(ut) - A_n(ut)$. Then :

$$B_n(ut) - A_n(ut) = 1 + \sum_{r=1}^n (-it)^r \sum_{1\le k_1<\cdots<k_r\le n} R_1^{k_1} R_{k_1+1}^{k_2}\cdots R_{k_{r-1}+1}^{k_r} = \sum_{r=0}^n (-it)^r \Delta_r^n.$$

Similarly, using as first step that $\sum_{1\le k_1<k_2} \eta'_{k_1} v_+(k_1-1)\theta^{k_1} v_+(k_2-k_1-1) = \sum_{0\le s<l<k_2} \rho_s R_{s+1}^l$ :

$$B_n(ut) = v_+(n) + \sum_{r=1}^n (-it)^r \sum_{0\le k_1<\cdots<k_{r+1}\le n} \rho_{k_1} R_{k_1+1}^{k_2}\cdots R_{k_r+1}^{k_{r+1}} = \sum_{r=0}^n (-it)^r \sum_{0\le k\le n} \rho_k\theta^k \Delta_r^{n-k}.$$

**Proposition 5.2**
*Set $2^{(k,l)} = 2$ if $k \ne l$ and $1$ if $k = l$. We have the following exact computations :*

*1.* $|B_n(ut) - A_n(ut)|^2 = \sum_{r=0}^n t^{2r} K_r(n)$*, with :*

$$K_r(n) = \sum_{1 \le l_1 < k_2 \le l_2 < \cdots < k_r \le l_r < k_{r+1} \le n+1} T_1^{l_1} \cdots T_{k_r}^{l_r} \rho_{k_{r+1}-1} 2^{H_r((k_i),(l_j))},$$

*where* $H_r((k_i),(l_j)) := \#\{1 \le i \le r \mid l_i + 1 < k_{i+1}\}$.

*2.* $|B_n(ut)|^2 = \sum_{r=0}^n t^{2r} L_r(n)$*, with* $L_r(n) = \sum_{0 \le k \le l \le n} \rho_k \rho_l 2^{(k,l)} \theta^l K_r(n-l)$.

*3.* $Re((B_n - A_n)\bar{B}_n)(ut) = \sum_{r=0}^n t^{2r} M_r(n)$*, with* $M_r(n) = \sum_{0 \le k \le n} \rho_k \theta^k K_r(n-k)$.

*4.* $Im(A_n(ut)\bar{B}_n(ut)) = \sum_{r=0}^{n-1} t^{2r+1} N_r(n)$*, with* $N_r(n) = \sum_{1 \le k \le l \le n} R_1^k 2^{(k,l)} \rho_l \theta^l K_r(n-l)$.

*When* $r > n$ *or* $r < 0$*, set* $K_r(n) = L_r(n) = M_r(n) = 0$*. Idem* $N_r(n) = 0$*,* $r \ge n$ *or* $r < 0$*.*

*Remark.* — Recall that $R_k^l$ and $T_k^l$ and therefore $K_r(n)$, $L_r(n)$, $M_r(n)$, $N_r(n)$ depend on $u \in S_+^{d-1}$ but that the dependence is omitted in the notations.

*Proof of the proposition :*
1. Since $B_n(ut) - A_n(ut) = \sum_{0 \le r \le n} (-it)^r \Delta_r^n$, this gives :

$$|B_n(ut) - A_n(ut)|^2 = (B_n(ut) - A_n(ut))\overline{(B_n(ut) - A_n(ut))} = \sum_{r=0}^n t^{2r} \sum_{p=-r}^r \Delta_{r+p}^n \Delta_{r-p}^n (-i)^{r+p} i^{r-p},$$

using the conventions for $\Delta_r^n$ concerning the value of $r$ with respect to $n$. Hence $|B_n(ut) - A_n(ut)|^2 = \sum_{r=0}^n t^{2r} K_r(n)$, with $K_0(n) = 1$ and :

$$K_r(n) = \sum_{p=-r}^r (-1)^p \Delta_{r+p}^n \Delta_{r-p}^n, \ r \ge 1.$$

We will show that :

$$K_1(n) = \sum_{1 \le k \le l \le n} T_1^k \rho_l 2^{(k,l)}, \tag{10}$$

together with the following recursive relation, for $r \ge 2$ :

$$K_r(n) = \sum_{1 \le k \le l \le n} T_1^k \rho_l \theta^l K_{r-1}(n-l) 2^{(k,l)}. \tag{11}$$

This then gives the announced formula. For the initial relation :

$$\begin{aligned} K_1(n) &= (\Delta_1^n)^2 - 2\Delta_2^n = \left( \sum_{1 \le k \le n} R_1^k \right)^2 - 2 \sum_{1 \le k < l \le n} R_1^k R_{k+1}^l \\ &= \sum_{1 \le k \le n} (R_1^k)^2 + 2 \sum_{1 \le k < l \le n} R_1^k (R_1^l - R_{k+1}^l). \end{aligned}$$

Observing that $R_1^k(R_1^l - R_{k+1}^l) = (R_1^k)^2(\rho_l/\rho_k) = T_1^k \rho_l$, this proves (10). Let us now turn to the proof of (11). Taking first general $p \ge 1$ and $q \ge 1$, we write :

$$\Delta_p^n \Delta_q^n = \sum_{\substack{1 \le k_1 < \cdots < k_p \le n \\ 1 \le k_1' < \cdots < k_q' \le n}} (R_1^{k_1} \cdots R_{k_{p-1}+1}^{k_p})(R_1^{k_1'} \cdots R_{k_{q-1}'+1}^{k_q'}).$$

Distinguishing the cases $k_1 = k_1'$, $k_1 < k_1'$ and $k_1' < k_1$, we decompose :

$$\Delta_p^n\Delta_q^n = \sum_{1\le k\le n}(R_1^k)^2\theta^k\Delta_{p-1}^{n-k}\theta^k\Delta_{q-1}^{n-k}$$
$$+ \sum_{\substack{1\le k_1<\cdots<k_p\le n\\ k_1<k_1'<\cdots<k_q'\le n}} R_1^{k_1}(R_{k_1+1}^{k_2}\cdots R_{k_{p-1}+1}^{k_p})(R_1^{k_1}\frac{\rho_{k_1'}}{\rho_{k_1}}+R_{k_1+1}^{k_1'})(R_{k_1'+1}^{k_2'}\cdots R_{k_{q-1}'+1}^{k_q'})$$
$$+ \sum_{\substack{1\le k_1'<\cdots<k_q'\le n\\ k_1'<k_1<\cdots<k_p\le n}} (R_1^{k_1'}\frac{\rho_{k_1}}{\rho_{k_1'}}+R_{k_1'+1}^{k_1})(R_{k_1+1}^{k_2}\cdots R_{k_{p-1}+1}^{k_p})(R_1^{k_1'}\cdots R_{k_{q-1}'+1}^{k_q'}).$$

Regrouping terms, this is rewritten as :

$$\begin{aligned}\Delta_p^n\Delta_q^n &= \sum_{1\le k\le n}(R_1^k)^2\left[\theta^k\Delta_{p-1}^{n-k}\sum_{k\le l\le n}\theta^l\Delta_{q-1}^{n-l}\frac{\rho_l}{\rho_k}+\theta^k\Delta_{q-1}^{n-k}\sum_{k<l\le n}\theta^l\Delta_{p-1}^{n-l}\frac{\rho_l}{\rho_k}\right]\\ &+ \sum_{1\le k\le n}R_1^k\left[\theta^k\Delta_{p-1}^{n-k}\theta^k\Delta_q^{n-k}+\theta^k\Delta_p^{n-k}\theta^k\Delta_{q-1}^{n-k}\right].\end{aligned}$$

Taking $r\ge 2$, insert the latter in $K_r(n)=\sum_{-r+1\le p\le r-1}(-1)^p\Delta_{r+p}^n\Delta_{r-p}^n+2(-1)^r\Delta_{2r}^n$ and get :

$$\begin{aligned}K_r(n) &= \sum_{1\le k\le n}(R_1^k)^2\sum_{-r+1\le p\le r-1}(-1)^p\left[\theta^k\Delta_{r+p-1}^{n-k}\sum_{k\le l\le n}\theta^l\Delta_{r-p-1}^{n-l}\frac{\rho_l}{\rho_k}+\theta^k\Delta_{r-p-1}^{n-k}\sum_{k<l\le n}\theta^l\Delta_{r+p-1}^{n-l}\frac{\rho_l}{\rho_k}\right]\\ &+ 2(-1)^r\Delta_{2r}^n+2\sum_{1\le k\le n}R_1^k\left[\sum_{-r+1\le p\le r-1}(-1)^p\theta^k\Delta_{r+p-1}^{n-k}\theta^k\Delta_{r-p}^{n-k}\right].\end{aligned}$$

The last line is $2\sum_{1\le k\le n}R_1^k[\sum_{-r+1\le p\le r}(-1)^p\theta^k\Delta_{r+p-1}^{n-k}\theta^k\Delta_{r-p}^{n-k}]$. The bracketed sum is 0, for instance when doing the change of variable $p\longmapsto -p+1$. Separating now the term with $k=l$ in the first sum above and recognizing $\theta^kK_{r-1}(n-k)$, we obtain :

$$K_r(n) = \sum_{1\le k\le n}(R_1^k)^2\left[\theta^kK_{r-1}(n-k)+2\sum_{-r+1\le p\le r-1}(-1)^p\theta^k\Delta_{r+p-1}^{n-k}\sum_{k<l\le n}\theta^l\Delta_{r-p-1}^{n-l}\frac{\rho_l}{\rho_k}\right].$$

Setting $m=n-k$ and $Z_r(m)=\sum_{-r\le p\le r}(-1)^p\Delta_{r+p}^m\sum_{1\le l\le m}\theta^l\Delta_{r-p}^{m-l}\rho_l$, we therefore have :

$$K_r(n)=\sum_{1\le k\le n}(R_1^k)^2\left[\theta^kK_{r-1}(n-k)+2\theta^kZ_{r-1}(n-k)\right].$$

We shall show that :

$$Z_r(m)=\sum_{1\le k\le m}\theta^kK_r(m-k)\rho_k,\ r\ge 1. \tag{12}$$

To complete the proof of (11), we simply apply this to $Z_{r-1}(n-k)$ in the previous equality. First of all, with $0\le p\le r-1$ :

$$\begin{aligned}\Delta_{r+p}^m\sum_{1\le l\le m}\theta^l\Delta_{r-p}^{m-l}\rho_l &= \sum_{\substack{1\le k_1<\cdots<k_{r+p}\le m\\ 1\le l_1<l_2<\cdots<l_{r-p+1}\le m}} R_1^{k_1}\cdots R_{k_{r+p-1}+1}^{k_{r+p}}R_{l_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}}\rho_{l_1}\\ &= \sum_{\substack{1\le k_1<\cdots<k_{r+p}\le m\\ k_1\le l_1<\cdots<l_{r-p+1}\le m}} R_1^{k_1}R_{k_1+1}^{k_2}\cdots R_{k_{r+p-1}+1}^{k_{r+p}}R_{l_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}}\rho_{l_1}\\ &+ \sum_{\substack{1\le l_1<\cdots<l_{r-p+1}\le m\\ l_1<k_1<\cdots<k_{r+p}\le m}} (R_1^{l_1}\frac{\rho_{k_1}}{\rho_{l_1}}+R_{l_1+1}^{k_1})R_{k_1+1}^{k_2}\cdots R_{k_{r+p-1}+1}^{k_{r+p}}R_{l_1+1}^{l_2}\cdots R_{l_{r-p}+1}^{l_{r-p+1}}\rho_{l_1}.\end{aligned}$$

Written in a more concise way :

$$\begin{aligned}\Delta^m_{r+p}\sum_{1\le l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l &= \sum_{1\le k\le m}R^k_1\left[\theta^k\Delta^{m-k}_{r+p-1}\sum_{k\le l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l+\theta^k\Delta^{m-k}_{r-p}\sum_{k<l\le m}\theta^l\Delta^{m-l}_{r+p-1}\rho_l\right]\\ &+ \sum_{1\le k\le m}\theta^k\Delta^{m-k}_{r+p}\theta^k\Delta^{m-k}_{r-p}\rho_k.\end{aligned}$$

This allows to write :

$$\begin{aligned}Z_r(m) &= (-1)^r\left[\Delta^m_{2r}\sum_{1\le l\le m}\rho_l+\sum_{1\le l\le m}\theta^l\Delta^{m-l}_{2r}\rho_l\right]+\sum_{1\le k\le m}\sum_{-r+1\le p\le r-1}(-1)^p\theta^k\Delta^{m-k}_{r+p}\theta^k\Delta^{m-k}_{r-p}\rho_k\\ &+ \sum_{1\le k\le m}R^k_1\sum_{-r+1\le p\le r-1}(-1)^p\left[\theta^k\Delta^{m-k}_{r+p-1}\sum_{k\le l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l+\theta^k\Delta^{m-k}_{r-p}\sum_{k<l\le m}\theta^l\Delta^{m-l}_{r+p-1}\rho_l\right].\end{aligned}$$

Recognizing some $\theta^kK_r(m-k)$, we get :

$$\begin{aligned}Z_r(m) &= (-1)^r\left[\Delta^m_{2r}\sum_{1\le l\le m}\rho_l-\sum_{1\le l\le m}\theta^l\Delta^{m-l}_{2r}\rho_l\right]+\sum_{1\le k\le m}\theta^kK_r(m-k)\rho_k\\ &+ \sum_{1\le k\le m}R^k_1\sum_{-r+1\le p\le r-1}(-1)^p\left[\theta^k\Delta^{m-k}_{r+p-1}\sum_{k\le l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l\right]\\ &+ \sum_{1\le k\le m}R^k_1\sum_{-r+2\le p\le r}(-1)^{p+1}\left[\theta^k\Delta^{m-k}_{r+p-1}\sum_{k<l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l\right].\end{aligned}$$

Consequently :

$$\begin{aligned}Z_r(m) &= \sum_{1\le k\le m}\theta^kK_r(m-k)\rho_k+(-1)^r\left[\Delta^m_{2r}\sum_{1\le l\le m}\rho_l-\sum_{1\le l\le m}\theta^l\Delta^{m-l}_{2r}\rho_l\right]\\ &+ (-1)^{r+1}\sum_{1\le k\le m}R^k_1\left(\theta^k\Delta^{m-k}_{2r-1}\sum_{k\le l\le m}\rho_l+\sum_{k<l\le m}\theta^l\Delta^{m-l}_{2r-1}\rho_l\right)\\ &+ \sum_{1\le k\le m}R^k_1\sum_{-r+1\le p\le r}(-1)^p\left[\theta^k\Delta^{m-k}_{r+p-1}\sum_{k\le l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l-\theta^k\Delta^{m-k}_{r+p-1}\sum_{k<l\le m}\theta^l\Delta^{m-l}_{r-p}\rho_l\right].\end{aligned}$$

The last line is $\sum_{1\le k\le m}R^k_1[\sum_{-r+1\le p\le r}(-1)^p\theta^k\Delta^{m-k}_{r+p-1}\theta^k\Delta^{m-k}_{r-p}\rho_k]$. For the same reason as before, the inside brackets are 0. Therefore it finally remains to show that the sum of the second and third terms is also 0, in other words that :

$$\Delta^m_{2r}\sum_{1\le l\le m}\rho_l-\sum_{1\le l\le m}\theta^l\Delta^{m-l}_{2r}\rho_l-\sum_{1\le k\le m}R^k_1\left(\theta^k\Delta^{m-k}_{2r-1}\sum_{k\le l\le m}\rho_l+\sum_{k<l\le m}\theta^l\Delta^{m-l}_{2r-1}\rho_l\right)=0.$$

Equivalently :

$$\sum_{1\le k\le m}R^k_1\theta^k\Delta^{m-k}_{2r-1}\sum_{1\le k<l}\rho_l-\sum_{1\le l\le m}\theta^l\Delta^{m-l}_{2r}\rho_l-\sum_{1\le k\le m}R^k_1\sum_{k<l\le m}\theta^l\Delta^{m-l}_{2r-1}\rho_l=0.$$

In the last term, replace $R_1^k$ by $(R_1^l - R_{k+1}^l)\rho_k/\rho_l$. It remains to show that :

$$-\sum_{1\le l\le m}\theta^l\Delta_{2r}^{m-l}\rho_l+\sum_{1\le k\le m}R_1^k\theta^k\Delta_{2r-1}^{m-k}\sum_{1\le l<k}\rho_l-\sum_{1\le k<l\le m}R_1^l\theta^l\Delta_{2r-1}^{m-l}\rho_k+\sum_{1\le k<l\le m}R_{k+1}^l\theta^l\Delta_{2r-1}^{m-l}\rho_k=0.$$

As this is true, this completes the proof of this first point.

2. Let us define $\tilde\Delta_r^n = \sum_{0\le k\le n}\rho_k\theta^k\Delta_r^{n-k}$, so that $B_n(ut) = \sum_{0\le r\le n}(-it)^r\tilde\Delta_r^n$. As for $|B_n(ut) - A_n(ut)|^2$ in the first point, we have :

$$|B_n(ut)|^2 = \sum_{0\le r\le n} t^{2r}L_r(n), \text{ where } L_r(n) = \sum_{-r\le p\le r}(-1)^p\tilde\Delta_{r+p}^n\tilde\Delta_{r-p}^n.$$

In order to compute $L_r(n)$, notice first that :

$$\tilde\Delta_{r+p}^n\tilde\Delta_{r-p}^n = \sum_{0\le k\le n}\rho_k\left[\theta^k\Delta_{r+p}^{n-k}\sum_{k\le l\le n}\theta^l\Delta_{r-p}^{n-l}\rho_l+\theta^k\Delta_{r-p}^{n-k}\sum_{k<l\le n}\theta^l\Delta_{r+p}^{n-l}\rho_l\right].$$

Replacing in $L_r(n)$, this allows to write, using the expressions of $K_r(n)$ and $Z_r(n)$ given in (12) :

$$\begin{aligned}
L_r(n) &= \sum_{0\le k\le n}\rho_k\sum_{-r\le p\le r}(-1)^p\left[\theta^k\Delta_{r+p}^{n-k}\sum_{k\le l\le n}\rho_l\theta^l\Delta_{r-p}^{n-l}+\theta^k\Delta_{r-p}^{n-k}\sum_{k<l\le n}\rho_l\theta^l\Delta_{r+p}^{n-l}\right]\\
&= \sum_{0\le k\le n}(\rho_k)^2\theta^kK_r(n-k)+2\sum_{0\le k\le n}(\rho_k)^2\theta^kZ_r(n-k) \qquad (13)\\
&= \sum_{0\le k\le n}\rho_k\left[\rho_k\theta^kK_r(n-k)+2\sum_{k<l\le n}\rho_l\theta^lK_r(n-l)\right] = \sum_{0\le k\le n}\rho_k\theta^kK_r(n-k)\sum_{0\le l\le k}2^{(l,k)}\rho_l.
\end{aligned}$$

This completes the proof of this point.

3. Directly, we obtain :

$$(B_n - A_n)(ut)\overline{B}_n(ut) = \sum_{0\le r\le n}(-it)^r\Delta_r^n\sum_{0\le r'\le n}(it)^{r'}\tilde\Delta_{r'}^n. \qquad (14)$$

When developing and taking the real part, only terms with $r + r'$ even intervene. This gives :

$$\mathrm{Re}((B_n - A_n)\overline{B}_n)(ut) = \sum_{0\le r\le n}t^{2r}\left[\sum_{-r\le p\le r}(-i)^{r+p}i^{r-p}\Delta_{r+p}^n\tilde\Delta_{r-p}^n\right] = \sum_{0\le r\le n}t^{2r}M_r(n),$$

with this time :

$$M_r(n) = \sum_{-r\le p\le r}(-1)^p\Delta_{r+p}^n\tilde\Delta_{r-p}^n.$$

Since $\tilde\Delta_r^n = \Delta_r^n + \sum_{1\le k\le n}\rho_k\theta^k\Delta_r^{n-k}$, using $K_r(n)$ and the value of $Z_r(n)$ in (12), we have :

$$M_r(n) = K_r(n) + Z_r(n) = \sum_{0\le k\le n}\rho_k\theta^kK_r(n-k).$$

This ends the proof of this point.

4. In the same way as for 3., when taking the imaginary part in (14), only terms with $r + r'$ odd come into play. Consequently :

$$\mathrm{Im}(A_n\overline{B}_n)(ut) = -\frac{1}{i}\sum_{0\le r\le n-1} t^{2r+1}\left[\sum_{-r-1\le p\le r}(-i)^{r+p+1}i^{r-p}\Delta^n_{r+p+1}\tilde{\Delta}^n_{r-p}\right] = \sum_{0\le r\le n-1} t^{2r+1}N_r(n),$$

with this time :

$$N_r(n) = \sum_{-r-1\le p\le r}(-1)^p\Delta^n_{r+p+1}\tilde{\Delta}^n_{r-p}.$$

Using again that $\tilde{\Delta}^n_r = \sum_{0\le k\le n}\rho_k\theta^k\Delta^{n-k}_r$, we get :

$$N_r(n) = \sum_{0\le k\le n}\sum_{-r-1\le p\le r}(-1)^p\Delta^n_{r+p+1}\theta^k\Delta^{n-k}_{r-p}\rho_k.$$

Notice that the term corresponding to $k=0$ equals 0, for symmetry reasons as before. It remains :

$$\begin{aligned}
N_r(n) &= (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta^{n-k}_{2r+1}\rho_k + \sum_{1\le l\le n}R^l_1\sum_{-r\le p\le r}(-1)^p\theta^l\Delta^{n-l}_{r+p}\sum_{l<k\le n}\theta^k\Delta^{n-k}_{r-p}\rho_k \\
&+ \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta^{n-k}_{r-p}\rho_k\sum_{k\le l\le n}R^l_1\theta^l\Delta^{n-l}_{r+p} \\
&= \sum_{1\le k\le n}R^k_1\rho_k\theta^kK_r(n-k) + \sum_{1\le l\le n}R^l_1\rho_l\theta^lZ_r(n-l) + O_r(n), \qquad (15)
\end{aligned}$$

where we introduce :

$$O_r(n) = (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta^{n-k}_{2r+1}\rho_k + \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta^{n-k}_{r-p}\rho_k\sum_{k<l\le n}R^l_1\theta^l\Delta^{n-l}_{r+p}.$$

To compute $O_r(n)$, in the last sum decompose $R^l_1 = R^k_1(\rho_l/\rho_k) + R^l_{k+1}$. As a result :

$$\begin{aligned}
O_r(n) &= \sum_{1\le k\le n}R^k_1\sum_{-r\le p\le r}(-1)^p\theta^k\Delta^{n-k}_{r-p}\sum_{k<l\le n}\theta^l\Delta^{n-l}_{r+p}\rho_l \\
&+ (-1)^{r+1}\sum_{1\le k\le n}\theta^k\Delta^{n-k}_{2r+1}\rho_k + \sum_{1\le k\le n}\sum_{-r\le p\le r}(-1)^p\theta^k\Delta^{n-k}_{r-p}\theta^k\Delta^{n-k}_{r+p+1}\rho_k \\
&= \sum_{1\le k\le n}R^k_1\theta^kZ_r(n-k)\rho_k + \sum_{1\le k\le n}\sum_{-r\le p\le r-1}(-1)^p\theta^k\Delta^{n-k}_{r-p}\theta^k\Delta^{n-k}_{r+p+1}\rho_k.
\end{aligned}$$

One more time, the last term is 0. Together with (15) and (12) we obtain :

$$N_r(n) = \sum_{1\le k\le n}\rho_kR^k_1(\theta^kK_r(n-k) + 2\theta^kZ_r(n-k)) = \sum_{1\le k\le l\le n}R^k_1\theta^lK_r(n-l)\rho_l2^{(k,l)}.$$

This gives the announced formula and concludes the proof of the proposition.

□

# 6 Proof of the theorem

## 6.1 Dominated variation

For $u\in S^{d-1}_+$, the inverse functions of $n\to\varphi_{u,+}(n)$ and $n\to\varphi_u(n)$ check a dominated variation property at infinity (Feller, 1969). Notice that the latter property holds for $\psi^{-1}_+$ and $\psi^{-1}_-$, as a consequence of the first point of lemma 4.6.

**Lemma 6.1**
*1. For any $x \geq 1$ and $K \geq 1$ :*

$$\psi^{-1}(Kx) \leq 2K^2\psi^{-1}(x).$$

*2. There exists a constant $C(\delta) > 0$, so that for any $u \in S_+^{d-1}$, any $x \geq 1$ and $K \geq 1$ :*

$$\varphi_{u,+}^{-1}(Kx) \leq \frac{2K^2}{\delta}\varphi_{u,+}^{-1}(x) \text{ and } \varphi_u^{-1}(Kx) \leq \frac{K^2}{C(\delta)}\varphi_u^{-1}(x).$$

*Proof of the lemma :*
1. Recall that $\psi^2(n) = n\left(w_+(n) \circ v_+^{-1}(n) + w_-(n) \circ v_-^{-1}(n)\right)$. For $x \geq 1$, let $n = \psi^{-1}(x)$, ie $\psi(n) \leq x < \psi(n+1)$. This implies that :

$$\psi(K^2(n+1)) \geq K\psi(n+1) > Kx.$$

Hence $\psi^{-1}(Kx) \leq K^2(n+1) \leq 2K^2 n = 2K^2\psi^{-1}(x)$.
2. Let $\kappa_{u,+}(n) = \sum_{1\leq k\leq l\leq n} T_k^l(u) = \sum_{0\leq k<l\leq n} \rho_k\rho_l(\zeta_{k+1}^l(u))^2$, setting $\zeta_k^l(u) = \sum_{s=k}^l \eta_s.u/\rho_s$, with $\zeta_k^l(u) = 0$ if $k > l$. We first claim that :

$$\frac{\kappa_{u,+}(n)}{v_+(n)} = \frac{\kappa_{u,+}(n-1)}{v_+(n-1)} + \frac{\rho_n}{v_+(n)v_+(n-1)}\left(\sum_{0\leq k<n}\rho_k\zeta_{k+1}^n(u)\right)^2.$$

In particular, $n \longmapsto \kappa_{u,+}(n)/v_+(n)$ is non-decreasing. Indeed :

$$\kappa_{u,+}(n) = \sum_{0\leq k<l\leq n}\rho_k\rho_l(\zeta_{k+1}^n(u))^2 + \sum_{0\leq k<l\leq n}\rho_k\rho_l(\zeta_{l+1}^n(u))^2 - 2\sum_{0\leq k<l\leq n}\rho_k\rho_l\zeta_{k+1}^n(u)\zeta_{l+1}^n(u).$$

This is rewritten as :

$$\begin{aligned}\kappa_{u,+}(n) &= \sum_{0\leq k<n}\rho_k(\zeta_{k+1}^n(u))^2\sum_{k<l\leq n}\rho_l + \sum_{1\leq l\leq n}\rho_l(\zeta_{l+1}^n(u))^2\sum_{0\leq k<l}\rho_k - \left(\sum_{0\leq k\leq n}\rho_k\zeta_{k+1}^n(u)\right)^2 \\ &+ \sum_{0\leq k\leq n}(\rho_k)^2(\zeta_{k+1}^n(u))^2.\end{aligned}$$

In other words :

$$\kappa_{u,+}(n) = v_+(n)\sum_{0\leq k<n}\rho_k(\zeta_{k+1}^n(u))^2 - \left(\sum_{0\leq k\leq n}\rho_k\zeta_{k+1}^n(u)\right)^2.$$

Next, directly from the definition of $\kappa_{u,+}(n)$, and then using the previous equality :

$$\kappa_{u,+}(n) - \kappa_{u,+}(n-1) = \rho_n\sum_{0\leq k<n}\rho_k(\zeta_{k+1}^n(u))^2 = \rho_n\frac{\kappa_{u,+}(n) + \left(\sum_{0\leq k<n}\rho_k\zeta_{k+1}^n(u)\right)^2}{v_+(n)}.$$

Observe that this is equivalent to the desired claim.

We next use that for all $n \geq 0$, $\delta \leq \rho_{n+1}/\rho_n \leq 1/\delta$, hence $v_+(n+1) \leq (2/\delta)v_+(n)$. As a result $v_+ \circ v_+^{-1}(n) \leq n \leq (2/\delta)v_+ \circ v_+^{-1}(n)$. Hence for $x \geq 1$ and $K \geq 1$ :

$$\kappa_{u,+} \circ v_+^{-1}(Kx) \geq \kappa_{u,+} \circ v_+^{-1}(x)\frac{v_+ \circ v_+^{-1}(Kx)}{v_+ \circ v_+^{-1}(x)} \geq \frac{\delta K}{2}\kappa_{u,+} \circ v_+^{-1}(x).$$

A similar property is verified for some symmetrically defined function $\kappa_{u,-} \circ v_-^{-1}$. Notice that :

$$\varphi_{u,+}^2(n) = \psi^2(n) + \kappa_{u,+} \circ v_+^{-1}(n) + \kappa_{u,-} \circ v_-^{-1}(n).$$

Notice that $\varphi_{u,+}(n) \to +\infty$, as $n \to +\infty$. As we showed in point one that $\psi^2(Kx) \geq K\psi^2(x)$, we obtain that for $x \geq 1$ and $K \geq 1$ :

$$\varphi_{u,+}(Kx) \geq \sqrt{(\delta K/2)}\,\varphi_{u,+}(x).$$

We conclude as in point one. Let $x \geq 1$ and $n = \varphi_{u,+}^{-1}(x)$ and $K \geq 1$. Then $\varphi_{u,+}(n) \leq x < \varphi_{u,+}(n+1)$, so :

$$\varphi_{u,+}((2K^2/\delta)(n+1)) \geq K\varphi_{u,+}(n+1) > Kx.$$

Consequently $\varphi_{u,+}^{-1}(Kx) \leq ((2K^2)/\delta)\varphi_{u,+}^{-1}(x)$.

It remains to show the same result for $\varphi_u$. This way, let $\kappa_u(-m,n) = \sum_{-m \leq k \leq l \leq n} T_k^l(u)$, for $m \geq 1$, $n \geq 1$. Then, the computation on $\kappa_{u,+}$ shows that :

$$n \longmapsto \frac{\kappa_u(-m,n)}{(v_-(m)/a_0) + v_+(n)} \text{ and } m \longmapsto \frac{\kappa_u(-m,n)}{(v_-(m)/a_0) + v_+(n)}$$

are non-decreasing. This furnishes that for some constant $C(\delta) > 0$ :

$$\kappa_u(-v_-^{-1}(Kx), v_+^{-1}(Kx)) \geq \frac{C(\delta)K}{2}\kappa_u(-v_-^{-1}(x), v_+^{-1}(x)).$$

As $\varphi_u^2(n) = \psi^2(n) + \kappa_u(-v_-^{-1}(n), v_+^{-1}(n))$, we conclude as before. This ends the proof of the lemma. □

### 6.2 Order of the real part of $1 - \chi_D(ut)$

With $u \in S_+^{d-1}$ and small $t > 0$, recall the decomposition $\chi_D(ut) = \varphi_0(ut)(p_0'\chi_D^+(ut) + q_0'\chi_D^-(ut))$ and also that :

$$\chi_D^+(ut) = f^+(ut) + O(R^+(t)) \text{ and } \chi_D^-(ut) = f^-(ut) + O(R^-(t)),$$

where the $O(\ )$ are uniform in $u \in S_+^{d-1}$ and where $R^+(t)$ and $R^-(t)$ have respective orders $1/\psi_+^{-1}(1/t)$ and $1/\psi_-^{-1}(1/t)$, by lemma 4.6.

**Lemma 6.2**
*Let $R(t) = R^+(t) + R^-(t)$.*

1. *We have $\chi_D(ut) = \varphi_0(ut)(p_0'f^+(ut) + q_0'f^-(ut)) + O(R(t))$.*
2. *We have $t^2 = O(R^+(t))$ and $t^2 = O(R^-(t))$.*
3. *We have $tIm(1 - f^+(ut)) = O(R^+(t))$ and $tIm(1 - f^-(ut)) = O(R^-(t))$.*
4. *We have $\chi_D(ut) = (1 + itm_0.ur_0/(1 - r_0))(p_0'f^+(ut) + q_0'f^-(ut)) + O(R(t))$ and*

$$Re(1 - \chi_D)(ut) = p_0'Re(1 - f^+(ut)) + q_0'Re(1 - f^-(ut)) + O(R(t)). \quad (16)$$

5. *There is a constant $c > 0$ so that for small $t > 0$, uniformly in $u \in S_+^{d-1}$ :*

$$Re(1 - \chi_D(ut)) \geq cR(t).$$

*Proof of the lemma :*
1. This follows from $\chi_D(ut) = \varphi_0(ut)(p'_0\chi^+_D(ut) + q'_0\chi^-_D(ut))$ and $\chi^\pm_D(ut) = f^\pm(ut) + O(R^\pm(t))$.

2. As $\psi^2_+(n) = nw_+ \circ v_+^{-1}(n)$, for some constant $c > 0$, $\psi_+(n^2) \geq cn$, so $\psi_+^{-1}(1/t) \leq c'/t^2$, $t > 0$, for some constant $c' > 0$. By lemma 4.6, $t^2 = O(\psi_+^{-1}(1/t)) = O(R_+(t))$, which gives the first property. The other one is proved in the same way.

3. We make use of proposition 5.2 and lemma 3.1. Taking any integer $n \geq 1$ and since $f^+(ut) = A_n(ut)/B_n(ut) + O(1/v_+(n))$ (where $O(\ )$ is independent on $u$ and $t$), we have :

$$\mathrm{Im}(f^+(ut)) \quad = \quad \frac{\mathrm{Im}(A_n(ut))\bar{B}_n(ut))}{|B_n(ut)|^2} + O(1/v_+(n)) = \frac{\sum_{0\leq r\leq n-1} t^{2r+1}N_r(n)}{\sum_{0\leq r\leq n} t^{2r}L_r(n)} + O(1/v_+(n)).$$

Now, see (13), $L_r(n) = \sum_{0\leq l\leq k\leq n} \rho_k\theta^k K_r(n-k)2^{(l,k)}\rho_l \geq \sum_{1\leq k\leq n} \rho_k v_+(k)\theta^k K_r(n-k)$, where the dependence in $u \in S^{d-1}_+$ is implicit, and :

$$N_r(n) = \sum_{1\leq k\leq l\leq n} R_1^k\theta^l K_r(n-l)\rho_l 2^{(k,l)} = \sum_{1\leq k\leq n}\left[\sum_{1\leq s\leq l\leq k} \frac{\eta_s.u}{\rho_s}\rho_l 2^{(l,k)}\right]\rho_k\theta^k K_r(n-k).$$

As the $\eta_n.u$ are uniformly bounded by some $C/2$ (as $n$ and $u \in S^{d-1}_+$ vary), we get :

$$|N_r(n)| \leq Cw_+(n)\sum_{1\leq k\leq n} v_+(k)\rho_k\theta^k K_r(n-k) \leq Cw_+(n)L_r(n).$$

We finally obtain $|\mathrm{Im}(f^+(ut))| \leq tw_+(n) + O(1/v_+(n))$. Let $n' = \psi_+^{-1}(1/t)$ and $n = v_+^{-1}(n')$. By definition of $\psi_+$, we have $n'w_+(n) \leq 1/t^2$. We obtain $|\mathrm{Im}(f^+(ut))| \leq 1/(tn') + O(1/n')$, which is the desired result. The situation for $t|\mathrm{Im}(f^-(ut))|$ is similar.

4. Write $\varphi_0(ut) = 1 + itm_0.ur_0/(1-r_0) + O(t^2)$, with $O(\ )$ uniform in $u \in S^{d-1}_+$. Using the first point of the lemma, we get :

$$\chi_D(ut) = \left(1 + \frac{itm_0.ur_0}{1-r_0}\right)(p'_0 f^+(ut) + q'_0 f^-(ut)) + O(R(t)),$$

with again an error term uniform in $u \in S^{d-1}_+$. Therefore :

$$1 - \chi_D(ut) = p'_0(1 - f^+(ut)) + q'_0(1 - f^-(ut)) - \frac{itm_0.ur_0}{1-r_0}(p'_0 f^+(ut) + q'_0 f^-(ut)) + O(R(t)).$$

Taking the real part :

$$\mathrm{Re}(1-\chi_D(ut)) = p'_0\mathrm{Re}(1-f^+(ut)) + q'_0\mathrm{Re}(1-f^-(ut)) + \frac{tm_0.ur_0}{1-r_0}\left(p'_0\mathrm{Im}(f^+(ut)) + q'_0\mathrm{Im}(f^-(ut))\right) + O(R(t)).$$

The third point of the lemma then gives (16).

5. By lemma 4.6, for a constant $c_1 > 0$ independent on $u \in S^{d-1}_+$, we have for small $t > 0$, $1 - |\chi^+_D(ut)| \geq c_1 R^+(ut)$. Idem, for some $c_2 > 0$, we get $1 - |\chi^-_D(ut)| \geq c_2 R^-(ut)$. As $\chi_D(ut) = \varphi_0(ut)(p'_0\chi^+_D(ut) + q'_0\chi^-_D(ut))$ and $|\varphi_0(ut)| \leq 1$ :

$$\begin{aligned}\mathrm{Re}(1-\chi_D(ut)) \geq 1 - |\chi_D(ut)| &\geq 1 - |p'_0\chi^+_D(ut) + q'_0\chi^-_D(ut)|\\ &\geq p'_0(1 - |\chi^+_D(ut)|) + q'_0(1 - |\chi^-_D(ut)|)\\ &\geq c_1p'_0R^+(t) + c_2q'_0R^-(t) \geq cR(t),\end{aligned}$$

for some constant $c > 0$. This completes the proof of the lemma.

□

*Remark.* — Notice that in [2] one always had $t = O(R^+(t))$. This is not true anymore here. For example if $\sum_{k\geq 1}(1/\rho_k) < \infty$, one may check that $R^+(t)$ can have order $t^2$, as $t \to 0$.

**Proposition 6.3**
*There is a constant $C \geq 1$ so that for $t > 0$ small enough, uniformly in $u \in S_+^{d-1}$ :*

$$\frac{1}{C} \leq \varphi_{u,+}^{-1}(1/t) Re(1 - \chi_D(ut)) \leq C.$$

*Proof of the proposition :*
We still fix $u \in S_+^{d-1}$ and $t > 0$. Recall that $f^+(ut) = \lim_{n\to+\infty} A_n(ut)/B_n(ut)$, where $(A_n(ut))$ and $(B_n(ut))$ check proposition 5.2. Fixing some $n \geq 1$, we use proposition 5.2 and lemma 3.1 :

$$\begin{aligned}
\mathrm{Re}(1 - f^+(ut)) &= \mathrm{Re}(1 - A_n(ut)/B_n(ut)) - \mathrm{Re}\left(\sum_{k>n} \frac{\rho_k}{B_k(ut)B_{k-1}(ut)}\right) \\
&\leq \frac{\mathrm{Re}((B_n(ut) - A_n(ut))\bar{B}_n(ut))}{|B_n(ut)|^2} + \frac{1}{v_+(n)} \\
&\leq \frac{v_+(n) + \sum_{1\leq r\leq n} t^{2r} M_r(n)}{v_+(n)^2 + \sum_{r=1}^n t^{2r} L_r(n)} + \frac{1}{v_+(n)} \leq \frac{1}{v_+(n)}\left(2 + \sum_{1\leq r\leq n} t^{2r} \frac{M_r(n)}{v_+(n)}\right),
\end{aligned}$$

where $L_r(n)$ and $M_r(n)$ depend on $u$. By the formula for $M_r(n)$ and $K_r(n)$ in proposition 5.2, $M_r(n) \leq (\sum_{1\leq k\leq l\leq n} T_k^l(u))^r 2^r v_+(n)$, for $r \geq 1$. Hence :

$$M_r(v_+^{-1}(n)) \leq n 2^r \varphi_{u,+}^{2r}(n).$$

As a result, for some constant $C > 0$ independent on $u$ and any $n \geq 1$ :

$$\mathrm{Re}(1 - f^+(ut)) \leq \frac{C}{n}\left[1 + \sum_{1\leq r\leq v_+^{-1}(n)} (2t^2)^r \varphi_{u,+}^{2r}(n)\right].$$

Choose $n = n_u(t) = \varphi_{u,+}^{-1}(1/(2t))$. In particular $\varphi_{u,+}^2(n) \leq 1/(4t^2)$. We arrive at :

$$\mathrm{Re}(1 - f^+(ut)) \leq \frac{C}{n}\left(1 + \sum_{r\geq 1} 2^{-r}\right) \leq \frac{2C}{n} = \frac{2C}{\varphi_{u,+}^{-1}(1/(2t))} \leq \frac{C'}{\varphi_{u,+}^{-1}(1/t)},$$

for some constant $C'$ independent on $u$, using lemma 6.1. Idem, $\mathrm{Re}(1 - f^-(ut)) \leq C'/\varphi_{u,+}^{-1}(1/t)$. Via now (16), using that $R^\pm(t) = O(1/\psi_\pm^{-1}(1/t)) = O(1/\varphi_{u,+}^{-1}(1/t))$, this shows the right-hand side inequality of the proposition.

Consider next the other direction. Starting in the same way, for any $n \geq 1$, via proposition 5.2 and lemma 3.1 (third point) :

$$\begin{aligned}
\mathrm{Re}(1 - f^+(ut)) &= \mathrm{Re}(1 - A_n(ut)/B_n(ut)) - \mathrm{Re}\left(\sum_{k>n} \frac{\rho_k}{B_k(ut)B_{k-1}(ut)}\right) \\
&\geq \frac{\mathrm{Re}((B_n(ut) - A_n(ut))\bar{B}_n(ut))}{|B_n(ut)|^2} - \frac{v_+(n)}{|B_n(ut)|^2} \\
&= \frac{v_+(n) + \sum_{1\leq r\leq n} t^{2r} M_r(n)}{|B_n(ut)|^2} - \frac{v_+(n)}{|B_n(ut)|^2} = \frac{\sum_{1\leq r\leq n} t^{2r} M_r(n)}{v_+(n)^2 + \sum_{1\leq r\leq n} t^{2r} L_r(n)}.
\end{aligned}$$

By prop. 5.2, $M_r(n) = \sum_{0\leq k\leq n} \rho_k \theta^k K_r(n-k)$ and $L_r(n) = \sum_{0\leq l\leq k\leq n} 2^{(l,k)} \rho_l \rho_k \theta^k K_r(n-k)$, so we have $L_r(n) \leq 2v_+(n) M_r(n)$. Hence :

$$\mathrm{Re}(1 - f^+(ut)) \geq \frac{1}{v_+(n)} \frac{\sum_{1\leq r\leq n} t^{2r} M_r(n)/v_+(n)}{1 + 2\sum_{1\leq r\leq n} M_r(n)/v_+(n)} \geq \frac{1}{v_+(n)} \frac{t^2 M_1(n)/v_+(n)}{1 + 2t^2 M_1(n)/v_+(n)}, \quad (17)$$

using in the last step that $x \longmapsto x/(1+2x)$ is increasing ($x>0$). As a result, for some constant $c>0$ independent on $u$ and all $n \geq 1$ :

$$\mathrm{Re}(1-f^+(ut)) \geq \frac{c}{n}\frac{ct^2M_1(v_+^{-1}(n))/n}{1+2ct^2M_1(v_+^{-1}(n))/n}.$$

Let $\kappa_{u,+}(m)=\sum_{1\leq k\leq l\leq m}T_k^l(u)$ and assume first that $\lim_{m\to+\infty}\kappa_{u,+}(m)=+\infty$. Note (using proposition 5.2) that $M_1(n)\geq \sum_{1\leq m\leq n}\rho_m\kappa_{u,+}(m)$. Therefore :

$$M_1(v_+^{-1}(n)) \geq \sum_{1\leq m\leq v_+^{-1}(n)} \rho_m\kappa_{u,+}(m).$$

Let $c_0\geq 2$ be such that for all $n$, $v_+(n+1)\leq c_0v_+(n)$. Set $m_u(t)=(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)$ and next choose $n_u(t)=c_0^2m_u(t)$. Let $s=v_+^{-1}(m_u(t))$ and $s'=v_+^{-1}(n_u(t))$. This gives :

$$v_+(s)\leq m_u(t)<v_+(s+1)\leq c_0v_+(s) \text{ and } v_+(s')\leq c_0^2m_u(t)<v_+(s'+1)\leq c_0v_+(s').$$

As a result, $c_0^2m_u(t)\geq v_+(s')-v_+(s)\geq (c_0-1)m_u(t)$ and $m_u(t)\geq v_+(s)\geq m_u(t)/c_0$. This furnishes the inequalities :

$$\frac{M_1(v_+^{-1}(n_u(t)))}{n_u(t)} \geq \frac{\sum_{s<m\leq s'}\rho_m\kappa_+(m)}{n_u(t)} \geq \kappa_{u,+}(s+1)\frac{v_+(s')-v_+(s)}{n_u(t)} \geq \frac{\alpha}{t^2} \text{ with } \alpha=(c_0-1)/c_0^2.$$

Consequently, with $\alpha'=(c^2\alpha)/(c_0^2(1+2c\alpha))$ :

$$\mathrm{Re}(1-f^+(ut)) \geq \frac{c}{n_u(t)}\frac{c\alpha}{1+2c\alpha} = \frac{\alpha'}{(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)}.$$

If now $m\longmapsto \kappa_{u,+}(m)$ is bounded, the previous inequality is valid as long as $(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)$ is defined. For smaller $t$, we have $(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)=+\infty$, so that the previous lower-bound is obvious in this case. Similarly, with $\kappa_{u,-}(m)=\sum_{-m\leq k\leq l\leq -1}T_k^l(u)$, we have :

$$\mathrm{Re}(1-f^-(ut)) \geq \alpha'/(\kappa_{u,-}\circ v_-^{-1})^{-1}(1/t^2).$$

To prove a lower bound, we use (16), giving for some constant $c_3>0$ independent on $u$ :

$$\mathrm{Re}(1-\chi_D(ut)) \geq p_0'\mathrm{Re}(1-f^+(ut))+q_0'\mathrm{Re}(1-f^-(ut))-c_3/\psi^{-1}(1/t). \tag{18}$$

Recall that $\varphi_{u,+}^2=\psi^2+\kappa_{u,+}\circ v_+^{-1}+\kappa_{u,-}\circ v_-^{-1}$. Then, for some constant $\beta>0$ independent on $u$ and $t$, we have :

$$\beta \leq \frac{\varphi_{u,+}^{-1}(1/t)}{\min\{\psi^{-1}(1/t),(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2),(\kappa_{u,-}\circ v_-^{-1})^{-1}(1/t^2)\}} \leq 1.$$

Fixing $t>0$, suppose for example that $(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)\leq(\kappa_{u,-}\circ v_-^{-1})^{-1}(1/t^2)$. This leads to the following discussion :

– If $(1/\psi^{-1}(1/t))\leq p_0'\mathrm{Re}(1-f^+(ut))/(2c_3)$ and $\psi^{-1}(1/t)\geq(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)$, then :

$$\mathrm{Re}(1-\chi_D(ut)) \geq (p_0'/2)\mathrm{Re}(1-f^+(ut)) \geq \frac{p_0'\alpha'/2}{(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)} \geq \frac{\beta p_0'\alpha'/2}{\varphi_{u,+}^{-1}(1/t)}.$$

– If $(1/\psi^{-1}(1/t))\leq p_0'\mathrm{Re}(1-f^+(ut))/(2c_3)$ and $\psi^{-1}(1/t)\leq(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)$, then, by lemma 6.2 and proposition 4.6, for absolute constants $c>0$ and $c'>0$ :

$$\mathrm{Re}(1-\chi_D(ut)) \geq cR(ut) \geq c'/\psi^{-1}(1/t) \geq \beta c'/\varphi_{u,+}^{-1}(1/t).$$

– If $(1/\psi^{-1}(1/t)) > p'_0\mathrm{Re}(1-f^+(ut))/(2c_3)$, then $\psi^{-1}(1/t) < (2c_3/(p'_0\alpha'))(\kappa_{u,+}\circ v_+^{-1})^{-1}(1/t^2)$. We obtain the inequality :

$$\frac{\beta}{\varphi_{u,+}^{-1}(1/t)} \leq \frac{1}{\psi^{-1}(1/t)\min\{p'_0\alpha'/(2c_3),1\}}.$$

We conclude as in the previous case, via $\mathrm{Re}(1-\chi_D(ut)) \geq cR(ut) \geq c'/\psi^{-1}(1/t)$. This completes the proof of the proposition.

□

## 6.3 Preliminaries for estimating the modulus of $1-\chi_D(ut)$

We still fix $u \in S_+^{d-1}$ and $t > 0$. We use proposition 5.2 concerning $f^+$ and its symmetric analogue for $f^-$. To precise the dependency with respect to $f^+$ or $f^-$, we put a superscript ($+$ or $-$) on $A_n$, $B_n$, etc. For example $f^+(ut) = \lim_{n\to+\infty} A_n^+(ut)/B_n^+(ut)$. Keeping the same sets of summation, the expressions corresponding to $K_r^-(n)$, etc, are deduced from proposition 5.2 by replacing $(q_k, p_k)$ by $(p_{-k}, q_{-k})$. Any $\rho_k$ becomes $\rho_{-k-1}q_0/p_0$. It is worth noticing that $T_k^l(u)$ is simply transformed into $T_{-l}^{-k}(u)$.

Let us begin with a formal computation on reversed continued fractions.

**Lemma 6.4**
*Let $n \geq 1$ and consider the formal reduced continued fraction :*

$$\frac{U_n}{V_n} = [(-c_1, d_1); (-c_2, d_2); \cdots; (-c_n, d_n)].$$

*Then the reduced reversed continued fraction :*

$$\frac{\tilde{U}_n}{\tilde{V}_n} = [(-1/c_n, d_n/c_n); (-1/c_{n-1}, d_{n-1}/c_{n-1}); \cdots; (-1/c_1, d_1/c_1)]$$

*verifies $V_n = c_1\cdots c_n\tilde{V}_n$.*

*Proof of the lemma :*
We have :

$$V_n = \langle e_1, \begin{pmatrix} d_n & -c_n \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d_1 & -c_1 \\ 1 & 0 \end{pmatrix} e_1\rangle$$

Transposing and next conjugating the matrices with $\mathrm{diag}(1,-1)$ :

$$\begin{aligned} V_n = \langle e_1, \begin{pmatrix} d_1 & 1 \\ -c_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d_n & 1 \\ -c_n & 0 \end{pmatrix} e_1\rangle &= \langle e_1, \begin{pmatrix} d_1 & -1 \\ c_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d_n & -1 \\ c_n & 0 \end{pmatrix} e_1\rangle \\ &= c_1\cdots c_n\langle e_1, \begin{pmatrix} d_1/c_1 & -1/c_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d_n/c_n & -1/c_n \\ 1 & 0 \end{pmatrix} e_1\rangle. \end{aligned}$$

Hence $V_n = c_1\cdots c_n\tilde{V}_n$. This proves the lemma.

□

Let us start from relation (6), $\chi_D(ut) = \varphi_0(ut)(p'_0\chi_D^+(ut) + q'_0\chi_D^-(ut)) = (\varphi_0(ut)/b_0)(\chi_D^+(ut) + a_0\chi_D^-(ut))$. This gives, using lemmas 4.6 and 6.2 and taking $t > 0$ small, independently on $u$ :

$$
\begin{aligned}
\chi_D(ut)-1 &= \frac{\varphi_0(ut)}{b_0}(\chi_D^+(ut)+a_0\chi_D^-(ut)-b_0/\varphi_0(ut)) \\
&= \frac{\varphi_0(ut)}{b_0}(f^+(ut)+a_0f^-(ut)-b_0/\psi_0(ut))+O(R(t)) \\
&= \frac{\varphi_0(ut)}{b_0}\left(\frac{A_n^+(ut)}{B_n^+(ut)}+a_0\frac{A_m^-(ut)}{B_m^-(ut)}-b_0/\psi_0(ut)\right) \\
&+ \frac{\varphi_0(ut)}{b_0}\left(\sum_{k>n}\frac{\rho_k}{B_k^+(ut)B_{k-1}^+(ut)}+\sum_{k>m}\frac{a_0^2\rho_{-k-1}}{B_k^-(ut)B_{k-1}^-(ut)}\right)+O(R(t)),
\end{aligned}
$$

with $O(\ )$ uniform in $u$ and arbitrary $n \geq 1$, $m \geq 1$. As a result :

$$
\begin{aligned}
\chi_D(ut)-1 &= \frac{\varphi_0(ut)}{b_0B_n^+(ut)B_m^-(ut)}\left(A_n^+(ut)B_m^-(ut)+a_0A_m^-(ut)B_n^+(ut)-(b_0/\psi_0(ut))B_n^+(ut)B_m^-(ut)\right) \\
&+ \frac{\varphi_0(ut)}{b_0}R_{-m,n}(ut)+O(R(t)), \qquad (19)
\end{aligned}
$$

with $|R_{-m,n}(ut)| \leq (v_+(n)/|B_n^+(ut)|^2)+a_0(v_-(m)/|B_m^-(ut)|^2)$, by proposition 3.1, and $O(\ )$ uniform in $u \in S_+^{d-1}$.

**Lemma 6.5**
*Let $n \geq 1$, $m \geq 1$ and the following reduced continued fraction :*

$$
\frac{\tilde{A}_{m+n+1}(ut)}{\tilde{B}_{m+n+1}(ut)} = [(-a_{-m}, b_{-m}/\psi_{-m}(ut)); (-a_{-m+1}, b_{-m+1}/\psi_{-m+1}(ut)); \cdots ; (-a_n, b_n/\psi_n(ut))].
$$

*Then* $\tilde{B}_{m+n+1}(ut) = -a_{-1}\cdots a_{-m}(A_n^+(ut)B_m^-(ut)+a_0A_m^-(ut)B_n^+(ut)-(b_0/\psi_0(ut))B_n^+(ut)B_m^-(ut))$.

*Proof of the lemma :*
Fix $m \geq 1$. Observe that the two functions $n \longmapsto -\tilde{B}_{m+n+1}(ut)/(a_{-1}\cdots a_{-m})$ and $n \longmapsto A_n^+(ut)B_m^-(ut)+a_0A_m^-(ut)B_n^+(ut)-(b_0/\psi_0(ut))B_n^+(ut)B_m^-(ut)$ check the same recursive relation $X_n = (b_n/\psi_n(ut))X_{n-1}-a_nX_{n-2}$, for $n \geq 1$. We just need to check that they coincide for the values $n=0$ and $n=1$.

First, $\tilde{B}_m(ut)/(a_{-1}\cdots a_{-m}) = B_m^-(ut)$ and :

$$
A_m^-(ut) = (1/a_{-1})\theta^{-1}B_{m-1}^-(ut) = \tilde{B}_{m-1}(ut)/(a_{-1}\cdots a_{-m}),
$$

by lemma 6.4. For $n=0$ we have $-\tilde{B}_{m+1}(ut)/(a_{-1}\cdots a_{-m})$ and $a_0A_m^-(ut)-(b_0/\psi_0(ut))B_m^-(ut)$. Since one has :

$$
\tilde{B}_{m+1}(ut) = (b_0/\psi_0(ut))\tilde{B}_m(ut)-a_0\tilde{B}_{m-1}(ut),
$$

this gives the result for $n=0$. For $n=1$, we have :

$$
\tilde{B}_{m+2}(ut) = \frac{b_1}{\psi_1(ut)}\tilde{B}_{m+1}(ut)-a_1\tilde{B}_m(ut) = \left(\frac{b_1}{\psi_1(ut)}\frac{b_0}{\psi_0(ut)}-a_1\right)\tilde{B}_m(ut)-\frac{b_1}{\psi_1(ut)}a_0\tilde{B}_{m-1}(ut).
$$

This has to be compared with $a_1B_m^-(ut)+a_0(b_1/\psi_1(ut))A_m^-(ut)-(b_0/\psi_0(ut))(b_1/\psi_1(ut))B_m^-(ut)$. This provides the conclusion of the lemma. □

As a consequence of this lemma we obtain :

$$\chi_D(ut) - 1 = -\frac{\varphi_0(ut)}{b_0}\left(a_0\rho_{-m-1}\frac{\tilde{B}_{m+n+1}(ut)}{B_n^+(ut)B_m^-(ut)} - R_{-m,n}(ut)\right) + O(R(t)). \quad (20)$$

Now it follows from proposition 5.2 that $|B_n^+(ut)|^2 = \sum_{r=0}^n t^{2r}L_r^+(n)$, with $H_r((k_i),(l_j)) := \#\{0 \le i \le r \mid l_i + 1 < k_{i+1}\}$ and :

$$L_r^+(n) = \sum_{0\le l_0<k_1\le l_1<\cdots<k_r\le l_r<k_{r+1}\le n+1} \rho_{l_0}T_{k_1}^{l_1}(u)\cdots T_{k_r}^{l_r}(u)\rho_{k_{r+1}-1}2^{H_r((k_i),(l_j))}.$$

As a result, setting $W_{-m,n}(ut) = a_0\rho_{-m-1}\tilde{B}_{m+n+1}(ut)$, we have :

$$|W_{-m,n}(ut)|^2 = \sum_{0\le r\le n+m+1} t^{2r}U_r(u),$$

with $H_r((k_i),(l_i)) = \#\{0 \le i \le r \mid l_i + 1 < k_{i+1}\}$ and :

$$\begin{aligned} U_r(u) = a_0^2\rho_{-m-1}^2 \quad \textstyle\sum_{-m-1\le l_0<k_1\le l_1<\cdots<k_s\le l_s<k_{s+1}\le n+1} \theta^{-m-1} \quad & \rho_{l_0+m+1}T_{k_1}^{l_1}(u)\cdots T_{k_s}^{l_s}(u)\theta^{-m-1} \\ & \times\rho_{k_{s+1}-1+m+1}2^{H_s((k_i),(l_i))}. \end{aligned}$$

After a cocycle simplification :

$$U_r(u) = a_0^2 \sum_{-m-1\le l_0<k_1\le l_1<\cdots<k_s\le l_s<k_{s+1}\le n+1} \rho_{l_0}T_{k_1}^{l_1}(u)\cdots T_{k_s}^{l_s}(u)\rho_{k_{s+1}-1}2^{H_s((k_i),(l_i))}. \quad (21)$$

### 6.4 Order of the modulus of $1 - \chi_D(ut)$

**Proposition 6.6**
*There is a constant $C \ge 1$ so that for $t > 0$ small enough, uniformly in $u \in S_+^{d-1}$ :*

$$\frac{1}{C} \le \varphi_u^{-1}(1/t)|1 - \chi_D(ut)| \le C.$$

*Proof of the proposition :*
Let us start from (20). Set :

$$F(ut) = \frac{W_{-m,n}(ut)}{B_n^+(ut)B_m^-(ut)} - R_{-m,n}(ut).$$

We then have $\chi_D(ut) - 1 = -(\varphi_0(ut)/b_0)F(ut) + O(R(t))$, where the last term is uniform in $u \in S_+^{d-1}$. By definition, $F(ut)$ does not depend on $m \ge 1$, $n \ge 1$ (see (20)). The latter are arbitrary for the moment. Using the upper-bound on $R_{-m,n}(ut)$, we get :

$$\begin{aligned} |F(ut)| &\ge \frac{|W_{-m,n}(ut)|}{|B_n^+(ut)||B_m^-(ut)|} - \frac{v_+(n)}{|B_n^+(ut)|^2} - a_0\frac{v_-(m)}{|B_m^-(ut)|^2} \\ &\ge \frac{1}{|B_n^+(ut)||B_m^-(ut)|}\left(|W_{-m,n}(ut)| - v_+(n)\frac{|B_m^-(ut)|}{|B_n^+(ut)|} - a_0v_-(m)\frac{|B_n^+(ut)|}{|B_m^-(ut)|}\right). \end{aligned}$$

Recall that $W_{-m,n}^2(ut) = \sum_{0\le r\le n+m+1} t^{2r}U_r(u)$, with $U_s(u)$ given by (21). In particular :

$$U_0(u) = a_0^2\left(\sum_{-m-1\le l\le n} \rho_l\right)^2 = (v_-(m) + a_0v_+(n))^2.$$

Introduce $Z^2_{-m,n}(t)$ such that :

$$|W_{-m,n}(ut)|^2 - (a_0^2|B_n^+(ut)|^2 + |B_m^-(ut)|^2) - 2a_0v_+(n)v_-(m) = Z^2_{-m,n}(ut).$$

Then $Z^2_{-m,n}(ut) = \sum_{1\le s\le n+m+1} t^{2s}V_s(u)$, where :

$$V_s(u) = a_0^2 \sum_{\substack{-m-1\le l_0<k_1\le l_1<\cdots<k_s\le l_s<k_{s+1}\le n+1\\ l_0<0<k_{s+1}}} \rho_{l_0}T_{k_1}^{l_1}(u)\cdots T_{k_s}^{l_s}(u)\rho_{k_{s+1}-1}2^{H_s((k_i),(l_i))}. \tag{22}$$

Observe now that :

$$\begin{aligned}\left(v_+(n)\frac{|B_m^-(ut)|}{|B_n^+(ut)|} + a_0v_-(m)\frac{|B_n^+(ut)|}{|B_m^-(ut)|}\right)^2 &= v_+^2(n)\frac{|B_m^-(ut)|^2}{|B_n^+(ut)|^2} + a_0^2v_-^2(m)\frac{|B_n^+(ut)|^2}{|B_m^-(ut)|^2} + 2a_0v_+(n)v_-(m)\\ &\le |B_m^-(ut)|^2 + a_0^2|B_n^+(ut)|^2 + 2a_0v_+(n)v_-(m)\\ &\le |W_{-m,n}(ut)|^2 - Z^2_{-m,n}(ut) \le |W_{-m,n}(ut)|^2. \end{aligned} \tag{23}$$

This allows to write :

$$\begin{aligned}|F(ut)| &\ge \frac{1}{|B_n^+(ut)||B_m^-(ut)|}\left(|W_{-m,n}(ut)| - v_+(n)\frac{|B_m^-(ut)|}{|B_n^+(ut)|} - a_0v_-(m)\frac{|B_n^+(ut)|}{|B_m^-(ut)|}\right)\\ &\ge \frac{|W_{-m,n}(ut)|^2 - \left(v_+(n)\frac{|B_m^-(ut)|}{|B_n^+(ut)|} + a_0v_-(m)\frac{|B_n^+(ut)|}{|B_m^-(ut)|}\right)^2}{2W_{-m,n}(ut)|B_n^+(ut)||B_m^-(ut)|}\\ &\ge \frac{Z^2_{-m,n}(ut)}{2|W_{-m,n}(ut)||B_n^+(ut)||B_m^-(ut)|}. \end{aligned} \tag{24}$$

We now give upper-bounds on $|W_{-m,n}(ut)|$ and $|B_n^+(ut)||B_m^-(ut)|$. Observe first that $L_r^+(n) \le V_r(u)v_+(n)/(a_0v_-(m))$, for $r \ge 1$, so that :

$$|B_n^+(ut)|^2 - v_+(n)^2 \le \frac{v_+(n)}{a_0v_-(m)}.$$

Similarly, $|B_m^-(ut)|^2 - v_-(m)^2 \le Z^2_{-m,n}(ut)v_-(m)/(a_0v_+(n))$. We obtain :

$$\begin{aligned}|W_{-m,n}(ut)|^2 &= (v_-(m) + a_0v_+(n))^2 + a_0^2(|B_n^+(ut)|^2 - v_+(n)^2) + (|B_m^-(ut)|^2 - v_-(m)^2) + Z^2_{-m,n}(ut)\\ &\le (v_-(m) + a_0v_+(n))^2 + (a_0v_+(n)/v_-(m) + v_-(m)/(a_0v_+(n)) + 1)Z^2_{-m,n}(ut)\\ &\le (v_-(m) + a_0v_+(n))^2\left[1 + \frac{1}{a_0v_-(m)v_+(n)}Z^2_{-m,n}(ut)\right]. \end{aligned} \tag{25}$$

In the same way :

$$\begin{aligned}|B_n^+(ut)|^2|B_m^-(ut)|^2 &= v_+(n)^2v_-(m)^2\left(1 + \sum_{1\le r\le n} t^{2r}\frac{L_r^+(n)}{v_+(n)^2}\right)\left(1 + \sum_{1\le r\le m} t^{2r}\frac{L_r^-(n)}{v_-(m)^2}\right)\\ &= v_+(n)^2v_-(m)^2\left(1 + \sum_{1\le s\le m+n} t^{2s}\sum_{0\le r\le s}\frac{L_r^+(n)L_{s-r}^-(m)}{v_+(n)^2v_-(m)^2}\right). \end{aligned}$$

Notice that $\sum_{0\le r\le s} L_r^+(n)L_{s-r}^-(m) \le V_s(u)v_+(n)v_-(m)/a_0$. Therefore :

$$|B_n^+(ut)|^2|B_m^-(ut)|^2 \leq v_+(n)^2 v_-(m)^2 \left(1+\frac{Z_{-m,n}^2(ut)}{a_0 v_+(n)v_-(m)}\right)$$

Inserting these two upper-bounds in (24) and using in the last step that the function $x \longmapsto /(1+x)$ is increasing, we obtain :

$$\begin{aligned} |F(ut)| &\geq \frac{1}{2(v_-(m)/a_0+v_+(n))}\frac{Z_{-m,n}^2(ut)/(a_0v_+(n)v_-(m))}{1+Z_{-m,n}^2(ut)/(a_0v_+(n)v_-(m))} \\ &\geq \frac{1}{2(v_-(m)/a_0+v_+(n))}\frac{t^2V_1(u)/(a_0v_+(n)v_-(m))}{1+t^2V_1(u)/(a_0v_+(n)v_-(m))}. \end{aligned}$$

Let us now focus on $V_1(u)$ that we write $V_1(u)=V_{u,1}(-m,n)$. Set $\kappa_u(r,s)=\sum_{r\leq k\leq l\leq s}T_k^l(u)$, for $r\leq s$. We assume first that $\sum_{-\infty<k\leq l<+\infty}T_k^l(u)=+\infty$. We obtain :

$$\begin{aligned} V_{u,1}(-m,n) &= a_0^2\sum_{\substack{-m-1\leq l_0<k_1\leq l_1<k_2\leq n+1\\ l_0<0<k_2}}\rho_{l_0}T_{k_1}^{l_1}(u)\rho_{k_2-1}2^{H_1((k_i),(l_i))} \\ &\geq a_0^2\sum_{-m\leq l_0\leq 0\leq k_2\leq n}\rho_{l_0-1}\rho_{k_2}\kappa_u(l_0,k_2). \end{aligned}$$

We next have the existence of a constant $c>0$ independent on $u\in S_+^{d-1}$ so that for all $n\geq 1$ :

$$|F(ut)|\geq \frac{c}{n}\frac{(ct^2/n^2)V_{u,1}(-v_-^{-1}(n),v_+^{-1}(n))}{1+(ct^2/n^2)V_{u,1}(-v_-^{-1}(n),v_+^{-1}(n))}.$$

Let $c_0\geq 2$ be such that for all $n\geq 0$, $v_+(n+1)\leq c_0v_+(n)$ and $v_-(n+1)\leq c_0v_-(n)$. Taking $t>0$, set $m_u(t)=\kappa_u(-v_-^{-1}(.),v_+^{-1}(.))^{-1}(1/t^2)$. Choose next $n_u(t)=c_0^2m_u(t)$. Let $r=v_+^{-1}(m_u(t))$, $s=v_-^{-1}(m_u(t))$ and $r'=v_+^{-1}(n_u(t))$, $s'=v_-^{-1}(n_u(t))$. This gives :

$$v_+(r)\leq m_u(t)<v_+(r+1)\leq c_0v_+(r) \text{ and } v_+(r')\leq c_0^2m_u(t)<v_+(r'+1)\leq c_0v_+(r').$$

As a result, $c_0^2m_u(t)\geq v_+(r')-v_+(r)\geq (c_0-1)m_u(t)$ and $m_u(t)\geq v_+(r)\geq m_u(t)/c_0$. In the same way, we have :

$$v_-(s)\leq m_u(t)<v_-(s+1)\leq c_0v_-(s) \text{ and } v_-(s')\leq c_0^2m_u(t)<v_+(s'+1)\leq c_0v_+(s').$$

Also, $c_0^2m_u(t)\geq v_-(s')-v_-(s)\geq (c_0-1)m_u(t)$ and $m_u(t)\geq v_-(s)\geq m_u(t)/c_0$. We obtain :

$$\begin{aligned} \frac{V_{u,1}(-v_-^{-1}(n_u(t)),v_+^{-1}(n_u(t)))}{n_u(t)^2} &\geq a_0^2\frac{\sum_{r<l\leq r',s<k\leq s'}\rho_{-k-1}\rho_l\kappa_u(-k,l)}{n_u(t)^2} \\ &\geq a_0\kappa_u(-s-1,r+1)\frac{(v_+(r')-v_+(r))(v_-(s')-v_-(s))}{n_u(t)^2} \\ &\geq \frac{a_0(c_0-1)^2m_u(t)^2}{t^2n_u(t)^2}=\frac{\alpha}{t^2}, \end{aligned}$$

where $\alpha=a_0(c_0-1)^2/c_0^2$. The conclusion for the moment is that there is a constant $c'>0$ independent on $u\in S_+^{d-1}$ so that for small $t>0$ :

$$|F(ut)|\geq \frac{c'}{\kappa_u(-v_-^{-1}(.),v_+^{-1}(.))^{-1}(1/t^2)}.$$

When $\sum_{-\infty<k\leq l<+\infty} T_k^l$ is bounded, the inequality is verified, as $\kappa_u(v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2) = +\infty$, for small enough $t > 0$. The previous lower-bound is then obvious in that case.

In order to draw the conclusion, recall that $\varphi_u^2(n) = \psi^2(n) + \kappa_u(-v_-^{-1}(n), v_+^{-1}(n))$ and $1 - \chi_D(ut) = (\varphi_0(ut)/b_0)F(ut) + O(R(t))$, with $O(\ )$ uniform in $u \in S_+^{d-1}$. Also, by lemma 6.2 :

$$|1-\chi_D(ut)| \geq \mathrm{Re}(1-\chi_D(ut)) \geq c_1 R(ut),$$

for some absolute constant $c_1 > 0$. Similarly, for constants $c_2 > 0$ and $c_3 > 0$, we have the inequalities $c_2 \leq R(t)\psi^{-1}(1/t) \leq c_3$. Then, for constants $\beta > 0$ and $c_4 > 0$ independent on $u \in S_+^{d-1}$, for small $t > 0$ :

$$\beta \leq \frac{\varphi_u^{-1}(1/t)}{\min\{\psi^{-1}(1/t), \kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2)\}}$$

and :

$$|1-\chi_D(ut)| \geq \frac{1}{2b_0}|F(ut)| - c_4 R(t) \geq \frac{c'}{2b_0\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2)} - \frac{c_4 c_3}{\psi^{-1}(1/t)}.$$

Fixing $t > 0$, we then have the following discussion :
– If $c'/(2b_0\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2)) \geq 2c_3c_4/\psi^{-1}(1/t)$ and $\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2) \leq \psi^{-1}(1/t)$ :

$$|1-\chi_D(ut)| \geq \frac{c'}{4b_0\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2)} \geq \frac{c'\beta}{4b_0\varphi_u^{-1}(1/t)}.$$

– If $c'/(2b_0\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2)) \geq 2c_3c_4/\psi^{-1}(1/t)$ and $\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2) > \psi^{-1}(1/t)$ :

$$|1-\chi_D(ut)| \geq c_1c_2/\psi^{-1}(1/t) \geq c_1c_2\beta/\varphi_u^{-1}(1/t).$$

– If $c'/(2b_0\kappa_u(-v_-^{-1}(.), v_+^{-1}(.))^{-1}(1/t^2)) < 2c_3c_4/\psi^{-1}(1/t)$, then for some absolute constant $c_5 > 0$ (independent on $u$), $1/\psi^{-1}(1/t) \geq c_5/\varphi_u^{-1}(1/t)$. Then, as above :

$$|1-\chi_D(ut)| \geq c_1c_2/\psi^{-1}(1/t) \geq c_1c_2c_5\varphi_u^{-1}(1/t).$$

This completes the proof of the lower bound. We next turn to the proof of the upper-bound. Let us start from the following inequality, for any $m \geq 1$, $n \geq 1$, using lemma 3.1 :

$$|1-\chi_D(ut)| \;\leq\; \frac{1}{b_0}|F(ut)| + O(R(t)) \leq \frac{|W_{-m,n}(ut)|}{b_0|B_n^+(ut)||B_m^-(ut)|} + \frac{v_+(n)}{|B_n^+(ut)|^2} + \frac{a_0v_-(m)}{|B_m^-(ut)|^2} + O(R(t)),$$

with $O(\ )$ uniform in $u \in S_+^{d-1}$. Observe that from the second line in (24) :

$$\frac{v_+(n)}{|B_n^+(ut)|^2} + \frac{a_0v_-(m)}{|B_m^-(ut)|^2} \leq \frac{1}{|B_n^+(ut)||B_m^-(ut)|}\left(v_+(n)\frac{|B_m^-(ut)|}{|B_n^+(ut)|} + a_0v_-(m)\frac{|B_n^+(ut)|}{|B_m^-(ut)|}\right) \leq \frac{|W_{-m,n}(ut)|}{|B_n^+(ut)||B_n^-(ut)|}.$$

Since $R(t) = O(1/\psi^{-1}(1/t)) = O(1/\varphi_u^{-1}(1/t))$, uniformly on $u \in S_+^{d-1}$, there exists some absolute constant $C > 0$ such that for small $t > 0$ and all $m \geq 1$ and $n \geq 1$ :

$$|1-\chi_D(ut)| \leq C\frac{|W_{-m,n}(ut)|}{|B_n^+(ut)||B_m^-(ut)|} + \frac{C}{\varphi_u^{-1}(1/t)}.$$

From (25) and lemma 3.1, we have :

$$\frac{|W_{-m,n}(ut)|}{|B_n^+(ut)||B_m^-(ut)|} \leq \frac{(v_-(m) + a_0v_+(n))\sqrt{1 + Z_{-m,n}^2(ut)/(a_0v_-(m)v_+(n))}}{v_+(n)v_-(m)}.$$

Let us recall that $Z^2_{-m,n}(ut) = \sum_{1\le s\le m+n+1} t^{2s}V_s(u)$, where $V_s(u)$ is given by relation (22), so checks $V_s(u) \le a_0 v_-(m)v_+(n)\kappa_u(-m,n)^s$, still setting $\kappa_u(-m,n) = \sum_{-m\le k\le l\le n} T_k^l(u)$. As a result, for another constant $C > 0$ independent on $u \in S_+^{d-1}$, small $t > 0$ and any $n \ge 1$ :

$$\begin{aligned} |1-\chi_D(ut)| &\le \frac{C}{n}\sqrt{1+\sum_{1\le s\le v_-^{-1}(n)+v_+^{-1}(n)+1} t^{2s}(\kappa_u(-v_-^{-1}(n), v_+^{-1}(n)))^s} + \frac{C}{\varphi_u^{-1}(1/t)} \\ &\le \frac{C}{n}\sqrt{1+\sum_{1\le s\le v_-^{-1}(n)+v_+^{-1}(n)+1} t^{2s}\varphi_u^{2s}(n)} + \frac{C}{\varphi_u^{-1}(1/t)}. \end{aligned}$$

Choose $n = \varphi_u^{-1}(1/2t)$. In particular, $\varphi_u(n) \le 1/(2t)$. This gives :

$$|1-\chi_D(ut)| \le \frac{C}{\varphi_u^{-1}(1/(2t))}\sqrt{1+\sum_{s\ge 1}(1/2)^{2s}} + \frac{C}{\varphi_u^{-1}(1/t)}.$$

By lemma 6.1, there is a constant $C'$ independent on $u \in S_+^{d-1}$ so that for small $t > 0$ :

$$|1-\chi_D(ut)| \le \frac{C'}{\varphi_u^{-1}(1/t)}.$$

This concludes the proof of the proposition. □

## 6.5 Conclusion

- *Theorem 2.4, corollary 2.5 and proposition 2.6.* By propositions 6.3, 6.6 and theorem 4.2, using that $\mathrm{Re}(1/a) = \mathrm{Re}(a)/|a|^2$, the random walk is recurrent if and only if, for some $\eta > 0$ :

$$\int_{(u,t)\in S_+^{d-1}\times(0,\eta)} \frac{(\varphi_u^{-1}(1/t))^2}{\varphi_{u,+}^{-1}(1/t)}\, t^{d-1}dudt = +\infty. \tag{26}$$

For fixed $u \in S_+^{d-1}$, we cut the interval $(0,\eta)$ in the contiguous intervals $[1/(n+1), 1/n]$, $n \ge n_0$. The latter have length of order $1/n^2$, so using finally lemma 6.1, the condition is equivalent to the one given in the statement of theorem 2.4.

Concerning proposition 2.6, we first show in the antisymmetric case that $\varphi_u^{-1}$ and $\varphi_{u,++}^{-1}$ have the same size, uniformly in $u \in S_+^{d-1}$. By lemma 6.1, it is enough to show that $\varphi_u \le C\varphi_{u,++}$. Observe that $p_0 = q_0$ and :

$$\begin{aligned} \varphi_u^2(n) &= \varphi_{u,+}^2(n) + \sum_{-v_-^{-1}(n)\le k\le 0\le l\le v_+^{-1}(n)} T_k^l(u) \\ &= \varphi_{u,+}^2(n) + \sum_{0\le k,l\le v_+^{-1}(n)} T_{\min(k,l)+1}^{\max(k,l)}(u) \le 4\varphi_{u,+}^2(n) \le 8\varphi_{u,++}^2(n). \end{aligned}$$

This completes the proof of this claim.

Concerning Corollary 2.5, we always have $\varphi_u^{-1} \le \varphi_{u,+}^{-1}$. Then :

$$\int_{S_+^{d-1}\times(0,\eta)} \frac{(\varphi_u^{-1}(1/t))^2}{\varphi_{u,+}^{-1}(1/t)}\, t^{d-1}dudt \le \int_{S_+^{d-1}\times(0,\eta)} \varphi_u^{-1}(1/t)t^{d-1}dudt. \tag{27}$$

In the antisymmetric case, both integrals have the same order. To complete the proofs of proposition 2.6 and corollary 2.5, we just need to show that the second term has the right order. For fixed

$u \in S_+^{d-1}$, up to decreasing $\eta > 0$, also taking $n_0$ independent on $u \in S_+^{d-1}$ (as $0 < \alpha \leq \varphi_u(1) \leq \beta$, for constants $\alpha$ and $\beta$, independent on $u \in S_+^{d-1}$) :

$$\sum_{n \geq n_0} \int_{1/\varphi_u(n+1)}^{1/\varphi_u(n)} \varphi_u^{-1}(1/t) t^{d-1}\, dt \leq \int_0^\eta \varphi_u^{-1}(1/t) t^{d-1}\, dt \leq \sum_{n \geq 1} \int_{1/\varphi_u(n+1)}^{1/\varphi_u(n)} \varphi_u^{-1}(1/t) t^{d-1}\, dt.$$

On each domain $(1/\varphi_u(n+1), 1/\varphi_u(n))$, we have $\varphi_u^{-1}(1/t) = n$. Hence $\int_0^\eta \varphi_u^{-1}(1/t) t^{d-1}\, dt$ has exact order :

$$\begin{aligned}
\sum_{n\geq 1} n \int_{1/\varphi_u(n+1)}^{1/\varphi_u(n)} t^{d-1}\, dt &= \frac{1}{d} \sum_{n \geq 1} n \left( \frac{1}{(\varphi_u(n))^d} - \frac{1}{(\varphi_u(n+1}) ^d \right) \\
&= \frac{1}{d} \lim_{N \to +\infty} \sum_{n=2}^{N} \frac{1}{(\varphi_u(n))^d} + \left( \frac{1}{(\varphi_u(1)^d} - \frac{N}{(\varphi_u(N+1))^d} \right). \\
&= \frac{1}{d} \lim_{N \to +\infty} \sum_{n=1}^{N} \left( \frac{1}{(\varphi_u(n))^d} - \frac{1}{(\varphi_u(N+1))^d} \right).
\end{aligned}$$

Remark that the right-hand side is bounded from above by $(1/d) \sum_{n \geq 1} 1/(\varphi_u(n))^d$. Hence :

- if $\int_{u \in S_+^{d-1}} \sum_{n\geq 1} (1/\varphi_u(n))^d < +\infty$, then the left-hand side in (27) is finite.
- if $\int_{u \in S_+^{d-1}} \sum_{n\geq 1} (1/\varphi_u(n))^d = +\infty$, using at the end Fatou's lemma :

$$\begin{aligned}
\int_{S_+^{d-1} \times (0,\eta)} \varphi_u^{-1}(1/t) t^{d-1} du dt &\geq C \int_{S_+^{d-1}} \lim_{N \to +\infty} \sum_{n=1}^{N} \left( \frac{1}{(\varphi_u(n))^d} - \frac{1}{(\varphi_u(N+1))^d} \right) du \\
&\geq C \int_{S_+^{d-1}} \sum_{n \geq 1} \frac{1}{(\varphi_u(n))^d}\, du = +\infty.
\end{aligned}$$

This completes the proofs of corollary 2.5 and of the first part of proposition 2.6. To complete the proof of the latter, we take $m_0 = 0$ and $m_n = -m_{-n} = c \neq 0$, $n \geq 1$. Then $\varphi_{u,++}^2(n)$ has immediately the same order as :

$$n w_+ \circ v_+^{-1}(n) + (c.u)^2 \sum_{1 \leq k < l \leq v_+^{-1}(n)} \rho_k \rho_l \left( \sum_{k \leq s \leq l} 1/\rho_s \right)^2.$$

We still denote by $\varphi_{u,++}^2(n)$ this quantity. Suppose now that $c_1 n^\alpha \leq \rho_n \leq c_2 n^\alpha$, $n \geq 0$. We reason up to multiplicative constants, using the notation $\asymp$.

– If $\alpha < -1$, the random walk is transient, as $(v_+(n))$ is bounded.

– If $\alpha = -1$, then $w_+(n) \asymp n^2$ and $v_+(n) \asymp \ln n$. As a result, $\varphi_{u,++}^2(n) \geq n e^{cn}$, for some $c > 0$, giving transience.

– Suppose that $-1 < \alpha < 1$. We show transience. We have $w_+(n) \asymp n^{1-\alpha}$ and $v_+(n) \asymp n^{1+\alpha}$.

$$\begin{aligned}
\sum_{1 \leq k < l \leq n} \rho_k \rho_l \left( \sum_{k \leq s \leq l} 1/\rho_s \right)^2 &\asymp \int_{1 \leq x \leq y \leq n} x^\alpha y^\alpha \left( \int_x^y t^{-\alpha}\, dt \right)^2 dx dy \\
&\asymp n^{2\alpha+2} \int_{1/n \leq x \leq y \leq 1} x^\alpha y^\alpha \left( \int_{nx}^{ny} t^{-\alpha}\, dt \right)^2 dx dy \\
&\asymp n^4 \int_{1/n \leq x \leq y \leq 1} x^\alpha y^\alpha \left( \int_x^y t^{-\alpha}\, dt \right)^2 dx dy \asymp n^4.
\end{aligned}$$

As a result $\varphi^2_{u,++}(n) \asymp n^{1+(1-\alpha)/(1+\alpha)} + (c.u)n^{4/(1+\alpha)}$, so $\varphi_{u,++}(n) \asymp n^{1/(1+\alpha)} + (c.u)n^{2/(1+\alpha)}$. We obtain that when $d=1$, $\varphi_{u,++}(n) \asymp n^{2/(1+\alpha)}$ and when $d \geq 2$, $\varphi^d_{u,++}(n) \geq cn^{d/(1+\alpha)}$. As the exponents are $> 1$ in each case, the random walk is transient, from corollary 2.5.

– Suppose next that $\alpha > 1$. Then $w_+(n) \asymp 1$, $v_+(n) \asymp n^{1+\alpha}$. If $d = 1$, then $\varphi^2_{u,++}(n) \leq C(n + (c,u)^2 n^2)$, so $\varphi_{u,++}(n) = O(n)$ and the random walk is recurrent. When $d = 2$ :

$$\varphi^2_{u,++}(n) \asymp n + (c.u)^2 \int_{1\leq x\leq y\leq n} x^\alpha y^\alpha \left(\int_x^y t^{-\alpha}\, dt\right)^2 dxdy.$$

The second term can be written as :

$$\begin{aligned} &\int_1^n x^\alpha\, dx \int_1^n x^\alpha \left(\int_x^{+\infty} t^{-\alpha}\, dt\right)^2 dx - \left(\int_1^n x^\alpha \int_x^{+\infty} t^{-\alpha} dt\, dx\right)^2 \\ \asymp\ &\left(\int_1^n x^\alpha\, dx\right)\left(\int_1^n x^{2-\alpha}\right) - \left(\int_1^n x\, dx\right)^2. \end{aligned}$$

Let $1 < \alpha < 3$. Then this term is equivalent to $((\alpha-1)^2/(\alpha+1)(3-\alpha))n^4$. As a result :

$$\varphi^2_{u,++}(n) \asymp n + (c.u)^2 n^{4/(1+\alpha)}.$$

In order to show transience we need to control the following quantity :

$$\sum_{n\geq 1} \int_{u\in S^1_+} \frac{1}{\varphi^2_{u,++}(n)} \asymp \sum_{n\geq 1} \int_0^{\pi/2} \frac{1}{n + \theta^2 n^{4/(1+\alpha)}}\, d\theta.$$

Setting $\theta = n^{1/2-2/(1+\alpha)}x$, it remains :

$$\sum_{n\geq 1} \frac{1}{n} n^{1/2-2/(1+\alpha)} \int_0^{(\pi/2)n^{2/(1+\alpha)-1/2}} \frac{1}{1+x^2}\, dx \asymp \sum_{n\geq 1} \frac{1}{n^{1/2+2/(1+\alpha)}} < +\infty,$$

as $1/2 + 2/(1+\alpha) > 1$. If $\alpha = 3$, then $\varphi^2_{u,++}(n) \asymp n + (c.u)^2 n \ln n \leq Cn\ln n$. When $\alpha > 3$, $\varphi^2_{u,++}(n) \asymp n + (c.u)^2 n \leq Cn$. In any case $\sum_{1\geq 1}(1/\varphi^2_{u,++}(n)) = +\infty$, giving recurrence.

– If $\alpha = 1$, then $w_+(n) \asymp \ln n$, $v_+(n) \asymp n^2$. When $d = 1$, $\varphi^2_{u,++}(n) \leq C(n\ln n + n^2(\ln n)^2)$, so $\varphi_{u,++}(n) = O(n\ln n)$ and the random walk is recurrent. When $d = 2$, notice that :

$$\varphi^2_{u,++}(n) \geq K(n\ln n + (c.u)^2 n^2).$$

In order to show transience, we just need to prove the finiteness of :

$$\sum_{n\geq 1} \int_0^{\pi/2} \frac{1}{n\ln n + \theta^2 n^2}\, d\theta = \sum_{n\geq 1} \int_0^{(\pi/2)\sqrt{n/\ln n}} \frac{dx}{1+x^2} \sqrt{(\ln n)/n}\, \frac{1}{n\ln n} < +\infty.$$

This completes the proof of the proposition.

- *Proposition 2.7.* Let $\tilde{\varphi}$ and $\tilde{\varphi_+}$ be the functions corresponding to the case when $m_n = 1$, $n \in \mathbb{Z}$. Set $D = \sum_{n\in\mathbb{Z}}(m_n/\rho_n)$. Observe that one always has in the present situation :

$$\varphi^2(n) = \sum_{-v_-^{-1}(n)\leq k\leq l\leq v_+^{-1}(n)} \rho_{k-1}\rho_l \left(\sum_{k\leq r\leq l}(m_r/\rho_r)\right)^2 \leq C \sum_{-v_-^{-1}(n)\leq k\leq l\leq v_+^{-1}(n)} \rho_{k-1}\rho_l \leq Cn^2.$$

Let $A > 0$ so that $v_\pm(n+1) \leq (A/2)v_\pm(n)$. In the case when $D \neq 0$, one also has :

$$\begin{aligned}\varphi^2(n) &\geq \sum_{-v_-^{-1}(n)\leq k\leq -v_-^{-1}(n/A),\, v_+^{-1}(n/A)\leq l\leq v_+^{-1}(n)} \rho_{k-1}\rho_l \left(\sum_{k\leq r\leq l}(m_r/\rho_r)\right)^2 \\ &\geq (n/A)(n/A)(D/2)^2,\end{aligned}$$

if $n$ is large enough. As a result $\varphi(n)$ and $\varphi^{-1}(n)$ have order $n$. The same is true for $\tilde{\varphi}$. It remains to show the finiteness of :

$$\sum_{n\geq 1}\frac{1}{\varphi_+^{-1}(n)} \leq C\sum_{n\geq 1}\frac{1}{\tilde{\varphi_+}^{-1}(n)} \leq C\sum_{n\geq 1}\frac{(\tilde{\varphi}^{-1}(n))^2}{n^2\tilde{\varphi_+}^{-1}(n)} < +\infty,$$

because the random walk is obviously transient when $m_n = 1$, $n \in \mathbb{Z}$.

When $D = 0$ and in the antisymmetric case, by corollary 2.5 the criterion reduces to :

$$\sum_{n\geq 1}\frac{1}{\varphi_+(n)} \geq \sum_{n\geq 1}\frac{1}{\varphi(n)} \geq c\sum_{n\geq 1}\frac{1}{n} = +\infty.$$

The random walk is recurrent. This completes the proof of the proposition.

### 6.6 Remarks

It seems necessary to interpret the recurrence criterion in order to use it in practice. When $\mu_n = \delta_c$, with $c \neq 0$, the integral in the criterion is finite, because the random walk is trivially transient. How does one may see it directly ? The question is not clear, even for $d = 1$ and $c = 1$.

It would be interesting to consider the case when the $(p_n, q_n, m_n)$ are a typical realization of an i.i.d. process with $m_n$ independent of $(p_n, q_n)$, $\mathbb{E}(\log(p_n/q_n)) = 0$, $\mathrm{Var}(\log(p_n/q_n)) > 0$, $\mathbb{E}(m_n) = 0$ and $\mathrm{var}(m_n) > 0$. One needs first of all to study in detail $(v_\pm(n))$. The random walk is without any doubt transient.

It would also be of interest to consider the analogous model in $\mathbb{Z} \times \mathbb{Z}^2$ in a $\mathbb{Z}^2$-invariant environment. If following the main strategy, the main difficulty in proving a characterization of the asymptotical behaviour is to detail the distribution of the local time during an excursion of simple random walk in the plane. There is no tree-structure behind, but a complicated graph with loops. A first step in this direction seems to be the following model in the plane :

$$P_{(m,n),(m,n\pm 1)} = 1/4,\ \mathbb{P}_{(m,n),(m+1,n)} = p(m,n)/2,\ \mathbb{P}_{(m,n),(m-1,n)} = q(m,n)/2,$$

with $p(m,n) + q(m,n) = 1$, for example making some hypothesis of stochastic homogeneity on the $(q(m,n), p(m,n))_{(m,n)\in\mathbb{Z}^2}$. The vertical component being recurrent, one may study the subsequence of return times on the horizontal axis. This random walk is a one-dimensional random walk in random medium with unbounded jumps. Very few results are known on such a random walk, cf Andjel [1], and they suppose the jump integrable, which is not the case here. The very first step in the proof (lemma 4.1) is already not clear.

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