Optimality Issues for a Class of Controlled Singularly Perturbed Stochastic Systems

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Abstract

The present paper aims at studying stochastic singularly perturbed control systems. We begin by recalling the linear (primal and dual) formulations for classical control problems. In this framework, we give necessary and sufficient support criteria for optimality of the measures intervening in these formulations. Motivated by these remarks, in a first step, we provide linearized formulations associated to the value function in the averaged dynamics setting. Second, these formulations are used to infer criteria allowing to identify the optimal trajectory of the averaged stochastic system.

Key words: Optimal stochastic control, singularly perturbed Brownian diffusions, occupation measures, linear programming.

AMS Classification: 93E20, 49J45, 49L25.

1 Preliminaries

1.1 Introduction

The present paper aims at studying stochastic singularly perturbed control systems. We begin by recalling the linear (primal and dual) formulations for classical control problems. In this framework, we give necessary and sufficient support criteria for optimality of the measures intervening in these formulations. Motivated by these remarks, in a first step, we provide linearized formulations associated to the value

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function in the averaged dynamics setting. Second, these formulations are used to infer criteria allowing to identify the optimal trajectory of the averaged stochastic system.

Linear programming techniques have proved to be very useful in dealing with deterministic and stochastic control problems. A wide literature is available on the subject both in the deterministic and in the stochastic setting ([1, 2, 3, 4, 5, 6, 7, 8]).

One of the advantages of transforming a nonlinear control problem into a linear optimization problem consists in the possibility of obtaining approximation results for the value function. Following the methods presented in [8] and [9] for the deterministic controlled dynamics, one can approximate the occupational measures by Dirac measures and construct an optimal feedback control. Moreover, when considering the ergodic control problem, e.g. [10], the study of the behavior of the value function is simplified whenever this value is expressed by a linear problem. Recently, linearized versions of the standard continuous infinite horizon discounted control problems have been provided in [9, 11].

When dealing with controlled perturbed dynamics, if the associated system is fully nonlinear, then it is very difficult to characterize the optimal trajectories using the classical methods. Indeed, these criteria involve Pontryagin's maximum principle which is difficult to study if one does not fully understand the averaged dynamics. We recall [12, 13, 14] and references therein dealing with this kind of problems.

We propose an alternative to these classical methods. Our approach consists in embedding the controlled trajectories into a space of probability measures satisfying a convenient constraint. This condition is given in terms of the coefficient functions (and involves the infinitesimal generator of the underlying process). The results allow to characterize the set of constraints as the closed convex hull of occupational measures associated to controls. We first consider general control problems with Lipschitz continuous running and final costs allowing to explain the approach. Using linearization techniques and the dual formulations, we characterize the optimal occupational measures by describing their support set. Next, we extend the linear formulations to singularly perturbed Brownian systems. Finally, we propose support criteria for the optimality of measures in this setting. To our best knowledge, this work is the first to propose a linearization approach to the existence of the optimal policy in the singularly perturbed setting. We emphasize that it does not require to effectively compute the averaged dynamics.

This paper is organized as follows. We briefly state our problem in Subsection 1.2. In Section 2, we present the main ingredients allowing to deal with classical control problems. We begin with recalling the linear formulations in this setting taken form [15]; see also [16]. In Subsection 2.2, we provide a support condition for the optimality of measures appearing in the primal linear formulation. We distinguish between the regular and the general case. The final section aims at presenting singularly perturbed control systems and the averaging method and some important results concerning the singularly perturbed systems and the value functions associated to this problem. We begin by recalling the basic assumptions and ingredients in Subsection 3.1. These results are mainly taken from [7]; see also [17]. Combined with the results in the classical framework, these ingredients allow one to infer linear formulations for the control problems with stochastic singularly perturbed systems in Subsection 3.2. Finally, in Subsection 3.3, we give some criteria for optimality in the singularly perturbed setting.

1.2 Singularly Perturbed Control Systems

In the following we shortly present our problem. We consider the following dynamics:

$$\begin{cases} dX_s^{x,y,u;\varepsilon} = f\left(X_s^{x,y,u;\varepsilon}, Y_s^{x,y,u;\varepsilon}, u_s\right) ds + \sigma\left(X_s^{x,y,u;\varepsilon}, Y_s^{x,y,u;\varepsilon}, u_s\right) dW_s, \\ dY_s^{x,y,u;\varepsilon} = \frac{1}{\varepsilon} g\left(X_s^{x,y,u;\varepsilon}, Y_s^{x,y,u;\varepsilon}, u_s\right) ds + \frac{1}{\sqrt{\varepsilon}} \beta\left(X_s^{x,y,u;\varepsilon}, Y_s^{x,y,u;\varepsilon}, u_s\right) dB_s, \\ X_0^{x,y,u;\varepsilon} = x, Y_0^{x,y,u;\varepsilon} = y, \end{cases} \tag{1}$$

for all $s \geq 0$, $(x, y) \in \mathbb{R}^M \times \mathbb{R}^N$ for some positive integers $M, N > 0$. Here, $\varepsilon > 0$ is a small real parameter. The regularity assumptions on the coefficient functions and the exact definition of our solutions will be made precise in the next paragraph. The evolutions of the two state variables X and Y of the system are of different scale. We call x the "slow" variable and y the "fast" variable.

The control space U is assumed to be a compact metric space. The functions $f : \mathbb{R}^M \times \mathbb{R}^N \times U \to \mathbb{R}^M$, $\sigma : \mathbb{R}^M \times \mathbb{R}^N \times U \to \mathbb{R}^{M \times d}$ and $g : \mathbb{R}^M \times \mathbb{R}^N \times U \to \mathbb{R}^N$, $\beta : \mathbb{R}^M \times \mathbb{R}^N \times U \to \mathbb{R}^{N \times d'}$ are assumed to be uniformly continuous on their domains and Lipschitz-continuous in (x, y), uniformly with respect to the control parameter $u \in U$. We consider the family of weak control processes :

$$\pi = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, (W, B), u\right)$$

is called a weakly-admissible control and for every $(x, y) \in \mathbb{R}^{M+N}$, $\left(X^{x,y,u^\pi;\varepsilon}, Y^{x,y,u^\pi;\varepsilon}, u^\pi\right)$ is called a weakly-admissible pair iff

(i) The quadruple $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ is a filtered probability space satisfying the usual assumptions;

(ii) The process W is a d-dimensional standard Brownian motion defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$; the process B is a d'-dimensional standard Brownian motion defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ and independent of W;

(iii) The process u is an $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking its values in U;

(iv) The process $\left(X^{x,y,u^\pi;\varepsilon}, Y^{x,y,u^\pi;\varepsilon}, u^\pi\right)$ is the unique solution of (1) on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ satisfying $X_0^{x,y,u^\pi;\varepsilon} = x$ and $Y_0^{x,y,u^\pi;\varepsilon} = y$.

The set of weakly-admissible controls is denoted by $\mathcal{U}^w$. We denote by $\left(X_{(\cdot)}^{x,y,u;\varepsilon}, Y_{(\cdot)}^{x,y,u;\varepsilon}\right)$ the solution of (1) starting from $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$ for some $\pi \in \mathcal{U}^w$. We wish to point out that taking weak control processes and their admissible pair amounts to considering weak solutions of our control system. To avoid confusion, the elements of some fixed $\pi \in \mathcal{U}^w$ will be denoted by $\left(\Omega^\pi, \mathcal{F}^\pi, (\mathcal{F}_t^\pi)_{t\geq 0}, \mathbb{P}^\pi, (W^\pi, B^\pi), u^\pi\right)$.

We let $h : \mathbb{R}^M \to \mathbb{R}$ be a given bounded function and $T > 0$ a finite time horizon and define the following payoff

$$C_{x,y;\varepsilon}(\pi) = \mathbb{E}^\pi \left[h\left(X_T^{x,y,u^\pi;\varepsilon}\right)\right], \tag{2}$$

for all $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$ and all $\pi \in \mathcal{U}^w$. The value function associated with (1) and (2) is

$$W_{\varepsilon,h}(x,y) = \inf_{\pi \in \mathcal{U}^w} C_{x,y;\varepsilon}(\pi), \tag{3}$$

for all $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$.

The asymptotic behavior of the value function (3) when $\varepsilon \to 0$ is a very interesting problem. Whenever the control system (1) has some stability property, it is possible to prove that the trajectories $\left(X_{(\cdot)}^{x,y,u^\pi;\varepsilon}, Y_{(\cdot)}^{x,y,u^\pi;\varepsilon}\right)$ of (1) converge towards some solution of some system obtained by formally replacing ε by 0 in (1). This is the so called Tikhonov approach which has been successfully developed in [18, 19], for instance.

When (1) is not stable, another approach consists in investigating relationships between the system (1) and a new differential equation

$$\begin{cases} dX_s^{x,y,u} = \bar{f}\left(X_s^{x,y,u}, \mu_s\right) ds + \bar{\sigma}\left(X_s^{x,y,u}, \mu_s\right) dW_s, \\ \qquad \mu_s \in D_{X_s^{x,y,u}} \text{ for (almost) all } s \in [0,T]. \end{cases} \tag{4}$$

obtained by an averaging method, that will be described later on. We emphasize that, in general, the averaged system is set-valued. We refer the reader to [14, 20] for averaging methods. It is important to notice that only the behavior of the "slow" variable $X_{(\cdot)}^{x,y,u^\pi;\varepsilon}$ is concerned by this approach.

2 Classical Control Problems

In this section, we present an occupation measure approach to the optimality problem in the framework of classical control problems. The basic idea is to embed the family of controlled trajectories in a larger family of probability measures. This later set has the advantage of being explicitly given by a linear constraint and is compact and convex. Using Lagrange duality techniques, we express the value function as a sup inf problem. The set of points realizing the infimum in this formulation gives a good candidate for the support of optimal measures. We distinguish between the regular case where the supremum is attained and the general case where (slightly) less general criteria can be obtained.

We let $\pi = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, u\right)$ be a weak control consisting of a complete probability space $(\Omega, \mathcal{F}, , \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions, a standard p-dimensional Brownian motion with respect to this filtration denoted W. We recall that an admissible control process u is any $\mathbb{F}-$progressively measurable process with values in the compact metric space U. We denote by $T > 0$ a finite time horizon and let $\mathcal{U}^w$ denote the class of all admissible (weak) controls on $[0, T]$. We consider the stochastic control system

$$\begin{cases} dX_s^{t,x,u} = b\left(X_s^{t,x,u}, u_s\right) ds + \rho\left(X_s^{t,x,u}, u_s\right) dW_s, \text{ for all } s \in [t, T], \\ X_t^{t,x,u} = x \in \mathbb{R}^m, \end{cases} \tag{5}$$

where $t \in [0, T]$. Throughout the section, we use the following standard assumption on the coefficient functions $b : \mathbb{R}^m \times U \longrightarrow \mathbb{R}^m$ and $\rho : \mathbb{R}^m \times U \longrightarrow \mathbb{R}^{m\times p}$:

$$\begin{cases} \text{(i) the functions } b \text{ and } \rho \text{ are bounded and uniformly continuous on } \mathbb{R}^m \times U, \\ \text{(ii) there exists a real constant } c > 0 \text{ such that} \\ |b(x,u) - b(y,u)| + |\rho(x,u) - \rho(y,u)| \leq c\,|x - y|, \end{cases} \tag{6}$$

for all $(x, y, u) \in \mathbb{R}^{2m} \times U$. Under the Assumption (6), for every $(t, x) \in [0, T] \times \mathbb{R}^m$ and every admissible control $\pi \in \mathcal{U}^w$, there exists a unique solution to (5) starting from (t, x) denoted by $X_\cdot^{t,x,u^\pi}$.

2.1 Lipschitz Continuous Cost Functionals

In this subsection, we recall the basic tools that allow to identify the primal and dual linear formulations associated to (finite horizon) stochastic control problems. The results can be found in [15] (see also [11] for the infinite time horizon).

To any $(t, x) \in [0, T[\times \mathbb{R}^m$ and any $\pi \in \mathcal{U}^w$, we associate the (expectation of the) occupation measures

$$\gamma^1_{t,T,x,\pi}(A \times B \times C) = \frac{1}{T-t}\mathbb{E}^\pi\left[\int_t^T 1_{A\times B\times C}\left(s, X_s^{t,x,u^\pi}, u_s^\pi\right) ds\right],\ \gamma^2_{t,T,x,\pi}(D) = \mathbb{E}^\pi\left[1_D\left(X_T^{t,x,u^\pi}\right)\right],$$

for all Borel subsets $A \times B \times C \times D \subset [t, T] \times \mathbb{R}^m \times U \times \mathbb{R}^m$. Also, we can define

$$\gamma^1_{T,T,x,\pi}(\cdot \times C) = \delta_{(T,x)}(\cdot) \times \mathbb{P}^\pi\left(u_T^\pi \in C\right),\ \gamma^2_{T,T,x,\pi} = \delta_x,$$

where δ denotes the Dirac measure. We denote by

$$\Gamma_{(b,\rho)}(t,T,x)=\left\{\gamma_{t,T,x,\pi}=\left(\gamma^1_{t,T,x,\pi},\gamma^2_{t,T,x,\pi}\right)\in\mathcal{P}\left([t,T]\times\mathbb{R}^m\times U\right)\times\mathcal{P}\left(\mathbb{R}^m\right):\pi\in\mathcal{U}^w\right\}.$$

Here, $\mathcal{P}(\mathcal{X})$ stands for the set of probability measures on the metric space $\mathcal{X}$. Due to the Assumption (6), there exists a positive constant C_0 (depending on $T>0$) such that, for every $(t,x)\in[0,T]\times\mathbb{R}^m$ and every $\pi\in\mathcal{U}^w$, one has

$$\sup_{s\in[t,T]}\mathbb{E}^\pi\left[\left|X_s^{t,x,u^\pi}\right|^4\right]\le C_0\left(|x|^4+1\right). \tag{7}$$

Therefore,

$$\begin{cases}\int_{\mathbb{R}^m}|y|^4\,\gamma^1_{t,T,x,\pi}\left([t,T],dy,U\right)\le C_0\left(|x|^4+1\right),\\ \int_{\mathbb{R}^m}|y|^4\,\gamma^2_{t,T,x,\pi}\left(dy\right)\le C_0\left(|x|^4+1\right).\end{cases} \tag{8}$$

We have chosen to give these estimates for the fourth-order moment in order to fit the framework of [7] (see Subsection 3.1 and Assumption (6)). We define

$$\Theta_{(b,\rho)}(t,T,x)=\left\{\begin{array}{c}\gamma\in\mathcal{P}\left([t,T]\times\mathbb{R}^m\times U\right)\times\mathcal{P}\left(\mathbb{R}^m\right):\forall\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^m\right),\\ \int_{[t,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m}\left[\begin{array}{c}(T-t)\,\mathcal{L}^v_{(b,\rho)}\phi(s,y)\\ +\phi(t,x)-\phi(T,z)\end{array}\right]\gamma^1(ds,dy,dv)\,\gamma^2(dz)=0.\end{array}\right\}, \tag{9}$$

where

$$\mathcal{L}^v_{(b,\rho)}\phi(s,y)=\frac{1}{2}Tr\left[(\rho\rho^*)(y,v)\,D^2\phi(s,y)\right]+\langle b(y,v),D\phi(s,y)\rangle+\partial_t\phi(s,y),$$

for all $(s,y)\in[0,T]\times\mathbb{R}^m$, $v\in U$ and all $\phi\in C^{1,2}\left([0,T]\times\mathbb{R}^m\right)$. The equality constraint appearing in the definition of $\Theta_{(b,\rho)}(t,T,x)$ is nothing else than Itô's formula applied to $\phi\left(s,X_s^{t,x,u^\pi}\right)$ on $[t,T]$ for regular test functions $\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^m\right)$. To see this, we can, alternatively, write it as

$$\phi(t,x)+(T-t)\int_{[t,T]\times\mathbb{R}^m\times U}\mathcal{L}^v_{(b,\rho)}\phi(s,y)\,\gamma^1(ds,dy,dv)=\int_{\mathbb{R}^m}\phi(T,z)\,\gamma^2(dz).$$

As a consequence,

$$\Gamma_{(b,\rho)}(t,T,x)\subset\Theta_{(b,\rho)}(t,T,x).$$

Moreover, the set $\Theta_{(b,\rho)}(t,T,x)$ is convex and a closed subset of $\mathcal{P}\left([t,T]\times\mathbb{R}^m\times U\right)\times\mathcal{P}\left(\mathbb{R}^m\right)$. For further details, the reader is referred to [15].

Let us suppose that $l_1:\mathbb{R}\times\mathbb{R}^m\times U\longrightarrow\mathbb{R}$, $l_2:\mathbb{R}^m\longrightarrow\mathbb{R}$ are bounded and uniformly continuous such that

$$|l_1(t,x,u) - l_1(s,y,u)| + |l_2(x) - l_2(y)| \leq c(|x-y| + |t-s|), \tag{10}$$

for all $(s,t,x,y,u) \in \mathbb{R}^2 \times \mathbb{R}^{2m} \times U$, and for some positive $c > 0$. We introduce the usual value function

$$\begin{aligned} V_{l_1,l_2,(b,\rho)}(t,x) &= \inf_{\pi \in \mathcal{U}^w} \mathbb{E}^{\pi}\left[\int_t^T l_1\left(s, X_s^{t,x,u^{\pi}}, u_s^{\pi}\right) ds + l_2\left(X_T^{tx,u^{\pi}}\right)\right] \qquad (11) \\ &= \inf_{\gamma \in \Gamma_{(b,\rho)}(t,T,x)} \left((T-t) \int_{[t,T]\times\mathbb{R}^m\times U} l_1(s,y,u)\,\gamma^1(ds,dy,du) + \int_{\mathbb{R}^m} l_2(y)\,\gamma^2(dy)\right), \end{aligned}$$

and the primal linearized value function

$$\Lambda_{l_1,l_2,(b,\rho)}(t,x) = \inf_{\gamma \in \Theta_{(b,\rho)}(t,T,x)} \left((T-t) \int_{[t,T]\times\mathbb{R}^m\times U} l_1(s,y,u)\,\gamma^1(ds,dy,du) + \int_{\mathbb{R}^m} l_2(y)\,\gamma^2(dy)\right), \tag{12}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^m$. We also consider the dual value function

$$\mu_{l_1,l_2,(b,\rho)}(t,x) = \sup\left\{\begin{array}{l} \mu \in \mathbb{R} : \exists \phi \in C_b^{1,2}([0,T]\times\mathbb{R}^m) \text{ s.t. } \forall (s,y,v,z) \in [t,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m, \\ \mu \leq (T-t)\left(\mathcal{L}_{(b,\rho)}^v \phi(s,y) + l_1(s,y,u)\right) + l_2(z) - \phi(T,z) + \phi(t,x), \end{array}\right\} \tag{13}$$

for all $(t,x) \in [0,T]\times\mathbb{R}^m$. The reader may want to note that this formulation corresponds to the Lagrange dual where the cost $(T-t)\,l_1(s,y,u) + l_2(z)$ is penalized by the constraint expression in the definition of $\Theta_{(b,\rho)}(t,T,x)$ (i.e. $(T-t)\,\mathcal{L}_{(b,\rho)}^v \phi(s,y) + \phi(t,x) - \phi(T,z)$). A second interpretation of this term comes from the theory of Hamilton-Jacobi-Bellman systems. The term $\mathcal{L}_{(b,\rho)}^v \phi(s,y) + l_1(s,y,u)$ comes from the Hamiltonian and $l_2(z) - \phi(T,z)$ is the final condition. Roughly speaking, one maximizes over viscosity subsolutions ϕ the value $\phi(t,x)$. This is coherent with Perron's preconization of the unique viscosity solution.

The following result is a slight generalization of [15, Theorem 4]. The proof is very similar and will be omitted.

Theorem 2.1 . *Under the Assumptions (6) and (10),*

$$V_{l_1,l_2,(b,\rho)} = \Lambda_{l_1,l_2,(b,\rho)} = \mu_{l_1,l_2,(b,\rho)}.$$

Since this result holds true for arbitrary (regular) functions l_1 and l_2, a standard separation argument yields:

Corollary 2.1 *The set of constraints* $\Theta_{(b,\rho)}(t,T,x)$ *is the closed, convex hull of* $\Gamma_{(b,\rho)}(t,T,x)$ *:*

$$\Theta_{(b,\rho)}(t,T,x) = \overline{co}\Gamma_{(b,\rho)}(t,T,x). \tag{14}$$

The closure is taken with respect to the usual (narrow) convergence of probability measures.

Remark 2.1 *1. Due to the inequality (8), Prohorov's theorem yields that* $co\Gamma(t,T,x)$ *is relatively compact and, thus,* $\Theta_{(b,\rho)}(t,T,x)$ *is compact. Moreover,*

$$\begin{cases} \int_{\mathbb{R}^m} |y|^4 \gamma^1([t,T],dy,U) \leq C_0\left(|x|^4+1\right), \\ \int_{\mathbb{R}^m} |y|^4 \gamma^2(dy) \leq C_0\left(|x|^4+1\right), \end{cases} \tag{15}$$

for all $\gamma = \left(\gamma^1,\gamma^2\right) \in \Theta_{(b,\rho)}(t,T,x)$.

2. In the applications intended in this paper, we will solely consider final costs (i.e. we take $l_1 = 0$*). However, the proofs rely on* Θ *being compact. This follows from the previous Corollary and its proof needs both final and running cost functions. This is the reason why we have chosen to give this (rather heavy) presentation.*

We equally mention the following result due to N. V. Krylov [21, Theorem 2.1]. It is both an essential ingredient in proving Theorem 2.1 and a tool for further developments.

Proposition 2.1 *There exists a constant* $C > 0$ *such that, for every* $\delta \in (0,1]$, *there exists a function* $V^\delta \in C_b^{1,2}\left(\left[0,T+\delta^2\right]\times\mathbb{R}^m\right)$ *such that*

$$\mathcal{L}^v_{(b,\rho)}\phi(s,y) + l_1(s,y,v) \geq 0,$$

for all $(s,y,v) \in \left[0,T+\delta^2\right]\times\mathbb{R}^m\times U$ *and*

$$(i)\ \left|V^\delta(t,\cdot) - l_2(\cdot)\right| \leq C\delta,\ \textit{for } t \in \left[T,T+\delta^2\right],\ \textit{and}$$
$$(ii)\ \left|V^\delta(\cdot) - V_{l_1,l_2,(b,\rho)}(\cdot)\right| \leq C\delta,\ \textit{on } [0,T]\times\mathbb{R}^N.$$

Remark 2.2 *(i) The constant* C *only depends on the Lipschitz constants and the bounds of* (b,ρ)*:*

$$C \leq c_0\left(1+|b|_\infty + Lip(b) + |\rho|_\infty + Lip(\rho)\right),$$

where c_0 *is a constant (depending, eventually on* T*).*

(ii) We assume that $l_1 = 0$. Then, the functions V^δ are obtained by the "shaking of coefficients" method as $V_\delta * \psi_\delta$, where $V_\delta = V_{0,l_2,(b^\delta,\rho^\delta)}$ with

$$b^\delta(x,u,v) = b(x+\delta v,u), \ \rho^\delta(x,u,v) = \rho(x+\delta v,u), u \in U, \ v \in \mathbb{R}^m, |v| \leq 1$$

and $(\psi_\delta)_\delta$ a sequence of standard mollifiers $\psi_\delta(y) = \frac{1}{\delta^m}\psi\left(\frac{y}{\delta}\right)$, $y \in \mathbb{R}^m$, $\delta > 0$, where $\psi \in C^\infty(\mathbb{R}^m)$ is a positive function such that

$$Supp(\psi) \subset \overline{B}(0,1) \ \text{ and } \int_{\mathbb{R}^m} \psi(x)dx = 1.$$

2.2 Characterization of Optimal Measures

In this subsection we present necessary and sufficient conditions for characterizing optimal occupational measures. We consider that $l_1 \equiv 0$, $T > 0$ is fixed and we set

$$\Theta(x) := \Theta_{(b,\rho)}(0,T,x), V_{l_2}(x) := V_{0,l_2,(b,\rho)}(0,x), \ \Lambda_{l_2}(x) := \Lambda_{0,l_2,(b,\rho)}(0,x), \ \eta_{l_2}(x) := \eta_{0,l_2,(b,\rho)}(0,x),$$

for simplicity. Recall that, with the above notations,

$$V_{l_2}(x) = \Lambda_{l_2}(x) = \eta_{l_2}(x),$$

for all initial data $x \in \mathbb{R}^m$ and

$$\eta_{l_2}(x) = \sup \left\{ \begin{array}{l} \eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m) \text{ s.t. } \forall (s,y,v,z) \in [0,T] \times \mathbb{R}^m \times U \times \mathbb{R}^m, \\ \eta \leq T\mathcal{L}^v \phi(s,y) + l_2(z) - \phi(T,z) + \phi(0,x) \end{array} \right\}, \tag{16}$$

for all $x \in \mathbb{R}^m$. As before, this formulation corresponds to the Lagrange dual where the cost $l_2(z)$ is penalized by the constraint expression in the definition of $\Theta(x)$ (i.e. $T\mathcal{L}^v\phi(s,y) - \phi(T,z) + \phi(0,x)$). Of course, for a fixed test function ϕ, one is interested in maximal η satisfying the previous inequality. With this in mind, we denote by

$$D_{l_2}(x) = \left\{ \begin{array}{l} (\eta,\phi) \in \mathbb{R} \times C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m) \text{ s.t.} \\ \eta = \inf\limits_{(s,y,v,z) \in [0,T] \times \mathbb{R}^m \times U \times \mathbb{R}^m} \{T\mathcal{L}^v\phi(s,y) + l_2(z) - \phi(T,z) + \phi(0,x)\} \end{array} \right\}, \tag{17}$$

for all $x \in \mathbb{R}^m$. By our assumptions, the coefficient functions are bounded and, thus, the set $D_{l_2}(x)$ is well defined.

The dual formulation yields

$$V_{l_2}(x) = \sup\{\eta, (\eta, \phi) \in D_{l_2}(x)\}. \tag{18}$$

2.2.1 The Regular Case

We introduce the following.

Definition 2.1 *Whenever $x \in \mathbb{R}^m$, we say that $(\bar{\eta}, \bar{\phi}) \in D_{l_2}(x)$ is an optimal pair whenever we have $V_{l_2}(x) = \bar{\eta}$.*

We denote by

$$\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x) = \left\{ \begin{array}{l} (s,y,v,z) \in [0,T] \times \mathbb{R}^m \times U \times \mathbb{R}^m, s.t. \\ \bar{\eta} = T\mathcal{L}^v \bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(0,x). \end{array} \right\} \tag{19}$$

We recall that the definition of $D_{l_2}(x)$ implies that $\bar{\eta} = \inf_{(s,y,v,z) \in [0,T] \times \mathbb{R}^m \times U \times \mathbb{R}^m} T\mathcal{L}^v \bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(0,x)$. It turns out that the support of optimal measures only takes into account those (s,y,v,z) which realize the infimum and this leads us to introducing $\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)$. Of course, neither the set of optimal pairs, nor $\Omega_{l_2,(\bar{\eta},\bar{\phi})}$ are a priori non empty. It is the case if $V_{0,l_2}(\cdot,\cdot)$ belongs to $C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m)$ and we consider the setting of the problem to be some invariant compact set $K \subset \mathbb{R}^m$. In this framework, one can guarantee that optimal pairs exists for every $x \in K$. Indeed, it suffices to consider $\bar{\phi} = V_{0,l_2}$ and get, using the fact that it is a (regular) subsolution of the associated HJB equation,

$$T\mathcal{L}^v \bar{\phi}(s,y) \geq 0, \ \ l_2(z) \geq \bar{\phi}(T,z),$$

for all $(s,y,v,z) \in [0,T] \times K \times U \times K$. Hence,

$$V_{l_2}(x) \leq T\mathcal{L}^v \bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(0,x),$$

for all $(s,y,v,z) \in [0,T] \times K \times U \times K$. The fact that $\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)$ is nonempty follows from the compactness of K.

Proposition 2.2 *Let $x \in \mathbb{R}^m$ be fixed and assume that $(\bar{\eta}, \bar{\phi}) \in D_{l_2}(x)$ is an optimal pair. Then, $\gamma \in \Theta(x)$ is optimal for $\Lambda_{l_2}(x)$ if and only if $\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)$ is nonempty and $\gamma\left(\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)\right) = 1$.*

Proof. The proof will be postponed to the Appendix. □

2.2.2 The General Framework

If the value function is not smooth, optimal pairs may not exist. However, if optimal pairs do not exist, one finds some sequence $(\eta_n, \phi_n) \in D_{l_2}(x)$ such that $(\eta_n)_n$ is strictly increasing and converging to $V_{l_2}(x)$. The functions ϕ_n can be chosen to be uniformly bounded (e.g. Theorem 2.1 in [21], see also Proposition 3 in [15]). We define the nonempty, closed sets

$$\Omega_{l_2}^n(x) = \left\{ \begin{array}{l} (s, y, v, z) \in [0, T] \times \mathbb{R}^m \times U \times \mathbb{R}^m, \text{s.t.} \\ V_{l_2}(x) + \sqrt{V_{l_2}(x) - \eta_n} \geq T\mathcal{L}^v \phi_n(s, y) + l_2(z) - \phi_n(T, z) + \phi_n(0, x). \end{array} \right\} \tag{20}$$

Following the regular case, one may be inclined to take η_n instead $V_{l_2}(x) + \sqrt{V_{l_2}(x) - \eta_n}$. Due to the fact that $\eta_n < V_{l_2}(x)$, this gives little information (especially when limit is involved). The penalty $\sqrt{V_{l_2}(x) - \eta_n}$ is decreasing and the choice of the square root is intended for technical reasons in Proposition 2.3. We also define the limit sets

$$\Omega_{l_2}^{in}(x) := \liminf_{n \to \infty} \Omega_{l_2}^n(x) = \underset{n \geq 1}{\cup} \underset{k \geq n}{\cap} \Omega_{l_2}^k(x), \ \Omega_{l_2}^{out}(x) := \limsup_{n \to \infty} \Omega_{l_2}^n(x) = \underset{n \geq 1}{\cap} \underset{k \geq n}{\cup} \Omega_{l_2}^k(x),$$
$$\Omega_{l_2}^{out,cl}(x) := \underset{n \geq 1}{\cap} cl\left(\underset{k \geq n}{\cup} \Omega_{l_2}^k(x)\right),$$

where cl is the usual Kuratowski closure operator.

Remark 2.3 *If an optimal pair $\left(V_{l_2}(x), \overline{\phi}\right)$ exists, we pick $\phi_n = \overline{\phi}$. In this case, $\eta_n = V_{l_2}(x)$. The sets $\Omega_{l_2}^n(x)$ coincide. Hence, $\Omega_{l_2}^{out}(x) = \Omega_{l_2}^{in}(x) = \Omega_{l_2,\left(V_{l_2}(x),\overline{\phi}\right)}(x)$ as in the previous case.*

We get the following characterization of the support of optimal measures.

Proposition 2.3 *Let us consider $x \in \mathbb{R}^m$.*

(i) If $\gamma \in \Theta(x)$ is optimal, then

$$\gamma\left(\Omega_{l_2}^{out,cl}(x)\right) = \gamma\left(\Omega_{l_2}^{out}(x)\right) = 1,$$

(i.e. the support of γ is included in $\Omega_{l_2}^{out}(x)$). In particular, when the limit of the sets exists (i.e. $\Omega_{l_2}^{in}(x) = \Omega_{l_2}^{out}(x)$), one gets

$$\sup_{n \geq 1} \gamma\left(\underset{k \geq n}{\cap} \Omega_{l_2}^k(x)\right) = 1.$$

(ii) Conversely, if $\gamma \in \Theta(x)$ is such that the supremum can be replaced with maximum (i.e. if there exists some n_0 such that $\gamma\left(\underset{k \geq n_0}{\cap} \Omega_{l_2}^k(x)\right) = 1$), then γ is optimal.

Proof. Again, the proof will be postponed to the Appendix. □

3 The Averaging Method

Motivated by the optimality results obtained in the classical framework, we develop linearization arguments for the control of singularly perturbed systems. We begin with some usual assumptions taken from [7]. The basic idea is that, under reasonable conditions, the value function for the averaged system can be seen as a limit of some standard value functions. This allows us to equally pass to the limit the dual value functions and get linear formulations in this perturbed framework. Next, we proceed similar to the standard case, by using the expression of the dual linear formulation. Since optimal pairs have no reason to exist, we proceed as in the second case described for classical control problems. Moreover, since in general, the dual formulation has not a $\sup\inf$ form (but rather some $\sup \lim_{\varepsilon\to 0} \inf$ form, where ε is the scaling parameter), we need to propose a particular choice for the test functions. This is done by using the shaking of coefficients idea of Krylov. The optimality results are closely connected to those already described for classical control problems.

3.1 General Considerations

All the assumptions and ideas of this preliminary part can be found in [7]. Let us shortly explain the behavior of the perturbed system (1) as $\varepsilon \to 0$. To this purpose, let us fix, for the time being, $\varepsilon > 0$ and the weak control $\pi = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, (W, B), u\right)$. If one makes the change of variables $\tau = \frac{s}{\varepsilon}$ in the system (1) and sets $\left(\tilde{X}_\tau, \tilde{Y}_\tau, \widetilde{u}_\tau\right) = (X_{\varepsilon\tau}, Y_{\varepsilon\tau}, u_{\varepsilon\tau})$, $B'_\tau = \frac{1}{\sqrt{\varepsilon}} B_{\varepsilon\tau}$, $W'_\tau = \frac{1}{\sqrt{\varepsilon}} W_{\varepsilon\tau}$ for $\tau \in [0, \frac{T}{\varepsilon}]$, one gets

$$\begin{cases} d\tilde{X}_\tau^{x,y,u} = \varepsilon f\left(\tilde{X}_\tau^{x,y,u}, \tilde{Y}_\tau^{x,y,u}, \widetilde{u}_\tau\right) d\tau + \sqrt{\varepsilon}\sigma\left(\tilde{X}_\tau^{x,y,u}, \tilde{Y}_\tau^{x,y,u}, \widetilde{u}_\tau\right) dW'_\tau, \\ d\tilde{Y}_\tau^{x,y,u} = g\left(\tilde{X}_\tau^{x,y,u}, \tilde{Y}_\tau^{x,y,u}, \widetilde{u}_\tau\right) d\tau + \beta\left(\tilde{X}_\tau^{x,y,u}, \tilde{Y}_\tau^{x,y,u}, \widetilde{u}_\tau\right) dB'_\tau, \end{cases} \tag{21}$$

When ε tends to 0, we are led to consider the following associated system:

$$dY_\tau^{x,y,u} = g\left(x, Y_\tau^{x,y,u}, u_\tau\right) d\tau + \beta\left(x, Y_\tau^{x,y,u}, u_\tau\right) dB'_\tau \tag{22}$$

for $\tau \in [0, +\infty)$, where x (resp. y) is a fixed $\mathbb{R}^M$(resp. $\mathbb{R}^N$)-valued random variable independent of B'. We denote by $y_{(\cdot)}^{y,u;x}$ the unique solution of (22) corresponding to the control u and to the initial value y. The framework will still be that of weak controls.

Assumption 1 *Following the approach in [7]; see also [17], throughout the paper, unless stated otherwise,* *we will assume that*

$$\begin{array}{l}\sup_{\varepsilon>0,t\in[0,T],\pi\in\mathcal{U}^w}\mathbb{E}^{\pi}\left[\left|Y_t^{x,y,u^{\pi};\varepsilon}\right|^4\right]<c\left(|x|^4+|y|^4+1\right),\\ \sup_{t\in[0,T],\pi\in\mathcal{U}^w}\mathbb{E}^{\pi}\left[\left|y_t^{x,y,u^{\pi}}\right|^4\right]<c\left(|x|^4+|y|^4+1\right),\end{array}\tag{A1}$$

for all initial data $(x,y)\in\mathbb{R}^M\times\mathbb{R}^N$.

For explicit conditions (e.g. asymptotic exponential stability for the fourth order moment) implying the above inequalities, the reader is referred to [7, Page 172].

Whenever $x\in\mathbb{R}^M$, we let

$$D_x:=\left\{\begin{array}{c}\mu\in\mathcal{P}\left(\mathbb{R}^N\times U\right):\\ \int\langle g(x,y,u),D\phi(y)\rangle+\frac{1}{2}Tr\left(\beta\beta^*(x,y,u)D^2\phi(x)\right)\mu(dydu)=0.\end{array}\right\}$$

It turns out that $x\rightsquigarrow D_x$ is an upper semicontinous set-valued function with nonempty, closed, convex values; see [7, Lemma 2.1].

The averaged system is given by

$$\left\{\begin{array}{l}d\overline{X}_s^{x,u}=\bar{f}\left(\overline{X}_s^{x,\mu},\mu_s\right)ds+\bar{\sigma}\left(\overline{X}_s^{x,\mu},\mu_s\right)dW_s,\\ \qquad\mu_s\in D_{\overline{X}_s^{x,\mu}}\text{ for (almost) all }s\in[0,T],\end{array}\right.\tag{23}$$

where $\bar{f}(x,\mu):=\int f(x,y,u)\mu(dydu),\overline{\sigma}(x,\mu):=\int\sigma(x,y,u)\mu(dydu)$ and the control processes are $\mathcal{P}\left(\overline{\mathbb{R}}^N\times U\right)$-valued. For further considerations on the compactness issues on $\mathcal{P}\left(\overline{\mathbb{R}}^N\times U\right)$, the reader is referred to [7, Section 2]. In particular, one can introduce a metric (denoted by d) on $\mathcal{P}\left(\overline{\mathbb{R}}^N\times U\right)$ which is consistent with the weak convergence of probability measures. The set of $\mathcal{P}\left(\overline{\mathbb{R}}^N\times U\right)$-valued weakly-admissible controls will be denoted by $\overline{\mathcal{U}}_N^w$.

Following [7, Assumption 2], we ask that

Assumption 2 *There exists some* $\omega_c\in C(\mathbb{R}_+;\mathbb{R}_+)$ *satisfying* $\lim_{S\to\infty}\omega_c(S)=0$ *such that, whenever* $x\in\mathbb{R}^M,y\in\mathbb{R}^N$ *satisfy* $|x|\leq c$ *and* $\mu\in D_x$*, there exists an admissible weak control* π *such that*

$$\mathbb{E}^{\pi}\left[d\left(\mu,\mu_{0,S,x,\pi}^1\right)\right]\leq\omega_c(S).$$

The measure $\mu_{0,S,x,\pi}^1$ is similar to the occupation measures $\gamma_{0,S,x,\pi}^1$ but it does not involve the expectation i.e.

$$\mu_{0,S,x,\pi}^1(B\times C)=\frac{1}{S}\int_0^S 1_{B\times C}\left(s,y_s^{y,u^{\pi};x},u_s^{\pi}\right)ds,$$

for all Borel subsets $B \times C \subset \mathbb{R}^N \times U$. The previous assumption is implied by classical mixing conditions in [7, Proposition 4.1], if one further assumes that the noise coefficient is control independent.

Additionally to the perturbed control problems $W_{\varepsilon,h}$ (given in Subsection 1.2), we consider the optimal control problem

$$W_h(x) = \inf_{\overline{\pi} \in \overline{\mathcal{U}}_N^w} \mathbb{E}^{\overline{\pi}} \left[h\left(\overline{X}_T^{x,\mu^{\overline{\pi}}} \right) \right], \tag{24}$$

for all initial data $x \in \mathbb{R}^M$.

We endow the space $\mathbb{R}^M \times \mathcal{P}\left(\overline{\mathbb{R}}^N \times U\right)$ with the metric $\widetilde{d}$ given by

$$\widetilde{d}\left(\left(x, \mu\right), \left(x', \mu'\right)\right) = \left|x - x'\right| + d\left(\mu, \mu'\right),$$

for all $\left(x, \mu\right), \left(x', \mu'\right) \in \mathbb{R}^M \times \mathcal{P}\left(\overline{\mathbb{R}}^N \times U\right)$. We introduce the set valued function with nonempty, convex, compact values

$$\mathbb{R}^M \ni x \rightsquigarrow Q_x := \left\{ \left(\overline{b}\left(x, \mu\right), \mu\right) : \mu \in D_x \right\}$$

and make the following (see [7, Assumption 3])

Assumption 3 *The set valued function Q is Lipschitz continuous on $\mathbb{R}^M$ (i.e. there exists $c_0 \in \mathbb{R}$ such that*

$$\widetilde{d}_{Hausdorff}\left(Q_x, Q_{x'}\right) \leq c_0 \left|x - x'\right|, \text{ for all } x, x' \in \mathbb{R}^M.$$

Here, $\widetilde{d}_{Hausdorff}$ denotes the Hausdorff distance constructed from $\widetilde{d}$).

Remark 3.1 *Both the Assumption 2 and Assumption 3 hold true if the system (22) satisfies an exponential ergodicity condition, uniformly with respect to the control process, using [7, Assumption 4; Proposition 5.2]. This condition can be obtained if dissipativity is assumed for the stochastic system (22). Alternatively, it is possible to adapt the arguments in [22] to deal with nonexpansive (yet nondissipative) systems. However, this generalization is not within the scopus of the present paper.*

Under the above conditions, using [7, Theorem 3.3 and Theorem 4.2] and [17, Theorem 5.1]), every partial limit of solutions $\left(X_{(\cdot)}^{x,y,u^{\pi_\varepsilon};\varepsilon}\right)_{\varepsilon>0}$ satisfies (23) and, conversely, for every solution $\overline{X}^{x,u^{\overline{\pi}}}$ of (23), one finds a suitable sequence $\left(X_{(\cdot)}^{x,y,u^{\pi_\varepsilon};\varepsilon}\right)_{\varepsilon>0}$ converging to $\overline{X}^{x,u^{\overline{\pi}}}$. Due to Assumption 2, the distance is given uniformly with respect to x within a compact set. To simplify our presentation, let us assume that

Assumption 4 *There exists some compact set $K \subset \mathbb{R}^M$ such that $K \times \mathbb{R}^N$ is invariant with respect to* (5).

For explicit criteria of invariance, the reader is referred to [23]; also see [24]. We note that these criteria only involve the coefficients f and σ.

If the cost functional h is bounded and uniformly continuous, the convergence of the value functions is a direct consequence of the convergence of trajectories. More precisely, we have $W_{\varepsilon,h} \to W_h$ with respect with the uniform convergence :

There exists $\omega \in C(\mathbb{R}_+;\mathbb{R}_+)$ satisfying $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$ such that

$$|W_{\varepsilon,h}(x,y) - W_h(x)| \le \omega(\varepsilon), \tag{25}$$

for all $x \in K$ and all $y \in \mathbb{R}^N$; see [7, Corollaries 3.4 and 4.3].

Remark 3.2 *The estimates in [7] show that ω depends on the bounds of the coefficient and cost functions and their continuity moduli, but not on the functions themselves. Thus, if $\delta > 0$ and $W_{\varepsilon,h,\delta}$ is the value function associated with the "shaked" problem (i.e. in which $\varphi \in \{f,\sigma,g,\beta\}$ are replaced with $\varphi^\delta(x,y,u,v) = \varphi(x+\delta v, y+\delta v', u), (v,v') \in \mathbb{R}^M \times \mathbb{R}^N, |(v,v')| \le 1$) under analogous assumptions, the inequality (25) holds true for some $W_{h,\delta}$ constructed as before replacing W_h. In particular,*

$$|W_{\varepsilon,h,\delta}(x,y) - W_{\varepsilon,h,\delta}(x,y')| \le 2\omega(\varepsilon),$$

for all $x \in K$ and all $y, y' \in \mathbb{R}^N$. Now, let us consider $(\psi_\delta)_\delta$ to be a sequence of standard mollifiers $\psi_\delta(x,y) = \frac{1}{\delta^{M+N}}\psi\left(\frac{x}{\delta},\frac{y}{\delta}\right)$, $(x,y) \in \mathbb{R}^{M+N}$, $\delta > 0$, where $\psi \in C^\infty\left(\mathbb{R}^{M+N}\right)$ is a positive function such that

$$Supp(\psi) \subset \overline{B}(0,1) \ \ and \ \int_{\mathbb{R}^{M+N}} \psi(x)dx = 1.$$

Then, using the Remark 2.2 (i) and (25), the convoluted function $W^\delta_{\varepsilon,h} := W_{\varepsilon,h,\delta} * \psi_\delta$ satisfy :

$$\begin{cases} \left|W^\delta_{\varepsilon,h}(x,y) - W_{\varepsilon,h}(x,y)\right| \le c_0\left(1+\frac{1}{\varepsilon}\right)\delta, \\ \left|W^\delta_{\varepsilon,h}(x,y) - W^\delta_{\varepsilon,h}(x,y')\right| \le 2c_0\left(1+\frac{1}{\varepsilon}\right)\delta + 2\left|W_{\varepsilon,h}(x,\cdot) - W_h(x)\right| \\ \le 2c_0\left(1+\frac{1}{\varepsilon}\right)\delta + 2\omega(\varepsilon) \end{cases} \tag{26}$$

where c_0 is independent of δ and ε. Moreover, since $D_x W^\delta_{\varepsilon,h} = \frac{1}{\delta} W_{\varepsilon,h,\delta} * D_x\psi_\delta$, one gets

$$\left|D_x W^\delta_{\varepsilon,h}(x,y) - D_x W^\delta_{\varepsilon,h}(x,y')\right| \le \frac{1}{\delta}2\omega(\varepsilon).$$

Similar assertions are valid for $\left|D^2_x W^\delta_{\varepsilon,h}(x,y) - D^2_x W^\delta_{\varepsilon,h}(x,y')\right|$. The approach equally works for the time dependent problem $W^\delta_{\varepsilon,h}(t,x,y), W_h(t,x)$; see Remark 2.2. Also, using [21, Theorem 2.1, Estimate 2.3], *one can prove that*

$$\left\|W^{\delta}_{\varepsilon,h}\right\|_{\infty}+\left\|\partial_t W^{\delta}_{\varepsilon,h}\right\|_{\infty}+\left\|DW^{\delta}_{\varepsilon,h}\right\|_{\infty}+\left\|D^2W^{\delta}_{\varepsilon,h}\right\|_{\infty}\leq c_0\frac{1}{\delta^2}, \tag{27}$$

where c_0 depends only on T (but not on δ).

3.2 Linear Formulations for the Averaged System

As previously, let us consider that $T>0$ is a fixed time horizon. We fix $\varepsilon>0$ and $(x_0,y_0)\in\mathbb{R}^M\times\mathbb{R}^N$. To every $\pi\in\mathcal{U}^w$, one can associate a couple of occupation measures $\gamma_{x_0,y_0,\pi;\varepsilon}=\left(\gamma^1_{x_0,y_0,\pi;\varepsilon},\gamma^2_{x_0,y_0,\pi;\varepsilon}\right)\in\mathcal{P}\left([0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right)$ defined by

$$\begin{cases}\gamma^1_{x_0,y_0,\pi;\varepsilon}\left(A\times B\times C\times D\right)=\frac{1}{T}\mathbb{E}^{\pi}\left[\int_0^T 1_{A\times B\times C\times D}\left(s,X_s^{x_0,y_0,u^{\pi};\varepsilon},Y_s^{x_0,y_0,u^{\pi};\varepsilon},u_s^{\pi}\right)ds\right],\\ \gamma^2_{x_0,y_0,\pi;\varepsilon}\left(E\times F\right)=\mathbb{E}^{\pi}\left[1_{E\times F}\left(X_T^{x_0,y_0,u^{\pi};\varepsilon},Y_T^{x_0,y_0,u^{\pi};\varepsilon}\right)\right],\end{cases}$$

for all Borel sets $A\subset[0,T]$, $B\subset\mathbb{R}^M$, $C\subset\mathbb{R}^N$ and $D\subset U$. The family of occupation measures associated to weak controls

$$\Gamma\left(x_0,y_0;\varepsilon\right):=\left\{\ \left(\gamma^1_{x_0,y_0,\pi;\varepsilon},\gamma^2_{x_0,y_0,\pi;\varepsilon}\right),\ \text{for all }\pi\in\mathcal{U}^w\ \right\} \tag{28}$$

can be embedded into a larger set

$$\Theta\left(x_0,y_0;\varepsilon\right)=\left\{\begin{array}{l}\left(\gamma^1,\gamma^2\right)\in\mathcal{P}\left([0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right)\\ \forall\phi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right),\\ \int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left(\begin{array}{c}\phi\left(0,x_0,y_0\right)+T\mathcal{L}^{u;\varepsilon}\phi\left(s,x,y\right)\\ -\phi\left(T,z,w\right)\end{array}\right)\gamma^1\left(dsdxdydu\right)\gamma^2\left(dzdw\right)=0.\end{array}\right\}, \tag{29}$$

where

$$\mathcal{L}^{u;\varepsilon}\phi\left(s,x,y\right)=\frac{1}{2}Tr\left[\left(\sigma\sigma^*\right)\left(x,y,u\right)D_x^2\phi\left(s,x,y\right)\right]+\frac{1}{2\varepsilon}Tr\left[\left(\beta\beta^*\right)\left(x,y,u\right)D_y^2\phi\left(s,x,y\right)\right]$$
$$+\left\langle f\left(x,y,u\right),D_x\phi\left(s,x,y\right)\right\rangle+\frac{1}{\varepsilon}\left\langle g\left(x,y,u\right),D_y\phi\left(s,x,y\right)\right\rangle+\partial_t\phi\left(s,x,y\right),$$

for all $\phi \in C^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N\right)$ and all $s \geq 0$, $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$, $u \in U$.

Remark 3.3 *Using similar arguments as in the previous sections, the set $\Theta\left(x_0, y_0; \varepsilon\right)$ contains all occupation measures issued from $\left(x_0, y_0\right)$ at time t. Moreover, it is also convex and relatively compact with respect to the weak convergence of probability measures (due to Prohorov's Theorem).*

Throughout the remaining of the paper, h is assumed to be bounded and Lipschitz-continuous. The linearized value function is given by

$$\Lambda_{\varepsilon,h}\left(x_0, y_0\right) = \inf_{\gamma=\left(\gamma^1,\gamma^2\right)\in\Theta\left(x_0,y_0;\varepsilon\right)} \int_{\mathbb{R}^M\times\mathbb{R}^N} h\left(z\right)\gamma^2\left(dzdw\right),$$

and its dual by

$$\eta_{\varepsilon,h}\left(x_0, y_0\right) = \sup\left\{\begin{array}{l} \eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N\right) \text{ s.t.} \\ \forall \left(s, x, y, v, z, w\right) \in [0,T] \times \mathbb{R}^M \times \mathbb{R}^N \times U \times \mathbb{R}^M \times \mathbb{R}^N, \\ \eta \leq T\mathcal{L}^{v;\varepsilon}\phi\left(s,x,y\right) + h\left(z\right) - \phi\left(T,z,w\right) + \phi\left(0,x_0,y_0\right). \end{array}\right\}, \tag{30}$$

for all $\left(x_0, y_0\right) \in \mathbb{R}^M \times \mathbb{R}^N$. This is a particular case of systems considered in Subsection 2.2. Hence, for every $\varepsilon > 0$, one gets, applying Theorem 2.1,

$$W_{\varepsilon,h}\left(x_0, y_0\right) = \Lambda_{\varepsilon,h}\left(x_0, y_0\right) = \eta_{\varepsilon,h}\left(x_0, y_0\right),$$

for all initial data $\left(x_0, y_0\right) \in \mathbb{R}^M \times \mathbb{R}^N$.

At this point, we wish to give the intuition leading to the linear formulation for the averaged problem : if one thinks of the y component as being some penalization term, as $\varepsilon \to 0$, the corresponding part in $\mathcal{L}^{u;\varepsilon}$ should be 0 on the support of admissible measures. For the remaining component, y would be indifferent. We denote by

$$\Theta\left(x_0, y_0\right) = \left\{\begin{array}{c} \gamma = \left(\gamma^1, \gamma^2\right) \in \mathcal{P}\left([0,T] \times \mathbb{R}^M \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^M \times \mathbb{R}^N\right) : \\ \exists \gamma_\varepsilon \in \Theta\left(x_0, y_0; \varepsilon\right), \gamma_\varepsilon \rightharpoonup \gamma \text{ along some subsequence } \varepsilon_n \underset{n\to\infty}{\to} 0 \end{array}\right\},$$

for all $\left(x_0, y_0\right) \in \mathbb{R}^M \times \mathbb{R}^N$. Whenever $\gamma_\varepsilon = \left(\gamma_\varepsilon^1, \gamma_\varepsilon^2\right) \in \Theta\left(x_0, y_0; \varepsilon\right)$ for all $\varepsilon > 0$, one can find a subsequence (still indexed by $\varepsilon > 0$, for notation purposes) and a probability measure γ such that $\gamma_\varepsilon \rightharpoonup \gamma$. This is done using (A1) and Prohorov's theorem. Hence, the set $\Theta\left(x_0, y_0\right)$ is nonempty. One can also prove that it is closed; see Corollary 14.

Proposition 3.1 *The following inclusion holds true*

$$\Theta(x_0,y_0)\subset$$

$$\left\{\begin{array}{l}\left(\gamma^1,\gamma^2\right)\in\mathcal{P}\left([0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right)\ \ s.t.\\ \forall\psi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\right)\ \ and\ \forall\phi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right),\\ \int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\begin{pmatrix}\psi(0,x_0)+T\mathcal{L}^{u,f}\psi(s,x,y)\\ -\psi(T,z)\end{pmatrix}\gamma^1(dsdxdydu)\,\gamma^2(dzdw)=0\ \ and\\ \int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\mathcal{L}^{u,g}\phi(s,x,y)\,\gamma^1(dsdxdydu)\,\gamma^2(dzdw)=0\end{array}\right\},$$

where

$$\mathcal{L}^{u,f}\psi(s,x,y)=\frac{1}{2}Tr\left[(\sigma\sigma^*)(x,y,u)\,D^2\psi(s,x)\right]+\langle f(x,y,u)\,,D_x\psi(s,x)\rangle+\partial_t\psi(s,x)$$

and

$$\mathcal{L}^{u,g}\phi(s,x,y)=\frac{1}{2}Tr\left[(\beta\beta^*)(x,y,u)\,D^2\phi(s,x,y)\right]+\langle g(x,y,u)\,,D_y\phi(s,x,y)\rangle\,,$$

for all $\phi\in C^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right)$, $\psi\in C^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\right)$ *and all* $s\geq 0$, $(x,y)\in\mathbb{R}^M\times\mathbb{R}^N$, $u\in U$.

Proof. Let us fix $\gamma\in\Theta(x_0,y_0)$ and $\gamma_\varepsilon=\left(\gamma_\varepsilon^1,\gamma_\varepsilon^2\right)\in\Theta(x_0,y_0;\varepsilon)$ such that $\gamma_\varepsilon\rightharpoonup\gamma$. Whenever $\psi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\right)$, the definition of $\Theta(x_0,y_0;\varepsilon)$ yields

$$\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left[\psi(0,x_0)+T\mathcal{L}^{u,f}\psi(s,x,y)-\psi(T,z)\right]\gamma_\varepsilon^1(dsdxdydu)\,\gamma_\varepsilon^2(dzdw)=0.$$

Moreover, if one considers any fixed (though arbitrary) $\phi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right)$, then

$$\begin{aligned}&\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\mathcal{L}^{u,g}\phi(s,x,y)\,\gamma_\varepsilon^1(dsdxdydu)\,\gamma_\varepsilon^2(dzdw)\\ &=-\varepsilon\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left[\phi(0,x_0,y_0)+T\mathcal{L}^{u,f}\phi(s,x,y)-\phi(T,z,w)\right]\gamma_\varepsilon^1(dsdxdydu)\,\gamma_\varepsilon^2(dzdw)\end{aligned}$$

and the conclusion follows by letting $\varepsilon\to 0$ and recalling that $\phi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right)$, resp. $\psi\in C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\right)$. □

We define the following linearized problem

$$\Lambda_h(x_0, y_0) = \inf_{\gamma=(\gamma^1,\gamma^2)\in\Theta(x_0,y_0)} \int_{\mathbb{R}^M\times\mathbb{R}^N} h(z)\,\gamma^2(dzdw),$$

and denote by

$$\eta_h(x_0) = \sup\left\{\begin{array}{l} \eta \in \mathbb{R} : \exists \alpha \in C(\mathbb{R}_+;\mathbb{R}_+),\ \lim\limits_{\varepsilon\to 0}\alpha(\varepsilon) = 0 \text{ s.t. } \forall \varepsilon > 0, \\ \exists \phi \in C_b^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N\right) \text{ s.t.} \\ \sup\limits_{y,y'\in\mathbb{R}^N} \|\phi(\cdot,\cdot,y) - \phi(\cdot,\cdot,y')\|_\infty \le \alpha(\varepsilon) \text{ and s.t.} \\ \forall (s,x,y,v,z) \in [0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M, \\ \eta \le T\mathcal{L}^{v;\varepsilon}\phi(s,x,y) + h(z) + \|-\phi(T,z,\cdot)\|_\infty + \|\phi(0,x_0,\cdot)\|_\infty \end{array}\right\}, \tag{31}$$

for all $(x_0, y_0) \in \mathbb{R}^M \times \mathbb{R}^N$.

Remark 3.4 *In the previous definition one can, equivalently, ask that* $\|\phi(\cdot,\cdot,\cdot) - \phi(\cdot,\cdot,y_0)\|_\infty \le \alpha(\varepsilon)$ *for some fixed* $y_0 \in \mathbb{R}^M$.

Consequently, we can formulate the main result of this section:

Theorem 3.1 *We assume (A1) and (25) to hold true. Moreover, we assume the invariance condition (4) to be satisfied. Then the following equalities hold true*

$$W_h(x_0) = \Lambda_h(x_0, y_0) = \eta_h(x_0),$$

for all $(x_0, y_0) \in K \times \mathbb{R}^N$.

Remark 3.5 *As we have hinted in the previous subsection, whenever the Assumptions 1 - 3 hold true, then (25) holds true. For further details, the reader is referred to [7]; see also [17].*

Proof. Let us fix $(x_0, y_0) \in K \times \mathbb{R}^N$. In a first step, we recall that there exists an optimal measure $\bar{\gamma}_{(x_0,y_0;)\varepsilon} = \left(\bar{\gamma}_\varepsilon^1, \bar{\gamma}_\varepsilon^2\right) \in \Theta(x_0, y_0; \varepsilon)$ such that

$$\Lambda_{\varepsilon,h}(x_0, y_0) = \int_{\mathbb{R}^M\times\mathbb{R}^N} h(z)\,\bar{\gamma}_\varepsilon^2(dzdw),$$

for all $\varepsilon > 0$. One can find a subsequence (still indexed by $\varepsilon > 0$, for notation purposes) and a probability measure γ such that $\overline{\gamma}_\varepsilon \rightharpoonup \gamma$ using (A1) and Prohorov's theorem. Consequently,

$$\begin{aligned}\Lambda_h (x_0, y_0) &\le \int\limits_{\mathbb{R}^M \times \mathbb{R}^N} h(z) \gamma^2 (dzdw) = \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^M \times \mathbb{R}^N} h(z) \bar{\gamma}_\varepsilon^2 (dzdw) \\ &= \lim_{\varepsilon \to 0} \Lambda_{\varepsilon;h} (x_0, y_0) = \lim_{\varepsilon \to 0} W_{\varepsilon,h} (x_0, y_0) = W_h (x_0). \end{aligned} \tag{32}$$

for all $(x_0, y_0) \in \mathbb{R}^M \times \mathbb{R}^M$. The converse inequality is similar.

We continue by considering $\gamma \in \Theta(x_0, y_0)$ and $\eta \in \mathbb{R}$ such that

$$\begin{aligned}&\exists \alpha \in C(\mathbb{R}_+; \mathbb{R}_+) \text{ with } \lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0, \text{ s.t. } \forall \varepsilon > 0, \ \exists \phi \in C_b^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N\right) \text{ s.t.} \\ &\sup_{y,y' \in \mathbb{R}^N} \|\phi(\cdot,\cdot,y) - \phi(\cdot,\cdot,y')\|_\infty \le \alpha(\varepsilon) \text{ and } \forall (s,x,y,v,z) \in [0,T] \times \mathbb{R}^M \times \mathbb{R}^N \times U \times \mathbb{R}^M, \\ &\eta \le T\mathcal{L}^{v;\varepsilon} \phi(s,x,y) + h(z) - \inf_{y' \in \mathbb{R}^N} \phi(T,z,y') + \sup_{y' \in \mathbb{R}^N} \phi(0,x_0,y')\end{aligned},$$

Then,

$$\eta \le T\mathcal{L}^{v;\varepsilon} \phi(s,x,y) + h(z) - \phi(T,z,w) + \phi(0,x_0,y_0) + 2\alpha(\varepsilon), \tag{33}$$

for all $\forall (s,x,y,v,z,w) \in [0,T] \times \mathbb{R}^M \times \mathbb{R}^N \times U \times \mathbb{R}^M \times \mathbb{R}^N$. By the definition of $\Theta(x_0, y_0)$, there exists some sequence $\gamma_\varepsilon \in \Theta(x_0, y_0; \varepsilon)$ converging to γ. By integrating with respect to γ_ε the inequality (33), we obtain that

$$\eta \le \int\limits_{\mathbb{R}^M \times \mathbb{R}^N} h(z) \gamma_\varepsilon^2 (dzdw) + 2\alpha(\varepsilon),$$

and, consequently, recalling that $\gamma \in \Theta(x_0, y_0)$, $\varepsilon > 0$ are arbitrary and $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$, it follows that

$$\eta_h (x_0) \le \Lambda_h (x_0, y_0). \tag{34}$$

Let $\varepsilon > 0$ be fixed. Using Proposition 2.1 (see Remark 3.2 for the specific details; in particular the inequality (26)), there exists a family of functions $W_{\varepsilon,h}^\delta \in C_b^{1,2}\left(\left[0, T+\delta^2\right] \times \mathbb{R}^{M+N}\right)$ such that, for every $(s,x,y,v,z,w) \in [t,T] \times \mathbb{R}^M \times \mathbb{R}^N \times U \times \mathbb{R}^M \times \mathbb{R}^N$,

$$\begin{aligned}&\mathcal{L}^{v;\varepsilon} W_{\varepsilon,h}^\delta (s,x,y) \ge 0 \text{ and} \\ &h(z) - W_{\varepsilon,h}^\delta (T,z,w) \ge h(z) - W_{\varepsilon,h}(T,z,w) - c_0\left(1 + \tfrac{1}{\varepsilon}\right)\delta \ge -c_0\left(1 + \tfrac{1}{\varepsilon}\right)\delta.\end{aligned}$$

Hence,

$$W_{\varepsilon,h}^{\delta}(0,x_0,y_0)-c_0\left(1+\frac{1}{\varepsilon}\right)\delta \le \mathcal{L}^{v;\varepsilon}W_{\varepsilon,h}^{\delta}(s,x,y)+h(z)-W_{\varepsilon,h}^{\delta}(T,z,w)+W_{\varepsilon,h}^{\delta}(0,x_0,y_0) \tag{35}$$
$$\le \mathcal{L}^{v;\varepsilon}W_{\varepsilon,h}^{\delta}(s,x,y)+h(z)-\inf_{w}W_{\varepsilon,h}^{\delta}(T,z,w)+\sup_{w}W_{\varepsilon,h}^{\delta}(0,x_0,w)$$

Thus, $W_{\varepsilon,h}^{\varepsilon^2}(0,x_0,y_0)-c_0\left(1+\frac{1}{\varepsilon}\right)\varepsilon^2\le\eta_h(x_0)$. The first inequality in (26) and (25) yield that

$$\left|W_{\varepsilon,h}^{\varepsilon^2}(0,x_0,y_0)-W_h(x_0)\right|\le\left|W_{\varepsilon,h}^{\varepsilon^2}(0,x_0,y_0)-W_{\varepsilon,h}(x_0)\right|+\left|W_{\varepsilon,h}(x_0,y_0)-W_h(x_0)\right|$$
$$\le c_0\left(1+\frac{1}{\varepsilon}\right)\varepsilon^2+\omega(\varepsilon). \tag{36}$$

Consequently, passing to the limit as $\varepsilon\to 0$, we get

$$W_h(x_0)\le\eta_h(x_0). \tag{37}$$

By combining the inequalities (34) and (37) and recalling we have already proven that $W_h(x_0)=\Lambda_h(x_0,y_0)$, we complete the proof. $\square$

Remark 3.6 *If the estimates in (26) are independent of ε (e.g. by imposing a dissipativity condition on (g,β)), then one can prove that Λ_h can be defined with respect to the (explicit) set appearing in Proposition 3.1.*

A careful look at the proof, especially (35) and (36), tells us that

$$W_h(x_0)=\lim_{n\to\infty}W_{\frac{1}{n},h}^{\frac{1}{n^2}}(0,x_0,y_0)$$
$$\le\liminf_{n\to\infty}\inf_{(s,x,y,v,z,w)\in[t,T]\times K\times\mathbb{R}^N\times U\times K\times\mathbb{R}^N}\begin{pmatrix}\mathcal{L}^{v;\frac{1}{n}}W_{\frac{1}{n},h}^{\frac{1}{n^2}}(s,x,y)+h(z)\\ -W_{\frac{1}{n},h}^{\frac{1}{n^2}}(T,z,w)+W_{\frac{1}{n},h}^{\frac{1}{n^2}}(0,x_0,y_0)\end{pmatrix} \tag{38}$$

In particular, we deduce that $\Theta(x_0,y_0)$ can be replaced with

$$\widetilde{\Theta}(x_0,y_0)=\left\{\begin{array}{c}\gamma=\left(\gamma^1,\gamma^2\right)\in\mathcal{P}\left([0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right):\\ \exists\gamma_n\in\Theta\left(x_0,y_0;\frac{1}{n}\right),\gamma_n\rightharpoonup\gamma\text{ along some subsequence}\end{array}\right\}. \tag{39}$$

Moreover, if γ_n is an optimal measure for $W_{\frac{1}{n},h}$, one can find a subsequence converging to an optimal measure in $\Theta(x_0,y_0)$. Hence, one can also replace $\Theta(x_0,y_0)$ with

$$\Theta^{opt}(x_0,y_0)=\left\{\begin{array}{c}\gamma=\left(\gamma^1,\gamma^2\right)\in\mathcal{P}\left([0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^M\times\mathbb{R}^N\right):\\ \exists\gamma_n\in\Theta\left(x_0,y_0;\frac{1}{n}\right),\ \gamma_n\text{ is optimal for }W_{\frac{1}{n},h}\\ \left(\text{i.e.}\displaystyle\int_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}h(z)\,\gamma_n(dsdxdydzdw)=W_{\frac{1}{n},h}(x_0,y_0)\right),\\ \gamma_n\rightharpoonup\gamma\text{ along some subsequence.}\end{array}\right\}.\tag{40}$$

3.3 Characterization of optimal trajectories for the averaged system

As already mentioned in the introduction, when the perturbed system is fully nonlinear it is very difficult to characterize the optimal trajectories using the Pontryagin maximum principle because we do not know exactly the form of the averaged dynamics. An alternative to this method is to look at the support of the occupational measures contained in the set $\Theta(x_0,y_0)$ in order to obtain optimal trajectories from every $x_0\in K$. Following the approach already introduced in Subsection 2.2, we denote by

$$D_{\varepsilon,h}(x_0,y_0)=\left\{\begin{array}{l}(\eta,\phi)\in\mathbb{R}\times C_b^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^M\times\mathbb{R}^N\right)\text{ s.t.}\\ \eta=\displaystyle\inf_{(s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left(\begin{array}{c}T\mathcal{L}^{v;\varepsilon}\phi(s,x,y)+h(z)\\ -\phi(T,z,w)+\phi(0,x_0,y_0)\end{array}\right)\end{array}\right\},\tag{41}$$

for all $(x_0,y_0)\in K\times\mathbb{R}^N$. We can write

$$W_{\varepsilon,h}(x_0,y_0)=\sup\left\{\eta,(\eta,\phi)\in D_{\varepsilon,h}(x_0,y_0)\right\}\text{ and}$$
$$W_h(x_0)=\sup\left\{\begin{array}{c}\limsup_{\varepsilon\to0}\eta_\varepsilon:(\eta_\varepsilon,\phi_\varepsilon)\in D_{\varepsilon,h}(x_0,y_0),\\ \lim_{\varepsilon\to0}\left\|\phi_\varepsilon(\cdot,\cdot,\cdot)-\phi_\varepsilon(\cdot,\cdot,y_0)\right\|_\infty=0\end{array}\right\}.$$

At this point, we pick $\left(\eta_n,W_{\frac{1}{n},h}^{\frac{1}{n^2}}\right)\in D_{\frac{1}{n},h}(x_0,y_0)$ and recall that

$$\left|W_{\varepsilon,h}^{\varepsilon^2}(\cdot,\cdot,\cdot)-W_{\varepsilon,h}^{\varepsilon^2}(\cdot,\cdot,y_0)\right|_\infty\le2c_0\left(1+\frac{1}{\varepsilon}\right)\varepsilon^2+2\omega(\varepsilon),$$

and the second inequality in (26). Then $W_h(x_0) \geq \limsup_{n\to\infty} \eta_n$. Combining this inequality with (38) yields

$$W_h(x_0) = \lim_{n\to\infty} \inf_{(s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N} \begin{pmatrix} \mathcal{L}^{v;\frac{1}{n}} W_{\frac{1}{n},h}^{\frac{1}{n^2}}(s,x,y) + h(z) \\ -W_{\frac{1}{n},h}^{\frac{1}{n^2}}(T,z,w) + W_{\frac{1}{n},h}^{\frac{1}{n^2}}(0,x_0,y_0) \end{pmatrix}.$$

Similar to the approach of Subsection 2.2, we introduce the following. Whenever $(x_0,y_0)\in K\times\mathbb{R}^N$, we denote by

$$\Omega_{\frac{1}{n},h}^{simple}(x_0,y_0) = \left\{ \begin{array}{l} (s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N \text{ s.t.} \\ W_h(x_0) + \sqrt{|W_h(x_0)-\eta_n|} \\ \geq \mathcal{L}^{v;\frac{1}{n}} W_{\frac{1}{n},h}^{\frac{1}{n^2}}(s,x,y) + h(z) - W_{\frac{1}{n},h}^{\frac{1}{n^2}}(T,z,w) + W_{\frac{1}{n},h}^{\frac{1}{n^2}}(0,x_0,y_0) \end{array} \right\} \tag{42}$$

$$\Omega_{\frac{1}{n},h}^{double}(x_0,y_0) = \left\{ \begin{array}{l} (s,x,y,v,z,w)\in[t,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N \text{ s.t.} \\ W_{\frac{1}{n},h}(x_0) + \sqrt{W_{\frac{1}{n},h}(x_0)-\eta_n} \\ \geq \mathcal{L}^{v;\frac{1}{n}} W_{\frac{1}{n},h}^{\frac{1}{n^2}}(s,x,y) + h(z) - W_{\frac{1}{n},h}^{\frac{1}{n^2}}(T,z,w) + W_{\frac{1}{n},h}^{\frac{1}{n^2}}(0,x_0,y_0) \end{array} \right\} \tag{43}$$

and by

$$\Omega_h^{in}(x_0,y_0) := \bigcup_{n\geq 1}\bigcap_{k\geq n} \Omega_{\frac{1}{k},h}^{double}(x_0,y_0),\ \Omega_h^{out,cl}(x_0,y_0) := \bigcap_{n\geq 1} cl\left(\bigcup_{k\geq n} \Omega_{\frac{1}{k},h}^{double}(x_0,y_0)\right).$$

In the simple superscript case we use the same kind of construction as in the classical framework, while in the "double" case, we also approximate the target value W_h by $W_{\frac{1}{n},h}$ (or, equivalently, by $W_{\frac{1}{m},h}$ and then take the diagonal $n=m$). We get the following criteria of optimality.

Proposition 3.2 *Let $(x_0,y_0)\in K\times\mathbb{R}^N$ be fixed.*

(i) If $\gamma_n \in \Theta\left(x_0,y_0;\frac{1}{n}\right)$ is a (sub)sequence such that

$$\lim_{n\to\infty} n^3\gamma_n\left(\left(\Omega_{\frac{1}{n},h}^{simple}(x_0,y_0)\right)^c\right) = 0,$$

then any limit of γ_n is optimal.

(ii) Every $\gamma\in\Theta^{opt}(x_0,y_0)$ is optimal for W_h and

$$\gamma\left(\Omega_h^{out,cl}(x_0,y_0)\right) = 1 \tag{44}$$

(i.e. the support of γ is included in $\Omega_h^{out,cl}(x_0,y_0)$). Moreover, if $\Omega_h^{in}(x_0,y_0) = \Omega_h^{out,cl}(x_0,y_0)$ (i.e. the

limit of $\Omega_{\frac{1}{n},h}(x_0,y_0)$ exists), then

$$\sup_{n\geq 1}\gamma\left(\underset{k\geq n}{\cap}\Omega^{double}_{\frac{1}{k},h}(x_0,y_0)\right)=1.$$

Remark 3.7 *(i) The sufficient condition (ii) relies on finding (near) optimal measures $\gamma_n\in\Theta\left(x_0,y_0;\frac{1}{n}\right)$.*

(ii) The reader is invited to note that the condition (44) is the same as in the classical framework, see Proposition 2.3.

Let us come back to the proof of Proposition 3.2.

Proof. (i) Let us fix $\Theta\left(x_0,y_0;\frac{1}{n}\right)\ni\gamma_n$ as in our assertion and converging (along some subsequence) to some γ. The inequality (27) yields

$$\mathcal{L}^{v;\frac{1}{n}}W^{\frac{1}{n^2}}_{\frac{1}{n},h}(s,x,y)\leq c(1+n)n^2,$$

for some constant c independent of n. Then, recalling the definition of $\Omega^{simple}_{\frac{1}{n},h}(x_0,y_0)$, we get

$$\begin{aligned}
W_h(x_0)&\leq\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}h(z)\,\gamma(dsdxdydzdw)\\
&=\lim_{n\to\infty}\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}h(z)\,\gamma_n(dsdxdydzdw)\\
&\leq\lim_{n\to\infty}\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}\left[\begin{array}{c}T\mathcal{L}^{v;\frac{1}{n}}W^{\frac{1}{n^2}}_{\frac{1}{n},h}(s,x,y)+h(z)\\-W^{\frac{1}{n^2}}_{\frac{1}{n},h}(T,z,w)+W^{\frac{1}{n^2}}_{\frac{1}{n},h}(0,x_0,y_0)\end{array}\right]\gamma_n(dsdxdydzdw)\\
&\leq\limsup_{n\to\infty}\left[W_h(x_0)+\sqrt{|W_h(x_0)-\eta_n|}+\left(3\|h\|_\infty+c(1+n)n^2\right)\gamma_n\left(\left(\Omega^{simple}_{\frac{1}{n},h}(x_0,y_0)\right)^c\right)\right]\\
&=W_h(x_0).
\end{aligned}$$

It follows that γ is optimal.

(ii) If $\gamma\in\Theta^{opt}(x_0,y_0)$, then $\gamma\in\widetilde{\Theta}(x_0,y_0)$ is the limit of some (sub)sequence $\gamma_n\in\Theta\left(x_0,y_0;\frac{1}{n}\right)$ of optimal measures for $W_{\frac{1}{n},h}(x_0,y_0)$, by using (40). It is obvious that

$$\begin{aligned}
&\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}h(z)\,\gamma(dsdxdydzdw)\\
&=\lim_{n\to\infty}\int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N}h(z)\,\gamma_n(dsdxdydzdw)=\lim_{n\to\infty}W_{\frac{1}{n},h}(x_0,y_0),
\end{aligned}$$

hence γ is optimal. Since $\gamma_n \in \Theta\left(x_0, y_0; \frac{1}{n}\right)$ is optimal for $W_{\frac{1}{n},h}(x_0, y_0)$, it follows that

$$
\begin{aligned}
W_{\frac{1}{n},h}(x_0) &= \int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N} h(z)\,\gamma_n(dsdxdydzdw)\\
&= \int\limits_{[0,T]\times\mathbb{R}^M\times\mathbb{R}^N\times U\times\mathbb{R}^M\times\mathbb{R}^N} \left[\begin{array}{c} T\mathcal{L}^{v;\frac{1}{n}} W^{\frac{1}{n^2}}_{\frac{1}{n},h}(s,x,y) + h(z) \\ -W^{\frac{1}{n^2}}_{\frac{1}{n},h}(T,z,w) + W^{\frac{1}{n^2}}_{\frac{1}{n},h}(0,x_0,y_0) \end{array}\right] \gamma_n(dsdxdydzdw)\\
&\geq \eta_n \gamma_n\left(\Omega^{double}_{\frac{1}{n},h}(x_0,y_0)\right) + \left(W_{\frac{1}{n},h}(x_0) + \sqrt{W_{\frac{1}{n},h}(x_0) - \eta_n}\right)\gamma_n\left(\Omega^{double}_{\frac{1}{n},h}(x_0,y_0)\right).
\end{aligned}
$$

Then

$$
\gamma_n\left(\Omega^{double}_{\frac{1}{n},h}(x_0,y_0)\right) \geq \frac{1}{1+\sqrt{W_{\frac{1}{n},h}(x_0) - \eta_n}},
$$

for all $n \geq 1$. Hence,

$$
\gamma_n\left(cl\left(\underset{k\geq n_0}{\cup}\Omega^{double}_{\frac{1}{k},h}(x_0,y_0)\right)\right) \geq \frac{1}{\left(1+\sqrt{W_{\frac{1}{n},h}(x_0) - \eta_n}\right)},
$$

for all $n \geq n_0$. Passing to the $\lim\limits_{n\to\infty}$, one gets $\gamma\left(cl\left(\underset{k\geq n_0}{\cup}\Omega^{double}_{\frac{1}{k},h}(x_0,y_0)\right)\right) = 1$, for all $n_0 \geq 1$ and the proof is complete. □

Remark 3.8 *(i) If a suitable monotonicity can be established for the approximating problems* $W_{\frac{1}{n},h}(x_0,y_0)$, *then, one can envisage the use of the dual formulation in Theorem 3.1 to infer necessary conditions similar to those in Proposition 2.3.*

(ii) When the inclusion (3.1) is an equality (see Remark 3.6 (i)), one can employ convex duality arguments to get another dual formulation for the limit value. One can, for instance, adapt the method of [11, Theorem 1]. This dual formulation would be very similar to the classical case and the ingredients of Proposition 2.3 apply. The main drawback in this approach is that, unlike the classical case, we have no information on the structure of the test functions ϕ_n *in the (almost-) optimal pairs.*

4 Perspectives

We wish to emphasize that the method allowing to deduce optimality criteria in the study of singularly perturbed control systems does not depend on particular properties of the Brownian setting. Instead, it strongly relies on the ability to linearize the approximating problems. In particular, this can be applied to larger classes of systems (e.g. Piecewise Deterministic Markov Processes presenting a mild path-dependance; see [25, 26] for the linearization techniques). For this particular class, little is available in the literature. The applications include but are not limited at multi-scale stochastic gene networks , reliability and traffic on random networks.

This paper is a first step in the study of optimal policies for singularly perturbed differential dynamics with random perturbations. This opens the way to compute strict optimal (or nearly-optimal) control policies following the approach of [27] for classical control problems. Also, numerical methods allowing to compute the optimal value function and, hence, the support set, are in progress. They follow the hints of [8] and rely on the dual linear formulation for the approximating problems.

We also wish to point out that, in all its generality and without further assumptions, the question of equivalent (necessary AND sufficient) criteria for optimality in the control of singularly perturbed control systems remains an open problem.

5 Conclusions

In this paper we have studied the optimality issues for a class of singularly perturbed controlled stochastic systems driven by a finite-dimensional Brownian motion. This is done via linear programming techniques by embedding the controlled trajectories for the scaled system in a larger class of probability measures. Using compactness techniques and passing to the limit we have achieved two goals. First, we have proposed linearized formulations (primal and dual) for the limit system whose dynamics are difficult to identify. Second, using these formulations, we have given a class of necessary and a class of sufficient criteria allowing to identify the optimal measures for the limit system. These conditions concern the support of the candidates to optimality belonging to the class of occupation measures.

The main advantage of the method is that it is independent of the knowledge of the limit differential dynamics which are often very difficult to obtain. The drawback of the method is that it relies on computing several approximating value functions or optimal measures for the approximating problems. Although the computational price might be high, this method is, to our best knowledge, the first method which does not rely on further information on the limit system (which might, itself be a high-cost issue).

Further numerical studies and application to different Markov-structured systems are in progress.

6 Appendix

6.1 Proof of Proposition 2.2

Proof. We begin with assuming that $\gamma \in \Theta(x)$ is such that $\gamma\left(\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)\right) = 1$, i.e. the support of γ is included in $\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)$. Then, by definition, we have the following equality

$$\bar{\eta} = T\mathcal{L}^v\bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x),$$

on $\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)$. Consequently, recalling the definition of $\Theta(x)$, one gets

$$V_{l_2}(x) = \int_{[0,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} \bar{\eta}\gamma(dsdxdudz) = \int_{[0,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} l_2(z)\,\gamma(dsdxdudz),$$

i.e. $\gamma \in \Theta(x)$ is optimal. Conversely, let us consider some optimal $\gamma \in \Theta(x)$. One writes

$$\begin{aligned}
&V_{l_2}(x) = \textstyle\int_{[t,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} l_2(z)\,\gamma(dsdxdudz) = \\
&\textstyle\int_{[0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left[T\mathcal{L}^v\bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)\right]\gamma(dsdxdudz).
\end{aligned}$$

for all optimal pairs $(\bar{\eta} = V_{l_2}(x), \bar{\phi}) \in D_{l_2}(x)$. By the definition of $D_{l_2}(x)$, it follows that

$$\left[T\mathcal{L}^v\bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)\right] \geq V_{l_2}(x).$$

Hence, in order for the previous equality to hold, one has

$$\left[T\mathcal{L}^v\bar{\phi}(s,y) + l_2(z) - \bar{\phi}(T,z) + \bar{\phi}(t,x)\right] = V_{l_2}(x),$$

γ-almost everywhere. Hence, the support of γ is included in $\Omega_{l_2,(\bar{\eta},\bar{\phi})}(x)$ and the proof is now complete.

□

Remark 6.1 *One can construct a set which is independent of the choice of optimal pairs $(\bar{\eta},\bar{\phi}) \in D_{l_2}(x)$. Indeed, in the case where the state space K is compact, the set $C_b^{1,2}([0,T]\times K)$ is compact. The family of optimal test functions is denoted by $Opt_{l_2}(x)$ and is totally bounded with respect to the usual topology of $C_b^{1,2}$. For every $n \geq 1$, we select a finite family $\left(\bar{\phi}_j^n\right)_{1\leq j\leq k^n} \subset Opt_{l_2}(x)$ such that, for every $\bar{\phi} \in Opt_{l_2}(x)$*

$$d\left(\bar{\phi},\bar{\phi}_j^n\right) \leq \frac{1}{n^2},$$

for some $1 \leq j \leq k^n$. The distance is given in the sense of $C_b^{1,2}([0,T] \times K)$ functions. We define

$$\Omega_{l_2}(x) := \underset{n\geq 1, 1\leq j\leq k^n}{\cap} \Omega_{l_2,(\bar{\eta},\bar{\phi}_j^n)}(x).$$

Due to the previous proposition, whenever γ is optimal, $\gamma(\Omega_{l_2}(x)) = 1$. The converse also holds true. If no invariant compact can be found for the system, a localization procedure can be developed starting from Remark 2.1.

6.2 Proof of Proposition 2.3

Proof. We begin with assuming that $\gamma \in \Theta(x)$ is optimal (should it exist). Then

$$V_{l_2}(x) = \int_{[t,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} l_2(z)\,\gamma(dsdxdudz) =$$
$$\int_{[0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} [T\mathcal{L}^v\phi_k(s,y) + l_2(z) - \phi_k(T,z) + \phi_k(0,x)]\,\gamma(dsdxdudz),$$

for all $k \geq 1$. The definition of $D_{l_2}(x)$ yields

$$[T\mathcal{L}^v\phi_k(s,y) + l_2(z) - \phi_k(T,z) + \phi_k(0,x)] \geq \eta_k,$$

for all $(s,y,v,z) \in [0,T] \times \mathbb{R}^N \times U \times \mathbb{R}^N$ and all $k \geq 1$. Thus,

$$V_{l_2}(x) \geq \eta_k\gamma\left(\Omega_{l_2}^k(x)\right) + \left(V_{l_2}(x) + \sqrt{V_{l_2}(x) - \eta_k}\right)\gamma\left(\left(\Omega_{l_2}^k(x)\right)^c\right).$$

It follows that

$$\left(1 + \sqrt{V_{l_2}(x) - \eta_k}\right)\gamma\left(\Omega_{l_2}^k(x)\right) \geq 1.$$

Passing to the $\limsup_{k\to\infty}$, one gets $\gamma\left(\Omega_{l_2}^{out}(x)\right) = 1$ and the proof is complete.

For the converse, let us assume that $\gamma\left(\underset{k\geq n_0}{\cap}\Omega_{l_2}^k(x)\right) = 1$. Then $\gamma\left(\Omega_{l_2}^k(x)\right) = 1$, for all $k \geq n_0$. It follows that

$$\begin{aligned}
V_{l_2}(x) &\leq \int_{[0,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} l_2(z)\,\gamma(dsdxdudz) \\
&= \int_{[0,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} (T\mathcal{L}^v\phi_k(s,y) + l_2(z) - \phi_k(T,z) + \phi_k(0,x))\,\gamma(dsdxdudz) \\
&\leq V_{l_2}(x) + \sqrt{V_{l_2}(x) - \eta_k},
\end{aligned}$$

for all $k \geq n_0$. Passing to the limit as $k \to \infty$, one gets that $\int_{[0,T]\times\mathbb{R}^m\times U\times\mathbb{R}^m} l_2(z)\,\gamma(dsdxdudz) = V_{l_2}(x)$, i.e. $\gamma \in \Theta(x)$ is optimal. $\square$

Remark 6.2 *If an optimal pair exists for our control problem then, due to the Remark 2.3, a measure $\gamma \in \Theta(x)$ is optimal if and only if* $\gamma\left(\Omega_{l_2}^{in/out}(x)\right) = 1$.

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