Vibration of structures containing compressible liquids with surface tension and sloshing effects. Reduced-order model

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Abstract This paper deals with the development of the linear vibration of a general viscoelastic structure, with a local wall acoustic impedance, containing an inviscid compressible liquid (but with an additional volume dissipative term), with surface tension (capillarity) and sloshing effects, and neglecting the effects of internal gravity waves and the elastogravity operator. The sloshing problems of incompressible liquids with capillarity effects in elastic structures exhibit a major difficulty induced by the boundary contact conditions on the triple line because the capillarity forces are forces per unit length while the elastic forces are forces per unit surface. The proposed framework has the following novel features: (i) introducing a new appropriate boundary condition for the contact angle condition compatible with a deformable structure considered here as viscoelastic, (ii) considering a compressible liquid while incompressibility hypothesis is generally used for FSI problems including capillarity phenomena, and (iii) constructing a reduced-order model for the computational coupled problem.

Keywords Linear vibration · Viscoelastic structure · Surface tension · Sloshing · Contact angle condition · Reduced-order-model

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1 Introduction

In this paper we are interested in developing a computational model of sloshing for compressible liquids in taking into account surface tension (capillarity) effects. Concerning the fundamental of the physics of capillarity phenomena, we refer the reader, for example, to [13,14], and to [33,44] for the classical theory of capillarity. Some developments of the behavior of liquids in microgravity environment are presented in [1,16,36,38,43].

General analyzes of sloshing problems for incompressible liquids in rigid structures can be found in [20,31].

For sloshing problems of incompressible liquids in viscoelastic structures with simple geometry and analyzed using semi-analytical approaches, we refer the reader, for instance, to [2] for the case without capillarity effects, and to [5] for the case with capillarity effects. Concerning the computational methods based on variational formulations, finite element discretization, and reduced-order models devoted to the sloshing problems of incompressible liquids without capillarity effects and in elastic structures, we refer the reader to [7,21–23,37,39,47,48].

For computational methods concerning the sloshing problems of incompressible liquids with capillarity effects in rigid structures, see [15,18,19,37,57]. In this framework, the difficult problem concerning the conditions of contact angle, has been discussed in literature [8,9,11,17,29,32,37,45,46, 49–51,56,58].

The sloshing problems of incompressible liquids with capillarity effects in elastic structures exhibit a major difficulty induced by the boundary contact conditions on the triple line because the capillarity forces are forces per unit length while the elastic forces are forces per unit surface. An attempt to solve this problem has been presented in [35] using an energy approach, and without deriving and discussing the local equations.

The sloshing problems of compressible liquids with capillarity effects in rigid structures has been analyzed by Finn [24–26], in which the classical capillarity equation is revisited, and in which a new equation is established in a more complete framework for which the compressibility must be taken into account for establishing a more coherent theory. Nevertheless, it should be noted that in such analysis, the internal gravity waves are not taken into account. For the interaction between internal gravity waves and compressibility, without capillarity effects, we refer the reader to [3, 27, 34].

The linear vibration of structures containing compressible liquids without surface tension and without sloshing effects have extensively been analyzed in the framework of computational formulations and reduced-order models (see, for instance, [28, 37, 40–42, 59]). For nonlinear sloshing problems of incompressible liquids in rigid structures, we refer the reader to [10, 12, 30]. General computational methods for nonlinear fluid-structure interaction problems can be found in [6, 4, 52–55].

This paper is devoted to linear sloshing problems of compressible liquids with capillarity effects in general viscoelastic structures, and neglecting the effects of internal gravity waves. As example of application, this type of phenomena is of prime importance for the dynamic stability analysis of geostationary satellites containing tanks partially filled with liquid propellants in microgravity environment. The boundary value problem is presented in introducing a new term in the local equations related to the condition of contact angle in the presence of an elastic wall. The variational formulation allowing for the justification of the presence of such an additional term is introduced. A reduced-order model that guaranties a fast convergence adapted to compressible liquids is developed.

Consequently, the novel features of the paper are the following: for the linear vibration of a deformable structure containing a liquid with capillarity phenomena (surface tension) and sloshing effects, (i) introducing a new appropriate boundary condition for the contact angle condition compatible with a deformable structure considered here as viscoelastic, (ii) considering a compressible liquid while incompressibility hypothesis is generally used when sloshing, capillarity phenomena and deformation of the structure are simultaneously taken into account, and (iii) constructing a reduced-order model for the computational coupled problem .

2 Fluid-structure system, notations and hypotheses

We consider the fluid-structure system whose configuration is defined in Figure 1, constituted of a linear damped structure Ω_S containing an inviscid compressible liquid, weakly dissipative, which occupies a domain Ω_L. The boundary of Ω_L is written as $\partial\Omega_L = \Gamma_L \cup \Gamma_Z \cup \Gamma$. The part Γ_Z of the internal fluid-structure interface is assumed to be dissipative and is modeled by a wall acoustic local impedance, while Γ_L is defined as the remaining part. It is assumed that $\Gamma \cap \Gamma_Z = \emptyset$ (this means that the wall acoustic impedance does not intersect the free surface Γ). The boundary Γ is the free surface for which the geometry corresponds to a stable static equilibrium of the compressible liquid submitted to external static forces (such as microgravity or gravity forces) and surface tension (capillarity). The boundary of Ω_S is written as $\partial\Omega_S = \Gamma_E \cup \Gamma_L \cup \Gamma_Z \cup \Gamma_G$. The boundary of bounded surface Γ is the curve denoted by γ, which is also the boundary of $\Gamma_{LZ} = \Gamma_L \cup \Gamma_Z$ (that is to say $\gamma = \partial\Gamma_{LZ}$ and also, $\gamma = \partial\Gamma_G$). The bounded domain whose boundary is $\Gamma \cup \Gamma_G$ is empty or is filled by a gas whose effects are neglected for sake of brevity (this is the domain above Ω_L in Figure 1). The external unit normal to $\partial\Omega_S$ is denoted as $\mathbf{n}^S$ while the one to $\partial\Omega_L$ is denoted as $\mathbf{n}$. The external unit normal to γ belonging to the tangent plane to surface Γ is denoted as $\boldsymbol{\nu}$.

We are interested in studying the linear vibration around the stable static equilibrium considered as the natural configuration. This means that the prestresses are not taken into account. In addition, the elastogravity operator introduced in [37, 47] is not considered here. These two effects can easily be added in the boundary value problem introduced hereinafter as an additional stiffness operator. The liquid is assumed to be homogeneous, and it is assumed that there is no interaction between compressibility and internal gravity waves [34]. The fluid-structure system is submitted to given forces (assumed to be in equilibrium), which are applied to the structure that is assumed to be in a free-free condition. The physical space is referred to a cartesian reference system and a generic point denoted by $\mathbf{x} = (x_1, x_2, x_3)$. The classical convention is used for summations over repeated Latin indices, but not over Greek indices. Let be $\mathbf{a}\cdot\mathbf{b} = a_j b_j$ in which $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. For any function $f(\mathbf{x})$, $f_{,j}(\mathbf{x})$ denotes the partial derivative $\partial f(\mathbf{x})/\partial x_j$ with respect to x_j. The gradient operator $\boldsymbol{\nabla} f$ is the vector $(f_{,1}, f_{,2}, f_{,3})$ and the 3D-Laplacian operator is such that $\boldsymbol{\nabla}^2 f = f_{,jj}$.

The linear vibration of the fluid-structure system is formulated in the frequency domain for which the angular frequency (in rad/s) is denoted by ω, and the pure imaginary complex number by i. Let $\mathbf{u}(\mathbf{x},\omega) = (u_1(\mathbf{x},\omega),\, u_2(\mathbf{x},\omega),\, u_3(\mathbf{x},\omega))$ be the displacement field of the structure, $\varepsilon_{ij}(\omega)$ be the strain tensor, and $\sigma_{ij}(\omega)$ be the stress tensor, at a given point $\mathbf{x}$ (in the sequel, $\mathbf{x}$ is removed from $\varepsilon_{ij}(\omega)$ and $\sigma_{ij}(\omega)$). Concerning the internal compressible liquid (acoustic fluid) Ω_L, let $p(\mathbf{x},\omega)$ be the pressure field, and let $\eta(\mathbf{x},\omega)$ be the normal displacement of the free surface Γ along $\mathbf{n}$ at point $\mathbf{x}$.

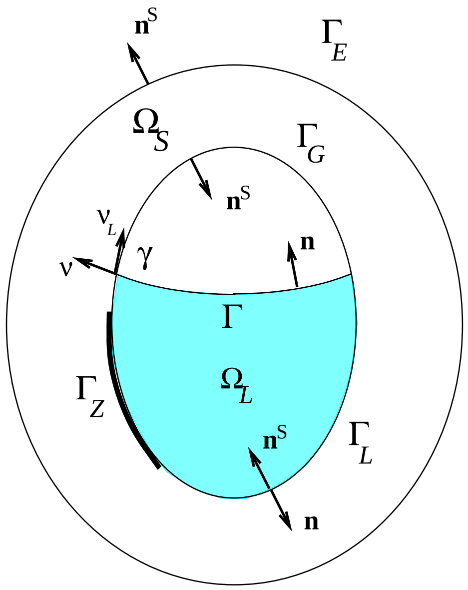

Fig. 1 Static equilibrium configuration of a fluid-structure system for a compressible liquid with surface tension and sloshing effects

3 Boundary value problem in $(p, \eta, \mathbf{u})$

The fluid-structure system is submitted to a surface force field $\mathbf{G}(\omega)$ applied on the structure, to a body force field $\mathbf{b}(\omega)$ applied in the structure. We are interested in studying the linear vibrations of the fluid-structure system around a stable static equilibrium, which is considered as a natural state at rest (the external structural forces are assumed to be in equilibrium). The boundary value problem is expressed in terms of structural displacement field $\mathbf{u}$, internal pressure field p and the normal displacement η of the free surface. For all real ω and for given $\mathbf{G}(\omega)$ and $\mathbf{b}(\omega)$ the problem consists in finding $\mathbf{u}(\omega)$, $p(\omega)$ and $\eta(\omega)$, such that

$$-\frac{\omega^2}{\rho_0 c_0^2}\, p - \mathrm{i}\,\omega \frac{\tau}{\rho_0} \boldsymbol{\nabla}^2 p - \frac{1}{\rho_0}\boldsymbol{\nabla}^2 p = 0 \quad \text{in} \quad \Omega_L \,, \tag{1}$$

$$(1 + \mathrm{i}\,\omega\,\tau)\,\frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0\, \mathbf{u}\cdot\mathbf{n} \quad \text{on} \quad \Gamma_L \,, \tag{2}$$

$$(1 + \mathrm{i}\,\omega\,\tau)\,\frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0\, \mathbf{u}\cdot\mathbf{n} - \mathrm{i}\,\omega\rho_0 \frac{p}{Z} \quad \text{on} \quad \Gamma_Z \,, \tag{3}$$

$$(1 + \mathrm{i}\,\omega\,\tau)\,\frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0\, \eta \quad \text{on} \quad \Gamma \,, \tag{4}$$

$$p = \rho_0\, \eta\, \mathbf{g}\cdot\mathbf{n} - \sigma_\Gamma \{(\frac{1}{R_1^2} + \frac{1}{R_2^2})\eta + \boldsymbol{\nabla}^2_\Gamma \eta\} \quad \text{on} \quad \Gamma \,, \tag{5}$$

$$\frac{\partial \eta}{\partial \boldsymbol{\nu}} = c_\eta\, \eta + \mathcal{J}\mathbf{u} \quad \text{on} \quad \gamma \,, \tag{6}$$

$$-\omega^2 \rho_S\, \mathbf{u} - \mathrm{div}\, \sigma(\mathbf{u}) = \mathbf{b} \quad \text{in} \quad \Omega_S \,, \tag{7}$$

$$\sigma(\mathbf{u})\, \mathbf{n}^S = \mathbf{G} \quad \text{on} \quad \Gamma_E \,, \tag{8}$$

$$\sigma(\mathbf{u})\, \mathbf{n}^S\, d\Gamma_{LZ} = p\, \mathbf{n}\, d\Gamma_{LZ} + \sigma_\Gamma\, (\mathcal{J}'\eta)\, d\mu_\gamma \quad \text{on} \quad \Gamma_{LZ} \,. \tag{9}$$

– Eq. (1) is the internal inviscid compressible liquid equation, with an additional small damping term, which corresponds to the classical Helmholtz equation with a dissipative term, in which ρ_0 is the constant mass density of the homogeneous liquid at equilibrium, c_0 the corresponding constant speed of sound, and the constant coefficient τ characterizes the dissipation in the internal liquid (as a function of the dynamic, kinematic, and second viscosities).
– Eq. (2) is the fluid-structure coupling condition of the internal inviscid compressible liquid (weakly dissipative) with the structure interface Γ_L.
– Eq. (3) is the fluid-structure coupling condition on the internal fluid-structure interface Γ_Z in which $Z(\mathbf{x}, \omega)$ is a local wall acoustic impedance that must satisfy appropriate conditions in order to ensure that the problem is correctly stated.
– Eq. (4) represents the kinematic equation for the free surface Γ.
– Eq. (5) corresponds to the free-surface constitutive equation of surface Γ, in which σ_Γ is the surface tension coefficient, $\mathbf{g}$ is the gravitational acceleration vector, R_1 and R_2 are the principal curvature radii, and where $\boldsymbol{\nabla}^2_\Gamma \eta$ denotes the surface Laplacian related to surface Γ (see [24,33,36–38,44]). If the surface tension effects are neglected, then Eq. (5) yields the classical sloshing free-surface boundary condition.
– Eq. (6), the first term in the right-hand side corresponds to the classical contact angle condition on γ in which c_η is the contact angle coefficient (see [9,11,25,33,37,45,50]), which is only valid for fixed rigid structure. In this new boundary condition proposed, there is an additional term that allows the structure deformation to be taken into account. In this term, $\mathcal{J}$ is a differential operator on manifold Γ_{LZ}, (i) defined on a set of sufficiently differentiable $\mathbb{R}^3$-valued functions that are the traces on Γ_{LZ} of functions on Ω_S, (ii) with values in a set of $\mathbb{R}$-valued functions that are defined on γ. A particular case for differential operator $\mathcal{J}$ is the one given in [37] (Section 4.3, page 80):

$$\mathcal{J}\mathbf{u} = E\, \mathbf{u}\cdot\mathbf{n}^S - \frac{\partial(\mathbf{u}\cdot\mathbf{n}^S)}{\partial \boldsymbol{\nu}_L} \,, \tag{10}$$

in which E is a real coefficient and where $\boldsymbol{\nu}_L$ is external unit normal to γ belonging to the tangent plane to surface Γ_{LZ}. If $\mathbf{u} = 0$ (for a fixed rigid structure), then the classical boundary condition on γ is obtained. If $\mathbf{u}$ is spanned by a rigid body motion of the structure (6 degrees of freedom), then the boundary condition is written as $\mathcal{J}\mathbf{u} = E\, \mathbf{u}\cdot\mathbf{n}^S$.
– Eq. (7) is the dynamic equation in the frequency domain for the structure, in which ρ_S is the structural mass den-

sity, and where $\{\mathrm{div}\,\sigma\}_i = \sigma_{ij,j}$. The tensor $\sigma_{ij}(\mathbf{u})$ corresponds to the constitutive equation for viscoelastic materials, defined by

$$\sigma_{ij}(\mathbf{u}) = a_{ijkh}(\omega)\;\varepsilon_{kh}(\mathbf{u}) + \mathrm{i}\,\omega\, b_{ijkh}(\omega)\;\varepsilon_{kh}(\mathbf{u})\,, \quad (11)$$

in which tensors $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ are fourth-order real tensors depending on $\mathbf{x}$ and ω, which verify even property with respect to ω, symmetry and positiveness properties, and an algebraically relationship due to the causality property and deduced from the use of the Hilbert transform (also called the Kramers and Kroning relationship), and where $\varepsilon_{kh}(\mathbf{u})$ is the strain tensor defined by

$$\varepsilon_{kh}(\mathbf{u}) = \frac{1}{2}(u_{k,h} + u_{h,k})\,. \quad (12)$$

- Eq. (8) corresponds to the boundary condition on Γ_E on which a given surface force field is applied.
- Eq. (9) is a new boundary condition on Γ_{LZ}, which takes into account the condition of contact angle in a presence of an elastic wall. The first term of the right-hand sides corresponds to the usual fluid-structure coupling condition of the internal inviscid compressible liquid (weakly dissipative) with the structure. Let us explain the second term in the right-hand side of this equation in which $d\Gamma_{LZ}$ is the surface measure on Γ_{LZ} such that $\int_{\Gamma_{LZ}} d\Gamma_{LZ} = |\Gamma_{LZ}|$ (area of surface Γ_{LZ}). Let $\mathbf{x} \mapsto f(\mathbf{x})$ be any real function defined on Γ_{LZ} such that its trace on γ is integrable on γ with respect to the curvilinear measure $d\gamma$ on γ such that $\int_\gamma d\gamma = |\gamma|$ (length of curve γ). Then, $d\mu_\gamma$ is a real measure on Γ_{LZ} such that $\int_{\Gamma_{LZ}} f(\mathbf{x})\, d\mu_\gamma(\mathbf{x}) = \int_\gamma f(\mathbf{x})\, d\gamma(\mathbf{x})$ (this means that the support of measure $d\mu_\gamma$ is γ). We have now to explain the meaning of the term $(\mathcal{J}'\eta)\, d\mu_\gamma$ which is defined on Γ_{LZ} by algebraic duality of the term $\mathcal{J}\mathbf{u}$ defined on γ and introduced in Eq. (6). We introduce the two following duality brackets,

$$< \mathcal{J}\mathbf{u}\,, \eta >_{d\gamma} = \int_\gamma (\mathcal{J}\mathbf{u})\, \eta\, d\gamma\,, \quad (13)$$

$$\ll \mathbf{u}\,, \mathcal{J}'\eta \gg_{d\mu_\gamma} = \int_{\Gamma_{LZ}} (\mathcal{J}'\eta)\cdot \mathbf{u}\, d\mu_\gamma\,. \quad (14)$$

The term $(\mathcal{J}'\eta)\, d\mu_\gamma$ is then defined by

$$\ll \mathbf{u}\,, \mathcal{J}'\eta \gg_{d\mu_\gamma} = < \mathcal{J}\mathbf{u}\,, \eta >_{d\gamma}\,. \quad (15)$$

Some details concerning the construction of the dissipative term introduced in Eq. (1), the local wall acoustic impedance introduced in Eq. (3), and the constitutive equation for the viscoelastic material introduced by Eq. (12), can be found in [42].

4 Computational fluid-structure model

Let $P(\omega)$, $H(\omega)$, and $U(\omega)$ be the complex vectors corresponding to the spatial discretization of fields $p(\mathbf{x},\omega)$, $\eta(\mathbf{x},\omega)$, and $\mathbf{u}(\mathbf{x},\omega)$. The discretization of the variational formulation of the boundary value problem in $(p,\eta,\mathbf{u})$ yields:

. for by Eqs. (1) to (4),

$$[A(\omega)]P(\omega) + \omega^2\,[C_{p\eta}]^T H(\omega) + \omega^2\,[C_{p\mathbf{u}}]^T U(\omega) = 0\,, \quad (16)$$

in which the complex symmetric matrix $[A(\omega)] = -\omega^2\,[M] + \mathrm{i}\,\omega[D] + [K] + \mathrm{i}\omega\,[A^Z(\omega)]$.

. for by Eqs. (5) and (6),

$$[C_{p\eta}]P(\omega) + ([K_g] + [K_c])H(\omega) + [C_{\eta\mathbf{u}}]U(\omega) = 0\,. \quad (17)$$

. for by Eqs. (7) to (9),

$$[C_{p\mathbf{u}}]P(\omega) + [C_{\eta\mathbf{u}}]^T H(\omega) + [A^S(\omega)]U(\omega) = F^S(\omega)\,, \quad (18)$$

in which $[A^S(\omega)] = -\omega^2\,[M^S] + \mathrm{i}\,\omega[D^S(\omega)] + [K^S(\omega)]$ is a complex symmetric matrix.

The matrices introduced in Eqs. (16) to (18) are defined in Section 5.

It can be proved that, for all $\omega > 0$, the problem defined by Eqs. (16) to (18) has a unique solution $(P(\omega), H(\omega), U(\omega))$.

5 Matrices of the discretized problem

In this section, we give the expressions of the real or the complex bilinear forms whose discretization allows the corresponding real or complex matrices to be constructed. For such a construction, we consider the fields $(p,\eta,\mathbf{u})$ and $(\delta p, \delta\eta, \delta\mathbf{u})$ as real (and not complex as it is the case throughout the paper).

Matrices related to the equations for the compressible liquid (weakly dissipative) in p.

. Symmetric real matrix $[K]$ is positive semidefinite with a kernel of dimension 1, and corresponds to $\frac{1}{\rho_0}\int_{\Omega_L} \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla}\delta p\, d\mathbf{x}$.

. Symmetric real matrix $[D] = \tau\,[K]$ is positive semidefinite with a kernel of dimension 1.

. Symmetric real matrix $[M]$ is positive definite, and corresponds to $\frac{1}{\rho_0 c_0^2}\int_{\Omega_L} p\,\delta p\, d\mathbf{x}$.

. Symmetric complex matrix $[A^Z(\omega)]$ comes from $\int_{\Gamma_Z}\frac{1}{Z(\omega)} p\,\delta p\, d\mathbf{x}$ in which $Z(\mathbf{x},\omega)$ is a complex-valued function.

Matrices related to the equations for the liquid free surface in η.

. Symmetric real matrix $[K_g]$ is positive definite, and corresponds to $\rho_0 \int_\Gamma \mathbf{g}\cdot\mathbf{n}\,\eta\,\delta\eta\,d\Gamma$.

. Symmetric real matrix $[K_c]$ is positive definite, and corresponds to $\sigma_\Gamma\int_\Gamma \boldsymbol{\nabla}_\Gamma\eta\,\cdot\,\boldsymbol{\nabla}_\Gamma\delta\eta\,d\Gamma - \sigma_\Gamma\int_\Gamma(\frac{1}{R_1^2}+\frac{1}{R_2^2})\eta\,\delta\eta\,d\Gamma - \sigma_\Gamma\int_\gamma c_\eta\eta\,\delta\eta\,d\gamma$.

Matrices related to the equations for the viscoelastic structure in $\boldsymbol{u}$.

. Symmetric real matrix $[M^S]$ is positive definite, and corresponds to $\int_{\Omega_S}\rho_S\,\mathbf{u}\cdot\delta\mathbf{u}\,d\mathbf{x}$.

. Symmetric real matrix $[D^S(\omega)]$ is positive semidefinite with a kernel of dimension 6, and corresponds to $\int_{\Omega_S} b_{ijkh}(\omega)\,\varepsilon_{kh}(\mathbf{u})\,\varepsilon_{ij}(\delta\mathbf{u})\,d\mathbf{x}$.

. Symmetric real matrix $[K^S(\omega)]$ is positive semidefinite with a kernel of dimension 6, and corresponds to $\int_{\Omega_S} a_{ijkh}(\omega)\,\varepsilon_{kh}(\mathbf{u})\,\varepsilon_{ij}(\delta\mathbf{u})\,d\mathbf{x}$.

Matrices related to the coupling terms.

. Rectangular real matrix $[C_{p\mathbf{u}}]$ corresponds to $-\int_{\Gamma_{LZ}} p\,\mathbf{n}\cdot\delta\mathbf{u}\,d\Gamma_{LZ}$.

. Rectangular real matrix $[C_{p\eta}]$ corresponds to $-\int_\Gamma p\,\delta\eta\,d\Gamma$.

. Rectangular real matrix $[C_{\eta\mathbf{u}}]$ corresponds to $-\sigma_\Gamma\int_\gamma(\mathcal{J}\mathbf{u})\,\delta\eta\,d\gamma$.

Vector of external forces.

. Complex vector $F^S(\omega)$ of external forces correspond to $\int_{\Gamma_E}\mathbf{G}\cdot\delta\mathbf{u}\,d\Gamma_E + \int_{\Omega_S}\mathbf{b}\cdot\delta\mathbf{u}\,d\mathbf{x}$.

6 Reduced-order computational model

6.1 Decomposition of the admissible space of the discretized coupled problem

The first step of the construction consists in establishing a decomposition of the admissible space $C_{P,H,U}$ of the discretized problem defined by Eqs. (16) to (18). This decomposition is illustrated in Figure 2. It can be shown that this admissible space can be decomposed in the following direct sum,

$$C_{P,H,U} = C_P \oplus C_H \oplus C_U\,, \tag{19}$$

in which

. C_P is the admissible space of the discretized problem in P (see Eq. (16)), related to the inviscid compressible liquid (without the additional dissipative term and without wall acoustic impedance) occupying domain Ω_L, with the boundary condition $\partial p/\partial\mathbf{n} = 0$ on $\partial\Omega_L\backslash\Gamma$ (fixed wall, *i.e.* $\mathbf{u} = 0$) and $p = 0$ on Γ. We then obtained the discretized equation,

$$[K]\,P - \omega^2\,[M]\,P = 0\,, \tag{20}$$

$$\text{with } P = 0 \text{ for the DOF related to } \Gamma\,. \tag{21}$$

. C_H is the admissible space of the discretized problem in H, related to the inviscid incompressible liquid, with sloshing and capillarity effects without the additional dissipative term and without wall acoustic impedance (*i.e.*, removing the terms $-\omega^2\,[M]$, $i\omega\,[D]$ and $i\omega\,[A^Z(\omega)]$ in the expression of $[A(\omega)]$ introduced in Eq. (16)), and with the boundary condition $\partial p/\partial\mathbf{n} = 0$ on $\partial\Omega_L\backslash\Gamma$ (fixed wall, *i.e.* $\mathbf{u} = 0$). We then obtained the following two discretized equations in (P,H) deduced from Eqs. (16) and (17), for which P must be eliminated to obtain the equation in H,

$$[K]\,P + \omega^2\,[C_{p\eta}]^T H = 0\,, \tag{22}$$

$$[C_{p\eta}]P + ([K_g] + [K_c])H = 0\,. \tag{23}$$

Since the kernel of $[K]$ is equal to 1, the elimination of P yields,

$$[K_{gc}]\,H - \omega^2\,[M_\Gamma]\,H = 0\,. \tag{24}$$

$$[L]\,H = 0\,, \tag{25}$$

in which $[L]$ is a real row matrix, and where $[K_{gc}]$ and $[M_\Gamma]$, under the constraints $[L]\,H = 0$, are positive-definite symmetric matrices that are constructed as a function of matrices $[K]$, $[C_{p\eta}]$, $[K_g]$ and $[K_c]$. For practical construction of these matrices, we refer the reader to Section 4.6 of Chapter 4 in Ref. [37].

. C_U is the admissible space of the discretized problem in U (see Eq. (18)), related to the viscoelastic structure Ω_S (at zero frequency, without dissipative term, and without given forces), coupled with the inviscid incompressible liquid (without the additional dissipative term and without wall acoustic impedance) occupying domain Ω_L, with the boundary condition $p = 0$ on Γ. We then obtained the following discretized equations in (P,U) deduced from Eqs. (16) and (18), for which P must be eliminated to obtain the equation in U,

$$[K]\,P + \omega^2\,[C_{p\mathbf{u}}]^T\,U = 0\,, \tag{26}$$

$$\text{with } P = 0 \text{ for the DOF related to } \Gamma\,, \tag{27}$$

$$[C_{p\mathbf{u}}]\,P + [K^S(0)]\,U - \omega^2\,[M^S]\,U = 0\,. \tag{28}$$

The elimination of P yields

$$[K^S(0)]\,U - \omega^2\,([M^S] + [M_A])\,U = 0\,, \tag{29}$$

in which $[M_A]$ is a positive symmetric matrix (called the added mass matrix) whose construction is given in Section 8.2.2 of Chapter 8 in Ref. [42].

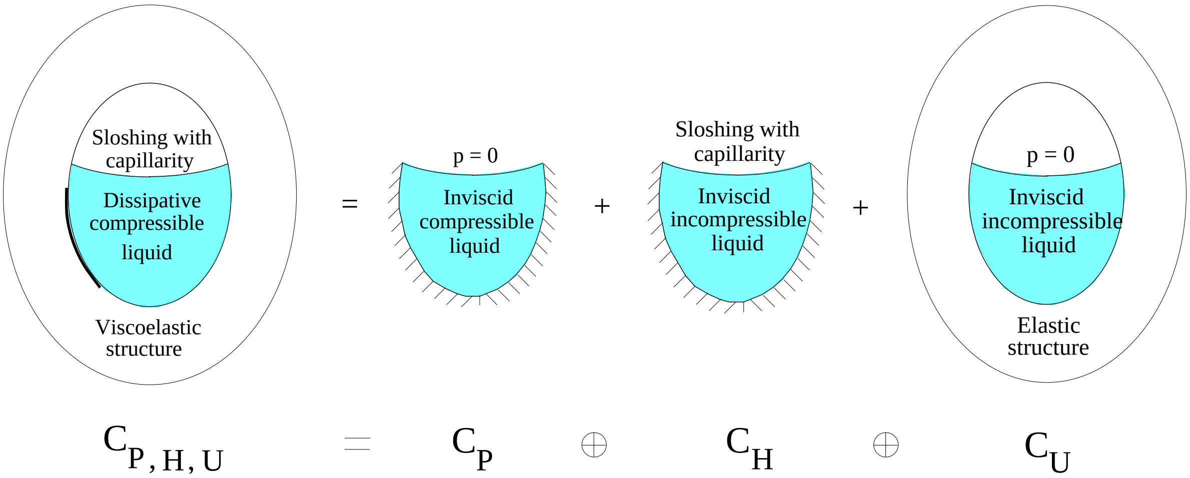

Fig. 2 Decomposition of the admissible space in direct sum for constructing the bases of the reduced-order model

6.2 Construction of the vector bases for the projection of the discretized coupled problem

(i) Vector basis of C_P for the internal inviscid compressible liquid with a zero-pressure free surface condition.

This vector basis is constituted of the *acoustic modes* of liquid occupying domain Ω_L, and are constructed in solving the generalized eigenvalue problem with constraints, deduced from Eqs. (20) and (21),

$$[K]\,P = \lambda_L\,[M]\,P\,, \tag{30}$$

$$\text{with } P = 0 \text{ for the DOF related to } \Gamma\,. \tag{31}$$

Let $[\mathcal{P}] = [P_1 \dots P_{N_L}]$ be the rectangular real matrix whose N_L columns are constituted of the eigenvectors associated with the N_L first smallest strictly positive eigenvalues.

(ii) Vector basis of C_H for the internal inviscid incompressible liquid with sloshing and capillarity effects.

This vector basis is constituted of the *sloshing modes* of liquid occupying domain Ω_L, and are constructed in solving the generalized eigenvalue problem with constraints, deduced from Eqs. (24) and (25),

$$[K_{gc}]\,H = \lambda_\Gamma\,[M_\Gamma]\,H\,. \tag{32}$$

$$[L]\,H = 0\,, \tag{33}$$

Let $[\mathcal{H}] = [H_1 \dots H_{N_\Gamma}]$ be the rectangular real matrix whose N_Γ columns are constituted of the eigenvectors associated with the N_Γ first smallest strictly positive eigenvalues.

(iii) Vector basis of C_U for the elastic structure with added mass induced by the internal inviscid incompressible liquid with a zero-pressure free surface condition.

This vector basis is constituted of the *elastic modes with added mass effects* of structure occupying domain Ω_S, and are constructed in solving the generalized eigenvalue problem, deduced from Eq. (29),

$$[K^S(0)]\,U = \lambda_S\,([M^S] + [M_A])\,U = 0\,. \tag{34}$$

Let $[\mathcal{U}] = [U_1 \dots U_{N_S}]$ be the rectangular real matrix whose N_S columns are constituted of the eigenvectors associated with the N_S first smallest positive eigenvalues including the zero eigenvalue with multiplicity 6 corresponding to the 6 rigid body modes.

Remark on the decomposition. Let us recall that in the dynamic substructuring technique for two coupled substructures through a coupling interface (such as the Craig and Bampton method), the elastic modes used for each substructure are the modes with fixed coupling interface (zero displacement). These two bases are completed by the boundary functions (lifting operator), and are introduced in the variational formulation of the coupled problem in order to derive the reduced-order model. In the presented formulation, the role of the boundary functions is played by the vector basis of space C_H for which a non zero pressure is derived on surface Γ (as it can be seen in Eq. (5)), and used in the variational formulation of the coupled problem.

6.3 Reduced-order computational model

The reduced-order model of order (N_L, N_Γ, N_S) is obtained by projecting Eqs. (16) to (18) as follows

$$P(\omega) = [\mathcal{P}]\,\mathbf{q}_P(\omega)\,, \tag{35}$$

$$H(\omega) = [\mathcal{H}]\,\mathbf{q}_H(\omega)\,, \tag{36}$$

$$U(\omega) = [\mathcal{U}]\,\mathbf{q}_U(\omega)\,, \tag{37}$$

We obtain the complex matrix equation for the reduced-order computational model,

$$[A_{\mathrm{FSI}}(\omega)] \begin{bmatrix} \mathbf{q}_P(\omega) \\ \mathbf{q}_H(\omega) \\ \mathbf{q}_U(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{f}^S(\omega) \end{bmatrix} . \tag{38}$$

For all positive frequency ω, Eq. (38) has a unique solution that is obtained in solving the complex reduced-matrix equation.

7 Conclusion

In this paper, we have constructed a reduced-order computational model in the frequency domain for the linear vibration of a general viscoelastic structure, with a local wall acoustic impedance, containing an inviscid compressible liquid (with an additional volume dissipative term), with surface tension (capillarity) and sloshing effects. The effects of internal gravity waves and the elastogravity operator, which have been neglected, can easily be taken into account. A new modeling has been proposed to take into account the boundary contact conditions on the triple line arising in the sloshing problems for a compressible liquid with capillarity effects in presence of a deformable structure. In addition, a reduced-order model has been proposed for the computational coupled system.

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