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Abstract. Here we aim at justifying rigorously different types of physically relevant diffusive limits for radiative flows. For simplicity, we consider the barotropic situation, and adopt the so-called $P1$-approximation of the radiative transfer equation. In the critical functional framework, we establish the existence of global-in-time strong solutions corresponding to small enough data, and exhibit uniform estimates with respect to the coefficients of the system. Combining with standard compactness arguments, this enables us to justify rigorously the convergence of the solutions to the expected limit systems.

Our results hold true in the whole space $\mathbb{R}^n$ as well as in a periodic box $\mathbb{T}^n$ with $n \geq 2$.

Keywords: Radiation hydrodynamics, Navier-Stokes system, diffusive limit, critical regularity, $P1$-approximation.

1. Introduction

We consider the barotropic version of a model of radiation hydrodynamics. Our main goal is to provide the rigorous justification of asymptotics that have been investigated formally and numerically by Lowrie, Morel and Hittinger [15], and mathematically by the second author and Š. Nečasová in [10, 11, 12] in the finite energy weak solutions framework.

The fluid is described by standard classical fluid mechanics for the mass density ϱ and the velocity field $\vec{u}$ as functions of the time $t \in \mathbb{R}_+$ and of the (Eulerian) spatial coordinate $x \in \Omega$ where Ω is either the whole space $\mathbb{R}^n$ or some periodic box $\mathbb{T}^n$ with $n \geq 2$.

Radiation acts through some *radiative momentum source* $\vec{S}_F$ which is given by

$$\vec{S}_F = \frac{1}{c} \int_0^\infty \int_{\mathcal{S}^{n-1}} \vec{\omega} S \, d\vec{\omega} \, d\nu,$$

where c is the light speed.

The radiative source $S = S(t, x, \vec{\omega}, \nu)$ depends on the direction vector $\vec{\omega} \in \mathcal{S}^{n-1}$ (where $\mathcal{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$), and on the frequency $\nu \geq 0$ of the photons, and is given by

$$S = \sigma_a \big(B(\nu, \varrho) - \mathcal{I} \big) + \sigma_s \big(\tilde{\mathcal{I}} - \mathcal{I} \big) \quad \text{where} \quad \tilde{\mathcal{I}} := \frac{1}{|\mathcal{S}^{n-1}|} \int_{\mathcal{S}^{n-1}} \mathcal{I} \, d\vec{\omega}.$$

The radiative intensity $\mathcal{I}$ obeys the transfer equation

$$\frac{1}{c} \partial_t \mathcal{I} + \vec{\omega} \cdot \nabla_x \mathcal{I} = S \qquad \text{in } (0, T) \times \Omega \times \mathcal{S}^{n-1} \times (0, \infty). \tag{1.1}$$

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In the present paper, as in [7, 8], we make the following simplifying assumptions

(1) Isotropy : the transport coefficients σ_a and σ_s are independent of $\vec{\omega}$;
(2) 'Gray' hypothesis : σ_a and σ_s are independent of ν;
(3) '$P1$ hypothesis' : the averaged radiative intensity $I := \int_0^\infty \mathcal{I}\, d\nu$ is given by the ansatz

$$I = I_0 + \vec{\omega}\cdot\vec{I_1}, \tag{1.2}$$

where I_0 and $\vec{I_1}$ are independent of $\vec{\omega}$ and ν.

Plugging (1.2) in (1.1), and computing the 0th and 1st order momentum with respect to $\vec{\omega}$, we find out the following evolution equations for I_0 and I_1 (keeping the same notation B for the distribution function averaged in ν)

$$\frac{1}{c}\,\partial_t I_0 + \frac{1}{n}\,\mathrm{div}_x\vec{I_1} = \sigma_a(\varrho)(B(\varrho) - I_0), \tag{1.3}$$

$$\frac{1}{c}\,\partial_t\vec{I_1} + \nabla_x I_0 = (\sigma_a(\varrho) + \sigma_s(\varrho))\vec{I_1}. \tag{1.4}$$

Besides, the radiative force is now given by

$$\vec{S}_F = \left(\frac{\sigma_a(\varrho)+\sigma_s(\varrho)}{n}\right)\vec{I_1}. \tag{1.5}$$

In order to identify the most relevant asymptotic regimes, we rewrite the equations in dimensionless form. To this end, introduce some reference hydrodynamical quantities (length, time, velocity, density, pressure): $\bar{L}$, $\bar{T}$, $\bar{U}, \bar{\varrho}$, $\bar{p}$, and reference radiative quantities (radiative intensity, absorption and scattering coefficients and equilibrium function): $\bar{I}$, $\bar{\sigma}_a$, $\bar{\sigma}_s$ and $\bar{B}$.

Let $Sr := \bar{L}/\bar{T}\bar{U}$, $Ma := \bar{U}/\sqrt{\bar{\varrho}\bar{p}}$ and $Re := \bar{U}\bar{\varrho}\bar{L}/\bar{\mu}$ be the Strouhal, Mach and Reynolds numbers corresponding to hydrodynamics. Let also define $\mathcal{C} := c/\bar{U}$, $\mathcal{L} := \bar{L}\bar{\sigma}_a$, $\mathcal{L}_s := \bar{\sigma}_s/\bar{\sigma}_a$, various dimensionless numbers corresponding to radiation. In all that follows, we assume our flow to be strongly under-relativistic so that $\mathcal{C}$ *is large.*

Choosing $\bar{B} = \bar{I}$, we discover that the evolution of the dimensionless unknowns (still denoted in the same way) is governed by the following system of equations

$$\begin{cases} Sr\,\partial_t\varrho + \mathrm{div}\,(\varrho\vec{u}) = 0,\\ Sr\,\partial_t(\varrho\vec{u}) + \mathrm{div}\,(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{Ma^2}\nabla p - \frac{1}{Re}\big(\mathrm{div}\,(\mu\nabla\vec{u}+{}^t\nabla\vec{u})+\nabla(\lambda\mathrm{div}\,\vec{u})\big) = \mathcal{L}\big(\frac{\sigma_a+\mathcal{L}_s\sigma_s}{n}\big)\vec{I_1},\\ \frac{Sr}{\mathcal{C}}\,\partial_t I_0 + \frac{1}{n}\,\mathrm{div}\vec{I_1} = \mathcal{L}\sigma_a\,(B - I_0)\,,\\ \frac{Sr}{\mathcal{C}}\,\partial_t\vec{I_1} + \nabla I_0 = -\mathcal{L}\,(\sigma_a + \mathcal{L}_s\sigma_s)\,\vec{I_1}, \end{cases}$$

where $\varrho = \varrho(t,x)\in\mathbb{R}_+$ and $\vec{u} = \vec{u}(t,x)$ stand for the density and pressure, respectively, $p = P(\rho)$ is the pressure, $\lambda = \lambda(\rho)$ and $\mu = \mu(\rho)$ are the viscosity coefficients. The given functions P, λ and μ are supposed sufficiently smooth, and we make the following strict ellipticity assumption

$$\nu := \lambda + 2\mu > 0 \quad\text{and}\quad \mu > 0.$$

In our recent work [8], we gave a mathematical justification of the low Mach number asymptotics. In the present paper, we investigate another type of physically relevant asymptotic regimes, which are of *diffusive* type. They correspond to the case where $\mathcal{C}$ is large and all the other dimensionless numbers, but $\mathcal{L}$ and $\mathcal{L}_s$, are of order 1. To make it more concrete, take

$$Ma = Sr = Re = 1,\quad \mathcal{C} = \varepsilon^{-1},\quad \bar{\varrho} = P'(\bar{\varrho}) = B'(\bar{\varrho}) = \sigma_a(\bar{\varrho}) = \sigma_s(\bar{\varrho}) = 1,$$

where ε is a small positive number, bound to tend to 0.

Because we shall focus on small perturbations of the reference density $\bar{\varrho} = 1$, it is convenient to introduce the new unknown $b := B(\varrho) - B(1)$. In this context, all the functions of ϱ may be written in terms of b. Setting $j_0 := I_0 - B(1)$ and $\vec{j}_1 := \vec{I}_1$, and using exponents to emphasize the dependency with respect to ε, we eventually get the following system

$$
(1.6)\qquad \begin{cases}
\partial_t b^\varepsilon + \vec{u}^\varepsilon \cdot \nabla b^\varepsilon + (1 + k_1(b^\varepsilon))\mathrm{div}\, \vec{u}^\varepsilon = 0, \\
\partial_t \vec{u}^\varepsilon + \vec{u}^\varepsilon \cdot \nabla \vec{u}^\varepsilon - (1 + k_2(b^\varepsilon))\mathcal{A}\vec{u}^\varepsilon + (1 + k_3(b^\varepsilon))\nabla b^\varepsilon = \frac{\mathcal{L}(1+\mathcal{L}_s)}{n}(1 + k_4(b^\varepsilon))\vec{j}_1^\varepsilon, \\
\varepsilon \partial_t j_0^\varepsilon + \frac{1}{n}\, \mathrm{div}\, \vec{j}_1^\varepsilon = \mathcal{L}(b^\varepsilon - j_0^\varepsilon), \\
\varepsilon \partial_t \vec{j}_1^\varepsilon + \nabla j_0^\varepsilon = -\mathcal{L}(1 + \mathcal{L}_s)\vec{j}_1^\varepsilon,
\end{cases}
$$

with $\mathcal{A} := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ and where $k_1,\ k_2,\ k_3,\ k_4$ are smooth functions vanishing at 0.

2. Formal asymptotics

Let us first present some formal computations so as to exhibit the limit equations we can get from (1.6) in different types of diffusive asymptotic regimes. We restrict to the case where the following necessary and sufficient linear stability condition (derived in [7]) is fulfilled

$$
(2.1)\qquad n\nu\mathcal{L} > \varepsilon\left(\frac{2 + \mathcal{L}_s}{1 + \mathcal{L}_s}\right)\cdot
$$

Note that (2.1) implies that $\liminf \mathcal{L}\varepsilon^{-1} > 0$ for ε going to 0.

In all that follows, it is assumed that $(b^\varepsilon, \vec{u}^\varepsilon, j_0^\varepsilon, \vec{j}_1^\varepsilon)$ converges to $(b, \vec{u}, j_0, \vec{j}_1)$ in some suitable space with enough regularity to pass to the limit in the nonlinear terms.

2.1. Case $\mathcal{L} \approx \varepsilon$ and $\mathcal{L}_s \to +\infty$.

Denoting by $\mathcal{P}$ the L^2 orthogonal projector on divergence free vector fields, we get

$$
(2.2)\qquad \mathcal{P}\vec{j}_1^\varepsilon(t) = e^{-\frac{\mathcal{L}}{\varepsilon}(1+\mathcal{L}_s)t}\,\mathcal{P}\vec{j}_1^\varepsilon(0).
$$

Hence $\mathcal{P}\vec{j}_1^\varepsilon$ tends to $\vec{0}$ for $\varepsilon \to 0$.

2.1.1. *Subcase* $\mathcal{L}^2\mathcal{L}_s \to 0$.

Setting $\mathcal{Q} := \mathrm{Id} - \mathcal{P}$, we see that the equation for j_0^ε entails that $\mathcal{Q}\vec{j}_1^\varepsilon = \mathcal{O}(\varepsilon)$. Next, the equation for $\mathcal{Q}\vec{j}_1^\varepsilon$ implies that ∇j_0^ε goes to $\vec{0}$, too, because $\varepsilon^2\mathcal{L}_s \to 0$. Assuming that j_0 decays to 0 at infinity, this yields $j_0 = 0$.

From the equation for $\vec{j}_1^\varepsilon$, we also get

$$
(2.3)\qquad -\mathcal{L}(1 + \mathcal{L}_s)\vec{j}_1^\varepsilon = \nabla j_0^\varepsilon + \mathcal{O}(\varepsilon).
$$

Hence $\varepsilon(1 + \mathcal{L}_s)\vec{j}_1^\varepsilon$ goes to $\vec{0}$ and $(b, \vec{u})$ thus satisfies the barotropic Navier-Stokes equations. In other words, the radiative effect becomes negligible in the asymptotic $\mathcal{L} \approx \varepsilon$ and $\varepsilon^2\mathcal{L}_s \to 0$ with $\mathcal{L}_s \to +\infty$

2.1.2. *Subcase* $\lim_{\varepsilon\to 0}\mathcal{L}^2\mathcal{L}_s\in(0,+\infty)$. This is the so-called *nonequilibrium diffusion regime.* The analysis of the previous paragraph shows that $\vec{j}_1^\varepsilon=\mathcal{O}(\varepsilon)$ (hence $\vec{j}_1=\vec{0}$) and that (2.3) holds true. The new fact is that the equation for j_0^ε combined with (2.3) implies that

$$\partial_t j_0^\varepsilon+\frac{\mathcal{L}}{\varepsilon}\big(j_0^\varepsilon-b^\varepsilon\big)-\frac{1}{n}\frac{\mathcal{L}}{\varepsilon}\frac{1}{\mathcal{L}^2\mathcal{L}_s}\Delta j_0^\varepsilon=\mathcal{O}(\varepsilon). \tag{2.4}$$

Now, if we assume that

$$\frac{\mathcal{L}}{\varepsilon}\to\frac{\kappa}{n\nu}\quad\text{and}\quad\mathcal{L}^2\mathcal{L}_s\to\frac{m}{\nu^2},$$

for some $m\in(0,+\infty)$ and $\kappa>1$ (see (2.1)), then $(b,\vec{u})$ satisfies the following compressible Navier-Stokes equations coupled with a parabolic equation

$$\begin{cases}\partial_t b+\vec{u}\cdot\nabla b+(1+k_1(b))\mathrm{div}\,\vec{u}=0,\\ \partial_t\vec{u}+\vec{u}\cdot\nabla\vec{u}-(1+k_2(b))\mathcal{A}\vec{u}+(1+k_3(b))\nabla b+\frac{1}{n}(1+k_4(b))\nabla j_0=\vec{0},\\ \partial_t j_0+\frac{\kappa}{n\nu}\big(j_0-b-\frac{\nu^2}{nm}\Delta j_0\big)=0.\end{cases} \tag{2.5}$$

2.1.3. *Subcase* $\mathcal{L}^2\mathcal{L}_s\to+\infty$. We still have $\vec{j}_1^\varepsilon=\mathcal{O}(\varepsilon)$, (2.3) and thus (2.4) holds true. Now, as $\mathcal{L}^2\mathcal{L}_s\to+\infty$ and $\mathcal{L}\approx\varepsilon$, the r.h.s. of (2.4) tends to 0. Therefore, if we assume as before that $\mathcal{L}/\varepsilon\to\kappa/(n\nu)$ then we find out that $(b,\vec{u},j_0)$ satisfies the following *degenerate nonequilibrium diffusion system*

$$\begin{cases}\partial_t b+\vec{u}\cdot\nabla b+(1+k_1(b))\mathrm{div}\,\vec{u}=0,\\ \partial_t\vec{u}+\vec{u}\cdot\nabla\vec{u}-(1+k_2(b))\mathcal{A}\vec{u}+(1+k_3(b))\nabla b+\frac{1}{n}(1+k_4(b))\nabla j_0=\vec{0},\\ \partial_t j_0+\frac{\kappa}{n\nu}(j_0-b)=0.\end{cases} \tag{2.6}$$

2.2. **Case** $\varepsilon\ll\mathcal{L}\ll 1$. Recall that we have (2.3) while the equation for j_0^ε implies that

$$\mathrm{div}\,\vec{j}_1^\varepsilon=n\mathcal{L}(b^\varepsilon-j_0^\varepsilon)+\mathcal{O}(\varepsilon). \tag{2.7}$$

Hence $\mathcal{Q}\vec{j}_1=0$ (as $\mathcal{L}\to 0$), and

$$\Delta j_0^\varepsilon+n\mathcal{L}^2(1+\mathcal{L}_s)(b^\varepsilon-j_0^\varepsilon)=\mathcal{O}(\varepsilon)+\mathcal{O}(\varepsilon\mathcal{L}(1+\mathcal{L}_s)). \tag{2.8}$$

Subcase $\mathcal{L}^2\mathcal{L}_s\to 0$. Then (2.8) implies that $\Delta j_0=0$ and thus $j_0\equiv 0$ (if one assumes that $j_0\to 0$ at ∞). Consequently, (2.3) implies that the radiative force in the velocity equation tends to 0 when ε goes to 0. Therefore $(b,\vec{u})$ just satisfies the classical compressible Navier-Stokes equation.

Subcase $\nu^2\mathcal{L}^2\mathcal{L}_s\to m\in(0,+\infty)$. We have $\vec{j}_1=\vec{0}$, and Relations (2.3), (2.8) imply that $(b,\vec{u},j_0)$ fulfills the following Navier-Stokes-Poisson system

$$\begin{cases}\partial_t b+\vec{u}\cdot\nabla b+(1+k_1(b))\mathrm{div}\,\vec{u}=0,\\ \partial_t\vec{u}+\vec{u}\cdot\nabla\vec{u}-(1+k_2(b))\mathcal{A}\vec{u}+(1+k_3(b))\nabla b+\frac{1}{n}(1+k_4(b))\nabla j_0=\vec{0},\\ -\nu^2\Delta j_0+mn(j_0-b)=0.\end{cases} \tag{2.9}$$

Subcase $\mathcal{L}^2\mathcal{L}_s\to+\infty$. Then (2.8) implies that $j_0=b$. Combining with (2.3), we thus find out that $(b,\vec{u})$ fulfills the following compressible Navier-Stokes equation with *modified pressure law*

$$\begin{cases}\partial_t b+\vec{u}\cdot\nabla b+(1+k_1(b))\mathrm{div}\,\vec{u}=0,\\ \partial_t\vec{u}+\vec{u}\cdot\nabla\vec{u}-(1+k_2(b))\mathcal{A}\vec{u}+\big(1+\frac{1}{n}+k_3(b)+\frac{1}{n}k_4(b)\big)\nabla b=\vec{0}.\end{cases} \tag{2.10}$$

2.3. **Case** $\nu\mathcal{L} \to \ell \in (0, +\infty)$**.**

Subcase $\nu^2\mathcal{L}^2\mathcal{L}_s \to m \in [0, +\infty)$. Passing to the limit in (2.8) gives

$$-\nu^2\Delta j_0 + n(\ell^2 + m)(j_0 - b) = 0. \tag{2.11}$$

So we get System (2.9) for $(b, j_0, \vec{u})$ with the last equation replaced by (2.11).

Subcase $\mathcal{L}_s \to +\infty$. Exactly as in the case $\mathcal{L} \to 0$, we get $j_0 = b$, $\vec{j}_1 = \vec{0}$, and $(b, \vec{u})$ satisfies (2.10).

2.4. **Case** $\mathcal{L} \to +\infty$**.** Relation (2.3) implies that $\vec{j}_1 = 0$, and thus, according to (2.7), we have $j_0 = b$. Therefore (2.3) implies that

$$\mathcal{L}(1 + \mathcal{L}_s)\vec{j}_1^{\,\varepsilon} \to \nabla b,$$

and $(b, \vec{u})$ thus satisfies (2.10).

To make a long story short, the above formal computations pointed out five types of asymptotic regimes. They are governed by

1. The ordinary compressible Navier-Stokes equations with null radiation (if $\mathcal{L} \to 0$ and $\mathcal{L}^2\mathcal{L}_s \to 0$);
2. The compressible Navier-Stokes equation with an extra pressure term see (2.10) (equilibrium diffusion regime corresponding to $\varepsilon \ll \mathcal{L}$ and $\mathcal{L}^2\mathcal{L}_s \to +\infty$, or $\mathcal{L} \to +\infty$);
3. The Navier-Stokes-Poisson equations (2.9) (or (2.11)) (case $\varepsilon \ll \mathcal{L} \lesssim 1$ and $\nu^2\mathcal{L}^2\mathcal{L}_s \to m \in (0, +\infty)$);
4. The compressible Navier-Stokes equations coupled with a parabolic equation (2.5) (nonequilibrium diffusion regime $\mathcal{L} \approx \varepsilon$ and $\nu^2\mathcal{L}_s\mathcal{L}^2 \to m \in (0, +\infty)$);
5. The compressible Navier-Stokes equations coupled with a damped equation (2.6) (degenerate nonequilibrium diffusion regime $\mathcal{L} \approx \varepsilon$ and $\mathcal{L}_s\mathcal{L}^2 \to +\infty$).

The rest of the paper is devoted to justifying rigorously the last four asymptotics globally in time in the framework of small solutions with critical regularity. In the next section, we introduce a few notations that will be needed to define our functional framework, and give an overview of the strategy. Section 4 is devoted to a fine analysis of the linearized equations (1.6) about $(0, \vec{0}, 0, \vec{0})$, which turns out to be essentially the key to proving global results and justifying the diffusive asymptotics we have in mind. The next three sections are devoted to the rigorous justification of the nonequilibrium diffusion regime $\mathcal{L} \approx \varepsilon$ and $\mathcal{L}_s\mathcal{L}^2 \gtrsim 1$, the equilibrium diffusion regime $\mathcal{L} \to +\infty$ and of the Poisson type diffusion regime ($\varepsilon \ll \mathcal{L} \lesssim 1$ and $\nu^2\mathcal{L}^2\mathcal{L}_s \to m \in (0, +\infty)$). In all of those sections, we establish a global-in-time existence result for the expected limit system, and for (1.6) supplemented with uniform estimates (for coefficients $\mathcal{L}$ and $\mathcal{L}_s$ satisfying the assumptions of the studied regime), and eventually show the convergence of the solutions of (1.6) to those of the expected limit system. Some estimates, of independent interest, for the solutions to a class of linear ODEs corresponding to the linearized equations of (1.6) in the Fourier space are postponed in the appendix.

3. Functional framework and overview of the method

The functional framework we shall work in is modeled on the linearized equations corresponding to (1.6), and is thus the same as in our first paper [7] devoted to the global well-posedness issue in critical regularity spaces for small perturbations of a stable constant state. The key to proving asymptotic results however, is to prescribe norms depending on the parameters ε, $\mathcal{L}$ and $\mathcal{L}_s$, so as to get optimal uniform estimates, enabling our justifying rigorously the different diffusive asymptotics exhibited above.

Let us first very briefly recall the definition of homogeneous Besov spaces $\dot{B}^s_{2,1}$ (the reader is referred to [1], Chap. 2 for more details). For simplicity, we focus on the $\mathbb{R}^n$ case (adapting the construction to the torus being quite straightforward). Fix some smooth radial bump function $\chi : \mathbb{R}^n \to [0,1]$ with $\chi \equiv 1$ on $B(0,1/2)$ and $\chi \equiv 0$ outside $B(0,1)$, nonincreasing with respect to the radial variable. Let $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$. The elementary spectral cut-off operator entering in the Littlewood-Paley decomposition is defined by

$$\dot{\Delta}_j u := \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}D)\mathcal{F}u), \qquad j \in \mathbb{Z}$$

where we denote by $\mathcal{F}$ the standard Fourier transform in $\mathbb{R}^n$.

For any $s \in \mathbb{R}$, the *homogeneous Besov space* $\dot{B}^s_{2,1}$ is the set of tempered distributions u so that

$$\|u\|_{\dot{B}^s_{2,1}} := \sum_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^2} < \infty,$$

and

$$\lim_{\lambda\to+\infty} \chi(\lambda D)u = 0 \quad \text{in} \quad L^\infty. \tag{3.1}$$

As pointed out in [7], scaling considerations that neglect low order terms of System (1.6) suggest that critical regularity is $\dot{B}^{\frac{n}{2}-1}_{2,1}$ for $\vec{u}_0$, $j_{0,0}$ and $\vec{j}_{1,0}$, and $\dot{B}^{\frac{n}{2}}_{2,1}$ for b_0. However, to handle lower order terms, one has to make additional assumptions for the low frequencies. To this end, it is convenient to introduce the following notation (where η stands for a positive parameter)

$$\|u\|^{\ell,\eta}_{\dot{B}^s_{2,1}} := \sum_{2^k\leq 2\eta} 2^{ks}\|\dot{\Delta}_k u\|_{L^2} \quad \text{and} \quad \|u\|^{h,\eta}_{\dot{B}^s_{2,1}} := \sum_{2^k\geq \eta/2} 2^{ks}\|\dot{\Delta}_k u\|_{L^2},$$

and also

$$u^{\ell,\eta} := \sum_{2^k\leq\eta} \dot{\Delta}_k u \quad \text{and} \quad u^{h,\eta} := \sum_{2^k>\eta} \dot{\Delta}_k u.$$

Note that $\|u^{\ell,\eta}\|_{\dot{B}^s_{2,1}} \leq C\|u\|^{\ell,\eta}_{\dot{B}^s_{2,1}}$ and $\|u^{h,\eta}\|_{\dot{B}^s_{2,1}} \leq C\|u\|^{h,\eta}_{\dot{B}^s_{2,1}}$. Because the Littlewood-Paley decomposition is not quite orthogonal, it is important to allow for a small overlap in the above definition of norms.

In some places, we will have to specify also the behavior for the middle frequencies, by considering for given $0 < \eta < \eta'$,

$$\|u\|^{m,\eta,\eta'}_{\dot{B}^s_{2,1}} := \sum_{\eta\leq 2^k\leq\eta'} 2^{ks}\|\dot{\Delta}_k u\|_{L^2}.$$

Broadly speaking, our strategy to justify the different types of diffusive limits is as follows:

- Step 1: We prove 'uniform estimates' for the global solutions to (1.6), uniform meaning that we want a bound independent of ε, but the norm itself may depend 'in a nice way' of the parameters ε, $\mathcal{L}$ and $\mathcal{L}_s$.
- Step 2: We show that the limit system is globally well-posed in the small data case.
- Step 3: We take advantage of estimates of Step 1 to exhibit weak compactness properties. Combining with the uniqueness result of Step 2, this allows to conclude to the convergence of the whole family of solutions of (1.6) to those of the limit system.

The most technical part is step 1, as it requires a fine analysis of the linearized equations of (1.6) about 0 that keeps track of the coefficients $\mathcal{L}$, $\mathcal{L}_s$ and ε. Schematically, in the Fourier space, one has to resort to different types of estimates for low, medium and high frequencies. The low frequency analysis is carried out by considering approximate eigenmodes of the system, that are constructed by a perturbative method from the (explicit) eigenmodes corresponding to null frequency. A part of the difficulty is that the 'fluid modes' are of parabolic type, hence the corresponding eigenvalues tend quadratically to 0 when the frequency size tends to 0 while the radiative modes are expected to be exponentially damped. The high frequency analysis is inspired by the corresponding one for the barotropic Navier-Stokes equations, after noticing that coupling between radiative and fluid unknowns occurs only through 0 order terms, and thus tend to be negligible for very high frequencies. Last but not least, medium frequency regime has to be looked at with the greatest care, as the low and high frequency regimes need not overlap. There is no general strategy for handling them, apart from guessing approximate eigenmodes of the system.

4. Uniform estimates for the linearized equations

In order to reduce the study to the case where the total viscosity $\nu := \lambda + 2\mu$ is 1, and to get a symmetric first order system for the radiative unknowns, let us set

$$\big(b, \vec{u}, j_0, \vec{j}_1\big)(t,x) := (b^\varepsilon, \vec{u}^\varepsilon, \sqrt{n}\, j_0^\varepsilon, \vec{j}_1^\varepsilon)(\nu t, \nu x). \tag{4.1}$$

Then $(b^\varepsilon, \vec{u}^\varepsilon, j_0^\varepsilon, \vec{j}_1^\varepsilon)$ satisfies (1.6) if and only if $(b, \vec{u}, j_0, \vec{j}_1)$ satisfies

$$\begin{cases} \partial_t b + \vec{u}\cdot\nabla b + (1+k_1(b))\mathrm{div}\,\vec{u} = 0, \\ \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} - (1+k_2(b))\widetilde{\mathcal{A}}\vec{u} + (1+k_3(b))\nabla b = \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}(1+k_4(b))\vec{j}_1, \\ \varepsilon\partial_t j_0 + \frac{1}{\sqrt{n}}\,\mathrm{div}\,\vec{j}_1 = \widetilde{\mathcal{L}}(b - \sqrt{n}\, j_0), \\ \varepsilon\partial_t \vec{j}_1 + \frac{1}{\sqrt{n}}\nabla j_0 = -\widetilde{\mathcal{L}}\mathcal{M}\vec{j}_1, \end{cases} \tag{4.2}$$

with

$$\mathcal{M} := 1 + \mathcal{L}_s, \quad \widetilde{\mathcal{L}} := \nu\mathcal{L} \ \text{ and } \ \widetilde{\mathcal{A}} := \nu^{-1}\mathcal{A}. \tag{4.3}$$

The corresponding linearized system reads

$$\begin{cases} \partial_t b + \mathrm{div}\,\vec{u} = f, \\ \partial_t \vec{u} - \widetilde{\mathcal{A}}\vec{u} + \nabla b - \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\vec{j}_1 = \vec{g}, \\ \varepsilon\partial_t j_0 + \frac{1}{\sqrt{n}}\,\mathrm{div}\,\vec{j}_1 + \widetilde{\mathcal{L}}(j_0 - \sqrt{n}\, b) = 0, \\ \varepsilon\partial_t \vec{j}_1 + \frac{1}{\sqrt{n}}\,\nabla j_0 + \widetilde{\mathcal{L}}\mathcal{M}\vec{j}_1 = \vec{0}. \end{cases} \tag{4.4}$$

On one hand, the coupling between the incompressible part of $\vec{u}$ and $\vec{j}_1$ that is $\mathcal{P}\vec{u}$ and $\mathcal{P}\vec{j}_1$ where $\mathcal{P}$ stands for the projector on divergence-free vector-fields is obvious as

$$\partial_t\mathcal{P}\vec{u} - \frac{\mu}{\nu}\Delta\mathcal{P}\vec{u} = \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\mathcal{P}\vec{j}_1, \tag{4.5}$$

and

$$\mathcal{P}\vec{j}_1(t) = e^{-\frac{\widetilde{\mathcal{L}}\mathcal{M}t}{\varepsilon}}\mathcal{P}\vec{j}_1(0),$$

hence in any functional space X we have

$$\widetilde{\mathcal{L}}\mathcal{M}\|\mathcal{P}\vec{j}_1\|_{L^1(X)} \leq \varepsilon\|\mathcal{P}\vec{j}_1(0)\|_X. \tag{4.6}$$

On the other hand, the coupling between $b,\ d := \Lambda^{-1}\mathrm{div}\,\vec{u},\ j_0$ and $j_1 := \Lambda^{-1}\mathrm{div}\,\vec{j}_1$ (where $\Lambda^s := (-\Delta)^{s/2}$) is quite complicated: in Fourier variables, we have

$$\frac{d}{dt}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix} + \begin{pmatrix}0 & \rho & 0 & 0\\ -\rho & \rho^2 & 0 & -\frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\\ -\frac{\sqrt{n}\widetilde{\mathcal{L}}}{\varepsilon} & 0 & \frac{\widetilde{\mathcal{L}}}{\varepsilon} & \frac{\rho}{\varepsilon\sqrt{n}}\\ 0 & 0 & -\frac{\rho}{\varepsilon\sqrt{n}} & \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\end{pmatrix}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{4.7}$$

The analysis that has been performed in [7] pointed out the following necessary and sufficient stability condition

$$\widetilde{\mathcal{L}} > \frac{\varepsilon}{n}(1+\mathcal{M}^{-1}). \tag{4.8}$$

So we shall make this assumption in all that follows. Of course one also has to keep in mind that $\mathcal{M} > 1$, a consequence of $\mathcal{M} := 1+\mathcal{L}_s$. For notational simplicity, we shall simply denote $\widetilde{\mathcal{L}}$ by $\mathcal{L}$ in the following computations.

4.1. **Estimates for small ρ.** In order to prove estimates in the case $0 \leq \rho \leq C_1$ (with $C_1 \geq \sqrt{1+n^{-1}}$), we shall use that (4.7) enters in the class of ODEs that has been considered in the Appendix. Indeed, it corresponds to (A.3) with

$$\varsigma = \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}, \quad \eta = \frac{\sqrt{n}\widetilde{\mathcal{L}}}{\varepsilon}, \quad \beta = \frac{\widetilde{\mathcal{L}}}{\varepsilon}, \quad \alpha = \frac{1}{\varepsilon\sqrt{n}}, \quad \gamma = \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\cdot \tag{4.9}$$

4.1.1. *The case $\mathcal{L} \gtrsim 1$ and $\mathcal{L}\varepsilon\mathcal{M} \gtrsim 1$.* We shall follow the first approach proposed in Appendix A. It corresponds to the following matrices $A_0,\ A_1,\ A_2$ and B_1

$$A_0 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&\frac{\mathcal{L}}{\varepsilon}&0\\0&0&0&\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\end{pmatrix}, \qquad A_1 = \begin{pmatrix}0&1&0&0\\-1-\frac{1}{n}&0&0&0\\0&0&0&\frac{1+\varepsilon^2}{\sqrt{n}\,\varepsilon}\\0&0&-\frac{1}{\sqrt{n}\,\varepsilon}&0\end{pmatrix},$$

$$B_1 = -\begin{pmatrix}0&0&0&\frac{\varepsilon}{n}\\0&0&\frac{1}{n^{3/2}}&0\\0&\sqrt{n}&0&0\\\frac{1}{\varepsilon}&0&0&0\end{pmatrix} \quad\text{and}\quad A_2 = \begin{pmatrix}0&0&0&0\\0&1&0&-\frac{\varepsilon}{n}\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Therefore we set

$$P := \begin{pmatrix}0&0&0&\frac{\varepsilon^2}{n\mathcal{L}\mathcal{M}}\\0&0&\frac{\varepsilon}{n^{3/2}\mathcal{L}}&0\\0&-\frac{\varepsilon\sqrt{n}}{\mathcal{L}}&0&0\\-\frac{1}{\mathcal{L}\mathcal{M}}&0&0&0\end{pmatrix}, \tag{4.10}$$

which corresponds to the change of unknowns

$$\begin{pmatrix} \widehat{\mathfrak{b}} \\ \widehat{\mathfrak{d}} \\ \widehat{\mathfrak{j}}_0 \\ \widehat{\mathfrak{j}}_1 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & \frac{\varepsilon^2}{n\mathcal{L}\mathcal{M}}\rho \\ -\frac{\varepsilon}{n\mathcal{L}}\rho & 1 & \frac{\varepsilon}{n^{3/2}\mathcal{L}}\rho & \frac{\varepsilon}{n} \\ -\sqrt{n} & -\frac{\sqrt{n}\,\varepsilon}{\mathcal{L}}\rho & 1 & -\frac{\varepsilon^2}{\sqrt{n}\,\mathcal{L}}\rho \\ -\frac{1}{\mathcal{L}\mathcal{M}}\rho & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{b} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix}. \tag{4.11}$$

According to (A.8), working with $(\widehat{a}, \widehat{d}, \widehat{j}_0, \widehat{j}_1)$ or $(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}}, \widehat{\mathfrak{j}}_0, \widehat{\mathfrak{j}}_1)$ is equivalent whenever

$$\rho \lesssim \mathcal{L} \min(\varepsilon^{-1}, \mathcal{M}). \tag{4.12}$$

Let us first compute the matrices PB_1, $[P, A_1]$ and A_3 appearing in (A.2)

$$PB_1 = \frac{\varepsilon}{n\mathcal{L}} \begin{pmatrix} -\mathcal{M}^{-1} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathcal{M}^{-1} \end{pmatrix},$$

$$[P, A_1] = \frac{1}{\mathcal{L}} \begin{pmatrix} 0 & 0 & -\frac{\varepsilon(1+\mathcal{M}^{-1})}{n^{3/2}} & 0 \\ 0 & 0 & 0 & \frac{1+\varepsilon^2(1+\mathcal{M}^{-1}(n+1))}{n^2} \\ \frac{1+\varepsilon^2(1+\mathcal{M}(1+n))}{\varepsilon\sqrt{n}\,\mathcal{M}} & 0 & 0 & 0 \\ 0 & -1-\mathcal{M}^{-1} & 0 & 0 \end{pmatrix},$$

$$A_3 = \frac{1}{n} \begin{pmatrix} 0 & \frac{\varepsilon^2}{\mathcal{L}^2} & 0 & \frac{\varepsilon^3}{n\mathcal{L}^2\mathcal{M}^2} \\ \frac{\varepsilon}{\mathcal{L}\mathcal{M}} - \frac{(1+n^{-1})\varepsilon^2}{\mathcal{L}^2\mathcal{M}^2} & 0 & \frac{\varepsilon}{\mathcal{L}\sqrt{n}} - \frac{\varepsilon^2}{\mathcal{L}^2 n^{3/2}} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon(1+\varepsilon^2)}{\sqrt{n}(\mathcal{L}\mathcal{M})^2} \\ 0 & 0 & -\frac{\varepsilon}{\mathcal{L}^2\sqrt{n}} & 0 \end{pmatrix}.$$

Because $\mathcal{L} \gtrsim 1$, we thus have $|A_3| \lesssim \frac{\varepsilon}{\mathcal{L}}\cdot$ Hence, up to a $\mathcal{O}(\varepsilon\rho^3/\mathcal{L})$ term, the equations for $(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})$ read

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \widehat{\mathfrak{b}} \\ \widehat{\mathfrak{d}} \end{pmatrix} + \rho \begin{pmatrix} 0 & 1 \\ -1-n^{-1} & 0 \end{pmatrix}\begin{pmatrix} \widehat{\mathfrak{b}} \\ \widehat{\mathfrak{d}} \end{pmatrix} + \rho^2 \begin{pmatrix} -\frac{\varepsilon}{n\mathcal{L}\mathcal{M}} & 0 \\ 0 & 1-\frac{\varepsilon}{n\mathcal{L}} \end{pmatrix}\begin{pmatrix} \widehat{\mathfrak{b}} \\ \widehat{\mathfrak{d}} \end{pmatrix} \\ = \rho^2 \begin{pmatrix} \frac{\varepsilon(1+\mathcal{M}^{-1})}{n^{3/2}\mathcal{L}} & 0 \\ 0 & \frac{\varepsilon}{n} - \frac{1+\varepsilon^2(1+\mathcal{M}^{-1}(n+1))}{n^2\mathcal{L}} \end{pmatrix}\begin{pmatrix} \widehat{\mathfrak{j}}_0 \\ \widehat{\mathfrak{j}}_1 \end{pmatrix}. \end{aligned}$$

In order to estimate $(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})$, we just follow the method of Appendix B, which requires Condition (4.8) and

$$\rho \le \frac{\sqrt{1+n^{-1}}}{1 - \frac{(1-\mathcal{M}^{-1})\varepsilon}{n\mathcal{L}}}\cdot \tag{4.13}$$

Keeping (4.12) in mind and noticing that

$$\tilde{\nu} = 1 - \frac{\varepsilon}{n\mathcal{L}}(1+\mathcal{M}^{-1}),$$

is of order 1 for small ε, we thus conclude that if

$$\rho \le \sqrt{1+n^{-1}} \quad \text{and} \quad \rho \lesssim \mathcal{L}\min(\varepsilon^{-1}, \mathcal{M}), \tag{4.14}$$

then

$$|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(t)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau\lesssim|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(0)|+\rho^2\int_0^t\Big(\frac{\varepsilon}{\mathcal{L}}|\widehat{\mathfrak{j}}_0|+\Big(\varepsilon+\frac{1}{\mathcal{L}}\Big)|\widehat{\mathfrak{j}}_1|\Big)d\tau+\frac{\varepsilon\rho^3}{\mathcal{L}}\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau. \tag{4.15}$$

Next, we see that the equations for $(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)$ read (omitting the $\mathcal{O}(\varepsilon\rho^3/\mathcal{L})$ term)

$$\frac{d}{dt}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}+\begin{pmatrix}\frac{\mathcal{L}}{\varepsilon}+\frac{\varepsilon\rho^2}{n\mathcal{L}} & 0\\ 0 & \frac{\mathcal{L}\mathcal{M}}{\varepsilon}+\frac{\varepsilon\rho^2}{n\mathcal{L}\mathcal{M}}\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}+\frac{\rho}{\sqrt{n}\,\varepsilon}\begin{pmatrix}0 & 1+\varepsilon^2\\ -1 & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}=\rho^2\begin{pmatrix}-\frac{1+\varepsilon^2(1+\mathcal{M}(1+n))}{\varepsilon\sqrt{n}\,\mathcal{L}\mathcal{M}} & 0\\ 0 & \frac{1+\mathcal{M}^{-1}}{\mathcal{L}}\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix}. \tag{4.16}$$

Therefore, computing

$$\frac{d}{dt}\Big(|\widehat{\mathfrak{j}}_0|^2+(1+\varepsilon^2)|\widehat{\mathfrak{j}}_1|^2\Big), \tag{4.17}$$

so as to eliminate the term in ρ, we end up with

$$|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)|+\frac{\mathcal{L}}{\varepsilon}\int_0^t|\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1|\,d\tau\lesssim|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|+\rho^2\int_0^t\Big(\Big(\frac{1}{\varepsilon\mathcal{L}\mathcal{M}}+\frac{\varepsilon}{\mathcal{L}}\Big)|\widehat{\mathfrak{b}}|+\frac{1}{\mathcal{L}}|\widehat{\mathfrak{d}}|\Big)d\tau+\frac{\varepsilon\rho^3}{\mathcal{L}}\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau. \tag{4.18}$$

Now, adding up (4.15) and (4.18), we easily conclude that if ε is small enough and

$$\mathcal{L}\min(1,\varepsilon\mathcal{M})\gtrsim 1, \tag{4.19}$$

then we have

$$|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau+\frac{\mathcal{L}}{\varepsilon}\int_0^t|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau\lesssim|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|, \tag{4.20}$$

whenever $0\leq\rho\leq\sqrt{1+n^{-1}}$.

Now, resuming to the $\widehat{\mathfrak{j}}_1$ equation in (4.16), and evaluating the first order term according to (4.20), we deduce that, in addition

$$\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|\widehat{\mathfrak{j}}_1|\,d\tau\lesssim|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|. \tag{4.21}$$

4.1.2. *The case $\varepsilon\ll\mathcal{L}\lesssim 1$ with $\varepsilon\mathcal{L}^2\mathcal{L}_s^2\gg 1$ and $\mathcal{L}^2\mathcal{L}_s\gg 1$.* If $\mathcal{L}\ll 1$ then plugging (4.15) in (4.18) does not allow to get (4.20) any longer. In order to overcome this, we shall follow the *second* approach proposed in Appendix A with coefficients defined as in (4.9): we set

$$P=\begin{pmatrix}0 & 0 & 0 & \frac{\varepsilon^2}{n\mathcal{L}\mathcal{M}}\\ 0 & 0 & \frac{\varepsilon}{n^{3/2}\mathcal{L}} & 0\\ 0 & -\frac{\varepsilon\sqrt{n}}{\mathcal{L}} & 0 & \frac{1+\varepsilon^2}{\sqrt{n}(1-\mathcal{M})\mathcal{L}}\\ -\frac{1}{\mathcal{L}\mathcal{M}} & 0 & \frac{1}{\sqrt{n}\,(1-\mathcal{M})\mathcal{L}} & 0\end{pmatrix},$$

and we thus have, remembering that $\mathcal{M}-1=\mathcal{L}_s$

$$V=\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\\ \widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}:=\begin{pmatrix}1 & 0 & 0 & \frac{\varepsilon^2}{n\mathcal{L}\mathcal{M}}\rho\\ -\frac{\varepsilon}{n\mathcal{L}}\rho & 1 & \frac{\varepsilon}{n^{3/2}\mathcal{L}}\rho & \frac{\varepsilon}{n}\\ -\sqrt{n} & -\frac{\sqrt{n}\,\varepsilon}{\mathcal{L}}\rho & 1 & -\frac{1+\varepsilon^2\mathcal{M}}{\sqrt{n}\,\mathcal{L}\mathcal{L}_s}\rho\\ \frac{\rho}{\mathcal{L}\mathcal{M}\mathcal{L}_s} & 0 & -\frac{\rho}{\sqrt{n}\,\mathcal{L}_s\mathcal{L}} & 1\end{pmatrix}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix}. \tag{4.22}$$

The determinant of the above matrix is

$$\Big(1+\frac{\varepsilon^2}{n\mathcal{L}^2}\rho^2\Big)\Big(1+\frac{\varepsilon^2}{n\mathcal{L}^2\mathcal{M}^2}\rho^2\Big)-\frac{1+\varepsilon^2}{n\mathcal{L}^2\mathcal{L}_s^2}\rho^2,$$

Hence working with $(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)$ or $(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)$ is equivalent whenever

$$\rho\lesssim\frac{\mathcal{L}}{\varepsilon}\quad\text{and}\quad\rho\leq\sqrt{n}\,\mathcal{L}\mathcal{L}_s. \tag{4.23}$$

Then following the computations of Appendix A, second approach, and setting $A_3:=(PA_0-A_1)P^2+A_2P$ leads to

$$\begin{aligned}\frac{d}{dt}V+\rho&\begin{pmatrix}0 & 1 & 0 & 0\\ -1-\frac{1}{n} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}V\\ &+\begin{pmatrix}-\frac{\varepsilon}{n\mathcal{L}\mathcal{M}}\rho^2 & 0 & 0 & 0\\ 0 & \big(1-\frac{\varepsilon}{n\mathcal{L}}\big)\rho^2 & 0 & 0\\ 0 & 0 & \frac{\mathcal{L}}{\varepsilon}+\big(\varepsilon+\frac{1+\varepsilon^2}{\varepsilon\mathcal{L}_s}\big)\frac{\rho^2}{n\mathcal{L}} & 0\\ 0 & 0 & 0 & \frac{\mathcal{L}\mathcal{M}}{\varepsilon}+\big(\frac{\varepsilon}{n\mathcal{L}\mathcal{M}}-\frac{1+\varepsilon^2}{n\varepsilon\mathcal{L}\mathcal{L}_s}\big)\rho^2\end{pmatrix}V\\ =\rho^2&\begin{pmatrix}0 & 0 & \frac{(1+\mathcal{M}^{-1})\varepsilon}{n^{3/2}\mathcal{L}} & 0\\ 0 & 0 & 0 & \frac{\varepsilon}{n}-\frac{1+\varepsilon^2}{n^2\mathcal{L}}-\frac{\varepsilon^2(1+n)}{n^2\mathcal{L}\mathcal{M}}\\ -\frac{1+\varepsilon^2}{\varepsilon\sqrt{n}\,\mathcal{L}_s\mathcal{L}}-\frac{(n+1)\varepsilon}{\sqrt{n}\,\mathcal{L}} & 0 & 0 & 0\\ 0 & -\frac{1}{\mathcal{L}\mathcal{L}_s\mathcal{M}} & 0 & 0\end{pmatrix}V\\ &\qquad+\rho^3(I+\rho P)A_3(I+\rho P)^{-1}V.\end{aligned} \tag{4.24}$$

Let us bound A_3 in order to determinate for which values of ρ the last term in (4.24) is indeed negligible. Just writing that $|A_3|\leq|P|^2(|P|\,|A_0|+|A_1|)+|A_2|\,|P|$ using the explicit values of A_0, A_1 and A_2 and

$$|P|\lesssim\frac{1}{\mathcal{L}}\max\Big(\varepsilon,\frac{1}{\mathcal{L}_s}\Big), \tag{4.25}$$

does not provide an accurate enough bound for A_3. Hence one has to go to further computations. Now, we get

$$PA_0P^2=\begin{pmatrix}0 & \frac{\varepsilon^2}{n\mathcal{L}^2\mathcal{L}_s} & 0 & \frac{\varepsilon}{n^2\mathcal{L}^2}\big(\frac{1+\varepsilon^2}{\mathcal{L}_s^2}-\frac{\varepsilon^2}{\mathcal{M}^2}\big)\\ \frac{1+\varepsilon^2}{n^2\mathcal{L}^2\mathcal{M}\mathcal{L}_s} & 0 & \frac{1}{n^{5/2}\mathcal{L}^2}\big(\frac{1+\varepsilon^2}{\mathcal{L}_s^2}-\varepsilon^2\big) & 0\\ 0 & -\frac{(1+\varepsilon^2)\mathcal{M}}{\sqrt{n}\,\mathcal{L}^2\mathcal{L}_s^2} & 0 & \frac{(1+\varepsilon^2)\mathcal{M}}{n^{3/2}\varepsilon\mathcal{L}_s\mathcal{L}^2}\big(\frac{\varepsilon^2}{\mathcal{M}^2}-\frac{1+\varepsilon^2}{\mathcal{L}_s^2}\big)\\ -\frac{\varepsilon(1+\varepsilon^2)}{n\mathcal{L}^2\mathcal{M}\mathcal{L}_s^2} & 0 & \frac{\varepsilon}{n^{3/2}\mathcal{L}_s\mathcal{L}^2}\big(\varepsilon^2-\frac{1+\varepsilon^2}{\mathcal{L}_s^2}\big) & 0\end{pmatrix},$$

$$A_2P=\begin{pmatrix}0&0&0&0\\ \frac{\varepsilon}{n\mathcal{L}\mathcal{M}}&0&\frac{\varepsilon}{n^{3/2}\mathcal{L}}&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}\quad\text{and}\quad A_1P^2=\begin{pmatrix}0&-\frac{\varepsilon^2}{n\mathcal{L}^2}&0&-\frac{\varepsilon(1+\varepsilon^2)}{n\mathcal{L}^2\mathcal{L}_s}\\ \frac{(1+n^{-1})\varepsilon^2}{n\mathcal{L}^2\mathcal{M}^2}&0&\frac{(1+n^{-1})\varepsilon^2}{n^{3/2}\mathcal{L}^2\mathcal{L}_s\mathcal{M}}&\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}.$$

Hence, given that $\varepsilon\ll\mathcal{L}\lesssim1$ and that $\mathcal{L}_s\approx\mathcal{M}$ in the regime that we are considering, one may conclude that

$$|A_3|\lesssim\max\Big(\frac{\varepsilon}{\mathcal{L}},\frac{1}{\mathcal{L}^2\mathcal{L}_s},\frac{1}{\varepsilon\mathcal{L}^2\mathcal{L}_s^2}\Big).\tag{4.26}$$

Note that we still have $\tilde\nu\to1$ for $\varepsilon\to0$. Hence applying the method of the appendix to handle $(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})$, we find out that if (4.8), (4.13) and (4.23) are fulfilled then

$$\begin{aligned}|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(t)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau\lesssim|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(0)|+\rho^2\int_0^t\Big(\frac{\varepsilon}{\mathcal{L}}|\widehat{\mathfrak{j}}_0|+\frac{1}{\mathcal{L}}|\widehat{\mathfrak{j}}_1|\Big)d\tau\\ +\rho^3\max\Big(\frac{\varepsilon}{\mathcal{L}},\frac{1}{\mathcal{L}^2\mathcal{L}_s},\frac{1}{\varepsilon\mathcal{L}^2\mathcal{L}_s^2}\Big)\int_0^t|V|\,d\tau.\end{aligned}$$

As regards the radiative modes, we have

$$\begin{aligned}|\widehat{\mathfrak{j}}_0(t)|+\bigg(\frac{\mathcal{L}}{\varepsilon}+\Big(\varepsilon+\frac{1+\varepsilon^2}{\varepsilon\mathcal{L}_s}\Big)\frac{\rho^2}{n\mathcal{L}}\bigg)\int_0^t|\widehat{\mathfrak{j}}_0|\,d\tau\leq|\widehat{\mathfrak{j}}_0(0)|+C\rho^2\int_0^t\Big(\frac{1}{\varepsilon\mathcal{L}\mathcal{L}_s}+\frac{\varepsilon}{\mathcal{L}}\Big)|\widehat{\mathfrak{b}}|\,d\tau\\ +C\rho^3\max\Big(\frac{\varepsilon}{\mathcal{L}},\frac{1}{\mathcal{L}^2\mathcal{L}_s},\frac{1}{\varepsilon\mathcal{L}^2\mathcal{L}_s^2}\Big)\int_0^t|V|\,d\tau,\end{aligned}$$

$$\begin{aligned}|\widehat{\mathfrak{j}}_1(t)|+\bigg(\frac{\mathcal{L}\mathcal{M}}{\varepsilon}+\Big(\frac{\varepsilon}{n\mathcal{L}\mathcal{M}}-\frac{1+\varepsilon^2}{n\varepsilon\mathcal{L}\mathcal{L}_s}\Big)\rho^2\bigg)\int_0^t|\widehat{\mathfrak{j}}_1|\,d\tau\leq|\widehat{\mathfrak{j}}_1(0)|+C\frac{\rho^2}{\mathcal{L}\mathcal{L}_s\mathcal{M}}\int_0^t|\widehat{\mathfrak{d}}|\,d\tau\\ +C\rho^3\max\Big(\frac{\varepsilon}{\mathcal{L}},\frac{1}{\mathcal{L}^2\mathcal{L}_s},\frac{1}{\varepsilon\mathcal{L}^2\mathcal{L}_s^2}\Big)\int_0^t|V|\,d\tau.\end{aligned}$$

From the above three inequalities, we get for any $A\in(0,1]$

$$\begin{aligned}|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(t)|+A|\widehat{\mathfrak{j}}_0(t)|+|\widehat{\mathfrak{j}}_1(t)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau+A\frac{\mathcal{L}}{\varepsilon}\int_0^t|\widehat{\mathfrak{j}}_0|\,d\tau+\frac{\mathcal{L}\mathcal{L}_s}{\varepsilon}\int_0^t|\widehat{\mathfrak{j}}_1|\,d\tau\\ \lesssim|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(0)|+A|\widehat{\mathfrak{j}}_0(0)|+|\widehat{\mathfrak{j}}_1(0)|+\rho^2\int_0^t\Big(\frac{\varepsilon}{\mathcal{L}}|\widehat{\mathfrak{j}}_0|+\frac{1}{\mathcal{L}}|\widehat{\mathfrak{j}}_1|\Big)d\tau+A\rho^2\int_0^t\Big(\frac{1}{\varepsilon\mathcal{L}\mathcal{L}_s}+\frac{\varepsilon}{\mathcal{L}}\Big)|\widehat{\mathfrak{b}}|\,d\tau\\ +\frac{\rho^2}{\mathcal{L}\mathcal{L}_s^2}\int_0^t|\widehat{\mathfrak{d}}|\,d\tau+\rho^3\max\Big(\frac{\varepsilon}{\mathcal{L}},\frac{1}{\mathcal{L}^2\mathcal{L}_s},\frac{1}{\varepsilon\mathcal{L}^2\mathcal{L}_s^2}\Big)\int_0^t|V|\,d\tau.\end{aligned}$$

Now, we notice that taking $A=c_0\min(1,\varepsilon\mathcal{L}\mathcal{L}_s)$ for a sufficiently small constant c_0 allows to absorb all the terms of the r.h.s. (but the data) by the l.h.s. provided we have $0\leq\rho\leq\sqrt{1+n^{-1}}$

$$\varepsilon\ll\mathcal{L}\lesssim1,\quad\varepsilon\mathcal{L}^2\mathcal{L}_s^2\gg1\ \text{ and }\ \mathcal{L}^2\mathcal{L}_s\gg1.\tag{4.27}$$

We thus conclude that for all $0\leq\rho\leq\sqrt{1+n^{-1}}$, we have

$$\begin{aligned}|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\min(1,\varepsilon\mathcal{L}\mathcal{L}_s)\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau+\frac{\mathcal{L}}{\varepsilon}\min(1,\varepsilon\mathcal{L}\mathcal{L}_s)\int_0^t|\widehat{\mathfrak{j}}_0|\,d\tau\\ +\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|\widehat{\mathfrak{j}}_1|\,d\tau\leq C|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\min(1,\varepsilon\mathcal{L}\mathcal{L}_s)\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|.\end{aligned}\tag{4.28}$$

Let us point out that in the case where $\mathcal{L}^2\mathcal{L}_s \approx 1$ (even if $\mathcal{L} \approx \varepsilon$ in fact) then the same computation will lead to (4.28), but only for $0 \le \rho \le c$, with c a small enough constant.

4.1.3. *The case* $\varepsilon \ll \mathcal{L} \lesssim \varepsilon^{1/2}$ *and* $\mathcal{L}^2\mathcal{L}_s \approx 1$. As the value $\sqrt{1+n^{-1}}$ will not play any particular role, we fix some $C_1 > c$ in the following computations. We want to get (4.28) for $\rho \in [c, C_1]$. To this end, we introduce ζ_0 and ζ_1 such that

$$\widehat{\zeta}_0 := \widehat{j}_0 - \frac{\sqrt{n}}{1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}}\widehat{b} \quad \text{and} \quad \widehat{\zeta}_1 := \widehat{j}_1 - \frac{\rho}{\sqrt{n}\,\mathcal{L}\mathcal{M}}\widehat{j}_0.$$

Then we discover that $(\widehat{b}, \widehat{d}, \widehat{\zeta}_0, \widehat{\zeta}_1)$ fulfills

$$\begin{cases} \partial_t\widehat{b} + \rho\widehat{d} = 0, \\ \partial_t\widehat{d} + \rho^2\widehat{d} - \rho\left(1 + \dfrac{1}{n\big(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\big)}\right)\widehat{b} = \dfrac{\rho}{n^{3/2}}\widehat{\zeta}_0 + \dfrac{\mathcal{L}\mathcal{M}}{n}\widehat{\zeta}_1, \\ \partial_t\widehat{\zeta}_0 + \dfrac{\mathcal{L}}{\varepsilon}\widehat{\zeta}_0 = -\dfrac{\rho}{\varepsilon\sqrt{n}}\widehat{\zeta}_1 - \sqrt{n}\dfrac{\rho}{1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}}\widehat{d}, \\ \partial_t\widehat{\zeta}_1 + \dfrac{\mathcal{L}}{\varepsilon}\left(\mathcal{M} - \dfrac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)\widehat{\zeta}_1 = \dfrac{\rho}{\sqrt{n}\,\varepsilon\mathcal{M}}\left(1+\dfrac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)\widehat{\zeta}_0. \end{cases}$$

Let ρ be in $[c, C_1]$. For the first two equations, performing the standard barotropic estimates (which rely on the use of $\mathcal{U}_\rho$ defined in (4.43)) leads to

$$|(\widehat{b},\widehat{d})(t)| + \int_0^t |(\widehat{b},\widehat{d})|\,d\tau \lesssim |(\widehat{b},\widehat{d})(0)| + \int_0^t |\widehat{\zeta}_0|\,d\tau + \mathcal{L}\mathcal{L}_s\int_0^t |\widehat{\zeta}_1|\,d\tau. \tag{4.29}$$

For $\widehat{\zeta}_0$, it is obvious that

$$|\widehat{\zeta}_0(t)| + \frac{\mathcal{L}}{\varepsilon}\int_0^t |\widehat{\zeta}_0|\,d\tau \le |\widehat{\zeta}_0(0)| + \frac{\rho}{\varepsilon\sqrt{n}}\int_0^t |\widehat{\zeta}_1|\,d\tau + \rho\sqrt{n}\int_0^t |\widehat{d}|\,d\tau, \tag{4.30}$$

and, as our conditions on $\mathcal{L}$ and $\mathcal{L}_s$ guarantee that $\rho \le \sqrt{n/2}\,\mathcal{L}\mathcal{M}$ for small enough ε, we also have

$$|\widehat{\zeta}_1(t)| + \frac{\mathcal{L}\mathcal{M}}{2\varepsilon}\int_0^t |\widehat{\zeta}_1|\,d\tau \le |\widehat{\zeta}_1(0)| + \frac{\rho}{\sqrt{n}\,\varepsilon\mathcal{M}}\left(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)\int_0^t |\widehat{d}|\,d\tau. \tag{4.31}$$

Putting together those three inequalities, we readily get for all $A, B > 0$, observing that $\rho^2 \approx \mathcal{L}^2\mathcal{M} \approx 1$

$$\begin{aligned} |(\widehat{b},\widehat{d})(t)| &+ A\frac{\varepsilon}{\mathcal{L}}|\widehat{\zeta}_0(t)| + B\varepsilon|\widehat{\zeta}_1(t)| + \int_0^t |(\widehat{b},\widehat{d})|\,d\tau + A\int_0^t |\widehat{\zeta}_0|\,d\tau + B\mathcal{L}\mathcal{M}\int_0^t |\widehat{\zeta}_1|\,d\tau \\ &\lesssim |(\widehat{b},\widehat{d})(0)| + A\frac{\varepsilon}{\mathcal{L}}|\widehat{\zeta}_0(0)| + B\varepsilon|\widehat{\zeta}_1(0)| + \mathcal{L}\mathcal{M}\int_0^t |\widehat{\zeta}_1|\,d\tau \\ &\quad + \int_0^t |\widehat{\zeta}_0|\,d\tau + \frac{A}{\mathcal{L}}\int_0^t |\widehat{\zeta}_1|\,d\tau + A\frac{\varepsilon}{\mathcal{L}}\int_0^t |\widehat{d}|\,d\tau + \frac{B}{\mathcal{M}}\int_0^t |\widehat{\zeta}_0|\,d\tau. \end{aligned}$$

It is now clear that if one takes first A large enough (independently of ε) and B much larger, all the integrals of the r.h.s. may be absorbed by the left-hand side, as $\mathcal{M}^{-1} \ll \rho$, and we thus get for all $\rho\in[c,C_1]$

$$|(\widehat{b},\widehat{d})(t)|+\frac{\varepsilon}{\mathcal{L}}|\widehat{\zeta}_0(t)|+\varepsilon|\widehat{\zeta}_1(t)|+\int_0^t|(\widehat{b},\widehat{d})|\,d\tau+\int_0^t|\widehat{\zeta}_0|\,d\tau+\mathcal{L}\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau\lesssim|(\widehat{b},\widehat{d})(0)|+\frac{\varepsilon}{\mathcal{L}}|\widehat{\zeta}_0(0)|+\varepsilon|\widehat{\zeta}_1(0)|. \tag{4.32}$$

Plugging this new inequality in (4.31), we easily deduce that

$$|\widehat{\zeta}_1(t)|+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau\lesssim|\widehat{\zeta}_1(0)|+\mathcal{L}|\widehat{\zeta}_0(0)|+\frac{1}{\varepsilon\mathcal{M}}|(\widehat{b},\widehat{d})(0)|,$$

then inserting this information and (4.32) in (4.30), we discover that

$$|\widehat{\zeta}_0(0)|+\frac{\mathcal{L}}{\varepsilon}\int_0^t|\widehat{\zeta}_0|\,d\tau\lesssim|\widehat{\zeta}_0(0)|+\mathcal{L}|\widehat{\zeta}_1(0)|+\left(1+\frac{1}{\varepsilon\mathcal{L}\mathcal{M}^2}\right)|(\widehat{b},\widehat{d})(0)|.$$

Therefore, putting (4.32) and the above two inequalities together, using that $|(\widehat{b},\widehat{d},\widehat{\zeta}_0,\widehat{\zeta}_1)|\approx|(\widehat{b},\widehat{d},\widehat{\jmath}_0,\widehat{\jmath}_1)|$ *and assuming in addition that* $\mathcal{L}\lesssim\varepsilon^{1/2}$, we conclude that

$$|(\widehat{b},\widehat{d},\frac{\varepsilon}{\mathcal{L}}\widehat{\jmath}_0,\widehat{\jmath}_1)(t)|+\int_0^t|(\widehat{b},\widehat{d})|\,d\tau+\frac{\mathcal{L}}{\varepsilon}\int_0^t|\widehat{\zeta}_0|\,d\tau+\frac{\mathcal{L}\mathcal{L}_s}{\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau\lesssim|(\widehat{b},\widehat{d},\frac{\varepsilon}{\mathcal{L}}\widehat{\jmath}_0,\widehat{\jmath}_1)(0)|\quad\text{for all } c\leq\rho\leq C_1. \tag{4.33}$$

Note that due to the expression of $\widehat{\zeta}_1$, one may replace $\widehat{\zeta}_1$ with $\widehat{\jmath}_1$ of $\widehat{\mathrm{j}}_1$ in the integral.

Still in the case $\mathcal{L}\lesssim\varepsilon^{1/2}$, we claim that we have the following inequality

$$|\rho\widehat{\jmath}_0(t)|+\frac{\mathcal{L}}{\varepsilon}\int_0^t|\rho\widehat{\zeta}_0|\,d\tau\lesssim|(\widehat{b},\widehat{d},\rho\widehat{\jmath}_0,\frac{\varepsilon}{\mathcal{L}}\widehat{\jmath}_0,\widehat{\jmath}_1)(0)|\quad\text{for all}\quad 0\leq\rho\leq C_1, \tag{4.34}$$

which turns out to be crucial in the justification of the asymptotics toward (2.9).

Indeed, inequality (4.30) does not require any assumption on ρ and thus implies that

$$|\rho\widehat{\zeta}_0(t)|+\frac{\mathcal{L}}{\varepsilon}\int_0^t|\rho\widehat{\zeta}_0|\,d\tau\leq|\rho\widehat{\zeta}_0(0)|+\frac{\rho^2}{\varepsilon\sqrt{n}}\int_0^t|\widehat{\zeta}_1|\,d\tau+\rho^2\sqrt{n}\int_0^t|\widehat{d}|\,d\tau.$$

For $0\leq\rho\leq c$ (resp. $c\leq\rho\leq C_1$), the last term may be bounded according to (4.28) (resp. (4.33)). Regarding the term with $\widehat{\zeta}_1$, we notice that

$$\widehat{\zeta}_1=\widehat{\mathrm{j}}_1+\frac{\rho}{\mathcal{L}\mathcal{L}_s\mathcal{M}}\left(\frac{\widehat{\jmath}_0}{\sqrt{n}}+\widehat{b}\right),$$

hence (4.28) and the fact that $\mathcal{L}^2\mathcal{L}_s\approx1$ guarantee that for $0\leq\rho\leq c$

$$\begin{aligned}\frac{\rho^2}{\varepsilon\sqrt{n}}\int_0^t|\widehat{\zeta}_1|\,d\tau&\lesssim\mathcal{L}\left(\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|\widehat{\mathrm{j}}_1|\,d\tau+\frac{\mathcal{L}^2}{\varepsilon}\int_0^t\rho^2|\widehat{\jmath}_0+\sqrt{n}\widehat{b}|\,d\tau\right)\\&\lesssim\mathcal{L}(1+\tfrac{\mathcal{L}^2}{\varepsilon})|(\widehat{b},\widehat{d},\varepsilon\mathcal{L}\mathcal{L}_s\widehat{\jmath}_0,\widehat{\jmath}_1)(0)|.\end{aligned}$$

In the case $\mathcal{L}\lesssim\varepsilon^{1/2}$, it is obvious that Inequality (4.33) implies that the above inequality is also true in the range $c\leq\rho\leq C_1$, which completes the proof of (4.34).

4.1.4. *The case $\mathcal{L}\approx\varepsilon$ and $\mathcal{L}_s\varepsilon^2\gtrsim 1$.* We saw that if (4.8) is fulfilled then (4.28) holds true on some small interval $[0,c]$. So we have to fill in the gap between c and $\sqrt{1+n^{-1}}$. As the value $\sqrt{1+n^{-1}}$ does not play any particular role, we fix some $C_1>c$ and look for estimates if $\rho\in[c,C_1]$. For simplicity, take $\mathcal{L}=\kappa\varepsilon/n$ (with $\kappa>1$ owing to (4.8)).

Setting $\widehat{\zeta}_1:=\widehat{j}_1-\frac{1}{\sqrt{n}\mathcal{L}\mathcal{M}}\rho\widehat{j}_0=\widehat{j}_1-\frac{\sqrt{n}}{\kappa\varepsilon\mathcal{M}}\rho\widehat{j}_0$ as before, we observe that

$$\left\{\begin{aligned}&\partial_t\widehat{b}+\rho\widehat{d}=0,\\&\partial_t\widehat{d}+\rho^2\widehat{d}-\rho\widehat{b}=\tfrac{\rho}{n^{3/2}}\widehat{j}_0+\tfrac{\kappa\varepsilon\mathcal{M}}{n^2}\widehat{\zeta}_1,\\&\partial_t\widehat{j}_0+\big(\tfrac{\kappa}{n}+\tfrac{\rho^2}{\kappa\varepsilon^2\mathcal{M}}\big)\widehat{j}_0+\tfrac{\rho}{\sqrt{n\varepsilon}}\widehat{\zeta}_1=\tfrac{\kappa}{\sqrt{n}}\widehat{b},\\&\partial_t\widehat{\zeta}_1+\big(\tfrac{\kappa\mathcal{M}}{n}-\tfrac{\rho^2}{\kappa\varepsilon^2\mathcal{M}}\big)\widehat{\zeta}_1=\tfrac{\rho}{\sqrt{n}\,\varepsilon\mathcal{M}}\big(1+\tfrac{\rho^2 n}{\kappa^2\varepsilon^2\mathcal{M}}\big)\widehat{j}_0-\tfrac{\rho}{\varepsilon\mathcal{M}}\widehat{b}.\end{aligned}\right.\tag{4.35}$$

Let us focus on the subsystem corresponding to the first three equations, namely

$$\left\{\begin{aligned}&\partial_t\widehat{b}+\rho\widehat{d}=0,\\&\partial_t\widehat{d}+\rho^2\widehat{d}-\rho\widehat{b}-\tfrac{\rho}{n^{3/2}}\widehat{j}_0=\widehat{f},\\&\partial_t\widehat{j}_0+\big(\tfrac{\kappa}{n}+\tfrac{\rho^2}{\kappa\varepsilon^2\mathcal{M}}\big)\widehat{j}_0-\tfrac{\kappa}{\sqrt{n}}\widehat{b}=\widehat{g}.\end{aligned}\right.\tag{4.36}$$

If we have the stronger condition $\varepsilon^2\mathcal{M}\to\infty$ then we rewrite System (4.36) as follows

$$\frac{d}{dt}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\end{pmatrix}+\begin{pmatrix}0&\rho&0\\-\rho&\rho^2&-\frac{\rho}{n^{3/2}}\\-\frac{\kappa}{\sqrt{n}}&0&\frac{\kappa}{n}\end{pmatrix}\begin{pmatrix}\widehat{b}\\ \widehat{d}\\ \widehat{j}_0\end{pmatrix}=\begin{pmatrix}0\\ \widehat{f}\\ -\frac{\rho^2}{\kappa\varepsilon^2\mathcal{M}}\widehat{j}_0+\widehat{g}\end{pmatrix}\cdot\tag{4.37}$$

The eigenvalues of the matrix M_ρ in the left-hand side are the roots of the polynomial $-X^3+a_1(\rho)X^2-a_2(\rho)X+a_3(\rho)$ with

$$a_1(\rho)=\frac{\kappa}{n}+\rho^2,\quad a_2(\rho)=\Big(1+\frac{\kappa}{n}\Big)\rho^2,\quad a_3(\rho)=\Big(1+\frac{1}{n}\Big)\frac{\rho^2\kappa}{n}\cdot$$

According to Routh-Hurwitz criterion, those roots have *positive* real part if and only if

$$a_1(\rho)>0,\qquad\begin{vmatrix}a_1(\rho)&1\\a_3(\rho)&a_2(\rho)\end{vmatrix}>0\quad\text{and}\quad\begin{vmatrix}a_1(\rho)&1&0\\a_3(\rho)&a_2(\rho)&a_1(\rho)\\0&0&a_3(\rho)\end{vmatrix}>0.$$

As $a_1(\rho)$ and $a_3(\rho)$ are positive, it suffices to check the second condition, that is

$$a_1(\rho)a_2(\rho)-a_3(\rho)=\Big(1+\frac{\kappa}{n}\Big)\rho^4+\frac{\rho^2\kappa}{n^2}(\kappa-1)>0,$$

and this is indeed the case for all $\rho>0$, as $\kappa>1$.

In particular, all the eigenvalues of the matrix M_ρ have positive real part if we assume ρ to belong to the compact set $[c,C_1]$. Therefore (see [7]) there exist two positive constants c_2 and C_2 depending only on c and C_1, so that the matrix M_ρ satisfies

$$\big|e^{-tM_\rho}\big|\leq C_2e^{-c_2t}\quad\text{for all}\ \ t\geq0\ \ \text{and}\ \ \rho\in[c,C_1].$$

By taking advantage of Duhamel's formula, we thus deduce that

$$|(\widehat{b},\widehat{d},\widehat{j}_0)(t)|+\int_0^t|(\widehat{a},\widehat{d},\widehat{j}_0)|\,d\tau\lesssim|(\widehat{b},\widehat{d},\widehat{j}_0)(0)|+\int_0^t|(\widehat{f},\widehat{g})|\,d\tau+\frac{1}{\varepsilon^2\mathcal{M}}\int_0^t|\widehat{j}_0|\,d\tau.$$

Of course, owing to the assumption $\varepsilon^2\mathcal{M}\to\infty$, the last term of the r.h.s. may be absorbed by the l.h.s., for ε going to 0. So we get

$$|(\widehat{b},\widehat{d},\widehat{j}_0)(t)|+\int_0^t|(\widehat{b},\widehat{d},\widehat{j}_0)|\,d\tau\lesssim|(\widehat{b},\widehat{d},\widehat{j}_0)(0)|+\int_0^t|(\widehat{f},\widehat{g})|\,d\tau. \tag{4.38}$$

In the case where $\varepsilon^2\mathcal{M}$ does not go to ∞ then we have to proceed slightly differently. If we assume (for simplicity) that $\varepsilon^2\mathcal{M}$ tends to some $m>0$, then the matrix M_ρ in (4.37) has to be changed in

$$N_\rho=\begin{pmatrix}0&\rho&0\\-\rho&\rho^2&-\frac{\rho}{n^{3/2}}\\-\frac{\kappa}{\sqrt{n}}&0&\frac{\kappa}{n}+\frac{\rho^2}{\kappa m}\end{pmatrix}.$$

The above analysis based on Routh-Hurwitz theorem still holds as the additional term has 'the good sign', and one may conclude, as before, that (4.38) is satisfied for all $\rho\in[c,C_1]$.

In every case $\varepsilon^2\mathcal{M}\gtrsim1$, resuming to (4.35), Inequality (4.38) allows us to get for all $\rho\in[c,C_1]$

$$|(\widehat{b},\widehat{d},\widehat{j}_0)(t)|+\int_0^t|(\widehat{b},\widehat{d},\widehat{j}_0)|\,d\tau\lesssim|(\widehat{b},\widehat{d},\widehat{j}_0)(0)|+\varepsilon\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau. \tag{4.39}$$

Next, from the equation of $\widehat{\zeta}_1$, we readily get

$$|\widehat{\zeta}_1(t)|+\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau\lesssim|\widehat{\zeta}_1(0)|+(\varepsilon\mathcal{M})^{-1}\int_0^t|(\widehat{b},\widehat{j}_0)|\,d\tau.$$

Hence, adding up to Inequality (4.39), we conclude that (4.28) is also true for all $\rho\in[c,C_1]$. This completes the proof of estimates in the low frequency regime $\rho\in[0,C_1]$.

4.2. **Estimates for middle frequencies.**

The case $\liminf\mathcal{M}=+\infty$. As in the previous paragraph, introduce $\widehat{\zeta}_1:=\widehat{j}_1-\frac{\rho}{\sqrt{n}\mathcal{L}\mathcal{M}}\widehat{j}_0$. The system fulfilled by $(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{\zeta}_1)$ reads

$$\begin{cases}\partial_t\widehat{b}+\rho\widehat{d}=0,\\ \partial_t\widehat{d}+\rho^2\widehat{d}-\rho\widehat{b}=\frac{\rho}{n^{3/2}}\widehat{j}_0+\frac{\mathcal{L}\mathcal{M}}{n}\widehat{\zeta}_1,\\ \partial_t\widehat{j}_0+\frac{1}{\varepsilon}\big(\mathcal{L}+\frac{\rho^2}{n\mathcal{L}\mathcal{M}}\big)\widehat{j}_0+\frac{\rho}{\sqrt{n}\,\varepsilon}\widehat{\zeta}_1=\sqrt{n}\frac{\mathcal{L}}{\varepsilon}\widehat{b},\\ \partial_t\widehat{\zeta}_1+\big(\frac{\mathcal{L}\mathcal{M}}{\varepsilon}-\frac{\rho^2}{n\varepsilon\mathcal{L}\mathcal{M}}\big)\widehat{\zeta}_1=\frac{\rho}{\sqrt{n}\mathcal{L}\mathcal{M}\varepsilon}\big(\mathcal{L}+\frac{\rho^2}{n\mathcal{L}\mathcal{M}}\big)\widehat{j}_0-\frac{\rho}{\varepsilon\mathcal{M}}\widehat{b}.\end{cases} \tag{4.40}$$

The subsystem corresponding to the first three equations is

$$\begin{cases}\partial_t\widehat{b}+\rho\widehat{d}=0,\\ \partial_t\widehat{d}+\rho^2\widehat{d}-\rho\widehat{b}-\frac{\rho}{n^{3/2}}\widehat{j}_0=\widehat{f},\\ \partial_t\widehat{j}_0+\frac{\mathcal{L}}{\varepsilon}\big(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\big)\widehat{j}_0-\sqrt{n}\frac{\mathcal{L}}{\varepsilon}\widehat{b}=\widehat{g},\end{cases} \tag{4.41}$$

with

$$\widehat{f}=\frac{\mathcal{L}\mathcal{M}}{n}\,\widehat{\zeta}_1\quad\text{and}\quad\widehat{g}=-\frac{\rho}{\sqrt{n}\,\varepsilon}\,\widehat{\zeta}_1. \tag{4.42}$$

Assume that $(\widehat{f},\widehat{g})\equiv(0,0)$ for a while and set

$$\mathcal{U}_\rho^2:=2|(\widehat{b},\widehat{d})|^2-2\rho\mathrm{Re}\,(\widehat{b}\,\overline{\widehat{d}})+|\rho\widehat{b}|^2. \tag{4.43}$$

On one hand, we have

$$\frac{1}{2}\frac{d}{dt}\mathcal{U}_\rho^2+\rho^2|\widehat{b}|^2+\rho^2|\widehat{d}|^2=\frac{\rho}{n^{3/2}}\mathrm{Re}\left((2\widehat{d}-\rho\widehat{b})\,\overline{\widehat{j_0}}\right),\tag{4.44}$$

and on the other hand,

$$\frac{1}{2}\frac{d}{dt}|\widehat{j_0}|^2+\frac{\mathcal{L}}{\varepsilon}\left(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)|\widehat{j_0}|^2=\sqrt{n}\frac{\mathcal{L}}{\varepsilon}\mathrm{Re}\,(\widehat{b}\,\overline{\widehat{j_0}}).$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\left(\mathcal{U}_\rho^2+\frac{\varepsilon}{n^2\mathcal{L}}|\rho\widehat{j_0}|^2\right)+\rho^2|(\widehat{b},\widehat{d})|^2+\frac{\rho^2}{n^2}\left(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)|\widehat{j_0}|^2=2\frac{\rho}{n^{3/2}}\mathrm{Re}\,(\widehat{d}\,\overline{\widehat{j_0}}).$$

Now, by using the fact that

$$2\frac{\rho}{n^{3/2}}\mathrm{Re}\,(\widehat{d}\,\overline{\widehat{j_0}})\leq\frac{1}{An}|\widehat{d}|^2+\frac{A\rho^2}{n^2}|\widehat{j_0}|^2,$$

and by taking $A=3/4$, we conclude that for $\rho^2\geq\frac{16n}{3(4n^2-1)}$, we have

$$\frac{d}{dt}\big(\mathcal{U}_\rho^2+\frac{\varepsilon}{n^2\mathcal{L}}|\rho\widehat{j_0}|^2\big)+\frac{\rho^2}{2n^2}|(\widehat{b},\widehat{d},\widehat{j_0})|^2\leq 0,$$

whence, because $\frac{16n}{3(4n^2-1)}\leq 1+\frac{1}{n}$, we get for some universal positive constants c_0 and C

$$|(\rho\widehat{b},\widehat{d})(t)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(t)|\leq Ce^{-c_0t}\left(|(\rho\widehat{b},\widehat{d})(0)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(0)|\right)\quad\text{for}\quad\rho\geq\sqrt{1+n^{-1}}.$$

Resuming to the equation fulfilled by $\widehat{j_0}$ in (4.41), the above inequality implies (still assuming that $\widehat{f}=\widehat{g}=0$) that

$$\frac{\mathcal{L}}{\varepsilon}\left(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)\int_0^t|\rho\widehat{j_0}|\,d\tau\leq|\rho\widehat{j_0}(0)|+\frac{\sqrt{n}\mathcal{L}}{\varepsilon}\int_0^t|\rho b|\,d\tau\lesssim\sqrt{\frac{\mathcal{L}}{\varepsilon}}|\rho\widehat{j_0}(0)|+\frac{\mathcal{L}}{\varepsilon}|(\widehat{d},\rho\widehat{b})(0)|,$$

then plugging this inequality in the equation for $\widehat{d}$, we get in addition

$$\rho^2\int_0^t|\widehat{d}|\,d\tau\lesssim|\widehat{d}(0)|+|\rho\widehat{b}(0)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(0)|.$$

Repeating the above computations in the case of general source terms $\widehat{f}$ and $\widehat{g}$, we conclude that the solution (b,d,j_0) to (4.41) satisfies for all $\rho\geq\sqrt{1+n^{-1}}$, assuming only that $\mathcal{L}\gtrsim\varepsilon$

$$\begin{aligned}|(\rho\widehat{b},\widehat{d})(t)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(t)|+\int_0^t|(\rho\widehat{b},\rho^2\widehat{d})|\,d\tau+\frac{\mathcal{L}}{\varepsilon}\left(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)\int_0^t|\rho\widehat{j_0}|\,d\tau\\ \leq C\left(|(\rho\widehat{b},\widehat{d})(0)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(0)|+\int_0^t\left(|\widehat{f}|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{g}|\right)d\tau\right)\cdot\end{aligned}$$

Resuming to the value of $\widehat{f}$ and $\widehat{g}$ in (4.42), we thus get for $\rho\geq\sqrt{1+n^{-1}}$ and $\mathcal{L}\gtrsim\varepsilon$

$$\begin{aligned}|(\rho\widehat{b},\widehat{d})(t)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(t)|+\int_0^t|(\rho\widehat{b},\rho^2\widehat{d})|\,d\tau+\frac{\mathcal{L}}{\varepsilon}\left(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\right)\int_0^t|\rho\widehat{j_0}|\,d\tau\\ \leq C\left(|(\rho\widehat{b},\widehat{d})(0)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j_0}(0)|+\left(\mathcal{L}\mathcal{M}+\frac{\rho^2}{\sqrt{\mathcal{L}\varepsilon}}\right)\int_0^t|\widehat{\zeta_1}|\,d\tau\right)\cdot\end{aligned}\tag{4.45}$$

Next, it is clear that the equation for $\widehat{\zeta}_1$ implies that whenever $\rho\leq\sqrt{\frac n2}\,\mathcal{L}\mathcal{M}$

$$|\widehat{\zeta}_1(t)|+\frac{\mathcal{L}\mathcal{M}}{2\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau\leq|\widehat{\zeta}_1(0)|+\frac{1}{\sqrt{n}\varepsilon\mathcal{M}}\biggl(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\biggr)\int_0^t|\rho\widehat{j}_0|\,d\tau+\frac{1}{\varepsilon\mathcal{M}}\int_0^t|\rho\widehat{b}|\,d\tau. \tag{4.46}$$

Hence, we get if $\mathcal{M}$ is large enough and $\rho^2\ll\mathcal{L}^{3/2}\mathcal{M}^2\varepsilon^{1/2}$

$$|\widehat{\zeta}_1(t)|+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau\leq C\biggl(|\widehat{\zeta}_1(0)|+\frac{1}{\varepsilon\mathcal{M}}\Bigl(|(\rho\widehat{b},\widehat{d})(0)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}\,|\rho\widehat{j}_0(0)|\Bigr)\biggr)\cdot \tag{4.47}$$

Then plugging that inequality in (4.45) implies that

$$\begin{aligned}|(\rho\widehat{b},\widehat{d})(t)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j}_0(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\int_0^t|\rho\widehat{b}|\,d\tau+\frac{\mathcal{L}}{\varepsilon}\biggl(1+\frac{\rho^2}{n\mathcal{L}^2\mathcal{M}}\biggr)\int_0^t|\rho\widehat{j}_0|\,d\tau\\ \leq C\biggl(|(\rho\widehat{b},\widehat{d})(0)|+\sqrt{\frac{\varepsilon}{\mathcal{L}}}|\rho\widehat{j}_0(0)|+\Bigl(\varepsilon+\frac{\varepsilon^{1/2}\rho^2}{\mathcal{M}\mathcal{L}^{3/2}}\Bigr)|\widehat{\zeta}_1(0)|\biggr),\end{aligned} \tag{4.48}$$

whenever $1+1/n\leq\rho^2\ll\mathcal{L}^{3/2}\mathcal{M}^2\varepsilon^{1/2}$.

Here is another method that gives decay estimates in the range $\mathcal{L}\sqrt{\mathcal{M}}\ll\rho\ll\mathcal{L}\mathcal{M}$ if $\mathcal{M}$ is large enough. From the first two equations of (4.40), we have

$$|(\rho\widehat{b},\widehat{d})(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\int_0^t|\rho\widehat{b}|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d})(0)|+\frac{\rho}{n^{3/2}}\int_0^t|\widehat{j}_0|\,d\tau+\frac{\mathcal{L}\mathcal{M}}{n}\int_0^t|\widehat{\zeta}_1|\,d\tau. \tag{4.49}$$

The equations for $\widehat{j}_0$ and $\widehat{\zeta}_1$ give, if $\rho\leq\sqrt{\frac n2}\,\mathcal{L}\mathcal{M}$

$$|\widehat{j}_0(t)|+\frac1\varepsilon\Bigl(\mathcal{L}+\frac{\rho^2}{n\mathcal{L}\mathcal{M}}\Bigr)\int_0^t|\widehat{j}_0|\,d\tau\leq|\widehat{j}_0(0)|+\frac{\rho}{\sqrt{n}\,\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau+\frac{\mathcal{L}\sqrt{n}}{\varepsilon}\int_0^t|\widehat{b}|\,d\tau, \tag{4.50}$$

$$|\widehat{\zeta}_1(t)|+\frac{\mathcal{L}\mathcal{M}}{2\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau\leq|\widehat{\zeta}_1(0)|+\frac{\rho}{\sqrt{n}\,\mathcal{L}\mathcal{M}\varepsilon}\Bigl(\mathcal{L}+\frac{\rho^2}{n\mathcal{L}\mathcal{M}}\Bigr)\int_0^t|\widehat{j}_0|\,d\tau+\frac{\rho}{\varepsilon\mathcal{M}}\int_0^t|\widehat{b}|\,d\tau. \tag{4.51}$$

Plugging (4.50) in (4.51), we discover if $\rho\ll\mathcal{L}\mathcal{M}$ that

$$|\widehat{\zeta}_1(t)|+\frac{\mathcal{L}\mathcal{M}}{4\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau\leq|\widehat{\zeta}_1(0)|+\frac{\rho}{\mathcal{L}\mathcal{M}}|\widehat{j}_0(0)|+\frac{1}{\varepsilon\mathcal{M}}\int_0^t\rho|\widehat{b}|\,d\tau. \tag{4.52}$$

Inserting that inequality in (4.49), we conclude that the last term of (4.49) may be absorbed by the l.h.s. if $\mathcal{M}$ is large enough. Now, Inequality (4.50) guarantees that

$$\begin{aligned}\frac{\rho}{n^{3/2}}\int_0^t|\widehat{j}_0|\,d\tau\leq\frac{1}{\sqrt{n}}\biggl(\frac{\rho\varepsilon\mathcal{L}\mathcal{M}}{n\mathcal{L}^2\mathcal{M}+\rho^2}\biggr)|\widehat{j}_0(0)|\\ +\biggl(\frac{\rho^2\mathcal{L}\mathcal{M}}{n^2\mathcal{L}^2\mathcal{M}+\rho^2 n}\biggr)\int_0^t|\widehat{\zeta}_1|\,d\tau+\biggl(\frac{\mathcal{L}^2\mathcal{M}}{n\mathcal{L}^2\mathcal{M}+\rho^2}\biggr)\int_0^t\rho|\widehat{b}|\,d\tau.\end{aligned}$$

Again, resuming to (4.49), we see that the second term in the r.h.s. may be absorbed by the l.h.s. This is also the case of the last one if $\rho^2\gg\mathcal{L}^2\mathcal{M}$. If all those conditions are fulfilled then we end up if $\mathcal{L}\sqrt{\mathcal{M}}\ll\rho\leq\sqrt{\frac n2}\,\mathcal{L}\mathcal{M}$ with

$$\begin{aligned}|(\rho\widehat{b},\widehat{d})(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\rho\int_0^t|\widehat{b}|\,d\tau+\rho\int_0^t|\widehat{j}_0|\,d\tau\\ +\mathcal{L}\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d})(0)|+\frac{\varepsilon\mathcal{L}\mathcal{M}}{\rho}|\widehat{j}_0(0)|+\varepsilon|\widehat{\zeta}_1(0)|.\end{aligned} \tag{4.53}$$

Resuming to (4.50) and (4.52), we thus easily deduce first that

$$|\widehat{\zeta}_1(t)| + \frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t |\widehat{\zeta}_1|\,d\tau \lesssim |\widehat{\zeta}_1(0)| + |\widehat{j}_0(0)| + \frac{1}{\varepsilon\mathcal{M}}|(\rho\widehat{b},\widehat{d})(0)|,$$

and next that

$$|\widehat{j}_0(t)| + \frac{1}{\varepsilon}\Big(\mathcal{L} + \frac{\rho^2}{\mathcal{L}\mathcal{M}}\Big)\int_0^t |\widehat{j}_0|\,d\tau \lesssim |\widehat{j}_0(0)| + \frac{\rho}{\mathcal{L}\mathcal{M}}|\widehat{\zeta}_1(0)| + \tfrac{\mathcal{L}}{\varepsilon\rho}|(\rho\widehat{b},\widehat{d})(0)|.$$

Our estimates and the definition of ζ_1 allow us to change ζ_1 to j_1 in (4.53). So finally, in the case $\varepsilon\mathcal{M} \gtrsim 1$ we get for some large enough C_1 and small enough c independent of ε

$$\begin{aligned}(4.54)\quad & |(\rho\widehat{b},\widehat{d},(1+\tfrac{\varepsilon\mathcal{L}\mathcal{M}}{\rho})\widehat{j}_0,\widehat{j}_1)(t)| + \int_0^t |\rho\widehat{b}|\,d\tau + \rho^2\int_0^t |\widehat{d}|\,d\tau \\ & + \tfrac{\rho}{\varepsilon\mathcal{L}\mathcal{M}}(1+\tfrac{\varepsilon\mathcal{L}\mathcal{M}}{\rho})\int_0^t |\rho\widehat{j}_0|\,d\tau + \frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t |\widehat{\zeta}_1|\,d\tau \lesssim |(\rho\widehat{b},\widehat{d},(1+\tfrac{\varepsilon\mathcal{L}\mathcal{M}}{\rho})\widehat{j}_0,\widehat{j}_1)(0)|,\end{aligned}$$

whenever $C_1\mathcal{L}\sqrt{\mathcal{M}} \le \rho \le c\mathcal{L}\mathcal{M}$.

The case $\varepsilon^2\mathcal{M} \le 1/2$ and $\mathcal{L} \gg \varepsilon$. Let $\zeta_0 := j_0 - \sqrt{n}\,b$. We start from

$$\begin{cases} \partial_t\widehat{b} + \rho\widehat{d} = 0, \\ \partial_t\widehat{d} + \rho^2\widehat{d} - \rho\widehat{b} = \frac{\mathcal{L}\mathcal{M}}{n}\widehat{j}_1, \\ \partial_t\widehat{\zeta}_0 + \frac{\mathcal{L}}{\varepsilon}\widehat{\zeta}_0 + \frac{\rho}{\sqrt{n}\,\varepsilon}\widehat{j}_1 - \sqrt{n}\,\rho\widehat{d} = 0, \\ \partial_t\widehat{j}_1 + \frac{\mathcal{L}\mathcal{M}}{\varepsilon}\widehat{j}_1 - \frac{\rho}{\sqrt{n}\,\varepsilon}\widehat{\zeta}_0 - \frac{\rho}{\varepsilon}\widehat{b} = 0. \end{cases}$$

In order to show the exponential decay, we set

$$(4.55)\qquad \mathcal{U}_\rho^2 := 2|\widehat{b}|^2 + 2|\widehat{d}|^2 + |\rho\widehat{b}|^2 - 2\rho\,\mathrm{Re}\,(\widehat{b}\overline{\widehat{d}}) \quad\text{and}\quad \mathcal{J}_\rho^2 := |\widehat{\zeta}_0|^2 + |\widehat{j}_1|^2.$$

We easily get for all $K \ge 0$,

$$\begin{aligned}(4.56)\quad & \frac{1}{2}\frac{d}{dt}\Big(\mathcal{U}_\rho^2 + K\mathcal{J}_\rho^2\Big) + \rho^2|(\widehat{b},\widehat{d})|^2 + K\frac{\mathcal{L}}{\varepsilon}\Big(|\widehat{\zeta}_0|^2 + \mathcal{M}|\widehat{j}_1|^2\Big) \\ & = 2\frac{\mathcal{L}\mathcal{M}}{n}\mathrm{Re}\,(\widehat{j}_1\,\overline{\widehat{d}}) + \rho\Big(\frac{K}{\varepsilon} - \frac{\mathcal{L}\mathcal{M}}{n}\Big)\mathrm{Re}\,(\widehat{j}_1\,\overline{\widehat{b}}) + K\sqrt{n}\,\mathrm{Re}\,(\rho\widehat{d}\,\overline{\widehat{\zeta}_0}).\end{aligned}$$

It is thus natural to take $K = \frac{\varepsilon}{n}\mathcal{L}\mathcal{M}$ to cancel out the second term of the r.h.s. For the first and the last terms, we write that

$$\begin{aligned} K\sqrt{n}\,\mathrm{Re}\,(\rho\widehat{d}\,\overline{\widehat{\zeta}_0}) &\le \tfrac{1}{4}\rho^2|\widehat{d}|^2 + nK^2|\widehat{j}_0 - \sqrt{n}\,\widehat{b}|^2, \\ \tfrac{2\mathcal{L}\mathcal{M}}{n}\mathrm{Re}\,(\widehat{j}_1\,\overline{\widehat{d}}) = \tfrac{2K}{\varepsilon}\mathrm{Re}\,(\widehat{j}_1\,\overline{\widehat{d}}) &\le \tfrac{2}{3}\rho^2|\widehat{d}|^2 + \tfrac{3K^2}{2\varepsilon^2\rho^2}|\widehat{j}_1|^2.\end{aligned}$$

Note that the last terms above may be absorbed by the l.h.s. of (4.56) if, say

$$nK \le \frac{\mathcal{L}}{2\varepsilon} \quad\text{and}\quad \frac{2K}{\varepsilon} \le \mathcal{L}\mathcal{M}\rho^2.$$

Given the value of K, the first condition is equivalent to $\varepsilon^2\mathcal{M} \le 1/2$, whereas the second one means that $\rho^2 n \ge 2$. Under this latter condition, we thus end up with

$$(4.57)\qquad \frac{d}{dt}\Big(\mathcal{U}_\rho^2 + \frac{\varepsilon\mathcal{L}\mathcal{M}}{n}\mathcal{J}_\rho^2\Big) + \frac{1}{6}\rho^2|(\widehat{b},\widehat{d})|^2 + \frac{\mathcal{L}^2\mathcal{M}}{n}\Big(|\widehat{\zeta}_0|^2 + \frac{1}{6}\mathcal{M}|\widehat{j}_1|^2\Big) \le 0,$$

which implies, according to (4.8), the following exponential decay estimate for some small enough $\kappa > 0$

$$\mathcal{U}_\rho^2(t) + \varepsilon\mathcal{L}\mathcal{M}\mathcal{J}_\rho^2(t) \le e^{-\kappa t}\Big(\mathcal{U}_\rho^2(0) + \frac{\varepsilon\mathcal{L}\mathcal{M}}{n}\mathcal{J}_\rho^2(0)\Big) \quad \text{if} \quad \rho \ge \sqrt{2/n}. \tag{4.58}$$

To exhibit the parabolic decay for d, we introduce $\widehat{\zeta}_1 := \widehat{j}_1 - (\sqrt{n}\mathcal{L}\mathcal{M})^{-1}\rho\widehat{j}_0$, and get

$$\begin{cases} \partial_t\widehat{b} + \rho\widehat{d} = 0, \\ \partial_t\widehat{d} + \rho^2\widehat{d} - \rho\big(1+\frac{1}{n}\big)\widehat{b} = \frac{\mathcal{L}\mathcal{M}}{n}\widehat{\zeta}_1 + \frac{\rho}{n^{3/2}}\widehat{\zeta}_0, \\ \partial_t\widehat{\zeta}_0 + \frac{1}{\varepsilon}\big(\mathcal{L} + \frac{\rho^2}{n\mathcal{L}\mathcal{M}}\big)\widehat{\zeta}_0 + \frac{\rho}{\sqrt{n}\,\varepsilon}\widehat{\zeta}_1 = \sqrt{n}\,\rho\widehat{d} - \frac{\rho^2}{\sqrt{n}\,\varepsilon\mathcal{L}\mathcal{M}}\widehat{b}, \\ \partial_t\widehat{\zeta}_1 + \big(\frac{\mathcal{L}\mathcal{M}}{\varepsilon} - \frac{\rho^2}{n\varepsilon\mathcal{L}\mathcal{M}}\big)\widehat{\zeta}_1 = \frac{\rho}{\sqrt{n}\,\mathcal{L}\mathcal{M}\varepsilon}\big(\mathcal{L} + \frac{\rho^2}{n\mathcal{L}\mathcal{M}}\big)\widehat{\zeta}_0 + \frac{\rho^3}{n\varepsilon\mathcal{L}^2\mathcal{M}^2}\widehat{b}. \end{cases}$$

We thus have for $\sqrt{2/n} \le \rho \le \sqrt{n/2}\,\mathcal{L}\mathcal{M}$

$$\begin{aligned} \int_0^t |\rho\widehat{b}|\,d\tau + \rho^2\int_0^t|\widehat{d}|\,d\tau &\lesssim |(\rho\widehat{b},\widehat{d})(0)| + \frac{\mathcal{L}\mathcal{M}}{n}\int_0^t|\widehat{\zeta}_1|\,d\tau + \frac{\rho}{n}\int_0^t|\widehat{\zeta}_0|\,d\tau, \\ \frac{1}{\varepsilon}\Big(\mathcal{L}+\frac{\rho^2}{n\mathcal{L}\mathcal{M}}\Big)\int_0^t|\widehat{\zeta}_0|\,d\tau &\le |\widehat{\zeta}_0(0)| + \frac{\rho}{\sqrt{n}\,\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau + \sqrt{n}\,\rho\int_0^t|\widehat{d}|\,d\tau + \frac{\rho}{\sqrt{n}\,\varepsilon\mathcal{L}\mathcal{M}}\int_0^t|\rho\widehat{b}|\,d\tau, \\ \frac{\mathcal{L}\mathcal{M}}{2\varepsilon}\int_0^t|\widehat{\zeta}_1|\,d\tau &\le |\widehat{\zeta}_1(0)| + \frac{\rho}{\sqrt{n}\,\varepsilon\mathcal{L}\mathcal{M}}\Big(\mathcal{L}+\frac{\rho^2}{n\mathcal{L}\mathcal{M}}\Big)\int_0^t|\widehat{\zeta}_0|\,d\tau + \frac{\rho^2}{n\varepsilon\mathcal{L}^2\mathcal{M}^2}\int_0^t|\rho\widehat{b}|\,d\tau. \end{aligned}$$

Combining the inequalities for $\widehat{\zeta}_0$ and $\widehat{\zeta}_1$, we easily get if $\rho \le c\mathcal{L}\mathcal{M}$ with c small enough

$$\begin{aligned} \mathcal{L}\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau &\lesssim \varepsilon|\widehat{\zeta}_1(0)| + \frac{\rho}{\mathcal{L}\mathcal{M}}\Big(\varepsilon|\widehat{\zeta}_0(0)| + \rho\varepsilon\int_0^t|\widehat{d}|\,d\tau + \frac{\rho}{\mathcal{L}\mathcal{M}}\int_0^t|\rho\widehat{b}|\,d\tau\Big), \\ \Big(\mathcal{L}+\frac{\rho^2}{\mathcal{L}\mathcal{M}}\Big)\int_0^t|\widehat{\zeta}_0|\,d\tau &\lesssim \varepsilon|\widehat{\zeta}_0(0)| + \frac{\rho\varepsilon}{\mathcal{L}\mathcal{M}}|\widehat{\zeta}_1(0)| + \rho\varepsilon\int_0^t|\widehat{d}|\,d\tau + \frac{\rho}{\mathcal{L}\mathcal{M}}\int_0^t|\rho\widehat{b}|\,d\tau. \end{aligned}$$

Now, the exponential decay pointed out in (4.58) allows to bound the last terms above, and we get

$$\mathcal{L}\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau \lesssim \mathcal{U}_\rho(0) + \sqrt{\varepsilon\mathcal{L}\mathcal{M}}\mathcal{J}_\rho(0) + \frac{\varepsilon}{\mathcal{L}\mathcal{M}}\int_0^t|\rho^2\widehat{d}|\,d\tau, \tag{4.59}$$

$$\Big(\mathcal{L}+\frac{\rho^2}{\mathcal{L}\mathcal{M}}\Big)\int_0^t|\widehat{\zeta}_0|\,d\tau \lesssim \varepsilon|(\widehat{\zeta}_0,\widehat{\zeta}_1)(0)| + \frac{\rho}{\mathcal{L}\mathcal{M}}\big(\mathcal{U}_\rho(0) + \sqrt{\varepsilon\mathcal{L}\mathcal{M}}\mathcal{J}_\rho(0)\big) + \rho\varepsilon\int_0^t|\widehat{d}|\,d\tau, \tag{4.60}$$

whereas using $\mathcal{U}_\rho^2$ allows to get directly

$$\rho^2\int_0^t|\widehat{d}|\,d\tau \lesssim |(\rho\widehat{b},\widehat{d})(0)| + \mathcal{L}\mathcal{M}\int_0^t|\widehat{j}_1|\,d\tau.$$

Using the definition of $\widehat{j}_1$ and, again, Inequality (4.58), we may replace $\widehat{j}_1$ with $\widehat{\zeta}_1$ as follows

$$\rho^2\int_0^t|\widehat{d}|\,d\tau \lesssim \mathcal{U}_\rho(0) + \sqrt{\varepsilon\mathcal{L}\mathcal{M}}\mathcal{J}_\rho(0) + \mathcal{L}\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau + \int_0^t|\rho\widehat{\zeta}_0|\,d\tau. \tag{4.61}$$

Then plugging (4.59) and (4.60) in (4.61) and observing that $\varepsilon \ll \mathcal{L}\mathcal{M}$, we get

$$\rho^2\int_0^t |\widehat{d}|\,d\tau \lesssim \mathcal{U}_\rho(0)+\sqrt{\varepsilon\mathcal{L}\mathcal{M}}\mathcal{J}_\rho(0) + \frac{\rho\mathcal{L}\mathcal{M}}{\mathcal{L}^2\mathcal{M}+\rho^2}\biggl(\varepsilon|(\widehat{\zeta}_0,\widehat{\zeta}_1)(0)| + \frac{\rho}{\mathcal{L}\mathcal{M}}\bigl(\mathcal{U}_\rho(0)+\sqrt{\varepsilon\mathcal{L}\mathcal{M}}\mathcal{J}_\rho(0)\bigr)+\rho\varepsilon\int_0^t|\widehat{d}|\,d\tau\biggr)\cdot$$

Because we assumed that $\mathcal{L} \gg \varepsilon$, the last term may be absorbed by the l.h.s. Using in addition that $\varepsilon^2\mathcal{M}\lesssim 1$, we end up with

$$\rho^2\int_0^t |\widehat{d}|\,d\tau \lesssim \mathcal{U}_\rho(0)+\sqrt{\varepsilon\mathcal{L}\mathcal{M}}\mathcal{J}_\rho(0).$$

Then resuming to (4.59), (4.60), we obtain

$$\rho\int_0^t|\widehat{\zeta}_0|\,d\tau+\mathcal{L}\mathcal{M}\int_0^t|\widehat{\zeta}_1|\,d\tau\lesssim \mathcal{U}_\rho(0). \tag{4.62}$$

Obviously, this inequality implies that

$$\mathcal{L}\mathcal{M}\int_0^t|\widehat{j}_1|\,d\tau\lesssim \mathcal{U}_\rho(0)+\sqrt{\varepsilon\mathcal{L}\mathcal{M}}\,\mathcal{J}_\rho(0)+\varepsilon|(\widehat{\zeta}_0,\widehat{\zeta}_1)(0)|. \tag{4.63}$$

Of course, we get the same inequality if replacing ζ_0 and ζ_1 with j_0 and j_1. So one can conclude that for $\sqrt{2/n}\le\rho\le c\mathcal{L}\mathcal{M}$, we have (4.58) and

$$\rho^2\int_0^t|\widehat{d}|\,d\tau+\mathcal{L}\mathcal{M}\int_0^t|\widehat{j}_1|\,d\tau+\rho\int_0^t|\widehat{j}_0|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d})(0)|+\sqrt{\varepsilon\mathcal{L}\mathcal{M}}|(\widehat{j}_0,\widehat{j}_1)(0)|. \tag{4.64}$$

4.3. **High frequencies.** We eventually come to the proof of decay estimates for $\rho\ge c\mathcal{L}\mathcal{M}$, where c is some given positive constant. We shall use that fact that the systems satisfied by (b,d) and by (j_0,j_1), respectively, tend to be uncoupled for $\rho\to+\infty$. As regards (b,d), the classical approach for the barotropic Navier-Stokes equation, based on the study of

$$\mathcal{U}_\rho^2:=2|(\widehat{b},\widehat{d})|^2-2\rho\mathrm{Re}\,(\widehat{b}\,\overline{\widehat{d}})+|\rho\widehat{b}|^2,$$

guarantees, if $\rho\ge c$, that

$$|(\rho\widehat{b},\widehat{d})(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\int_0^t\rho|\widehat{b}|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d})(0)|+\frac{\mathcal{L}\mathcal{M}}{n}\int_0^t|\widehat{j}_1|\,d\tau. \tag{4.65}$$

Next, from the system fulfilled by $(\widehat{j}_0,\widehat{j}_1)$, we get

$$\begin{aligned}\frac12\frac{d}{dt}\Bigl(|\widehat{j}_0|^2+|\widehat{j}_1|^2\Bigr)+\frac{\mathcal{L}}{\varepsilon}|\widehat{j}_0|^2+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}|\widehat{j}_1|^2&=\sqrt{n}\frac{\mathcal{L}}{\varepsilon}\mathrm{Re}\,(\widehat{b}\,\overline{\widehat{j}_0}),\\ \frac{d}{dt}\mathrm{Re}\,(\widehat{j}_0\,\overline{\widehat{j}_1})+\frac{\mathcal{L}}{\varepsilon}(1+\mathcal{M})\mathrm{Re}\,(\widehat{j}_0\,\overline{\widehat{j}_1})+\frac{\rho}{\varepsilon\sqrt{n}}|\widehat{j}_1|^2-\frac{\rho}{\varepsilon\sqrt{n}}|\widehat{j}_0|^2&=\sqrt{n}\frac{\mathcal{L}}{\varepsilon}\mathrm{Re}\,(\widehat{b}\,\overline{\widehat{j}_1}).\end{aligned}$$

Therefore, for any $\kappa>0$

$$\frac12\frac{d}{dt}\biggl(|\widehat{j}_0|^2+|\widehat{j}_1|^2-\frac{2\kappa\mathcal{L}\mathcal{M}}{\rho}\mathrm{Re}\,(\widehat{j}_0\,\overline{\widehat{j}_1})\biggr)+\Bigl(1+\frac{\kappa}{\sqrt{n}}\mathcal{M}\Bigr)\frac{\mathcal{L}}{\varepsilon}|\widehat{j}_0|^2+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\Bigl(1-\frac{\kappa}{\sqrt{n}}\Bigr)|\widehat{j}_1|^2 = \frac{\kappa\mathcal{L}^2\mathcal{M}}{\rho\varepsilon}(1+\mathcal{M})\mathrm{Re}\,(\widehat{j}_0\,\overline{\widehat{j}_1})+\sqrt{n}\,\frac{\mathcal{L}}{\varepsilon}\mathrm{Re}\,\Bigl(\widehat{b}\,\Bigl(\overline{\widehat{j}_0}-\frac{\kappa\mathcal{L}\mathcal{M}}{\rho}\overline{\widehat{j}_1}\Bigr)\Bigr)\cdot$$

For $\rho\ge c\mathcal{L}\mathcal{M}$, it is clear that our choosing κ small enough implies that

- the first term of the r.h.s. may be absorbed by the second and third ones of the l.h.s.,

- we have $|\widehat{j}_0|^2+|\widehat{j}_1|^2-\frac{2\kappa\mathcal{L}\mathcal{M}}{\rho}\mathrm{Re}\,(\widehat{j}_0\,\overline{\widehat{j}_1})\approx|\widehat{j}_0|^2+|\widehat{j}_1|^2,$
- we have $\bigl(1+\frac{\kappa}{\sqrt{n}}\mathcal{M}\bigr)\frac{\mathcal{L}}{\varepsilon}|\widehat{j}_0|^2+\bigl(1-\frac{\kappa}{\sqrt{n}}\bigr)\frac{\mathcal{L}\mathcal{M}}{\varepsilon}|\widehat{j}_1|^2\geq\frac{\kappa}{2}\frac{\mathcal{L}\mathcal{M}}{\sqrt{n}\,\varepsilon}(|\widehat{j}_0|^2+|\widehat{j}_1|^2).$

Therefore, we end up with the following inequality

$$\varepsilon|(\widehat{j}_0,\widehat{j}_1)(t)|+\mathcal{L}\mathcal{M}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau\lesssim\varepsilon|(\widehat{j}_0,\widehat{j}_1)(0)|+\mathcal{L}\int_0^t|\widehat{b}|\,d\tau. \tag{4.66}$$

Combining with (4.65), we conclude that if

$$\rho\geq c\max(1,\mathcal{L}\mathcal{M})\quad\text{and}\quad\rho\geq C\mathcal{L}, \tag{4.67}$$

for a large enough constant C then

$$|(\rho\widehat{b},\widehat{d})(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\int_0^t\rho|\widehat{b}|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d},\varepsilon\widehat{j}_0,\varepsilon\widehat{j}_1)(0)|, \tag{4.68}$$

$$|(\widehat{j}_0,\widehat{j}_1)(t)|+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau\lesssim\frac{\mathcal{L}}{\rho\varepsilon}|(\rho\widehat{b},\widehat{d})(0)|+|(\widehat{j}_0,\widehat{j}_1)(0)|, \tag{4.69}$$

whence

$$|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\int_0^t\rho|\widehat{b}|\,d\tau+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|. \tag{4.70}$$

The only case where the condition $\rho\geq C\mathcal{L}$ may be stronger than $\rho\geq c\mathcal{L}\mathcal{M}$ is when $\mathcal{M}$ is bounded. From our study for small ρ's, we must assume that $\mathcal{L}\mathcal{M}\gtrsim\varepsilon^{-1}$, and thus $c\mathcal{L}\mathcal{M}$ is still much larger than $\sqrt{2/n}$. Therefore, one may take advantage of (4.58) to bound the r.h.s. of (4.66), and combining with (4.65), we get for $\rho\approx\mathcal{L}$

$$\begin{aligned}|(\rho\widehat{b},\widehat{d},\sqrt{\varepsilon\mathcal{L}}\widehat{j}_0,\sqrt{\varepsilon\mathcal{L}}\widehat{j}_1)(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\int_0^t\rho|\widehat{b}|\,d\tau\\+\mathcal{L}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d},\sqrt{\varepsilon\mathcal{L}}\widehat{j}_0,\sqrt{\varepsilon\mathcal{L}}\widehat{j}_1)(0)|.\end{aligned} \tag{4.71}$$

5. The non-equilibrium diffusion regime

This section is devoted to the study of the so-called *non-equilibrium diffusion asymptotics*. Assuming that for some $\kappa>1$ and $m>0$, we have

$$\frac{\mathcal{L}}{\varepsilon}\to\frac{\kappa}{n\nu}\quad\text{and}\quad\mathcal{L}^2\mathcal{L}_s\to\frac{m}{\nu^2}, \tag{5.1}$$

we want to prove the convergence of the solutions of (1.6) to those of (2.5) or (2.6) if $m<+\infty$ or $m=+\infty$, respectively, when ε goes to 0.

The first subsection concerns the proof of global existence with 'uniform' estimates for the radiative Navier-Stokes equations (1.6) in the asymptotic (5.1), in the small data case with critical regularity. Next, still for small critical data, we establish the global existence for the limit systems (2.5) and (2.6). In the last part of the present section, we combine the uniform estimates with compactness arguments in order to justify the convergence of (1.6) to (2.5) or (2.6).

5.1. **Global existence and uniform estimates for** (1.6) **with $\mathcal{L}\approx\varepsilon$ and $\mathcal{L}^2\mathcal{L}_s\gtrsim 1$.** In order to get a global-in-time existence statement for (1.6) in the non-equilibrium diffusion regime, we first put together the estimates that we obtained in the previous section, in the case $\mathcal{L}\approx\varepsilon$ and $\mathcal{L}^2\mathcal{L}_s\gtrsim 1$. Even though localizing the linearized equations by means of Littlewood-Paley operators allows to get essentially optimal estimates for the linearized equations of (1.6), it is not enough for our purpose, owing to the convection term $\vec{u}\cdot\nabla b$ that may cause a loss of one derivative. The difficulty may be overcome by paralinearizing the whole system, as explained below. After that, it is easy to prove global in time estimates for the solutions to (1.6) just by combining the estimates for the paralinearized system, and standard product laws in Besov spaces to handle the other nonlinear terms.

5.1.1. *Linear estimates.* Performing the change of variables (4.1) reduces the study to that of the linear system (4.4).

Low frequencies estimates. Using (4.28), the comment that follows, (4.39) and the fact that $|(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)|\approx|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathrm{j}}_0,\widehat{\mathrm{j}}_1)|$, we get for the solution $(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)$ to (4.7),

$$(5.2)\quad |(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)|\,d\tau+\int_0^t|\widehat{\mathrm{j}}_0|\,d\tau+\mathcal{M}\int_0^t|\widehat{\mathrm{j}}_1|\,d\tau \\ \leq C|(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for all }\ 0\leq\rho\leq C_1,$$

with[1] $\widehat{\mathrm{j}}_0:=\widehat{j}_0-\sqrt{n}\,\widehat{b}-\sqrt{n}\,\frac{\varepsilon}{\widetilde{\mathcal{L}}}\,\rho\widehat{d}$ and $\widehat{\mathrm{j}}_1:=\widehat{j}_1-\frac{\rho}{\sqrt{n}\,\widetilde{\mathcal{L}}\mathcal{M}}\widehat{j}_0+\frac{\rho\widehat{b}}{\widetilde{\mathcal{L}}\mathcal{L}_s\mathcal{M}}\cdot$

Middle frequencies estimates. If $\widetilde{\mathcal{L}}^2\mathcal{L}_s\approx 1$ then using (4.54), and the definition of $\widehat{\zeta}_1$ versus that of $\widehat{\mathrm{j}}_1$, we get

$$(5.3)\quad |(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|(\widehat{d},\widehat{j}_0)|\,d\tau+\rho\int_0^t|\widehat{b}|\,d\tau+\mathcal{M}\int_0^t|\widehat{\mathrm{j}}_1|\,d\tau \\ \leq C|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for all }\ C_1\leq\rho\leq c\widetilde{\mathcal{L}}\mathcal{M}.$$

If $\widetilde{\mathcal{L}}^2\mathcal{L}_s\to+\infty$ then (4.47) and (4.48) ensure that

$$(5.4)\quad |(\rho\widehat{b},\widehat{d},\rho\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\rho\int_0^t|(\widehat{b},\widehat{j}_0)|\,d\tau+\mathcal{M}\int_0^t|\widehat{\mathrm{j}}_1|\,d\tau \\ \leq C|(\rho\widehat{b},\widehat{d},\rho\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for all }\ \sqrt{1+n^{-1}}\leq\rho\leq\varepsilon\nu\sqrt{\mathcal{M}},$$

and, according to (4.54)

$$(5.5)\quad |(\rho\widehat{b},\widehat{d},(1+\tfrac{\varepsilon^2\nu\mathcal{M}}{\rho})\widehat{j}_0,\widehat{j}_1)(t)|+\int_0^t|\rho\widehat{b}|\,d\tau+\rho^2\int_0^t|\widehat{d}|\,d\tau \\ +(1+\tfrac{\rho}{\varepsilon^2\nu\mathcal{M}})\int_0^t|\rho\widehat{j}_0|\,d\tau+\mathcal{M}\int_0^t|\widehat{\mathrm{j}}_1|\,d\tau\lesssim|(\rho\widehat{b},\widehat{d},(1+\tfrac{\varepsilon^2\nu\mathcal{M}}{\rho})\widehat{j}_0,\widehat{j}_1)(0)|,$$

whenever $C_1\varepsilon\nu\sqrt{\mathcal{M}}\leq\rho\leq c\varepsilon\nu\mathcal{M}$.

[1]Note that the last term of j_0 given by (4.22) is negligible for $\rho\lesssim 1$ and may thus be omitted.

Hence

$$(5.6)\quad |(\rho\widehat{b},\widehat{d},\max(1,\min(\rho,\rho^{-1}\varepsilon^2\nu\mathcal{M}))\widehat{j}_0,\widehat{j}_1)(t)|+\rho\int_0^t|(\widehat{b},\min(1,\tfrac{\rho^2}{\varepsilon^2\nu\mathcal{M}})\widehat{j}_0)|\,d\tau$$
$$+\rho^2\int_0^t|\widehat{d}|\,d\tau+\mathcal{M}\int_0^t|\widehat{\mathrm{j}}_1|\,d\tau\leq C|(\rho\widehat{b},\widehat{d},\max(1,\min(\rho,\rho^{-1}\varepsilon^2\nu\mathcal{M}))\widehat{j}_0,\widehat{j}_1)(0)|.$$

High frequencies estimates. Using (4.70), we have

$$(5.7)\quad |(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\rho\int_0^t|\widehat{b}|\,d\tau+\mathcal{M}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau$$
$$\leq C|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for}\quad \rho\geq c\widetilde{\mathcal{L}}\mathcal{M}.$$

For notational simplicity, we shall slightly abusively change c and C_1 to 1 in all the following computations.

Optimal estimates in Besov spaces. If $\widetilde{\mathcal{L}}^2\mathcal{L}_s\approx 1$ then localizing (4.4) with nonzero source terms f and $\vec g$ according to Littlewood-Paley operator $\dot\Delta_k$, using (4.5) and following the computations leading to (5.2), (5.3) and (5.7) (combined with Fourier-Plancherel theorem) we end up for all $s\in\mathbb{R}$ with[2]

$$(5.8)\quad \|(\vec u,j_0,\vec j_1)(t)\|_{\dot B^s_{2,1}}+\|b(t)\|^{\ell,1}_{\dot B^s_{2,1}}+\|b(t)\|^{h,1}_{\dot B^{s+1}_{2,1}}+\int_0^t\|\vec u\|_{\dot B^{s+2}_{2,1}}\,d\tau$$
$$+\int_0^t\|(b,j_0,\vec j_1)\|^{\ell,1}_{\dot B^{s+2}_{2,1}}\,d\tau+\int_0^t\|b\|^{h,1}_{\dot B^{s+1}_{2,1}}\,d\tau+\int_0^t\|\mathrm{j}_0\|^{\ell,1}_{\dot B^s_{2,1}}\,d\tau$$
$$+\mathcal{M}\int_0^t\|\vec{\mathrm{j}}_1\|^{\ell,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot B^s_{2,1}}\,d\tau+\int_0^t\|j_0\|^{m,1,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot B^{s+2}_{2,1}}\,d\tau+\mathcal{M}\int_0^t\|(j_0,\vec j_1)\|^{h,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot B^s_{2,1}}\,d\tau$$
$$\lesssim\|(\vec u,j_0,\vec j_1)(0)\|_{\dot B^s_{2,1}}+\|b(0)\|^{\ell,1}_{\dot B^s_{2,1}}+\|b(0)\|^{h,1}_{\dot B^{s+1}_{2,1}}+\int_0^t\left(\|f\|^{\ell,1}_{\dot B^s_{2,1}}+\|f\|^{h,1}_{\dot B^{s+1}_{2,1}}+\|\vec g\|_{\dot B^s_{2,1}}\right)d\tau,$$

with

$$(5.9)\qquad \mathrm{j}_0:=j_0-\sqrt{n}\,b-\sqrt{n}\frac{\varepsilon}{\widetilde{\mathcal{L}}}\mathrm{div}\,\vec u\quad\text{and}\quad \vec{\mathrm{j}}_1:=\vec j_1+\frac{1}{\sqrt{n}\,\widetilde{\mathcal{L}}\mathcal{M}}\nabla j_0-\frac{1}{\widetilde{\mathcal{L}}\mathcal{L}_s\mathcal{M}}\nabla b.$$

According to our previous work in [7], the critical regularity framework corresponds to $s=n/2-1$. Therefore, the following quantities will play an important role

$$\|(b,\vec u,j_0,\vec j_1)\|_X:=\|b\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}+\|b\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|(\vec u,j_0,\vec j_1)\|_{\dot B^{\frac n2-1}_{2,1}},$$

and

$$\|(b,\vec u,j_0,\vec j_1)\|_Y:=\sup_{t\geq0}\|(b,\vec u,j_0,\vec j_1)(t)\|_X+\int_{\mathbb{R}_+}\Big(\|b\|^{\ell,1}_{\dot B^{\frac n2+1}_{2,1}}+\|b\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec u\|_{\dot B^{\frac n2+1}_{2,1}}\Big)d\tau$$
$$+\int_{\mathbb{R}_+}\Big(\mathcal{M}\|\vec{\mathrm{j}}_1\|^{\ell,\varepsilon\nu\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}}+\|\mathrm{j}_0\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}+\|j_0\|^{m,1,\varepsilon\nu\mathcal{M}}_{\dot B^{\frac n2+1}_{2,1}}+\mathcal{M}\|(j_0,\vec j_1)\|^{h,\varepsilon\nu\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}}\Big)d\tau.$$

We denote by X and Y the corresponding functional spaces (where time continuity is imposed rather than just boundedness) and agree that $Y(t)$ stands for the restriction of Y to the interval $[0,t]$.

[2]Further explanations on the method will be supplied to the reader in the next paragraph.

In the case $\varepsilon^2\mathcal{L}_s \to +\infty$, we have to change slightly the definition of the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ as the middle frequencies obey (5.6). Consequently, we change $\|\cdot\|_X$ to $\|\cdot\|_{X_\infty}$ with

$$\begin{aligned}\|(b,\vec u,j_0,\vec j_1)\|_{X_\infty} :=& \|b\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}} + \|b\|^{h,1}_{\dot B^{\frac n2}_{2,1}} + \|(\vec u,\vec j_1)\|_{\dot B^{\frac n2-1}_{2,1}}\\ &+\|j_0\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}} + \|j_0\|^{h,\varepsilon\nu\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}},+ \sum_{1\le 2^k\le\varepsilon\mathcal{M}} 2^{k\frac n2}\max(1,2^k,\min(2^{-k}\varepsilon^2\nu\mathcal{M}))\|\dot\Delta_k j_0\|_{L^2}\end{aligned}$$

and $\|\cdot\|_Y$ to

$$\begin{aligned}\|(b,\vec u,j_0,\vec j_1)\|_{Y_\infty} :=& \sup_{t\ge0}\|(b,\vec u,j_0,\vec j_1)(t)\|_{X_\infty} + \int_{\mathbb{R}_+}\Big(\|b\|^{\ell,1}_{\dot B^{\frac n2+1}_{2,1}} + \|b\|^{h,1}_{\dot B^{\frac n2}_{2,1}} + \|\vec u\|_{\dot B^{\frac n2+1}_{2,1}}\Big)d\tau\\ &+\int_{\mathbb{R}_+}\Big(\mathcal{M}\|\vec j_1\|^{\ell,\varepsilon\nu\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}} + \|j_0\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}} + \mathcal{M}\|(j_0,\vec j_1)\|^{h,\varepsilon\nu\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}}\Big)d\tau\\ &+\int_{\mathbb{R}_+}\sum_{1\le 2^k\le\varepsilon\mathcal{M}} 2^{k\frac n2}\min(1,2^{2k}\varepsilon^{-2}\nu^{-1}\mathcal{M}^{-1})\|\dot\Delta_k j_0\|_{L^2}d\tau.\end{aligned}$$

To prove global estimates for the nonlinear system (1.6), the natural next step would be to take advantage of (5.8) with $s=n/2-1$ and all the nonlinear terms in the r.h.s. Unfortunately, this does not work for the convection term $\vec u\cdot\nabla b$ causes a loss of one derivative (indeed, if b is in $\dot B^{\frac n2}_{2,1}$ then $\vec u\cdot\nabla b$ cannot be smoother than $\dot B^{\frac n2-1}_{2,1}$). A nowadays standard way to overcome the difficulty is to paralinearize (1.6), that is to add to (4.4) the principal parts of the convection terms. This is the aim of the next paragraph.

5.1.2. *The paralinearized system.* Before introducing the paralinearized system associated to (1.6), let us shortly recall the definition of the paraproduct, according to the pioneering paper [2] by J.-M. Bony. The (homogeneous) *paraproduct* between two tempered distributions U and V satisfying (3.1) is given by

$$T_UV := \sum_k \dot S_{k-1}U\dot\Delta_kV \quad\text{with}\quad \dot S_{k-1} := \chi(2^{k-1}D).$$

We also introduce

$$T'_VU := \sum_k \dot S_{k+2}V\,\dot\Delta_kU,$$

and, observe that, at least formally

$$UV = T_UV + T'_VU.$$

To some extent, if U is smooth enough then T_UV may be seen as the *principal part* of the product UV. This motivates our considering the following system

$$\left\{\begin{aligned}&\partial_tb + T_{\vec v}\cdot\nabla b + \operatorname{div}\vec u = F,\\ &\partial_t\vec u + T_{\vec v}\cdot\nabla\vec u - \widetilde{\mathcal{A}}\vec u + \nabla b - \tfrac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\vec j_1 = \vec G,\\ &\partial_tj_0 + \tfrac{1}{\varepsilon\sqrt n}\operatorname{div}\vec j_1 + \tfrac{\widetilde{\mathcal{L}}}{\varepsilon}(j_0-\sqrt n\,b) = 0,\\ &\partial_t\vec j_1 + \tfrac{1}{\varepsilon\sqrt n}\nabla j_0 + \tfrac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\vec j_1 = \vec 0,\end{aligned}\right. \tag{5.10}$$

where $\mathcal{A} = \mu\Delta + (\lambda+\mu)\nabla\operatorname{div}$, $\mathcal{M} := 1+\mathcal{L}_s$, $\vec v$ and $\vec G$ are given time dependent vector-fields, and F is a given real valued function.

Proposition 5.1. *For any smooth solution $(b, \vec{u}, j_0, \vec{j}_1)$ we have the following a priori estimate if $0 < m < +\infty$*

$$\|(b,\vec{u},j_0,\vec{j}_1)\|_{Y(t)} \leq C\biggl(\|(b,\vec{u},j_0,\vec{j}_1)(0)\|_X + \int_0^t \|\nabla\vec{v}\|_{L^\infty}\|(b,\vec{u},j_0,\vec{j}_1)\|_X\,d\tau$$
$$+\int_0^t \|(\nabla F,\vec{G})\|^{h,1}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\,d\tau + \int_0^t \|(F - T_{\vec{v}}\cdot\nabla b, \vec{G} - T_{\vec{v}}\cdot\nabla\vec{u})\|^{\ell,1}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\,d\tau\biggr)\cdot$$

A similar inequality holds if $m = +\infty$, with $X_\infty(t)$ and $Y_\infty(t)$ instead of $X(t)$ and $Y(t)$.

Proof: Localizing System (5.10) by means of $\dot{\Delta}_k$ yields

$$\left\{\begin{array}{l}\partial_t\dot{\Delta}_k b + \dot{\Delta}_k(T_{\vec{v}}\cdot\nabla b) + \operatorname{div}\dot{\Delta}_k\vec{u} = \dot{\Delta}_k F,\\ \partial_t\dot{\Delta}_k\vec{u} + \dot{\Delta}_k(T_{\vec{v}}\cdot\nabla\vec{u}) - \widetilde{\mathcal{A}}\dot{\Delta}_k\vec{u} + \nabla\dot{\Delta}_k b - \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\dot{\Delta}_k\vec{j}_1 = \dot{\Delta}_k\vec{G},\\ \partial_t\dot{\Delta}_k j_0 + \frac{1}{\varepsilon\sqrt{n}}\operatorname{div}\dot{\Delta}_k\vec{j}_1 + \frac{\widetilde{\mathcal{L}}}{\varepsilon}(\dot{\Delta}_k j_0 - \sqrt{n}\dot{\Delta}_k b) = 0,\\ \partial_t\dot{\Delta}_k\vec{j}_1 + \frac{1}{\varepsilon\sqrt{n}}\nabla\dot{\Delta}_k j_0 + \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\dot{\Delta}_k\vec{j}_1 = \vec{0}.\end{array}\right.\tag{5.11}$$

The important point is that in order to obtain all the estimates corresponding to $\rho \geq C_1$, one only has to resort to combinations between fluid unknowns on one side, and radiative unknowns, on the other side. This will enable us to use exactly the same energy method for (5.10) as for (4.4), in the middle and high frequency regimes, without introducing unwanted parts of convection terms in the inequalities.

1. Low frequencies: $2^k \leq C_1$.

Including the para-convection terms in the source terms of (5.11) and repeating the computations leading to (5.2), we get after taking L^2 norms and using Fourier-Plancherel theorem

$$\|\dot{\Delta}_k(b,\vec{u},j_0,\vec{j}_1)(t)\|_{L^2} + 2^{2k}\int_0^t \|\dot{\Delta}_k(b,\vec{u})\|_{L^2}\,d\tau + \int_0^t \|\dot{\Delta}_k \mathrm{j}_0\|_{L^2}\,d\tau$$
$$+\nu\mathcal{L}_s\int_0^t \|\dot{\Delta}_k\vec{\mathrm{j}}_1\|_{L^2}\,d\tau \lesssim \|\dot{\Delta}_k(b,\vec{u},j_0,\vec{j}_1)(0)\|_{L^2}$$
$$+\int_0^t \|\dot{\Delta}_k(F - T_{\vec{u}}\cdot\nabla b)\|_{L^2}\,d\tau + \int_0^t \|\dot{\Delta}_k(\vec{G} - T_{\vec{u}}\cdot\nabla\vec{u})\|_{L^2}\,d\tau.\tag{5.12}$$

2. Medium frequencies: $C_1 \leq 2^k \leq c\widetilde{\mathcal{L}}\mathcal{L}_s$.

Keeping in mind the proof of (5.3), we see that it is suitable to introduce

$$\vec{\zeta}_1 := \vec{j}_1 + \frac{1}{\sqrt{n}\,\widetilde{\mathcal{L}}\mathcal{M}}\nabla j_0.$$

Now, because we have

$$\left\{\begin{array}{l}\partial_t\dot{\Delta}_k b + \operatorname{div}\dot{\Delta}_k\vec{u} = \dot{\Delta}_k F,\\ \partial_t\dot{\Delta}_k\vec{u} - \widetilde{\mathcal{A}}\dot{\Delta}_k\vec{u} + \nabla\dot{\Delta}_k b = \frac{1}{n^{3/2}}\nabla\dot{\Delta}_k j_0 + \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\dot{\Delta}_k\vec{\zeta}_1 + \dot{\Delta}_k\vec{G},\end{array}\right.$$

we easily get by computing

$$\frac{1}{2}\frac{d}{dt}\Bigl(2\|(\dot{\Delta}_k b, \dot{\Delta}_k\vec{u})\|^2_{L^2} + \|\dot{\Delta}_k\nabla b\|^2_{L^2} + 2(\dot{\Delta}_k\nabla b|\dot{\Delta}_k\vec{u})_{L^2}\Bigr),\tag{5.13}$$

and by using Lemma 4.1 in [7] to handle the para-convection terms, the following inequality for all $2^k \geq C_1$

$$\|(\dot\Delta_k\nabla b, \dot\Delta_k\vec u)(t)\|_{L^2} + 2^{2k}\int_0^t \|\dot\Delta_k\vec u\|_{L^2}\,d\tau + \int_0^t \|\dot\Delta_k\nabla b\|_{L^2}\,d\tau \lesssim \|(\dot\Delta_k\nabla b, \dot\Delta_k\vec u)(0)\|_{L^2}$$
$$+\int_0^t \|(\dot\Delta_k\nabla F, \dot\Delta_k\vec G)\|_{L^2}\,d\tau + \int_0^t \|\nabla\dot\Delta_k j_0\|_{L^2}\,d\tau + \widetilde{\mathcal L}\mathcal M\int_0^t \|\nabla\dot\Delta_k\vec j_1\|_{L^2}\,d\tau$$
$$+\sum_{k'\sim k}\int_0^t \|\nabla\vec v\|_{L^\infty}\|(\dot\Delta_{k'}\nabla b, \dot\Delta_{k'}\vec u)\|_{L^2}\,d\tau.$$

Then looking at the equations satisfied by $\dot\Delta_k j_0$ and $\dot\Delta_k\vec\zeta_1$ (in the spirit of (4.40)), we derive inequalities similar to (4.50) and (4.51) for $\|\dot\Delta_k j_0\|_{L^2}$ and $\|\dot\Delta_k\vec\zeta_1\|_{L^2}$, and thus following the computations leading to (4.54), we end up in the case $m < +\infty$ with

$$\|\dot\Delta_k(\nabla b, \vec u, \nabla j_0, \vec j_1)(t)\|_{L^2} + 2^{2k}\int_0^t \|(\dot\Delta_k\vec u, \dot\Delta_k j_0)\|_{L^2}\,d\tau + \int_0^t \|\dot\Delta_k\nabla b\|_{L^2}\,d\tau \tag{5.14}$$
$$+\nu\mathcal L_s\int_0^t \|\dot\Delta_k\vec\zeta_1\|_{L^2}\,d\tau \lesssim \|\dot\Delta_k(\nabla b, \vec u, \nabla j_0, \vec j_1)(0)\|_{L^2} + \int_0^t \|\dot\Delta_k(\nabla F, \vec G)\|_{L^2}\,d\tau$$
$$+\sum_{k'\sim k}\int_0^t \|\nabla\vec v\|_{L^\infty}\|\dot\Delta_{k'}(\nabla b, \vec u)\|_{L^2}\,d\tau.$$

Comparing the definition of $\vec\zeta_1$ and $\vec{\jmath}_1$, we see that one may replace $\vec\zeta_1$ with $\vec{\jmath}_1$ above, if $C_1 \leq 2^k \leq c\widetilde{\mathcal L}\mathcal L_s$.

The obvious modifications to be done if $m = +\infty$ are left to the reader.

3. High frequencies: $2^k \geq c\widetilde{\mathcal L}\mathcal L_s$.

Again, we compute (5.13) to bound the fluid unknowns. In addition, to handle radiative unknowns, we compute for some small enough κ (see the proof of (4.66)) the following quantity

$$\frac12\frac{d}{dt}\big(\|\dot\Delta_k j_0\|_{L^2}^2 + \|\dot\Delta_k\vec j_1\|_{L^2}^2 - \kappa\widetilde{\mathcal L}\mathcal M\,2^{-2k}(\dot\Delta_k j_0|\dot\Delta_k\mathrm{div}\,\vec j_1)_{L^2}\big).$$

Combining the computations leading to (5.7) with Fourier-Plancherel theorem and Lemma 4.1 in [7] eventually yields

$$\|\dot\Delta_k(\nabla b, \vec u, j_0, \vec j_1)(t)\|_{L^2} + 2^{2k}\int_0^t \|\dot\Delta_k\vec u\|_{L^2}\,d\tau + 2^k\int_0^t \|\dot\Delta_k b\|_{L^2}\,d\tau + \nu\mathcal L_s\int_0^t \|(\dot\Delta_k j_0, \dot\Delta_k\vec j_1)\|_{L^2}\,d\tau$$
$$\lesssim \|\dot\Delta_k(\nabla b, \vec u, j_0, \vec j_1)(0)\|_{L^2} + \int_0^t \|\dot\Delta_k(\nabla F, \vec G)\|_{L^2}\,d\tau + \sum_{k'\sim k}\int_0^t \|\nabla\vec v\|_{L^\infty}\|\dot\Delta_{k'}(\nabla b, \vec u)\|_{L^2}\,d\tau.$$

Finally, multiplying (5.12), (5.14) and the above inequality by $2^{k(\frac n2-1)}$ and summing up over k completes the proof of the proposition. □

5.1.3. *A global existence result.* According to the computations of the previous paragraph and to the change of variables (4.1), it is suitable to introduce the following norms for getting global solutions with uniform estimates in the case[3] $m < +\infty$

$$\|(b, \vec u, j_0, \vec j_1)\|_{X_\varepsilon^\nu} := \|b\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2-1}_{2,1}} + \nu\|b\|^{h,\nu^{-1}}_{\dot B^{\frac n2}_{2,1}} + \|(\vec u, j_0, \vec j_1)\|_{\dot B^{\frac n2-1}_{2,1}},$$

[3]Writing out the corresponding definition if $m = +\infty$ is left to the reader.

and

$$\|(b,\vec{u},j_0,\vec{j}_1)\|_{Y^\nu_\varepsilon} := \sup_{t\geq 0}\|(b,\vec{u},j_0,\vec{j}_1)(t)\|_{X^\nu} + \int_{\mathbb{R}_+}\Big(\nu\|b\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2+1}_{2,1}} + \|b\|^{h,\nu^{-1}}_{\dot B^{\frac n2}_{2,1}} + \nu\|\vec{u}\|_{\dot B^{\frac n2+1}_{2,1}}\Big)d\tau$$
$$+\int_{\mathbb{R}_+}\Big(\nu^{-1}\mathcal{M}\|\vec{\mathrm{j}}_1\|^{\ell,\varepsilon\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}} + \nu^{-1}\|\mathrm{j}_0\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2-1}_{2,1}} + \nu\|j_0\|^{m,\nu^{-1},\varepsilon\mathcal{M}}_{\dot B^{\frac n2+1}_{2,1}} + \nu^{-1}\mathcal{M}\|(j_0,\vec{j}_1)\|^{h,\varepsilon\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}}\Big)d\tau,$$

with $\mathrm{j}_0 := j_0 - b - \frac{\varepsilon}{\mathcal{L}}\mathrm{div}\,\vec{u}$ and $\vec{\mathrm{j}}_1 := j_1 + \frac{1}{\mathcal{L}\mathcal{M}}\nabla j_0 - \frac{1}{\mathcal{L}\mathcal{L}_s\mathcal{M}}\nabla b$.

Of course, if $(b',\vec{u}',j_0',\vec{j}_1')$ and $(b,\vec{u},j_0,\vec{j}_1)$ are interrelated through (4.1) and $\nu\mathcal{L}$ is used for $(b',\vec{u}',j_0',\vec{j}_1')$ instead of $\mathcal{L}$, then we have

$$\|(b',\vec{u}',j_0',\vec{j}_1')\|_{X^1_\varepsilon} = \nu^{-1}\|(b,\vec{u},j_0,\vec{j}_1)\|_{X^\nu_\varepsilon} \quad\text{and}\quad \|(b',\vec{u}',j_0',\vec{j}_1')\|_{Y^1_\varepsilon} = \nu^{-1}\|(b,\vec{u},j_0,\vec{j}_1)\|_{Y^\nu_\varepsilon}.$$

Theorem 5.1. *Assume that* $\mathcal{L}\approx 1$, *that* $\liminf \varepsilon^{-1}n\nu\mathcal{L} > 1$ *and that* $\mathcal{L}^2\mathcal{L}_s\approx 1$. *There exists a positive constant* η *depending only on* μ/ν, n *and on the pressure law such that if* ε *is small enough and the data* $(b_0^\varepsilon,\vec{u}_0^\varepsilon,j_{0,0}^\varepsilon,\vec{j}_{1,0}^\varepsilon)$ *satisfy*

$$\|(b_0^\varepsilon,\vec{u}_0^\varepsilon,j_{0,0}^\varepsilon,\vec{j}_{1,0}^\varepsilon)\|_{X^\nu_\varepsilon} \leq \eta\nu, \tag{5.15}$$

then System (1.6) *admits a unique global solution* $(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon,\vec{j}_1^\varepsilon)$ *in* Y^ν_ε. *In addition, we have*

$$\|(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon,\vec{j}_1^\varepsilon)\|_{Y^\nu_\varepsilon} \leq C\|(b_0^\varepsilon,\vec{u}_0^\varepsilon,j_{0,0}^\varepsilon,\vec{j}_{1,0}^\varepsilon)\|_{X^\nu_\varepsilon}. \tag{5.16}$$

A similar result holds true if $\mathcal{L}^2\mathcal{L}_s\to+\infty$.

Proof: Performing the change of variables proposed in (4.1) reduces the proof to the case $\nu=1$ (changing $\mathcal{L}$ into $\widetilde{\mathcal{L}}:=\nu\mathcal{L}$). Hence we consider a smooth enough solution to (4.2), and show that one may close the estimates globally[4] under Assumption (5.15).

Let us set $U_0 := \|(b_0^\varepsilon,\vec{u}_0^\varepsilon,j_{0,0}^\varepsilon,\vec{j}_{1,0}^\varepsilon)\|_{X^1_\varepsilon}$ and $U(t) := \|(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon,\vec{j}_1^\varepsilon)\|_{Y^1_\varepsilon(t)}$. In what follows, we drop exponents ε for notational simplicity. Finally, to shorten the presentation, we just treat the case where $\mathcal{L}^2\mathcal{L}_s\approx 1$.

Now applying Proposition 5.1 with $\vec{v}=\vec{u}$, $F := -T'_{\nabla b}\cdot\vec{u} - k_1(b)\mathrm{div}\,\vec{u}$ and

$$\vec{G} := -T'_{\nabla\vec{u}}\cdot\vec{u} + k_2(b)\widetilde{\mathcal{A}}\vec{u} - k_3(b)\nabla b + \frac{\widetilde{\mathcal{L}}\mathcal{M}}{n}\,k_4(b)\vec{j}_1,$$

yields for all $t\geq 0$

$$U(t)\leq C\bigg(U_0+\int_0^t\|\nabla\vec{u}\|_{L^\infty}\|(b,\vec{u},j_0,\vec{j}_1)\|_{X^1_\varepsilon}\,d\tau$$
$$+\int_0^t\Big(\|F\|_{\dot B^{\frac n2}_{2,1}} + \|F-T_{\vec{u}}\cdot\nabla b\|_{\dot B^{\frac n2-1}_{2,1}} + \|\vec{G}\|_{\dot B^{\frac n2-1}_{2,1}} + \|T_{\vec{u}}\cdot\nabla\vec{u}\|_{\dot B^{\frac n2-1}_{2,1}}\Big)\,d\tau\bigg). \tag{5.17}$$

Using standard continuity results for the paraproduct and remainder, and composition estimates leads to

$$\|T'_{\nabla b}\cdot\vec{u}\|_{\dot B^{\frac n2}_{2,1}} \leq C\|\nabla b\|_{\dot B^{\frac n2-1}_{2,1}}\|\vec{u}\|_{\dot B^{\frac n2+1}_{2,1}},$$
$$\|k_1(b)\mathrm{div}\,\vec{u}\|_{\dot B^{\frac n2}_{2,1}} \leq C\|b\|_{\dot B^{\frac n2}_{2,1}}\|\mathrm{div}\,\vec{u}\|_{\dot B^{\frac n2}_{2,1}}.$$

[4]Existence follows from spectral truncation as in e.g. [1], Chap. 10, and is thus omitted. As for uniqueness, we refer to [7].

Hence we have

$$\int_0^t \|F\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\,d\tau \le CU^2(t). \tag{5.18}$$

We also have

$$\begin{aligned}
\|T_{\vec{u}}\cdot\nabla\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} &\le C\|\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\nabla\vec{u}\|_{\dot{B}^{\frac{n}{2}}_{2,1}},\\
\|T'_{\nabla\vec{u}}\cdot\vec{u}\|_{\dot{B}^{\frac{n}{2}}_{2,1}} &\le C\|\nabla\vec{u}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}},\\
\|k_2(b)\widetilde{\mathcal{A}}\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} &\le C\|b\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla^2\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}},\\
\|k_3(b)\nabla b\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} &\le C\|b\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla b\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}.
\end{aligned}$$

Bounding $\widetilde{\mathcal{L}}\mathcal{M}k_4(b)\vec{j}_1$ is slightly more involved as it is not true that the low frequencies of $\vec{j}_1$ are bounded in $L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{2}-1}_{2,1})$. However, one may write that

$$\widetilde{\mathcal{L}}\mathcal{M}\vec{j}_1 = \widetilde{\mathcal{L}}\mathcal{M}\vec{j}_1^{\,h,\widetilde{\mathcal{L}}\mathcal{M}} + \widetilde{\mathcal{L}}\mathcal{M}\vec{\mathrm{j}}_1^{\,\ell,\widetilde{\mathcal{L}}\mathcal{M}} + \mathcal{L}_s^{-1}\nabla b^{\ell,\widetilde{\mathcal{L}}\mathcal{M}} - n^{-1/2}\nabla j_0^{\ell,\widetilde{\mathcal{L}}\mathcal{M}}.$$

Therefore

$$\begin{aligned}
\widetilde{\mathcal{L}}\mathcal{M}\|k_4(b)\vec{j}_1\|_{L^1_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} \lesssim{}& \widetilde{\mathcal{L}}\mathcal{M}\big(\|\vec{j}_1\|^{h,\widetilde{\mathcal{L}}\mathcal{M}}_{L^1_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} + \|\vec{\mathrm{j}}_1\|^{\ell,\widetilde{\mathcal{L}}\mathcal{M}}_{L^1_t(\dot{B}^{\frac{n}{2}-1}_{2,1})}\big)\|b\|_{L^\infty_t(\dot{B}^{\frac{n}{2}}_{2,1})}\\
&+\|b\|_{L^2_t(\dot{B}^{\frac{n}{2}}_{2,1})}\big(\mathcal{L}_s^{-1}\|\nabla b\|^{\ell,\widetilde{\mathcal{L}}\mathcal{M}}_{L^2_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} + \|\nabla j_0\|^{\ell,\widetilde{\mathcal{L}}\mathcal{M}}_{L^2_t(\dot{B}^{\frac{n}{2}-1}_{2,1})}\big).
\end{aligned}$$

Hence

$$\int_0^t \big(\|T_{\vec{u}}\cdot\nabla\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\vec{G}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big)\,d\tau \le CU^2(t). \tag{5.19}$$

Finally

$$\begin{aligned}
\|\vec{u}\cdot\nabla b\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \le{}& C\|\vec{u}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla b\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}},\\
\|k_1(b)\mathrm{div}\,\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \le{}& C\|b\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\mathrm{div}\,\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}.
\end{aligned}$$

Therefore, by Cauchy-Schwarz inequality

$$\int_0^t \|F - T_{\vec{u}}\cdot\nabla b\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\,d\tau \le CU^2(t). \tag{5.20}$$

Inserting (5.18), (5.19), (5.20) in (5.17) and remembering that $\dot{B}^{\frac{n}{2}}_{2,1}\hookrightarrow L^\infty$ (to ensure that, say, $|b|\le 1/2$ if $\|b\|_{\dot{B}^{\frac{n}{2}}_{2,1}}$ is small enough), we end up with

$$U(t) \le C(U_0 + U^2(t)) \quad \text{for all} \quad t\ge 0.$$

By a standard bootstrap argument, we easily deduce that

$$U(t)\le 2CU_0 \quad \text{for all} \quad t\ge 0,$$

provided the data have been chosen so that $4C^2U_0\le 1$. $\square$

5.2. **Study of the limit system.** In this paragraph, we prove the existence and uniqueness of strong (small) solutions with critical regularity for Systems (2.5) and (2.6). We shall give a common proof that works for both systems.

Before giving the global existence statement, let us introduce the solution space

- If $m \in (0, +\infty)$ (that is for System (2.5)) then Initial data will be taken in the space $\mathcal{X}^\nu$ which is the set of triplets $(b, \vec{u}, j_0)$ satisfying

$$\|(b,\vec{u},j_0)\|_{\mathcal{X}^\nu} := \|b\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|j_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \nu^{-1}\|j_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-2}_{2,1}} < \infty,$$

and the solution space $\mathcal{Y}^\nu$ will be the set of triplets $(b, \vec{u}, j_0)$ in $\mathcal{C}_b(\mathbb{R}_+; \mathcal{X}^\nu)$ satisfying

$$\|(b,\vec{u},j_0)\|_{\mathcal{Y}^\nu} := \sup_{t\geq 0}\|(b,\vec{u},j_0)(t)\|_{\mathcal{X}^\nu} + \int_{\mathbb{R}^+}\Big(\|j_0-b\|_{\dot{B}^{\frac{n}{2}}_{2,1}} + \nu\|b\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \nu\|\vec{u}\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \|j_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}}\Big)d\tau.$$

- If $m = +\infty$ (that is for System (2.6)), Initial data will be taken in the space $\mathcal{X}^\nu_\infty$ which is the set of triplets $(b, \vec{u}, j_0)$ satisfying

$$\|(b,\vec{u},j_0)\|_{\mathcal{X}^\nu_\infty} := \|b\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|j_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}} < \infty,$$

and the solution space $\mathcal{Y}^\nu_\infty$ will be the set of triplets $(b, \vec{u}, j_0)$ in $\mathcal{C}_b(\mathbb{R}_+; \mathcal{X}^\nu_\infty)$ satisfying

$$\|(b,\vec{u},j_0)\|_{\mathcal{Y}^\nu_\infty} := \sup_{t\geq 0}\|(b,\vec{u},j_0)(t)\|_{\mathcal{X}^\nu_\infty} + \int_{\mathbb{R}^+}\Big(\|j_0-b\|_{\dot{B}^{\frac{n}{2}}_{2,1}} + \nu\|b\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \nu\|\vec{u}\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \|j_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}}\Big)d\tau < \infty.$$

Theorem 5.2. *There exist two positive constants c and C so that if*

$$\|(b_0,\vec{u}_0,j_{0,0})\|_{\mathcal{X}^\nu} \leq c\nu \quad (\textit{case}\ \ m < +\infty), \tag{5.21}$$

$$\textit{or}\qquad \|(b_0,\vec{u}_0,j_{0,0})\|_{\mathcal{X}^\nu_\infty} \leq c\nu \quad (\textit{case}\ \ m = +\infty), \tag{5.22}$$

then System (2.5) *(resp.* (2.6)*) admits a unique solution in the space $\mathcal{Y}^\nu$ (resp. $\mathcal{Y}^\nu_\infty$) satisfying in addition,*

$$\|(b,\vec{u},j_0)\|_{\mathcal{Y}^\nu} \leq C\|(b_0,\vec{u}_0,j_{0,0})\|_{\mathcal{X}^\nu} \quad \textit{if}\ \ m < +\infty, \tag{5.23}$$

$$\|(b,\vec{u},j_0)\|_{\mathcal{Y}^\nu_\infty} \leq C\|(b_0,\vec{u}_0,j_{0,0})\|_{\mathcal{X}^\nu_\infty} \quad \textit{if}\ \ m = +\infty. \tag{5.24}$$

Proof: Set $\widetilde{\kappa} := \kappa/n$ and $\widetilde{m} := mn$. As usual, it is enough to treat the case $\nu = 1$ as performing the change of unknowns

$$(b, u, j_0)(t, x) = (\widetilde{b}, \widetilde{u}, \widetilde{j}_0)(\nu^{-1}t, \nu^{-1}x),$$

gives Systems (2.5) or (2.6) for $(\widetilde{b}, \widetilde{u}, \widetilde{j}_0)$ with $\nu = 1$ and $\widetilde{\mathcal{A}} := \mathcal{A}/\nu$ and, obviously

$$\|(b,u,j_0)(t)\|_{\mathcal{X}^\nu} = \nu\|(\widetilde{b},\widetilde{u},\widetilde{j}_0)(\nu^{-1}t)\|_{\mathcal{X}^1} \quad\text{and}\quad \|(b,u,j_0)\|_{\mathcal{Y}^\nu} = \nu\|(\widetilde{b},\widetilde{u},\widetilde{j}_0)\|_{\mathcal{Y}^1}.$$

Let us start with the study of the linearized equations with no source term, namely

$$\left\{\begin{array}{l} \partial_t b + \operatorname{div}\vec{u} = 0,\\ \partial_t\vec{u} - \widetilde{\mathcal{A}}\vec{u} + \nabla b + n^{-1}\nabla j_0 = \vec{0},\\ \partial_t j_0 + \widetilde{\kappa}(j_0 - \widetilde{m}^{-1}\Delta j_0 - b) = 0. \end{array}\right. \tag{5.25}$$

The divergence-free part $\mathcal{P}\vec{u}$ of the velocity satisfies

$$\partial_t\mathcal{P}\vec{u} - \mu\Delta\mathcal{P}\vec{u} = \vec{0},$$

while the coupling between $b,\ d:=\Lambda^{-1}\mathrm{div}\,\vec{u}$ and j_0 is described by

$$\begin{cases}\partial_t b+\Lambda d=0,\\ \partial_t d-\Delta d-\Lambda b-n^{-1}\Lambda j_0=0,\\ \partial_t j_0+\widetilde{\kappa}(j_0-\widetilde{m}^{-1}\Delta j_0-b)=0.\end{cases}\tag{5.26}$$

Note that the stability of a similar system has already been established in the previous section for $\kappa>1$ (or, equivalently, $\widetilde{\kappa}>1/n$).

Linear estimates for low frequencies. We introduce $\zeta_0:=j_0-b-\widetilde{\kappa}^{-1}\Lambda d$ and notice that

$$\begin{cases}\partial_t\widehat{b}+\rho\widehat{d}=0,\\ \partial_t\widehat{d}+\rho^2\big(1-\frac{1}{\widetilde{\kappa}n}\big)\widehat{d}-\big(1+\frac{1}{n}\big)\rho\widehat{b}=\frac{1}{n}\rho\widehat{\zeta}_0,\\ \partial_t\widehat{\zeta}_0+\big(\widetilde{\kappa}+(\frac{\widetilde{\kappa}}{\widetilde{m}}+\frac{1}{\widetilde{\kappa}n})\rho^2\big)\widehat{\zeta}_0=-\Big(\big(1+\frac{1}{n}\big)\frac{1}{\widetilde{\kappa}}+\frac{\widetilde{\kappa}}{\widetilde{m}}\Big)\rho^2\widehat{b}+\Big(\big(1-\frac{1}{\widetilde{\kappa}n}\big)\frac{1}{\widetilde{\kappa}}-\frac{1}{\widetilde{m}}\Big)\rho^3\widehat{d}.\end{cases}$$

On one hand, because $\widetilde{\kappa}n>1$, the method described in the appendix (see in particular (B.7)) allows to write that, omitting the dependence with respect to $\widetilde{\kappa}$

$$|(\widehat{b},\widehat{d})(t)|+\rho^2\int_0^t|(\widehat{b},\widehat{d})|\,d\tau\lesssim|(\widehat{b},\widehat{d})(0)|+\rho\int_0^t|\widehat{\zeta}_0|\,d\tau.$$

On the other hand, the last equation directly gives

$$|\widehat{\zeta}_0(t)|+\Big(\widetilde{\kappa}+(\frac{\widetilde{\kappa}}{\widetilde{m}}+\frac{1}{\widetilde{\kappa}n})\rho^2\Big)\int_0^t|\widehat{\zeta}_0|\,d\tau\le|\widehat{\zeta}_0(0)|+C\Big(1+\frac{1}{m}\Big)\rho^2\int_0^t|(\widehat{b},\rho\widehat{d})|\,d\tau.$$

Hence plugging the second inequality in the first one

$$|(\widehat{b},\widehat{d})(t)|+\rho^2\int_0^t|(\widehat{b},\widehat{d})|\,d\tau\lesssim|(\widehat{b},\widehat{d})(0)|+\frac{\rho}{1+(1+m^{-1})\rho^2}\bigg(|\widehat{\zeta}_0(0)|+(1+m^{-1})\rho^2\int_0^t|(\widehat{b},\rho\widehat{d})|\,d\tau\bigg)\cdot$$

It is clear that the last term may be absorbed by the integral of the l.h.s. if $\rho\ll\frac{m}{1+m}\cdot$ Hence we eventually get for some small enough $\rho_\ell>0$

$$|(\widehat{b},\widehat{d},\rho\widehat{\zeta}_0)(t)|+\rho^2\int_0^t|(\widehat{b},\widehat{d})|\,d\tau+\int_0^t|\rho\widehat{\zeta}_0|\,d\tau\lesssim|(\widehat{b},\widehat{d},\rho\widehat{\zeta}_0)(0)|\quad\text{if}\quad\rho\le\Big(\frac{m}{1+m}\Big)\rho_\ell.\tag{5.27}$$

Linear estimates for high frequencies. We set $\delta:=d-\Lambda^{-1}b$ and notice that

$$\begin{cases}\partial_t\widehat{b}+\widehat{b}=-\rho\widehat{\delta},\\ \partial_t\widehat{\delta}+(\rho^2-1)\widehat{\delta}=\rho^{-1}\widehat{b}+n^{-1}\rho\widehat{j}_0,\\ \partial_t\widehat{j}_0+\widetilde{\kappa}\big(1+\frac{\rho^2}{\widetilde{m}}\big)\widehat{j}_0=\widetilde{\kappa}\widehat{b}.\end{cases}$$

Therefore

$$|\widehat{\delta}(t)|+(\rho^2-1)\int_0^t|\widehat{\delta}|\,d\tau\le|\widehat{\delta}(0)|+\rho^{-1}\int_0^t|\widehat{b}|\,d\tau+\frac{1}{n}\int_0^t\rho|\widehat{j}_0|\,d\tau.$$

At the same time

$$|\widehat{b}(t)|+\int_0^t|\widehat{b}|\,d\tau\le|\widehat{b}(0)|+\rho\int_0^t|\widehat{\delta}|\,d\tau,$$

$$|\widehat{j}_0(t)|+\widetilde{\kappa}\Big(1+\frac{\rho^2}{\widetilde{m}}\Big)\int_0^t|\widehat{j}_0|\,d\tau\le|\widehat{j}_0(0)|+\widetilde{\kappa}\int_0^t|\widehat{b}|\,d\tau.$$

Hence

$$|\widehat{\delta}(t)|+(\rho^2-1)\int_0^t|\widehat{\delta}|\,d\tau\leq|\widehat{\delta}(0)|+\Bigl(\frac{1}{\rho}+\frac{\rho}{n}\Bigr)|\widehat{b}(0)|$$
$$+\Bigl(\frac{n^{-1}\widetilde{\kappa}^{-1}}{1+\widetilde{m}^{-1}\rho^2}\Bigr)\rho|\widehat{j}_0(0)|+\Bigl(1+\frac{\rho^2}{n(1+\widetilde{m}^{-1}\rho^2)}\Bigr)\int_0^t|\widehat{\delta}|\,d\tau.$$

Therefore there exists a constant ρ_h depending only on m and n (with $n\geq 2$ if $m=+\infty$) such that for $\rho\geq\rho_h$, we have

$$|(\rho\widehat{b},\widehat{\delta})(t)|+\min(\rho,m\rho^{-1})|\widehat{j}_0(t)|+\rho\int_0^t|(\widehat{b},\widehat{j}_0)|\,d\tau+\rho^2\int_0^t|\widehat{\delta}|\,d\tau\lesssim|(\rho\widehat{b},\widehat{\delta},\rho\widehat{j}_0)(0)|. \tag{5.28}$$

Of course, one may replace δ with d in (5.28).

Linear estimates for medium frequencies. The stability argument used just below (4.37) allows to write that there exist two constants c and C depending continuously on $1/m$, such that if $\rho\in[\frac{m}{m+1}\rho_\ell,\rho_h]$ then

$$|(\widehat{b},\widehat{d},\widehat{j}_0)(t)|\leq Ce^{-ct}|(\widehat{b},\widehat{d},\widehat{j}_0)(0)|. \tag{5.29}$$

Estimates for the paralinearized system. The previous steps allow us to get handy estimates for the following paralinearized version of System (2.6)

$$\begin{cases}\partial_t b+T_{\vec v}\cdot\nabla b+\operatorname{div}\vec u=F,\\ \partial_t\vec u+T_{\vec v}\cdot\nabla\vec u-\widetilde{\mathcal{A}}\vec u+\nabla b+n^{-1}\nabla j_0=\vec G,\\ \partial_t j_0+\widetilde{\kappa}(j_0-\widetilde{m}^{-1}\Delta j_0-b)=0.\end{cases} \tag{5.30}$$

More precisely, following the steps leading to (5.27), (5.28) and (5.29), introducing $\zeta_0:=j_0-b-\widetilde{\kappa}^{-1}\operatorname{div}\vec u$, and arguing as in Subsection 5.1.2 we end up with[5]

$$\|j_0(t)\|^{\ell,1}_{\dot B^{\frac n2}_{2,1}}+\|(b,\vec u)(t)\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}+\int_0^t\bigl(\|j_0-b\|^{\ell,1}_{\dot B^{\frac n2}_{2,1}}+\|(b,\vec u)\|^{\ell,1}_{\dot B^{\frac n2+1}_{2,1}}\bigr)\,d\tau \tag{5.31}$$
$$\lesssim\|j_0(0)\|^{\ell,1}_{\dot B^{\frac n2}_{2,1}}+\|(b,\vec u)(0)\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}+\int_0^t\Bigl(\|F-T_{\vec v}\cdot\nabla b\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}+\|\vec G-T_{\vec v}\cdot\nabla\vec u\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}\Bigr)d\tau.$$

For high frequencies, we get, in the case $m=+\infty$

$$\|(b,j_0)(t)\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec u(t)\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}+\int_0^t\bigl(\|(b,j_0)\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec u\|^{h,1}_{\dot B^{\frac n2+1}_{2,1}}\bigr)\,d\tau \tag{5.32}$$
$$\lesssim\|(b,j_0)(0)\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec u(0)\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}+\int_0^t\Bigl(\|F\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec G\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}\Bigr)d\tau$$
$$+\int_0^t\|\nabla\vec v\|_{L^\infty}\bigl(\|(b,j_0)\|_{\dot B^{\frac n2}_{2,1}}+\|\vec u\|_{\dot B^{\frac n2-1}_{2,1}}\bigr)\,d\tau,$$

[5]Here we do not track the dependency with respect to m.

and if $0<m<+\infty$

$$\|b(t)\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|j_0(t)\|^{h,1}_{\dot B^{\frac n2-2}_{2,1}}+\|\vec u(t)\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}+\int_0^t\bigl(\|(b,j_0)\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec u\|^{h,1}_{\dot B^{\frac n2+1}_{2,1}}\bigr)\,d\tau \tag{5.33}$$
$$\lesssim\|b(0)\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|j_0(0)\|^{h,1}_{\dot B^{\frac n2-2}_{2,1}}+\|\vec u(0)\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}+\int_0^t\Bigl(\|F\|^{h,1}_{\dot B^{\frac n2}_{2,1}}+\|\vec G\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}\Bigr)d\tau$$
$$+\int_0^t\|\nabla\vec v\|_{L^\infty}\bigl(\|b\|_{\dot B^{\frac n2}_{2,1}}+\|j_0\|^h_{\dot B^{\frac n2-2}_{2,1}}+\|\vec u\|_{\dot B^{\frac n2-1}_{2,1}}\bigr)\,d\tau.$$

Proof of existence. We only establish global-in-time a priori bounds in the space $\mathcal{Y}^1$ or $\mathcal{Y}^1_\infty$ for the solutions to (2.5) or (2.6) with data satisfying (5.21) or (5.22). Our proof is based on (5.31), (5.32) and (5.33) with $\vec v=\vec u$

$$F=-T'_{\nabla b}\cdot\vec v-k_1(b)\mathrm{div}\,\vec u\quad\text{and}\quad\vec G=-T'_{\nabla\vec u}\cdot\vec u+k_2(b)\widetilde{\mathcal{A}}\vec u-k_3(b)\nabla b-n^{-1}k_4(b)\nabla j_0.$$

Bounding $\|F-T_{\vec v}\cdot\nabla b\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}$ and $\|F\|^{h,1}_{\dot B^{\frac n2}_{2,1}}$ relies on (5.18) and (5.20). As regards $\vec G$, the computations that we did in the proof of Theorem 5.1 ensure that the first three terms may be bounded as in (5.19). To handle the last term, $k_4(b)\nabla j_0$, in the case[6] $m<+\infty$ we use the decomposition

$$k_4(b)\nabla j_0=k_4(b)\nabla b+k_4(b)\nabla(j_0-b).$$

The first term may be bounded quadratically exactly as $k_3(b)\nabla b$. As for the last term, we may write

$$\|k_4(b)\nabla(j_0-b)\|_{L^1_t(\dot B^{\frac n2}_{2,1})}\lesssim\|b\|_{L^\infty_t(\dot B^{\frac n2}_{2,1})}\|\nabla(j_0-b)\|_{L^1_t(\dot B^{\frac n2-1}_{2,1})},$$

hence it is also bounded by $C\|(b,\vec u,j_0)\|^2_{\mathcal{Y}^1(t)}$.

This enables to conclude that we do have for all $t\in\mathbb{R}_+$

$$\|(b,\vec u,j_0)\|_{\mathcal{Y}^1(t)}\le C\bigl(\|(b,\vec u,j_0)(0)\|_{\mathcal{X}^1}+\|(b,\vec u,j_0)\|^2_{\mathcal{Y}^1(t)}\bigr).$$

This obviously yields (5.24) if (5.22) is fulfilled.

Proof of uniqueness. It works the same as for the standard barotropic Navier-Stokes equations: we look at the system satisfied by the difference $(\delta b,\delta\vec u,\delta j_0)$ between two solutions $(b^1,\vec u^1,j_0^1)$ and $(b^2,\vec u^2,j_0^2)$ of (2.5), namely (denoting $K_i=1+k_i$ for $i=1,2,3,4$)

$$\begin{cases}\partial_t\delta b+\vec u^2\cdot\nabla\delta b=-\delta\vec u\cdot\nabla b^1+(K_1(b^1)-K_1(b^2))\mathrm{div}\,\vec u^2-K_1(b^1)\mathrm{div}\,\delta\vec u,\\ \partial_t\delta\vec u+\vec u^2\cdot\nabla\delta\vec u+\delta\vec u\cdot\nabla\vec u^1-(K_2(b^2)-K_2(b^1))\mathcal{A}\vec u^2-K_2(b^1)\mathcal{A}\delta\vec u+(K_3(b^2)-K_3(b^1))\nabla b^2\\ \qquad\qquad+K_3(b^1)\nabla\delta b+n^{-1}(K_4(b^2)-K_4(b^1))\nabla j_0^1+n^{-1}K_4(b^2)\nabla\delta j_0=\vec 0,\\ \partial_t\delta j_0+\widetilde\kappa\bigl(\delta j_0-\delta b-\frac{1}{\widetilde m}\Delta\delta j_0\bigr)=0.\end{cases}$$

Now, exactly as for the barotropic Navier-Stokes equations, it is possible to bound δb, $\delta\vec u$ and δj_0 just resorting to basic estimates for the transport and heat equations. However, the hyperbolic nature of the first equation forces us to estimate $(\delta b,\delta\vec u,\delta j_0)$ with one less derivative, namely in

$$L^\infty(0,T;\dot B^{\frac n2-1}_{2,1})\times\bigl(L^\infty(0,T;\dot B^{\frac n2-2}_{2,1})\cap L^1(0,T;\dot B^{\frac n2}_{2,1})\bigr)^n\times L^1(0,T;\dot B^{\frac n2-1}_{2,1}).$$

In dimension $n=3$ combining estimates for the transport and the heat equation allows to get uniqueness on a small time interval, then on the whole $\mathbb{R}_+$ by induction. In dimension

[6]The case $m=+\infty$ does not require that decomposition.

$n=2$, this is slightly more involved as some product laws do not work correctly if estimating $(\delta b,\delta\vec u,\delta j_0)$ in the above space (some regularity exponents become too negative). Nevertheless this may be overcome by combining logarithmic interpolation and Osgood lemma (see e.g. [6] for more details). This completes the proof of the theorem. □

Remark 5.1. *If $0<m<+\infty$ then one may alternately assume that j_0 is in $\dot B^{\frac n2}_{2,1}$. Taking advantage of the parabolic smoothing given by the equation for j_0, it is not difficult to get a solution $(b,\vec u,j_0)$ with*

$$\widetilde m\int_{\mathbb{R}_+}\|j_0\|_{\dot B^{\frac n2+2}_{2,1}}\leq C\Bigl(\|b_0\|_{\dot B^{\frac n2-1}_{2,1}}+\nu\|b_0\|_{\dot B^{\frac n2}_{2,1}}+\|\vec u_0\|_{\dot B^{\frac n2-1}_{2,1}}+\|j_0\|_{\dot B^{\frac n2}_{2,1}}\Bigr)\cdot$$

5.3. **Weak convergence.** Here we justify the weak convergence of (1.6) to (2.5) or (2.6) under the assumption that $\liminf\varepsilon^2\mathcal{L}_s>0$ and that $\mathcal{L}$ tends to $\frac{\kappa\varepsilon}{n\nu}$ for some $\kappa>1$.

Theorem 5.3. *Let the family of data $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)_{0<\varepsilon<1}$ satisfy Condition (5.15). Assume in addition that*

$$\mathcal{L}^2\mathcal{L}_s\nu^2\to m\in(0,+\infty]\quad\text{and}\quad\frac{n\nu\mathcal{L}}{\varepsilon}\to\kappa\in(1,+\infty).\tag{5.34}$$

Then the global solution $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ given by Theorem 5.1 satisfies

$$\vec j_1^\varepsilon\to\vec 0\quad\text{in}\quad L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1}),$$

and, if $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon)\rightharpoonup(b_0,\vec u_0,j_{0,0})$ then $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon)$ converges weakly to the unique solution $(b,\vec u,j_0)$ of (2.5) supplemented with initial data $(b_0,\vec u_0,j_{0,0})$.

Proof: From (5.16) we gather that

$$(\vec{\mathrm{j}}_1^\varepsilon)^{\ell,\mathcal{LM}}=\mathcal{O}(\mathcal{M}^{-1})\quad\text{and}\quad(\vec j_1^\varepsilon)^{h,\mathcal{LM}}=\mathcal{O}(\mathcal{M}^{-1})\ \text{ in }\ L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}).$$

Therefore, taking advantage of the boundedness of the low frequencies of ∇b^ε and ∇j_0^ε in $L^1(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, and of the fact that

$$\vec j_1^\varepsilon=\vec{\mathrm{j}}_1^\varepsilon-\frac{1}{\mathcal{LM}}\nabla j_0^\varepsilon+\frac{1}{\mathcal{L}\mathcal{L}_s\mathcal{M}}\nabla b^\varepsilon,$$

we get

$$\vec j_1^\varepsilon=\mathcal{O}(\varepsilon)\quad\text{in}\quad L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1}).\tag{5.35}$$

Next, we observe that (5.16) implies that (b^ε) and $(\vec u^\varepsilon)$ are bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}\cap\dot B^{\frac n2}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac n2+1}_{2,1}+\dot B^{\frac n2}_{2,1})$ and $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac n2+1}_{2,1})$, respectively. Note that this implies that $\vec u^\varepsilon$ is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$. Because

$$\partial_t b^\varepsilon=-\vec u^\varepsilon\cdot\nabla b^\varepsilon-k_1(b^\varepsilon)\mathrm{div}\,\vec u^\varepsilon,$$

and the product maps $\dot B^{\frac n2-1}_{2,1}\times\dot B^{\frac n2}_{2,1}$ in $\dot B^{\frac n2-1}_{2,1}$, we thus get in addition that $\partial_t b^\varepsilon$ is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$, and thus (b^ε) is bounded in $\mathcal{C}^{\frac12}(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$. Interpolating with the bound in $\mathcal{C}_b(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, we thus have (b^ε) bounded in $\mathcal{C}^{\frac\alpha2}(\mathbb{R}_+;\dot B^{\frac n2-\alpha}_{2,1})$ for all $\alpha\in[0,1]$. Then combining locally compact Besov embeddings and Ascoli theorem allows to conclude that there exists b

in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}\cap\dot B^{\frac n2}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac n2+1}_{2,1}+\dot B^{\frac n2}_{2,1})$ and a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ going to 0 so that, for all $\phi\in\mathcal{S}$ and all $\alpha\in(0,1]$

$$\phi\, b^{\varepsilon_k}\longrightarrow\phi\, b\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+;\dot B^{\frac n2-\alpha}_{2,1}).\tag{5.36}$$

From (5.16), we readily get for some sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ tending to 0

$$\vec u^{\varepsilon_k}\rightharpoonup\vec u\quad\text{in}\quad L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac n2+1}_{2,1})\quad\text{weak }*,\tag{5.37}$$

which, combined with (5.36) is clearly enough to pass to the limit in the mass equation. Next, we see that (5.16) implies that (j_0^ε) is bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$. Hence there exists $j_0\in L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ and a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ going to 0 so that

$$j_0^{\varepsilon_k}\rightharpoonup j_0\quad\text{in}\quad L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})\quad\text{weak }*.$$

Because

$$\frac1\varepsilon\mathrm{div}\,\vec j_1^\varepsilon=-\frac{\mathcal{L}}{\varepsilon}\biggl(\frac{1}{\mathcal{L}^2(1+\mathcal{L}_s)}\biggr)\Delta j_0^\varepsilon-\frac{1}{\mathcal{L}(1+\mathcal{L}_s)}\partial_t\mathrm{div}\,\vec j_1^\varepsilon,\tag{5.38}$$

and (5.35) implies that $\partial_t\vec j_1^\varepsilon\to0$ in the sense of distributions, we deduce that

$$\frac1\varepsilon\mathrm{div}\,\vec j_1^\varepsilon\to-\frac{\kappa\nu}{nm}\Delta j_0\quad\text{in}\quad\mathcal{S}'.$$

Note that the right-hand side is 0 if $m=+\infty$. Therefore (b,j_0) satisfies the third line of (2.5) (case $m<+\infty$) or (2.6) (case $m=+\infty$).

Let us finally pass to the limit in the second equation of (1.6). The main difficulty is that, owing to the radiative term which is only bounded in a L^1-in-time type space (namely $L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ or so), one cannot take advantage of some suitable bound of $\partial_t\vec u^\varepsilon$ so as to glean some equicontinuity and then resort to Ascoli theorem. To overcome this, we use the fact that, owing to (5.38)

$$\partial_t\biggl(\vec u^\varepsilon+\frac\varepsilon n(1+k_4(b^\varepsilon))\,\vec j_1^\varepsilon\biggr)=-\vec u^\varepsilon\cdot\nabla\vec u^\varepsilon+(1+k_2(b^\varepsilon))\mathcal{A}\vec u^\varepsilon$$
$$-(1+k_3(b^\varepsilon))\nabla b^\varepsilon-\frac1n\,(1+k_4(b^\varepsilon))\,\nabla j_0^\varepsilon+\frac\varepsilon n k_4'(b^\varepsilon)\partial_t b^\varepsilon\,\vec j_1^\varepsilon.$$

Now, because $(\vec u^\varepsilon)$ is bounded in the space $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})\cap L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, (j_0^ε) is bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ and (b^ε) is bounded in $(L^2\cap L^\infty)(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, product laws in Besov spaces ensure that the first four terms of the r.h.s. are bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,1})$ (only in $L^2(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,\infty})$ if $n=2$). The same property holds true for the last term for $(\partial_t b^\varepsilon)$ is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ and $(\vec j_1^\varepsilon)$ is bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$. Using locally compact Besov embedding and Ascoli theorem, one can now conclude that there exists some $\vec v$ in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ so that for all ϕ in $\mathcal{S}$ and $\alpha\in(0,1)$, we have, up to extraction

$$\phi\Bigl(\vec u^\varepsilon+\frac\varepsilon n(1+k_4(b^\varepsilon))\,\vec j_1^\varepsilon\Bigr)\longrightarrow\phi\vec v\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+;\dot B^{\frac n2-1-\alpha}_{2,1}).$$

Of course, combining with (5.35), we discover that $\vec v=\vec u$. Hence we also have

$$\phi\vec u^\varepsilon\longrightarrow\phi\vec v\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+\dot B^{\frac n2-1-\alpha}_{2,1})\quad\text{for all}\quad\phi\in\mathcal{S}.$$

It is now easy to conclude that the second line of (2.5) is fulfilled by $(b,\vec{u},j_0)$.

Finally, that the whole family $(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon)$ (and not only subsequences) converges to $(b,\vec{u},j_0)$ stems from the fact that the solution to (2.5) or (2.6) is unique. □

Remark 5.2. *It is also possible to justify the strong convergence of the solutions of* (1.6) *to* (2.5) *or* (2.6) *using* (5.35) *and performing the difference between* $(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon)$ *and the solution* $(b,\vec{u},j_0)$ *to the limit system. Again, taking advantage of the decay properties of* $\mathfrak{j}_1^\varepsilon$ *is crucial. Note however that, exactly as in the proof of uniqueness, owing to the hyperbolic nature of the density equation, one cannot prove the strong convergence in the solution space. There is a loss of one derivative that may be partially compensated by combining with uniform estimates. As we do not think this approach to bring much compared to weak compactness, we leave the details to the reader.*

6. The equilibrium diffusion regime

This section is devoted to the mathematical justification of the *equilibrium diffusion regime* given by (2.10). To avoid useless technicality, we focus on the case where

$$\mathcal{L}\to+\infty \quad\text{and}\quad \varepsilon\mathcal{L}\mathcal{M}\approx 1. \tag{6.1}$$

6.1. **Linear estimates.** Let us gather the estimates we proved for (4.7) for the above asymptotics in Section 4.

Regarding low frequencies, one may combine (4.20) and (4.21) to get

$$\begin{aligned}|(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|(\widehat{b},\widehat{d})|\,d\tau+\frac{\widetilde{\mathcal{L}}}{\varepsilon}\int_0^t|\mathfrak{j}_0|\,d\tau+\frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t|\mathfrak{j}_1|\,d\tau&\\ \leq C|(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for}\quad 0\leq\rho\leq\sqrt{1+n^{-1}},&\end{aligned}\tag{6.2}$$

with $\widetilde{\mathcal{L}}:=\nu\mathcal{L}$, $\widehat{\mathfrak{j}}_0:=\widehat{j}_0-\sqrt{n}\widehat{b}-\sqrt{n}\,\frac{\varepsilon}{\widetilde{\mathcal{L}}}\rho\widehat{d}$ and $\widehat{\mathfrak{j}}_1=\widehat{j}_1-\frac{\rho}{\widetilde{\mathcal{L}}\mathcal{M}}\widehat{b}.$

For middle frequencies, we have according to (4.58) and (4.64)

$$\begin{aligned}|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\int_0^t|\rho\widehat{b}|\,d\tau+\rho^2\int_0^t|\widehat{d}|\,d\tau+\rho\int_0^t|\widehat{j}_0|\,d\tau+\widetilde{\mathcal{L}}\mathcal{M}\int_0^t|\widehat{j}_1|\,d\tau&\\ \leq C|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for}\quad\sqrt{2/n}\leq\rho\leq c\widetilde{\mathcal{L}}\mathcal{M},&\end{aligned}\tag{6.3}$$

and (4.70) gives, if $\mathcal{M}$ is large enough

$$\begin{aligned}|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)|+\rho^2\int_0^t|\widehat{d}|\,d\tau+\rho\int_0^t|\widehat{b}|\,d\tau+\frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau&\\ \leq C|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)|\quad\text{for}\quad\rho\geq c\widetilde{\mathcal{L}}\mathcal{M}.&\end{aligned}\tag{6.4}$$

If $\mathcal{M}$ is bounded then we must assume that $\rho\geq C_1\widetilde{\mathcal{L}}\mathcal{M}$ for some $C_1>c$. However, we have (4.71) and $\mathcal{M}$ bounded implies that $\varepsilon\widetilde{\mathcal{L}}\approx 1$. Therefore (6.3) is satisfied up to $\rho\leq C_1\widetilde{\mathcal{L}}\mathcal{M}$.

For the whole system (4.4) with nonzero source terms f and $\vec{g}$, we thus obtain (taking slightly abusively $c = C_1 = 1$ for notational simplicity)

$$
\begin{aligned}
(6.5)\quad & \|(\vec{u}, j_0, \vec{j}_1)(t)\|_{\dot{B}^s_{2,1}} + \|b(t)\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|b(t)\|^{h,1}_{\dot{B}^{s+1}_{2,1}} + \int_0^t \|\vec{u}\|_{\dot{B}^{s+2}_{2,1}}\,d\tau + \frac{\widetilde{\mathcal{L}}}{\varepsilon}\int_0^t \|\mathrm{j}_0\|^{\ell,1}_{\dot{B}^s_{2,1}}\,d\tau \\
&+ \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t \|\vec{\mathrm{j}}_1\|^{\ell,1}_{\dot{B}^s_{2,1}}\,d\tau + \int_0^t \|(b, j_0, \vec{j}_1)\|^{\ell,1}_{\dot{B}^{s+2}_{2,1}}\,d\tau + \int_0^t \|j_0\|^{m,1,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot{B}^{s+1}_{2,1}}\,d\tau + \widetilde{\mathcal{L}}\mathcal{M}\int_0^t \|\vec{j}_1\|^{m,1,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot{B}^s_{2,1}}\,d\tau \\
&\qquad + \int_0^t \|b\|^{h,1}_{\dot{B}^{s+1}_{2,1}}\,d\tau + \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t \|(j_0, \vec{j}_1)\|^{h,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot{B}^s_{2,1}}\,d\tau \lesssim \|(\vec{u}, j_0, \vec{j}_1)(0)\|_{\dot{B}^s_{2,1}} \\
&\qquad\qquad + \|b(0)\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|b(0)\|^{h,1}_{\dot{B}^{s+1}_{2,1}} + \int_0^t \big(\|f\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|f\|^{h,1}_{\dot{B}^{s+1}_{2,1}} + \|\vec{g}\|_{\dot{B}^s_{2,1}}\big)\,d\tau,
\end{aligned}
$$

with

$$
\mathrm{j}_0 := j_0 - \sqrt{n}b - \sqrt{n}\,\frac{\varepsilon}{\widetilde{\mathcal{L}}}\mathrm{div}\,\vec{u} \quad\text{and}\quad \vec{\mathrm{j}}_1 = \vec{j}_1 + \frac{1}{\widetilde{\mathcal{L}}\mathcal{M}}\nabla b.
$$

Back to the original variables, that linear analysis induces us to introduce the following norms

$$
\|(b, \vec{u}, j_0, \vec{j}_1)\|_{\widetilde{X}^\nu_\varepsilon} := \|b\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|(j_0, \vec{j}_1)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \quad\text{and}
$$

$$
\begin{aligned}
\|(b, \vec{u}, j_0, \vec{j}_1)\|_{\widetilde{Y}^\nu_\varepsilon} &:= \sup_{t\geq 0}\|(b, \vec{u}, j_0, \vec{j}_1)(t)\|_{\widetilde{X}^\nu_\varepsilon} + \nu\int_{\mathbb{R}_+}\bigg(\|(b, j_0, \vec{j}_1)\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \|\vec{u}\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\bigg)d\tau \\
&\quad + \int_{\mathbb{R}_+}\bigg(\|b\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \frac{\mathcal{L}\mathcal{M}}{\varepsilon}\|\vec{\mathrm{j}}_1\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \frac{\mathcal{L}}{\varepsilon}\|\mathrm{j}_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\bigg)\,d\tau \\
&\quad + \int_{\mathbb{R}_+}\bigg(\|j_0\|^{m,\nu^{-1},\mathcal{L}\mathcal{M}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \mathcal{L}\mathcal{M}\|\vec{j}_1\|^{m,\nu^{-1},\mathcal{L}\mathcal{M}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \frac{\mathcal{L}\mathcal{M}}{\varepsilon}\|(j_0, \vec{j}_1)\|^{h,\mathcal{L}\mathcal{M}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\bigg)d\tau,
\end{aligned}
$$

with $\mathrm{j}_0 := j_0 - b - \frac{\varepsilon}{\mathcal{L}}\mathrm{div}\,\vec{u}$ and $\vec{\mathrm{j}}_1 := \vec{j}_1 + \frac{1}{\mathcal{L}\mathcal{M}}\nabla b$.

We denote by $\widetilde{X}^\nu_\varepsilon$ and $\widetilde{Y}^\nu_\varepsilon$ the corresponding functional spaces (where time continuity is imposed rather than just boundedness). Of course, we still have

$$
\|(b', \vec{u}', j_0', \vec{j}_1')\|_{\widetilde{X}^1_\varepsilon} = \nu^{-1}\|(b, \vec{u}, j_0, \vec{j}_1)\|_{\widetilde{X}^\nu_\varepsilon} \quad\text{and}\quad \|(b', \vec{u}', j_0', \vec{j}_1')\|_{\widetilde{Y}^1_\varepsilon} = \nu^{-1}\|(b, \vec{u}, j_0, \vec{j}_1)\|_{\widetilde{Y}^\nu_\varepsilon},
$$

through the change of variables (4.1), if we replace $\mathcal{L}$ by $\widetilde{\mathcal{L}}$ in the left-hand side.

6.2. **The paralinearized equations.** In the equilibrium diffusion limit case the estimates for the paralinearized system

$$
(6.6)\qquad \begin{cases}
\partial_t b + T_{\vec{v}}\cdot\nabla b + \mathrm{div}\,\vec{u} = F, \\
\partial_t\vec{u} + T_{\vec{v}}\cdot\nabla\vec{u} - \mathcal{A}\vec{u} + \nabla b - \frac{\mathcal{L}(1+\mathcal{L}_s)}{n}\vec{j}_1 = \vec{G}, \\
\partial_t j_0 + \frac{\mathrm{div}\,\vec{j}_1}{n\varepsilon} + \frac{\mathcal{L}}{\varepsilon}(j_0 - b) = 0, \\
\partial_t\vec{j}_1 + \frac{\nabla j_0}{\varepsilon} + \frac{\mathcal{L}(1+\mathcal{L}_s)}{\varepsilon}\vec{j}_1 = \vec{0}.
\end{cases}
$$

recast as follows

Proposition 6.1. *For any smooth solution* $(b,\vec u,j_0,\vec j_1)$ *we have the following a priori estimate for* (5.10)

$$\|(b,\vec u,j_0,\vec j_1)\|_{\widetilde Y^\nu_\varepsilon(t)}\leq C\biggl(\|(b,\vec u,j_0,\vec j_1)(0)\|_{\widetilde X^1_\varepsilon}+\int_0^t\|\nabla\vec v\|_{L^\infty}\|(b,\vec u,j_0,\vec j_1)\|_{\widetilde X^\nu_\varepsilon}\,d\tau$$
$$+\int_0^t\|(\nabla F,\vec G)\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau+\int_0^t\|(F-T_{\vec v}\cdot\nabla b,\vec G-T_{\vec v}\cdot\nabla\vec u)\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau+\int_0^t\|T_{\vec v}\cdot\nabla b-F\|^{m,1,\widetilde{\mathcal L}\mathcal M}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau\biggr)\cdot$$

Proof: Except in the middle frequencies range, the proof goes along the lines of the corresponding result in the non-equilibrium case. Still assuming that $\nu=1$ and replacing $\mathcal L$ with $\widetilde{\mathcal L}=\nu\mathcal L$ then, working directly on the localized paralinearized system (6.6), and combining Inequalities (6.2) to (6.4) with estimates for the para-convection terms gives
1. Low frequencies: $2^k\leq C_1$.

$$\|\dot\Delta_k(b,\vec u,j_0,\vec j_1)(t)\|_{L^2}+2^{2k}\int_0^t\|\dot\Delta_k(b,\vec u,j_0,\vec j_1)\|_{L^2}\,d\tau+\frac{\widetilde{\mathcal L}\mathcal M}{\varepsilon}\int_0^t\|\dot\Delta_k\vec{\jmath}_1\|_{L^2}\,d\tau$$
$$+\frac{\widetilde{\mathcal L}}{\varepsilon}\int_0^t\|\dot\Delta_k\jmath_0\|_{L^2}\,d\tau\lesssim\|\dot\Delta_k(b,\vec u,j_0,\vec j_1)(0)\|_{L^2}$$
$$+\int_0^t\|\dot\Delta_k(F-T_{\vec v}\cdot\nabla b)\|_{L^2}\,d\tau+\int_0^t\|\dot\Delta_k(\vec G-T_{\vec v}\cdot\nabla\vec u)\|_{L^2}\,d\tau.$$

2. Medium frequencies: $C_1\leq 2^k\leq c\widetilde{\mathcal L}\mathcal M$.

One has to keep in mind that in order to derive (6.3) from (4.58) and (4.64), one has to consider the system that is fulfilled by $(b,\vec u,\zeta_0,\vec j_1)$ with $\zeta_0:=j_0-\sqrt n\,b$. In particular, a part of the the paraconvection term of b enters in the equation for ζ_0 as we have

$$\partial_t\zeta_0+\frac{\widetilde{\mathcal L}}{\varepsilon}\zeta_0+\frac{1}{\varepsilon\sqrt n}\mathrm{div}\,\vec j_1-\sqrt n\mathrm{div}\,\vec u=\sqrt n(T_{\vec v}\cdot\nabla b-F).$$

Therefore, following the computations leading to (4.58) and (4.64), and using Lemma 4.1 in [7] to bound the convection terms coming from the equations for b and $\vec u$, we end up with

$$\|\dot\Delta_k(\nabla b,\vec u,j_0,\vec j_1)(t)\|_{L^2}+2^{2k}\int_0^t\|\dot\Delta_k\vec u\|_{L^2}\,d\tau+2^k\int_0^t\|(\dot\Delta_k b,\dot\Delta_k j_0)\|_{L^2}\,d\tau$$
$$+\mathcal L(1+\mathcal L_s)\int_0^t\|\dot\Delta_k\vec j_1\|_{L^2}\,d\tau\lesssim\|\dot\Delta_k(\nabla b,\vec u,j_0,\vec j_1)(0)\|_{L^2}+\int_0^t\|\dot\Delta_k(\nabla F,\vec G)\|_{L^2}\,d\tau$$
$$+\int_0^t\|\dot\Delta_k(T_{\vec v}\cdot\nabla b-F)\|_{L^2}\,d\tau+\sum_{k'\sim k}\int_0^t\|\nabla\vec v\|_{L^\infty}\|\dot\Delta_{k'}(\nabla b,\vec u)\|_{L^2}\,d\tau.$$

3. High frequencies: $2^k\geq c\widetilde{\mathcal L}\mathcal M$. We get

$$\|\dot\Delta_k(\nabla b,\vec u,j_0,\vec j_1)(t)\|_{L^2}+\int_0^t\bigl(2^{2k}\dot\Delta_k\vec u\|_{L^2}+2^k\|\dot\Delta_k b\|_{L^2}\bigr)\,d\tau+\widetilde{\mathcal L}\mathcal M\varepsilon\int_0^t\|\dot\Delta_k(j_0,\vec j_1)\|_{L^2}\,d\tau$$
$$\lesssim\|\dot\Delta_k(\nabla b,\vec u,j_0,\vec j_1)(0)\|_{L^2}+\int_0^t\|\dot\Delta_k(\nabla F,\vec G)\|_{L^2}\,d\tau$$
$$+\sum_{k'\sim k}\int_0^t\|\nabla\vec v\|_{L^\infty}\|\dot\Delta_{k'}(\nabla b,\vec u)\|_{L^2}\,d\tau.$$

Putting together all those inequalities completes the proof. □

6.3. **A global existence result.** Our global existence result with uniform estimates reads

Theorem 6.1. *There exists a positive constant η depending only on μ/ν, n and on the pressure law such that if $\varepsilon\in(0,1)$ and the data $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)$ satisfy*

$$\|(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)\|_{\widetilde X_\varepsilon^\nu}\leq\eta\nu, \tag{6.7}$$

then System (1.6) *admits a unique global solution $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ in $\widetilde Y_\varepsilon^\nu$. In addition, we have*

$$\|(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)\|_{\widetilde Y_\varepsilon^\nu}\leq C\|(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)\|_{\widetilde X_\varepsilon^\nu}. \tag{6.8}$$

The proof relies on Proposition 6.1. Note in particular that the 'new' last term in the estimate of (6.1) does not entail a loss of derivative as we simply have

$$\|T_{\vec v}\cdot\nabla b-F\|^{m,1,\widetilde{\mathcal{LM}}}_{L^1_t(\dot B^{\frac n2-1}_{2,1})}\leq\|\vec v\cdot\nabla b\|_{L^1_t(\dot B^{\frac n2-1}_{2,1})}\lesssim\|\vec v\|_{L^2_t(\dot B^{\frac n2}_{2,1})}\|b\|_{L^2_t(\dot B^{\frac n2}_{2,1})}.$$

The rest of the proof works exactly the same as in the non-equilibrium case. □

6.4. **Weak convergence.** Here we justify weak convergence to (2.10) when assumption (6.1) is fulfilled.

Theorem 6.2. *Let the family of data $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)_{0<\varepsilon<1}$ satisfy* (6.7)*. Then the global solution $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ in $\widetilde Y_\varepsilon^\nu$ given by Theorem 6.1 satisfies*

$$\vec j_1^\varepsilon\to\vec 0\quad in\quad L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1}),$$

and if $(b_0^\varepsilon,\vec u_0^\varepsilon)\rightharpoonup(b_0,\vec u_0)$ then $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon)$ converges weakly to $(b,\vec u,b)$ where $(b,\vec u)$ stands for the unique solution of

$$\begin{cases}\partial_t b+\vec u\cdot\nabla b+k_1(b)\operatorname{div}\vec u=0,\\ \partial_t\vec u+\vec u\cdot\nabla\vec u-k_2(b)\mathcal A\vec u+\big(k_3(b)+n^{-1}k_4(b)\big)\nabla b=\vec 0.\end{cases} \tag{6.9}$$

supplemented with initial data $(b_0,\vec u_0)$.

Proof: Let $\vec{\mathrm{j}}_1^\varepsilon=\vec j_1^\varepsilon+\frac{\nabla b^\varepsilon}{\mathcal{LM}}\cdot$ From (6.8), we have

$$\frac{\mathcal{LM}}{\varepsilon}\|\vec{\mathrm{j}}_1^\varepsilon\|^{\ell,\nu^{-1}}_{L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})}+\nu\|\nabla b^\varepsilon\|^{\ell,\nu^{-1}}_{L^1(\mathbb R_+;\dot B^{\frac n2}_{2,1})}+\mathcal{LM}\|\vec j_1^\varepsilon\|^{h,\nu^{-1}}_{L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})}\leq C\eta\nu.$$

Hence, given (6.1), we deduce that

$$\vec{\mathrm{j}}_1^\varepsilon=\mathcal O(\varepsilon)\quad\text{in}\quad L^1(\mathbb R_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1}). \tag{6.10}$$

Using the equation of $\vec j_1^\varepsilon$, this gives

$$\nabla j_0^\varepsilon+\mathcal L(1+\mathcal L_s)\vec j_1^\varepsilon\to0\quad\text{in the sense of distributions.} \tag{6.11}$$

As in the non-equilibrium case, (6.8) implies that the families (b^ε) and $(\vec u^\varepsilon)$ are bounded in $L^\infty(\mathbb R_+;\dot B^{\frac n2-1}_{2,1}\cap\dot B^{\frac n2}_{2,1})\cap L^1(\mathbb R_+;\dot B^{\frac n2+1}_{2,1}+\dot B^{\frac n2}_{2,1})$ and $L^\infty(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})\cap L^1(\mathbb R_+;\dot B^{\frac n2+1}_{2,1})$, respectively. Hence $(\partial_t b^\varepsilon)$ is bounded in $L^2(\mathbb R_+;\dot B^{\frac n2-1}_{2,1})$ and we can thus deduce that there exists b in $L^\infty(\mathbb R_+;\dot B^{\frac n2-1}_{2,1}\cap\dot B^{\frac n2}_{2,1})\cap L^1(\mathbb R_+;\dot B^{\frac n2+1}_{2,1}+\dot B^{\frac n2}_{2,1})$ and a sequence $(\varepsilon_k)_{k\in\mathbb N}$ going to 0 so that, for all $\phi\in\mathcal S$ and all $\alpha\in(0,1]$

$$\phi\,b^{\varepsilon_k}\longrightarrow\phi\,b\quad\text{in}\quad L^\infty(\mathbb R_+;\dot B^{\frac n2-\alpha}_{2,1}). \tag{6.12}$$

For $(\vec u^\varepsilon)$, we still have the weak convergence result given by (5.37), which suffices to pass to the limit in the mass equation.

Next, we observe that (6.8) implies that (j_0^ε) is bounded in $L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2+1}_{2,1})$. Hence, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ going to 0 so that $j_0^{\varepsilon_k}\rightharpoonup j_0$ in the sense of distributions. Moreover, we have

$$j_0^\varepsilon-b^\varepsilon=-\mathcal{L}^{-1}n^{-1}\mathrm{div}\,\vec{j}_1^\varepsilon-\varepsilon\mathcal{L}^{-1}\partial_t j_0^\varepsilon.$$

Remembering (6.10) and (6.1), we see that the first term of the r.h.s. is $\mathcal{O}(\varepsilon)$ in a suitable space. The second one also tends to 0 in the sense of distributions as $\varepsilon\mathcal{L}^{-1}\to0$. Hence

$$j_0=b. \tag{6.13}$$

In order to pass to the limit in the velocity equation, we proceed as in the non-equilibrium case. First we use (5.37) and next, the fact that

$$\partial_t\Bigl(\vec{u}^\varepsilon+\frac{\varepsilon}{n}\,k_4(b^\varepsilon)\,\vec{j}_1^\varepsilon\Bigr)=-\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon+k_2(b^\varepsilon)\mathcal{A}\vec{u}^\varepsilon-k_3(b^\varepsilon)\nabla b^\varepsilon+\frac{\varepsilon}{n}k_4'(b^\varepsilon)\partial_t b^\varepsilon\,\vec{j}_1^\varepsilon-\frac1n\,k_4(b^\varepsilon)\,\nabla j_0^\varepsilon.$$

Taking advantage of (6.8) and of product laws in Besov spaces, we readily obtain that the four first terms of the r.h.s. are bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,1})$ (or only in $L^2(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,\infty})$ if $n=2$). To handle the last term, we observe that according to (6.8), ∇j_0^ε is bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,1})$ hence the term $k_4(b^\varepsilon)\nabla j_0^\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,1})$ (or $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-2}_{2,\infty})$ if $n=2$).

As in the non-equilibrium case, it is now easy to conclude that there exists some $\vec{v}$ in $L^\infty(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$ so that for all ϕ in $\mathcal{S}$ and $\alpha\in(0,1)$, we have

$$\phi\Bigl(\vec{u}^{\varepsilon_k}+\frac{\varepsilon^k}{n}\vec{j}_1^{\varepsilon_k}\Bigr)\longrightarrow\phi\vec{v}\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+;\dot B^{\frac n2-1-\alpha}_{2,1}).$$

Of course, combining with (6.10), this implies that $\vec{v}=\vec{u}$, and $(b,\vec{u})$ thus satisfies the second line of (6.9).

Finally, that the whole family $(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon,\vec{j}_1^\varepsilon)$ converges to $(b,\vec{u},b,\vec{0})$ stems from the fact that the solution to (6.9) is unique (note that it is just the standard barotropic Navier-Stokes equations with a *modified* but still stable pressure law). □

7. The Poisson diffusion regime

This section is devoted to the study of the asymptotics regime where

$$\varepsilon\ll\mathcal{L}\lesssim\varepsilon^{1/2}\quad\text{and}\quad\mathcal{L}^2\mathcal{L}_s\approx1. \tag{7.1}$$

According to the formal computations of Section 2, we expect the solutions of (1.6) to tend to those of the Navier-Stokes-Poisson sytem (2.9).

The general scheme of the proof that we here propose is the same as in the study of the other asymptotics: we first perform a fine analysis of the linearized equations so as to check the long-time stability and exhibit the quantities that are likely to be bounded uniformly when $\varepsilon\to0$, then tackle the proof of the global existence. We rapidly justify that the limit system is globally well-posed in a functional framework that is consistent with the analysis we used for (1.6), and eventually take advantage of compactness arguments so as to prove the expected convergence result. As in the other regimes, the fact that the limit system has a unique solution will guarantee that the whole family of solutions to (1.6) converges to the solution to (2.9).

7.1. **Linear analysis of** (1.6) **in the Poisson regime** (7.1)**.** We here gather the estimates for (4.7) that have been obtained in Section 4. Recall that $\widetilde{\mathcal{L}} := \nu\mathcal{L}$.

Small frequencies. Using (4.28), (4.33), (4.34) and the fact that $|(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)| \approx |(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|$ and that the last term in the original definition of $\widehat{\mathfrak{j}}_0$ in (4.22) has a negligible contribution with respect to $\widehat{\mathfrak{j}}_1$, we get

$$
\begin{aligned}
(7.2)\quad |(\widehat{b},\widehat{d},\tfrac{\varepsilon}{\widetilde{\mathcal{L}}}\widehat{j}_0,\rho\widehat{j}_0,\widehat{j}_1)(t)| &+ \rho^2\int_0^t |(\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)|\,d\tau + \int_0^t |\widehat{\mathfrak{j}}_0|\,d\tau + \frac{\widetilde{\mathcal{L}}}{\varepsilon}\int_0^t |\rho\widehat{\zeta}_0| \\
&+ \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t |\widehat{\mathfrak{j}}_1|\,d\tau \leq C|(\widehat{b},\widehat{d},\tfrac{\varepsilon}{\widetilde{\mathcal{L}}}\widehat{j}_0,\rho\widehat{j}_0,\widehat{j}_1)(0)| \quad \text{for all } \ 0\leq\rho\leq C_1,
\end{aligned}
$$

with $\widehat{\mathfrak{j}}_0 := \widehat{j}_0 - \sqrt{n}\,\widehat{b} - \sqrt{n}\frac{\varepsilon}{\widetilde{\mathcal{L}}}\rho\widehat{d}$, $\widehat{\zeta}_0 := \widehat{j}_0 - \frac{\sqrt{n}}{1+\frac{\rho^2}{n\widetilde{\mathcal{L}}^2\mathcal{M}}}\widehat{b}$ and $\widehat{\mathfrak{j}}_1 := \widehat{j}_1 - \frac{\rho}{\sqrt{n}\,\widetilde{\mathcal{L}}\mathcal{M}}\widehat{j}_0 + \frac{\rho\widehat{b}}{\widetilde{\mathcal{L}}\mathcal{L}_s\mathcal{M}}\cdot$

Middle frequencies. Combining (4.54) and the definition of $\widehat{\zeta}_1$ versus that of $\widehat{\mathfrak{j}}_1$, we get

$$
\begin{aligned}
(7.3)\quad |(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)| &+ \rho\int_0^t|\widehat{b}|\,d\tau + \rho^2\int_0^t|\widehat{d}|\,d\tau + \rho^2\frac{\widetilde{\mathcal{L}}}{\varepsilon}\int_0^t|\widehat{j}_0|\,d\tau \\
&+ \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t|\widehat{\mathfrak{j}}_1|\,d\tau \lesssim |(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)| \quad \text{for } \ C_1\leq\rho\leq c\widetilde{\mathcal{L}}\mathcal{M}.
\end{aligned}
$$

Large frequencies. Finally, using (4.70), we have

$$
\begin{aligned}
(7.4)\quad |(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(t)| &+ \rho^2\int_0^t|\widehat{d}|\,d\tau + \rho\int_0^t|\widehat{b}|\,d\tau + \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t|(\widehat{j}_0,\widehat{j}_1)|\,d\tau \\
&\leq C|(\rho\widehat{b},\widehat{d},\widehat{j}_0,\widehat{j}_1)(0)| \quad \text{for } \ \rho\geq c\widetilde{\mathcal{L}}\mathcal{M}.
\end{aligned}
$$

Therefore, localizing (4.4) (with nonzero source terms f and $\vec{g}$) according to Littlewood-Paley operator $\dot{\Delta}_k$, using (4.5), following the computations leading to the above three inequalities and using Fourier-Plancherel theorem, we end up with the following inequality for all $s\in\mathbb{R}$

$$
\begin{aligned}
(7.5)\quad &\|(\vec{u},\vec{j}_1)(t)\|_{\dot{B}^s_{2,1}} + \|b(t)\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|b(t)\|^{h,1}_{\dot{B}^{s+1}_{2,1}} + \frac{\varepsilon}{\widetilde{\mathcal{L}}}\|j_0(t)\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|j_0(t)\|^{\ell,1}_{\dot{B}^{s+1}_{2,1}} + \|j_0(t)\|^{h,1}_{\dot{B}^s_{2,1}} \\
&+ \int_0^t\|\vec{u}\|_{\dot{B}^{s+2}_{2,1}}\,d\tau + \int_0^t\|(b,j_0,\vec{j}_1)\|^{\ell,1}_{\dot{B}^{s+2}_{2,1}}\,d\tau + \frac{\widetilde{\mathcal{L}}}{\varepsilon}\int_0^t\|\zeta_0\|^{\ell,1}_{\dot{B}^{s+1}_{2,1}}\,d\tau + \int_0^t\|\mathfrak{j}_0\|^{\ell,1}_{\dot{B}^s_{2,1}}\,d\tau \\
&+ \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t\|\vec{\mathfrak{j}}_1\|^{\ell,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot{B}^s_{2,1}}\,d\tau + \frac{\widetilde{\mathcal{L}}}{\varepsilon}\int_0^t\|j_0\|^{m,1,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot{B}^{s+2}_{2,1}} + \int_0^t\|b\|^{h,1}_{\dot{B}^{s+1}_{2,1}}\,d\tau + \frac{\widetilde{\mathcal{L}}\mathcal{M}}{\varepsilon}\int_0^t\|(j_0,\vec{j}_1)\|^{h,\widetilde{\mathcal{L}}\mathcal{M}}_{\dot{B}^s_{2,1}}\,d\tau \\
&\lesssim \|(\vec{u},\vec{j}_1)(0)\|_{\dot{B}^s_{2,1}} + \|b(0)\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|b(0)\|^{h,1}_{\dot{B}^{s+1}_{2,1}} + \frac{\varepsilon}{\widetilde{\mathcal{L}}}\|j_0(0)\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|j_0(0)\|^{\ell,1}_{\dot{B}^{s+1}_{2,1}} \\
&\qquad + \|j_0(0)\|^{h,1}_{\dot{B}^s_{2,1}} + \int_0^t\big(\|f\|^{\ell,1}_{\dot{B}^s_{2,1}} + \|f\|^{h,1}_{\dot{B}^{s+1}_{2,1}} + \|\vec{g}\|_{\dot{B}^s_{2,1}}\big)\,d\tau,
\end{aligned}
$$

with $\mathfrak{j}_0 := j_0 - \sqrt{n}\,b - \sqrt{n}\,\frac{\varepsilon}{\widetilde{\mathcal{L}}}\operatorname{div}\vec{u}$,

$$
\zeta_0 := j_0 - \sqrt{n}\Big(\mathrm{Id} - \frac{1}{n\widetilde{\mathcal{L}}^2\mathcal{M}}\Delta\Big)^{-1} b \quad \text{and} \quad \vec{\mathfrak{j}}_1 := \vec{j}_1 + \frac{1}{\sqrt{n}\,\widetilde{\mathcal{L}}\mathcal{M}}\nabla j_0 - \frac{1}{\widetilde{\mathcal{L}}\mathcal{L}_s\mathcal{M}}\nabla b.
$$

As in the previous sections, owing to the convection term in the equation for b, the above inequality does not allow to prove the global existence for (1.6), and one has to consider the paralinearized system (5.10). Adapting the proof of Proposition 5.1, we get the following

Proposition 7.1. *If the coefficients $\mathcal{L}$, $\mathcal{L}_s$ and ε fulfill* (7.1) *then for any smooth solution* $(b,\vec u,j_0,\vec j_1)$ *to* (5.10)*, one has the following inequality*

$$\|(b,\vec u,j_0,\vec j_1)(t)\|_{\check Y^\nu_\varepsilon(t)}\leq C\biggl(\|(b,\vec u,j_0,\vec j_1)(0)\|_{\check X^\nu_\varepsilon}+\int_0^t\|\nabla\vec v\|_{L^\infty}\|(b,\vec u,j_0,\vec j_1)\|_{\check X^\nu_\varepsilon}\,d\tau$$
$$+\int_0^t\|(\nabla F,\vec G)\|^{h,1}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau+\int_0^t\|(F-T_{\vec v}\cdot\nabla b,\vec G-T_{\vec v}\cdot\nabla\vec u)\|^{\ell,1}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau\biggr),$$

with

$$\|(b,\vec u,j_0,\vec j_1)\|_{\check X^\nu_\varepsilon}:=\|(\vec u,\vec j_1)\|_{\dot B^{\frac n2-1}_{2,1}}+\|b\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2-1}_{2,1}}+\nu\|b\|^{h,\nu^{-1}}_{\dot B^{\frac n2}_{2,1}}+\frac{\varepsilon}{\mathcal{L}\nu}\|j_0\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2-1}_{2,1}}+\nu\|j_0\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2}_{2,1}}+\|j_0\|^{h,\nu^{-1}}_{\dot B^{\frac n2-1}_{2,1}},$$

and

$$\|(b,\vec u,j_0,\vec j_1)(t)\|_{\check Y^\nu_\varepsilon(t)}:=\sup_{0\leq\tau\leq t}\|(b,\vec u,j_0,\vec j_1)(\tau)\|_{\check X^\nu_\varepsilon}$$
$$+\nu\int_0^t\Bigl(\|\vec u\|_{\dot B^{\frac n2+1}_{2,1}}+\|(b,\vec j_1)\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2+1}_{2,1}}+\tfrac{\mathcal{L}\nu}{\varepsilon}\|j_0\|^{m,\nu^{-1},\mathcal{L}\mathcal{M}}_{\dot B^{\frac n2+1}_{2,1}}\Bigr)\,d\tau+\frac{\mathcal{L}}{\varepsilon}\int_0^t\|\zeta_0\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2}_{2,1}}\,d\tau$$
$$+\nu^{-1}\int_0^t\|\mathrm{j}_0\|^{\ell,\nu^{-1}}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t\|\vec{\mathrm{j}}_1\|^{\ell,\mathcal{L}\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau+\int_0^t\|b\|^{h,1}_{\dot B^{\frac n2}_{2,1}}\,d\tau+\frac{\mathcal{L}\mathcal{M}}{\varepsilon}\int_0^t\|(j_0,\vec j_1)\|^{h,\mathcal{L}\mathcal{M}}_{\dot B^{\frac n2-1}_{2,1}}\,d\tau.$$

Above, we set

$$\mathrm{j}_0:=j_0-b-\frac{\varepsilon}{\mathcal{L}}\mathrm{div}\,\vec u,\quad \zeta_0:=j_0-\left(\mathrm{Id}-\frac{1}{n\mathcal{L}^2\mathcal{M}}\Delta\right)^{-1}b\quad\textit{and}\quad \vec{\mathrm{j}}_1:=\vec j_1+\frac{1}{\mathcal{L}\mathcal{M}}\nabla j_0-\frac{1}{\mathcal{L}\mathcal{L}_s\mathcal{M}}\nabla b.$$

7.2. **Uniform global well-posedness in the Poisson regime.** In this paragraph, we sketch the proof of the following global existence result.

Theorem 7.1. *There exists a positive constant η depending only on μ/ν, n and on the pressure law such that if $\varepsilon\in(0,1)$ and if the coefficients $\mathcal{L}$ and $\mathcal{L}_s$ fulfill* (7.1) *then any data* $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)$ *satisfying*

$$\|(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)\|_{\check X^\nu_\varepsilon}\leq\eta\nu, \tag{7.6}$$

generates a unique global solution $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ *in* $\check Y^\nu_\varepsilon$ *to System* (1.6)*.*

Furthermore, we have

$$\|(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)\|_{\check Y^\nu_\varepsilon}\leq C\|(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)\|_{\check X^\nu_\varepsilon}. \tag{7.7}$$

Proof. Assuming with no loss of generality that $\nu=1$, the proof relies on Proposition 7.1 with, dropping the indices ε for better readability, $\vec v=\vec u$

$$F:=-T'_{\nabla b}\cdot\vec u-k_1(b)\mathrm{div}\,\vec u\quad\text{and}\quad \vec G:=-T'_{\nabla\vec u}\cdot\vec u+k_2(b)\widetilde{\mathcal{A}}\vec u-k_3(b)\nabla b+\frac{\mathcal{L}\mathcal{M}}{n}k_4(b)\vec j_1.$$

Let us just explain how to handle the last term, as it cannot be bounded exactly as in the proof of Theorems 5.1 or 6.1 due to the difference between the spaces $\check Y^\nu_\varepsilon$ and $\widetilde Y^\nu_\varepsilon$. We use the fact that

$$\mathcal{L}\mathcal{M}\vec j_1=\mathcal{L}\mathcal{M}\vec{\mathrm{j}}_1+\mathcal{L}_s^{-1}\nabla b-\nabla j_0,$$

and thus

$$\mathcal{LM}k_4(b)\vec{j}_1 = \mathcal{LM}k_4(b)(\vec{j}_1^{h,\mathcal{LM}} + \vec{j}_1^{\ell,\mathcal{LM}}) + \mathcal{L}_s^{-1}k_4(b)\nabla b^{\ell,\mathcal{LM}} - k_4(b)\nabla j_0^{\ell,\mathcal{LM}}.$$

It is clear that

$$\|k_4(b)\vec{j}_1^{h,\mathcal{LM}}\|_{L^1(\dot{B}^{\frac{n}{2}-1}_{2,1})} \lesssim \|b\|_{L^\infty(\dot{B}^{\frac{n}{2}}_{2,1})}\|\vec{j}_1^{h,\mathcal{LM}}\|_{L^1(\dot{B}^{\frac{n}{2}-1}_{2,1})} \lesssim (\mathcal{LM})^{-1}\|(b,\vec{u},j_0,\vec{j}_1)\|_{\check{Y}^1_\varepsilon},$$

that the second term in the r.h.s. may be bounded in the same way, and that the third one can be bounded as the pressure term $k_3(b)\nabla b$. For the last term, one just has to observe that the definition of $\|\cdot\|_{\check{Y}^1_\varepsilon}$ guarantees that

$$\|\nabla j_0\|^{\ell,\mathcal{LM}}_{L^1(\dot{B}^{\frac{n}{2}}_{2,1})} \lesssim \|(b,\vec{u},j_0,\vec{j}_1)\|_{\check{Y}^1_\varepsilon}, \tag{7.8}$$

and that

$$\|k_4(b)\|_{L^\infty(\dot{B}^{\frac{n}{2}}_{2,1})} \lesssim \|b\|_{L^\infty(\dot{B}^{\frac{n}{2}}_{2,1})}.$$

The rest of the proof is standard, and thus left to the reader. □

7.3. **Study of the limit system.** We introduce the following norms

$$\|(b,\vec{u},j_0)\|_{\check{\mathcal{X}}^\nu} := \|b\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\vec{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|j_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu^3\|j_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}+2}_{2,1}},$$

and

$$\begin{aligned}\|(b,\vec{u},j_0)\|_{\check{\mathcal{Y}}^\nu} := \sup_{t\geq 0}\|(b,\vec{u},j_0)(t)\|_{\check{\mathcal{X}}^\nu}\\ +\nu\int_{\mathbb{R}_+}\big(\|(b,j_0)\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \|\vec{u}\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\big)\,dt + \int_{\mathbb{R}_+}\big(\|b\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \nu^2\|j_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}+2}_{2,1}}\big)\,dt.\end{aligned}$$

Theorem 7.2. *Let the data $(b_0,\vec{u}_0,j_{0,0})$ satisfy for a small enough constant $c>0$*

$$\|b_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\vec{u}_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq c\nu, \tag{7.9}$$

and the compatibility condition

$$j_{0,0} - \frac{\nu^2}{nm}\Delta j_{0,0} = b_0.$$

Then System (2.9) admits a unique global solution $(b,\vec{u},j_0)$ in the space $\check{\mathcal{Y}}^\nu$, satisfying in addition for a large enough constant C independent of ν

$$\|(b,\vec{u},j_0)\|_{\check{\mathcal{Y}}^\nu} \leq C\big(\|b_0\|^{\ell,\nu^{-1}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \nu\|b_0\|^{h,\nu^{-1}}_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\vec{u}_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big). \tag{7.10}$$

Proof. We just sketch the proof as it is very similar to the standard one for the barotropic Navier-Stokes equations. As usual, it suffices to treat the case $\nu=1$.

The first step is to analyse the linearized system

$$\begin{cases}\partial_t b + \operatorname{div}\vec{u} = f,\\ \partial_t\vec{u} - \widetilde{\mathcal{A}}\vec{u} + \nabla b + \frac{1}{n}\nabla j_0 = g,\\ \big(\mathrm{Id} - \frac{1}{nm}\Delta\big)j_0 = b.\end{cases} \tag{7.11}$$

To this end, we set $d=(-\Delta)^{-1/2}\mathrm{div}\,\vec u$ and observe that in the Fourier space, $(\widehat b,\widehat d)$ fulfills the following ODE if $f=g=0$

$$\begin{cases}\partial_t\widehat b+\rho\widehat d=0,\\ \partial_t\widehat d+\rho^2\widehat d-\rho a_\rho\widehat b=0\end{cases}$$

with $\rho:=|\xi|$ and $a_\rho:=1+\frac{m}{\rho^2+nm}\cdot$ Of course, j_0 may be computed from b by the relation

$$\widehat{j_0}=\frac{nm}{\rho^2+nm}\widehat b.$$

Introducing the following Lyapunov and diffusion functionals

$$\mathcal{L}_\rho^2=2a_\rho|\widehat b|^2+2|\widehat d|^2+|\rho\widehat b|^2-2\mathrm{Re}\,(\rho\widehat b\overline{\widehat d})\quad\text{and}\quad\mathcal{H}_\rho^2=\rho^2(a_\rho|\widehat b|^2+|\widehat d|^2),$$

we see that

$$\frac12\frac{d}{dt}\mathcal{L}_\rho^2+\mathcal{H}_\rho^2=0.$$

Because we have $a_\rho^{\pm1}\leq c_0$ for some c_0 independent of ρ, one can thus conclude exactly as in the standard barotropic case that for all $t\geq0$ and $\rho\geq0$

$$|(\widehat b,\rho\widehat b,\widehat d)(t)|+\min(1,\rho)\int_0^t|\rho\widehat b|\,d\tau+\rho^2\int_0^t|\widehat d|\,d\tau\lesssim|(\widehat b,\rho\widehat b,\widehat d)(0)|.$$

Back to (7.11), one may combine Fourier-Plancherel theorem and Duhamel formula to get the following estimate for all $s\in\mathbb{R}$

$$\begin{aligned}\|(b,\nabla b,\vec u)(t)\|_{\dot B^s_{2,1}}&+\|j_0(t)\|_{\dot B^s_{2,1}\cap\dot B^{s+3}_{2,1}}+\int_0^t\big(\|\vec u\|_{\dot B^{s+2}_{2,1}}+\|(b,j_0)\|^{\ell,1}_{\dot B^{s+2}_{2,1}}\big)\,d\tau\\&+\int_0^t\big(\|b\|^{h,1}_{\dot B^{s+1}_{2,1}}+\|j_0\|^{h,1}_{\dot B^{s+3}_{2,1}}\big)\,d\tau\leq C\bigg(\|(b_0,\nabla b_0,\vec u_0)\|_{\dot B^s_{2,1}}+\int_0^t\|(f,\nabla f,g)\|_{\dot B^s_{2,1}}\,d\tau\bigg).\end{aligned}$$

However, because of the convection term in the equation for b, this does not allow to prove estimates for the nonlinear system (2.9). Therefore, mimicking the standard approach for the compressible Navier-Stokes equation we 'paralinearize' the system and get the following result

Proposition 7.2. *The solutions to the following paralinearized system*

$$\begin{cases}\partial_t b+T_{\vec v}\cdot\nabla\vec u+\mathrm{div}\,\vec u=f,\\ \partial_t\vec u+T_{\vec v}\cdot\nabla\vec u-\widetilde{\mathcal{A}}\vec u+\nabla b+\frac1n\nabla j_0=\vec g,\\ \big(\mathrm{Id}-\frac{1}{nm}\Delta\big)j_0=b.\end{cases}$$

fulfill the following a priori estimate

$$\begin{aligned}\|(b,\nabla b,\vec u)(t)\|_{\dot B^s_{2,1}}&+\|j_0(t)\|_{\dot B^s_{2,1}\cap\dot B^{s+3}_{2,1}}+\int_0^t\big(\|\vec u\|_{\dot B^{s+2}_{2,1}}+\|(b,j_0)\|^{\ell,1}_{\dot B^{s+2}_{2,1}}\big)\,d\tau\\&+\int_0^t\big(\|b\|^{h,1}_{\dot B^{s+1}_{2,1}}+\|j_0\|^{h,1}_{\dot B^{s+3}_{2,1}}\big)\,d\tau\leq C\bigg(\|(b_0,\nabla b_0,\vec u_0)\|_{\dot B^s_{2,1}}+\int_0^t\|(f,\nabla f,g)\|_{\dot B^s_{2,1}}\,d\tau\\&\qquad+\int_0^t\|\nabla\vec v\|_{L^\infty}\|(b,\nabla b,\vec u)\|_{\dot B^s_{2,1}}\,d\tau\bigg).\end{aligned}$$

Now, in order to estimate the solutions of the nonlinear system (2.9), it suffices to apply the above proposition with $\vec v=\vec u$

$$f=-T'_{\vec u}\cdot\nabla b-k_1(b)\mathrm{div}\,\vec u\quad\text{and}\quad \vec g=-T'_{\vec u}\cdot\nabla\vec u+k_2(b)\widetilde{\mathcal{A}}\vec u-k_3(b)\nabla b-n^{-1}k_4(b)\nabla j_0.$$

All the terms but the last one of $\vec g$ are already present in the barotropic Navier-Stokes equations, and may be bounded quadratically in terms of $\|(b,\vec u,j_0)\|_{\check{\mathcal{Y}}^1}$. Now, we have

$$\|k_4(b)\nabla j_0\|_{\dot B^{\frac n2-1}_{2,1}}\lesssim\|b\|_{\dot B^{\frac n2}_{2,1}}\|j_0\|_{\dot B^{\frac n2}_{2,1}},$$

hence

$$\|k_4(b)\nabla j_0\|_{L^1_t(\dot B^{\frac n2-1}_{2,1})}\lesssim\|b\|_{L^2_t(\dot B^{\frac n2}_{2,1})}\big(\|j_0\|^{\ell,1}_{L^2_t(\dot B^{\frac n2}_{2,1})}+\|j_0\|^{h,1}_{L^2_t(\dot B^{\frac n2+2}_{2,1})}\big),$$

and one can thus conclude that whenever the solution $(b,\vec u,j_0)$ exists we have

$$\|(b,\vec u,j_0)\|_{\check{\mathcal{Y}}^1(t)}\leq C\big(\|b_0\|_{\dot B^{\frac n2-1}_{2,1}\cap\dot B^{\frac n2}_{2,1}}+\|\vec u_0\|_{\dot B^{\frac n2-1}_{2,1}}+\|(b,\vec u,j_0)\|^2_{\check{\mathcal{Y}}^1(t)}\big),$$

which allows to get (7.10) if (7.9) is fulfilled with a small enough c. □

7.4. **Weak convergence.** Here we justify weak convergence to (2.9) when assumption (7.1) is fulfilled and, in addition

$$\nu^2\mathcal{L}^2\mathcal{L}_s\to m\in(0,+\infty).\tag{7.12}$$

Theorem 7.3. *Let the family of data $(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon,\vec j_{1,0}^\varepsilon)_{0<\varepsilon<1}$ satisfy (7.6). Then the global solution $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon,\vec j_1^\varepsilon)$ in $\check Y^\nu_\varepsilon$ given by Theorem 7.1 satisfies*

$$\vec j_1^\varepsilon=\mathcal{O}(\mathcal{L})\quad in\quad L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1}),\tag{7.13}$$

and, up to extraction, $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon)$ converges weakly to some solution $(b,\vec u,j_0)$ in $\check{\mathcal{Y}}^\nu$ of System (2.9) when ε goes to 0.

If in addition

$$(b_0^\varepsilon,\vec u_0^\varepsilon,j_{0,0}^\varepsilon)\rightharpoonup(b_0,\vec u_0,j_{0,0})\quad with\quad -\nu^2\Delta j_{0,0}+nm(j_{0,0}-b_0)=0,\tag{7.14}$$

then the whole family $(b^\varepsilon,\vec u^\varepsilon,j_0^\varepsilon)$ converges to the unique solution $(b,\vec u,j_0)$ corresponding to the initial data $(b_0,\vec u_0,j_{0,0})$, given by Theorem 7.2.

Proof: Let us first prove (7.13). From (7.7), we already know that $(\vec{\mathrm{j}}_1^\varepsilon)^{\ell,\mathcal{LM}}$ and $(\vec j_1^\varepsilon)^{h,\mathcal{LM}}$ are $\mathcal{O}(\varepsilon\mathcal{L})$ in $L^1(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$. Now, we have

$$\vec j_1^\varepsilon=\vec{\mathrm{j}}_1^\varepsilon-\frac{1}{\mathcal{L}\mathcal{L}_s}\nabla j_0^\varepsilon+\frac{1}{\mathcal{L}\mathcal{L}_s\mathcal{M}}\nabla b^\varepsilon.$$

It is easy to see that the last term is $\mathcal{O}(\mathcal{L}^3)$ in $L^1(\mathbb{R}_+;\dot B^{\frac n2}_{2,1}+\dot B^{\frac n2-1}_{2,1})$, and that, according to (7.8) and $\mathcal{L}^2\mathcal{L}_s\approx1$, the last but one term is $\mathcal{O}(\mathcal{L})$ in $L^1(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$, which completes the proof of (7.13).

Next, let us turn our attention to the convergence of j_0^ε. First, (7.7) and the definition of $\|\cdot\|_{\check Y^\nu_\varepsilon}$ ensure that $(j_0^\varepsilon)^{h,\mathcal{LM}}$ is bounded in, say, $L^2(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1})$. Next, using the bound for the middle frequencies of j_0 and for the low frequencies of ζ_0, we discover that $(j_0^\varepsilon)^{\ell,\mathcal{LM}}$ is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac n2}_{2,1})$. Hence, up to an omitted extraction

$$j_0^\varepsilon\rightharpoonup j_0\quad\text{weak}\ *\quad\text{in}\quad L^2(\mathbb{R}_+;\dot B^{\frac n2-1}_{2,1}+\dot B^{\frac n2}_{2,1}).\tag{7.15}$$

Now, taking the divergence of the equation of $\vec{j}_1^\varepsilon$, then using the equation of j_0^ε gives

$$\Delta j_0^\varepsilon = -\mathcal{L}(1+\mathcal{L}_s)\big(n\mathcal{L}(b^\varepsilon - j_0^\varepsilon) - \varepsilon n\partial_t j_0^\varepsilon\big) - \varepsilon\partial_t \mathrm{div}\, j_1^\varepsilon. \tag{7.16}$$

Given (7.13), one can assert that the last term tends to 0 in the sense of distributions. We also know that, up to an omitted extraction, $j_0^\varepsilon \to j_0$ in the sense of distributions, hence given that $\mathcal{L}(1+\mathcal{L}_s)\varepsilon \to 0$, the term with $\partial_t j_0^\varepsilon$ also tends to 0. Finally, exactly as in the cases treated before, (b^ε) is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1}\cap\dot{B}^{\frac n2}_{2,1})$ hence weakly converges to some $b\in L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1}\cap\dot{B}^{\frac n2}_{2,1})$. As (7.12) has been assumed, passing to the limit in (7.16) gives

$$\nu^2\Delta j_0 = -nm\big(b-j_0\big).$$

Passing to the limit in the equation of b goes along the lines of the non-equilibrium case we notice that $(\partial_t b^\varepsilon)$ is bounded in $L^2(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$ and we thus have, up to an omitted extraction

$$\phi\, b^\varepsilon \longrightarrow \phi\, b \quad\text{in}\quad L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-\alpha}_{2,1})\quad\text{for all}\ \ \alpha\in(0,1). \tag{7.17}$$

As (7.7) also implies that $(\vec{u}^\varepsilon)$ is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac n2+1}_{2,1})$, we have $\vec{u}^\varepsilon \rightharpoonup \vec{u}$ weakly * in that space, which is enough to justify the first equation of (2.9).

In order to pass to the limit in the velocity equation, we use again the fact that

$$\partial_t\bigg(\vec{u}^\varepsilon + \frac{\varepsilon}{n}\,k_4(b^\varepsilon)\,\vec{j}_1^\varepsilon\bigg) = -\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon + k_2(b^\varepsilon)\mathcal{A}\vec{u}^\varepsilon - k_3(b^\varepsilon)\nabla b^\varepsilon + \frac{\varepsilon}{n}k_4'(b^\varepsilon)\partial_t b^\varepsilon\,\vec{j}_1^\varepsilon - \frac1n\,k_4(b^\varepsilon)\,\nabla j_0^\varepsilon.$$

As in the other asymptotic regimes, the first four terms of the r.h.s. are bounded in $L^2(\mathbb{R}_+;\dot{B}^{\frac n2-2}_{2,1})$ (or in $L^2(\mathbb{R}_+;\dot{B}^{\frac n2-2}_{2,\infty})$ if $n=2$). To handle the last term, we observe that according to (7.7) and (7.15), (∇j_0^ε) is bounded in $L^2(\mathbb{R}_+;\dot{B}^{\frac n2-2}_{2,1}+\dot{B}^{\frac n2-1}_{2,1})$. Because (b^ε) is bounded in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1}\cap\dot{B}^{\frac n2}_{2,1})$, this implies that $k_4(b^\varepsilon)\nabla j_0^\varepsilon$ is bounded in $L^2(\mathbb{R}_+;\dot{B}^{\frac n2-2}_{2,1})$ (or $L^2(\mathbb{R}_+;\dot{B}^{\frac n2-2}_{2,\infty})$ if $n=2$), and thus $\partial_t\big(\vec{u}^\varepsilon+\frac{\varepsilon}{n}\,k_4(b^\varepsilon)\,\vec{j}_1^\varepsilon\big)$ is bounded in the same space.

As in the already studied cases, we conclude that there exists some $\vec{u}$ in $L^\infty(\mathbb{R}_+;\dot{B}^{\frac n2-1}_{2,1})$ so that for all ϕ in $\mathcal{S}$ and $\alpha\in(0,1)$, we have

$$\phi\Big(\vec{u}^\varepsilon+\frac{\varepsilon}{n}\vec{j}_1^\varepsilon\Big)\longrightarrow\phi\vec{u}\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+;\dot{B}^{\frac n2-1-\alpha}_{2,1}).$$

Finally, in the case where (7.14) is fulfilled, the limit system (2.9) supplemented with initial data $(b_0,\vec{u}_0,j_{0,0})$ possesses a unique solution $(b,\vec{u},j_0)$ given by Theorem 7.2, and the whole family $(b^\varepsilon,\vec{u}^\varepsilon,j_0^\varepsilon)$ thus converges to $(b,\vec{u},j_0)$. □

Appendix A. Estimates for a toy linear differential equation

The appendix is devoted to the proof of decay estimates for the solutions to systems of ODEs of the form

$$\partial_t U + A_0 U + \rho\,(A_1+B_1)\,U + \rho^2 A_2 U = 0, \tag{E}$$

where ρ is a nonnegative parameter, and A_0, A_1, B_1 and A_2 are given $N\times N$ matrices. We have in mind System (4.7) in which case, after suitable change of unknowns (see (A.4)), A_0 is a degenerate nonnegative diagonal matrix, A_2 has nonnegative eigenvalues and A_1 is skewsymmetric up to some positive diagonal symmetrizer.

A.1. **A general approach.** The basic idea is to set $V := (I+\rho P)U$ where P is a suitable matrix, so as to eliminate the bad first order term $\rho B_1 U$. Now, whenever $(I+\rho P)$ is invertible, the equation for V reads

$$\partial_t V + A_0 V + \rho\big(A_1 + B_1 + [P, A_0]\big)V + \rho^2\big([A_0, P]P + [P, A_1] + [P, B_1] + A_2\big)V$$
$$+\rho^3 (I+\rho P)\big((A_1+B_1)P^2 - A_0 P^3 - A_2 P\big)(I+\rho P)^{-1} V = 0.$$

Therefore, if one can find some matrix P so that

$$[A_0, P] = B_1, \tag{A.1}$$

then we have

$$\partial_t V + A_0 V + \rho A_1 V + \rho^2\left(A_2 + PB_1 + [P, A_1]\right)V = \rho^3 (I+\rho P)\, A_3\, (I+\rho P)^{-1} V, \tag{A.2}$$

where $A_3 := (PA_0 - A_1)P^2 + A_2 P$.

The gain is clear as the matrix B_1 now appears at order 2 instead of order 1. Hence the system for V is more likely to be tractable for small enough ρ as we shall see below.

A.2. **Application to the linearized system for barotropic radiative flows.** The system we are interested in reads

$$\frac{d}{dt}\begin{pmatrix} \widehat{a} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix} + \begin{pmatrix} 0 & \rho & 0 & 0 \\ -\rho & \rho^2 & 0 & -\varsigma \\ -\eta & 0 & \beta & \alpha\rho \\ 0 & 0 & -\alpha\rho & \gamma \end{pmatrix}\begin{pmatrix} \widehat{a} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.3}$$

where all the coefficients of the matrix are positive.

To bound the solutions of (A.3) for small enough ρ (under some stability condition that we will discover below), we propose two different approaches, the first one being appropriate to handle the case where β and γ are of the same order of magnitude, and the second one, to the case where $\beta/\gamma \ll 1$ or $\gamma/\beta \ll 1$ (of course only $\gamma \geq \beta$ is relevant as far as (4.7) is concerned).

A.2.1. *First approach.* Making the change of unknown

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\varsigma}{\gamma} \\ -\frac{\eta}{\beta} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \widehat{a} \\ \widehat{d} \\ \widehat{j}_0 \\ \widehat{j}_1 \end{pmatrix}, \tag{A.4}$$

and setting $\tilde{\alpha} := \alpha + \frac{\varsigma\eta}{\beta\gamma}$, we see that U satisfies a system of type (E) with

$$A_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1-\frac{\alpha\varsigma\eta}{\beta\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\alpha} \\ 0 & 0 & -\alpha & 0 \end{pmatrix},$$

$$B_1 := -\begin{pmatrix} 0 & 0 & 0 & \frac{\varsigma}{\gamma} \\ 0 & 0 & \frac{\alpha\varsigma}{\gamma} & 0 \\ 0 & \frac{\eta}{\beta} & 0 & 0 \\ \frac{\alpha\eta}{\beta} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{\varsigma}{\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the above matrices may be written in block form as follows

$$B_1=\begin{pmatrix}0 & B_1^1\\ B_1^2 & 0\end{pmatrix},\quad A_0=\begin{pmatrix}0 & 0\\ 0 & \Delta\end{pmatrix},\quad A_1=\begin{pmatrix}A_1^1 & 0\\ 0 & A_1^2\end{pmatrix},\quad P=\begin{pmatrix}P^{11} & P^{12}\\ P^{21} & P^{22}\end{pmatrix}.$$

Computing the commutator

$$[A_0,P]=\begin{pmatrix}0 & -P^{12}\Delta\\ \Delta P^{21} & [\Delta,P^{22}]\end{pmatrix},\tag{A.5}$$

we see (A.1) is satisfied if

$$P^{11}:=0,\quad P^{22}:=0,\quad P^{12}:=-B_1^1\Delta^{-1},\quad P^{21}:=\Delta^{-1}B_1^2.$$

In other words

$$P=\begin{pmatrix}0 & 0 & 0 & \frac{\varsigma}{\gamma^2}\\ 0 & 0 & \frac{\alpha\varsigma}{\beta\gamma} & 0\\ 0 & -\frac{\eta}{\beta^2} & 0 & 0\\ -\frac{\alpha\eta}{\beta\gamma} & 0 & 0 & 0\end{pmatrix},\tag{A.6}$$

which, remembering (A.4), corresponds to the following change of unknowns

$$V=\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\\ \widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}:=\begin{pmatrix}1 & 0 & 0 & \frac{\varsigma}{\gamma^2}\rho\\ -\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho & 1 & \frac{\alpha\varsigma}{\beta\gamma}\rho & \frac{\varsigma}{\gamma}\\ -\frac{\eta}{\beta} & -\frac{\eta}{\beta^2}\rho & 1 & -\frac{\varsigma\eta}{\beta^2\gamma}\rho\\ -\frac{\alpha\eta}{\beta\gamma}\rho & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\widehat{a}\\ \widehat{d}\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix}.\tag{A.7}$$

Note that the determinant of the matrix $(I+\rho P)$ is

$$\Big(1+\frac{\alpha\varsigma\eta}{\beta^3\gamma}\rho^2\Big)\Big(1+\frac{\alpha\varsigma\eta}{\beta\gamma^3}\rho^2\Big),$$

and is thus of order 1 whenever ρ satisfies the smallness condition

$$\rho^2\lesssim\frac{\beta\gamma}{\alpha\varsigma\eta}\,\min(\beta^2,\gamma^2).\tag{A.8}$$

In order to go further in the estimates of V, we compute

$$PB_1=\begin{pmatrix}-B_1^1\Delta^{-1}B_1^2 & 0\\ 0 & \Delta^{-1}B_1^2B_1^1\end{pmatrix}=\begin{pmatrix}-\frac{\alpha\varsigma\eta}{\beta\gamma^2} & 0 & 0 & 0\\ 0 & -\frac{\alpha\varsigma\eta}{\beta^2\gamma} & 0 & 0\\ 0 & 0 & \frac{\alpha\varsigma\eta}{\beta^2\gamma} & 0\\ 0 & 0 & 0 & \frac{\alpha\varsigma\eta}{\beta\gamma^2}\end{pmatrix}\quad\text{and}$$

$$\begin{aligned}[P,A_1]&=\begin{pmatrix}0 & -B_1^1\Delta^{-1}A_1^2+A_1^1B_1^1\Delta^{-1}\\ \Delta^{-1}B_1^2A_1^1-A_1^2\Delta^{-1}B_1^2 & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & 0 & -\frac{\alpha\varsigma}{\gamma}(\frac{1}{\beta}+\frac{1}{\gamma}) & 0\\ 0 & 0 & 0 & \frac{\alpha\tilde\alpha\varsigma}{\beta\gamma}+\frac{\varsigma}{\gamma^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma})\\ \frac{\alpha\tilde\alpha\eta}{\beta\gamma}+\frac{\eta}{\beta^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma}) & 0 & 0 & 0\\ 0 & -\frac{\alpha\eta}{\beta}(\frac{1}{\beta}+\frac{1}{\gamma}) & 0 & 0\end{pmatrix}.\end{aligned}$$

Finally, $A_3 := (PA_0 - A_1)P^2 + A_2P$ reads

$$A_3 = \frac{\alpha\varsigma\eta}{\beta\gamma}\begin{pmatrix} 0 & \frac{1}{\beta^2} & 0 & -\frac{\varsigma}{\gamma^3} \\ \frac{1}{\gamma}-\frac{1}{\gamma^2}\left(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\right) & 0 & \frac{1}{\eta}-\frac{\alpha\varsigma}{\beta^2\gamma} & 0 \\ 0 & 0 & 0 & \frac{\tilde\alpha}{\gamma^2} \\ 0 & 0 & -\frac{\alpha}{\beta^2} & 0 \end{pmatrix}. \tag{A.9}$$

Therefore, resuming to (A.2), we conclude that

$$\frac{d}{dt}V + A_0V + \rho A_1 V + \rho^2\big(PB_1 + A_2\big)V = \rho^2[A_1, P]V + \mathcal{O}(\rho^3).$$

Of course, the remainder term $\mathcal{O}(\rho^3)$ strongly depends on the coefficients of the system. We shall see below that the structure of $[A_1, P]$ will enable us to treat $\rho^2[A_1, P]$ and the nondiagonal term of A_2 as small error terms as well.

Let us focus on the system satisfied by $(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})$ for a while. We have

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix} + \rho\begin{pmatrix} 0 & 1 \\ -1-\frac{\alpha\varsigma\eta}{\beta\gamma} & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix} + \rho^2\begin{pmatrix} -\frac{\alpha\varsigma\eta}{\beta\gamma^2} & 0 \\ 0 & 1-\frac{\alpha\varsigma\eta}{\beta^2\gamma}\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix} \\ = \rho^2\begin{pmatrix} \frac{\alpha\varsigma}{\gamma}\big(\frac{1}{\beta}+\frac{1}{\gamma}\big) & 0 \\ 0 & \frac{\varsigma\nu}{\gamma}-\frac{\alpha\tilde\alpha\varsigma}{\beta\gamma}-\frac{\varsigma}{\gamma^2}\big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big)\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix} + \mathcal{O}(\rho^3).\end{aligned} \tag{A.10}$$

For small enough ρ, optimal estimates may be proved by taking advantage of the results of Appendix B. Indeed, denoting by $\widehat{F}_\rho$ the r.h.s. of (A.10), we see from (B.6) that if we set

$$\mathcal{U}_\rho^2 := \left(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\right)|\widehat{\mathfrak{b}}|^2 + |\widehat{\mathfrak{d}}|^2 - \rho\left(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\Big(\frac{1}{\gamma}-\frac{1}{\beta}\Big)\right)\mathrm{Re}\,(\widehat{\mathfrak{b}}\,\overline{\widehat{\mathfrak{d}}}),$$

then, under the following necessary and sufficient stability condition

$$\tilde\nu := 1 - \frac{\alpha\varsigma\eta}{\beta\gamma}\left(\frac{1}{\beta}+\frac{1}{\gamma}\right) > 0, \tag{A.11}$$

we have (see (B.4) and (B.5))

$$\mathcal{U}_\rho \approx |(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})| \quad\text{and}\quad \frac{d}{dt}\mathcal{U}_\rho^2 + \frac{\tilde\nu}{3}\rho^2\mathcal{U}_\rho^2 \lesssim \mathcal{U}_\rho|\widehat{F}_\rho|, \tag{A.12}$$

whenever

$$\rho \leq \frac{\sqrt{1+\frac{\alpha\varsigma\eta}{\beta\gamma}}}{1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big(\frac{1}{\gamma}-\frac{1}{\beta}\big)}. \tag{A.13}$$

So finally, we get for some appropriate constant $C = C(\alpha, \beta, \gamma, \varsigma, \eta)$

$$|(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})(t)| + \tilde\nu\rho^2\int_0^t |(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})|\,d\tau \leq C\left(|(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})(0)| + \rho^2\int_0^t |(\widehat{\mathfrak{j}}_0, \widehat{\mathfrak{j}}_1)|\,d\tau + \rho^3\int_0^t |(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}}, \widehat{\mathfrak{j}}_0, \widehat{\mathfrak{j}}_1)|\,d\tau\right),$$

which, if $\rho \ll \tilde\nu$, may be simplified into

$$|(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})(t)| + \tilde\nu\rho^2\int_0^t |(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})|\,d\tau \leq C\left(|(\widehat{\mathfrak{b}}, \widehat{\mathfrak{d}})(0)| + \rho^2\int_0^t |(\widehat{\mathfrak{j}}_0, \widehat{\mathfrak{j}}_1)|\,d\tau\right). \tag{A.14}$$

The modified radiative modes $\mathfrak{j}_0$ and $\mathfrak{j}_1$ fulfill

$$\text{(A.15)}\quad \frac{d}{dt}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}+\rho\begin{pmatrix}0&\tilde\alpha\\ -\alpha&0\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}+\begin{pmatrix}\beta+\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho^2&0\\ 0&\gamma+\frac{\alpha\varsigma\eta}{\beta\gamma^2}\rho^2\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{j}}_0\\ \widehat{\mathfrak{j}}_1\end{pmatrix}$$
$$=\rho^2\begin{pmatrix}-\frac{\alpha\tilde\alpha\eta}{\beta\gamma}-\frac{\eta}{\beta^2}(1+\frac{\alpha\varsigma\eta}{\beta\gamma})&0\\ 0&\frac{\alpha\eta}{\beta}(\frac{1}{\beta}+\frac{1}{\gamma})\end{pmatrix}\begin{pmatrix}\widehat{\mathfrak{b}}\\ \widehat{\mathfrak{d}}\end{pmatrix}+\mathcal{O}(\rho^3).$$

Therefore we easily get

$$\frac12\frac{d}{dt}\biggl(|\widehat{\mathfrak{j}}_0|^2+\frac{\tilde\alpha}{\alpha}|\widehat{\mathfrak{j}}_1|^2\biggr)+\biggl(\beta+\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho^2\biggr)|\widehat{\mathfrak{j}}_0|^2+\frac{\tilde\alpha}{\alpha}\biggl(\gamma+\frac{\alpha\varsigma\eta}{\beta\gamma^2}\rho^2\biggr)|\widehat{\mathfrak{j}}_1|^2$$
$$\leq C\bigl(\rho^2|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|+\rho^3|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\bigr).$$

Then, integrating and assuming that $\rho\ll1$ yields

$$\text{(A.16)}\qquad (|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)|+\min(\beta,\gamma)\int_0^t|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau\leq C\biggl(|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|+\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau\biggr)\cdot$$

Combining with (A.14), we can conclude that there exists some positive constants ρ_0 and C depending only on $(\alpha,\beta,\gamma,\varsigma,\eta)$ so that for all

$$\text{(A.17)}\qquad 0\leq\rho\leq\min(1,\tilde\nu)\,\rho_0,$$

we have

$$\text{(A.18)}\quad |(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(t)|+\tilde\nu|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)|+\tilde\nu\rho^2\int_0^t|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau+\tilde\nu\min(\beta,\gamma)\int_0^t|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau$$
$$\leq C\bigl(|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(0)|+\tilde\nu|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|\bigr).$$

A.2.2. *Second approach.* In the case where β and γ are not of the same order of magnitude, Inequality (A.18) is not fully satisfactory, first because we would like to have a control on $\beta\int_0^t|\widehat{\mathfrak{j}}_0|\,d\tau$ and $\gamma\int_0^t|\widehat{\mathfrak{j}}_1|\,d\tau$ rather than just on $\min(\beta,\gamma)\int_0^t|(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau$ and, second, because the range for which (A.18) holds true tends to shrink to 0 if $\beta\ll\gamma$ or $\gamma\ll\beta$.

In this paragraph, we propose another approach to handle (A.3) in the case $\beta\neq\gamma$, still based on rewriting the system in the form (A.2), but with a different definition of A_1 and B_1 (A_0 and A_2 being unchanged). More precisely, we now set

$$A_1:=\begin{pmatrix}0&1&0&0\\ -1-\frac{\alpha\varsigma\eta}{\beta\gamma}&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}\quad\text{and}\quad B_1:=\begin{pmatrix}0&0&0&-\frac{\varsigma}{\gamma}\\ 0&0&-\frac{\alpha\varsigma}{\gamma}&0\\ 0&-\frac{\eta}{\beta}&0&\tilde\alpha\\ -\frac{\alpha\eta}{\beta}&0&-\alpha&0\end{pmatrix}.$$

Then writing the matrices coming into play in block form, we see according to (A.5), that a possible choice for P is

$$P^{11}:=0,\quad P^{12}:=-B_1^1\Delta^{-1},\quad P^{21}:=\Delta^{-1}B_1^2,\quad P^{22}:=\frac{1}{\beta-\gamma}\begin{pmatrix}0&\tilde\alpha\\ \alpha&0\end{pmatrix}\ \text{ with }\ \tilde\alpha:=\alpha+\frac{\varepsilon\eta}{\beta\gamma}\cdot$$

With this new definition of P, we have

$$PB_1=\begin{pmatrix} -\frac{\alpha\varsigma\eta}{\beta\gamma^2} & 0 & -\frac{\alpha\varsigma}{\gamma^2} & 0\\ 0 & -\frac{\alpha\varsigma\eta}{\beta^2\gamma} & 0 & \frac{\alpha\tilde\alpha\varsigma}{\beta\gamma}\\ \frac{\alpha\tilde\alpha\eta}{\beta(\gamma-\beta)}0 & 0 & \frac{\alpha\varsigma\eta}{\beta^2\gamma}+\frac{\alpha\tilde\alpha}{\gamma-\beta} & 0\\ 0 & \frac{\alpha\eta}{\beta(\gamma-\beta)} & 0 & \frac{\alpha\varsigma\eta}{\beta\gamma^2}+\frac{\alpha\tilde\alpha}{\beta-\gamma}\end{pmatrix}$$

$$\text{and}\quad [P,A_1]=\begin{pmatrix} 0 & 0 & -\frac{\alpha\varsigma}{\beta\gamma} & 0\\ 0 & 0 & 0 & \frac{\varsigma}{\gamma^2}\big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big)\\ \frac{\eta}{\beta^2}\big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big) & 0 & 0 & 0\\ 0 & -\frac{\alpha\eta}{\beta\gamma} & 0 & 0\end{pmatrix}.$$

Therefore setting

$$V=\begin{pmatrix}\widehat{\mathfrak b}\\ \widehat{\mathfrak d}\\ \widehat{\mathfrak j}_0\\ \widehat{\mathfrak j}_1\end{pmatrix}:=\begin{pmatrix} 1 & 0 & 0 & \frac{\varsigma}{\gamma^2}\rho\\ -\frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho & 1 & \frac{\alpha\varsigma}{\beta\gamma}\rho & \frac{\varsigma}{\gamma}\\ -\frac{\eta}{\beta} & -\frac{\eta}{\beta^2}\rho & 1 & \big(\frac{\tilde\alpha}{\beta-\gamma}-\frac{\varsigma\eta}{\beta^2\gamma}\big)\rho\\ \frac{\alpha\eta}{\gamma(\gamma-\beta)}\rho & 0 & \frac{\alpha}{\beta-\gamma}\rho & 1\end{pmatrix}\begin{pmatrix}\widehat a\\ \widehat d\\ \widehat{j}_0\\ \widehat{j}_1\end{pmatrix}, \tag{A.19}$$

it is clear that working with $(\widehat a,\widehat d,\widehat{j}_0,\widehat{j}_1)$ or $(\widehat{\mathfrak b},\widehat{\mathfrak d},\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)$ is equivalent whenever

$$\rho\leq C|\gamma-\beta|,$$

for some positive constant C depending continuously on the coefficients of the system.

Putting together the previous computations, we see that V fulfills

$$\frac{d}{dt}V+\begin{pmatrix} -\frac{\alpha\varsigma\eta}{\beta\gamma^2}\rho^2 & 0 & 0 & 0\\ 0 & \big(1-\frac{\alpha\varsigma\eta}{\beta^2\gamma}\big)\rho^2 & 0 & 0\\ 0 & 0 & \beta+(\frac{\alpha\varsigma\eta}{\beta^2\gamma}+\frac{\alpha\tilde\alpha}{\gamma-\beta})\rho^2 & 0\\ 0 & 0 & 0 & \gamma+\big(\frac{\alpha\varsigma\eta}{\beta\gamma^2}+\frac{\alpha\tilde\alpha}{\beta-\gamma}\big)\rho^2\end{pmatrix}V$$
$$+\rho\begin{pmatrix} 0 & 1 & 0 & 0\\ -1-\frac{\alpha\varsigma\eta}{\beta\gamma} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}V$$
$$=\rho^2\begin{pmatrix} 0 & 0 & \frac{\alpha\varsigma}{\gamma}\big(\frac1\beta+\frac1\gamma\big) & 0\\ 0 & 0 & 0 & \frac{\varsigma}{\gamma}-\frac{\alpha\tilde\alpha\varsigma}{\beta\gamma}-\frac{\varsigma}{\gamma^2}\big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big)\\ \frac{\alpha\tilde\alpha\eta}{\beta(\beta-\gamma)}-\frac{\eta}{\beta^2}\big(1+\frac{\alpha\varsigma\eta}{\beta\gamma}\big) & 0 & 0 & 0\\ 0 & \frac{\alpha\eta}{\gamma(\beta-\gamma)} & 0 & 0\end{pmatrix}$$
$$+\rho^3(I+\rho P)A_3(I+\rho P)^{-1}V,$$

with $A_3:=(PA_0-A_1)P^2+A_2P$ satisfying $|A_3|\leq C\big(1+|\gamma-\beta|^{-3}\big)$.

Next, arguing exactly as to handle (A.10), we discover that under the stability condition (A.11) and for ρ satisfying (A.13) (and of course also $\rho\leq c|\beta-\gamma|^3$), we have

$$|(\widehat{\mathfrak b},\widehat{\mathfrak d})(t)|+\tilde\nu\rho^2\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d})|\,d\tau\leq C\bigg(|(\widehat{\mathfrak b},\widehat{\mathfrak d})(0)|$$
$$+\rho^2\int_0^t|(\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau+\rho^3\bigg(1+\frac{1}{|\gamma-\beta|^3}\bigg)\int_0^t|(\widehat{\mathfrak b},\widehat{\mathfrak d},\widehat{\mathfrak j}_0,\widehat{\mathfrak j}_1)|\,d\tau\bigg)\cdot \tag{A.20}$$

Now, in contrast with the first method, we can bound $\widehat{\mathfrak{j}}_0$ and $\widehat{\mathfrak{j}}_1$ independently from one another from the equation satisfied by $\widehat{\mathfrak{j}}_0$, we readily get

$$
\begin{aligned}
(\text{A.21}) \quad |\widehat{\mathfrak{j}}_0(t)| + \Big(\beta + \frac{\alpha\varsigma\eta}{\beta^2\gamma}\rho^2\Big)\int_0^t |\widehat{\mathfrak{j}}_0|\,d\tau \leq |\widehat{\mathfrak{j}}_0(0)| + \frac{C\rho^2}{|\beta-\gamma|}\int_0^t |\widehat{\mathfrak{b}}|\,d\tau \\
+ C\rho^3\Big(1+\frac{1}{|\gamma-\beta|^3}\Big)\int_0^t |(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau,
\end{aligned}
$$

while $\widehat{\mathfrak{j}}_1$ satisfies

$$
\begin{aligned}
(\text{A.22}) \quad |\widehat{\mathfrak{j}}_1(t)| + \Big(\gamma + \Big(\frac{\alpha\varsigma\eta}{\beta\gamma^2} + \frac{\alpha\tilde\alpha}{\beta-\gamma}\Big)\rho^2\Big)\int_0^t |\widehat{\mathfrak{j}}_1|\,d\tau \leq |\widehat{\mathfrak{j}}_1(0)| + \frac{C\rho^2}{|\beta-\gamma|}\int_0^t |\widehat{\mathfrak{d}}|\,d\tau \\
+ C\rho^3\Big(1+\frac{1}{|\gamma-\beta|^3}\Big)\int_0^t |(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}},\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)|\,d\tau.
\end{aligned}
$$

Putting inequalities (A.20), (A.21) and (A.22) together, it is now easy to conclude that

$$
\begin{aligned}
(\text{A.23}) \quad |(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(t)| + \tilde\nu|\gamma-\beta||(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(t)| + \tilde\nu\rho^2\int_0^t |(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})|\,d\tau \\
+ \tilde\nu|\gamma-\beta|\Big(\beta\int_0^t |\widehat{\mathfrak{j}}_0|\,d\tau + \gamma\int_0^t |\widehat{\mathfrak{j}}_1|\,d\tau\Big) \leq C\big(|(\widehat{\mathfrak{b}},\widehat{\mathfrak{d}})(0)| + \tilde\nu|\gamma-\beta||(\widehat{\mathfrak{j}}_0,\widehat{\mathfrak{j}}_1)(0)|\big),
\end{aligned}
$$

if, for some small enough constant c depending continuously on α, β, γ and ς, we have

$$
(\text{A.24}) \qquad \rho \leq c\min\Big(1,\, \tilde\nu,\, |\gamma-\beta|^2,\, \tilde\nu|\gamma-\beta|^3\Big)\cdot
$$

Appendix B. Optimal decay estimates for a toy system

For the reader convenience, we here recall some results that have been obtained in our recent work [8] for the following linear system of ordinary differential equations

$$
(\text{B.1}) \qquad \left\{\begin{array}{l} \partial_t X + a\rho Y - b\rho^2 X = A,\\ \partial_t Y - c\rho X + d\rho^2 Y = B.\end{array}\right.
$$

Above, ρ stands for a given nonnegative small parameter and a, b, c and d are four real numbers satisfying the stability condition

$$
(\text{B.2}) \qquad a>0, \quad c>0 \quad \text{and} \quad d-b>0.
$$

Routine computations show that the following *Lyapunov functional* $\mathcal{L}_\rho^2 := c|X|^2 + a|Y|^2 - \rho(d+b)\mathrm{Re}\,(X\bar Y)$ satisfies the relation

$$
\begin{aligned}
(\text{B.3}) \quad \frac12\frac{d}{dt}\mathcal{L}_\rho^2 + \Big(\frac{d-b}{2}\Big)\rho^2\big(c|X|^2+a|Y|^2\big) + \Big(\frac{b^2-d^2}{2}\Big)\rho^3\mathrm{Re}\,(X\bar Y) \\
= \mathrm{Re}\,\big(cA\bar X + aB\bar Y - \rho(b+d)(B\bar X + A\bar Y)\big).
\end{aligned}
$$

Now, observe that whenever $\rho \leq \frac{\sqrt{ac}}{|b+d|}$, we have

$$
\Big|\Big(\frac{b^2-d^2}{2}\Big)\rho^3\mathrm{Re}\,(X\bar Y)\Big| \leq \Big(\frac{d-b}{4}\Big)\rho^2\big(c|X|^2+a|Y|^2\big),
$$

and

$$
(\text{B.4}) \qquad \frac12\big(c|X|^2+a|Y|^2\big) \leq \mathcal{L}_\rho^2 \leq \frac32\big(c|X|^2+a|Y|^2\big),
$$

which leads if $A \equiv B \equiv 0$ to

$$\frac{d}{dt}\mathcal{L}_\rho^2 + \left(\frac{d-b}{3}\right)\mathcal{L}_\rho^2 \leq 0, \tag{B.5}$$

and thus

$$\mathcal{L}_\rho(t) \leq e^{-(\frac{d-b}{6})\rho^2 t}\mathcal{L}_\rho(0). \tag{B.6}$$

Combining with (B.4) and Duhamel's formula, we deduce that for general source terms A and B we have

$$\sqrt{c|X(t)|^2 + a|Y(t)|^2} \leq \sqrt{3}\, e^{-(\frac{d-b}{6})\rho^2 t}\Bigg(\sqrt{c|X(0)|^2 + a|Y(0)|^2} + \int_0^t e^{(\frac{d-b}{6})\tau}\sqrt{c|A|^2 + a|B|^2}\, d\tau\Bigg)\cdot \tag{B.7}$$

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