Approximate and Approximate Null-Controllability of a Class of Piecewise Linear Markov Switch Systems

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Abstract

We propose an explicit, easily-computable algebraic criterion for approximate null-controllability of a class of general piecewise linear switch systems with multiplicative noise. This gives an answer to the general problem left open in [13]. The proof relies on recent results in [4] allowing to reduce the dual stochastic backward system to a family of ordinary differential equations. Second, we prove by examples that the notion of approximate controllability is strictly stronger than approximate null-controllability. A sufficient criterion for this stronger notion is also provided. The results are illustrated on a model derived from repressed bacterium operon (given in [19] and reduced in [5]).

1 Introduction

This short paper aims at giving an answer to an approximate (null-)controllability problem left open in [13]. We deal with Markovian systems of switch type consisting of a couple mode/ trajectory denoted by (Γ, X). The mode component Γ evolves as a pure jump Markov process and cannot be controlled. It corresponds to spikes inducing regime switching. The second component X obeys a controlled linear stochastic differential equation (SDE) with respect to the compensated random measure associated to Γ. The linear coefficients governing the dynamics depend on the current mode.

The controllability problem deals with criteria allowing one to drive the X_T component arbitrarily close to acceptable targets. An extensive literature on controllability is available in different frameworks: finite-dimensional deterministic setting (Kalman's condition, Hautus test [14]), infinite dimensional settings (via invariance criteria in [22], [6], [21], [17], [16], etc.), Brownian-driven control systems (exact terminal-controllability in [20], approximate controllability in [3], [9], mean-field Brownian-driven systems in [12], infinite-dimensional setting in [8], [23], [1], [10], etc.), jump systems ([11], [13], etc.). We refer to [13] for more details on the literature as well as applications one can address using switch models.

The paper [13] provides some necessary and some sufficient conditions under which approximate controllability towards null target can be achieved. In all generality, the conditions are either too strong (sufficient) or too weak (only necessary). Equivalence is obtained in [13] for particular cases : (i) Poisson-driven systems with mode-independent coefficients and (ii) continuous switching. In the present paper, we extend the work of [13] and give explicit equivalence criterion for the general switching case. The approach relies, in a first step, as it has already been the case in [13, Theorem 1], on duality techniques (briefly presented in Subsection 2.1). However, the intuition on this new criterion and its proof are extensively based on the recent ideas in [4]. The dual backward stochastic system associated to controllability is interpreted as a system of (backward) ordinary differential equations in Proposition 12. Reasoning on this new system provides the necessary and sufficient criterion for approximate null-controllability for general switching systems with mode-dependent multiplicative noise (Theorem 6 whose proof relies on Propositions 13 and 14). As a by-product, we considerably simplify the proofs of [13, Criteria 3 and 4] (in Subsection 2.3). Second, we give some elements on the stronger notion of

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(general) approximate controllability. While the notions of approximate and approximate null-controllability are known to coincide for Poisson-driven systems with mode-independent coefficients, we give an example (Example 9) showing that this is no longer the case for general switching systems. Furthermore, we show that the condition exhibited in [13, Proposition 3] in connection to approximate null-controllability is actually sufficient for general approximate controllability (see Condition 10). The proof follows, once again, from the deterministic reduction inspired by [4]. The theoretical results are illustrated on a model derived from repressed bacterium operon (given in [19] and reduced in [5]).

We begin with presenting the problem, the standing assumptions and the main results: the duality abstract characterization in Theorem 2, the explicit criterion in Theorem 6. We give a considerably simplified proof of the results in [13] in Subsection 2.3. We discuss the difference between null and full approximate controllability in Subsection 2.4, Example 9 and give a sufficient criterion for the stronger notion of approximate controllability (Criterion 10). Section 3 focuses on an example derived from [19] (see also [5]). The proofs of the results and the technical constructions allowing to prove Theorem 6 are gathered in Section 4.

2 The Control System and Main Results

We briefly recall the construction of a particular class of pure jump, non explosive processes on a space Ω and taking their values in a metric space $(E, \mathcal{B}(E))$. Here, $\mathcal{B}(E)$ denotes the Borel σ-field of E. The elements of the space E are referred to as modes. These elements can be found in [7] in the particular case of piecewise deterministic Markov processes (see also [2]). To simplify the arguments, we assume that E is finite and we let $p \geq 1$ be its cardinal. The process is completely described by a couple (λ, Q), where $\lambda : E \longrightarrow \mathbb{R}_+$ and the measure $Q : E \longrightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ stands for the set of probability measures on $(E, \mathcal{B}(E))$ such that $Q(\gamma, \{\gamma\}) = 0$. Given an initial mode $\gamma_0 \in E$, the first jump time satisfies $\mathbb{P}^{0,\gamma_0}(T_1 \geq t) = \exp(-t\lambda(\gamma_0))$. The process $\Gamma_t := \gamma_0$, on $t < T_1$. The post-jump location γ^1 has $Q(\gamma_0, \cdot)$ as conditional distribution. Next, we select the inter-jump time $T_2 - T_1$ such that $\mathbb{P}^{0,\gamma_0}\left(T_2 - T_1 \geq t \ / \ T_1, \gamma^1\right) = \exp\left(-t\lambda\left(\gamma^1\right)\right)$ and set $\Gamma_t := \gamma^1$, if $t \in [T_1, T_2)$. The post-jump location γ^2 satisfies $\mathbb{P}^{0,\gamma_0}\left(\gamma^2 \in A \ / \ T_2, T_1, \gamma^1\right) = Q\left(\gamma^1, A\right)$, for all Borel set $A \subset E$. And so on. To simplify arguments on the equivalent ordinary differential system, following [4, Assumption (2.17)], we will assume that the system stops after a non-random, fixed number $M > 0$ of jumps i.e. $\mathbb{P}^{0,\gamma_0}(T_{M+1} = \infty) = 1$. The reader is invited to note (see Remark 5) that, for large M, the criteria given in the main result (Theorem 6) no longer depend on M (due to the finite dimension of the mode and state spaces).

We look at the process Γ under $\mathbb{P}^{0,\gamma_0}$ and denote by $\mathbb{F}^0$ the filtration $\left(\mathcal{F}_{[0,t]} := \sigma\{\Gamma_r : r \in [0,t]\}\right)_{t \geq 0}$. The predictable σ-algebra will be denoted by $\mathcal{P}^0$ and the progressive σ-algebra by $Prog^0$. As usual, we introduce the random measure q on $\Omega \times (0,\infty) \times E$ by setting $q(\omega, A) = \sum_{k \geq 1} 1_{\left(T_k(\omega), \Gamma_{T_k(\omega)}(\omega)\right) \in A}$, for all $\omega \in \Omega$, $A \in \mathcal{B}(0,\infty) \times \mathcal{B}(E)$. The compensated martingale measure is denoted by $\widetilde{q}$. (For our readers familiar with [13], we emphasize that the notation is slightly different, the counting measure q corresponds to p in the cited paper and the martingale measure $\widetilde{q}$ replaces q in the same reference. Further details on the compensator are given in Subsection 4.1.)

We consider a switch system given by a process $(X(t), \Gamma(t))$ on the state space $\mathbb{R}^N \times E$, for some $N \geq 1$ and the family of modes E. The control state space is assumed to be some Euclidian space $\mathbb{R}^d$, $d \geq 1$. The component $X(t)$ follows a controlled differential system depending on the hidden variable γ. We will deal with the following model (A is implicitly assumed to be 0 after the last jump).

$$dX_s^{x,u} = \left[A(\Gamma_s) X_s^{x,u} + Bu_s\right] ds + \int_E C(\Gamma_{s-}, \theta) X_{s-}^{x,u} \widetilde{q}(ds, d\theta), \ s \geq 0, \ X_0^{x,u} = x. \tag{1}$$

The operators $A(\gamma) \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times d}$ and $C(\gamma, \theta) \in \mathbb{R}^{N \times N}$, for all $\gamma, \theta \in E$. For linear operators, we denote by ker their kernel and by Im the image (or range) spaces. Moreover, the control process $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is an $\mathbb{R}^d$-valued, $\mathbb{F}^0-$ progressively measurable, locally square integrable process. The space of all such processes will be denoted by $\mathcal{U}_{ad}$ and referred to as the family of admissible control processes. The explicit structure of such processes can be found in [18, Proposition 4.2.1], for instance. Since the control process does not (directly) intervene in the noise term, the solution of the above system can be explicitly computed with $\mathcal{U}_{ad}$ processes instead of the (more usual) predictable processes.

2.1 The Duality Abstract Characterization of Approximate Null-Controllability

We begin with recalling the following approximate controllability concepts.

Definition 1 *The system (1) is said to be approximately controllable in time $T>0$ starting from the initial mode $\gamma_0\in E$, if, for every $\mathcal{F}_{[0,T]}$-measurable, square integrable $\xi\in\mathbb{L}^2\left(\Omega,\mathcal{F}_{[0,T]},\mathbb{P}^{0,\gamma_0};\mathbb{R}^N\right)$, every initial condition $x\in\mathbb{R}^N$ and every $\varepsilon>0$, there exists some admissible control process $u\in\mathcal{U}_{ad}$ such that $\mathbb{E}^{0,\gamma_0}\left[\left|X_T^{x,u}-\xi\right|^2\right]\leq\varepsilon$. The system (1) is said to be approximately null-controllable in time $T>0$ if the previous condition holds for $\xi=0$ ($\mathbb{P}^{0,\gamma_0}$-a.s.).*

At this point, let us consider the backward (linear) stochastic differential equation

$$\left\{\begin{array}{l} dY_t^{T,\xi}=\left[-A^*\left(\Gamma_t\right)Y_t^{T,\xi}-\int_E\left(C^*\left(\Gamma_t,\theta\right)+I\right)Z_t^{T,\xi}\left(\theta\right)\lambda\left(\Gamma_t\right)Q\left(\Gamma_t,d\theta\right)\right]dt+\int_E Z_t^{T,\xi}\left(\theta\right)q\left(dt,d\theta\right), \\ Y_T^{T,\xi}=\xi\in\mathbb{L}^2\left(\Omega,\mathcal{F}_{[0,T]},\mathbb{P}^{0,\gamma_0};\mathbb{R}^N\right). \end{array}\right. \tag{2}$$

Classical arguments on the controllability operators and the duality between the concepts of controllability and observability lead to the following characterization (cf. [13, Theorem 1]).

Theorem 2 ([13, Theorem 1]) *The necessary and sufficient condition for approximate null-controllability (resp. approximate controllability) of (1) with initial mode $\gamma_0\in E$ is that any solution $\left(Y_t^{T,\xi},Z_t^{T,\xi}\left(\cdot\right)\right)$ of the dual system (2) for which $Y_t^{T,\xi}\in\ker B^*$, $\mathbb{P}^{0,\gamma_0}\otimes Leb$ almost everywhere on $\Omega\times[0,T]$ should equally satisfy $Y_0^{T,\xi}=0$, $\mathbb{P}^{0,\gamma_0}-$almost surely (resp. $Y_t^{T,\xi}=0$, $\mathbb{P}^{0,\gamma_0}\otimes Leb-a.s.$).*

Remark 3 *Concerning the operator A, it is assumed to be a switched matrix but it could also depend on (t,Γ_t) or on all the times and marks prior to t. This is why, we implicitly assumed that $A=0$ after the last jump (M^{th}) occurs. Similar assertions are true for C (otherwise, the backward equation (2) should be written with the compensator $\widehat{q}$ replacing $\lambda\left(\Gamma_t\right)Q\left(\Gamma_t,d\theta\right)$.) The reader may also look at the end of Subsection 4.1.*

2.2 Main Result : An Iterative Invariance Criterion

Before stating the main result of our paper, we need the following invariance concepts (cf. [6], [22]).

Definition 4 *We consider a linear operator $\mathcal{A}\in\mathbb{R}^{N\times N}$ and a family $\mathcal{C}=\left(\mathcal{C}_i\right)_{1\leq i\leq k}\subset\mathbb{R}^{N\times N}$.*

(i) A set $V\subset\mathbb{R}^N$ is said to be $\mathcal{A}$- invariant if $\mathcal{A}V\subset V$.

(ii) A set $V\subset\mathbb{R}^N$ is said to be $(\mathcal{A};\mathcal{C})$- invariant if $\mathcal{A}V\subset V+\sum_{i=1}^k\operatorname{Im}\mathcal{C}_i$.

We construct a mode-indexed family of linear subspaces of $\mathbb{R}^N$ denoted by $\left(V_\gamma^{M,n}\right)_{0\leq n\leq M,\ \gamma\in E}$ by setting

$$\mathcal{A}^*\left(\gamma\right):=A^*\left(\gamma\right)-\int_E\left(C^*\left(\gamma,\theta\right)+I\right)\lambda(\gamma)Q(\gamma,d\theta)\text{ and }V_\gamma^{M,M}=\ker B^*, \tag{3}$$

for all $\gamma\in E$, and computing, for every $0\leq n\leq M-1$,

(4) $V_\gamma^{M,n}$ the largest $\left(\mathcal{A}^*\left(\gamma\right);\left[\left(C^*(\gamma,\theta)+I\right)\Pi_{V_\theta^{M,n+1}}:\theta\in E,\ Q\left(\gamma,\theta\right)>0\right]\right)$ – invariant subspace of $\ker B^*$.

Here, Π_V denotes the orthogonal projection operator onto the linear space $V\subset\mathbb{R}^N$. Whenever there is no confusion at risk, having fixed the maximal number of jumps $M\geq1$, we drop the dependency on M (i.e. we write V_γ^n instead of $V_\gamma^{M,n}$ for all $0\leq n\leq M$).

Remark 5 *(i) A simple recurrence argument shows that $V_\gamma^{M,n}\subset V_\gamma^{M,m}$, for every $0\leq n\leq m\leq M$. Furthermore, $V_\gamma^{M,M-n}=V_\gamma^{M',M'-n}$, for all $0\leq n\leq M\leq M'$. Moreover, since the dimension of $\ker B^*$ cannot exceed N, $V_\gamma^{M,0}=V_\gamma^{\min(M,N^p),0}$.*

(ii) This spaces do not depend on the choice of the controllability horizon $T>0$. Therefore, if the approximate (null-)controllability is described by these sets, it is independent of the time horizon.

The main result of the paper is the following.

Theorem 6 *The switch system (1) is approximately null-controllable (in time $T>0$) with γ_0 as initial mode, if and only if the generated set $V_{\gamma_0}^0$ reduces to $\{0\}$.*

The proof is postponed to Section 4. This proof uses the reduction of backward equations with respect to Marked point processes to a system of ordinary differential equations given in [4]. In order to formulate this system (see Proposition 12), we need to explain some concepts and notations in Subsection 4.1. To prove necessity of the condition, one uses convenient feedback controls and the equivalence between invariance and the concept of feedback invariance (see Proposition 13). Sufficiency (given by Proposition 14) follows from (time-) invariance of convenient linear subspaces with respect to ordinary differential dynamics.

2.3 Comparison With [13]

We begin with giving a different (and simpler) proof of (some of) the results in [13]. Besides the general (abstract) characterization of approximate and approximate null-controllability, explicit invariance criteria were given in two specific settings.

(i) In **the case without multiplicative noise** $C = 0$, one notes that the subspaces V_γ^n (for $0 \le n < M$) do not depend on n. They reduce, in fact, to the largest $\mathcal{A}^*(\gamma)$-invariant subspace of $\ker B^*$. Moreover, in this framework, $\mathcal{A}^*(\gamma)$-invariance and $A^*(\gamma)$-invariance coincide and Theorem 6 yields the following.

Criterion 7 ([13, Criterion 4]) *The system (1) is approximately null-controllable (with initial mode $\gamma_0 \in E$) if and only if the largest subspace of* $\ker B^*$ *which is $A^*(\gamma_0)$ - invariant is reduced to the trivial subspace $\{0\}$ for all $\gamma_0 \in E$.*

(ii) In **the case of Poisson-driven systems with mode-independent coefficients** A and C, one works with the mode-independent operator $\mathcal{A}^* := A^* - \int_E (C^*(\theta) + I) \lambda Q(d\theta)$. The reader familiar with [13, Criterion 3] will note that the necessary and sufficient criterion concerns a notion of strict invariance. We get the same condition provided the system has the possibility to stabilize (the maximal number of jumps $M \ge N+1$ is allowed to exceed the dimension of the state space). Moreover, without loss of generality, one assumes that E is the support of Q.

Criterion 8 ([13, Criterion 3]) *Let us assume that $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times d}$ are fixed and $C(\theta) \in \mathbb{R}^{N\times N}$, for all $\theta \in E$ and that $\lambda(\gamma) Q(\gamma, d\theta)$ is independent of $\gamma \in E$. Moreover, we assume that $M \ge N+1$. Then the associated system is approximately null-controllable if and only if the largest subspace $V_0 \subset \ker B^*$ which is $(A^*; [C^*(\theta) \Pi_{V_0} : \theta \in E])$-invariant is reduced to $\{0\}$.*

Proof. The reader will note that the V^n spaces in (4) no longer depend on $\gamma \in E$. They are obviously non-decreasing in n (see Remark 5). Since $\ker B^* \subset \mathbb{R}^N$, it follows that, provided that $M \ge N+1$, one has $V^0 = V^1$ (indeed, $V^0 \subset V^1 \subset ... \subset V^M = \ker B^*$ and, whenever $V^k = V^{k+1}$, for some $0 \le k \le M-1$, it follows (by definition), that $V^j = V^k$, for all $j \le k$. On the other hand, the inclusions cannot always be strict if $M \ge N+1$ by recalling that the dimension of $\ker B^*$ cannot exceed N). Moreover, this space V^0 is the largest subspace $V_0 \subset \ker B^*$ which is $(\mathcal{A}^*; [(C^*(\theta) + I) \Pi_{V_0} : \theta \in E])$-invariant which is the same as $(A^*; [C^*(\theta) \Pi_{V_0} : \theta \in E])$-invariant. The proof is complete by invoking Theorem 6. ■

2.4 Approximate or Approximate Null-Controllability

Using Riccati techniques, one proves (see [13, Criterion 3]) that, for Poisson-driven systems with mode-independent coefficients, approximate controllability and approximate null-controllability properties coincide. However, in the case of actual switching systems, the two notions have no reason to and do not coincide. This is illustrated by the following example.

Example 9 *We consider the space dimension $N = 4$, the control dimension $d = 2$, $E = \{0, 1\}$, a switching rate $\lambda = 1$ and a transition probability $Q(\gamma, 1-\gamma) = 1$, for $\gamma \in E$. Moreover, we consider, for $\gamma \in \{0,1\}$,*

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ A(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1+\gamma & 0 & 0 & 0 \\ 0 & 2-\gamma & 0 & 0 \end{pmatrix},\ C(\gamma, 1-\gamma) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \gamma & 0 & -1 & 0 \\ 0 & 1-\gamma & 0 & -1 \end{pmatrix}.$$

The reader is invited to note that $\ker B^* = span\{e^3, e^4\}$ (standard vectors of the basis of $\mathbb{R}^4$). Thus, simple computations yield $V_0^1 \subset span\{e^4\}$, $V_1^1 \subset span\{e^3\}$. Hence, $V_0^0 = V_1^0 = \{0\}$ and the system is approximately null-controllable starting from every initial mode (if $M \ge 2$). However, if one considers $\gamma_0 = 0$, assumes the mode can jump twice $M = 2$ and sets $\xi := 1_{T_1 \le T < T_2} e^3 - 1_{T_2 \le T} e^3$, then one easily notes that $(Y_t, Z_t) := \left(1_{T_1 \le t \le T, t < T_2} e^3 - 1_{T_2 \le t \le T} e^3, (1_{t \le T \wedge T_1} - 2 \times 1_{T_1 < t \le T}) e^3\right)$ obey the equation (2). To this purpose, it suffices to note that $A^*(\Gamma_t) Y_t + (C^*(\Gamma_t, 1-\Gamma_t) + I) Z_t = 0$ on $[0, T \wedge T_2]$. For every $u \in \mathcal{U}_{ad}$, Itô's formula (e.g. [15, Chapter II, Section 5, Theorem 5.1]) applied to the inner product $\left\langle X_\cdot^{0,u}, Y_\cdot \right\rangle$ on $[0,T]$ yields $\mathbb{E}^{0,0}\left[\left\langle X_T^{0,u}, \xi \right\rangle\right] = \mathbb{E}^{0,\gamma_0}\left[\int_0^T \langle u_t, B^* Y_t \rangle dt\right] = 0$. In particular, this implies that $\mathbb{E}^{0,0}\left[\left|X_T^{0,u} - \xi\right|^2\right] \ge \mathbb{E}^{0,0}\left[|\xi|^2\right] > 0$ and, thus, the system (1) is not approximately controllable (towards ξ).

In fact, the reader may note that the null-controllability property strongly depends on the initial mode (through the computation of $V^0_{\gamma_0}$ as last step). A sufficient criterion (already available in [13]) is that the largest subspace of $\ker B^*$ which is $(\mathcal{A}^*(\gamma_0);[(C^*(\gamma_0,\theta)+I)\Pi_{\ker B^*}:\theta\in E, Q(\gamma_0,\theta)>0])$ −invariant should be reduced to $\{0\}$. It turns out that asking this condition to hold true for all $\gamma_0\in E$ actually implies approximate controllability. (The proof is postponed to Section 4.)

Condition 10 *Let us assume that the largest* $(\mathcal{A}^*(\gamma);[(C^*(\gamma,\theta)+I)\Pi_{\ker B^*}:Q(\gamma,\theta)>0])$*-invariant subspace of* $\ker B^*$ *is reduced to* $\{0\}$*, for every* $\gamma\in E$*. Then, for every* $T>0$ *and every* $\gamma_0\in E$*, the system (1) is approximately controllable in time* $T>0$*.*

Remark 11 *The reader is invited to note that the notion of* $(\mathcal{A}^*(\gamma);[(C^*(\gamma,\theta)+I)\Pi_{\ker B^*}:Q(\gamma,\theta)>0])$ *-invariance and that of* $(A^*(\gamma);[(C^*(\gamma,\theta)+I)\Pi_{\ker B^*}:Q(\gamma,\theta)>0])$ *-invariance coincide for subspaces of* $\ker B^*$*. Second, according to [13, Criterion 3], the notions of approximate and approximate null-controllability coincide in the context of Poisson-driven systems with mode-independent coefficients. Then, a careful look at [13, Example 4] provides an example of system which is approximately controllable without satisfying the sufficient condition given before.*

3 Towards Applications

A model. We will explain how the previous method can be applied in the study of stochastic gene networks. To this purpose, we consider the following reaction system describing a repressed bacterium operon model introduced in [19].

$$D+R\overset{K_1}{\rightleftarrows}DR,\ D+RNAP\overset{K_2}{\rightleftarrows}DRNAP,\ DRNAP\overset{k_3}{\to}TrRNAP,\ TrRNAP\overset{k_4}{\to}RBS+D+RNAP$$
$$RBS\overset{k_5}{\to}\varnothing,\ RBS+Rib\overset{K_6}{\rightleftarrows}RibRBS,\ RibRBS\overset{k_7}{\to}ElRib+RBS,\ ElRib\overset{k_8}{\to}Protein$$
$$Protein\overset{k_9}{\to}FoldedProtein,\ Protein\overset{k_{10}}{\to}\varnothing,\ FoldedProtein\overset{k_{11}}{\to}\varnothing.$$

Partitioning and simplifying. The authors of [5] propose a partition of "species" according to which only $ElRib, Protein$ and $FoldedProtein$ are continuous. The averaging procedures in [5, Figure 4] simplify the model to

$$\begin{aligned}&D^{**}\overset{K_3^*}{\rightleftarrows}TrRNAP\overset{k_4^*}{\to}RBS^*\overset{k_5^*}{\to}\varnothing,\ RBS^*\overset{k_7^*}{\to}ElRib+RBS^*,\\&ElRib\overset{k_8}{\to}Protein,\ Protein\overset{k_9}{\to}FoldedProtein,\ Protein\overset{k_{10}}{\to}\varnothing,\ FoldedProtein\overset{k_{11}}{\to}\varnothing.\end{aligned}\tag{5}$$

Due to the conservation law of $[D,R,DR,RNAP,DRNAP,TrRNAP]$ one should have something like $D^{**}+TrRNAP\simeq 1$.

It is known ([5, Page 21]) that "RBS^* presents infrequent bursts of activity leading to rapid production of $ElRib$" and "RBS^* rapidly switches to 0 by the reaction $RBS^*\to\varnothing$". To take into account these elements and keep the conservation law, we proceed as follows :

(1) as RBS^* switches to 0, D^{**} will be reset to 1 (hence, $D^{**}+TrRNAP+RBS^*=1$);

(2) bursts (given by the reaction having k_7^* as speed) will be considered as part of the stochastic updating of the continuous species and will have null-mean (i.e. they will multiply the martingale measure generated by the mode switching mechanism). In our toy-model, as RBS^* switches to 1, stochastic bursts on $ElRib$ will affect (in multiplicative way) the synthesis of $Protein$ (i.e. the reaction $ElRib\overset{k_8}{\to}Protein$).

A toy mathematical system. The first condition leads to a mode space $E=\{e^1,e^2,e^3\}$ consisting of the standard vector basis of $\mathbb{R}^3$, with a jump intensity λ and a transition measure $\left(Q\left(e^i,\{e^j\}\right)=Q_{i,j}\right)_{1\le i,j\le 3}$ given by

$$\lambda(\gamma)=\left\langle\begin{pmatrix}k_3^*\\k_{-3}^*+k_4^*\\k_5^*\end{pmatrix},\gamma\right\rangle>0,\ \text{for all }\gamma\in E,\ Q=\begin{pmatrix}0&1&0\\\frac{k_{-3}^*}{k_{-3}^*+k_4^*}&0&\frac{k_4^*}{k_{-3}^*+k_4^*}\\1&0&0\end{pmatrix}.\tag{6}$$

We are going to assume that the positive reaction speeds k_7^*,k_8,k_9 and k_{11} depend on the mode γ (note that RBS^* is part of γ and intervenes to get $ElRib$) and, maybe, of external one-dimensional control parameters (temperature or catalysts). Since all the reactions concerning continuous components have one reactant, the resulting ODE will be linear (see [5, Eq. (28)]). A first order model for the control will give $dx_t = [A(\Gamma_t)x_t + Bu_t]\,dt$, where A is given by (7). Furthermore, in our toy model, let us assume that the external control focuses on regulation of *ElRib* (i.e. $B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t = e^1$). We add to that the bursts (see item (2) above) to finally get a (toy-)model of type (1) for which, for every $\gamma, \theta \in E$,

$$(7)\qquad B = e^1,\ A(\gamma) = \begin{pmatrix} -k_8(\gamma) & 0 & 0 \\ k_8(\gamma_t) & -k_9(\gamma) & 0 \\ 0 & k_9(\gamma_t) & -k_{11}(\gamma) \end{pmatrix},\ C(\gamma,\theta) = \begin{pmatrix} 0 & 0 & 0 \\ k_7^*(\gamma) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ k_7^*(\gamma) = 1_{e^3}(\gamma).$$

Approximate null-controllability. The largest subspace of $\ker B^*$ which is $\left(A^*\left(e^2\right) - \lambda\left(e^2\right)I; \Pi_{\ker B^*}\right)$-invariant reduces to *span* $\left(e^3\right)$ and the largest subspace of $\ker B^*$ which is $\left(A^*\left(e^1\right) - \lambda\left(e^1\right)I; \Pi_{span(e^3)}\right)$invariant is $\{0\}$ (recall that k_8 and k_9 are reaction speeds and, thus, are strictly positive and so is λ). Due to the structure of the transition measure Q, as soon as $M \geq 2$, the system is approximately null-controllable starting from e^1. Nevertheless, the space $\ker B^*$ being $\mathcal{A}^*\left(e^3\right) - \left(k_8\left(e^3\right) - k_5^*\right)\left(C^*\left(e^3, e^1\right) + I\right)$-invariant, constructions similar to Example 9 show that, provided e^3 is reachable in M jumps, the system is not approximately controllable.

4 Proof of the Results

4.1 Technical Preliminaries

Before giving the reduction of our backward stochastic equation to a system of ODE, we need to introduce some notations making clear the stochastic structure of several concepts : final data, predictable and càdlàg adapted processes and compensator of the initial random measure. The notations in this subsection follow the ordinary differential approach from [4]. Since we are only interested in what happens on $[0,T]$, we introduce a cemetery state $(\infty, \overline{\gamma})$ which will incorporate all the information after $T \wedge T_M$. It is clear that the conditional law of T_{n+1} given (T_n, Γ_{T_n}) is now composed by an exponential part on $[T_n \wedge T, T]$ and an atom at ∞. Similarly, the conditional law of $\Gamma_{T_{n+1}}$ given $(T_{n+1}, T_n, \Gamma_{T_n})$ is the Dirac mass at $\overline{\gamma}$ if $T_{n+1} = \infty$ and given by Q otherwise. Finally, under the assumption $\mathbb{P}^{0,\gamma_0}(T_{M+1} = \infty) = 1$, after T_M, the marked point process is concentrated at the cemetery state.

We set $\overline{E}_T := ([0,T] \times E) \cup \{(\infty, \overline{\gamma})\}$. For every $n \geq 1$, we let $\overline{E}_{T,n} \subset \left(\overline{E}_T\right)^{n+1}$ be the set of all marks of type $e = ((t_0, \gamma_0), ..., (t_n, \gamma_n))$, where

$$(8)\qquad t_0 = 0,\ (t_i)_{0 \leq i \leq n} \text{ are non-decreasing; } t_i < t_{i+1}, \text{ if } t_i \leq T;\ (t_i, \gamma_i) = (\infty, \overline{\gamma}), \text{ if } t_i > T,\ \forall 0 \leq i \leq n-1,$$

and endow it with the family of all Borel sets $\mathcal{B}_n$. For these sequences, the maximal time is denoted by $|e| := t_n$. Moreover, by abuse of notation, we set $\gamma_{|e|} := \gamma_n$. Whenever $T \geq t > |e|$, we set

$$(9)\qquad e \oplus (t,\gamma) := ((t_0,\gamma_0), ..., (t_n,\gamma_n), (t,\gamma)) \in \overline{E}_{T,n+1}.$$

By defining

$$(10)\qquad e_n := ((0,\gamma_0), (T_1, \Gamma_{T_1}), ..., (T_n, \Gamma_{T_n})),$$

we get an $\overline{E}_{T,n}-$valued random variable, corresponding to our mode trajectories.

The final data ξ is $\mathcal{F}_{[0,T]}-$measurable and, thus, for every $n \geq 0$, there exists a $\mathcal{B}_n/\mathcal{B}\left(\mathbb{R}^N\right)-$measurable function $\overline{E}_{T,n} \ni e \mapsto \xi^n(e) \in \mathbb{R}^N$ such that:

$$(11)\qquad \text{If } |e| = \infty, \text{ then } \xi^n(e) = 0. \text{ Otherwise, on } T_n(\omega) \leq T < T_{n+1}(\omega),\ \xi(\omega) = \xi^n(e_n(\omega)).$$

A càdlàg process Y continuous except, maybe, at switching times T_n is given by the existence of a family of $\mathcal{B}_n \otimes \mathcal{B}([0,T])/\mathcal{B}\left(\mathbb{R}^N\right)$-measurable functions y^n such that, for all $e \in \overline{E}_{T,n}$, $y^n(e,\cdot)$ is continuous on $[0,T]$ and constant on $[0, T \wedge |e|]$ and

$$(12)\qquad \text{If } |e| = \infty, \text{ then } y^n(e,\cdot) = 0. \text{ Otherwise, on } T_n(\omega) \leq t < T_{n+1}(\omega),\ Y_t(\omega) = y^n(e_n(\omega), t),\ t \leq T.$$

Similar, an $\mathbb{R}^N-$valued **$\mathbb{F}$-predictable process** Z defined on $\Omega \times [0,T] \times E$ is given by the existence of a family of $\mathcal{B}_n \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(E)/\mathcal{B}\left(\mathbb{R}^N\right)$ $-$measurable functions z^n satisfying

$$(13)\ \text{If } |e| = \infty, \text{ then } z^n(e,\cdot,\cdot) = 0. \text{ On } T_n(\omega) < t \leq T_{n+1}(\omega),\ Z_t(\omega,\gamma) = z^n(e_n(\omega), t, \gamma),\ \text{for } t \leq T,\ \gamma \in E.$$

To deduce the form of **the compensator**, one simply writes $\widehat{q}(\omega, dt, d\gamma) := \sum_{n\geq 0} \widehat{q}^n_{e_n(\omega)}(dt, d\gamma)\, 1_{T_n(\omega)<t\leq T_{n+1}(\omega)\wedge T}$ such that

$$
\left\{
\begin{array}{l}
\text{If } n \geq M, \text{ then } \widehat{q}^n_e(dt, d\gamma) = \delta_{\overline{\gamma}}(d\gamma)\,\delta_\infty(dt). \text{ If } n \leq M-1, \\
\widehat{q}^n_e(dt, d\gamma) := \lambda(\gamma_{|e|})Q(\gamma_{|e|}, d\gamma)1_{|e|<\infty, t\in[|e|,T]}Leb(dt) + \delta_{\overline{\gamma}}(d\gamma)\,\delta_\infty(dt)\,1_{(|e|<\infty, t>T)\cup|e|=\infty},
\end{array}
\right. . \tag{14}
$$

Let us now concentrate on the specific form of the jump contribution Z (to the BSDE (2)). We consider a càdlàg process Y continuous except, maybe, at switching times T_n. Then, as explained before, this can be identified with a family (y^n). We construct, for every $n \geq 0$,

$$
\widehat{y}^{n+1}(e, t, \gamma) := y^{n+1}(e \oplus (t, \gamma), t)\, 1_{|e|<t} \tag{15}
$$

and $Y_{T_{n+1}}$ can be obtained by simple integration of the previous quantity with respect to the conditional law of $(T_{n+1}, \Gamma_{T_{n+1}})$ knowing $\mathcal{F}_{T_n}$. Then, Z is given by $z^n(e, t, \gamma) := \widehat{y}^{n+1}(e, t, \gamma) - y^n(e, t)$.

The **coefficient** function $A(\Gamma_t)$ is adapted and can be seen as follows: if $|e| = \infty$, then $A = 0$; otherwise, one works with $A(\gamma_{|e|})$. Similar constructions hold true for C. In fact, the results of the present paper can be generalized to more general path-dependence of the coefficients.

4.2 Reduction to a System of Linear ODEs

We consider the family of (ordinary) differential equations

$$
\left\{
\begin{array}{l}
y^M(e_M(\omega), \cdot) = \xi^M(e_M(\omega)). \text{ For } n \leq M-1,\ y^n(e_n(\omega), T) = \xi^n(e_n(\omega)), \\
dy^n(e_n(\omega), t) = -A^*\left(\gamma_{|e_n(\omega)|}\right) y^n(e_n(\omega), t)\, dt \\
\qquad\qquad - \int_E \left(C^*\left(\gamma_{|e_n(\omega)|}, \theta\right) + I\right)\left(\widehat{y}^{n+1}(e_n(\omega), t, \theta) - y^n(e_n(\omega), t)\right)\widehat{q}^n_{e_n(\omega)}(dt, d\theta) \\
(= -\mathcal{A}^*\left(\gamma_{|e_n(\omega)|}\right) y^n(e_n(\omega), t)\, dt \\
- \sum_{\theta\in E} \lambda(\gamma_{|e_n(\omega)|})Q(\gamma_{|e_n(\omega)|}, \theta)\left(C^*\left(\gamma_{|e_n(\omega)|}, \theta\right) + I\right) y^{n+1}(e_n(\omega) \oplus (t, \theta), t)\, dt),
\end{array}
\right. \tag{16}
$$

where we have used the notation (3). The following result adapts [4, Lemma 7] to our case.

Proposition 12 *A càdlàg adapted process Y given by a family of functions (y^n) as in (12) is solution to (2) if and only if, for $\mathbb{P}$-almost all ω and all $0 \leq n \leq M$, it satisfies the system (16).*

The proof is quasi-identical to the one of [4, Lemma 7]. The only difference in our case is the presence of the term $-A^*\left(\gamma_{|e_n(\omega)|}\right) y^n(e_n(\omega), t)\, dt$ which is, of course, classical. The results of [4, Lemma 7] apply directly if one assumes that $\lambda(\gamma) > 0$ for all $\gamma \in E$ (that is if there exists no absorbing state). Otherwise, we actually get an ODE of type $dy^n(e_n(\omega), t) = -A^*\left(\gamma_{|e_n(\omega)|}\right) y^n(e_n(\omega), t)\, dt$.

4.3 An Iterative Invariance-Based Criterion (Proof of Theorem 6)

As already hinted in [13], the (approximate) controllability properties can be expressed with respect to invariance conditions. The equivalence between the dual (backward) stochastic equation (2) and the (backward) ordinary differential system (16) yields the following approximate controllability criterion.

Proposition 13 *If the system (1) is approximately null-controllable with γ_0 as initial mode, then the generated set $V^0_{\gamma_0}$ reduces to $\{0\}$.*

Proof. Using classical results on the different notions of invariance (e.g. [22, Theorem 3.2], see also [6, Lemma 4.6]), invariance is equivalent to feedback invariance. Thus, one gets the existence of a family of operators $F^n_{\gamma,\theta} \in \mathcal{L}\left(V^n_\gamma; V^{n+1}_\theta\right)$ such that $\left(\mathcal{A}^*(\gamma) + \sum_{\theta\in E,\ Q(\gamma,\theta)>0}\left(C^*(\gamma,\theta) + I\right)F^n_{\gamma,\theta}\right)V^n_\gamma \subset V^n_\gamma$, for all $n \geq 0$. We begin with picking (an arbitrary) $v_0 \in V^0_{\gamma_0}$ and define $\zeta^0(t_0, \gamma_0) = v_0$. We proceed by setting, for every $n \geq 1$ and $e_n \in \overline{E}_{T,n}$, ϕ_{n,e_n} to be the unique solution of the ordinary differential system

$$
\left\{
\begin{array}{l}
d\phi_{n,e_n}(t) = -\left(\mathcal{A}^*\left(\gamma_{|e_n|}\right) + \sum_{\theta\in E,\ Q(\gamma^n,\theta)>0}\left(C^*(\gamma_{|e_n|}, \theta) + I\right)F^n_{\gamma_{|e_n|},\theta}\right)\phi_{n,e_n}(t)\, dt, \quad |e_n| \leq t \leq T \\
\phi_{n,e_n}(|e_n|) = \left\{
\begin{array}{l}
\zeta^n(e_n), \text{ if } |e_n| < \infty, \\
0, \text{ otherwise.}
\end{array}
\right. \text{ and} \\
\zeta^{n+1}(e_n \oplus (t, \theta)) = \left\{
\begin{array}{l}
\frac{1}{\lambda(\gamma_{|e_n|})Q(\gamma_{|e_n|},\theta)} F^n_{\gamma_{|e_n|},\theta}\phi_n(e_n, t)\, 1_{T\geq t>|e_n|}, \text{ if } \lambda(\gamma_{|e_n|})Q(\gamma_{|e_n|}, \theta) > 0, \\
0, \text{ otherwise}
\end{array}
\right.
\end{array}
\right.
$$

One also sets $\phi_{n,e_n}(t) = \phi_{n,e_n}(|e_n| \vee t)$ to extend the solution for $t \in [0,T]$. Then, one easily notes that $\phi_{n,e_n}(t) \in \ker B^*$, for all $1 \le n \le M$, all $e_n \in \overline{E}_{T,n}$ and all $t \in [0,T]$. Moreover, a simple glance at the construction shows that by setting $y^n(e_n,t) := \phi_{n,e_n}(t)$, for $1 \le n \le M$, all $e_n \in \overline{E}_{T,n}$ and all $t \in [0,T]$, one gets the solution of (16) with the particular choice of the final data ξ such that $\xi^n(e_n) = \phi_{n,e_n}(T)$. Since we have assumed the system (1) to be approximately null-controllable, Theorem 2 and Proposition 12 yield $v_0 = 0$. The proof is complete by recalling that $v_0 \in V^0_{\gamma_0}$ is arbitrary. ∎

At this point, the reader may want to note that these considerations involve one equation at the time. The invariant space obtained is then employed for the next equation and gives a coherent character to the system. The basic idea is to provide some kind of local in time invariance of the sets concerned. In [13], this is done using Riccati techniques. But, except for special cases, the solvability of these stochastic schemes is far from obvious. Due to the ordinary differential structure of the equivalent system (16), we are able to elude these techniques and work directly on the deterministic systems.

Proposition 14 *Conversely, if the generated set $V^0_{\gamma_0}$ reduces to $\{0\}$, then the system (1) is approximately null-controllable with γ_0 as initial mode.*

Proof. We begin with a solution of (2) for which Y belongs to $\ker B^*$. We prove by descending recurrence that $y^n(e,t) \in V^n_{\gamma_{|e|}}$, for all $t \in [0,T]$ and all $e \in \overline{E}_{T,n}$ (starting from γ_0), where we use the structure (12). The assertion is obvious for $n = M$ since, by notation, $V^M_{\cdot} = \ker B^*$. We assume it to hold true for $n+1 \le M$ and prove it for $n \ge 0$. By equation (16), one has

$$dy^n(e,t) = \left(-\mathcal{A}^*\left(\gamma_{|e|}\right) y^n(e,t) - \sum_{\theta \in E} \lambda(\gamma_{|e|}) Q(\gamma_{|e|},\theta) \left(C^*\left(\gamma_{|e|},\theta\right) + I\right) y^{n+1}\left(e \oplus (t,\theta), t\right) \right) dt.$$

We have assumed that $y^n(e,t) \in \ker B^*$ and, thus, $[I - \Pi_{\ker B^*}] y^n(e,t) = 0$. We infer that

$$\mathcal{A}^*\left(\gamma_{|e|}\right) y^n(e,t) + \sum_{\theta \in E} \lambda(\gamma_{|e|}) Q(\gamma_{|e|},\theta) \left(C^*\left(\gamma_{|e|},\theta\right) + I\right) y^{n+1}\left(e \oplus (t,\theta), t\right) \in \ker B^*.$$

Hence, using the recurrence assumption, $y^n(e,t)$ is (for almost all $t \in [0,T]$), an element of the linear space

$$W^0 := \left\{ \begin{array}{c} v \in \ker B^* : \exists w^\theta \in V^{n+1}_\theta, \text{ for all } \theta \in E \text{ s.t. } Q\left(\gamma_{|e|},\theta\right) > 0 \text{ satisfying} \\ \mathcal{A}^*\left(\gamma_{|e|}\right) v + \sum\limits_{\theta \in E,\ Q\left(\gamma_{|e|},\theta\right) > 0} \left(C^*\left(\gamma_{|e|},\theta\right) + I\right) w^\theta \in \ker B^* \end{array} \right\}.$$

By repeating our argument, we prove that $y^n(e,t)$ is (for almost all $t \in [0,T]$), an element of the linear space

$$W^{m+1} := \left\{ \begin{array}{c} v \in W^m : \exists w^\theta \in V^{n+1}_\theta, \text{ for all } \theta \in E \text{ s.t. } Q\left(\gamma_{|e|},\theta\right) > 0 \text{ satisfying} \\ \mathcal{A}^*\left(\gamma_{|e|}\right) v + \sum\limits_{\theta \in E,\ Q\left(\gamma_{|e|},\theta\right) > 0} \left(C^*\left(\gamma_{|e|},\theta\right) + I\right) w^\theta \in W^m \end{array} \right\},$$

for every $m \ge 0$. Then, $W := \underset{0 \le m \le N}{\cap} W^m$ is an $\left(\mathcal{A}^*\left(\gamma_{|e|}\right); \left[\left(C^*\left(\gamma_{|e|},\theta\right) + I\right) \Pi_{V^{n+1}_\theta} : Q\left(\gamma_{|e|},\theta\right) > 0\right]\right)$−invariant subspace of the (at most N-dimensional) space $\ker B^*$. Therefore, we have proven that $y^n(e,t) \in V^n_{\gamma_{|e|}}$. To complete our argument, one only needs to recall that, by assumption, $V^0_{\gamma_0} = \{0\}$ and use Theorem 2 and Proposition 12. ∎

4.4 Proof of Sufficiency Condition 10 for Approximate Controllability

Proof of Condition 10. In light of the Theorem [13, Theorem 1] and Proposition 12, one only needs to show that the only solution of (16) remaining in $\ker B^*$ is constant 0. One proceeds as in the Proof of Proposition 14 starting with a solution of (2) for which Y belongs to $\ker B^*$ and showing that $y^n(e,t) \in V^n_{\gamma_{|e|}} \subset \ker B^*$, for all $t \in [0,T]$ (recall that y^n is continuous). One recalls that V^n_γ is $\left(\mathcal{A}^*(\gamma); \left[\left(C^*(\gamma,\theta) + I\right) \Pi_{V^{n+1}_\theta} : Q(\gamma,\theta) > 0\right]\right)$-invariant, for every $\gamma \in E$. Hence, a fortiori, $V^n_{\gamma_{|e|}}$ is $\left(\mathcal{A}^*\left(\gamma_{|e|}\right); \left[\left(C^*(\gamma_{|e|},\theta) + I\right) \Pi_{\ker B^*} : Q\left(\gamma_{|e|},\theta\right) > 0\right]\right)$-invariant. Our assumption implies that $V^n_{\gamma_{|e|}} = \{0\}$ and approximate controllability follows. ∎

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