# Functional Versions of Lp-Affine Surface Area and Entropy Inequalities

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# Functional versions of $L_p$-affine surface area and entropy inequalities. *

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**Abstract**

In contemporary convex geometry, the rapidly developing $L_p$-Brunn-Minkowski theory is a modern analogue of the classical Brunn-Minkowski theory. A central notion of this theory is the $L_p$-affine surface area of convex bodies. Here, we introduce a functional analogue of this concept, for log-concave and $s$-concave functions. We show that the new analytic notion is a generalization of the original $L_p$-affine surface area. We prove duality relations and affine isoperimetric inequalities for log-concave and $s$-concave functions. This leads to a new inverse log-Sobolev inequality for $s$-concave densities.

## 1 Introduction.

In recent years, functional versions of several isoperimetric type inequalities from convex geometry have been established [15, 25, 32, 35, 36, 46]. A natural extension of convexity theory is the study of log-concave functions. One of the most important discoveries in these recent investigations in this direction is the functional version of the famous Blaschke-Santaló inequality [3, 7, 20, 28]. Another important inequality from convex geometry is the affine isoperimetric inequality [8, 40, 42]. In [4], Artstein-Avidan, Klartag, Schütt and Werner obtained the functional form of this inequality for log-concave functions which turned out to be a reverse log-Sobolev inequality.

In this paper, we define analytic versions of several important geometric invariants and obtain new inequalities. For example, we introduce a functional analogue of $L_p$-affine surface area and establish an analytic inequality corresponding to the $L_p$ affine isoperimetric inequality. The $L_p$-affine surface area, a notion of the $L_p$-Brunn-Minkwoski theory, is an extension of affine surface area, the case $p = 1$, to all other $p \in \mathbb{R}$. It was first introduced for $p > 1$ by Lutwak in the groundbreaking paper [33] and extended later in [39] and [47] to all other $p$. For a convex body $K$ in $\mathbb{R}^n$ with the origin in its interior and real $p \neq -n$, it is, if the integral exists, defined as

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} \, d\mu_K(x), \tag{1}$$

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where $\partial K$ is the boundary of $K$, $N_K(x)$ is the outer unit normal vector at $x \in \partial K$ and $\kappa_K(x)$ is the (generalized) Gauss curvature in $x \in \partial K$. The $L_p$-affine surface area and its related inequalities play a major role in convex and differential geometry [2, 22, 24, 30, 31, 34, 43, 45, 47, 48, 50, 51, 52]. Therefore, it is desirable to have functional versions of this notion available. We provide exactly that and introduce a functional version of $L_p$-affine surface area for $s$-concave and log-concave functions.

In Theorem 2, we establish a duality relation for the analytic $L_p$-affine surface area of log-concave functions and deduce in Corollary 3 corresponding $L_p$-affine isoperimetric inequalities. Those are the counterparts to the ones that hold for convex bodies. In fact, we show that the $L_p$-affine isoperimetric inequalities for convex bodies can be obtained from the ones for log-concave functions. This is explained in Section 3.3.

Finally, we generalize the notion of $L_p$-affine surface area to $s$-concave functions for $s > 0$. We establish in Theorem 4 a duality relation which enables us to prove the corresponding $L_p$-affine inequalities and a reverse log-Sobolev inequality for $s$-concave functions.

The characterization of equality in the reverse log-Sobolev inequality of [4] had remained open in [4]. We start our paper by providing this equality characterization, along with a simple and short proof of the reverse log-Sobolev inequality of [4].

## 2 Equality characterization in the reverse log-Sobolev inequality.

We start by giving a short proof of the reverse log-Sobolev inequality for log concave functions due to Artstein, Klartag, Schütt and Werner [4]. We first recall the usual log-Sobolev inequality. Let $\gamma_n$ be the standard Gaussian measure on $\mathbb{R}^n$. The log-Sobolev inequality, due to Gross [23] (see also [17, 49]), asserts that for every probability measure $\mu$ on $\mathbb{R}^n$ that is absolutely continuous with respect to Lebesgue measure,

$$H\left(\mu \mid \gamma_n\right) \leq \frac{1}{2} I\left(\mu \mid \gamma_n\right),$$

where $H$ and $I$ denote the relative entropy and Fisher information, respectively,

$$H(\mu \mid \gamma_n) = \int_{\mathbb{R}^n} \log\left(\frac{d\mu}{d\gamma_n}\right) d\mu, \quad I(\mu \mid \gamma_n) = \int_{\mathbb{R}^n} \left|\nabla \log\left(\frac{d\mu}{d\gamma_n}\right)\right|^2 d\mu$$

and $|\cdot|$ is the Euclidean norm. It is well known (see for instance [6]) that this inequality can be slightly improved to

$$H(\mu \mid \gamma_n) \leq \frac{C(\mu)}{2} + \frac{n}{2}\log\Big(1 + \frac{I(\mu \mid \gamma_n) - C(\mu)}{n}\Big), \tag{2}$$

where

$$C(\mu) = \int_{\mathbb{R}^n} |x|^2 \, d\mu - n$$

is the gap between the second moment of $\mu$ and that of the Gaussian. The usual log-Sobolev inequality is recovered from (2), using the inequality $\log(1 + x) \leq x$. Inequality (2) can be written in a more concise way. Put $\psi = -\log(d\mu/dx)$ and let

$$S(\mu) = \int_{\mathbb{R}^n} \psi \, d\mu = -H(\mu \mid dx) = -H(\mu \mid \gamma_n) + \frac{C(\mu)}{2} + \frac{n}{2}\log(2\pi \mathrm{e})$$

be the Shannon entropy of $\mu$. Then $S(\gamma_n) = \frac{n}{2}\log(2\pi \mathrm{e})$ so that

$$H(\mu \mid \gamma_n) - \frac{C(\mu)}{2} = S(\gamma_n) - S(\mu).$$

Moreover one has

$$I(\mu \mid \gamma_n) = \int_{\mathbb{R}^n} |x - \nabla\psi(x)|^2 d\mu = C(\mu) + n + \int_{\mathbb{R}^n} \left(|\nabla\psi(x)|^2 - 2\langle x, \nabla\psi(x)\rangle\right) d\mu.$$

Hence inequality (2) is equivalent to

$$2\left(S(\gamma_n) - S(\mu)\right) \le n \log\left(\frac{2n - 2\int_{\mathbb{R}^n}\langle x, \nabla\psi(x)\rangle d\mu + \int_{\mathbb{R}^n}|\nabla\psi(x)|^2 d\mu}{n}\right).$$

If $\mathrm{e}^{-\psi}$ is $C^2$ on $\mathbb{R}^n$ and if $\lim_{x_i\to\pm\infty} x_i e^{-\psi} = 0$ and $\lim_{x_i\to\pm\infty} \frac{\partial\psi}{\partial x_i} e^{-\psi} = 0$ for all $1 \le i \le n$, then $\int_{\mathbb{R}^n}\langle x, \nabla\psi(x)\rangle d\mu = n$ and $\int_{\mathbb{R}^n}|\nabla\psi(x)|^2 d\mu = \int_{\mathbb{R}^n}\Delta\psi\, d\mu$ so that inequality (2) is equivalent to

$$2\left(S(\gamma_n) - S(\mu)\right) \le n \log\left(\frac{\int_{\mathbb{R}^n}\Delta\psi\, d\mu}{n}\right), \tag{3}$$

where $\Delta$ is the Laplacian.

Recall that a measure $\mu$ with density $e^{-\psi}$ with respect to the Lebesgue measure is called log-concave if $\psi\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a convex function. For such a convex function $\psi$ we define $\Omega_\psi$ to be the interior of the convex domain of $\psi$, that is

$$\Omega_\psi = \mathrm{int}\left(\{x \in \mathbb{R}^n, \psi(x) < +\infty\}\right).$$

In this paper, we always consider convex functions $\psi$ such that $\Omega_\psi \neq \emptyset$. We will use the classical Legendre transform of $\psi$,

$$\psi^*(y) = \sup_x\left(\langle x, y\rangle - \psi(x)\right). \tag{4}$$

In the general case, when $\psi$ is neither smooth nor strictly convex, the gradient of $\psi$, denoted by $\nabla\psi$, exists almost everywhere by Rademacher's theorem (see, e.g., [10]), and a theorem of Alexandrov [1] and Busemann and Feller [11] guarantees the existence of the Hessian, denoted by $\nabla^2\psi$, almost everywhere in $\Omega_\psi$. Recall also that

$$\psi(x) + \psi^*(y) \ge \langle x, y\rangle$$

for every $x, y \in \mathbb{R}^n$, with equality if and only if $x$ is in the domain of $\psi$ and $y \in \partial\psi(x)$, the sub differential of $\psi$ at $x$. In particular

$$\psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x)\rangle - \psi(x), \quad \text{a.e. in } \Omega_\psi. \tag{5}$$

More information about duality transforms of convex functions can be found in [37, 41, 44].

For log-concave measures the following reverse form of inequality (3) holds.

**Theorem 1.** *Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$ with density $\mathrm{e}^{-\psi}$ with respect to the Lebesgue measure. Then*

$$\int_{\mathbb{R}^n} \log\left(\det(\nabla^2\psi)\right) d\mu \le 2\left(S(\gamma_n) - S(\mu)\right). \tag{6}$$

*Equality holds if and only if $\mu$ is Gaussian (with arbitrary mean and positive definite covariance matrix).*

Inequality (6) is due to Artstein-Avidan, Klartag, Schütt and Werner [4] under additional smoothness assumptions. The equality conditions were also left open.

In the following we give a new simple and short proof of inequality (6) which allows for a complete characterization of the equality cases. Our proof is based on the functional form of the Blaschke-Santaló inequality [3, 7, 20, 28]. Let $f, g$ be non-negative integrable functions on $\mathbb{R}^n$ satisfying

$$f(x)g(y) \le \mathrm{e}^{-\langle x,y\rangle}, \quad \forall x, y \in \mathbb{R}^n.$$

If $f$ has its barycenter at 0, that is $\int_{\mathbb{R}^n} xf(x)dx = 0$, then

$$\Big(\int_{\mathbb{R}^n} f\, dx\Big) \times \Big(\int_{\mathbb{R}^n} g\, dx\Big) \le (2\pi)^n. \tag{7}$$

There is equality if and only if there exists a positive definite matrix $A$ and $C > 0$ such that, a.e. in $\mathbb{R}^n$,

$$f(x) = C\,\mathrm{e}^{-\langle Ax,x\rangle/2}, \quad g(y) = \frac{\mathrm{e}^{-\langle A^{-1}y,y\rangle/2}}{C}.$$

*Proof of Theorem 1.* Without loss of generality, we may assume that the function $\psi$ is lower semi-continuous. Both terms of the inequality are invariant under translations of the measure $\mu$, so we can assume that $\mu$ has its barycenter at 0. Then by the functional Santaló inequality above

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\psi^*}\, dx \le (2\pi)^n. \tag{8}$$

Let $\Omega_\psi, \Omega_{\psi^*}$ be the interiors of the domains of $\psi$ and $\psi^*$, respectively. If $\psi$ is $C^2$-smooth and strictly convex, then the map $\nabla\psi\colon \Omega_\psi \to \Omega_{\psi^*}$ is smooth and bijective. So by the change of variable formula,

$$\int_{\mathbb{R}^n} e^{-\psi^*(y)}\, dy = \int_{\Omega_{\psi^*}} e^{-\psi^*(y)}\, dy = \int_{\Omega_\psi} e^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x))\, dx. \tag{9}$$

As noted above, in the general case, the gradient $\nabla\psi$ and the Hessian $\nabla^2\psi$ of $\psi$ exist almost everywhere in $\Omega_\psi$ so that the right hand side of (9) is still well defined. Although it is clear (take $\psi(x) = |x|$ in $\mathbb{R}$) that this equality may fail in general, a result of McCann [37, Corollary 4.3 and Proposition A.1] shows that

$$\int_{\Omega_\psi} \mathrm{e}^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x))\, dx = \int_{X_{\psi^*}} \mathrm{e}^{-\psi^*(y)}\, dy, \tag{10}$$

where $X_{\psi^*}$ is the set of vectors of $\Omega_{\psi^*}$ at which $\nabla^2\psi^*$ exists and is invertible. Together with (8) we get

$$\int_{\Omega_\psi} \mathrm{e}^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x))\, dx \le (2\pi)^n.$$

By (5), the previous inequality thus becomes

$$\int_{\Omega_\psi} \mathrm{e}^{-\langle x,\nabla\psi(x)\rangle + \psi(x)} \det(\nabla^2\psi(x))\, dx \le (2\pi)^n,$$

which can be rewritten as

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\langle x, \nabla\psi(x)\rangle + 2\psi(x)} \det(\nabla^2\psi(x))\, d\mu \leq (2\pi)^n. \tag{11}$$

Taking the logarithm and using Jensen's inequality (recall that $\mu$ is assumed to be a probability measure) we obtain

$$-\int_{\mathbb{R}^n} \langle x, \nabla\psi(x)\rangle\, d\mu + 2S(\mu) + \int_{\mathbb{R}^n} \log\bigl(\det(\nabla^2\psi)\bigr)\, d\mu \leq n\log(2\pi).$$

We will need some version of the Gauss-Green (or Stokes) formula and refer to [14] as a general reference and for recent results on this subject. Let $v$ be the vector flow $v(x) = e^{-\psi(x)}x$. By the convexity and lower semi-continuity of $\psi$, $v$ is continuous and locally Lispchitz on $\overline{\Omega_\psi}$, the closure of $\Omega_\psi$. Assume first that $\Omega_\psi$ is bounded. Then by the Gauss-Green formula [16, 18], we have

$$\int_{\Omega_\psi} \mathrm{div}(v(x))dx = \int_{\partial\Omega_\psi} \langle v(x), N_{\Omega_\psi}(x)\rangle d\sigma_{\Omega_\psi},$$

where $N_{\Omega_\psi}(x)$ is an exterior normal to the convex set $\Omega_\psi$ at the point $x$ and $\sigma_{\Omega_\psi}$ is the Hausdorff measure restricted to $\partial\Omega_\psi$. Hence

$$\begin{aligned}\int_{\mathbb{R}^n} \langle x, \nabla\psi(x)\rangle\, d\mu &= \int_{\Omega_\psi} \langle x, \nabla\psi(x)\rangle e^{-\psi(x)} dx \\ &= \int_{\Omega_\psi} \mathrm{div}(x) e^{-\psi(x)} dx - \int_{\partial\Omega_\psi} \langle x, N_{\Omega_\psi}(x)\rangle \mathrm{e}^{-\psi(x)} d\sigma_{\Omega_\psi}.\end{aligned}$$

This formula also holds true for an unbounded domain $\Omega_\psi$ by a simple truncation argument and by the fast decay of log-concave integrable functions. Since $\Omega_\psi$ is convex, the barycenter 0 of $\mu$ is in $\Omega_\psi$. Thus $\langle x, N_{\Omega_\psi}(x)\rangle \geq 0$ for every $x \in \partial\Omega_\psi$ and $\mathrm{div}(x) = n$ hence

$$\int_{\mathbb{R}^n} \langle x, \nabla\psi(x)\rangle\, d\mu \leq n.$$

This finishes the proof of the inequality. Let us move on to the equality case. It is easily checked that there is equality in Theorem 1 for Gaussian measures. On the other hand, the above proof shows that if there is equality in (6), then there must be equality in (8). Thus, by the equality case of the functional Santaló inequality, $\mu$ is Gaussian. □

# 3 A functional $L_p$-affine surface area.

## 3.1 General theorems.

We first give a definition that generalizes the notion of $L_p$-affine surface area of convex bodies to a functional setting. Generalizations of a different nature were given in [12] and [13].

**Definition 1.** *For measurable $F_1, F_2 \colon \mathbb{R} \to (0, +\infty)$, $\lambda \in \mathbb{R}$ and a convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, let $X_\psi$ be the set of points of $\Omega_\psi$ at which its Hessian $\nabla^2\psi$ in the sense of Alexandrov is defined and invertible. We define*

$$as_\lambda(F_1, F_2, \psi) = \int_{X_\psi} \Bigl(F_1(\psi(x))\Bigr)^{1-\lambda} \Bigl(F_2(\langle x, \nabla\psi(x)\rangle - \psi(x))\Bigr)^{\lambda} \Bigl(\det \nabla^2\psi(x)\Bigr)^{\lambda} dx. \tag{12}$$

Since $\det(\nabla^2\psi(x)) = 0$ outside $X_\psi$, the integral may be taken on $\Omega_\psi$ for $\lambda > 0$. Definition 1 is motivated by two important facts. First, we prove that for a particular choice of $F_1$, $F_2$ and $\psi$ it coincides with the usual $L_p$-affine surface area of a convex body. This is the content of Theorem 3. Second, in the case of log-concave functions, for $F_1(t) = F_2(t) = e^{-t}$ the functional affine surface area $as_1(F_1, F_2, \psi)$ becomes

$$as_1(F_1, F_2, \psi) = \int_{X_\psi} e^{-\psi^*(\nabla\psi(x))}\det\nabla^2\psi(x)dx = \int_{\Omega_\psi} e^{-\psi^*(\nabla\psi(x))}\det\nabla^2\psi(x)dx$$

and is of particular interest. This is discussed in Subsection 3.2.

Our main result is the duality formula of Theorem 2. A special case is the identity (10) which was the starting point of the short proof of the reverse log-Sobolev inequality presented in Section 2.

Notice also that for any linear invertible map $A$ on $\mathbb{R}^n$, one has

$$as_\lambda(F_1, F_2, \psi \circ A) = |\det A|^{2\lambda-1}as_\lambda(F_1, F_2, \psi), \tag{13}$$

which corresponds to an $SL(n)$ invariance with a degree of homogeneity of $(2\lambda-1)$. This is easily checked using $\nabla_x(\psi \circ A) = A^t\nabla_{Ax}\psi$ and $\nabla_x^2(\psi \circ A) = A^t\nabla_{Ax}^2\psi A$.

We shall use Corollary 4.3 and Proposition A.1 of [37], where McCann showed a general change of variable formula, namely for every Borel function $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\int_{X_\psi} f(\nabla\psi(x))\det\nabla^2\psi(x)dx = \int_{X_{\psi^*}} f(y)dy. \tag{14}$$

The same holds true for every integrable function $f : \mathbb{R}^n \to \mathbb{R}$. Identity (14) is obvious when $\psi$ satisfies some regularity assumptions, like $C^2$. It suffices to make the change of variable $y = \nabla\psi(x)$. However, the proofs are more delicate in a general setting.

**Theorem 2.** *Let $\lambda \in \mathbb{R}$, let $F_1, F_2 \colon \mathbb{R} \to \mathbb{R}_+$ and let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex. If $\lambda < 0$ or $\lambda > 1$, assume moreover that $F_1 \circ \psi > 0$ on $X_\psi$ and $F_2 \circ \psi^* > 0$ on $X_{\psi^*}$. Then*

$$as_\lambda(F_1, F_2, \psi) = as_{1-\lambda}(F_2, F_1, \psi^*).$$

*Proof.* Without loss of generality, we can assume that $\psi$ is lower semi-continuous so that $\psi = (\psi^*)^*$. By (5),

$$as_\lambda(F_1, F_2, \psi) = \int_{X_\psi} (F_1 \circ \psi(x))^{1-\lambda}(F_2 \circ \psi^*(\nabla\psi(x)))^\lambda(\det\nabla^2\psi(x))^\lambda dx.$$

By Proposition A.1 in [37],

$$x = \nabla\psi^* \circ \nabla\psi(x) \ \text{ and } \ \nabla^2\psi^*(\nabla\psi(x)) = (\nabla^2\psi(x))^{-1}, \quad \forall x \in X_\psi,$$

so that $as_\lambda(F_1, F_2, \psi)$ is equal to

$$\int_{X_\psi} (F_1 \circ \psi \circ \nabla\psi^*(\nabla\psi(x)))^{1-\lambda}(F_2 \circ \psi^*(\nabla\psi(x)))^\lambda(\det\nabla^2\psi^*(\nabla\psi(x)))^{1-\lambda}\det\nabla^2\psi(x)dx.$$

Using (14), we obtain

$$as_\lambda(F_1, F_2, \psi) = \int_{X_{\psi^*}} (F_1 \circ \psi \circ \nabla\psi^*(y))^{1-\lambda}(F_2 \circ \psi^*(y))^\lambda(\det\nabla^2\psi^*(y))^{1-\lambda}dy.$$

Since $(\psi^*)^* = \psi$, we conclude the proof using (5) with $\psi^*$ instead of $\psi$. $\square$

**Corollary 1.** *The function $\lambda \mapsto \log(as_\lambda(F_1,F_2,\psi))$ is convex on $\mathbb{R}$. Moreover,*

$$\forall \lambda \in [0,1], \;\; as_\lambda(F_1,F_2,\psi) \le \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-\lambda} \left(\int_{X_{\psi^*}} F_2 \circ \psi^*\right)^{\lambda}.$$

*Equality holds trivially if $\lambda = 0$ and $\lambda = 1$. If $F_1 \circ \psi > 0$ on $X_\psi$ and $F_2 \circ \psi^* > 0$ on $X_{\psi^*}$, then*

$$\forall \lambda \notin [0,1], \;\; as_\lambda(F_1,F_2,\psi) \ge \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-\lambda} \left(\int_{X_{\psi^*}} F_2 \circ \psi^*\right)^{\lambda}.$$

*Proof.* The convexity of $\lambda \mapsto \log(as_\lambda(F_1,F_2,\psi))$ is a consequence of Hölder's inequality. For the inequalities we use Hölder's inequality and also the duality relation of Theorem 2 with $\lambda = 1$, $as_1(F_1,F_2,\psi) = as_0(F_2,F_1,\psi^*) = \int_{X_{\psi^*}} F_2 \circ \psi^*$. $\square$

We define the non-increasing function $F : \mathbb{R} \to \mathbb{R}_+$ by

$$F(t) = \sup_{\frac{t_1+t_2}{2} \ge t} \sqrt{F_1(t_1)F_2(t_2)}. \tag{15}$$

Notice that if $F_1 = F_2$ is a log-concave, non-increasing function then $F = F_1 = F_2$.

**Corollary 2.** *Let $F_1, F_2 \colon \mathbb{R} \to \mathbb{R}_+$ and let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. For $z \in \mathbb{R}^n$, let $\psi_z(x) = \psi(x+z)$. Then there exists $z \in \mathbb{R}^n$ such that*

$$\forall \lambda \in [0,1], \quad as_\lambda(F_1,F_2,\psi_z) \le \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda} \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-2\lambda}.$$

*Equality holds trivially if $\lambda = 0$. If $F_1 \circ \psi > 0$ on $X_\psi$ and $F_2 \circ \psi^* > 0$ on $X_{\psi^*}$ then*

$$\forall \lambda < 0, \quad as_\lambda(F_1,F_2,\psi_z) \ge \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda} \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-2\lambda}.$$

*If $F$ is decreasing, $\lambda \neq 0$ and $\int_{X_\psi} F_1 \circ \psi \neq 0$, then there is equality in each of these inequalities if and only if there exists $c \in \mathbb{R}_+$, $a \in \mathbb{R}$ and a positive definite matrix $A$ such that, for every $x \in \mathbb{R}^n$ and $t \ge 0$,*

$$\psi_z(x) = \langle Ax, x\rangle + a, \quad F_1(t+a) = c\, F(t) \quad \text{and} \quad F_2(t-a) = \frac{F(t)}{c}.$$

**Remark.** Notice that if $\psi$ is even then one may choose $z = 0$. Moreover, the inequality of Corollary 2, together with the duality relation of Theorem 2, yields another inequality which is stronger as the one of Corollary 2, when $\lambda \in [\frac{1}{2}, 1]$. Indeed, by Theorem 2 and Corollary 2, we have for all $\lambda \in [0,1]$ that

$$as_\lambda(F_1,F_2,\psi) = as_{1-\lambda}(F_2,F_1,\psi^*) \le \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2(1-\lambda)} \left(\int_{X^*_\psi} F_2 \circ \psi^* dx\right)^{2\lambda-1}. \tag{16}$$

If $\lambda \in [\frac{1}{2}, 1]$, we can now apply the functional Blaschke-Santaló inequality (20) and get that

$$as_\lambda(F_1, F_2, \psi) \le \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda} \left(\int_{X_\psi} F_1 \circ \psi dx\right)^{1-2\lambda},$$

which is the inequality of Corollary 2. Thus, for $\lambda \in [\frac{1}{2}, 1]$, (16) is as strong as the inequality of Corollary 2.

*Proof.* We recall a general form of the functional Blaschke-Santaló inequality [20, 29]. Let $f$ be a non-negative integrable function on $\mathbb{R}^n$. There exists $z_0 \in \mathbb{R}^n$ such that for every measurable $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and every $g : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$f(z_0 + x)g(y) \le \left(\rho^2(\langle x, y\rangle)\right), \tag{17}$$

for every $x, y \in \mathbb{R}^n$ with $\langle x, y\rangle > 0$, we have

$$\int_{\mathbb{R}^n} f\, dx \int_{\mathbb{R}^n} g\, dx \le \left(\int_{\mathbb{R}^n} \rho(|x|^2) dx\right)^2. \tag{18}$$

If $f$ is even, a result of Ball [7] asserts that one may choose $z_0 = 0$. Moreover, if there exists $g$ satisfying (17) and equality holds in (18), then there exists $c > 0$ and a positive definite matrix $T$, such that for every $x \in \mathbb{R}^n$,

$$f(z_0 + x) = c\rho\left(|Tx|^2\right) \quad \text{and} \quad g(y) = \frac{1}{c}\rho\left(|T^{-1}x|^2\right). \tag{19}$$

For $z \in \mathbb{R}^n$, let us denote $\psi_z^* = (\psi_z)^*$. Since $F$ is non-increasing, we have by (4), for every $x, y, z \in \mathbb{R}^n$ such that $\langle x, y\rangle > 0$,

$$F_1(\psi_z(x))F_2(\psi_z^*(y)) \le F^2\left(\frac{\psi_z(x) + \psi_z^*(y)}{2}\right) \le F^2\left(\frac{\langle x, y\rangle}{2}\right).$$

By the functional Blaschke-Santaló inequality there exists $z_0 \in \mathbb{R}^n$ such that

$$\left(\int F_1 \circ \psi\right)\left(\int F_2 \circ \psi_{z_0}^*\right) \le \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^2. \tag{20}$$

Applying Corollary 1 to $\psi_{z_0}$, we deduce that for $\lambda \in [0, 1]$,

$$\begin{aligned} as_\lambda(F_1, F_2, \psi_{z_0}) &\le \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-\lambda} \left(\int_{X_{\psi_{z_0}^*}} F_2 \circ \psi_{z_0}^*\right)^{\lambda} \\ &\le \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda} \left(\int_{X_\psi} F_1 \circ \psi\right)^{1-2\lambda}. \end{aligned}$$

For $\lambda < 0$ we deduce from (20) that

$$\left(\int_{X_\psi} F_1 \circ \psi\right)^{\lambda} \left(\int_{X_{\psi_{z_0}^*}} F_2 \circ \psi_{z_0}^*\right)^{\lambda} \ge \left(\int_{\mathbb{R}^n} F\left(\frac{|x|^2}{2}\right) dx\right)^{2\lambda}$$

and we conclude by using the second part of Corollary 1.

In order to characterize the equality case, we suppose that $\int_{X_\psi} F_1 \circ \psi \neq 0$ which means that the expressions in the inequality are not identically zero. For $\lambda \neq 0$, if there is equality in one of the inequalities of Corollary 2, it follows from the proof that we have equality in the functional Blaschke-Santaló inequality. Thus by (19), there exists $c > 0$ and an invertible linear map $T$, such that for every $x \in \mathbb{R}^n$,

$$F_1 \circ \psi_{z_0}(x) = c\ F\left(\frac{|Tx|^2}{2}\right) \quad \text{and} \quad F_2 \circ \psi_{z_0}^*(x) = \frac{1}{c} F\left(\frac{|T^{-1}x|^2}{2}\right).$$

Let us define $\varphi(x) = \psi(T^{-1}x + z_0)$. Then we have

$$F_1 \circ \varphi(x) = c\ F\left(\frac{|x|^2}{2}\right) \quad \text{and} \quad F_2 \circ \varphi^*(x) = \frac{1}{c} F\left(\frac{|x|^2}{2}\right). \tag{21}$$

Hence

$$F\left(\frac{|x|^2}{2}\right) = \sqrt{F_1 \circ \varphi(x) F_2 \circ \varphi^*(x)} \leq F\left(\frac{\varphi(x) + \varphi^*(x)}{2}\right) \leq F\left(\frac{|x|^2}{2}\right).$$

Since $F$ is decreasing, we deduce that $\varphi(x) + \varphi^*(x) = |x|^2$. It is a classical fact that this implies that $\varphi(x) = |x|^2/2 + a$. See for example the argument given in the proof of Theorem 8 in [20]. Defining $A = T^*T/2$, we get that $\psi_{z_0}(x) = \langle Ax, x\rangle + a$, for every $x \in \mathbb{R}^n$. From (21) we deduce that for every $t \geq 0$,

$$F_1(t + a) = c\ F(t) \quad \text{and} \quad F_2(t - a) = \frac{1}{c} F(t).$$

Thus all stated conditions are proved. Reciprocally, if these conditions are fulfilled, a simple computation shows that there is equality. □

### 3.2 Application for log-concave functions.

We define $F_1$ and $F_2$ on $\mathbb{R}$ by $F_1(t) = F_2(t) = e^{-t}$ . Then $F(t) = e^{-t}$ as well and we use the simplified notation

$$as_\lambda(\psi) = as_\lambda(e^{-t}, e^{-t}, \psi) = \int_{X_\psi} e^{(2\lambda-1)\psi(x) - \lambda\langle x, \nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx. \tag{22}$$

As before, we can replace $X_\psi$ by $\Omega_\psi$ for $\lambda > 0$. Observe that for the Euclidean norm $|\cdot|$,

$$as_\lambda\left(\frac{|\cdot|^2}{2}\right) = (2\pi)^{\frac{n}{2}}\ . \tag{23}$$

We call a real valued map $\Phi$ on the set of convex functions $\psi$ a valuation (see e.g., [4]), if

$$\Phi(\psi_1) + \Phi(\psi_2) = \Phi(\max(\psi_1, \psi_2)) + \Phi(\min(\psi_1, \psi_2)),$$

provided $\min(\psi_1, \psi_2)$ is convex. Then it is not difficult to see (see e.g., [12]) that $as_\lambda$ is a valuation and that it is homogeneous of degree $(2\lambda - 1)n$, since, by (13), for any linear invertible map $A$ on $\mathbb{R}^n$ and all convex $\psi$

$$as_\lambda(\psi \circ A) = |\det A|^{2\lambda-1} as_\lambda(\psi).$$

For convex bodies with the origin in their interiors, such upper semi-continuous valuations were characterized as $L_p$-affine surface areas in [30] and [31] which motivated us to call $as_\lambda(\psi)$ the $L_\lambda$-affine surface area of $\psi$. This is further justified by Theorem 3 of the next section (where we also recall the definition of $L_p$-affine surface area for convex bodies), and by the identity (31) of Section 4.

From Theorem 2 and Corollary 1 we get that $\lambda \mapsto \log(as_\lambda(\psi))$ is convex and that

$$\forall \lambda \in \mathbb{R}, \; as_\lambda(\psi) = as_{1-\lambda}(\psi^*). \tag{24}$$

The following isoperimetric inequalities are a direct consequence of Corollary 2 and a result of [28] which states that the Santaló point $z_0$ in the functional Blaschke-Santaló inequality (7) can be taken equal to 0 when $\int_{\mathbb{R}^n} xe^{-\psi(x)}dx = 0$ or $\int_{\mathbb{R}^n} xe^{-\psi^*(x)}dx = 0$.

**Corollary 3.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a convex function such that $\int_{\mathbb{R}^n} xe^{-\psi(x)}dx = 0$ or $\int_{\mathbb{R}^n} xe^{-\psi^*(x)}dx = 0$. Then*

$$\forall \lambda \in [0,1], \quad as_\lambda(\psi) \le (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi}\right)^{1-2\lambda},$$
$$\forall \lambda \in (-\infty,0], \quad as_\lambda(\psi) \ge (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi}\right)^{1-2\lambda}.$$

*Equality holds in both inequalities for $\lambda \neq 0$, if and only if there exists $a \in \mathbb{R}$ and a positive definite matrix $A$ such that $\psi(x) = \langle Ax, x\rangle + a$, for every $x \in \mathbb{R}^n$.*

**Remark.** *(i) To emphasize the isoperimetric character of these inequalities, note that with (23), the inequalities are equivalent to*

$$\forall \lambda \in [0,1], \quad \frac{as_\lambda(\psi)}{as_\lambda\left(\frac{|\cdot|^2}{2}\right)} \le \left(\frac{\int_{X_\psi} e^{-\psi}}{\int_{\mathbb{R}^n} e^{-\frac{|\cdot|^2}{2}}}\right)^{1-2\lambda}$$

*and*

$$\forall \lambda < 0, \quad \frac{as_\lambda(\psi)}{as_\lambda\left(\frac{|\cdot|^2}{2}\right)} \ge \left(\frac{\int_{X_\psi} e^{-\psi}}{\int_{\mathbb{R}^n} e^{-\frac{|\cdot|^2}{2}}}\right)^{1-2\lambda}.$$

*(ii) It follows from Corollary 3 and the functional Blaschke Santaló inequality that*

$$\forall \lambda \in [0,1/2], \quad as_\lambda(\psi)as_\lambda(\psi^*) \le (2\pi)^n.$$

There are several other direct consequences of Corollary 3 that should be noticed. As observed already, we have for every $\lambda > 0$,

$$as_\lambda(\psi) = \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx.$$

Since $\int_{X_\psi} e^{-\psi} \le \int_{\mathbb{R}^n} e^{-\psi}$ we deduce from Corollary 3 that for any $\lambda \in (0,1/2]$,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \le (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}. \tag{25}$$

This inequality also holds trivially for $\lambda = 0$. Moreover, by Theorem 2, we know that $as_\lambda(\psi) = as_{1-\lambda}(\psi^*)$. Since the inequalities of Corollary 3 are also valid when $\int_{\mathbb{R}^n} xe^{-\psi^*(x)}dx = 0$, we deduce from (25) that if $\lambda \in [1/2, 1]$,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx = as_\lambda(\psi)$$

$$= as_{1-\lambda}(\psi^*) \leq (2\pi)^{n(1-\lambda)} \left(\int_{\mathbb{R}^n} e^{-\psi^*}\right)^{2\lambda-1}.$$

By the functional Blaschke-Santaló inequality (see (20)), we know that $\int_{\mathbb{R}^n} e^{-\psi} \int_{\mathbb{R}^n} e^{-\psi^*} \leq (2\pi)^n$ and we conclude that for all $\lambda \in [1/2, 1]$,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \leq (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}.$$

For $\lambda < 0$ or $\lambda > 1$, an important case concerns $C^2$ strictly convex functions $\psi$. In such a situation $X_\psi = \Omega_\psi$ and $X_{\psi^*} = \Omega_{\psi^*}$ and we deduce from Corollary 2 that for all $\lambda < 0$,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \geq (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}.$$

For all $\lambda > 1$, we go back to Corollary 1 and deduce that

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx = as_\lambda(\psi) \geq \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi^*}\right)^{\lambda}.$$

By the asymptotic functional reverse Santaló inequality [21] (see also [27] in the even case), there exists a constant $c > 0$ such that $\int_{\mathbb{R}^n} e^{-\psi} \int_{\mathbb{R}^n} e^{-\psi^*} \geq c^n$. Therefore, for all $\lambda > 1$,

$$\int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \geq c^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}.$$

Thus, we have proved the following:

**Corollary 4.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a convex function such that $\int_{\mathbb{R}^n} xe^{-\psi(x)}dx = 0$ or $\int_{\mathbb{R}^n} xe^{-\psi^*(x)}dx = 0$. Then*

$$\forall\lambda \in [0,1], \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \leq (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}.$$

*Moreover, if $\psi \in C^2(\Omega_\psi)$ is strictly convex, then*

$$\forall\lambda < 0, \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \geq (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}$$

*and there exists an absolute constant $c > 0$ such that*

$$\forall\lambda > 1, \int_{\Omega_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \nabla^2\psi(x)\right)^\lambda dx \geq c^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{1-2\lambda}.$$

These are the complete analogues of the $L_p$-affine surface area inequalities from [33, 26, 47] which will be discussed in more detail in the next subsection.

### 3.3 The case of convex bodies.

We continue to study the case $F_1(t) = F_2(t) = e^{-t}$. Additionally, we consider the case of 2-homogeneous proper convex functions $\psi$, that is $\psi(\lambda x) = \lambda^2 \psi(x)$ for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. Such functions $\psi$ are necessarily (and this is obviously sufficient) of the form $\psi(x) = \|x\|_K^2/2$ for a certain convex body $K$ with 0 in its interior. Here, $\|\cdot\|_K$ is the gauge function of the convex body $K$,

$$\|x\|_K = \min\{\alpha \geq 0 : \ x \in \alpha K\} = \max_{y \in K^\circ} \langle x, y\rangle = h_{K^\circ}(x).$$

Differentiating with respect to $\lambda$ at $\lambda = 1$, we get

$$\langle x, \nabla \psi(x)\rangle = 2\psi(x).$$

Thus, for 2-homogeneous functions $\psi$, formula (22) further simplifies to

$$as_\lambda(\psi) = \int_{X_\psi} \left(\det \nabla^2 \psi(x)\right)^\lambda e^{-\psi(x)} dx. \tag{26}$$

The following theorem indicates why we call $as_\lambda(\psi)$ the $L_\lambda$-affine surface area of $\psi$. First we recall that for $p \in \mathbb{R}$, $p \neq -n$, the $L_p$-affine surface area of a convex body $K$ in $\mathbb{R}^n$ with the origin in its interior is defined [26, 33, 47] by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x). \tag{27}$$

Here, $N_K(x)$ is the outer unit normal at $x \in \partial K$, $\mu_K$ is the usual surface area measure on $\partial K$ and $\kappa_K(x)$ is the Gauss curvature at $x$. We denote by $(\partial K)_+$ the points of $\partial K$ where the Gauss curvature is strictly positive.

**Theorem 3.** *Let $K$ be a convex body in $\mathbb{R}^n$ containing the origin in its interior. For any $p \geq 0$, let $\lambda = \frac{p}{n+p}$. Then*

$$as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) = \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|}\ as_p(K).$$

*Moreover, if $(\partial K)_+$ has full measure in $\partial K$, then the same relation holds true for every $p \neq -n$.*

**Remark.** *For all $p$, $as_p(B_2^n) = n|B_2^n|$. Therefore, together with (23), Theorem 3 can be rewritten as*

$$\frac{as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)}{as_\lambda\left(\frac{|\cdot|^2}{2}\right)} = \frac{as_p(K)}{as_p(B_2^n)}.$$

*Proof.* We will use formula (26) for $\psi = \frac{\|\cdot\|_K^2}{2}$. By a result of Hug [26, Theorem 2.2], the function $\psi$ is twice differentiable at almost every point of $\partial K$ and we have that

$$\det\left(\nabla^2 \psi(x)\right) = \frac{\kappa_K(x)}{\langle N_K(x), x\rangle^{n+1}}.$$

Note in particular that $X_\psi$ coincides with the the radial extension of the points of $\partial K$ where the Gauss curvature is strictly positive, namely $(\partial K)_+$. Now we integrate in polar coordinates with respect to the normalized cone measure $\sigma_K$ of $K$. Thus, if we write $x = r\theta$, with $\theta \in \partial K$, then $dx = n|K| r^{n-1} dr d\sigma_K(\theta)$. We also use that the map $x \mapsto \det \nabla^2\psi(x)$ is 0-homogeneous. Therefore we obtain from (26),

$$\begin{aligned} as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) &= n|K|\int_0^{+\infty} r^{n-1}e^{\frac{-r^2}{2}}dr\int_{(\partial K)_+}\left(\det\nabla^2\psi(\theta)\right)^\lambda\, d\sigma_K(\theta)\\ &= (2\pi)^{\frac{n}{2}}\frac{|K|}{|B_2^n|}\int_{(\partial K)_+}\left(\frac{\kappa_K(x)}{\langle N_K(x),x\rangle^{n+1}}\right)^\lambda\, d\sigma_K(\theta). \end{aligned}$$

The relation between the normalized cone measure $\sigma_K$ and the Hausdorff measure $\mu_K$ on $\partial K$ is given by

$$d\sigma_K(x) = \frac{\langle x, N_K(x)\rangle d\mu_K(x)}{n|K|}.$$

Thus, with $\lambda = \frac{p}{n+p}$,

$$\begin{aligned} as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|}\int_{(\partial K)_+}\left(\frac{\kappa(x)}{\langle x,N_K(x)\rangle^{n+1}}\right)^\lambda\langle x,N_K(x)\rangle d\mu_K(x)\\ &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|}as_p(K), \end{aligned}$$

when $\lambda \in [0,1)$ or when $(\partial K)_+$ is of full measure in $\partial K$. □

Let us conclude this section with several observations. First, observe that

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|_K^2}{2}}dx = 2^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2}\right)|K|.$$

Combining this with Theorem 3 and Corollary 3, we recover the $L_p$-affine isoperimetric inequalities for convex bodies. Namely, for a convex body $K$ with the origin in its interior, we have for $\lambda \in [0,1)$, which corresponds to $p \in [0,\infty)$ ($\lambda$ and $p$ are related by $\lambda = \frac{p}{n+p}$),

$$\frac{as_p(K)}{as_p(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if $K$ is an ellipsoid. For $\lambda \in (-\infty, 0]$, which corresponds to $p \in (-n, 0]$, we use Corollary 4 and get that for any $C_2^+$ convex body $K$, i.e., $\partial K$ is $C^2$ with strictly positive Gauss curvature everywhere,

$$\frac{as_p(K)}{as_p(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if $K$ is an ellipsoid. If $\lambda \ge 1$, which corresponds to $p \in [-\infty, -n)$, then

$$c^{\frac{np}{n+p}}\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \le \frac{as_p(K)}{as_p(B_2^n)},$$

where $c$ is a universal constant. For $p \geq 1$ these inequalities were proved by Lutwak [33] and for all other $p$ by Werner and Ye [53].

Second, the functional definition $as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)$ and $as_p(K)$ may differ for $p < 0$. Indeed, if $\partial K \setminus (\partial K)_+$ has positive measure then $as_p(K) = +\infty$ while it can happen that the corresponding functional definition is finite. A simple example is the convex hull of the point $(-e_1)$ with the half unit sphere $\{\sum x_i^2 = 1, x_1 \geq 0\}$.

Note that $\left(\frac{\|\cdot\|_K^2}{2}\right)^* = \frac{\|\cdot\|_{K^\circ}^2}{2}$, where $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \forall x \in K\}$ is the polar body of $K$. Thus the functional duality relation (24) implies the identity

$$\forall \lambda \in \mathbb{R}, \ as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right) = as_{1-\lambda}\left(\frac{\|\cdot\|_{K^\circ}^2}{2}\right).$$

Together with Theorem 3 and for $\lambda = p/(n+p)$, we recover the classical duality relation

$$as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$$

for any $p > 0$. Moreover, this is also valid for any $p \neq -n$ when $(\partial K)_+$ has full measure in $\partial K$. This duality relation was proved in [26] for $p > 0$ and for all $p \neq -n$ in [53], under additional regularity assumption when $p < 0$.

# 4 The $L_p$-affine surface area for $s$-concave functions.

The purpose of this section is to generalize Definition 1, the functional version of $L_p$-affine surface area, to the context of $s$-concave functions for $s > 0$. One possibility is to consider $F_1(t) = F_2(t) = F^{(s)}(t) = (1 - st)_+^{1/s}$, where $a_+ = \max\{a, 0\}$. Since $F^{(s)}$ is log-concave and non-increasing, one has according to (15), $F = F^{(s)}$ and when $s \to 0$, it recovers the previous case of $F(t) = e^{-t}$. However, when $\psi$ is convex, $F \circ \psi$ and $F \circ \psi^*$ do not satisfy a nice duality relation. Therefore, instead of the Legendre duality, we use a different duality transform coming from the natural duality for $s$-concave functions studied in [3, 38].

## 4.1 The $s$-concave duality.

We need some additional notation to explain the definition. Let $s \in (0, +\infty)$ and $f : \mathbb{R}^n \to \mathbb{R}_+$. Following Borell [9], we say that $f$ is $s$-concave if for every $\lambda \in [0, 1]$ and all $x$ and $y$ such that $f(x) > 0$ and $f(y) > 0$,

$$f((1-\lambda)x + \lambda y) \geq \left((1-\lambda)f(x)^s + \lambda f(y)^s\right)^{1/s}.$$

Since $s > 0$, one may equivalently assume that $f^s$ is concave on its support. For the construction of the duality, we assume that $f$ is upper semi-continuous, $f(0) > 0$ and that $f$ is bounded. We denote this class of functions by $Conv_s^+(\mathbb{R}^n)$. Let $S_f$ be the convex set $\{x : f(x) > 0\}$. We define the $(s)$-Legendre dual of $f$ (see [3, 38]) as

$$f^\circ_{(s)}(y) = \inf_{x \in S_f} \frac{(1 - s\langle x, y\rangle)_+^{1/s}}{f(x)}.$$

Equivalently one may define a function $\psi$ on $S_f$ by

$$\psi(x) = \frac{1 - f^s(x)}{s}, \quad x \in S_f, \tag{28}$$

extend it by continuity to the closure of $S_f$ and by $+\infty$ outside the closure of $S_f$. We associate with it a new dual function $\psi^\star_{(s)}$ defined by

$$\psi^\star_{(s)}(y) = \sup_{x \in S_f} \frac{\langle x, y\rangle - \psi(x)}{1 - s\psi(x)} \tag{29}$$

where $y \in S_{f^\circ_{(s)}} = \frac{1}{s} S_f^\circ = \{\frac{z}{s} : \forall x \in S_f, \langle x, z\rangle < 1\}$. Since $f$ is $s$-concave, upper semi-continuous, $\psi$ is convex, lower semi-continuous. And $f > 0$ on $S_f$, hence $\psi^\star_{(s)}$ is well defined. The $(s)$-Legendre dual of $f$ is now given by

$$f^\circ_{(s)}(y) = \left(1 - s\psi^\star_{(s)}(y)\right)^{1/s}, \quad \forall y \in S_{f^\circ_{(s)}},$$

where $S_{f^\circ_{(s)}} = \{y : 1 - s\psi^\star_{(s)}(y) > 0\}$. We extend it by continuity at the boundary and by 0 outside the closure of $S_{f^\circ_{(s)}}$. It is done in such a way that $f^\circ_{(s)}$ is upper semi-continuous. Since $f \in Conv_s^+(\mathbb{R}^n)$ then $f^\circ_{(s)} \in Conv_s^+(\mathbb{R}^n)$. It is not difficult to see that as for the Legendre transform, $(f^\circ_{(s)})^\circ_{(s)} = f$ or equivalently that $(\psi^\star_{(s)})^\star_{(s)} = \psi$.

There is an implicit equation between the classical Legendre function $\psi^*$ and the $(s)$-Legendre function $\psi^\star_{(s)}$ given by

$$\forall y \in S_{f^\circ_{(s)}}, \ \left(1 - s\psi^\star_{(s)}(y)\right)\left(1 + s\psi^*\left(\frac{y}{1 - s\psi^\star_{(s)}(y)}\right)\right) = 1. \tag{30}$$

Our definition of the $L_\lambda$ affine surface area of an $s$-concave function is the following.

**Definition 2.** *For any $s > 0$, let $f$ be an $s$-concave function and $\psi$ be the convex function associated with it by (28). For any $\lambda \in \mathbb{R}$, let*

$$as_\lambda^{(s)}(\psi) = \frac{1}{1 + ns} \int_{X_\psi} \frac{(1 - s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)} \left(\det \nabla^2 \psi(x)\right)^\lambda}{(1 + s(\langle x, \nabla\psi(x)\rangle - \psi(x)))^{\lambda\left(n+\frac{1}{s}+1\right)-1}} \, dx.$$

Note that $as_\lambda^{(s)}$ does not correspond to Definition 1 for particular functions $F_1$ and $F_2$. As in the log-concave case, we call $as_\lambda^{(s)}$ the $L_\lambda$-affine surface area of an $s$-concave function $f$. This is motivated by two reasons. As in Theorem 2 , we prove in Theorem 4 a satisfactory duality relation, from which we deduce a reverse log-Sobolev inequality for $s$-concave measures. Moreover, in the case $s = 1/k > 0$ where $k$ is an integer, this functional affine surface area corresponds to an $L_p$-affine surface area of a convex body associated with $f$ in dimension $n + k$. Indeed, as in [3], this convex body $K_s(f)$ in $\mathbb{R}^{n+\frac{1}{s}}$ is given by

$$K_s(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{\frac{1}{s}} : \frac{x}{\sqrt{s}} \in S_f, \ |y| \le f^s\left(\frac{x}{\sqrt{s}}\right)\right\}.$$

The $L_\lambda$-affine surface area of $f$ is a multiple of the $L_p$-affine surface area of $K_s(f)$ with $p = \left(n + \frac{1}{s}\right) \ \frac{\lambda}{1-\lambda}$,

$$(1+ns)\ as_\lambda^{(s)}(\psi) = \frac{as_p\left(K_s(f)\right)}{s^{\frac{n}{2}}\,\mathrm{vol}_{\frac{1}{s}-1}\left(S^{\frac{1}{s}-1}\right)}. \tag{31}$$

Identity (31) follows from Proposition 5 in [12].
Finally, we note that, as it is the case for log-concave functions, the $L_\lambda$-affine surface area for $s$-concave functions is also invariant under the action of $SL(n)$ and is homogeneous.

**Theorem 4.** *Let $f \in Conv_s^+(\mathbb{R}^n)$ with associated convex function $\psi$. Let $\lambda \in \mathbb{R}$ then*

$$as_{1-\lambda}^{(s)}(\psi^\star_{(s)}) = as_\lambda^{(s)}(\psi).$$

*Proof.* Let us start with the case when $f$ is sufficiently smooth, say $\psi$ is twice continuously differentiable with strictly positive definite Hessian on $\Omega_\psi$. Then $X_\psi = \Omega_\psi$ and

$$as_\lambda^{(s)}(\psi) = \frac{1}{1+ns} \int_{\Omega_\psi} \frac{(1-s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)}\left(\det\nabla^2\psi(x)\right)^\lambda}{(1+s(\langle x, \nabla\psi(x)\rangle - \psi(x)))^{\lambda\left(n+\frac{1}{s}+1\right)-1}}\, dx. \tag{32}$$

A simple computation shows that the supremum in (29) is attained at the point $x \in S_f$ such that

$$y = \frac{1-s\langle x, y\rangle}{1-s\psi(x)}\, \nabla\psi(x) = (1-s\psi^\star_{(s)}(y))\nabla\psi(x).$$

From (30), we have

$$\frac{1}{1-s\psi^\star_{(s)}(y)} = 1 + s\psi^*\left(\frac{y}{1-s\psi^\star_{(s)}(y)}\right) = 1 + s\psi^*(\nabla\psi(x)). \tag{33}$$

Therefore, we have that the supremum in (29) is attained at the point $x \in S_f$, that is,

$$\psi^\star_{(s)}(y) = \frac{\langle x, y\rangle - \psi(x)}{1-s\psi(x)},$$

if and only if

$$y = \frac{\nabla\psi(x)}{1+s\psi^*(\nabla\psi(x))} = \frac{\nabla\psi(x)}{1+s(\langle\nabla\psi(x), x\rangle - \psi(x))}.$$

We define the change of variable

$$\frac{\nabla\psi(x)}{1+s(\langle\nabla\psi(x), x\rangle - \psi(x))} = T_\psi(x). \tag{34}$$

A straightforward computation shows that

$$d_x T_\psi = \frac{1}{1+s\psi^*(\nabla\psi(x))}\left(\mathrm{Id} - \frac{s}{1+s\psi^*(\nabla\psi(x))} x \otimes \nabla\psi(x)\right)\nabla^2\psi(x).$$

Since

$$\det\left(\mathrm{I}d - \frac{s}{1+s\psi^*(\nabla\psi(x))} x \otimes \nabla\psi(x)\right) = 1 - \frac{s}{1+s\psi^*(\nabla\psi(x))}\langle x, \nabla\psi(x)\rangle$$

we see that the Jacobian of $T_\psi$ at $x$ is given by

$$dy = |\det d_x T_\psi|\, dx = \frac{1 - s\psi(x)}{\left(1 + s(\langle \nabla\psi(x), x\rangle - \psi(x))\right)^{n+1}} \ \det\nabla^2\psi(x)\ dx. \tag{35}$$

As the duality $(\psi^\star_{(s)})^\star_{(s)} = \psi$ holds, we see that $T_\psi \circ T_{\psi^\star_{(s)}} = \mathrm{I}d$ and $T_{\psi^\star_{(s)}} \circ T_\psi = \mathrm{I}d$ from which it is easy to deduce that for $y = T_\psi(x)$,

$$\det\,(d_x T_\psi)\det\left(d_y T_{\psi^\star_{(s)}}\right) = 1. \tag{36}$$

We now make the change of variable $y = T_\psi(x)$ in formula (32). From (33) and the fact that $(\psi^\star_{(s)})^\star_{(s)} = \psi$, we have

$$\frac{1}{1 - s\psi^\star_{(s)}(y)} = 1 + s(\langle \nabla\psi(x), x\rangle - \psi(x)) \text{ and } \frac{1}{1 - s\psi(x)} = 1 + s(\langle \nabla\psi^\star_{(s)}(y), y\rangle - \psi^\star_{(s)}(y)).$$

Combining this with (35) and (36), we obtain

$$\det\nabla^2\psi(x)\left(\frac{1 - s\psi^\star_{(s)}(y)}{1 + s(\langle \nabla\psi^\star_{(s)}(y), y\rangle - \psi^\star_{(s)}(y)}\right)^{n+2} \det\nabla^2\psi^\star_{(s)}(y) = 1. \tag{37}$$

Consequently, with $y = T_\psi(x)$,

$$\begin{aligned}
(1+ns)\ as^{(s)}_\lambda(\psi) &= \int_{\Omega_\psi} \frac{(1 - s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)-1}\left(\det\nabla^2\psi(x)\right)^{\lambda-1}}{(1 + s(\langle x, \nabla\psi(x)\rangle - \psi(x)))^{(\lambda-1)(n+1)+\frac{\lambda}{s}-1}}\ |\det d_x T_\psi|\, dx \\
&= \int_{\Omega_{\psi^\star_{(s)}}} \frac{\left(1 - s\psi^\star_{(s)}(y)\right)^{(n+2)(1-\lambda)+(\lambda-1)(n+1)+\frac{\lambda}{s}-1}\left(\det\nabla^2\psi^\star_{(s)}(y)\right)^{1-\lambda}}{\left(1 + s(\langle y, \nabla\psi^\star_{(s)}(y)\rangle - \psi^\star_{(s)}(y))\right)^{(n+2)(1-\lambda)+\left(\frac{1}{s}-1\right)(1-\lambda)-1}} dy \\
&= \int_{\Omega_{\psi^\star_{(s)}}} \frac{\left(1 - s\psi^\star_{(s)}(y)\right)^{\lambda\left(\frac{1}{s}-1\right)}\left(\det\nabla^2\psi^\star_{(s)}(y)\right)^{1-\lambda}}{\left(1 + s(\langle y, \nabla\psi^\star_{(s)}(y)\rangle - \psi^\star_{(s)}(y))\right)^{\left(n+1+\frac{1}{s}\right)(1-\lambda)-1}} dy \\
&= (1+ns)\ as^{(s)}_{1-\lambda}(\psi^\star_{(s)}).
\end{aligned}$$

This concludes the proof in the smooth case.

For the general case, we need several observations. By (5), we have a.e. in $\Omega_\psi$,

$$(1 + s(\langle x, \nabla\psi(x)\rangle - \psi(x))) = 1 + s\psi^*(\nabla\psi(x)).$$

Therefore, we can use a result of Mc Cann [37], see (14), to get

$$\begin{aligned}
(1+ns)\ as^{(s)}_\lambda(\psi) &= \int_{X_\psi} \frac{(1 - s\psi(x))^{\left(\frac{1}{s}-1\right)(1-\lambda)}\left(\det\nabla^2\psi(x)\right)^{\lambda}}{(1 + s\psi^*(\nabla\psi(x)))^{\lambda\left(n+\frac{1}{s}+1\right)-1}}\ dx \\
&= \int_{X_{\psi^*}} \frac{(1 - s\psi(\nabla\psi^*(z)))^{\left(\frac{1}{s}-1\right)(1-\lambda)}\left(\det\nabla^2\psi^*(z)\right)^{1-\lambda}}{(1 + s\psi^*(z))^{\lambda\left(n+\frac{1}{s}+1\right)-1}}\ dz.
\end{aligned} \tag{38}$$

We want to make the change of variable $z = T(y) = \frac{y}{1-s\psi^\star_{(s)}(y)}$. Observe that T is an injective map on $\Omega_{\psi^\star_{(s)}}$. Indeed, let $y_1$ and $y_2$ so that $T(y_1) = T(y_2)$, that is

$$\frac{y_1}{1 - s\psi^\star_{(s)}(y_1)} = \frac{y_2}{1 - s\psi^\star_{(s)}(y_2)}.$$

From (30), we deduce that $1 - s\psi^\star_{(s)}(y_1) = 1 - s\psi^\star_{(s)}(y_2)$ hence we have $y_1 = y_2$. From (30), our change of variable $z = T(y)$ is equivalent to $y = \frac{z}{1+s\psi^*(z)}$. Therefore, a similar computation to (35) gives a.e. in $\Omega_{\psi^\star_{(s)}}$,

$$|\det d_yT| = \frac{1 + s(\psi^\star_{(s)})^*(\nabla\psi^\star_{(s)}(y))}{(1 - s\psi^\star_{(s)}(y))^{n+1}}. \tag{39}$$

Similarly to proposition A.1 in [37], $T$ maps $X_{\psi^\star_{(s)}}$ to $X_{\psi^*}$ and the Alexandrov derivatives satisfy

$$\left(\frac{1 - s\psi^\star_{(s)}(y)}{1 + s(\psi^\star_{(s)})^*(\nabla\psi^\star_{(s)}(y))}\right)^{n+2} \det\nabla^2\psi^\star_{(s)}(y) = \det\nabla^2\psi^*(z). \tag{40}$$

Since $(\psi^\star_{(s)})^\star_{(s)} = \psi$, we deduce from (30) that

$$\forall x \in S_f,\ (1 - s\psi(x))\left(1 + s(\psi^\star_{(s)})^*\left(\frac{x}{1 - s\psi(x)}\right)\right) = 1.$$

Using (30) and the definition of $T$, we get that a.e. in $\Omega_{\psi^*}$,

$$\frac{\nabla\psi^*(z)}{1 - s\psi(\nabla\psi^*(z))} = \nabla\psi^\star_{(s)}(y), \quad \text{for} \quad z = Ty$$

which shows that for $z = Ty$,

$$1 - s\psi(\nabla\psi^*(z)) = \frac{1}{1 + s(\psi^\star_{(s)})^*(\nabla\psi^\star_{(s)}(y))}. \tag{41}$$

We have almost all the tools in hand to make the change of variable $z = T(y)$ in (38). We compute

$$\int_{X_{\psi^*}} \frac{(1 - s\psi(\nabla\psi^*(z)))^{\left(\frac{1}{s}-1\right)(1-\lambda)} \left(\det\nabla^2\psi^*(z)\right)^{1-\lambda}}{(1 + s\psi^*(z))^{\lambda\left(n+\frac{1}{s}+1\right)-1}}\ dz$$

by an approximation argument. This will be to ensure that $T$ is a Lipschitz map so that we can use the area formula, see Theorem 3.2.3 in [19]. We have for any $y_1, y_2 \in X_{\psi^\star_{(s)}}$,

$$|Ty_1 - Ty_2| \le \frac{|y_1 - y_2|}{1 - s\psi^\star_{(s)}(y_1)} + \frac{s|y_2|}{(1 - s\psi^\star_{(s)}(y_1))(1 - s\psi^\star_{(s)}(y_2))} |\psi^\star_{(s)}(y_2) - \psi^\star_{(s)}(y_1)|. \tag{42}$$

Since $\psi^\star_{(s)}$ is convex on $S_{f^\circ_{(s)}}$ we deduce that for any $\varepsilon \in (0,1)$, it is Lipschitz (with a Lipschitz constant depending on $\varepsilon$) on the set of points in $S_{f^\circ_{(s)}}$ which are at distance at least $\varepsilon$ from the boundary of $S_{f^\circ_{(s)}}$. Let us denote by $\mathcal{Y}_\varepsilon$ the intersection of this set with $X_{\psi^\star_{(s)}}$. Hence we integrate on $z \in T(\mathcal{Y}_\varepsilon) \cap B(0,R) =: X^{\varepsilon,R}_{\psi^*}$ where $B(0,R)$ is a

Euclidean ball of radius $R$. And we will let $\varepsilon$ go to zero and $R$ go to infinity. Let $X^{\varepsilon,R}_{\psi^\star_{(s)}} := T^{-1}(X^{\varepsilon,R}_{\psi^*})$, the set of $y \in X_{\psi^\star_{(s)}}$ such that $z = Ty$ for $z \in X^{\varepsilon,R}_{\psi^*}$. We have $1 - s\psi^\star_{(s)}(0) > 0$ hence it is strictly positive on a neighborhood of the origin in $S_{f^\circ_{(s)}}$. And we deduce from the relation $z = Ty = \frac{y}{1-s\psi^\star_{(s)}(y)}$ that for any $y \in X^{\varepsilon,R}_{\psi^\star_{(s)}}$, $1 - s\psi^\star_{(s)}(y)$ is uniformly bounded from below by a positive constant. Since $f \in Conv_s^+(\mathbb{R}^n)$ hence $f^o_{(s)} \in Conv_s^+(\mathbb{R}^n)$ and $1 - s\psi^\star_{(s)}$ is bounded from above. Since $z \in B(0,R)$ and $y = z(1-s\psi^\star_{(s)}(y))$, we conclude that there exists $R' > 0$ such that for any $y \in X^{\varepsilon,R}_{\psi^\star_{(s)}}$, $|y| \le R'$. Moreover $\psi^\star_{(s)}$ is Lipschitz on $X^{\varepsilon,R}_{\psi^\star_{(s)}}$. Hence we conclude from (42) that $T$ is a Lipschitz map on $X^{\varepsilon,R}_{\psi^\star_{(s)}}$.

We can apply the area formula, see Theorem 3.2.3 in [19] and make the change of variable $z = T(y)$ in the following integral:

$$\int_{X^{\varepsilon,R}_{\psi^*}} \frac{(1 - s\psi(\nabla\psi^*(z)))^{\left(\frac{1}{s}-1\right)(1-\lambda)} \left(\det\nabla^2\psi^*(z)\right)^{1-\lambda}}{(1 + s\psi^*(z))^{\lambda\left(n+\frac{1}{s}+1\right)-1}} \, dz.$$

We deduce from (39), (40), (41) that it is equal to

$$\int_{X^{\varepsilon,R}_{\psi^\star_{(s)}}} \frac{\left(1 - s\psi^\star_{(s)}(y)\right)^{\lambda\left(\frac{1}{s}-1\right)} \left(\det\nabla^2\psi^\star_{(s)}(y)\right)^{1-\lambda}}{\left(1 + s(\langle y, \nabla\psi^\star_{(s)}(y)\rangle - \psi^\star_{(s)}(y))\right)^{\left(n+1+\frac{1}{s}\right)(1-\lambda)-1}} dy.$$

Letting $\varepsilon$ going to zero and $R$ going to infinity, we conclude from (38) that

$$(1 + ns)\; as^{(s)}_\lambda(\psi) = \int_{X_{\psi^\star_{(s)}}} \frac{\left(1 - s\psi^\star_{(s)}(y)\right)^{\lambda\left(\frac{1}{s}-1\right)} \left(\det\nabla^2\psi^\star_{(s)}(y)\right)^{1-\lambda}}{\left(1 + s(\langle y, \nabla\psi^\star_{(s)}(y)\rangle - \psi^\star_{(s)}(y))\right)^{\left(n+1+\frac{1}{s}\right)(1-\lambda)-1}} dy.$$

This finishes the proof of the duality relation in the general case. □

## 4.2 Consequences of the duality relation.

In this section, we assume that $f$ satisfies more regularity assumptions: $f$ is twice continuously differentiable on $S_f$, its Hessian is non zero on $S_f$, $\lim_{x\to\partial S_f} f^s(x) = 0$ and the origin belongs to the interior of $S_f$. With these assumptions, $X_\psi = S_f$ and $X_{\psi^\star_{(s)}} = S_{f^\circ_{(s)}}$ and we remark that the definition of $as^{(s)}_\lambda(\psi)$ implies that

$$as^{(s)}_0(\psi) = \int_{S_f} f(x)dx \quad \text{and} \quad as^{(s)}_1(\psi) = \int_{S_{f^\circ_{(s)}}} f^\circ_{(s)}(y)dy. \tag{43}$$

Indeed,

$$\begin{aligned} as^{(s)}_0(\psi) &= \frac{1}{1+ns} \int_{S_f} (1 - s\psi(x))^{\frac{1}{s}-1} \left(1 + s(\langle\nabla\psi(x), x\rangle - \psi(x))\right) \, dx \\ &= \frac{1}{1+ns} \int_{S_f} f(x) \left(1 - s\frac{\langle\nabla f(x), x\rangle}{f(x)}\right) \, dx = \int_{S_f} f(x)dx, \end{aligned} \tag{44}$$

where the last equality follows from Stokes formula and the fact that $\lim_{x\to\partial S_f} f^s(x) = 0$. The second relation follows from the duality relation proved in Theorem 4.

In a way similar to the proof of Corollary 1 and Theorem 1, it is possible to deduce from Theorem 4 some isoperimetric inequalities and a general reverse log-Sobolev inequality in the $s$-concave setting.

**Proposition 1.** *Let $f$ be an $s$-concave function which satisfies the regularity assumptions stated at the beginning of Section 4.2 and let $\psi$ be its associated convex function. Then*

$$\forall \lambda \in [0,1], \quad as_\lambda^{(s)}(\psi) \le \left(\int_{\mathbb{R}^n} f\ dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} f_{(s)}^\circ\ dx\right)^{\lambda};$$
$$\forall \lambda \notin [0,1], \quad as_\lambda^{(s)}(\psi) \ge \left(\int_{\mathbb{R}^n} f\ dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} f_{(s)}^\circ\ dx\right)^{\lambda}.$$

*Proof.* We use Hölder's inequality and (43) to prove the first inequality:

$$\begin{aligned} as_\lambda^{(s)}(\psi) &\le \frac{1}{1+ns}\Bigg[\left(\int_{\mathbb{R}^n} \left(1-s\psi(x)\right)^{\frac{1}{s}-1}\left(1-s\psi(x)+s\langle x,\nabla\psi(x)\rangle\right)\ dx\right)^{1-\lambda} \\ &\qquad \left(\int_{\mathbb{R}^n} \frac{\det\nabla^2\psi(x)}{\left(1-s\psi(x)+s\langle x,\nabla\psi(x)\rangle\right)^{n+\frac{1}{s}}}\ dx\right)^{\lambda}\Bigg] \\ &= \left(\int_{\mathbb{R}^n} f\ dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} f_{(s)}^\circ\ dx\right)^{\lambda}. \end{aligned}$$

Similarly, one can prove the second inequality. □

The next theorem establishes the reverse log-Sobolev inequality for $s$-concave functions. There, we put

$$d\mu = \left(1-s\psi(x)\right)^{\left(\frac{1}{s}-1\right)}\left(1+s(\langle\nabla\psi(x),x\rangle-\psi(x))\right)\ \frac{dx}{1+ns}.$$

If $\int_{\mathbb{R}^n} f(x)dx = 1$, then by (44), $\mu$ is a probability measure on $\mathbb{R}^n$. We let $S(\mu) = \int -\log\left(\frac{d\mu}{dx}\right)d\mu$ be the Shannon entropy of $\mu$.

**Theorem 5.** *Let $f$ be an $s$-concave function which satisfies the regularity assumptions stated at the beginning of Section 4.2 and let $\psi$ be its associated convex function. Assume moreover that $f$ is even and that $\int_{\mathbb{R}^n} f(x)dx = 1$. Then*

$$\begin{aligned} \int_{\mathbb{R}^n} \log\left(\det\left(\nabla^2\psi(x)\right)\right)d\mu &\le \int_{\mathbb{R}^n} \log\left(\left(1+s(\langle x,\nabla\psi(x)\rangle-\psi(x))\right)^{\frac{1}{s}+n}\right)d\mu - S(\mu) \\ &\quad + \log\left(\left(\frac{\pi}{s}\right)^n \frac{(1+ns)\left(\Gamma(1+\frac{1}{2s})\right)^2}{\left(\Gamma(1+\frac{n}{2}+\frac{1}{2s})\right)^2}\right). \end{aligned} \tag{45}$$

*There is equality if and only if there is a positive definite matrix $A$ such that $f(x) = c_0\left(1-s\,|Ax|^2\right)^{\frac{1}{2s}}$, where $c_0 = \left(\frac{\pi}{s}\right)^{-\frac{n}{2}}\left(\frac{\Gamma(1+\frac{1}{2s})}{\Gamma(1+\frac{n}{2}+\frac{1}{2s})}\right)^{-1}$.*

**Remark.** Since $S(\gamma_n) = \log (2\pi e)^{\frac{n}{2}}$, the right hand side of inequality (45) tends to $2\left[S(\gamma_n) - S(\mu)\right]$ for $s \to 0$ and we recover the inequality of Theorem 1.

*Proof.* The proof follows the line of the proof of Theorem 1 presented in Section 2. By definition (29) of $\psi^\star_{(s)}$, we have for all $x \in S_f$ and for all $y \in \frac{1}{s} S_f^\circ$ that

$$f(x) f^\circ_{(s)}(y) = (1 - s\psi(x))^{\frac{1}{s}} \; (1 - s\psi^\star_{(s)}(y))^{\frac{1}{s}} \leq (1 - s\langle x, y\rangle)^{\frac{1}{s}}.$$

We let $\rho(t) = (1 - st)_+^{\frac{1}{2s}}$. As $f \equiv 0$ outside $S_f$ and $f^\circ_{(s)} \equiv 0$ outside $\frac{1}{s} S^\circ_f$, the functions $f$ and $f^\circ_{(s)}$ satisfy the assumption (17) with $z_0 = 0$ because $f$ is even. It follows from (18) that

$$\left(\int_{\mathbb{R}^n} f dx\right)\left(\int_{\mathbb{R}^n} f^\circ_{(s)} dx\right) \leq \left(\int_{\mathbb{R}^n} (1 - s|x|^2)_+^{\frac{1}{2s}} dx\right)^2 = \left(\frac{\pi}{s}\right)^n \frac{\left(\Gamma(1 + \frac{1}{2s})\right)^2}{\left(\Gamma(1 + \frac{n}{2} + \frac{1}{2s})\right)^2}. \qquad (46)$$

By Theorem 4, we have $\int_{\mathbb{R}^n} f^\circ_{(s)} dx = as_0^{(s)}(\psi^\star_{(s)}) = as_1^{(s)}(\psi)$ which means that

$$\begin{aligned}
\int_{\mathbb{R}^n} f^\circ_{(s)} dx =& \frac{1}{1+ns} \int_{X_\psi} \frac{\det \nabla^2 \psi(x)}{\left(1 + s(\langle x, \nabla\psi(x)\rangle - \psi(x))\right)^{\left(n + \frac{1}{s}\right)}} \, dx \\
=& \frac{1}{1+ns} \int_{X_\psi} \frac{\det \nabla^2 \psi(x)}{\left(1 + s(\langle x, \nabla\psi(x)\rangle - \psi(x))\right)^{\left(n + \frac{1}{s}\right)}} \, \frac{dx}{d\mu(x)} \, d\mu(x).
\end{aligned}$$

Since $\int_{\mathbb{R}^n} f dx = 1$, $\mu$ is a probability measure and we get from Jensen's inequality

$$\begin{aligned}
\log\left(\int_{\mathbb{R}^n} f^\circ_{(s)} dx\right) \quad &\geq \quad S(\mu) - \log(1 + ns) + \int_{\mathbb{R}^n} \log\left(\det \nabla^2 \psi\right) d\mu \\
&- \quad \int_{\mathbb{R}^n} \log\left(\left(1 + s(\langle x, \nabla\psi(x)\rangle - \psi)\right)^{\frac{1}{s} + n}\right) d\mu.
\end{aligned}$$

Therefore, with (46) and as $\int_{\mathbb{R}^n} f dx = 1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \log\left(\det\left(\nabla^2 \psi\right)\right) d\mu \leq & \int_{\mathbb{R}^n} \log\left(\left(1 + s(\langle x, \nabla\psi(x)\rangle - \psi)\right)^{\frac{1}{s} + n}\right) d\mu - S(\mu) \\
& + \log(1 + ns) + \log\left(\left(\frac{\pi}{s}\right)^n \frac{\left(\Gamma(1 + \frac{1}{2s})\right)^2}{\left(\Gamma(1 + \frac{n}{2} + \frac{1}{2s})\right)^2}\right).
\end{aligned}$$

If equality holds in (45), then, in particular, equality holds in the Blaschke-Santaló inequality (46). It was proved in [20] that this happens if and only if, in our situation, $f(x) = c_0 \left(1 - s\,|Ax|^2\right)^{\frac{1}{2s}}$, for a positive definite matrix $A$ and where $c_0$ is chosen such that $\int_{\mathbb{R}^n} f dx = 1$. On the other hand, it is easy to see that equality holds in (45), when $f(x) = c\left(1 - s\,|Ax|^2\right)^{\frac{1}{2s}}$, for a positive definite matrix $A$ and a positive constant $c$. □

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