Small Ball Estimates for Quasi-Norms

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Abstract. This note contains two types of small ball estimates for random vectors in finite dimensional spaces equipped with a quasi-norm. In the first part, we obtain bounds for the small ball probability of random vactors under some smoothness assumptions on their density functions. In the second part, we obtain Littlewood-Offord type estimates for quasi-norms. This generalizes a result which was previously obtained in [FS07, RV09].

1. Introduction

Let $E = \big(\mathbb{R}^n, \|\cdot\|\big)$ be an n-dimensional space equipped with a quasi-norm $\|\cdot\|$, and let X be a random vector in E. The present note is concerned with small ball estimates of X, i.e., estimates of the form

$$\mathbb{P}\big(\|X\| \le t\big) \le \varphi(t), \tag{1.1}$$

where $\varphi(t) \to 0$ as $t \to 0$.

Estimates of the form (1.1) have been studied under different assumptions on E and X. One direction is the case when $E = \ell_2^n$, i.e., when $\|\cdot\| = |\cdot|_2$ is the Euclidean norm, and X is assumed to be log-concave or, more generally, κ-concave. Recall that a log-concave vector is a vector that satisfies that for every $A, B \subseteq \mathbb{R}^n$ and every $\lambda \in [0,1]$,

$$\mathbb{P}\big(X \in \lambda A + (1-\lambda)B\big) \ge \mathbb{P}\big(X \in A\big)^{\lambda} \cdot \mathbb{P}\big(X \in B\big)^{1-\lambda}.$$

For such vectors it was shown in [Pao12] that

$$\mathbb{P}\big(|X|_2 \le \sqrt{n}t\big) \le \big(Ct\big)^{C'\sqrt{n}},$$

and this result was later generalized in [AGL$^+$12] to κ-concave vectors.

Another direction which has been studied is the case when X is a gaussian vector, and $\|\cdot\|$ is a *general* norm. For example, in [LO05] it was shown that if X is a centered gaussian vector and $\|\cdot\|$ is a norm on $\mathbb{R}^n$ with unit ball K such that its n-dimensional gaussian measure,

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denoted $\gamma_n(K)$, is less than 1/2, then

$$\mathbb{P}\big(\|X\| \le t\big) \le \big(2t\big)^{\frac{\omega^2}{4}} \gamma_n(K),$$

where ω is the inradius of K. See also [LS01] for an earlier survey of the subject.

Finally, let us mention that small ball estimates play a rôle in other problems, such as invertibility of random matrices and convex geometry. See e.g. [RV09] and [PP13].

While the above results have a more geometric flavor, in the present note we will try to present a more analytic approach. For a random vector X, let ϕ_X be its characteristic function ,i.e.,

$$\phi_X(\xi) = \mathbb{E} \exp\big(i\langle \xi, X\rangle\big).$$

Recall the following result:

Theorem 1.1. [FG11, Theorem 3.1] *Assume that $\|\cdot\|$ is a quasi-norm on $\mathbb{R}^n$ with unit ball K. Then*

$$\mathbb{P}\big(\|X\| \le t\big) \le |K| \left(\frac{t}{2\pi}\right)^n \int_{\mathbb{R}^n} \big|\phi_X(\xi)\big| d\xi. \tag{1.2}$$

Theorem 1.1 says that one can obtain small estimates by estimating the L_1 norm of the characteristic function of the random vector. Moreover, one can consider a "smoothed" version of Theorem 1.1: consider instead of X the random vector $X + tG$, where G is a standard gaussian vector in $\mathbb{R}^n$ which is independent of X. Since $\|\cdot\|$ is a quasi-norm on $\mathbb{R}^n$, there exists a constant $C_K > 0$ such that

$$\|x + y\| \le C_K(\|x\| + \|y\|),\ x, y \in \mathbb{R}^n. \tag{1.3}$$

Therefore,

$$\begin{aligned}\mathbb{P}\big(\|X + tG\| \le 2C_K t\big) &\ge \mathbb{P}\big(\|X\| \le t \wedge \|G\| \le 1\big) \\ &= \mathbb{P}\big(\|X\| \le t\big) \cdot \mathbb{P}\big(\|G\| \le 1\big) \\ &= \mathbb{P}\big(\|X\| \le t\big) \cdot \gamma_n(K),\end{aligned}$$

where $\gamma_n(\cdot)$ is the n-dimensional gaussian measure. Thus, Theorem 1.1 implies

$$\mathbb{P}\big(\|X\| \le t\big) \le \frac{\mathbb{P}\big(\|X + tG\| \le 2C_K t\big)}{\gamma_n(K)} \le \frac{|K|}{\gamma_n(K)} \left(\frac{C_K t}{\pi}\right)^n \int_{\mathbb{R}^n} \big|\phi_{X+tG}(\xi)\big| d\xi.$$

Using the independence of X and G,

$$\mathbb{P}\big(\|X\| \le t\big) \le \frac{|K|}{\gamma_n(K)} \left(\frac{C_K t}{\pi}\right)^n \int_{\mathbb{R}^n} \big|\phi_X(\xi)\big| \phi_{tG}(\xi) d\xi$$
$$= \frac{|K|}{\gamma_n(K)} \left(C'_K t\right)^n \int_{\mathbb{R}^n} \big|\phi_X(\xi)\big| e^{-\frac{t^2|\xi|_2^2}{2}} d\xi. \tag{1.4}$$

Inequality (1.4) enables one to obtain small ball estimates in cases where (1.2) cannot be applied. We use it for two different sets of examples. In the first set of examples we consider continuous random vector under certain assumptions on their characteristic functions (which are nothing but the Fourier transform of their density functions). This is discussed in Section 2. In the second set of examples, we consider random vectors X of the form

$$X = \sum_{i=1}^{N} \alpha_i a_i,$$

where the a_i are fixed vectors in $\mathbb{R}^n$ and the α_i's are i.i.d. random vectors that satisfy a certain anti-concentration condition. This problems and its applications have been studied by many authors, first in the one dimensional case (i.e., when n = 1) and later in the multidimensional case. See [FS07, RV08, RV09, TV09a, TV09b, TV12, Ngu12, NV13] and the reference therein for more information on this subject. In the case $E = \ell_2^n$, the problem of finding a small ball estimate have been previously considered in [FS07, RV09] and is called a Littlewood-Offord type estimate. Here such an estimate is obtained for a general quasi-norm. This is discussed in Section 3.

Notation. In this note C, C', etc. always denote absolute constants. $\|\cdot\|$ denotes a quasi-norm with unit ball K. $|\cdot|_2$ denotes the euclidean norm on $\mathbb{R}^n$. $B(x,r)$ denotes the closed ball around x with radius r with respect to the euclidean norm. $\gamma_n(\cdot)$ denotes the n-dimensional gaussian measure.

2. Small ball estimates for continuous random vectors

In this section we consider continuous random vectors, i.e., vectors with density function f_X. For such vectors we have

$$\phi_X(\xi) = \mathbb{E} \exp\big(i\langle \xi, X\rangle\big) = \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} f_X(x) dx = \hat{f}(\xi).$$

We can rewrite (1.4) in the following way:

$$\mathbb{P}\big(\|X\| \le t\big) \le \frac{|K|}{\gamma_n(K)} \left(C'_K t\right)^n \int_{\mathbb{R}^n} \big|\hat{f}_X(\xi)\big| e^{-\frac{t^2|\xi|_2^2}{2}} d\xi.$$

This suggest that small ball estimates are related to weighted norms of $\hat{f}_X$ which are in turn known to be related to smoothness properties of f_X. First, we consider vectors with independent coordinates and later we prove small ball estimates in terms of Sobolev norms.

2.1. **Distributions with independent coordinates.** Let $f \in L_1(\mathbb{R})$ and define

$$\|f\|_{BV(\mathbb{R})} \stackrel{\text{def}}{=} \sup \left\{ \sum_{k=1}^{\infty} \left| f(x_{k+1}) - f(x_k) \right| : \{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R} \right\}.$$

We say that $f \in BV(\mathbb{R})$ is $\|f\|_{BV(\mathbb{R})} < \infty$. It is known that $|\xi| \cdot |\hat{f}(\xi)| \le \|f\|_{BV(\mathbb{R})}$. Thus, if $f \in BV(\mathbb{R}) \bigcap L_1(\mathbb{R})$ then

$$|\hat{f}(\xi)| \le \min\left(\frac{\|f\|_{BV(\mathbb{R})}}{|\xi|}, \|f\|_{L_1(\mathbb{R})} \right). \tag{2.1}$$

Using (2.1) we can prove the following theorem.

Theorem 2.1. *Assume that $X = (X_1, \dots, X_n)$ is a random vector with independent coordinates, such that $f_{X_i} \in BV(\mathbb{R})$. Let $t \le \min_{1 \le j \le n} \left\{ \frac{1}{\|f_{X_j}\|_{BV(\mathbb{R})}} \right\}$. Then*

$$\mathbb{P}(\|X\| \le t) \le \frac{|K|}{\gamma_n(K)} (C_K' t)^n \prod_{j=1}^{n} \left[\|f_{X_j}\|_{BV(\mathbb{R})} \log\left(\frac{e^4}{t\|f_{X_j}\|_{BV(\mathbb{R})}} \right) \right],$$

where C_K' is the constant from (1.4).

Proof. Since f_{X_j} is a density function, $\|f_{X_j}\|_{L_1(\mathbb{R})} = 1$. Also, since $f_{X_j} \in BV(\mathbb{R})$, we get by (2.1),

$$\phi_{X_j}(\xi) \le \min\left(\frac{\|f_{X_j}\|_{BV(\mathbb{R})}}{|\xi|}, 1 \right).$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}} \phi_{X_j}(\xi) e^{-\frac{t^2\xi^2}{2}} d\xi &\le \int_{|\xi| \le \|f_{X_j}\|_{BV(\mathbb{R})}} d\xi + \int_{|\xi| > \|f_{X_j}\|_{BV(\mathbb{R})}} \frac{\|f_{X_j}\|_{BV(\mathbb{R})}}{|\xi|} e^{-\frac{t^2\xi^2}{2}} d\xi \\
&= 2\|f_{X_j}\|_{BV(\mathbb{R})} + \|f_{X_j}\|_{BV(\mathbb{R})} \int_{|\xi| > t\|f_{X_j}\|_{BV(\mathbb{R})}} e^{-\frac{\xi^2}{2}} \frac{d\xi}{|\xi|} \\
&\le 2\|f_{X_j}\|_{BV(\mathbb{R})} + \|f_{X_j}\|_{BV(\mathbb{R})} \int_{|\xi| \in \left(t\|f_{X_j}\|_{BV(\mathbb{R})}, 1\right)} \frac{d\xi}{|\xi|} + \|f_{X_j}\|_{BV(\mathbb{R})} \int_{|\xi|>1} e^{-\frac{\xi^2}{2}} d\xi \\
&\le 4\|f_{X_j}\|_{BV(\mathbb{R})} + \|f_{X_j}\|_{BV(\mathbb{R})} \log\left(\frac{1}{t\|f_{X_j}\|_{BV(\mathbb{R})}} \right) \\
&= \|f_{X_j}\|_{BV(\mathbb{R})} \log\left(\frac{e^4}{t\|f_{X_j}\|_{BV(\mathbb{R})}} \right).
\end{aligned} \tag{2.2}$$

Now, by the independence of the coordinates, we get

$$\begin{aligned}\mathbb{P}\left(\|X\| \le t\right) &\overset{(1.4)}{\le} \frac{|K|}{\gamma_n(K)} \left(C'_K t\right)^n \int_{\mathbb{R}^n} \phi_X(\xi)\, e^{-\frac{t^2|\xi|_2^2}{2}} d\xi \\ &= \frac{|K|}{\gamma_n(K)} \left(C'_K t\right)^n \prod_{j=1}^n \left(\int_{\mathbb{R}} \phi_{X_j}(\xi)\, e^{-\frac{t^2\xi^2}{2}} d\xi \right) \\ &\overset{(2.2)}{\le} \frac{|K|}{\gamma_n(K)} \left(C'_K t\right)^n \prod_{j=1}^n \left(\|f_{X_j}\|_{BV(\mathbb{R})} \log\left(\frac{e^4}{t\|f_{X_j}\|_{BV(\mathbb{R})}} \right) \right).\end{aligned}$$

□

Note that when X is isotropic and log-concave, then by a result from [Fra97], we have that $\|f_X\|_\infty \le e^n f_X(0)$. If we assume in addition that X has independent coordinates, then we also have $f_X(0) = \prod_{j=1}^n f_{X_j}(0) \le C^n$, and so we get

$$\mathbb{P}\left(\|X\| \le t\right) \le |K| f_X(0)\, (et)^n \le |K|\, (Cet)^n.$$

2.2. **Small ball estimates and Sobolev norm.** Recall the definition of Sobolev norm: if $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$, then

$$\|f\|_{\beta,p} = \|f\|_{H_{\beta,p}(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{\beta/2} \hat{f} \right) \right\|_{L_p(\mathbb{R}^n)}. \tag{2.3}$$

Theorem 2.2. *Assume that X is a random vector in $\mathbb{R}^n$. Assume that $1 < p \le 2$. Then for every quasi-norm $\|\cdot\|$ on $\mathbb{R}^n$ with unit ball K,*

$$\mathbb{P}\left(\|X\| \le t\right) \le C_K'^n \frac{|K|}{\gamma_n(K)} \|f_X\|_{\beta,p} \cdot M(\beta,p,n,t) \tag{2.4}$$

If $pt^2 \le 2$, then

$$M(\beta,p,n,t) \le \begin{cases} 2^{\frac{n}{2p}-\frac{\beta}{2}} |\mathbb{S}^{n-1}|^{1/p} \Gamma\left(\frac{n-\beta p}{2}\right)^{1/p} p^{\frac{\beta}{2}-\frac{n}{2p}} \cdot t^{\beta+\frac{n}{p'}} & 2 < n-\beta p, \\ 2^{\frac{n}{2p}-\frac{\beta}{2}} |\mathbb{S}^{n-1}|^{1/p} \left(\log\left(\frac{2e}{pt^2}\right)\right)^{1/p} p^{\frac{\beta}{2}-\frac{n}{2p}} \cdot t^{\beta+\frac{n}{p'}} & 0 < \beta p < n-\beta \le 2, \\ |\mathbb{S}^{n-1}|^{1/p} \left(\log\left(\frac{2e}{pt^2}\right)\right)^{1/p} t^n & n-\beta p \le 0, \end{cases}$$

where $p' = p/(p-1)$. Otherwise, if $pt^2 \ge 2$, then

$$M(\beta,p,n,t) \le \begin{cases} |\mathbb{S}^{n-1}|^{1/p} \left(2e^{-\frac{pt^2}{18}} + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) \right)^{1/p} t^n & 2 \le pt^2 \le n-\beta p, \\ 3^{1/p} |\mathbb{S}^{n-1}|^{1/p} e^{-\frac{t^2}{18p}} t^n & n-\beta p \le pt^2 \le n, \\ |\mathbb{S}^{n-1}|^{1/p} \left(\frac{2n}{pt^2} \log\left(\frac{ept^2}{n}\right)\right)^{\frac{n}{2p}} t^n & n \le pt^2. \end{cases}$$

The main tool in the proof of Theorem 2.2 is the following lemma.

Lemma 2.3. *Let $p \in [1,2]$. If $pt^2 \le 2$ then*

$$\frac{\left\| (1+|\xi|^2)^{-\frac{\beta}{2}}\, e^{-\frac{t^2|\xi|^2}{2}} \right\|_{L_p(\mathbb{R}^n)}^p}{|\mathbb{S}^{n-1}|} \le \begin{cases} \Gamma\left(\frac{n-\beta p}{2}\right)\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} & \beta p < n-2 \\ \log\left(\frac{2e}{pt^2}\right)\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} & n-2 \le \beta p < n \\ \log\left(\frac{2e}{pt^2}\right) & \beta p \ge n. \end{cases}$$

Otherwise, if $pt^2 \ge 2$ then

$$\frac{\left\| (1+|\xi|^2)^{-\frac{\beta}{2}}\, e^{-\frac{t^2|\xi|^2}{2}} \right\|_{L_p(\mathbb{R}^n)}^p}{|\mathbb{S}^{n-1}|} \le \begin{cases} 2e^{-\frac{pt^2}{18}} + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}}\Gamma\left(\frac{n-\beta p}{2}\right) & 2 \le pt^2 \le n-\beta p, \\ 3e^{-\frac{pt^2}{18}} & n-\beta p \le pt^2 \le n, \\ \left(\frac{2n}{pt^2}\log\left(\frac{ept^2}{n}\right)\right)^{n/2} & n \le pt^2. \end{cases}$$

As part of the proof of Lemma 2.3, we need the following.

Proposition 2.4. *Assume that $x \ge \alpha \ge 1$. Then*

$$\int_x^\infty r^{\alpha-1}e^{-r}dr \le \frac{2^{\alpha+1}x^\alpha e^{-x}}{\alpha}.$$

Proof. We have

$$\begin{aligned} \int_x^\infty r^{\alpha-1}e^{-r}dr &= e^{-x}\int_0^\infty (u+x)^{\alpha-1}e^{-u}du \\ &= e^{-x}\left[\int_0^x (u+x)^{\alpha-1}e^{-u}du + \int_x^\infty (u+x)^{\alpha-1}e^{-u}du\right]. \end{aligned} \tag{2.5}$$

Now,

$$\int_0^x (u+x)^{\alpha-1}e^{-u}du \le \int_0^x (u+x)^{\alpha-1}du = \frac{x^\alpha\left(2^\alpha-1\right)}{\alpha} \le \frac{2^\alpha x^\alpha}{\alpha}.$$

For the second integral, since $x+u \le 2u$ we have

$$\int_x^\infty (u+x)^{\alpha-1}e^{-u}du \le 2^{\alpha-1}\int_x^\infty u^{\alpha-1}e^{-u}du.$$

Altogether, we get in (2.5),

$$\int_x^\infty r^{\alpha-1}e^{-r}dr \le \frac{2^\alpha x^\alpha e^{-x}}{\alpha} + 2^{\alpha-1}e^{-x}\int_x^\infty r^{\alpha-1}e^{-r}dr.$$

Since $x \ge \alpha$, we have, $2^{\alpha-1}e^{-x} \le 1/2$, which completes the proof. □

We can now proceed to the proof of Lemma 2.3

Proof of Lemma 2.3. To estimate the norm, notice that $(1+|\xi|^2)^{-\beta/2} \le \min\left(1, |\xi|^{-\beta}\right)$, and so using polar coordinates

$$\left\| \left(1+|\xi|^2\right)^{-\beta/2} e^{-\frac{t^2|\xi|_2^2}{2}} \right\|_{L_p(\mathbb{R}^n)}^p \le |\mathbb{S}^{n-1}| \int_0^\infty r^{n-1} \min\left(1, r^{-\beta p}\right) e^{-\frac{pt^2 r^2}{2}} dr$$

Now,

$$\begin{aligned}\int_0^\infty r^{n-1} \min\left(1, r^{-\beta p}\right) e^{-\frac{pt^2 r^2}{2}} dr &= \int_0^1 r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr + \int_1^\infty r^{n-1-\beta p} e^{-\frac{pt^2 r^2}{2}} dr \\ &= \int_0^1 r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr + \frac{1}{2}\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_{\frac{pt^2}{2}}^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr.\end{aligned} \tag{2.6}$$

Case 1: Assume $pt^2 \le 2$. To bound the first term in (2.6), use the assumption that $pt^2 \le 2$ and the trivial bound

$$\int_0^1 r^{n-1} e^{-r} dr \le \int_0^1 r^{n-1}\ dr = \frac{1}{n}. \tag{2.7}$$

To bound the second term, note first that

$$\int_{\frac{pt^2}{2}}^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr = \int_{\frac{pt^2}{2}}^1 r^{\frac{n-\beta p}{2}-1} e^{-r} dr + \int_1^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr. \tag{2.8}$$

Since $pt^2 \le 2$,

$$\int_{\frac{pt^2}{2}}^1 r^{\frac{n-\beta p}{2}-1} e^{-r} dr \le \int_{\frac{pt^2}{2}}^1 r^{\frac{n-\beta p}{2}-1} dr = \begin{cases} \frac{2}{n-\beta p}\left(1-\left(\frac{pt^2}{2}\right)^{\frac{n-\beta p}{2}}\right) & \beta p \ne n, \\ \log\left(\frac{2}{pt^2}\right) & \beta p = n, \end{cases} \tag{2.9}$$

and also

$$\int_1^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr \le \begin{cases} 1 & \frac{n-\beta p}{2} - 1 \le 0, \\ \Gamma\left(\frac{n-\beta p}{2}\right) & \frac{n-\beta p}{2} - 1 > 0. \end{cases} \tag{2.10}$$

Plugging (2.9) and (2.10) into (2.8), we get

$$\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_{\frac{pt^2}{2}}^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr \le \begin{cases} \frac{2}{n-\beta p}\left(\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} - 1\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) & \beta p < n-2, \\ \frac{2}{n-\beta p}\left(\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} - 1\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} & n-2 \le \beta p < n, \\ \log\left(\frac{2e}{pt^2}\right) & \beta p = n, \\ \frac{2}{\beta p - n}\left(1 - \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}}\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} & \beta p > n. \end{cases}$$

Now, if $a \geq 1$ then we have

$$\left|\frac{a^x - 1}{x}\right| \leq \begin{cases} a^x \log a & 0 < x \leq 1, \\ a^x & x \geq 1. \end{cases}$$

Thus, when $\beta p < n-2$ we have

$$\begin{aligned} \frac{2}{n-\beta p}\left(\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} - 1\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) &\leq \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \left(1 + \Gamma\left(\frac{n-\beta p}{2}\right)\right) \\ &\leq 2\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right). \end{aligned}$$

When $n-2 \leq \beta p < n$ we have

$$\begin{aligned} \frac{2}{n-\beta p}\left(\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} - 1\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} &\leq \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \log\left(\frac{2}{pt^2}\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \\ &= \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \log\left(\frac{2e}{pt^2}\right). \end{aligned}$$

Also, when $\beta p > n$ we use the fact that when $0 < a \leq 1$ and $x > 0$,

$$\frac{1-a^x}{x} \leq \log\left(\frac{1}{a}\right),$$

and get

$$\begin{aligned} \frac{2}{\beta p - n}\left(1 - \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}}\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \leq \log\left(\frac{2}{pt^2}\right) + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} &\leq \log\left(\frac{2}{pt^2}\right) + 1 \\ &= \log\left(\frac{2e}{pt^2}\right). \end{aligned}$$

Altogether,

$$\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_{\frac{pt^2}{2}}^{\infty} r^{\frac{n-\beta p}{2}-1} e^{-r} dr \leq \begin{cases} \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) & \beta p < n-2, \\ \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \log\left(\frac{2e}{pt^2}\right) & n-2 \leq \beta p < n, \\ \log\left(\frac{2e}{pt^2}\right) & \beta p \geq n. \end{cases}$$

Plugging this into (2.6) and using (2.7), we get

$$\int_0^\infty r^{n-1} \min\left(1, r^{-p\beta}\right) e^{-\frac{pt^2 r^2}{2}} dr \leq \begin{cases} \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) & \beta p < n-2 \\ \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \log\left(\frac{2e}{pt^2}\right) & n-2 \leq \beta p < n \\ \log\left(\frac{2e}{pt^2}\right) & \beta p \geq n, \end{cases}$$

which completes the proof in the case $pt^2 \le 2$.

Case 2: Assume $pt^2 \ge 2$**.** Assume also in this case that $n \ge 2$. To estimate the first term in (2.6), we consider two different cases. If $pt^2 \ge n$, choose

$$r_0 = \sqrt{\frac{n}{pt^2} \log\left(\frac{pt^2}{n}\right)},$$

and then

$$\begin{aligned}\int_0^1 r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr &= \int_0^{r_0} r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr + \int_{r_0}^1 r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr \le \int_0^{r_0} r^{n-1}\, dr + \int_{r_0}^1 r e^{-\frac{pt^2 r^2}{2}} dr \\ &\le \frac{r_0^n}{n} + \frac{1}{pt^2} e^{-\frac{pt^2 r_0^2}{2}} = \frac{1}{n}\left(\frac{n}{pt^2}\log\left(\frac{pt^2}{n}\right)\right)^{n/2} + \frac{1}{pt^2}\left(\frac{n}{pt^2}\right)^{n/2} \\ &\le \left(\frac{n}{pt^2}\log\left(\frac{ept^2}{n}\right)\right)^{n/2}.\end{aligned}$$

Otherwise, if $2 \le pt^2 \le n$, choose

$$r_0 = e^{-\frac{pt^2}{n}}.$$

Note that we have, say, $r_0 \ge 1/3$. Then, since $1 - e^{-x} \le x$,

$$\begin{aligned}\int_0^1 r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr &\le \frac{r_0^n}{n} + \frac{1}{pt^2}\left(e^{-\frac{pt^2 r_0^2}{2}} - e^{-\frac{pt^2}{2}}\right) \le \frac{1}{n} e^{-pt^2} + \frac{e^{-\frac{pt^2 r_0^2}{2}}}{pt^2} \cdot \frac{pt^2(1 - r_0^2)}{2} \\ &\le \frac{1}{n} e^{-pt^2} + \frac{pt^2}{n} e^{-\frac{pt^2}{18}} \le 2e^{-\frac{pt^2}{18}}.\end{aligned}$$

For the first term in (2.6) we thus have (assuming that $n \ge 2$),

$$\int_0^1 r^{n-1} e^{-\frac{pt^2 r^2}{2}} dr \le \begin{cases} \left(\frac{n}{pt^2}\log\left(\frac{ept^2}{n}\right)\right)^{n/2} & pt^2 \ge n, \\ 2e^{-\frac{pt^2}{18}} & pt^2 \le n. \end{cases} \tag{2.11}$$

If $n - \beta p \le 2$ then

$$\int_1^\infty r^{n-\beta p-1} e^{-\frac{pt^2 r^2}{2}} dr \le \int_1^\infty r e^{-\frac{pt^2 r^2}{2}} dr = \frac{2e^{-\frac{pt^2}{2}}}{pt^2} \le e^{-\frac{pt^2}{2}}. \tag{2.12}$$

Otherwise, if $n - \beta p \ge 2$, then again we consider two different cases. If $pt^2 \le n - \beta p$, we have

$$\begin{aligned}\frac{1}{2}\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_{\frac{pt^2}{2}}^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr &\le \frac{1}{2}\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_0^\infty r^{\frac{n-\beta p}{2}-1} e^{-r} dr \\ &= \frac{1}{2}\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right).\end{aligned} \tag{2.13}$$

Otherwise, suppose that we still have $n-\beta p \geq 2$, but now $pt^2 \geq n-\beta p$. Then by Proposition 2.4, we have

$$\frac{1}{2}\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_{\frac{pt^2}{2}}^{\infty} r^{\frac{n-\beta p}{2}-1} e^{-r} dr \leq \frac{2^{\frac{n-\beta p}{2}} e^{-\frac{pt^2}{2}}}{n-\beta p} \leq 2^{\frac{n-\beta p}{2}-1} e^{-\frac{pt^2}{2}}. \tag{2.14}$$

Altogether, combining (2.12), (2.13) and (2.14), we obtain

$$\frac{1}{2}\left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \int_{\frac{pt^2}{2}}^{\infty} r^{\frac{n-\beta p}{2}-1} e^{-r} dr \leq \begin{cases} 2^{\frac{n-\beta p}{2}} e^{-\frac{pt^2}{2}} & 2 \leq n-\beta p \leq pt^2, \\ \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) & 2 \leq pt^2 \leq n-\beta p, \\ e^{-\frac{pt^2}{2}} & n-\beta p \leq 2 \leq pt^2. \end{cases} \tag{2.15}$$

Plugging (2.11) and (2.15) into (2.6) gives

$$\int_0^\infty r^{n-1} \min\left(1, r^{-\beta p}\right) e^{-\frac{pt^2 r^2}{2}} dr \leq \begin{cases} 2e^{-\frac{pt^2}{18}} + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) & 2 \leq pt^2 \leq n-\beta p, \\ 2e^{-\frac{pt^2}{18}} + 2^{\frac{n-\beta p}{2}} e^{-\frac{pt^2}{2}} & 2 \leq n-\beta p \leq pt^2 \leq n, \\ \left(\frac{n}{pt^2} \log\left(\frac{ept^2}{n}\right)\right)^{n/2} + 2^{\frac{n-\beta p}{2}} e^{-\frac{pt^2}{2}} & 2 \leq n-\beta p \leq n \leq pt^2, \\ 2e^{-\frac{pt^2}{18}} + e^{-\frac{pt^2}{2}} & n-\beta p \leq 2 \leq pt^2 \leq n, \\ \left(\frac{n}{pt^2} \log\left(\frac{ept^2}{n}\right)\right)^{n/2} + e^{-\frac{pt^2}{2}} & n-\beta p \leq 2 \leq n \leq pt^2. \end{cases}$$

In order to simplify the last expression, first notice that when $n \leq pt^2$, we have

$$e^{-\frac{pt^2}{2}} \leq \left(\frac{n}{pt^2} \log\left(\frac{ept^2}{n}\right)\right)^{n/2}.$$

Also, we have that whenever $pt^2 \geq n-\beta p \geq 2$, since we have that $1-\log 2 > 1/4$ we get the following estimate,

$$2^{\frac{n-\beta p}{2}} e^{-\frac{pt^2}{2}} \leq e^{-\frac{pt^2}{2}(1-\log 2)} \leq e^{-\frac{pt^2}{8}} \leq e^{-\frac{pt^2}{18}}.$$

Hence, we conclude that,

$$\int_0^\infty r^{n-1} \min\left(1, r^{-\beta p}\right) e^{-\frac{pt^2 r^2}{2}} dr \leq \begin{cases} 2e^{-\frac{pt^2}{18}} + \left(\frac{2}{pt^2}\right)^{\frac{n-\beta p}{2}} \Gamma\left(\frac{n-\beta p}{2}\right) & 2 \leq pt^2 \leq n-\beta p, \\ 3e^{-\frac{pt^2}{18}} & n-\beta p \leq pt^2 \leq n, \\ \left(\frac{2n}{pt^2} \log\left(\frac{ept^2}{n}\right)\right)^{n/2} & n \leq pt^2. \end{cases}$$

□

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. we have,

$$\int_{\mathbb{R}^n} |\hat{f}_X(\xi)| e^{-\frac{t^2|\xi|_2^2}{2}} d\xi \le \left\| \left(1+|\xi|^2\right)^{\beta/2} \hat{f}_X \right\|_{L_{p'}(\mathbb{R}^n)} \left\| \left(1+|\xi|^2\right)^{-\beta/2} e^{-\frac{t^2|\xi|_2^2}{2}} \right\|_{L_p(\mathbb{R}^n)}.$$

Since $1 < p \le 2$, $\mathcal{F} : L_p \to L_{p'}$ is bounded with norm 1. Hence,

$$\begin{aligned} \left\| \left(1+|\xi|^2\right)^{\beta/2} \hat{f}_X \right\|_{L_{p'}(\mathbb{R}^n)} &= \left\| \mathcal{F}\left(\mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{\beta/2} \hat{f}_X \right) \right) \right\|_{L_{p'}(\mathbb{R}^n)} \\ &\le \left\| \mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{\beta/2} \hat{f}_X \right) \right\|_{L_p(\mathbb{R}^n)} \\ &\overset{(2.3)}{=} \|f_X\|_{\beta,p}. \end{aligned}$$

Altogether,

$$\int_{\mathbb{R}^n} |\hat{f}_X(\xi)| e^{-\frac{t^2|\xi|_2^2}{2}} d\xi \le \left\| \left(1+|\xi|^2\right)^{-\beta/2} e^{-\frac{t^2|\xi|_2^2}{2}} \right\|_{L_p(\mathbb{R}^n)} \|f\|_{\beta,p}.$$

Now use Lemma 2.3. □

3. Littlewood-Offord type estimates

Let $a_1, \ldots, a_N$ be (deterministic) vectors in $\mathbb{R}^n$, and denote by A the $N \times n$ matrix whose rows are $a_1, \ldots, a_N$. Let $\delta_1, \ldots, \delta_N$ be i.i.d random variables for which there exists $b \in (0,1)$ such that

$$\sup_{x \in \mathbb{R}} \mathbb{P}\left(|\delta_i - x| \le 1\right) \le 1 - b. \tag{3.1}$$

Now, consider the random vector

$$X = \sum_{k=1}^N \delta_k a_k. \tag{3.2}$$

As in [FS07, RV09], the small ball estimate of X involves the least common denominator of the matrix A. Thus, for $\alpha > 0$ and $\gamma \in (0,1)$, define

$$\mathrm{LCD}_{\alpha,\gamma}(A) \overset{\mathrm{def}}{=} \inf\left\{ |\theta|_2 : \theta \in \mathbb{R}^n, d_2\left(A\theta, \mathbb{Z}^n\right) < \min\left(\gamma |A\theta|_2, \alpha\right) \right\}. \tag{3.3}$$

Theorem 3.1. *Let X be defined as in* (3.2)*, and assume that the $N \times n$ matrix A satisfies $|A\theta|_2 \ge |\theta|_2$ for all θ in $\mathbb{R}^n$. Assume also that $t \ge \frac{\sqrt{n}}{\mathrm{LCD}_{\alpha,\gamma}(A)}$. Then*

$$\mathbb{P}\left(\|X\| \le t\right) \le \frac{|K|}{\gamma_n(K)} \left(\frac{C_K}{\pi}\right)^n \left(\left(\frac{t}{\gamma\sqrt{b}}\right)^n + \exp\left(-b\alpha^2\right) \right),$$

where C_K is again the quasi-norm constant from (1.3)*. In particular, for any $p > 0$,*

$$\mathbb{P}\left(|X|_p \le t n^{1/p}\right) \le (C \cdot C_p)^n \left(\left(\frac{t}{\gamma\sqrt{b}}\right)^n + \exp\left(-b\alpha^2\right) \right),$$

where $C_p = \min\left\{2^{1/p-1}, 1\right\}$.

The first step of the proof is to estimate the small ball probability using the integer structure of the vectors a_i. To do that, for a given $\theta \in \mathbb{R}^n$, define

$$f(\theta) \stackrel{\text{def}}{=} \inf_{m \in \mathbb{Z}^N} \left| \frac{z}{t} A\theta - m \right|_2. \tag{3.4}$$

Lemma 3.2 (Small ball estimate in terms of integer structure)**.** *Let X be a random vector as in* (3.2) *and let $t > 0$. Then*

$$\mathbb{P}\left(\|X\| \le t\right) \le \frac{|K|}{\gamma_n(K)} \left(C_K' t\right)^n \cdot \sup_{z \ge \frac{1}{2\pi}} \int_{\mathbb{R}^n} e^{-4bf(\theta)^2 - |\theta|_2^2/2} d\theta.$$

Proof. By (1.4) we have

$$\mathbb{P}\left(\|X\| \le t\right) \le \frac{|K|}{\gamma_n(K)} \left(C_K' t\right)^n \int_{\mathbb{R}^n} \left|\phi_X(\xi)\right| e^{-\frac{t^2|\xi|_2^2}{2}} d\xi.$$

Setting $\theta = t\xi$,

$$t^n \int_{\mathbb{R}^n} \left|\phi_X(\xi)\right| e^{-\frac{t^2|\xi|_2^2}{2}} d\xi = \int_{\mathbb{R}^n} \left|\phi_X(\theta/t)\right| e^{-\frac{|\theta|_2^2}{2}} d\theta. \tag{3.5}$$

Using the definition of X, and the independence of $\delta_1, \ldots, \delta_N$, we have

$$\left|\phi_X\left(\theta/t\right)\right| = \mathbb{E}\exp\left(i\left\langle \sum_{i=1}^N \delta_i a_i, \theta/t \right\rangle\right) = \prod_{k=1}^N \mathbb{E}\exp\left(i\delta_k \frac{\langle a_k, \theta\rangle}{t}\right) = \prod_{k=1}^N \left|\phi_\delta\left(\frac{\langle \theta, a_k\rangle}{t}\right)\right|, \tag{3.6}$$

where δ is an independent copy of $\delta_1, \ldots, \delta_N$. In order to estimate the right side of (3.6), follow the conditioning argument that was used in [FS07, RV09]. Let δ' be an independent copy of δ, and denote by $\bar{\delta}$ the symmetric random variable $\delta - \delta'$. We have, $\left|\phi_\delta\left(\xi\right)\right|^2 = \mathbb{E}\cos\left(\xi\bar{\delta}\right)$. Using the inequality $|x| \le \exp\left(-\left(1 - x^2\right)/2\right)$, which is valid for all $x \in \mathbb{R}$, we obtain

$$\left|\phi_\delta\left(\xi\right)\right| \le \exp\left(-\frac{\left(1 - \mathbb{E}\cos\left(\xi\bar{\delta}\right)\right)}{2}\right). \tag{3.7}$$

By assumption (3.1) it follows that $\mathbb{P}\left(|\bar{\delta}| \ge 1\right) \ge b$. Therefore, by conditioning on $\bar{\delta}$, we get

$$1 - \mathbb{E}\cos\left(\xi\bar{\delta}\right) \ge \mathbb{P}\left(|\bar{\delta}| \ge 1\right) \cdot \mathbb{E}\left(1 - \cos\left(\xi\bar{\delta}\right) \middle| |\bar{\delta}| \ge 1\right) \ge b \cdot \mathbb{E}\left(1 - \cos\left(\xi\bar{\delta}\right) \middle| |\bar{\delta}| \ge 1\right).$$

By the fact that $1 - \cos\theta \ge \frac{2}{\pi^2}\theta^2$, for any $|\theta| \le \pi$, we have for any $\theta \in \mathbb{R}$,

$$1 - \cos\theta \ge \frac{2}{\pi^2} \min_{m \in \mathbb{Z}} |\theta - 2\pi m|^2.$$

Hence,

$$1-\mathbb{E}\cos\left(\xi\bar{\delta}\right)\geq\frac{2b}{\pi^2}\cdot\mathbb{E}\left(\min_{m\in\mathbb{Z}}\left|\xi\bar{\delta}-2\pi m\right|^2\middle|\left|\bar{\delta}\right|\geq 1\right)=8b\cdot\mathbb{E}\left(\min_{m\in\mathbb{Z}}\left|\xi\bar{\delta}-m\right|^2\middle|\left|\bar{\delta}\right|\geq 1/2\pi\right).$$

Plugging this into (3.7) gives

$$\left|\phi_\delta(\xi)\right|\leq\exp\left(-4b\mathbb{E}\left(\min_{m\in\mathbb{Z}}\left|\xi\bar{\delta}-m\right|^2\middle|\left|\bar{\delta}\right|\geq 1/2\pi\right)\right). \tag{3.8}$$

Replacing the conditional expectation with supremum over all the possible values $z\geq 1/2\pi$ and using Jensen's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\phi_X\left(\theta/t\right)\right|e^{-|\theta|_2^2/2}d\theta &\overset{(3.6)}{=}\int_{\mathbb{R}^n}\prod_{k=1}^N\left|\phi_\delta\left(\frac{\langle\theta,a_k\rangle}{t}\right)\right|e^{-|\theta|_2^2/2}d\theta\\ &\overset{(3.8)}{\leq}\int_{\mathbb{R}^n}\exp\left(-4b\cdot\mathbb{E}\left(\sum_{k=1}^N\min_{m\in\mathbb{Z}}\left|\frac{\langle\theta,a_k\rangle}{t}\bar{\delta}-m\right|^2\middle|\left|\bar{\delta}\right|\geq 1/2\pi\right)-|\theta|_2^2/2\right)d\theta\\ &\leq\mathbb{E}\left[\int_{\mathbb{R}^n}\exp\left(-4b\min_{m\in\mathbb{Z}^N}\left|\frac{\bar{\delta}}{t}A\theta-m\right|_2^2-|\theta|_2^2/2\right)d\theta\middle|\left|\bar{\delta}\right|\geq 1/2\pi\right]\\ &\leq\sup_{z\geq 1/2\pi}\int_{\mathbb{R}^n}\exp\left(-4bf\left(\theta\right)^2-|\theta|_2^2/2\right)d\theta.\end{aligned}$$

Using (3.5) the result follows. □

Define the set

$$T_s\overset{\text{def}}{=}\left\{\theta\in\mathbb{R}^n:f(\theta)\leq s\right\}.$$

The next step in the proof is to rewrite the integral that appears in Lemma 3.2 in the following way:

$$\begin{aligned}\int_{\mathbb{R}^n}\exp\left(-4bf\left(\theta\right)^2\right)\exp\left(-|\theta|_2^2/2\right)d\theta &=\int_{\mathbb{R}^n}\int_{s\geq f(\theta)}8bs\exp\left(-4bs^2\right)ds\exp\left(-|\theta|_2^2/2\right)d\theta\\ &=(2\pi)^{n/2}\int_0^\infty 8bs\exp\left(-4bs^2\right)\gamma_n\left(T_s\right)ds,\end{aligned} \tag{3.9}$$

which means that we have to bound $\gamma_n(T_s)$. To do that, we start with the following covering lemma.

Lemma 3.3 (Covering of T_s)**.** *Let $\alpha>0$ and $\gamma\in(0,1)$. Assume that $t\geq\frac{\sqrt{n}}{\mathrm{LCD}_{\alpha,\gamma}(A)}$. Assume also that $|A\theta|_2\geq|\theta|_2$ for all $\theta\in\mathbb{R}^n$. If $0\leq s\leq\alpha/2$, then there exist vectors $\{x_i\}_{i\in I}\subseteq\mathbb{R}^n$ such that*

$$T_s\subseteq\bigcup_{i\in I}B(x_i,r)\text{ and }|x_i-x_{i'}|_2\geq R,\forall i\neq i', \tag{3.10}$$

where $r = \frac{2st}{\gamma z}$ *and* $R = \frac{\sqrt{n}}{z}$. *Moreover, for any* $j \geq 1$,

$$\operatorname{card}\left(\{i \in I : jR \leq |x_i|_2 < (j+1)\,R\}\right) \leq n2^n\,(j+1)^{n-1}\,. \tag{3.11}$$

Proof. Let $\theta_1, \theta_2 \in T_s$. By (3.4), there exists $p_1, p_2 \in \mathbb{Z}^N$ such that

$$\left|\frac{z}{t}A\theta_1 - p_1\right| \leq s \quad \text{and} \quad \left|\frac{z}{t}A\theta_2 - p_2\right| \leq s.$$

By the triangle inequality,

$$\left|\frac{z}{t}A\,(\theta_1 - \theta_2) - (p_1 - p_2)\right| \leq 2s,$$

which means that $d_2\left(A\tau, \mathbb{Z}^N\right) \leq 2s \leq \alpha$, where $\tau = z\,(\theta_1 - \theta_2)\,/t$. By (3.3) this implies that either

$$|\tau|_2 \geq \mathrm{LCD}_{\alpha,\gamma}\,(A)\,,$$

or

$$\alpha \geq 2s \geq d_2\left(A\tau, \mathbb{Z}^N\right) \geq \min\left(\gamma|A\tau|_2, \alpha\right) = \gamma|A\tau|_2.$$

By the assumptions that $|A\tau|_2 \geq |\tau|_2$ and $\mathrm{LCD}_{\alpha,\gamma}\,(A) \geq \sqrt{n}/t$, we conclude that

$$\text{either} \quad |\theta_1 - \theta_2|_2 \geq \frac{\sqrt{n}}{z} =: R \quad \text{or} \quad |\theta_1 - \theta_2|_2 \leq \frac{2st}{\gamma z} =: r.$$

Hence, T_s can be covered by a union of euclidean balls of radius r whose centers are R-separated, which proves (3.10). Next, for $j \geq 1$, let

$$M_j \stackrel{\text{def}}{=} \operatorname{card}\left(\{i \in I : jR \leq |x_i|_2 \leq (j+1)R\}\right).$$

To estimate M_j, use a well-known volumetric argument. Indeed, since $\{x_i\}_{i\in I}$ are R-separated, we know that the Euclidean balls $B\,(x_i, R/2)$ are disjoint and contained in the shell

$$\{y \in \mathbb{R}^n : (j - 1/2)\,R \leq |y|_2 \leq (j + 3/2)\,R\}\,.$$

Hence, taking the volume,

$$M_j\left(\frac{R}{2}\right)^n \leq R^n\left((j+3/2)^n - (j-1/2)^n\right) = R^n\,(j+1/2)^n\left(\left(1 + \frac{1}{2j+1}\right)^n - \left(1 - \frac{1}{2j+1}\right)^n\right).$$

Since for every $x \in (0,1)$, we have $(1+x)^n - (1-x)^n \leq 2nx\,(1+x)^{n-1}$, we conclude that

$$M_j \leq n2^n\,(j+1)^{n-1}\,.$$

□

Using the covering lemma, we can now prove the required volume estimate.

Corollary 3.4. *Let r and R be as in Lemma 3.3. If $R \geq 2r$, then*

$$\gamma_n(T_s) \leq \left(\frac{Cr}{R}\right)^n = \left(\frac{2Cts}{\gamma\sqrt{n}}\right)^n.$$

Proof. Let $y \in \mathbb{R}^n$, we have

$$\gamma_n(B(y,r)) = \frac{1}{(2\pi)^{n/2}} \int_{|y-x|_2 \leq r} e^{-\frac{|x|_2^2}{2}} dx.$$

Since $|x|_2^2 + |x-y|_2^2 = \frac{1}{2}\left(|y|_2^2 + |2x-y|_2^2\right) \geq \frac{1}{2}|y|_2^2$,

$$\gamma_n(B(y,r)) \leq \frac{1}{(2\pi)^{n/2}} e^{-\frac{|y|_2^2}{4}} \int_{|y-x|_2 \leq r} e^{\frac{|y-x|_2^2}{2}} dx.$$

Therefore, if $|y|_2 \geq R \geq 2r$,

$$\gamma_n(B(y,r)) \leq \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|y|_2^2}{4}\right) e^{r^2/2} |B(0,r)| \leq \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|y|_2^2}{8}\right) |B(0,r)|. \tag{3.12}$$

Assume that s is such that $r \leq R/2$, i.e. $4ts \leq \gamma\sqrt{n}$, then by (3.10)

$$\gamma_n(T_s) \leq \sum_{i \in I} \gamma_n(B(x_i,r)) \leq \sum_{j=0}^{\infty} \sum_{i \in I : jR \leq |x_i|_2 < (j+1)R} \gamma_n(B(x_i,r)).$$

Also, for $j \geq 1$, we have by (3.11)

$$\operatorname{card}\left(\{i \in I : jR \leq |x_i|_2 < (j+1)R\}\right) \leq C^n j^{n-1}.$$

By (3.12),

$$\gamma_n(B(x_i,r)) \leq \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{j^2R^2}{8}\right) |B(0,r)|.$$

Hence

$$\begin{aligned}\gamma_n(T_s) &\leq \gamma_n(B(0,r)) + \sum_{j=1}^{\infty} \left(\frac{C^2}{2\pi}\right)^{n/2} j^{n-1} \exp\left(-\frac{j^2R^2}{8}\right) |B(0,r)| \\ &\leq \frac{|B(0,r)|}{(2\pi)^{n/2}} \left(1 + C^n \sum_{j=1}^{\infty} j^{n-1} \exp\left(-\frac{j^2R^2}{8}\right)\right). \end{aligned} \tag{3.13}$$

The function $v \mapsto v^{n-1} e^{-v^2R^2/8}$ is decreasing for $v \geq 2\sqrt{n}/R$. By comparing series with integrals,

$$\begin{aligned}\sum_{j=1}^{\infty} j^{n-1} \exp\left(-\frac{j^2R^2}{8}\right) &\leq \left(\frac{2\sqrt{n}}{R}\right)^n + \int_0^{\infty} v^{n-1} e^{-v^2R^2/8} dv \\ &= \left(\frac{2\sqrt{n}}{R}\right)^n + \frac{8^{n/2}}{R^n} \int_0^{\infty} u^{\frac{n-1}{2}-1} e^{-u} du \leq \left(\frac{Cn^{1/2}}{R}\right)^n. \end{aligned}$$

Since $z \geq 1/2\pi$, we have $R \leq 2\pi\sqrt{n}$, so that

$$\left(1 + C^n \sum_{j=1}^{\infty} j^{n-1} \exp\left(-\frac{j^2 R^2}{8}\right)\right) \leq \left(\frac{C_1 n^{1/2}}{R}\right)^n.$$

Moreover, it is well-known that $|B(0,r)| \leq C_2^n n^{-n/2} r^n$ which implies by (3.13) that

$$\gamma_n(T_s) \leq \left(\frac{Cr}{R}\right)^n = \left(\frac{2Cts}{\gamma\sqrt{n}}\right)^n.$$

□

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2 and (3.9), to have a small ball estimate it is enough to evaluate the integral

$$\int_0^\infty 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds.$$

We have,

$$\begin{aligned}
&\int_0^\infty 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds \\
&\quad = \int_0^{\alpha/2} 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds + \int_{\alpha/2}^\infty 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds \\
&\quad \leq \int_0^{\alpha/2} 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds + \exp\left(-b\alpha^2\right).
\end{aligned}$$

Assume first that $\alpha \leq 2\gamma\sqrt{n}/t$ so that for any $t \leq \alpha/2$ we have $R \geq 2r$. By Corollary 3.4,

$$\gamma_n(T_s) \leq \left(\frac{2Cts}{\gamma\sqrt{n}}\right)^n,$$

and so

$$\begin{aligned}
\int_0^{\alpha/2} 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds &\leq \int_0^{\alpha/2} 8bs \exp\left(-4bs^2\right) \left(\frac{2Cts}{\gamma\sqrt{n}}\right)^n ds \\
&\leq 8b \left(\frac{2Ct}{\gamma\sqrt{n}}\right)^n \int_0^\infty s^{n+1} e^{-4bs^2} ds \\
&= \left(\frac{Ct}{\gamma\sqrt{b}\sqrt{n}}\right)^n \int_0^\infty u^{n/2} e^{-u} du \\
&\leq \left(\frac{C't}{\gamma\sqrt{b}}\right)^n.
\end{aligned}$$

Assume otherwise that $\alpha \geq 2\gamma\sqrt{n}/t \stackrel{\text{def}}{=} \alpha_0$. Then, as before,

$$\int_0^\infty 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds \leq \int_0^{\alpha_0/2} 8bs \exp\left(-4bs^2\right) \gamma_n(T_s)\, ds + \exp\left(-b\alpha_0^2\right).$$

For $s \leq \alpha_0/2$ we do exactly the same computation as in the first case and obtain

$$\int_0^\infty 8bs \exp\left(-4bs^2\right) \gamma_n\left(T_s\right) dt \leq \left(\frac{C's}{\gamma\sqrt{b}}\right)^n + \exp\left(-b\alpha_0^2\right).$$

In that case, we also have

$$\exp\left(-b\alpha_0^2\right) = \exp\left(-4b\gamma^2 n/t^2\right) \leq \left(\frac{Ct}{\gamma\sqrt{b}}\right)^n.$$

That concludes the fact that

$$\int_{\mathbb{R}^n} \exp\left(-4bf\left(\theta\right)^2\right) \exp\left(-|\theta|_2^2/2\right) d\theta \leq \left(\frac{Ct}{\gamma\sqrt{b}}\right)^n + \exp\left(-b\alpha^2\right).$$

Using Lemma 3.2, Theorem 3.1 follows. □

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