Orthogonal polynomials and diffusions operators

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Abstract

Generalizing the work of [5, 41], we give a general solution to the following problem: describe the triplets (Ω, g, μ) where $g = (g^{ij}(x))$ is the (co)metric associated to the symmetric second order differential operator $L(f) = \frac{1}{\rho}\sum_{ij} \partial_i(g^{ij}\rho\partial_j f)$, defined on a domain Ω of $\mathbb{R}^d$ and such that L is expandable on a basis of orthogonal polynomials on $\mathcal{L}^2(\mu)$, and $d\mu = \rho(x)dx$ is some admissible measure. Up to affine transformations, we find 11 compact domains Ω in dimension $d = 2$. We also give some aspects of the non-compact cases in this dimension.

Contents

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1 Introduction

In this paper, we investigate the following question: for a given set $\Omega \subset \mathbb{R}^d$, can we describe a probability measure $\mu(dx)$ on Ω, absolutely continuous with respect to the Lebesgue measure, and an elliptic diffusion operator

$$\mathbf{L}(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f,$$

defined on Ω such that there exists orthonormal basis for $\mathcal{L}^2(\mu)$, formed by orthogonal polynomials ordered according to the total degree, which are eigenvectors of the operator $\mathbf{L}$.

In dimension 1, given the measure μ, there is a unique family of associated orthogonal polynomials, up a choice of sign. Some of them share extra properties, and as such are widely used in many areas. This is in particular the case of Hermite, Laguerre and Jacobi polynomials, which correspond respectively to the measures with density $Ce^{-x^2/2}$ on $\mathbb{R}$, $C_a x^{a-1}e^{-x}$, $a > 0$, on $[0, \infty)$ and $C_{a,b}(1-x)^{a-1}(1+x)^{b-1})$, $a, b > 0$, on $[-1, 1]$ (where $C, C_a, C_{a,b}$ are normalizing constants which play no role here). In those 3 cases, and only in those ones, the associated polynomials are eigenvectors of some second order differential operator $\mathbf{L}$: see the classification of [5, 41]. Those families have been extensively studied, since they play a central role in probability, analysis, partial differential equations, geometry... (see e.g. [17, 55, 20, 18, 19, 52, 53], see also [21],[45] and references therein).

The differential operator $\mathbf{L}$ may be replaced by some other generator of Markov semigroup (finite difference, or q-difference operators) and the orthogonal polynomials eigenfunctions are Hahn, Kravtchouk, Charlier, Meixner (see [42]). But even in dimension 1, no classification had ben done for such families, beyond the case of diffusions (see [5]).

The aim of this paper is to generalize the dimension 1 classification for differential operators to higher dimensions, and in particular in dimension 2, to give a precise description of the differential operators, the measures and the domains concerned.

For bounded sets $\Omega \subset \mathbb{R}^d$ with piecewise smooth boundary, one may reduce the problem to some algebraic question about the boundary. Only in dimension 2, this problem may be solved entirely, and we provide the complete list of 11 different bounded sets $\Omega \subset \mathbb{R}^2$ which, up to affine transformations, are the only ones on which this problem have a solution. We also provide a description of the associated measures and operators. Under stronger requirements on the sets, we also provide a list of the 7 non compact models which solve the problem in dimension 2.

Further extension to higher dimensional models are provided, although a classification seems out of reach with the techniques provided here.

Orthogonal polynomials are a long standing subject of investigation in mathematics. They describe natural Hilbert bases in $\mathcal{L}^2(\mu)$ spaces, where μ is a probability measure on some measurable set Ω in $\mathbb{R}^d$ for which polynomials are dense. As a way to describe functions $f : \Omega \mapsto \mathbb{R}$, they are widely used in analysis to describe many problems in partial differential equations, especially when they present some quadratic non linearities: since products are in general easy to compute in such polynomial bases, approximation schemes which consist in restricting the approximation of functions to a finite number of components in those bases are easy to implement in practice.

In higher dimension, there are several choices for a base of orthogonal polynomials. As such, they are far less simple to describe. However, many families have been described in various settings. In particular, extension of the previous families (in the sense that they are eigenvectors of differential operators) have been described by many authors (see Koornwinder [30, 31, 32, 33, 6, 36, 35], Krall-Scheffer [37], Heckman [27], Rösler [46]; see also [2] for a generalization of the Rodrigues formula). For a general overview on orthogonal polynomials of several variables, we refer to Koornwinder [34] , Suetin [51] and to the book of Dunkl and Xu [16].

In general, in dimension $d \geq 2$, one orders polynomials by their total degree: if $\mathcal{P}_n^d$ denotes the set of polynomials in d variables with degree less than n, we are looking for an Hilbert basis of $\mathcal{L}^2(\mu)$ such that for each n, we get a finite-dimensional basis of $\mathcal{P}_n^d$. This basis is not unique in general. This is what we call an orthogonal polynomial basis, and is the object of our study. Observe that other partial ordering of the set of polynomials would lead to a different classification.

On the other hand, these polynomial bases are not always the best choice to develop functions or to obtain good approximation schemes. This is in particular the case in probability theory, when one is concerned with symmetric diffusion processes as they naturally appear as solutions of stochastic differential equations. Indeed, a Markov diffusion process $(X_t)_{t\geq 0}$, with continuous trajectories on an open set of $\mathbb{R}^d$ or a manifold, has a law entirely characterized by the family of Markov kernels $(P_t)_{t\geq 0}$:

$$P_t(f)(x) = \mathbb{E}(f(X_t)/X_0 = x)\,, x \quad x \in \mathbb{R}^d\,,$$

where f is in a suitable class of functions. The infinitesimal generator $\mathbf{L}$ associated to $(P_t)_{t\geq 0}$ is defined by

$$\mathbf{L}f = \lim_{t\to\infty} \frac{P_t f - f}{t}\,,$$

whenever this limit exists.

This operator governs the semigroup in the sense that if $F(x,t) = P_t(f)(x)$, then F is the solution of the heat equation

$$\partial_t F = \mathbf{L}F,\ F(x,0) = f(x).$$

It is quite difficult in general to obtain a complete description of P_t in terms of the operator $\mathbf{L}$, which is in general the only datum that one has at hand from the description of (X_t), for example as the solution of a stochastic differential equation.

This operator $\mathbf{L}$ is a second order differential operator with no zero order component, moreover semi-elliptic, of the form

$$L(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f\,. \tag{1.1}$$

Although not easy to compute explicitly, the operator P_t, which describes the law of the random variable X_t, has a nice expression at least when $\mathbf{L}$ is self-adjoint with respect to some measure μ, and when the spectrum is discrete (μ is then said to be the reversible measure for (X_t)). When μ has a density ρ which is $\mathcal{C}^1$ with respect to the Lebesgue measure, and the coefficients g^{ij} are also assumed to be at least $\mathcal{C}^1$, then this latter case amounts to look for operators $\mathbf{L}$ of the form

$$\mathbf{L}(f) = \frac{1}{\rho}\sum_{ij} \partial_i(g^{ij}\rho\partial_j f). \tag{1.2}$$

In this paper, we shall restrict our attention to operators which are elliptic in the interior of the support of μ. Such an operator described in (1.2) will be called a symmetric diffusion operator.

In the case under study, the spectral decomposition leads to some more or less explicit representation. Namely, if there is an orthonormal basis (e_n) of $\mathcal{L}^2(\mu)$ composed of eigenvectors of $\mathbf{L}$,

$$\mathbf{L}e_n = -\lambda_n e_n,$$

then one has

$$P_t(f)(x) = \int f(y)p_t(x,y)d\mu(y),$$

where

$$p_t(x,y) = \sum_n e^{-\lambda_n t} e_n(x)e_n(y).$$

For fixed x, the function $p_t(x,y)$ represents the density with respect to $\mu(dy)$ of the law of X_t when $X_0 = x$. Of course, this representation is a bit formal, since that one has to insure first that this series converges, which requires P_t to be trace class, or Hilbert-Schmidt. However, even if it is quite rare that the eigenvalues λ_n and the eigenvectors e_n are explicitly known, it can be of great help to know that such a decomposition exists: it provides a good approximation of P_t when t goes to infinity, and as such allows to control convergence to equilibrium. But even when one explicitly knows the eigenvectors and eigenvalues, it is not always easy to extract many useful informations from the previous description. It is even not immediate to check in general that the previous expansion leads to non negative functions.

Even when $\mathbf{L}$ is elliptic and symmetric,it's knowledge, given on say smooth function compactly supported in Ω, is not enough to describe the associated semigroup P_t or any self-adjoint extension of $\mathbf{L}$. One requires in general some boundary conditions. This requirement will be useless in our context, since we shall impose the eigenvectors to be polynomials. As a counterpart, this will impose some boundary condition on the operator itself.

As mentioned earlier, we are interested in the description of the situation when the previous eigenvector expansion coincides with a family of orthogonal polynomials associated with the reversible measure. Although the situation is well known and described in dimension 1, such description is not known in higher dimension, apart from some generic families. At least when the set Ω is relatively compact with $\mathcal{C}^1$ piecewise boundary, and when the reversible measure μ has a $\mathcal{C}^1$ density with respect to the Lebesgue measure, we may turn the complete description of this situation into some problem of algebraic nature: the operators and the measures are entirely described by the boundary of Ω, which is some algebraic surface with degree at most $2d$ in dimension d. Then, we completely solve this problem in dimension 2, leading, up to affine transformations, to the 11 different possible boundaries: the square, the circle, the triangle, the coaxial parabolas, the parabola with one tangent and one secant, the parabola with two tangents, the nodal cubic,the cuspidal cubic with one secant line, the cuspidal cubic with one tangent, the swallow tail and the deltoid.

Once the boundary is known, the possible measures are completely described from some parameters (as many parameters than irreducible components in the minimal equation of the boundary of Ω). It turns out that in many situations, for some half integer values of these parameters, the associated operator has a natural geoemetric interpretation in terms of Lie group action on symmetric spaces. We then provide explicitly many of these interpretations whenever they are at hand.

We also show that when $\Omega = \mathbb{R}^2$ (that is when the density ρ of μ is everywhere positive), the only possible measures are Gaussian. Under some extra hypothesis, we also provide some classification of the non compact models. Further extensions to higher dimension are also provided.

The paper is organized as follows. In Section 2, after some rapid overview of the dimension 1 case, we describe the general setting in any dimension, and, when the set Ω is relatively compact with piecewise smooth boundary, show how to reduce the description to the classification of some algebraic surfaces in $\mathbb{R}^d$. We also describe the various associated measures from the description of the boundary of Ω.

Then, Section 3 is devoted to the classification of the compact dimension 2 models, which leads to 11 different cases up to affine transformations. Section 4 provides a more detailed description of the 11 models, with some insight on their geometric content for various values of the parameters. Section 5 describes the case where no boundary is present, and the main result of this section is that the only possible measures are Gaussian ones. Section 6 describes the non compact cases under some extra assumption which extends the natural condition of the compact case. Finally, Section 7 provides some way of constructing 3-dimensional models from 2-dimensional ones.

2 Diffusions associated with orthogonal polynomials

2.1 Dimension 1

As mentioned previously, the one dimension case has been totally described for a long time (see e.g. [5, 41]). We recall here briefly the framework and results.

Let μ be a finite measure absolutely continuous with respect to the Lebesgue measure on an open interval I of $\mathbb{R}$ with $\mathcal{C}^1$ density ρ (we may assume μ is a probability measure), for which polynomials are dense in $\mathcal{L}^2(\mu)$ (this is automatic when I is bounded, but it is enough for that in general that μ has some exponential moment). Let $(Q_n)_{n\geq 0}$ be the family of orthogonal polynomials obtained from the sequence $(x^n)_{n\geq 0}$ by orthonormalization, e.g. by the Gram–Schmidt process (the normalization of Q_n plays no role in what follows). Assume furthermore that some elliptic diffusion operator $\mathbf{L}$ of type (1.2) exists on I (and therefore $\rho(x)\,dx$ is its reversible measure), such that for some sequence (λ_n) of real numbers,

$$\mathbf{L}Q_n = -\lambda_n Q_n\,.$$

Then up to affine transformations, I, μ and $\mathbf{L}$ may be reduced to one of the three following cases:

1. The Ornstein–Uhlenbeck operator on $I = \mathbb{R}$

$$H = \frac{d^2}{dx^2} - x\frac{d}{dx}\,,$$

the measure μ is Gaussian centered: $\mu(dx) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$. The family $(Q_n)_n$ are the Hermite polynomials, $\lambda_n = n$.

2. The Laguerre operator (or squared radial generalized Ornstein–Uhlenbeck operator) on $I = \mathbb{R}_+^*$

$$L_a = x\frac{d^2}{dx^2} + (a-x)\frac{d}{dx}\,,\quad a>0,$$

the measure $\mu_a(dx) = C_a x^{a-1}e^{-x}\,dx$. The family $(Q_n)_n$ are the Laguerre polynomials, $\lambda_n = n$.

3. The Jacobi operator on $I = (-1,1)$

$$J_{a,b} = (1-x^2)\frac{d^2}{dx^2} - \big(a-b+(a+b)x\big)\frac{d}{dx}\,,\quad a,b>0$$

the measure $\mu_{a,b}(dx) = C_{a,b}(1-x)^{a-1}(1+x)^{b-1}\,dx$, the family $(Q_n)_n$ are the Jacobi polynomials, $\lambda_n = n(n+a+b-1)$.

The first two families appear as limits of the Jacobi case. For example, when we chose $a=b=n/2$ and let then n go to ∞, and scale the space variable x into $x/\sqrt{n}$, the measure $\mu_{a,a}$ converges to the Gauss measure, the Jacobi polynomials converge to the Hermite ones, and $\frac{2}{n}J_{a,a}$ converges to H.

In the same way, the Laguerre setting is obtained from the Jacobi one fixing b, changing x into $\frac{2x}{a}-1$, and letting a go to infinity. Then $\mu_{a,b}$ converges to μ_b, and $\frac{1}{a}J_{a,b}$ converges to L_b.

Also, when a is a half-integer, the Laguerre operator may be seen as the image of the Ornstein–Uhlenbeck operator in dimension d. Indeed, as the product of one dimensional Ornstein–Uhlenbeck operators, the latter has generator $H_d = \Delta - x.\nabla$. It's reversible measure is $e^{-|x|^2/2}dx/(2\pi)^{d/2}$, it's eigenvectors are the products $Q_{k_1}(x_1)\cdots Q_{k_d}(x_d)$, and it's associated process $X_t = (X_t^1,\cdots,X_t^d)$, is formed of independent one dimensional Ornstein-Uhlenbeck processes. Then, if one considers $R(x) = |x|^2$, then one may observe that, for any smooth function $F:\mathbb{R}_+\mapsto\mathbb{R}$,

$$H_d\big(F(R)\big) = 2L_a(F)(R),$$

where $a = d/2$. In the probabilist interpretation, this amounts to observe that if X_t is a d-dimensional Ornstein–Uhlenbeck process, then $|X_{t/2}|^2$ is a Laguerre process with parameter $a = d/2$.

In the same way, when $a = b = d/2$, $J_{a,a}$ may be seen as the Laplace operator Δ_{S^n} on the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ acting on functions depending only on the first coordinate (or equivalently on functions invariant under the rotations leaving $(1, 0, \cdots, 0)$ invariant), which may be interpreted as the fact that the first coordinate of a Brownian motion on the unit sphere is a diffusion process with generator $J_{d/2,d/2}$. A similar interpretation is valid for $J_{p/2,q/2}$ for some integers p and q. Namely, when one looks at the unit sphere $\mathbb{S}^{p+q-1} \in \mathbb{R}^{p+q}$, and consider functions on $\mathbb{S}^{p+q-1}$ depending only on $X = x_1^2 + \cdots x_p^2$. Then, setting $Y = 2X - 1 : \mathbb{S}^{p+q-1} \mapsto [-1, 1]$, for any smooth function $f : [-1, 1] \mapsto \mathbb{R}$, $\Delta_{\mathbb{S}^{p+q-1}} f(Y) = 4J_{p,q}(f)(Y)$. Once again, the associated Jacobi process may be seen as the image of a Brownian motion on the $(p+q-1)$-dimensional sphere through the function $Y = 2X - 1$. This interpretation comes from Zernike and Brinkman [8] and Braaksma and Meulenbeld [7] (see also Koornwinder [29]). We shall come back to such interpretations in Section 4. Jacobi polynomials also play a central role in the analysis on compact Lie groups. Indeed, for (a, b) taking the various values of $(q/2, q/2)$, $((q-1)/2, 1)$, $(q-1, 2)$, $(2(q-1), 4)$ and $(4, 8)$ the Jacobi operator $J_{a,b}$ appears as the radial part of the Laplace-Beltrami (or Casimir) operator on the compact rank 1 symmetric spaces, that is spheres, real, complex and quaternionic projective spaces, and the special case of the projective Cayley plane (see Sherman [48]).

2.2 General setting

We now state our problem in full generality, and describe the framework we are looking for. In this section, we describe the general problem (DOP, Definition 2.1) as stated above, and we further consider a more constrained one (SDOP, Definition 2.5). It turns out that they are equivalent whenever the domain Ω is bounded, and that the latter is much more easy to handle.

Let Ω be some open set in $\mathbb{R}^d$, with piecewise $\mathcal{C}^1$ boundary. Let $\mathbf{L}$ be a diffusion operator with smooth coefficients on Ω, i.e. $\mathbf{L}$, acting on smooth compactly supported function in Ω, writes

$$\mathbf{L}(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f, \tag{2.3}$$

where g^{ij} and b^i are smooth functions on Ω, and the matrix (g^{ij}) is symmetric, positive definite for any $x \in \Omega$. This last assumption of ellipticity could be relaxed to the weaker one of hypoellipticity, but many of the examples studied below rely in an essential way on it. Moreover, diffusion operators (operators such that the associated semigroups are Markov operators) require at least that $\mathbf{L}$ are semi elliptic, therefore that the matrices (g^{ij}) are non negative. In the sequel, we shall use the square field operator, in constant use throughout (see [4]).

$$\mathbf{\Gamma}(f, g) = \sum_{ij} g^{ij}\partial_i f \partial_j g = \frac{1}{2}\Big(\mathbf{L}(fg) - f\mathbf{L}(g) - g\mathbf{L}(f)\Big), \tag{2.4}$$

and observe that for any smooth function $\Phi : \mathbb{R}^k \mapsto \mathbb{R}$ and any k-uple of smooth functions $\mathbf{f} = (f_1, \cdots, f_k)$ $f_i : \Omega \mapsto \mathbb{R}$, one has

$$\mathbf{L}\big(\Phi(f_1, \cdots, f_k)\big) = \sum_{ij} \partial^2_{ij}\Phi(\mathbf{f})\mathbf{\Gamma}(f_i, f_j) + \sum_i \partial_i\Phi(\mathbf{f})\mathbf{L}(f_i). \tag{2.5}$$

We also consider some probability measure $\mu(dx) = \rho(x)dx$ with smooth density ρ on Ω for which polynomials are dense in $\mathcal{L}^2(\mu)$. This last assumption is automatic as soon as Ω is relatively compact (in which case polynomials are even dense in any $\mathcal{L}^p(\mu)$, $1 \leq p < \infty$). It would require some extra-assumption on μ in the general case. For example, it is enough for this to hold to require that μ has some exponential moments: $\int_\Omega e^{e\|x\|}d\mu(x) < \infty$ for some $\epsilon > 0$, in which case polynomials are as before dense in every $\mathcal{L}^p(\mu)$, $1 \leq p < \infty$.

The fundamental question is to study whether there exists an orthonormal basis (P_n) of polynomials in $\mathcal{L}^2(\mu)$ which are eigenvectors for $\mathbf{L}$, that is that there exist some real numbers (λ_n) with $\mathbf{L}P_n = -\lambda_n P_n$. Such eigenvalues (λ_n) turn out to be necessarily non negative (this is a general property of symmetric diffusion operators, as a direct consequence on the non-negativity of $\mathbf{\Gamma}$).

Recall that $\mathcal{P}^d_n$ denotes the vector space of polynomials in d variables and total degree smaller than n, and denote by $\mathcal{H}^d_n$ the space of polynomials with total degree n, orthogonal to $\mathcal{P}^d_{n-1}$ in $\mathcal{P}^d_n$. Then

$$\dim \mathcal{P}^d_n = \binom{n+d}{n}, \text{ and } \dim \mathcal{H}^d_n = \binom{n+d-1}{n}.$$

The choice of an orthonormal basis made of polynomials in $\mathcal{L}^2(\mu)$ amounts to the choice of a basis for $\mathcal{H}^d_n$, for any n.

From expression (2.3) one sees that $\mathbf{L}$ maps $\mathcal{P}^d_n$ into $\mathcal{P}^d_n$ and $\mathcal{H}^d_n$ into $\mathcal{H}^d_n$. Moreover, when $P \in \mathcal{P}^d_n$ and $Q \in \mathcal{P}^d_m$, $\mathbf{\Gamma}(P, Q) \in \mathcal{P}^d_{n+m}$.

The restriction of $\mathbf{L}$ to $\mathcal{P}^d_n$ being symmetric for any n, one has, for any pair (P, Q) of polynomials

$$\int_\Omega P\mathbf{L}(Q)d\mu = \int Q\mathbf{L}(P)d\mu. \tag{2.6}$$

Using (2.6) with $Q = 1$ leads to $\int_\Omega \mathbf{L}(P)\, d\mu = 0$ for any polynomial. Applying this to PQ together with the definition of the operator $\mathbf{\Gamma}$, one gets, for any pair (P, Q) of polynomials

$$\int_\Omega P\mathbf{L}(Q)\, d\mu = \int_\Omega Q\mathbf{L}(P)\, d\mu = -\int_\Omega \mathbf{\Gamma}(P, Q)\, d\mu. \tag{2.7}$$

Applying with $P = Q = P_n$, since $\mathbf{\Gamma}(P_n, P_n) \geq 0$, one sees that $\lambda_n \geq 0$.

From equation (2.7), we see that the restriction of $\mathbf{L}$ to polynomials is entirely determined by $\mathbf{\Gamma}$ (hence by the matrices $(g^{ij}(x))_{x\in\Omega}$), and the measure μ.

This leads us to state our problem in the following way

Definition 2.1 (DOP problem)**.** *Let Ω be an open set with piecewise smooth boundary (which may be empty), $\mu(dx) = \rho(x)dx$ a probability measure with smooth positive density on Ω, such that polynomials are dense in $\mathcal{L}^2(\mu)$, and let $\mathbf{L}$ an elliptic diffusion operator with smooth coefficients on Ω. We say that $(\Omega, \mathbf{L}, \mu)$ is a solution to the Diffusion-Orthogonal Polynomials problem (in short DOP problem) if there exists a complete basis of $\mathcal{L}^2(\mu)$ formed with orthogonal polynomials which are at the same time eigenvectors for the operator $\mathbf{L}$.*

Let us start with few elementary remarks. The first basic but important observation is that our problem is invariant under affine transformations:

Proposition 2.2. *If $(\Omega, \mathbf{L}, \mu)$ is a solution to the DOP problem, and if $A = (A^1, \dots, A^d)$ is an affine invertible transformation of $\mathbb{R}^d$, so is $(\Omega_1, \mathbf{L}_1, \mu_1)$, where $\Omega_1 = A(\Omega)$, μ_1 is the image measure through A of μ and*

$$\mathbf{L}_1(f) = \mathbf{L}(f \circ A) \circ (A^{-1}).$$

Proof — Affine transformations map polynomials onto polynomials with the same degree. It suffices then to see that the associated operator $\mathbf{L}_1(f) = \mathbf{L}(f \circ A) \circ A^{-1}$ is again a diffusion operator, which has a family of orthogonal polynomials as eigenvectors. Moreover, orthogonality for the measure μ is carried to orthogonality for the measure μ_1.

∎

Moreover, the following Proposition shows that solutions to the DOP problem are stable under products

Proposition 2.3. *If $(\Omega_1, \mathbf{L}_1, \mu_1)$ and $(\Omega_2, \mathbf{L}_2, \mu_2)$ are solutions to the DOP problem in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively, then $(\Omega_1 \times \Omega_2, \mathbf{L}_1 \oplus \mathbf{L}_2, \mu_1 \otimes \mu_2)$ is also a solution.*

Proof — Here $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$ denotes the operator acting separately on x and y: $\mathbf{L}f(x,y) = \mathbf{L}_x f + \mathbf{L}_y f$. Similarly, $\mu_1 \otimes \mu_2$ is the product measure. The proof is then immediate: if $(P_k^{(1)})$ and $(P_q^{(2)})$ are the associated families of orthogonal polynomials, the polynomials associated to $\mathbf{L}$ are $P_{k,q}(x,y) = P_k^{(1)}(x)P_q^{(2)}(y)$.

∎

Next, we describe the general form of the coefficients of the operator $\mathbf{L}$.

Proposition 2.4. *If $\mathbf{L}$ is a solution to the DOP problem, in the representation* (2.3) *of $\mathbf{L}$, for any $i = 1, \cdots, d$, $b^i(x) \in \mathcal{P}_1^d$ and for any $i, j = 1, \cdots, d$, $g^{ij}(x) \in \mathcal{P}_2^d$.*

Proof — Since $\mathbf{L}$ maps $\mathcal{P}_n^d$ into $\mathcal{P}_n^d$ for any $n \in \mathbb{N}$, this follows from the fact that $b^i(x) = \mathbf{L}(x_i)$ and $g^{ij}x) = \Gamma(x_i, x_j)$.

∎

The integration by parts formula (2.7) is valid for polynomials only. It may be interesting (and crucial) to extend it to any smooth compactly supported functions. Tis leads us to the Strong Diffusion Orthogonal Polynomials problem.

Definition 2.5 (SDOP problem)**.** *The triple $(\Omega, \mathbf{L}, \mu)$ satisfies the Strong Diffusion Orthogonal Polynomial problem (SDOP in short) if it satisfies the DOP problem (Definition 2.1) and in addition, for any f and g smooth and compactly supported in $\mathbb{R}^d$, one has*

$$\int_\Omega f\mathbf{L}(g)\, d\mu = \int_\Omega g\mathbf{L}(f)\, d\mu. \tag{2.8}$$

Observe that we do not restrict formula (2.8) to functions f and g compactly supported in Ω. However, restricting (2.8) to such functions, and writing $\mu(dx) = \rho(x)\, dx$, for any f smooth and compactly supported in Ω, $\mathbf{L}(f)$ may be defined by formula

$$\mathbf{L}(f) = \frac{1}{\rho} \sum_{ij} \partial_i \Big(g^{ij} \rho \partial_j f \Big), \tag{2.9}$$

and therefore $\mathbf{L}$ is entirely determined from the (co)metric $g = (g^{ij})$ and the measure density $\rho(x)$. We therefore talk about the triple (Ω, g, ρ) as a solution of the SDOP problem.

Moreover, when (2.8) is valid, for any pair of smooth functions compactly supported in Ω, we also have as before

$$\int_\Omega f\mathbf{L}(g)\, d\mu = -\int_\Omega \Gamma(f, g)\, d\mu. \tag{2.10}$$

Proposition 2.6. *If (Ω, g, μ) is a solution to the SDOP problem, then there exist d polynomials $L_i \in \mathcal{P}_1^d$, $i = 1, \cdots, d$ (that is polynomials with degree at most 1) such that, for any $x \in \Omega$ and any $i = 1, \cdots, d$*

$$\sum_j g^{ij}\partial_j \log(\rho(x)) = L^i(x). \tag{2.11}$$

Observe that given that $g^{ij}(x)$ belong to $\mathcal{P}_2^d$, Proposition 2.6 is equivalent to the fact that the coefficients $b^i(x)$ belong to $\mathcal{P}_1^d$.

Indeed, the distinction between DOP and SDOP solution is only relevant in the non compact case.

Proposition 2.7. *Whenever Ω is relatively compact, any solution of the DOP problem is a solution of the SDOP problem.*

Proof — We just have to whow that for relatively compact sets Ω, equation (2.10) is satisfied for any pair (f, g) of smooth compactly supported functions. Since Ω is relatively compact, for any f smooth and compactly supported in $\mathbb{R}^d$ (and not necessarily in Ω), we first choose some compact K which contains both the support of f and Ω. There exists a sequence (P_n) of polynomials such that (P_n) and $(\mathbf{L}P_n)$ converge uniformly on K to f and $\mathbf{L}f$ respectively. Indeed, one first choses on K a polynomial sequence (R_n) converging uniformly on K to $\partial^{2d}_{11\cdots dd}f$, such that each appropriate integral of R_n converges uniformly in K to the corresponding derivative of f. The functions g^{ij} and b^i being polynomials, they are bounded on K. Therefore, (P_n) and $(\mathbf{L}P_n)$ converge uniformly, and hence in $\mathcal{L}^2(\mu)$, to f and $\mathbf{L}(f)$ respectively.

Choosing such approximations (P_n) and (Q_n) for f and g respectively, we also see that $\Gamma(P_n, Q_n)$ converges uniformly on K to $\Gamma(f, g)$. Then, it is clear that formula (2.7) extends immediately to the pair (f, g)). ∎

From now on, we shall call Δ the determinant of the matrix (g^{ij}). Since each $g^{ij} \in \mathcal{P}_2^d$, we see that $\Delta \in \mathcal{P}_{2d}^d$, and we decompose it into irreducible real factors.

$$\Delta = \Delta_1^{m_1} \cdots \Delta_p^{m_p}.$$

Furthermore, for every irreducible real factor Δ_j which may be factorized in $\mathbb{C}[X, Y]$, we write this factorization as $\Delta_j = (\mathcal{R}_j + i\mathcal{I}_j)(\mathcal{R}_j - i\mathcal{I}_j)$, where $\mathcal{R}_j$ and $\mathcal{I}_j$ are real. By convention, we shall write $\mathcal{I}_i = 0$ when the factor Δ_i is complex-irreducible.

As a consequence of Proposition 2.6, we get

Corollary 2.8. *For any solution of the SDOP problem, the density ρ is an analytic function on Ω. The only points where ρ may vanish or be infinite lie in the algebraic set $\{\Delta = 0\}$.*

(We shall see a complete description of the density measures ρ in Proposition 2.13.)

Proof — Let (g_{ij}) be the inverse matrix of (g^{ij}). Since for any (ij), $g^{ij} \in \mathcal{P}_2^d$, then for any (ij), g_{ij} is a rational function, whose denominator may vanish only on $\{\Delta = 0\}$. From equation (2.11), one sees that $\partial_j \log \rho = \sum_j g_{ij} L^j$, where $L^j \in \mathcal{P}_1^d$. Therefore, $\log \rho$ may only vanish or become infinite on $\{\Delta = 0\}$. Every partial derivative of $\log \rho$ is a rational function, and hence ρ is analytic.

∎

We can now state the main result of this section. Recall that $\Delta_1, \cdots, \Delta_p$ are the real irreducible components of $\Delta = \det(g^{ij})$. Since no one of them may vanish in Ω, we always assume that they are positive in Ω.

Theorem 2.9.

1. *Suppose that (Ω, g, ρ) is a solution to the SDOP problem. Then the boundary $\partial\Omega$ lies in the algebraic hyper-surface $\{\Delta = 0\}$. Moreover, if $\Delta_1 \cdots \Delta_r$ denotes the irreducible equation of $\partial\Omega$, then for any $k = 1, \cdots, r$, and any $i = 1, \cdots, d$, there exists some polynomial $S_k^i \in \mathcal{P}_1^d$ such that, for any $i = 1, \cdots, d$*
$$\sum_j g^{ij} \partial_j \Delta_k = S_k^i \Delta_k.$$
2. *Conversely, assume that Ω is relatively compact and has a boundary $\partial\Omega$ which is an algebraic hyper-surface with irreducible equation $\hat{\Delta} = \Delta_1 \cdots \Delta_r = 0$. Suppose moreover that there exists a (co)metric $(g^{ij})(x)$ positive definite on Ω such that for any (i,j), $g^{ij} \in \mathcal{P}_2^d$ and, for any k, for any $i = 1, \cdots, d$, there exists some $S_k^i \in \mathcal{P}_1^k$ such that*
$$\sum_j g^{ij} \partial_j \Delta_k = S_k^i \Delta_k. \tag{2.12}$$
Then, for any real numbers $a_1, \cdots, a_r$ for which $\rho(x) = \Delta_1^{a_1} \cdots \Delta_r^{a_r}$ is integrable on Ω, $(\Omega, g, C\rho)$ is a solution to the DOP problem, where C is the normalizing constant which turns $C\rho(x)dx$ into a probability measure on Ω.
3. *Property of point 2 still holds if one replaces the family of equations (2.12) by*
$$\sum_{ij} g^{ij} \partial_j \hat{\Delta} = S\hat{\Delta}, \tag{2.13}$$
for some $S \in \mathcal{P}_1^d$ and where $\hat{\Delta}$ is the irreducible equation of $\partial\Omega$.

Remark 2.10. *In theorem 2.9, saying that $\partial\Omega$ has $\Delta_1 \cdots \Delta_r = 0$ as irreducible equation means first that every real polynomial Δ_k is complex-irreducible, that $\partial\Omega \subset \{\Delta_1 \cdots \Delta_r = 0\}$, and moreover that $\partial\Omega$ is not included in any of the subsets $\{\prod_{j \neq i} \Delta_j = 0\}$ for any $i = 1, \cdots, r$. But it may (and will in general) be only a part of the set $\{\Delta_1 \cdots \Delta_r = 0\}$.*

Proof — Let us first prove the first point. From Corollary 2.8, one may consider some regular point x_0 on the boundary $\partial\Omega$ where $0 < \rho(x_0) < \infty$. Then ρ is continuous in a neighborhood of x_0, and we may assume that $\rho(x)$ is smooth and positive in some ball $B(x_0, r)$. We may also assume that x_0 does not belong to any of the sets $\Delta_i = 0$, where Δ_i is one of the real irreducible components of Δ which is not complex irreducible (such points belong to an algebraic set of codimension at least 2). We may also assume that x_0 does not belong to any singular set of $\{\Delta_i = 0\}$ for any i.

Then, we know that, for any function f smooth and compactly supported in $B(x_0, r)$, one has, from Proposition 2.7,

$$\int_\Omega f\mathbf{L}(x_1)\,\rho(x)dx + \int_\Omega \mathbf{\Gamma}(f, x_1)\,\rho(x)dx = 0.$$

Consider then the 1-form ω_f given by

$$\omega_f = \sum_j g^{1j} f\rho\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n.$$

One has $d\omega_f = \rho(x)\big(\mathbf{\Gamma}(f, x_1) + f\mathbf{L}(x_1)\big)dx$ and therefore $\int_\Omega d\omega_f = 0$. From Stokes formula, we deduce that

$$\int_{\partial\Omega} \omega_f = \int_{\partial\Omega} g^{1j} n_j f\rho dx = 0,$$

where n_j is the normal vector on the boundary. This being valid for any f smooth and compactly supported in $B(x_0, r)$, we deduce that $g^{1j} n_j = 0$ on $\partial\Omega \cap B(x_0, r)$. We may do the same operation for any coordinate x_i replacing x_1, and we deduce that for any i,

$$\sum_j g^{ij} n_j = 0 \text{ on } \partial\Omega \cap B(x_0, r). \tag{2.14}$$

From this we see that the normal vector to the boundary belongs to the kernel of the matrix g^{ij}, and therefore that $x_0 \in \{\Delta = 0\}$. Moreover, x_0 being a regular point, $\partial\Omega \cap B(x_0, r)$ must belong to some regular component of some algebraic set $\Delta_k = 0$ (where Δ_k is indeed complex irreducible), and hence that the normal to the boundary to $\partial\Omega$ at x_0 is parallel to $(\partial_j \Delta_k)$, since x_0 is a regular point of $\{\Delta_k = 0\}$. We may therefore replace n_j by $\partial_j \Delta_k$ in the boundary equation (2.14). Then, the polynomial $\sum_i g^{ij} \partial_j \Delta_k$ which is of degree $\deg(\Delta_k) + 1$ vanishes on $\{\Delta_k = 0\}$ in a neighborhood of x_0, hence everywhere. From the Hilbert's Nullstellensatz, and since Δ_k is complex irreducible, this shows that there exists some polynomial S_k with $\deg(S_k) \le 1$ such that $\sum_i g^{ij} \partial_j \Delta_k = S_k^i \Delta_k$: the first assertion is proved.

For the converse assertion, assume that equation (2.12) is true for any i and any $\Delta_k \in \partial\Omega$. Then, the metric g^{ij} has a non trivial vector in its kernel on any regular point of the boundary $\partial\Omega$. Therefore, its determinant Δ vanishes on those points, and $\hat{\Delta}$ divides Δ. (One sees then that the equations (2.12) may only be satisfied when $\deg(\hat{\Delta}) \le 2d$.) Moreover, we have on any regular point of the boundary, and for any $i = 1, \cdots, d$, $g^{ij} n_j = 0$: this shows that for any pair (f, g) of smooth functions compactly supported in $\mathbb{R}^d$, and for the operator $\mathbf{L}_0(f) = \sum_{ij} \partial_i (g^{ij} \partial_j f)$, one has

$$\int_\Omega f\mathbf{L}_0(g)\, dx = \int_\Omega g\mathbf{L}_0(f)\, dx.$$

This is the case in particular for any pair (P, Q) of polynomials, since we may extend them outside Ω into smooth compactly supported functions. But from the form of $\mathbf{L}_0$, it is clear that $\mathbf{L}_0$ maps $\mathcal{P}_k^d$ into $\mathcal{P}_k^d$: therefore, on $\mathcal{P}_k^d$, with the Euclidean structure inherited from the $\mathcal{L}^2$ norm associated with the Lebesgue measure, $\mathbf{L}_0$ is a symmetric operator which may be diagonalized in some orthonormal basis. This being done for any k provides an Hilbert basis of $\mathcal{L}^2(\Omega, dx)$ of orthogonal polynomials which are eigenvectors for $\mathbf{L}_0$.

Moreover, the boundary equations (2.12) show that for the measures with densities $\rho(x) = \Delta_1^{a_1} \cdots \Delta_r^{a_r}$, the associated operator $\mathbf{L}_\rho = \frac{1}{\rho} \sum_{ij} \partial_i \big(g^{ij} \rho \partial_j\big)$ also maps $\mathcal{P}_k^d$ into $\mathcal{P}_k^d$. It remains to show that it is symmetric on the set of polynomials. For this, we first observe that if the measure $\rho(x)dx$ has to be finite on Ω, then $a_i > -1$ for any i, and, considering the behavior of Δ near a regular point $x_0 \in \partial\Omega$, one sees that $g^{ij} n_i \rho$ goes to 0 when x goes to the boundary $\partial\Omega$ (for this last point, replace (n_i) by the unit vector parallel to $\partial_i \Delta$ near the boundary). This is enough to assert that for any pair (f, g) of smooth functions, one has

$$\int_\Omega f \mathbf{L}_\rho(g)\, \rho(x) dx = \int_\Omega g \mathbf{L}_\rho(f)\, \rho(x) dx.$$

Then, the associated operator $\mathbf{L}_\rho$ is symmetric on $\mathcal{P}_k^d$ for the Euclidean structure inherited from $\mathcal{L}^2(\rho dx)$, and the same argument that we used for the Lebesgue measure holds.

For the last point, first observe that (2.12) may be written as

$$\sum_j g^{ij} \partial_j \log \Delta_k = S_k^i,$$

and adding those equations on k leads to (2.13). Conversely, formula (2.13) reads

$$\sum_j g^{ij} \partial_j \big(\sum_k \log \Delta_k \big) = S^i,$$

and taking the limit of a regular point $x_0 \in \{\Delta_k = 0\}$, one sees that $\sum_j g^{ij} \partial_j \Delta_k = 0$ on the set $\{\Delta_k = 0\}$, and by Hilbert's Nullstellensatz, one gets that $\sum_j g^{ij} \partial_j \Delta_k = S_k^i \Delta_k$ for some $S_k^i \in \mathcal{P}_1^d$.

∎

Remark 2.11. *The equation $\hat{\Delta} = 0$ of the boundary being given, the problem of finding a symmetric matrix $(g^{ij})(x)$ formed with second degree polynomials and first degree polynomial S^i such that*

$$g^{ij} \partial_j \hat{\Delta} = S^i \hat{\Delta}$$

is a linear problem in the g^{ij} and S^i coefficients (there are $d(d+1)^2(d+2)/4 + d(d+1)$ such coefficients). In order to get a non trivial solution, the determinant of this equation (which depends only on $\hat{\Delta}$) has to be 0.

Unfortunately, the number of variables involved is so high that in practise this condition is useless, except in low dimension ($d = 2, 3$, e.g.) for which it provides an easy way to determine if a given set Ω is a good candidate for our problem. Classifying all these sets is a much more difficult question, due to the affine invariance of the problem. But provided $\hat{\Delta}$ is given and is an admissible solution, finding the metric g is in general an easy problem.

Remark 2.12. *When the boundary equation has maximal degree $2d$, then it is proportional to the determinant of the metric Δ. In this case, if $\Delta^{-1/2}$ is integrable on the domain , then the Laplace-Beltrami operator associated with the co-metric g is a solution of the DOP problem on Ω. It turns out that in any example where it is the case, the associated curvature (in dimension 2 the scalar curvature) is constant, and even either 0 either positive. We do not know for the moment how to prove it directly from the equation of the metric, even if one restricts the attention to the scalar curvature.*

We turn now to the general description of the admissible measures.

Proposition 2.13 (General form of the measure)**.** *Suppose that the determinant Δ of (g^{ij}) writes $\Delta = \Delta_1^{m_1} \cdots \Delta_p^{m_p}$, where Δ_i are real irreducible. Then, there exist real constants (α_i, β_j), and some polynomial Q with $\deg(Q) \leq 2n - \deg(\Delta)$, such that*

$$\rho = \prod_i |\Delta_i|^{\alpha_i} \exp\Big(\frac{Q}{\Delta_1^{m_1-1} \cdots \Delta_p^{m_p-1}} + \sum_j \beta_j \arctan \frac{\mathcal{I}_j}{\mathcal{R}_j}\Big). \tag{2.15}$$

Proof —

Let us start with the first point. With $h = \log \rho$, one has from equation (2.11)

$$\partial_i h = \sum_j g_{ij} L^j, \tag{2.16}$$

where g is the inverse matrix of g and $L^i \in \mathcal{P}_1^d$. But $g = \Delta^{-1}\hat{g}$, where $\hat{g}$ is matrix of co-factors of g. Then, $\hat{g}_{ij}$ is a polynomial which belongs to $\mathcal{P}_{2d-2}^d$, and therefore $\partial_i h = \Delta^{-1} C_i$, where $C_i \in \mathcal{P}_{2d-1}^d$.

Let us extend the differential form dh to an holomorphic one ω in the complex domain $\mathcal{C}^d \setminus \{\Delta = 0\}$. By Alexander duality (see [1, 44, 38], if $\hat{\Delta}_q$, $q = 1, \cdots, m$, denote the irreducible complex factors of Δ, then there exist complex numbers $(\gamma_1, \cdots, \gamma_m)$ such that $\omega - \sum_p \gamma_p d\log(\hat{\Delta}_p) = dF_0$ is exact. We chose a generic coordinate system such that $\deg_{x_i} \Delta = \deg \Delta$ for any $i = 1, \cdots, d$. This remains true for any small perturbation of the coordinate system. From the form of ω, we know that

$$\partial_i F_0 = \frac{C_i}{\Delta} - \sum_p \gamma_p \frac{\partial_i \hat{\Delta}_p}{\hat{\Delta}_p} = \frac{\hat{C}_i}{\Delta},$$

with $\deg \hat{C}_i \leq 2d - 1$.

From elementary calculus, we know that, when fixing all variables x_j for $j \neq i$, $Q = F_0 \prod_p \hat{\Delta}_p^{m_p-1}$ is then a polynomial in x_i with degree at most $n_0 = 2d - \deg \Delta$. Therefore $\frac{\partial^{|k|}}{\partial_1^{k_1} \cdots \partial_d^{k_d}} Q = 0$ as soon as $\max k_i > n_0$. Hence, we know that Q is a polynomial, and that $\deg_{x_i} Q \leq n_0$, for any $i = 1, \cdots, d$. Moreover, since the same remains true for any small perturbation of the coordinate system, this shows that $\deg Q \leq n_0$.

We now deal with the real form of ρ. Whenever there is a real irreducible factor Δ_p of Δ which is $\mathbb{C}$-reducible, its irreducible decomposition in $\mathbb{C}^d$ writes $\Delta_p = (\mathcal{R}_p + i\mathcal{I}_p)(\mathcal{R}_p - i\mathcal{I}_p)$, and the corresponding factors in ρ must be of the form

$$\gamma_p \log(\mathcal{R}_p + i\mathcal{I}_p) + \bar{\gamma}_p \log(\mathcal{R}_p - i\mathcal{I}_p),$$

which writes in real form

$$\alpha_p \log \Delta_p + \beta_p \arctan(\frac{\mathcal{I}_p}{\mathcal{R}_p}) .$$

∎

3 The bounded solutions in dimension 2

In this Section, we concentrate on the DOP problem in dimension 2 for bounded domains. The central result of this section is the following

Theorem 3.1. *In $\mathbb{R}^2$, up to affine transformations, there are exactly 11 relatively compact sets of which there exist a solution for the DOP problem: the triangle, the square, the disk, and the areas bounded by two co-axial parabolas, by one parabola and two tangent lines, by one parabola, a tangent line and the line parallel to the axis of the parabola, by a nodal cubic, by a cuspidal cubic and one tangent, by a cuspidal cubic and a line parallel to the axis of the cubic (that is a line passing through the infinite point of the cubic), by a swallow tail, or by a deltoid curve (see Section 4 for more details).*

Since we look at bounded domains, we may therefore reduce to the SDOP problem, and we solve the algebraic problem described in Section 2.2 in the particular case of dimension 2. For basic references on algebraic curves, see Walker [54].

We thus are reduced to the following. Let

$$g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a symmetric matrix whose coefficients a, b, c are polynomials of degree 2 in $\mathbb{R}[x, y]$. Let $\hat{\Delta}$ a square free polynomial in $\mathbb{R}[x, y]$ which is factor of $\Delta := ac - b^2$ such that for each irreducible factor Δ_1 of $\hat{\Delta}$,

$$a\partial_1\Delta_1 + b\partial_2\Delta_1 = L_1\Delta_1 \tag{3.17}$$

$$b\partial_1\Delta_1 + c\partial_2\Delta_1 = L_2\Delta_1 \tag{3.18}$$

where $\deg L_i \leq 1, i = 1, 2$. We want to describe such $a, b, c, \hat{\Delta}$.

3.1 A preliminary study of Newton polygons of a, b, c and Δ

In this section, we assume $\deg \Delta_1 \geq 3$ and Δ_1, g are in $\mathbb{C}[x, y]$. Indeed we shall also use projective coordinates $(X : Y : Z)$ such that $x = \frac{X}{Z}, y = \frac{Y}{Z}$ and denote L_∞ the line $Z = 0$. Let γ be an analytic branch of the curve $\Delta_1 = 0$ at some finite or infinite point, i.e. γ is a germ at 0 of a non-constant meromorphic mapping $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (\xi(t), \eta(t))$ such that $\Delta_1(\xi(t), \eta(t)) = 0$. Let $v_\gamma : \mathbb{C}[x, y] \to \mathbb{Z} \cup \{\infty\}$ be the corresponding valuation, i.e. $v_\gamma(f) = \mathrm{ord}_t f\big(\xi(t), \eta(t)\big)$ where

$$\mathrm{ord}_t u(t) = \begin{cases} n & \text{if } u(t) = \sum_{k \geq n} u_k t^k \text{ and } u_n \neq 0, \\ \infty & \text{if } u(t) = 0 \end{cases}$$

In this section we suppose that γ is a fixed branch of Δ_1. In particular, it is not a branch of a multiple component of Δ. We denote $p = v_\gamma(x) = \mathrm{ord}_t(\xi)$ and $q = v_\gamma(y) = \mathrm{ord}_t(\eta)$ (we assume here that x and y are parametrized in Puiseux series in $\xi(t)$ and $\eta(t)$).

Lemma 3.2.
(a) Suppose that none of $\xi(t)$, $\eta(t)$ is constant. Then

$$v_\gamma(a) - v_\gamma(b) = v_\gamma(b) - v_\gamma(c) = \mathrm{ord}_t\dot{\xi} - \mathrm{ord}_t\dot{\eta}. \tag{3.19}$$

(b) Suppose that $\eta(t)$ is constant. Then $v_\gamma(b) = v_\gamma(c) = \infty$, i.e., b and c vanish identically on γ.

Proof — (a) Differentiating the identity $\Delta_1(\xi, \eta) = 0$, we obtain

$$\dot{\xi}\partial_1\Delta_1 + \dot{\eta}\partial_2\Delta_1 = 0$$

Hence,

$$v_\gamma(\partial_1\Delta_1) + \mathrm{ord}_t\dot{\xi} = v_\gamma(\partial_2\Delta_1) + \mathrm{ord}_t\dot{\eta}. \tag{3.20}$$

By (3.17) and (3.18) we have

$$v_\gamma(\partial_1\Delta_1) + v_\gamma(a) = v_\gamma(\partial_2\Delta_1) + v_\gamma(b) \quad \text{and} \quad v_\gamma(\partial_1\Delta_1) + v_\gamma(b) = v_\gamma(\partial_2\Delta_1) + v_\gamma(c). \tag{3.21}$$

Let us show that $v_\gamma(\partial_2\Delta_1) \neq \infty$. Indeed, otherwise both Δ_1 and $\partial_2\Delta_1$ vanish identically along γ, hence Δ_1 divides $\partial_1\Delta_1$ which implies ξ is constant. Similarly we show that $v_\gamma(\partial_1\Delta) \neq \infty$.

Suppose that one of a, b, or c vanishes identically along γ. Then (3.21) implies that all of them vanish identically along γ. Then Δ_1 divides a, b, and c. Hence, Δ_1^2 divides $\Delta = ac - b^2$ which is impossible because γ is not a branch of a multiple component of Δ.

Thus, each of $v_\gamma(a)$, $v_\gamma(b)$, $v_\gamma(c)$ is finite, and (3.19) follows from (3.20), (3.21).

(b) Immediate from (3.20). ∎

As usually, for a polynomial $u = \sum u_{kl}x^k y^l$, we define its Newton polygon $\mathcal{N}(u)$ as the convex hull in $\mathbb{R}^2$ of the set $\{(k,l) \mid u_{kl} \neq 0\}$.

We have $v_\gamma(x^k y^l) = L_\gamma(k,l)$ where L_γ is the linear form $L_\gamma(r,s) = pr + qs$ and $p = v_\gamma(x)$, $q = v_\gamma(y)$.. Thus, for any polynomial $u(x,y)$ we have $v_\gamma(u) \geq \min_{\mathcal{N}(u)} L_\gamma$ and if the minimum of L_γ is attained at a single vertex of $\mathcal{N}(u)$, then then $v_\gamma(u) = \min_{\mathcal{N}(u)} L_\gamma$

The notation of the style $p = \begin{bmatrix} \bullet & & \\ \circ & \bullet & \\ \bullet & \circ & \circ \end{bmatrix}$ (any combination of $\circ$ and $\bullet$) means that p is a linear combination of monomials corresponding to the $\bullet$'s. For example, $b = \begin{bmatrix} \bullet & & \\ \circ & \bullet & \\ \bullet & \circ & \circ \end{bmatrix}$ means that $b_{01} = b_{10} = b_{20} = 0$ (the coefficients of y, x, and x^2) and the other coefficients may or may not be zero.

In the following Lemma, we investigate the different values for (p,q) that will be needed in our case: since Δ_1 is of degree less than 4, $1 \leq p < q \leq 4$. The cases p or q negative correspond to points at infinity and can be reduced to the positive case by projective change of coordinates. For a listing of all the kind of singularities, see Table 1 in Section 3.2.

Lemma 3.3.
(a) If $(p,q) = (1,2)$, then $b = \begin{bmatrix} \bullet & & \\ \bullet & \bullet & \\ \circ & \bullet & \bullet \end{bmatrix}$ and $c = \begin{bmatrix} \bullet & & \\ \bullet & \bullet & \\ \circ & \circ & \bullet \end{bmatrix}$.
(b) If $(p,q) = (1,3)$, then $b = \begin{bmatrix} \bullet & & \\ \bullet & \bullet & \\ \circ & \circ & \bullet \end{bmatrix}$ and $c = \begin{bmatrix} \bullet & & \\ \circ & \bullet & \\ \circ & \circ & \circ \end{bmatrix}$, in particular, $\mathrm{mult}_{(0,0)}\Delta \geq 2$.
(c) If $(p,q) = (1,4)$, then $a = \begin{bmatrix} \bullet & & \\ \bullet & \bullet & \\ \circ & \circ & \bullet \end{bmatrix}$, $b = \begin{bmatrix} \bullet & & \\ \circ & \bullet & \\ \circ & \circ & \circ \end{bmatrix}$, and $c = \begin{bmatrix} \bullet & & \\ \circ & \circ & \\ \circ & \circ & \circ \end{bmatrix}$,

(d) If $(p,q) = (-1,0)$*, then* $b = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & \circ\end{bmatrix}$ *and* $c = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & \circ\end{bmatrix}$.

(e) If $(p,q) = (-1,1)$*, then* $b = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \circ & \circ\end{bmatrix}$ *and* $c = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \circ & \circ\end{bmatrix}$.

(f). $p = -1$ *and* $2 \leq q < \infty$ *is impossible.*

(g) If $(p,q) \in \{(-2,-1),\ (-3,-2),\ (-4,-3)\}$*, then* $b = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & \circ\end{bmatrix}$ *and* $c = \begin{bmatrix}\bullet & & \\ \bullet & \circ & \\ \bullet & \bullet & \circ\end{bmatrix}$.

(h) If $(p,q) = (-2,1)$*, then* $b = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \circ & \circ\end{bmatrix}$ *and* $c = \begin{bmatrix}\bullet & & \\ \circ & \circ & \\ \circ & \circ & \circ\end{bmatrix}$.

(i) $(p,q) = (3,4)$ *is impossible.*

(j) $(p,q) = (-3,-1)$ *corresponds to a cusp* A_2 *at infinity, with* L_∞ *tangent to the cusp. It leads to a non compact model, and will further be eliminated below (case* (i_2) *in further Corollary 3.8).*

Proof — (a) If $(p,q) = (1,2)$, then $v_\gamma(\dot{\xi}) = 0$ and $v_\gamma(\dot{\eta}) = 1$. Hence, by (3.19) we have $v_\gamma(b) = v_\gamma(a) + 1 \geq 1$ and $v_\gamma(c) = v_\gamma(a) + 2 \geq 2$ and the result follows from the fact that $v_\gamma(1) = 0$, $v_\gamma(x) = 1$, and $v_\gamma(x^k y^l) \geq 1$ when $(k,l) \notin \{(0,0),(0,1)\}$.

(b) If $(p,q) = (1,3)$, then $v_\gamma(\dot{\xi}) = 0$ and $v_\gamma(\dot{\eta}) = 2$. Hence, by (3.19) we have $v_\gamma(b) = v_\gamma(a) + 2 \geq 2$ and $v_\gamma(c) = v_\gamma(b) + 2 = v_\gamma(a) + 4 \geq 4$. The values of v_γ on the monomials of degree ≤ 2 are:

$$v_\gamma(1) = 0,\ \ v_\gamma(x) = 1,\ \ v_\gamma(x^2) = 2,\ \ v_\gamma(y) = 3,\ \ v_\gamma(xy) = 4,\ \ v_\gamma(y^2) = 6. \tag{3.22}$$

Thus, $v_\gamma(b) \geq 2$ implies $b_{00} = b_{01} = 0$ and $v_\gamma(c) \geq 4$ implies $c_{00} = c_{10} = c_{20} = c_{01} = 0$. In particular, $\mathrm{mult}_{(0,0)} b \geq 1$ and $\mathrm{mult}_{(0,0)} c \geq 2$. Hence, $\mathrm{mult}_{(0,0)}(b^2 - ac) \geq 2$

(c) If $(p,q) = (1,4)$, then $v_\gamma(\dot{\xi}) = 0$ and $v_\gamma(\dot{\eta}) = 3$. Hence, by (3.19) we have

$$v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = 3 \tag{3.23}$$

The values of v_γ on the monomials of degree ≤ 2 are:

$$v_\gamma(1) = 0,\ \ v_\gamma(x) = 1,\ \ v_\gamma(x^2) = 2,\ \ v_\gamma(y) = 4,\ \ v_\gamma(xy) = 5,\ \ v_\gamma(y^2) = 8.$$

Hence, we have $\{v_\gamma(a), v_\gamma(b), v_\gamma(c)\} \subset \{0,1,2,4,5,8\}$. Under this condition, (3.23) is possible only for $v_\gamma(a) = 2$, $v_\gamma(b) = 5$, $v_\gamma(c) = 8$ and the result follows.

(d) $(p,q) = (-1,0)$. Similar to (a)–(c). If $\dot{\eta} = 0$, we use Lemma 3.2(b).

(e) If $(p,q) = (-1,1)$, then by (3.19) we have $v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = 2$, hence $v_\gamma(a) = -2$, $v_\gamma(b) = 0$, $v_\gamma(c) = 2$ and the result follows. Note that $v_\gamma(1) = v_\gamma(xy) = 0$ but the condition $v_\gamma(c) = 2$ does not imply $c_{00} = c_{11} = 0$ because it is possible that that the initial term of $c_{11}\xi\eta$ cancels against $-c_{00}$.

(f) We have $v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = q + 1$ and $v_\gamma(x^2, x, 1, xy, y, y^2) = (-2, -1, 0, q-1, q, 2q)$. Thus, $v_\gamma(a,b,c) = (-2, q-1, 2q)$, i.e. $b = \begin{bmatrix}\bullet & & \\ \bullet & \bullet & \\ \circ & \circ & \circ\end{bmatrix}$ and $c = \begin{bmatrix}\bullet & & \\ \circ & \circ & \\ \circ & \circ & \circ\end{bmatrix}$, Therefore, $\Delta = y^2 f(x,y)$. This is impossible because γ cannot be a branch of a polynomial of degree ≤ 2.

(g,h) The proof is similar to the previous cases.

(i) We have $v_\gamma\{a,b,c\} \subset v_\gamma\{1, x, y, x^2, xy, y^2\} = \{0,3,4,6,7,8\}$. Combining this with $v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = \mathrm{ord}_t\dot{\eta} - \mathrm{ord}_t\dot{\xi} = 3 - 2 = 1$, we obtain $v_\gamma(a) = 6$, $v_\gamma(b) = 7$, $v_\gamma(c) = 8$, i.e. $a = \begin{bmatrix}\bullet & & \\ \circ & \bullet & \\ \circ & \circ & \bullet\end{bmatrix}$, $b = \begin{bmatrix}\bullet & & \\ \circ & \bullet & \\ \circ & \circ & \circ\end{bmatrix}$, $c = \begin{bmatrix}\bullet & & \\ \circ & \circ & \\ \circ & \circ & \circ\end{bmatrix}$. Thus, $\mathrm{mult}_0(\Delta) = 4$, i.e., $\Delta = 0$ is a union of four lines. Contradiction. ∎

Corollary 3.4.
(a). Δ cannot have a singularity of type E_6 at a finite point.
(b). Suppose that γ is a singular branch of Δ_1 of type A_2 at a point $P \in L_\infty$ and L_∞ is not tangent to γ at P. Then there is another branch of Δ at P.

Proof —
The point (a) follows from Lemma 3.3(i).

Point (b) corresponds to $(p, q) = (-2, 1)$ (whereas $(-3, -2)$ corresponds to a cusp on L_∞ tangent to L_∞). We are therefore in case Lemma 3.3(h). Hence $ac - b^2$ has terms of type $(1, y, y^2, y^3, y^4, xy, xy^2, xy^3, x^2y^2)$ hence with valuations $(0, 1, 2, 3, 4, -1, 0, 1, -2)$ hence the only remaining terms are in $(1, y, xy^2, xy^3)$, which gives in $(z, y) = (\frac{Z}{X}, \frac{Y}{X}) = (z^4, zy^2, z^3y, y^3)$: we conclude observing that the lower part of the Newton polygon has at least two edges. ∎

Lemma 3.5. *Let γ be a flex or planar branch of Δ_1 at P. Then*
(a) if $P \in L_\infty$, then γ is tangent to L_∞.
(b) if $P \notin L_\infty$, then $\operatorname{mult}_P(\Delta) > \operatorname{mult}_P(\Delta_1)$, in particular, this is impossible when Δ is irreducible.
(c) All possible planar points are in L_∞.

Proof — (a) Follows from Lemma 3.3(f).

(b) Let us choose affine coordinates so that P is the origin and the axis $y = 0$ is tangent to γ. Thus, $\operatorname{ord}_t\gamma = (1, q)$, $q \geq 3$. By Lemma 3.3(b,c), we have $\operatorname{mult}_{(0,0)}\Delta \geq 2$, i.e., there is another branch β of Δ passing through the origin. Since $\deg_x \Delta(x, 0) \leq 4$ the multiplicities of the intersection of the axis $y = 0$ with γ and β are 3 (i.e., $q = 3$) and 1 respectively.

It remains to prove that β cannot be a branch of $\hat{\Delta}$. Suppose it is. Let us choose coordinates so that the axis $x = 0$ is tangent to β. Then Lemma 3.3(a,b) applied to β implies
$$a_{00} = a_{01} = 0. \tag{3.24}$$
(we swap $x \leftrightarrow y$ and $a \leftrightarrow c$ in Lemma 3.3). Hence, $v_\gamma(a) \geq 1$. By (3.19), we have
$$v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = 2 \tag{3.25}$$
(see the proof of Lemma 3.3(b)). Recall that the values of v_γ on monomials are given by (3.22). Hence, $\{v_\gamma(a), v_\gamma(b)v_\gamma(c)\} \subset \{0, 1, 2, 3, 4, 6\}$. Combining this with (3.25) and $v_\gamma(a) \geq 1$, we obtain $v_\gamma(a) = 2$, $v_\gamma(b) = 4$, $v_\gamma(c) = 6$. By (3.24), this implies $\operatorname{mult}_{(0,0)}(a) = \operatorname{mult}_{(0,0)}(b) = \operatorname{mult}_{(0,0)}(c) = 2$, hence $\operatorname{mult}_{(0,0)}(\Delta) = 4$. This means that Δ is a union of four lines passing through the origin. Contradiction. ∎

Lemma 3.6. *Let γ be a branch of Δ_1 at $P \in L_\infty$. Suppose that there exists a line L passing through P which is tangent to a branch β of Δ_1 at a finite point Q. Suppose also that $\beta, \gamma \not\subset L$. Then*
(a) β is smooth at Q.
(b) $\operatorname{mult}_P(\Delta) \geq 2$.

Proof — It is enough to treat the case when γ is smooth and not a branch of L. Let us choose coordinates so that L is the axis $y = 0$ and and β is tangent to L at the origin. Then all possibilities for γ are covered by Lemma 3.3(d)–(g) and in all these cases we have $b, c = \left[\begin{smallmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & \circ\end{smallmatrix}\right]$, i.e., $b_{20} = c_{20} = 0$.

(a) Let $\beta = (\xi, \eta)$ and $(p, q) = \mathrm{ord}_t(\xi, \eta)$. Suppose that β is singular. Then $\min(p, q) \geq 2$. We have also $q > p$ (because L is tangent to β) and $q = (L.\beta) \leq 3$ (because $(L.\beta) + (L.\gamma) \leq 4$). Thus, $(p, q) = (2, 3)$ hence, by (3.19), we have $v_\beta(c) - v_\beta(b) = v_\beta(b) - v_\beta(a) = 1$. Combining this fact with $v_\beta(1, x, y, x^2, xy, y^2) = (0, 2, 3, 4, 5, 6)$ and $b_{20} = c_{20} = 0$, we obtain $v_\beta(a, b, c) = (4, 5, 6)$, i.e., $a = \begin{bmatrix}\bullet&&\\ \circ&\bullet&\\ \circ&\circ&\bullet\end{bmatrix}$, $b = \begin{bmatrix}\bullet&&\\ \circ&\bullet&\\ \circ&\circ&\circ\end{bmatrix}$, $c = \begin{bmatrix}\bullet&&\\ \circ&\circ&\\ \circ&\circ&\circ\end{bmatrix}$. Thus, Δ is homogeneous. Contradiction.

(b) Combining $b_{20} = c_{20} = 0$ with Lemma 3.3(a) applied to β, we obtain $b = \begin{bmatrix}\bullet&&\\ \bullet&\bullet&\\ \circ&\bullet&\circ\end{bmatrix}$, $c = \begin{bmatrix}\bullet&&\\ \bullet&\bullet&\\ \circ&\circ&\circ\end{bmatrix}$. Thus, it is enough to show that $c_{11} = 0$. Indeed, if γ is tangent to L_∞, this is already proven in Lemma 3.3(f,g). Otherwise we have $v_\gamma(c) > v_\gamma(xy)$, and the condition $c_{00} = c_{10} = 0$ implies that the value $v_\gamma(xy)$ cannot be attained on other monomials of c. ∎

3.2 The duals of quartic curves

Let C be an irreducible algebraic curve in $\mathbb{P}^2$ of degree $d \geq 2$. Let $\check{C}$ be the dual curve in $\check{\mathbb{P}}^2$ is the set of all lines in $\mathbb{P}^2$ endowed with the natural structure of the projective plane, and $\check{C}$ is the set of all lines in $\mathbb{P}^2$ which are tangent to C.

If $t \to \gamma(t)$ is a local analytic branch of C, then we denote the *dual branch* of $\check{C}$ by $\check{\gamma}$. It is defined by $t \mapsto \check{\gamma}(t)$ where $\check{\gamma}(t)$ is the line which is tangent to C at $\gamma(t)$.

Let γ be a local branch of C. Let us choose affine coordinates (X, Y) so that γ is given by $X = \xi(t)$, $Y = \eta(t)$, $\xi(0) = \eta(0) = 0$. Then the equation of the line $\check{\gamma}(t)$ is $(X - \xi)\dot{\eta} - (Y - \eta)\dot{\xi} = 0$. Thus, in the standard homogeneous coordinates on $\check{\mathbb{P}}^2$ corresponding to the coordinate chart (X, Y), the dual branch $\check{\gamma}$ has a parametrization of the form

$$t \mapsto (\dot{\eta} \,:\, -\dot{\xi} \,:\, \dot{\xi}\eta - \xi\dot{\eta}) \tag{3.26}$$

and we obtain the following fact.

Lemma 3.7. *Let γ be a local branch of C and $\check{\gamma}$ the dual branch of $\check{C}$. Let (X, Y) be an affine chart such that γ has the form $X = \xi(t)$, $Y = \eta(t)$ with $0 < p < q$ where $p = \mathrm{ord}_t\xi$ and $q = \mathrm{ord}_t\eta$. Then, in suitable affine coordinates $(\check{X}, \check{Y})$ on $\check{\mathbb{P}}^2$, the branch $\check{\gamma}$ has the form $\check{X} = \check{\xi}(t)$, $\check{Y} = \check{\eta}(t)$ with $\mathrm{ord}_t\check{\xi} = q - p$ and $\mathrm{ord}_t\check{\eta} = q$.* □

For a point $P \in C$, we denote the *local genus* of C at P by δ_P or $\delta_P(C)$. It is defined as the number of double points appearing in a generic perturbation of all local branches of C at P. We have

$$2\delta_P = \mu + r - 1 = \sum m_i(m_i - 1)$$

where μ is the Milnor number and r is the number of local branches of C at P, and $m_1, m_2, \dots$ is the sequence of the multiplicities of all infinitely near points of P. If P is a non-singular point of C, then $\delta_P = 0$. It easily follows from the definition that

$$\delta_P = \delta_P^0 + \sum_{i=1}^{r} \delta_P(\gamma_i) \qquad \text{where} \quad \delta_P^0 = \sum_{1 \leq i < j \leq r} (\gamma_i \cdot \gamma_j), \tag{3.27}$$

$\gamma_1, \dots, \gamma_r$ are local branches of C at P, and $(\gamma_i \cdot \gamma_j)$ is the intersection number of γ_i and γ_j at P. Let g be the genus of C. By the genus formula, we have

$$2g + 2 \sum_{P \in C} \delta_P = (d - 1)(d - 2). \tag{3.28}$$

Combyning (3.27) and (3.28), we obtain

$$2g + 2n + 2\sum_{\gamma} \delta(\gamma) = (d-1)(d-2) \tag{3.29}$$

where γ runs over all local branches of C and $n = \sum_{p \in C} \delta_P^0$.

For a local branch γ of a curve C at a point P, we denote the multiplicity of γ at P by $m(\gamma)$. If γ is parametrized by $X = \xi(t)$, $Y = \eta(t)$ in some local coordinates X, Y, then $m(\gamma) = \min(\mathrm{ord}_t \xi, \mathrm{ord}_t \eta)$. We set also $\epsilon(\gamma) = 2\delta(\gamma) + m(\gamma) - 1$. Let $\check{d}$ be the degree of $\check{C}$. In this notation, the first Plücker formula (the class formula) takes the form

$$\check{d} = d(d-1) - 2n - \sum_{\gamma} \epsilon(\gamma) \tag{3.30}$$

and the second Plücker formula (the Riemann-Hurwitz formula for a generic projection of $\check{C}$ onto a line) is

$$2 - 2g = 2\check{d} - d - \sum_{\gamma} (m(\check{\gamma}) - 1) \tag{3.31}$$

(in the both formulas γ runs over all local branches of C).

If $d = 4$, then $\sum \delta(\gamma) \le 3$ by (3.29), hence all singular branches are of the types A_2, A_4, A_6 and E_6 (recall that A_k and E_6 are given by $v^2 = u^{k+1}$ and $v^3 = u^4$ is suitable curvilinear local coordinates). In Table 1, we list all types of local branches $\gamma(t) = (\xi(t), \eta(t))$, $\mathrm{ord}_t \xi = p$, $\mathrm{ord}_t \eta = q$, $p < q$, and their invariants contributing to (3.29), (3.30), and (3.31) (we use Lemma 3.7 to compute $\check{p} = \mathrm{ord}_t \check{\xi}$ and $\check{q} = \mathrm{ord}_t \check{\eta}$).

Table 1

	(p,q)	$(\check{p},\check{q})$	$\delta(\gamma)$	$\epsilon(\gamma)$	$m(\check{\gamma}) - 1$
generic point	(1,2)	(1,2)	0	0	0
flex point	(1,3)	(2,3)	0	0	1
planar point	(1,4)	(3,4)	0	0	2
A_2	(2,3)	(1,3)	1	3	0
A_4	(2,4)	(2,4)	2	5	1
A_6	(2,4)	(2,4)	3	7	1
E_6	(3,4)	(1,4)	3	8	0

Thus, denoting the number of branches of the respective types by f (flex), p (planar), a_2, a_4, a_6, and e_6, we rewrite (3.29) – (3.31) as

$$\begin{gathered} g + n + a_2 + 2a_4 + 3a_6 + 3e_6 = 3, \\ \check{d} = 12 - 2n - 3a_2 - 5a_4 - 7a_6 - 8e_6, \\ 2 - 2g = 2\check{d} - 4 - f - 2p - a_4 - a_6. \end{gathered}$$

Eliminating g and $\check{d}$, we obtain

$$f + 2p = 24 - 8a_2 - 15a_4 - 21a_6 - 22e_6 - 6n. \tag{3.32}$$

Since all the ingredients (including g) are non-negative, we obtain the following fact.

Lemma 3.8. *Suppose that C is an irreducible quartic curve in $\mathbb{P}^2$ which has at most one smooth non-generic (i.e., flex or planar) local branch. Then C is rational (i.e., $g = 0$), all singularities of C are irreducible (i.e., $n = 0$), and one of the following cases occur:*

- (i) (tricuspidal quartic) *C has three singular points of type A_2 and no smooth non-generic branches (i.e., $f = p = 0$). The dual curve $\check{C}$ is a nodal cubic.*
- (ii) (swallow tail) *C has two singular points of type A_2 and one planar point (i.e., $f = 0$, $p = 1$). The degree of $\check{C}$ is 4, it has one singular point of type E_6 and two flex points. The equation of $\check{C}$ in suitable affine coordinates is $y = x^4 + x^2$.*
- (iii) *Each of C and $\check{C}$ has two singular points of types A_2 and A_4 and one flex point (i.e., $f = 1$, $p = 0$), the degree of $\check{C}$ is 4,*
- (iv) *Each of C and $\check{C}$ has one singular point of type E_6 and one planar point (i.e., $f = 0$, $p = 1$). The degree of $\check{C}$ is 4. The equation of $\check{C}$ in suitable affine coordinates is $y = x^4$.*

In each of the cases (i)–(iv) the formulated conditions uniquely determine the curve C up to automorphism of $\mathbb{CP}^2$.

Proof —
Substituting each nonnegative solution of $g + n + a_2 + 2a_4 + 3a_6 + 3e_6 = 3$ into (3.32), we see that the only cases when $f + p = 1$ are:

$$(i)\quad \check{d} = 3,\ a_2 = 3; \quad (iii)\ \ \check{d} = 4,\ a_2 = a_4 = 1,\ f = 1,$$
$$(ii)\quad \check{d} = 4,\ a_2 = 2,\ p = n = 1, \qquad (iv)\ \ \check{d} = 4,\ e_6 = p = 1.$$

Let us show that these cases are uniquely realizable. In cases (ii) and (iv) this follows from the fact that $\check{C}$ has the singularity E_6, hence it has the equation $y = f(x)$, $\deg_x f = 4$, in suitable coordinates. By affine changes of coordinates, this equation reduces to $y = x^4$ or $y = x^4 + x^2$.

In case (i), the dual curve is a nodal cubic. It is unique up to projective transformation, thus C is also unique.

In case (iii), let us choose homogeneous coordinates $(X : Y : Z)$ so that A_2 and A_4 are at $(0 : 0 : 1)$ and $(0 : 1 : 0)$ respectively and the lines $Y = 0$ and $Z = 0$ are tangent to C at these points. Let $F(X, Y, Z) = 0$ be the equation of C, The choice of the coordinates provides $F = u_{30}X^3Z + G$ where $G = u_{40}X^4 + u_{21}X^2YZ + u_{02}Y^2Z^2$. Moreover, the fact that F has a single branch at $(0 : 1 : 0)$ implies that G is a complete square. Hence, rescaling the coordinate, we can obtain $u_{30} = u_{40} = u_{02} = 1$, $u_{11} = 2$.

∎

Corollary 3.9. *Suppose that C is an irreducible quartic curve in $\mathbb{C}^2$ which satisfies the restrictions provided by Lemmas 3.3(h,i), 3.5 and 3.6, i.e.:*

- *any smooth non-generic branch of C is tangent to the infinite line L_∞;*
- *if C meets L_∞ transversally at a point P, then there is no line through P (except, maybe, L_∞) which is tangent to C at a smooth point;*
- *if C has a cusp A_2 at a point $P \in L_\infty$, and L_∞ is not the tangent to C at P, then C has another branch through P;*

- *C does not have a singularity of type E_6 at a finite point.*

Then one of the cases (i) or (ii) of Lemma 3.8 occur and the position of C with respect to the infinite line L_∞ is one of:

- (i_1) (deltoid *or* $(1,3)$-hypocycloid) *L_∞ is the bitangent of C.*
- (i_2) *L_∞ is the tangent at a cusp. This case is not possible.*
- (ii) (swallow tail) *L_∞ is the tangent at the planar point.*

In each of the cases (i_1) and (ii) the affine curve C is unique up to affine transformation of $\mathbb{C}^2$. In suitable affine coordinates, C is parametrized by

- (i_1) $X = 2\cos\theta + 2\sin 2\theta$, $Y = 2\sin\theta - 2\sin 2\theta$
- (ii) $X = t(t^2-3)$, $Y = t^2(t^2-2)$

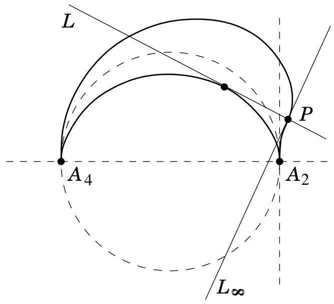


Figure 1: A_2A_4 — .

The real quartic with A_2 and A_4
Proof — Since $\deg C = 4$, there is no room for more than one non-generic tangency with L_∞. Thus one of the cases (i)–(iv) of Lemma 3.8 occur. We consider them separately.

(i) Let P be a smooth point of C. Riemann-Hurwitz for the projection from P implies that there existe a unique line L_P through P tangent to C at another (smooth or singular) point.

Suppose that (i_2) does not hold. Then, by Lemma 3.3(h), all infinite points of C are smooth. Let P be one of them. Let Q be the point where L_P is tangent to C. Lemma 3.6 implies $L_P = L_\infty$, i.e., $Q \in L_\infty$. Then, again by Lemma 3.6, we have $L_Q = L_\infty$, thus (i_1) takes place.

Suppose (i_2) holds. Then let P be the cusp at infinity and Q a cusp at finite distance. Then P does not belong to the tangent at Q. Choosing coordinates we can assume that there are two branches, one with $(p,q) = (-3,-1)$, the other one with $(p,q) = (3,2)$. With Lemma 3.2, we conclude that Δ is not irreducible, which is a contradiction since Δ_1 is of degree 4.

(ii) No other choice for L_∞.

(iii) Let P be the flex point. Then L_∞ is tangent to C at P by Lemma 3.5. Applying Riemann-Hurwitz formula to the projection from P, we see that there exists a line L through P which is tangent to C at a smooth point. Contradiction with Lemma 3.6. *Remark.* The existence of such L can be also derived from the unicity of C up to projective transformations. Indeed, we can realize C as a real curve in $\mathbb{R}^2$ obtained by a small perturbation of a double circle: $(x^2+y^2)^2 = \varepsilon y^3(x+1)$, $0 < \varepsilon \ll 1$. Then L is clearly visible in Figure 1.

(*iv*) Impossible by Lemma 3.5 and Lemma 3.4(a). $\square$ $\blacksquare$

3.3 Cubic factor of Δ

In this section we suppose that $\Delta = \Delta_3\Delta_1$ where Δ_3 is an irreducible cubic factor of $\mathbf{\Gamma}$. Let C be the quartic curve defined by $\Delta = 0$ and let C_3 and C_1 be the respective irreducible components of C (if $\deg \Delta = 3$, then $C_1 = L_\infty$).

Lemma 3.10. *C_3 is rational.*

Proof — Othewise C_3 has nine flex points. They cannot all be on $C_1 \cup L_\infty$. So, this contradicts to Lemma 3.5. $\square$ $\blacksquare$

By an isomorphism of $\mathbb{CP}^2$, any rational cubic can by identified either with the *nodal cubic* $y^2 = x^3 - x^2$ or with the *cuspidal cubic* $y^2 = x^3$.

The nodal cubic has three flex points lying on the same line. The cuspidal cubic has a single flex point.

Lemma 3.11. *Suppose that C_3 is a nodal cubic. Then $\mathbf{\Gamma} = \Delta_3$, the line L_∞ is tangent to C_3 at a flex point, and C_1 is the line passing through all the three flex points of C_3.*

Proof — Let L_0 be the line passing through all the flex points of C_3. Then $L_0 \neq L_\infty$ by Lemma 3.5(a). Thus, at least two flex points are not on L_∞, hence Lemma 3.5(b) implies that a non-trivial component of $\Delta/\mathbf{\Gamma}$ passes through them. Hence, $C_1 = L_0$. and $\mathbf{\Gamma} = \Delta_3$.

Suppose that C_3 has more than one point at the infinity. Then there is a point P such that $(C_3.L_\infty)_P = 1$. Then P is not a flex point by Lemma 3.5. Hence, Riemann-Hurwitz formula for the projection from P implies that there exists a line L through P which is tangent to C at some other point Q. If Q were finite, then Lemma 3.6(b) would imply that C_1 passes through P. This is impossible by Bezout's theorem because C_1 has already three intersections with C_3 at the flex points. Thus, $Q \in L_\infty$. Applying the same arguments to Q, we obtain a contradiction.

Thus, C_3 has a single point P at the infinity. It remains to show that P is not the node of C_3. Suppose it is. Choose coordinates so that $P = (1:0:0)$, the axis $x = 0$ is the tangent at a flex point, and the tangents at P are L_∞ and the axis $y = 0$. Then C_3 admots a parametrization $x = \xi(t) = (t-1)^3/t$, $y = \eta(t) = t$.

Applying Lemma 3.3(a,g) to the branches of C_3 at P, we obtain $b = \left[\begin{smallmatrix}\bullet & & \\ \bullet & \bullet & \\ \bullet & \circ & \circ\end{smallmatrix}\right]$ and $c = \left[\begin{smallmatrix}\bullet & & \\ \bullet & \circ & \\ \bullet & \circ & \circ\end{smallmatrix}\right]$. Let γ be the branch of $\mathbf{\Gamma}$ at P tangent to the axis $y = 0$. By Lemma 3.2 we have $v_\gamma(c) - v_\gamma(b) = 2$. The values of v_γ on the monomials involved in b and c are $v_\gamma(1, xy, y, y^2) = (0, 0, 1, 2)$. Hence $c_{00} = c_{01} = 0$, i.e., $c = c_{02}y^2$. It follows that $c_{02} \neq 0$ (otherwise Δ would be equal to b^2), so we can assume that $c_{02} = 1$.

Thus, the identity $b(\xi,\eta)\dot\eta = c(\xi,\eta)\dot\xi$ takes form

$$b_{00} + b_{01}t + b_{02}t^2 + b_{11}(t^3 - 3t^2 + 3t - 1) = t^2(2t - 3 + t^{-2}).$$

Equating the coefficients of $t^3, t^2, t, 1$, we find $b_{11} = 2$, $b_{02} = 3$, $b_{01} = -6$, $b_{00} = 3$, i.e., $b = 3(y-1)^2 + 2xy$. and hence $b(\xi,\eta) = 2t^3 - 3t^2 + 1$. Substituting all these into $a(\xi,\eta)\dot\eta = b(\xi,\eta)\dot\xi$, we obtain

$$a_{20}(t^4 + \cdots + t^{-2}) = (2t^3 - 3t^2 + 1)(2t - 3 + t^{-2}) = 4t^4 + \cdots + t^{-2}.$$

Contradiction. $\square$ ∎

Lemma 3.12. *Suppose that C_3 is a cuspidal cubic. Then L_∞ is tangent to C_3 at some point P. Let F be the flex point of C_3.*
(a) If P is the cusp, then $\mathbf{\Gamma} = \Delta_3$ and $P \in C_1$.
(b) If $P = F$, then C_1 is any line. If, moreover, $\mathbf{\Gamma} = \Delta$, then either $F \in C_1$ or C_1 is tangent to C_3.
(c) If P is not as above, then $\mathbf{\Gamma} = \Delta_3$ and C_1 is the line (PF).

Proof — Let us prove that L_∞ is tangent to C_3. Suppose, it is not. Then, $C_3 \cap L_\infty \subset C_1$. Indeed, let $Q \in C_3 \cap L_\infty$. If Q is the cusp of C_3, then $Q \in C_1$ by Corollary 3.4(b). If Q is a smooth point og C_3, then $Q \neq F$ by Lemma 3.5(a) and Riemann-Hurwitz formula for the projection from Q implies that there is a line through Q tangent to C_3, hence $Q \in C_1$ by Lemma 3.6(b). Thus, $C_3 \cap L_\infty \subset C_1$. Since $C_3 \cap L_\infty$ contains at least two points, this implies $C_1 = L_\infty$ which is impossible because $F \notin L_\infty$ by Lemma 3.5(a) and $F \in C_1$ by Lemma 3.5(b)

So, let P be the point where C_3 is tangent to L_∞.

(a) Follows from Lemma 3.5(b).

(b) Suppose that $\mathbf{\Gamma} = \Delta$ and $F \notin C_1$. Let $Q = C_1 \cap L_\infty$. Let L be a line through Q tangent to C_3 at a finite point. Then $L = C_1$ by Lemma 3.6(b).

(c) By Lemma 3.5(b), we have $F \in C_1$ and $\mathbf{\Gamma} \neq \Delta_3$. Moreover, Riemann-Hurwitz formula for the projection from Q implies that there is a line through Q tangent to C_3, hence $Q \in C_1$ by Lemma 3.6(b). $\square$ ∎

Remark 3.13. *The cusp at infinity may be reduced to the case $y^3 = x$ with $\Delta = x(y^3 - x)$. It leads to a non compact domain. Moreover, because of the form of the measure, it is not possible even in the non compact case.*

Proposition 3.14. *Suppose that C_3 is a nodal cubic. Then, in suitable affine coordinates we have*

$$a = 4x(x+1), \qquad b = 2y(3x+2), \qquad c = 3x^2 + 9y^2 + 4,$$

$$\mathbf{\Gamma} = \Delta_3 = x^3 + x^2 - y^2, \qquad \Delta_1 = 4(3x+4).$$

Proof — By Lemma 3.11, we can choose coordinates so that C_3 is parametrized by $x = \xi(t) = t^2 - 1$ and $y = \eta(t) = t^3 - t$. ∎

Remark 3.15. *The case where the node is at infinity may be eliminated by direct calculus.*

3.4 Quadratic factor of Δ

In this section we suppose that $\Delta = \Delta_2 \tilde{\Delta}_2$ where Δ_2 is an irreducible quadratic factor of Δ and a component of the boundary. Then we look at all possibles values of a, b, c such that (3.17) and (3.18) are fulfilled. We may reduce by affine transformation to the three cases:

$$\Delta_2 = 1 - x^2 - y^2 \tag{3.33}$$

$$\Delta_2 = 1 - xy \tag{3.34}$$

$$\Delta_2 = y - x^2 \tag{3.35}$$

By direct computation, it is easy to see that the matrix (g) are given by

1. In case (3.33)
$$g = \Delta_2 \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} + r \begin{pmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \end{pmatrix}$$
2. In case (3.34)
$$g = \Delta_2 \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} + r \begin{pmatrix} x^2 & -1 \\ -1 & y^2 \end{pmatrix}$$
3. In case (3.35)
$$g = \Delta_2 \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} + r \begin{pmatrix} x & 2y \\ 2y & 4xy \end{pmatrix} + (\lambda + \mu y) \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}$$

We may easily eliminate (3.34) which leads to no compact solutions. In case (3.33), by rotation we may reduce to the case $\beta = 0$. And in case (3.35), to have a compact solution then (3.17) and (3.18) must be satisfied for $\tilde{\Delta}_2 = \Delta/\Delta_2$ and by affine transformations, we are reduced to three cases: two coaxial parabolas (with the limit case of a parabola and a line), a parabola with a tangent and a secant, and a parabola with two tangents.

4 The bounded 2-dimensional models

4.1 Generalities

In this section, we will explore separately the various 2 dimensional compact models. It turns out that for some values (in general half-integer) of the parameters appearing in the measure, one may produce a geometric interpretation, coming in general from Lie groups or symmetric spaces, as it is the case for the one dimensional Jacobi operator (Section 2.1). We do not pretend to present all the possible origins of the various models, but we provide some insight whenever they are at hand and relatively easy to produce. Moreover, these geometric interpretation may lead to natural higher dimensional models for the DOP problem.

Recall that the boundary of Ω is an algebraic curve of degree at most 4. When the degree is 4, this boundary is $\{\Delta = 0\}$ where Δ is the determinant of the matrix (g^{ij}). Among the admissible measures, one may chose $\rho(x) = \Delta^{-1/2}$, which corresponds to the Laplace-Beltrami operator associated with the (co-)metric g. It turns out that in every such example, this Laplace-Beltrami operator has constant curvature, either 0 or positive. We did not succeed in proving this fact in the general setting (and we do not even know if is is true in higher dimension). However, when the boundary has degree less than 4, it is not always true that the curvature is constant (see Section 4.8). But even in this latter case, when the measure has density $\Delta_1^{-1/2}$, where Δ_1 is the irreducible equation of the boundary (while in this model Δ has degree 4 and Δ_1 degree 3), there exists a natural interpretation coming from a 4 dimensional sphere.

Then, one may interpret the associated model as some quotient of the Euclidean or spherical Laplace operator through some discrete or continuous symmetry subgroups. When the curvature is 0 (Section 4.7 and Section 4.12), this shows some relation with root systems and the associated Hall polynomials [39], with connection with Hecke algebras. See also Araki [3] and Harish-Chandra [22, 23]. Many other natural geometric interpretations come from spherical functions on rank 2 symmetric spaces (see Helgason [28], Heckman et al. [25, 24, 43, 26]). For references on Dunkl operators, we also refer to Dunkl [15] or more recently to Rösler [47].

From a general point of view, there is a dictionary linking the angles of the reflection associated to the symmetries and the type of singularities of the boundary of Ω: double points, cusps and double tangents correspond respectively to angles $\pi/2$, $\pi/3$, $\pi/4$.

It turns out that many of the models described above have some nice geometric interpretation in terms of compact homogeneous spaces $M = G/H$: we try to interpret the given operator as the Laplace-Beltrami operator Δ_G on G acting on some specific functions $(X, Y) : G \mapsto \mathbb{R}^2$.

In this whole section, the identification will be made with the Laplace operator acting on the d-dimensional sphere, on the d-dimensional Euclidean space or on some classical Lie group such as $SO(d)$ and $SU(d)$. For the sake of clarity, we recall here some well known formulas and facts on these operators. The general principle is the following. When $\mathbf{L}$ is a Laplace-Beltrami operator (or more generally any second order differential operator with no 0-order term) on some model space E, recall that the associated carré du champ is defined by

$$\mathbf{\Gamma}_L(f, g) = \frac{1}{2}\Big(\mathbf{L}(fg) - f\mathbf{L}(g) - g\mathbf{L}(f)\Big),$$

and $\mathbf{L}$ satisfies the change of variable formula (2.5). Then, we we are looking for pairs (X^1, X^2) of real functions $E \mapsto \mathcal{R}$ such that $\mathbf{L}(X^i) = L^i(X^1, X^2)$, and $\mathbf{\Gamma}(X^i, X^j) = G^{ij}(X^1, X^2)$, where L^i are degree 1 polynomials and G^{ij} are degree 2 polynomials in the two variables (X^1, X^2). Then, from the change of variable formula (2.5), for any smooth function $\Phi : \mathcal{R}^2 \mapsto \mathcal{R}$, one has $\mathbf{L}\big(\Phi(X^1, X^2)\big) = \mathbf{L}_1(\Phi)(X^1, X^2)$, where

$$\mathbf{L}_1(f) = \sum_{ij} G^{ij}(x)\partial^2_{ij} f + \sum_i L^i(x)\partial_i f.$$

We shall say that such an operator $\mathbf{L}_1$ is the image measure of $\mathbf{L}$ through (X^1, X^2). Moreover, it is immediate that, if μ is the reversible measure for $\mathbf{L}$, then $\mathbf{L}_1$ has reversible measure the image of μ through (X^1, X^2).

Indeed, what is immediate from the study of the various models is the knowledge of $\mathbf{\Gamma}(X^i, X^j) = g^{ij}$ and the density measure ρ. From (1.2), it is then immediate that

$$L^i(x) = \sum_j \partial_j g^{ij} + g^{ij}\partial_j \log \rho. \tag{4.36}$$

Through an affine change of coordinates, one is reduced to find two eigenvectors X^1 and X^2 of $\mathbf{L}$ for which $\mathbf{\Gamma}(X^i, X^j)$ satisfy a quadratic relation $\mathbf{\Gamma}(X^i, X^j) = G^{ij}(X^i, X^j)$. In this respect, similar problems are studied (although mainly in dimension 1) in the study of isoparametric surfaces (see Cartan [10, 9, 11, 12]).

It can be quite hard to find from which model space a given model comes from. Spectral analysis can be useful: indeed, for any polynomial $P(x, y)$ and whenever $P(X^1, X^2) \in \mathcal{L}^2(\mu)$, the spectrum of $\mathbf{L}_1$ is embedded in the discrete spectrum of $\mathbf{L}$. But, as it happens in Section 4.7 and Section 4.12, it could be that the reversible measure μ for $\mathbf{L}$ has infinite mass, and that X^1 and X^2 are eigenvectors for $\mathbf{L}$ which are not in $\mathcal{L}^2(\mu)$. However, whenever $\mathbf{L}_1$ is the image of some geometric operator $\mathbf{L}$ on some compact model space E, the spectrum of $\mathbf{L}_1$ is imbedded in the spectrum of

L. Nevertheless, this could be misleading in some specific situation. For example, on the unit sphere $\mathbb{S}^d$ imbedded in $\mathbb{R}^d$ with the induced Riemanian metric, the spectrum of the associated Laplace-Beltrami operator $\Delta_{\mathbb{S}^d}$ is $\{-k(k+d-1), k \in \mathbb{N}\}$. But then, for any integer p, the spectrum of $p^2\Delta_{\mathbb{S}^d}$ is included into $\{k(k+p(d-1), k \in \mathbb{N}\}$, which is the spectrum of a sphere with dimension $p(d-1)+1$, and therefore, since in general we know $\mathbf{L}_1$ only up to some scaling factor, we are not even able to determine the dimension of the sphere it may come from (if ever). We already saw this phenomenon in the case of Jacobi operators with parameter (p,p) for which we have two distinct geometric interpretation, one coming from $\mathbb{S}^p$ and another one coming from $\mathbb{S}^{2p-1}$ (Section 2.1).

As mentioned above, for the purpose of the description of our 11 models, we shall mainly use a few model spaces, namely Euclidean, spheres, $SO(d)$ and $SU(d)$. In order to be able to carry the identification described above from these models, it is worth to describe the Laplace-Beltrami (or Casimir) operators for those models, in a simple way leading to the further interpretations. In some cases, it is also useful to extend the operators $\mathbf{L}$ and $\mathbf{\Gamma}$ to complex valued functions, and we shall do that without further notice.

If E_d is an d-dimensional Euclidean space, and x^i the coordinates in some orthonormal basis, one has for the Euclidean Laplace operator Δ_{E_d}:

$$\Delta_{E_d}(x^i) = 0, \ \mathbf{\Gamma}_{E_d}(x^i, x^j) = \delta^{ij},$$

and the dimension does not appear in these relations, hence we can omit the subscript d. These descriptions of course do no depend of the chosen orthonormal basis in E_d.

On the unit sphere $\mathbb{S}^d$ imbedded into $\mathbb{R}^{d+1}$, and for the restriction to $\mathbb{S}^d$ of the same Euclidean coordinates x^i, one has for the Laplace operator $\Delta_{\mathbb{S}^d}$

$$\Delta_{\mathbb{S}^d}(x^i) = -dx^i, \ \mathbf{\Gamma}_{\mathbb{S}^d}(x^i, x^j) = \delta^{ij} - x^i x^j. \tag{4.37}$$

As previously, we can write $\mathbf{\Gamma}_{\mathbb{S}}$ since it does not depend on the dimension. When F is the restriction to the unit sphere in $\mathbb{R}^{d+1}$ of some smooth function in the Euclidean space , $\Delta_{\mathbb{S}^d}(F)$ and $\mathbf{\Gamma}_S(F)$ may be computed from the related quantities Δ_E and $\mathbf{\Gamma}_E$ in the ambient Euclidean space, since they are the restriction to the sphere of the quantities

$$\Delta_E(F) - (r\partial_r)^2 F - (d-1) r\partial_r F, \ \mathbf{\Gamma}_S(F) = \mathbf{\Gamma}_E(S) - (r\partial_r F)^2 \tag{4.38}$$

where $r\partial_r F = \sum_i x^i \partial_i F$.

As mentioned above the spectrum of $-\Delta_{\mathbb{S}^d}$ is $\{k(k+d-1), k \in \mathbb{N}\}$. The eigenspace associated with $-k(k+d-1)$ consists of the restriction to the sphere of degree k harmonic homogeneous polynomials (see Stein and Weiss [50]).

Beyond the case of spheres, we shall also use Casimir operators on the semi-simple groups $SU(d)$ and $SO(d)$. Once again, in order to describe them, we consider the entries x^{ij} ($SO(d)$ case) and z^{ij} ($SU(d)$ case) as functions on the group (complex valued in the latter case) and describe the operators through the action of $\mathbf{L}$ and $\mathbf{\Gamma}$ on them.

For $SO(d)$, up to some constant, we have,

$$\Delta_{SO(d)}(x^{ij}) = -(d-1)x^{ij}, \ \mathbf{\Gamma}_{SO(d)}(x^{kl}, x^{pq}) = \delta^{kp}\delta^{lq} - x^{kq}x^{pl}. \tag{4.39}$$

For $SU(d)$ the formulas are similar.

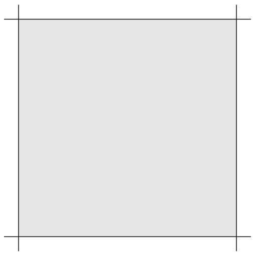
$$\Delta_{SU(d)}(z^{ij}) = -2\frac{(d-1)(d+1)}{d} z^{ij}, \tag{4.40}$$

and also

$$\mathbf{\Gamma}_{SU(d)}(z^{kl}, z^{pq}) = -2z^{kq}z^{pl} + \frac{2}{d}z^{kl}z^{pq}, \ \mathbf{\Gamma}_{SU(d)}(z^{kl}, \bar{z}^{pq}) = 2(\delta^{kp}\delta^{lq} - \frac{1}{d}z^{kl}\bar{z}^{pq}). \tag{4.41}$$

In order not to get confused in the notations, we shall use upper case letters (X, Y) instead of (X^1, X^2) for the coordinate system in the different 2-dimensional models Ω, and lower case letters (x^i) for the coordinates on the geometric model it comes from.

4.2 The square or rectangle



This is the simplest model. By affine transformation, we may chose the square to be $[-1, 1] \times [-1, 1]$. The metric is

$$G = \begin{pmatrix} 1 - X^2 & 0 \\ 0 & 1 - Y^2 \end{pmatrix}$$

and the density of the measure is

$$\rho(X, Y) = C(1 - X)^a(1 + X)^b(1 - Y)^c(1 + Y)^d.$$

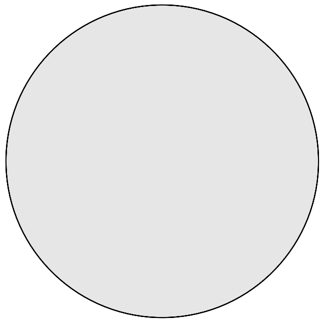
This corresponds to the products of dimension 1 Jacobi polynomials. We recall that for any $p, q \in \mathbb{N}$, the one dimensional Jacobi operator $\mathbf{L}_{p,q}$ with reversible measure $(1 - X)^{(p-2)/2}(1 + X)^{(q-2)/2}$ can be realized on a $p + q - 1$ dimensional sphere $\{(x^1)^2 + \ldots + (x^{p+q})^2 = 1\}$ through the function $X = 2\big((x^1)^2 + \ldots + (x^p)^2\big) - 1$. Hence we have

$$\Delta_{\mathbb{S}^{p+q-1}}\big(h(X)\big) = 4\mathbf{L}_{p,q}(h)(X).$$

Since the boundary is degree four, among the admissible density measures is $\det(g)^{-1/2}$, and the metric is then the Euclidean metric, through the change of coordinates $X = \cos(x^1)$, $Y = \cos(x^2)$. Then, the operator is nothing else than the Laplace operator on $\mathbb{R}^2$, acting on functions which are invariant under the symmetries with respect to the lines $\{x^1 = k\pi\}$, $\{x^2 = k'\pi\}$, which is the square lattice in $\mathbb{R}^2$. Of course, this is covered by the first case since when $p = q = 1$, the sphere is nothing else than the 1-dimensional torus.

Therefore, this square model for half integer values of the coefficients (a, b, c, d) may be seen as images of products of spheres. We already mentioned also the various interpretations coming from compact rank 1 symmetric spaces.

4.3 The circle



We may chose Ω to be the unit disk in $\mathbb{R}^2$. In this case, the metric is not unique, and, up to scaling, depends on 2 free parameters. Up to some rotation,

$$G_{a,b,c} = (1 - X^2 - Y^2)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + c\begin{pmatrix} 1 - X^2 & -XY \\ -XY & 1 - Y^2 \end{pmatrix}.$$

Ellipticity imposes $c \geq 0$, and whenever $c \neq 0$, we may reduce by homogeneity to $c = 1$. We concentrate only on this case.

Ellipticity condition also imposes $a > -1$, $b > -1$. When $a, b \neq 0$, the determinant Δ writes $(1 - X^2 - Y^2)P_2(X, Y)$, where P_2 has degree 2 and is irreducible (and is constant whenever $a = b = 0$). Comparing Proposition 2.13 and formula (2.11) with the value of the determinant, it is easily seen that the only admissible measures have density

$$\rho_p(X, Y) = C(1 - X^2 - Y^2)^p.$$

This remains the case even when $ab = 0$, although in this case P_2 is real reducible. In complex notations, with $Z = X + iY$, the operator associated with $c = 1$ and measure with density $C(1 - X^2 - Y^2)^p$ may be described from

$$\mathbf{L}_{p,a,b,1}(Z) = -\big(2p + 3 + (a + b)(p + 1)\big)Z - (a - b)(p + 1)\bar{Z},$$

$$\mathbf{\Gamma}_{a,b,1}(Z, Z) = (a - b)(1 - Z\bar{Z}) - Z^2,\ \mathbf{\Gamma}_{a,b,1}(Z, \bar{Z}) = (a + b + 2) - (a + b + 1)Z\bar{Z},$$

with of course the conjugate values for $\mathbf{L}_{p,a,b,1}(\bar{Z})$ and $\mathbf{\Gamma}_{a,b,1}(\bar{Z}, \bar{Z})$.

When $a = b = 0$, this model is well known. The metric has constant curvature, and the operator corresponds for $p = -1/2$, to the Laplace operator on $\mathbb{S}^2 = \{(x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$, acting on functions which are invariant under the symmetry $x^3 \mapsto -x^3$. If one consider the unit disk as a local chart for the upper half-sphere, this is nothing else that the Laplace operator acting on functions of (x^1, x^2). The spectrum is then the spectrum of the sphere. The eigenvalues are $\lambda_k = -k(k + 1)$.

When $p = (d-3)/2$, $d \in \mathbb{N}, d \geq 3$, $a = b = 0$, this still corresponds to a Laplace operator on spheres $\mathbb{S}^d$. More precisely, if one considers the Laplace operator $\Delta_{\mathbb{S}^d}$ on the unit sphere $\mathbb{S}^d = \{(x^1)^2 + \ldots + (x^{d+1})^2 = 1\}$, and consider a function depending only on $(x^1, x^2) = (X, Y)$, one gets

$$\Delta_{\mathbb{S}^d}\big(f(x^1, x^2)\big) = \mathbf{L}_{(d-3)/2,0,0,1}(f)(x^1, x^2).$$

It is therefore the image of $\Delta_{\mathbb{S}^d}$ through the projection $x \mapsto (x_1, x_2)$.

In this case, one may also get some other interpretation, from spheres $\mathbb{S}^{2d-1}$: on the unit sphere $\mathbb{S}^{2d-1} \subset \mathbb{R}^{2d}$, let Z_k be the complex functions which are the restrictions to the sphere of the linear forms $Z_k(x) = \langle e_k, x\rangle + i\langle e_{d+k}, x\rangle$. Then, consider the complex function $U = Z_1^2 + \cdots + Z_d^2$. One may see that

$$\Delta_{\mathbb{S}^{2d-1}}(U) = -4dU,\ \Delta_{\mathbb{S}^{2d-1}}(\bar{U}) = -4d\bar{U},$$

and

$$\mathbf{\Gamma}(U, U) = -4U^2, \mathbf{\Gamma}(U, \bar{U}) = 8 - 4U\bar{U}.$$

Therefore, going to the real forms $U = X + iY$, one has

$$\mathbf{\Gamma}(X, X) = 4(1 - X^2), \mathbf{\Gamma}(Y, Y) = 4(1 - Y^2),\ \mathbf{\Gamma}(X, Y) = -4XY.$$

This corresponds to the operator $4\mathbf{L}_{(d-3)/2,0,0,1}$.

One may also obtain similar forms using $U = \sum_i Z_i Z_i'$, where Z_i and Z_i' are defined in a similar way but on the product of two spheres $\mathbb{S}^{2d-1}$ with the product metric, which leads to $2\mathbf{L}_{d-2,0,0,1}$.

For the other metrics, the situation is more complicated. Still restricting to $c = 1$, the condition for the metric to be non negative on the disk is $a > -1$ and $b > -1$. Even in the case $a = b$, the Laplace operator associated to the metric is no longer a solution to our problem, and one may check that the metric has not constant curvature.

If we restrict our attention to the diagonal case $a = b$, then the equation simplifies. Up to some scaling, the operator may be considered as the sum of the previous operator with $a = b = 0$ and $a(x\partial_y - y\partial_x)^2$, which corresponds to a circular Brownian motion in the plane. But we may construct this in a more geometric way as follows. For $-1 < a < 0$, and density measure $(1 - X^2 - Y^2)^{(p-1)/2}$, one may consider a sphere with radius r, where $a = -r^2/(1 + r^2)$ and dimension $d = 2p + 3$. Then, we chose e_1 and e_2 two vectors in $\mathbb{R}^{d+1}$ which are orthogonal and norm 1, and consider the complex linear forms $Z_1(x) = \langle e_1, x\rangle + i\langle e_2, x\rangle$, that we restrict on the sphere $\mathbb{S}^d$. It satisfies, for the Laplace operator on the sphere

$$\Delta_{\mathbb{S}^d}(Z_1) = -\frac{d}{r^2}Z_1,\ \mathbf{\Gamma}_{\mathbb{S}^d}(Z_1, Z_1) = \frac{1}{r^2}Z_1^2, \mathbf{\Gamma}_{\mathbb{S}^d}(Z_1, \bar{Z}_1) = -\frac{1}{r^2}(2 - Z_1\bar{Z}_1).$$

Consider now the product $\mathbb{S}^1 \times \mathbb{S}^d$, with the product structure and Laplacian $\mathbf{L}$. With the function on $z = e^{i\theta}$ on $\mathbb{S}^1$, we look at the function $Z = zZ_1$. We have, for the product structure

$$\mathbf{L}(Z) = -(\frac{d}{r^2} + 1)Z,\ \mathbf{\Gamma}(Z, Z) = -(1 + \frac{1}{r^2})Z^2,\ \mathbf{\Gamma}(Z, \bar{Z}) = \frac{2}{r^2} + (1 - \frac{1}{r^2})Z\bar{Z}.$$

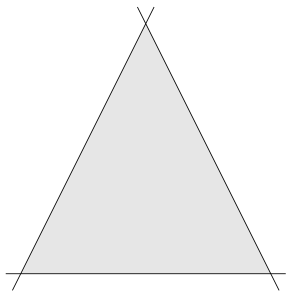
Then, the image of $\dfrac{r^2}{1 + r^2}\mathbf{L}$ through Z is $\mathbf{L}_{p,a,a,1}$. However, the case $a \neq b$ remains mysterious.

This model has an immediate d-dimensional extension. On the unit disk in $\mathbb{R}^d$, one may consider the operator

$$(1-|x|^2)D(a_1,\dots a_n)+(g_0^{ij}(x)), \tag{4.42}$$

where $|x|$ denotes the Euclidean norm, $(g_0^{ij}) = (\delta^{ij} - x^i x^j)$ corresponds to the projection of the spherical metric of $\mathbb{S}^d$ onto an hyperplane, and $D(a_1,\dots,a_n)$ is any diagonal matrix. In fact, it is quite easy to check with this co-metric the boundary condition $g^{ij}\partial_j P = 2(a_i-1)x_i P$, where $P = 1-|x|^2$ is the boundary equation. The condition for the metric to be non negative on the unit ball is again that $a_i > -1$ for any i. Again, when $D(a_1,\dots,a_n)=0$, the choice of the measure $(1-|x|^2)^{(q-1)/2}$ corresponds to a Laplace operator on the $q+d$ sphere. Indeed, adding squares of infinitesimal rotations in various directions, with different coefficients, provide a larger class of matrices (g^{ij}) solutions of the problem for the boundary $|x|^2=1$.

4.4 The triangle



By affine transformation, one may reduce to the case where the triangle is delimited by the lines $X=0$, $X+Y=1$, $Y=0$, such that the domain Ω is the 2-dimensional simplex $\{X\geq 0, Y\geq 0, X+Y\leq 1\}$. Then,the metric depends again on three parameters

$$G_{a,b,c} = \begin{pmatrix} cX(1-X)+aX(1-X-Y) & -cXY \\ -cXY & cY(1-Y)+bY(1-X-Y) \end{pmatrix}.$$

The density of the measure depends on 3 parameters

$$\rho_{p,q,r}(X,Y) = CX^pY^q(1-X-Y)^r\,,$$

which leads to a family of operators $\mathbf{L}_{p,q,r,a,b,c}$, for which

$$\mathbf{L}_{p,q,r,a,b,c}(X) = -X\big((a+1)(r+2)+(1+a)p+q+1\big) - a(1+p)Y + (a+1)(p+1),$$

and a similar form for Y exchanging X and Y, p and q, a and b. This model is closely related to the circle one. We first observe that if we take the circle model, and let the operator act on functions on $x^2=X$, $y^2=Y$, we find the operator on the triangle acting on the variable (X,Y) (the simplex is clearly the image of the disk under this transformation). We obtain in this way the complete family of metrics, but only the measures $\rho_{0,0,r}$, which are the image measures of the measures on the unit disk with density $(1-X^2-Y^2)^r$.

For other values of the measure $\rho_{p,q,r}$, provided p, q are half integers integers, one may use the d-dimensional model on the unit ball. As for the circle case, we restrict our study to $c=1$. Setting $p=\frac{p_1}{2}-1$, $q=\frac{q_1}{2}-2$ and $d=p_1+q_1$, consider the operator given on the unit ball in $\mathbb{R}^d$ by the metric described in formula (4.42), together with the measure $(1-|x|^2)^r$ with $a_1=\ldots=a_{p_1}=a$, $a_{p_1+1}=\ldots=a_d=b$. Now, for this operator $\mathbf{L}^{\mathbb{B}}_{d,r,a,b}$, we consider $X=\sum_1^{p_1}x_i^2$, $Y=\sum_{i=p_1+1}^d x_i^2$: then the image of $\mathbf{L}^{\mathbb{B}}_{d,r,a,b}$ through (X,Y) is $4\mathbf{L}_{p,q,r,a,b,1}$, as easily checked comparing for both cases $\mathbf{L}(X)$, $\mathbf{L}(Y)$, $\mathbf{\Gamma}(X,X)$, $\mathbf{\Gamma}(X,Y)$ and $\mathbf{\Gamma}(Y,Y)$.

Therefore, we see that the triangle case may be interpreted, at least for half integers values of the measure parameters, as images of the unit ball operators, in exactly the same way that one dimensional non symmetric Jacobi operators may be obtained from spheres.

Once again, those operator have an immediate d dimensional extension on the d-dimensional simplex $\{x^i\geq 0, i=1,\ldots,d,\ \sum_i x^i\leq 1\}$, with the (co)-metric G^{ij} given by

$$G^{ij}=x^i\Big(\delta_{ij}\big(1+\alpha_i(1-\sum_j x^j)\big)-x^j\Big),$$

where α_i are constants.

Remark 4.1. *In the same way that we have some other interpretation on the circle coming from complex representations, one may have some other interpretations on the triangle in some particular case. For example, on $\mathbb{S}^5$, consider the complex linear forms $Z_1=x^1+ix^2$, $Z_2=x^3+ix^4$, $Z_3=x^5+ix^6$ restricted to the sphere, and the function*

$$Z=Z_1\bar Z_2+Z_2\bar Z_3+Z_3\bar Z_1.$$

One may check that, for the Laplace $\Delta_{\mathbb{S}^5}$ operator on the sphere, one has

$$\Delta_{\mathbb{S}^5}(Z)=-12Z, \mathbf{\Gamma}_{\mathbb{S}^5}(Z,Z)=4\bar Z-4Z^2, \mathbf{\Gamma}_{\mathbb{S}^5}(Z,\bar Z)=4-4Z\bar Z,$$

which corresponds, through the change of variables $Z=1-\frac{3}{2}(X+Y)+i\frac{\sqrt3}{2}(Y-X)$, then the image of $\frac14\Delta_{\mathbb{S}^5}$ under (X,Y) is $\mathbf{L}_{0,0,0,0,0,1}$.

In the same spirit, one may consider the unit sphere in $\mathbb{R}^{2p_1+r+1}$, where a point in $\mathbb{R}^{2p+r}$ is represented as $(X_1,X_2,X_3)\in\mathbb{R}^p\times\mathbb{R}^p\times\mathbb{R}^r$, the pair $(X,Y)=(\|X_1\|^2+\|X_2\|^2, X_1\cdot X_2)$ satisfy the relations

$$\Delta_{\mathbb{S}^{2p+r}}(X)=-2(2p+r+1)X+2p,\ \Delta_{\mathbb{S}^{2p+r}}(Y)=-2(2p+r+1)Y,$$

and

$$\mathbf{\Gamma}_{\mathbb{S}^{2p+r}}(X)=4X(1-X),\ \mathbf{\Gamma}_{\mathbb{S}^{2p+r}}(X,Y)=4Y(1-X),\ \mathbf{\Gamma}_{\mathbb{S}^{2p+r}}(Y)=X-4Y^2.$$

Let us then chose $X_1=\frac12(X+2Y)$, $Y_1=\frac12(X-2Y)$. We get

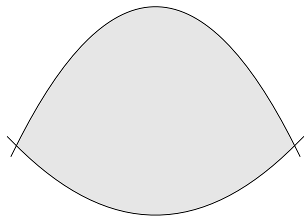
$$\Delta_{\mathbb{S}^{2p_1+r}}X_1=-2(2p_1+r+1)X_1+p_1,\ \Delta_{\mathbb{S}^{2p_1+r}}Y_1=-2(2p_1+r+1)Y_1+p_1,$$

and

$$\mathbf{\Gamma}_{\mathbb{S}^{2p_1+r}}(X_1)=4X_1(1-X_1), \mathbf{\Gamma}_{\mathbb{S}^{2p_1+r}}(Y_1)=4Y_1(1-Y_1),\ \mathbf{\Gamma}_{\mathbb{S}^{2p_1+r}}(X_1,Y_1)=-4X_1Y_1.$$

Then, we see that the image of $\frac14\Delta_{\mathbb{S}^{2p+r}}$ through (X_1,Y_1) is $\mathbf{L}_{(p-1)/2,(p-1)/2,(r-1)/2,0,0,1}$.

4.5 The coaxial parabolas.



Up to affine transforms, the domain may be bounded by the two parabolas $Y = X^2 - 1$ et $Y = 1 - aX^2$. Here $a > -1$ is the condition for the domain to be compact. Indeed, this forms a one parameter family up to affine transformations, but may be reduced to a single model up to some non-linear transform.

The (co)-metric is

$$G_a = \begin{pmatrix} 1 - \frac{1}{2}(1+a)X^2 & X((1-a) - (1+a)Y) \\ X((1-a) - (1+a)Y) & 2(1+a)(1-Y^2) - 4aX^2 \end{pmatrix}.$$

When $a \neq 0$, the boundary has a equation of degree 4, and therefore the Laplace operator associated with the metric is an admissible solution, corresponding to the measure $\det(G_a)^{-1/2}$. It turns out that the associated metric has scalar curvature $1 + a$, and therefore the operator is locally a spherical Laplacian.

In fact, the (non-affine) change of coordinates $Y_1 = Y + aX^2$, $X_1 = \sqrt{1+a}X$ allows us to reduce, up to a scaling parameter, to the case $a = 0$, and then the domain is bounded by the parabola $Y = X^2 - 1$ and $Y = 1$. In this case, one gets

$$G_0 = \begin{pmatrix} 2 - X^2 & 2X((1-Y) \\ 2X(1-Y) & 4(1-Y^2) \end{pmatrix}.$$

Even though the boundary has no longer degree 4 in this case, the Laplace operator is still an admissible solution. In fact, the determinant of the metric is still equal to the irreducible equation of the boundary even in this case. This particular model is known in the literature as the parabolic biangle (see Koornwinder and Schwartz [36]).

For symmetry reasons, we prefer to consider the case $a = 1$, in which case

$$G_1 = \begin{pmatrix} 1 - X^2 & -2XY \\ -2XY & 4(1 - X^2 - Y^2) \end{pmatrix}.$$

When the density of the measure is $\det(G_1)^{-1/2}$, the operator may be directly seen as the image of the Laplace operator on $\mathbb{S}^2 = \{(x^1)^2 + (x^2)^2 + (x^3)^2 = 1\} \subset \mathbb{R}^3$ through (X, Y), where $X = x^3$ and $Y = 2x^1x^2$. Then, the associated operator is nothing else than the spherical Laplace operator acting on functions which are invariant under the symmetries through the hyperplanes $\{x^1 = x^2\}$ and $\{x^1 = -x^2\}$ (their angle is $\pi/2$, which corresponds to the double points of the boundary of the domain).

For the other measures densities with $a = 1$, if we set $\rho = (Y + X^2 - 1)^{(p-1)/2}(1 - Y - X^2)^{(q-1)/2}$ with p and q integers, we obtain an operator $\mathbf{L}_{p,q}$, for which we have we have

$$\mathbf{L}_{p,q}(X) = -(2 + p + q)X, \ \mathbf{L}_{p,q}(Y) = -(6 + 2(p + q))Y - 2(p - q),$$

and this operator is an image of a sphere $\mathbb{S}^{p+q+2} \subset \mathbb{R}^{p+q+3}$, through the following functions: with $x = (x^1, \cdots, x^d)$ with $d = p + q + 3$, choose $X = x_d$, $Y = (x^1)^2 + \ldots (x^{p+1})^2 - (x^{p+2})^2 - \cdots - (x^{p+q+2})^2$. Using formulae (4.37), it is easily checked that they satisfy the required values for $\mathbf{L}_{p,q}(X), \mathbf{L}_{p,q}L(Y)$, $\mathbf{\Gamma}_1(X, X)$, $\mathbf{\Gamma}_1(X, Y)$, and $\mathbf{\Gamma}_1(Y, Y)$.

Although the general case may be reduced to $a = 0$, this latter case offers a more general admissible family of (co)-metrics, namely

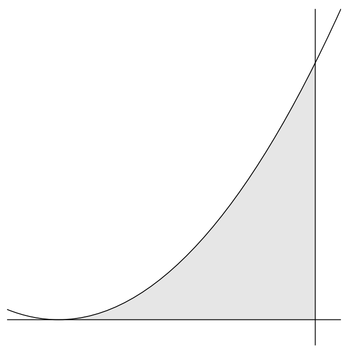
$$G_\alpha = \begin{pmatrix} 2 - X^2 + \alpha(1 + Y - X^2) & 2X(1 - Y) \\ 2X(1 - Y) & 4(1 - Y^2) \end{pmatrix},$$

with $\alpha \geq -1$, and the special case $\alpha = -1$ where

$$G_{-1} = (1 - Y) \begin{pmatrix} 1 & 2X \\ 2X & 4(1 + Y) \end{pmatrix}.$$

In those cases, the associated metric does not have constant curvature.

4.6 The parabola with one tangent and one secant.



Remember that in this case the secant lines cut the line at infinity at the same point than the parabola. Then, up to affine transformation, we may chose the domain Ω with boundaries delimited by the equations

$$Y = X^2, \ Y = 0, \ X = 1.$$

Up to scaling, there is just one (co)-metric which is a solution of the problem, which is

$$G = \begin{pmatrix} 4X(1 - X) & 8Y(1 - X) \\ 8Y(1 - X) & 16Y(X - Y) \end{pmatrix}.$$

Once again, the boundary has degree 4, and the Laplace operator corresponding to the associated metric is a solution, which corresponds to a metric with constant scalar curvature equal to 2 (that is why we chosed this normalization of the metric), and therefore it may be realized on a unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. For the general case with measure density $(X^2 - Y)^p Y^q (1-X)^r$, providing a family $\mathbf{L}_{p,q,r}$ of operators, for which

$$\mathbf{L}_{p,q,r}(X) = 4(2p+2q+3) - 4(4+2p+2q+r)X, \ \mathbf{L}_{p,q,r}(Y) = 16(q+1)X - 8(5+2p+2q+r)Y.$$

The Laplace operator corresponds to $\mathbf{L}_{-1/2,-1/2,-1/2}$, and is the image of $\Delta_{\mathbb{S}^2} = \{(x_1)^2 + (x_2)^2 + (x_3)^2\} = 1$ through (X, Y) where $X = x_1^2 + x_2^2$ and $Y = 4x_1^2 x_2^2$, corresponding to functions on the sphere which are invariant under the symmetries with respect to the hyperplanes $\{x^3 = 0\}$, $\{x^1 = 0\}$, $\{x^2 = 0\}$ and $\{x^1 = \pm x^2\}$. The fundamental domain on the sphere for this group action is a triangle with two $\pi/2$ angles, and one $\pi/4$ angle, which corresponds to the two double points and the bi-tangent point.

For other values of the parameters (p, q, r) we shall set $p = (p_1 - 1)/2, q = (q_1 - 1)/2, r = (r_1 - 1)/2$.

In the case $p_1 = 0$, one may get a geometric interpretation for this model: we look at a unit sphere in $\mathbb{R}^d$, with $d = 2q_1 + r_1 + 3$, and consider the functions T_1, T_2 which are the restrictions to the sphere of the functions $(x^1)^2 + \ldots + (x^{q+1})^2$ and $(x^{q+2})^2 + \ldots + (x^{2(q+1)})^2$. Then we choose

$$X = T_1 + T_2, \ Y = 4T_1T_2.$$

Then, $\mathbf{L}_{-1/2,q,r}$ is the image of $\Delta_{\mathbb{S}^{d-1}}$ through (X, Y).

For $q_1 \geq 1$ and $p_1 = 2^n, n = 1, 2, 3$, we may construct the operator as an image of a sphere in some appropriate dimension. Namely, consider the unit sphere in $\mathbb{R}^{2p_1q_1+r+1}$. For those values of p_1, we may construct (when $p_1 \geq 2$) $p_1 - 1$ orthogonal transformations ℓ_i on $\mathbb{R}^{p_1}$, such that, for any $X \in \mathbb{R}^{p_1}$, with $\|X\| = 1$, $\{X, \ell_1(X), \cdots \ell_{p_1-1}(X)\}$ forms a orthonormal basis. This is done through the complex, quaternionic or octionionic multiplications (say from the left) by the basis elements of the algebra, which provides orthonormal transformations of the space which satisfy the required conditions (although in the octonionic case it is not just a simple application of the algebra rule due to the non-associativity of the product), see Conway and Smith [13]. Indeed, this property fails for higher order Cayley-Dickson algebras.

We consider then a point in $\mathbb{R}^{2q_1p_1}$ as a pair (X_1, Y_1) of vectors in $\mathbb{R}^{q_1} \otimes \mathbb{R}^{p_1}$. The operators ℓ_i lift to this $\mathbb{R}^{p_1} \otimes \mathbb{R}^{q_1}$ into orthogonal transformations which produce mutually orthogonal vectors. Then we consider $X = \|X_1\|^2 + \|Y_1\|^2$ and $Y = 4(\|X_1\|^2\|Y_1\|^2 - |X_1 \cdot Y_1|^2)$. The norm $\|X\|^2$ denotes the usual quadratic norm of X in $\mathbb{R}^{p_1q_1}$ and

$$|X_1 \cdot Y_1|^2 = (X_1 \cdot Y_1)^2 + \sum_{i=1}^{p_1-1} (X_1 \cdot \ell_i(Y_1))^2.$$

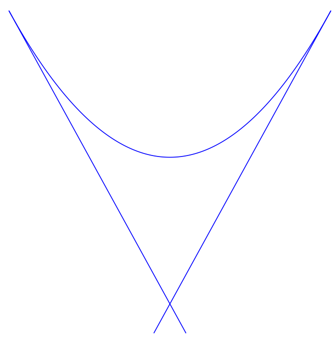
(This notation is reminiscent of the case $p_1 = 2$ this corresponds to complex length of the vector, but is in general not exactly the product in the algebra).

Then, it may be checked that the restriction of the functions X and Y to the unit sphere in $\mathbb{R}^{2p_1q_1+r+1}$ satisfy the relations required for $\boldsymbol{\Gamma}(X, X)$, $\boldsymbol{\Gamma}(X, Y)$ and $\boldsymbol{\Gamma}(Y, Y)$. Indeed, once we have remarked that X and Y are homogeneous with degree respectively 2 and 4 in $\mathbb{R}^{2p_1q_1+r+1}$, and for this value of X, everything boils down to verify that, for the Euclidean operator $\mathbf{\Gamma}_E$ in $\mathbb{R}^{2p_1q_1}$, one has $\mathbf{\Gamma}_E(Y,Y) = 16XY$, which is quite easy to check. Then, one also checks that

$$\Delta_{bS^d}(X) = 4p_1q_1 - 2(d+1)X,\ \Delta_{\mathbb{S}^d}(Y) = 8p_1(q_1-1)X - 4(d+3)Y,$$

wich corresponds to $\mathbf{L}_{p,q,r}$ with $p = (p_1-1)/2$, $q = (p_1q_1-3)/2$, $r = d+8-2p_1(q_1+1)$.

4.7 The parabola with two tangents



Here, the domain Ω is delimited by the equations

$$Y = X^2,\ Y = 2X-1,\ Y = -2X-1.$$

With this boundary, up to scaling, the (co)-metric is unique and is

$$G = \begin{pmatrix} Y+1-2X^2) & 2X(1-Y) \\ 2X(1-Y) & 4(2X^2-Y-Y^2) \end{pmatrix}.$$

Once again, the boundary has degree 4, the Laplace operator corresponding to this (co)-metric is a solution, and has constant curvature 0. For the general density measure $(X^2-Y)^{p_1}(Y-2X+1)^{p_2}(Y+2X+1)^{p_3}$, we get an operator $\mathbf{L}_{p_1,p_2,p_3}$ with

$$\begin{aligned} \mathbf{L}_{p_1,p_2,p_3}(X) &= 2(p_3-p_2) - 2X(3+2p_1+p_2+p_3), \\ \mathbf{L}_{p_1,p_2,p_3}(Y) &= -2(1+2p_1) + 4X(p_3-p_2) - 2Y(5+2p_1+2p_2+2p_3). \end{aligned}$$

Wen $p_1 = p_2 = p_3 = -1/2$, this corresponds to the image of a 2-dimensional Euclidean Laplacian, constructed from the root system B_2 as follows. Consider in $\mathbb{R}^2$, with canonic basis (e_1, e_2), the 4 roots $\lambda_j = \pm\sqrt{2}e_i$, and the 4 roots $\mu_j = \pm\sqrt{2}e_i \pm \sqrt{2}e_j$ (the factor $\sqrt{2}$ is there to fit with the final values of X and Y). Then, let $X(x,y) = \frac{1}{4}\sum_{j=1}^4 \exp(i\lambda_j.(x,y)) = (\cos(\sqrt{2}x) + \cos(\sqrt{2}y))/2,\ Y(x,y) = \frac{1}{4}\sum_{j=1}^4 \exp(i\mu_j.(x,y)) = \cos(\sqrt{2}x)\cos(\sqrt{2}y)$. Then, it is directly checked that $\Gamma_{\mathbb{R}^2}(X,X)$, $\Gamma_{\mathbb{R}^2}(X,Y)$, $\Gamma_{\mathbb{R}^2}(Y,Y)$, $\Delta_{\mathbb{R}^2}(X)$, $\Delta_{\mathbb{R}^2}(Y)$ satisfy the relations required for $\mathbf{L}_{-1/2,-1/2,-1/2}$. This is just one example of the family of Jack polynomials associated with root systems (see MacDonald [40]). Following Koornwinder [30, 31], one may find other representations for symmetric rank 2 spaces with restricted root systems B_2 (which include for example $SO(5)$ and $SO(d+2)/SO(d)$). For a reference on this model, see also Sprinkhuizen-Kuyper [49]. For the sake of completeness, we give below some naive representations of those models coming from the Laplace-Beltrami operator on $SO(d)$ described in (4.39). One may find more complete descriptions of those models in Doumerc's thesis [14]. Moreover, this allows us to show how to deal in a convenient way with matrix operators.

For a given operator on square matrices in dimension d, such as the one described in (4.39) or (4.40) and (4.41), one may consider the image of the operator on the spectrum, determined by the coefficients $a_0, \cdots, a_{d-1}$ of the characteristic polynomial $P(\lambda) = \det(M - \lambda \mathrm{Id})$. Of course, for small values of d, one may perform computations by hand, but it is perhaps worth to describe general methods.

The first task is to compute the various derivatives with respect to the entries M_{ij} of M of the various coefficients of $P(\lambda)$. One may start from the comatrix $\hat{M} = \hat{M}_{ij}$ for which $\hat{M}^t = \det(M)M^{-1}$ (where M^t is the transposed of M) and satisfies $\partial_{M_{ij}} \hat{M}_{ij} = 0$. Together with $\partial_{M_{ij}} M^{-1}_{kl} = -M^{-1}_{ki} M^{-1}_{jl}$, we get $\partial_{M_{ij}} \log\det(M) = M^{-1}_{ji}$ (which are valid on the dense domain where $\det(M) \neq 0$).

Now, for an operator on matrices satisfying $\mathbf{L}(M_{ij}) = -\mu M_{ij}$ and $\mathbf{\Gamma}(M_{ij}, M_{kl}) = \delta_{ik}\delta_{jl} - M_{il}M_{jk}$, denoting $M(\lambda)$ the matrix $M - \lambda \mathrm{Id}$, the previous formulae together with the change of variable formula (2.5) leads to

$$\begin{aligned}\mathbf{L}\big(\log P(\lambda)\big) = \quad & -\mu \mathrm{trace}\Big(M(0)M(\lambda)^{-1}\Big) - \mathrm{trace}\Big(M^{-1}(\lambda)M^t(\lambda)^{-1}\Big) \\ & + \Big(\mathrm{trace}\,(M(0)M(\lambda)^{-1})\Big)^2,\end{aligned}$$

$$\begin{aligned}\mathbf{\Gamma}\Big(\log\big(P(\lambda_1)\big), \log\big(P(\lambda_2)\big)\Big) = \quad & \mathrm{trace}\Big(M^t(\lambda_1)^{-1}M(\lambda_2)^{-1}\Big) \\ & -\mathrm{trace}\Big(M(\lambda_1)^{-1}M(0)M(\lambda_2)^{-1}M(0)\Big).\end{aligned}$$

For the special case of $SO(d)$ where $\mu = d-1$ and $M^t = M^{-1}$, one gets

$$\Delta_{SO(d)}\big(\log P(\lambda)\big) = -\frac{d\lambda^2}{1-\lambda^2} + \lambda\frac{\partial_\lambda P}{P}\big(\frac{1+\lambda^2}{1-\lambda^2} - d\big) + \Big(\lambda\frac{\partial_\lambda P}{P}\Big)^2,$$

together with

$$\begin{aligned}\mathbf{\Gamma}_{SO(d)}\big(\log P(\lambda_1), \log P(\lambda_2)\big) = \quad & \frac{1}{1-\lambda_1\lambda_2}\Big(d\lambda_1\lambda_2 - \lambda_1\frac{P'}{P}(\lambda_1) - \lambda_2\frac{P'}{P}(\lambda_2)\Big) \\ & + \frac{1}{\lambda_1 - \lambda_2}\Big(\lambda_1^2\frac{\partial_\lambda P}{P}(\lambda_1) - \lambda_2^2\frac{\partial_\lambda P}{P}(\lambda_2)\Big),\end{aligned}$$

which lead to the very simple formula

$$\Delta_{SO(d)}\big(P(\lambda)\big) = -(d-1)\lambda P'(\lambda) + \lambda^2 P''(\lambda).$$

For $d = 4$, writing $P(\lambda) = \lambda^4 + X\lambda^3 + Y\lambda^2 + 1$, one gets

$$\Delta_{SO(4)}(X) = -3X, \Delta_{SO(4)}(Y) = -4Y$$

and

$$\mathbf{\Gamma}_{SO(4)}(X) = 4 - X^2 + 2Y,\ \mathbf{\Gamma}_{SO(4)}(X,Y) = -X(Y-6),\ \mathbf{\Gamma}_{SO(4)}(Y) = 4 - X^2 + 2Y.$$

From this, we see that $\mathbf{L}_{1/2,-1/2,-1/2}$ is the image of $2\Delta_{SO(4)}$ through $(\frac{X}{4}, \frac{Y-2}{4})$.

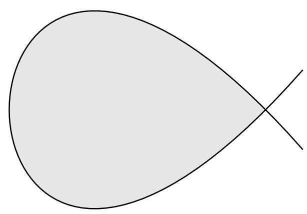
For $SO(5)$, setting $P(\lambda) = \lambda^5 + X\lambda^4 + Y\lambda^3 + Y\lambda^2 + X\lambda + 1$, and $X = 4X_1 + 1$, $Y = 4X_1 + 4Y_1 + 2$, one sees with the same method that the image of $2\Delta_{SO(5)}$ through (X_1, Y_1) is $\mathbf{L}_{1/2,-1/2,1/2}$.

One may also project $\Delta_{SO(d)}$ on any $p \times q$ sub matrix $\hat{M}$ (it is obvious from formulae (4.39) that the operator projects). It is less obvious that it also projects on the square $q \times q$ matrices $N = \hat{M}^t \hat{M}$, and produces on the entries N_{ij} of those matrices the operator defined by $\Delta_{SO(d)}(N_{ij}) = -2dN_{ij} + 2p\delta_{ij}$ and

$$\mathbf{\Gamma}_{SO(d)}(N_{ij}, N_{kl}) = N_{ik}\delta_{jl} + N_{il}\delta_{jk} + N_{jk}\delta_{il} + N_{jl}\delta_{ik} - 2(N_{ik}N_{jl} + N_{ik}N_{jl}).$$

Again, this projects on the spectrum of such matrices. In particular, when $q = 2$, one may chose as variables trace $(N) = X + 1$ and $4\det(N) = Y + 2X + 1$, and, for $p \geq 2$ and $d \geq p + 2$, the image of $\frac{1}{2}\Delta_{SO(d)}$ through (X, Y) is $\mathbf{L}_{0,(d-3-p)/2,(p-3)/2}$. For $p = 1$, the image is obviously degenerate, and concentrated on the boundary $\{Y + 2X + 1 = 0\}$, while for $d = p + 1$, it concentrates on $\{Y - 2X + 1 = 0\}$, as would do the image measure setting $p = 1 + \epsilon$ or $p = d - 1 + \epsilon$ and letting $\epsilon \to 0$. In more complicated setting, one may follow this remark to detect algebraic relations between various components of the system under study.

4.8 The nodal cubic



In this situation, we may choose the equation of the boundary to be $Y^2 = X^2(1 - X)$. There is a unique metric up to scaling

$$G = \begin{pmatrix} 4X(1-X) & 2Y(2-3X) \\ 2Y(2-3X) & 4X - 3X^2 - 9Y^2 \end{pmatrix}.$$

The boundary is degree 3, and in this situation the measure density $\rho(x) = \det(G)^{-1/2}$ is not an admissible measure (it does not satisfy equationı (2.11), as one may check directly). Also, the metric has a non constant curvature. The general form of the density measure is $\rho_p(X, Y) = \big(X^2(1 - X) - Y^2\big)^p$, for which we have

$$\mathbf{L}(X) = 2\big(4(p + 1) - (7 + 6p)\big)X\,, \quad \mathbf{L}(Y) = -6(4 + 3p)Y\,.$$

It turns out that for $p = -1/2$, the operator may be interpreted from a 3-dimensional sphere, through a projection which is very close to the Hopf fibration. Indeed, on the unit sphere $\mathbb{S}^3 = \{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$, consider the functions$X = (x^1)^2 + (x^2)^2$ and $Y = \big((x^1)^2 - (x^2)^2\big)x^3 + 2x^1x^2x^4$, we may check directly that they satisfy the required equations on $\Delta_{\mathbb{S}^3}(X)$, $\Delta_{\mathbb{S}^3}(Y)$, $\mathbf{\Gamma}_{\mathbb{S}^3}(X, X)$, $\mathbf{\Gamma}_{\mathbb{S}^3}(X, Y)$ and $\mathbf{\Gamma}_{\mathbb{S}^3}(Y, Y)$.

To understand which functions on the sphere are of the form $f(X,Y)$, one may represent the sphere in complex notations as $\{|z_1|^2+|z_2|^2=1\}$, where $(z_1,z_2)\in\mathbb{C}^2$, that we write in polar coordinates as $z_j=\rho_j\exp(i\theta_j)$. We then see that

$$(X,Y)=\big(\rho_1^2,\rho_1^2\rho_2\cos(2\theta_1-\theta_2)\big).$$

Then, (X,Y) is invariant under

$$(z_1,z_2)\mapsto(e^{i\theta}z_1,e^{2i\theta}z_2).$$

Moreover, the quotient of the sphere under this action is parametrized by $(|z_1|^2,z_1^2\bar{z}_2)=(t,re^{i\phi})=T\in\mathbb{R}_+\times\mathbb{C}$, with the relation $r^2=t^2(1-t)$, and therefore this image is homeomorphic to by the product $\partial\Omega\times S^1\simeq S^2$. This set Ω_1 is naturally imbedded into $\mathbb{R}^3$ with a symmetry $x_3\mapsto -x_3$. The domain Ω is then the projection of T onto the hyperplane $\{x_3=0\}$, and the function on $\mathbb{S}^3$ which are of the form $g(X,Y)$ are the functions on this quotient set Ω_1 which are moreover invariant under the symmetry $x_3\mapsto -x_3$.

For other values of the measure parameter, writing $p=(q-1)/2$, and for $q=2^n-1$, $n=1,2,3$, we shall use similar interpretations than the one described in Section 4.6. Let us write a point in $\mathbb{R}^{3(q+1)+1}$ as U,V,W, with $U,V\in\mathbb{R}^{q+1}$ and $W=(w_0,w_1,\cdots,w_{q+1})\in\mathbb{R}^{q+2}$. Consider then $X=\|U\|^2+\|V\|^2$. For the values $q+1=2^n$, as seen above, there exist (for $n\geq 1$) q orthogonal transformations ℓ_j in $\mathbb{R}^{q+1}$ such that, for any unit vector $V\in\mathbb{R}^{q+1}$ $\{V,\ell_1(V),\ell_2(V),\cdots,\ell_q(V)\}$ is an orthonormal basis. Then, on $\mathbb{R}^{2(q+1)}$ one considers the bilinear applications $B_1(U,V)=2U\cdot V, B_2(U,V)=2U\cdot\ell_1(V),\cdots,B_{q+1}=2U\cdot\ell_q(V)$, for which it is immediate that $\sum_{j=1}^{q+1}B_j^2=4\|U\|^2\|V\|^2$. Let also $B_0(U,V)=\|U\|^2-\|V\|^2$, such that $\sum_{i=0}^{q+1}B_i^2=X^2$. Also, for the Euclidean Laplacian on $\mathbb{R}^{2(q+1)}$, one has $\Delta_{\mathbb{E}}B_i=0$, $\mathbf{\Gamma}_{\mathbb{E}}(B_i,B_j)=4\delta_{ij}(\|U\|^2+\|V\|^2)$, $i,j=0,\cdots,q+1$.

We then consider the function $Y=\sum_{i=0}^{q+1}w_iB_i$. For the Euclidean Laplace operator in $\mathbb{R}^{3(q+1)}$, one easily checks that $\mathbf{\Gamma}_{\mathbb{E}}(X,Y)=4Y$ and that $\mathbf{\Gamma}_{\mathbb{E}}(Y,Y)=X^2+4X\|W\|^2$. The comparison (4.38) of spherical Laplace operator and Euclidean one shows that the restrictions of X and Y of the Laplace operator on $\mathbb{R}^{3(q+1)}$ satisfy the required relations for $\mathbf{L}_p$, with $p=(q-1)/2$.

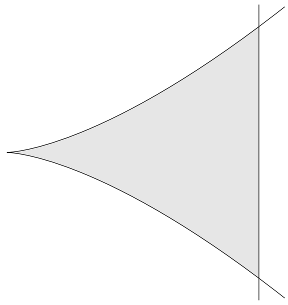
It is perhaps worth to observe that in the above construction, the bilinear applications $B_0,B_1,\cdots,B_{q+1}$, considered as functions on $\mathbb{R}^{2(q+1)}$ are harmonic and satisfy $\mathbf{\Gamma}_{\mathbb{E}}(B_i,B_i)=4\delta_{ij}(\|U\|^2+\|V\|^2)$, and $\sum_i B_i^2=(\|U\|^2+\|V\|^2)^2$. Their restriction $\mathbb{S}^{2q+1}$ satisfy then the same relations (up to some factor 4) than the coordinates on a unit sphere $\mathbb{S}^{q+1}$. Any construction performed on those spheres may be then carried to $\mathbb{S}^{2q+1}$, just replacing X_i by B_i.

4.9 The cuspidal cubic with one secant line

We may choose the boundary equation to be $(Y^2-X^3)(X-1)=0$. Up to scaling, the associated metric is unique and we have

$$G=\begin{pmatrix}4X(1-X), & 6Y(1-X)\\ 6Y(1-X) & 9(X^2-Y^2)\end{pmatrix}.$$

Since the boundary has degree 4, the Laplace operator associated with this metric belongs to the admissible solutions and we may check that the associated metric has constant scalar curvature 2 and therefore may be realized from the unit sphere $\mathbb{S}^2$.



The general density measure is $\rho_{p_1,p_2} = (1-X)^{p_1}(X^3-Y^2)^{p_2}$, for which we have for the associated operator $\mathbf{L}_{p_1,p_2}$

$$\mathbf{L}_{p_1,p_2}(X) = -2(7+2p_1+6p_2)X + 10 + 12p_2, \quad \mathbf{L}_{p_1,p_2}(Y) = -3(8+2p_1+6p_2)Y.$$

For the Laplacian case, $\mathbf{L}_{-1/2,-1/2}$ is the image of $\Delta_{\mathbb{S}^2}$ through

$$X = (x^1)^2 + (x^2)^2, \; Y = x^1\big((x^1)^2 - 3(x^2)^2\big).$$

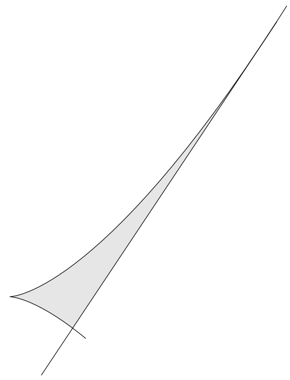
The functions $F(X,Y)$ are the functions on the unit sphere which are invariant under $x^3 \mapsto -x^3$ and such that the projection $z = x^1 + ix^2 = \rho e^{i\theta}$ on the hyperplane $\{x^3 = 0\}$ depend only on ρ and $\cos(3\theta)$. These are the functions which are invariant under symmetries through the hyperplanes $H = \{x^2 = 0\}$ and the two hyperplanes having an angle $\pm\pi/3$ with H. The fundamental domain for these symmetries on the sphere is a triangle with angles $(\pi/3, \pi/2, \pi/2)$, which correspond to one cusp and two double points.

For the other density measures, we may set $q = 2p_1+1$, $q = 2p_2+1$, and consider the unit sphere in $\mathbb{R}^{2+3p} \times \mathbb{R}^{q+1}$. For a point $(U,V) \in \mathbb{R}^{2+3p} \times \mathbb{R}^{q+1}$, we set $X = \|U\|^2$ and we chose for Y some homogeneous degree 3 harmonic polynomial $P(U)$. Then, the various formulae for $\mathbf{L}_{p_1,p_2}(X)$, $\mathbf{L}_{p_1,p_2}(Y)$, $\mathbf{\Gamma}(X)$, $\mathbf{\Gamma}(X,Y)$ and $\mathbf{\Gamma}(Y)$ are satisfied as soon as $\mathbf{\Gamma}_{\mathbb{E}}(Y) = 9\|U\|^4$, where $\mathbf{\Gamma}_{\mathbb{E}}$ denotes the Euclidean operator $\mathbf{\Gamma}$. This problem has been studied by Cartan [10] where he proved that such polynomials exist only for $p = 0, 1, 2, 4, 8$. Beyond the case $p = 0$ (the above example), this corresponds respectively to real, complex, quaternionic and octonionic structures. Such a function (for $p = 1, 2, 4$) may be for example represented as follows: consider a symmetric 3 matrix with trace 0 and respectively real, complex and quaternionic entries. On this space of matrices, one may consider the Euclidean structure given by $\|M\|^2 = \operatorname{trace}(M^t M)$, and, for this structure, the function $Y : M \mapsto \det(M)$, satisfies $\mathbf{\Gamma}_{\mathbb{E}}(M) = \|M\|^2$, as one may check by direct computation (care has to be taken however when computing the determinant of the quaternionic matrix, respecting the order of multiplication). The case $p = 0$ corresponds to diagonal matrices, and the Cayley case is slightly more complex.

4.10 The cuspidal cubic with one tangent

We may choose the boundary equation to be

$$(Y^2 - X^3)(2(Y-1) - 3(X-1)) = 0.$$



Then, up to scaling, there is a unique solution

$$G = \begin{pmatrix} 8(X+Y-2X^2) & 12(Y-2XY+X^2) \\ 12(Y-2XY+X^2) & 18(X-Y)(X+2Y) \end{pmatrix}.$$

The boundary being degree 4, the density measure $\det(G)^{-1/2}$ belongs to the admissible solutions. Therefore, the Laplace operator associated with this (co)-metric is an admissible solution. The scalar curvature is 2, and therefore we may realize this Laplace operator as an image of the spherical Laplacian $\Delta_{\mathbb{S}^2}$.

For the general density measure $\rho = (Y^2 - X^3)^{(p-1)/2}(3X - 2Y - 1)^{(q-1)/2}$

$$\mathbf{L}_{p,q}(X) = -4\big[2X(7+6p+3q)-(5+6p)\big],\ \mathbf{L}_{p,q}(Y) = -6\big[2Y(8+6p+3q)-X(7+6p)\big].$$

For the case $p = q = -1/2$, which corresponds to the Laplace operator, one may see that the operator is the image of a two-dimensional sphere, where X is a degree 4 polynomial and Y has degree 6. Indeed, it is worth to represent X and Y as $X = -\frac{1}{3}(t_1t_2 + t_2t_3 + t_3t_1)$ and $Y = \frac{1}{2}t_1t_2t_3$ with $t_1 + t_2 + t_3 = 0$, which reflects the fact that $Y^2 - X^3$ is the discriminant of the polynomial $T^3 - 3XT + 2Y$.

A solution is given by $t_i = 3(x^i)^2 - 1$, and one may check that all the relations concerning $\mathbf{L}_{-1/2,-1/2}X$, $\mathbf{L}_{-1/2,-1/2}Y$, $\mathbf{\Gamma}(X,X)$, $\mathbf{\Gamma}(X,Y)$ and $\mathbf{\Gamma}(Y,Y)$ are satisfied for this choice.

From this representation, it is clear that X and Y are invariant under the symmetries through the hyperplanes $\{x_i = 0\}$ and $\{x_i = x_j\}$. The fundamental domain for those reflexions is a triangle on the sphere, defined by the hyperplane coordinates, cut along it's three medians, with angles $\pi/2, \pi/3, \pi/4$. This corresponds to one double point, one cusp and one double tangent.

For the general case, setting $p = (p_1 - 1)/2, q = (q_1 - 1)/2$, one may look for a sphere in $\mathbb{R}^d$, with $d = 6p_1 + 3q_1 + 3$. In the particular case $p_1 = 0$, we may take a unit sphere in $\mathbb{R}^{3d}$, with $d = q_1 + 1$, and consider

$$t_1 = \sum_{i=1}^{d}(x^i)^2 - 1/3,\ t_2 = \sum_{i=d+1}^{2d}(x^i)^2 - 1/3,\ t_3 = \sum_{i=2d+1}^{3d}(x^i)^2 - 1/3,$$

and let

$$X = -3(t_1t_2 + t_2t_3 + t_3t_1),\ Y = \frac{27}{2}t_1t_2t_3,$$

then, for the spherical Laplace operator $\mathbf{\Gamma}(X,X)$, $\mathbf{\Gamma}(X,Y)$ and $\mathbf{\Gamma}(Y,Y)$, $\mathbf{L}_{p,q}(X)$ and $\mathbf{L}_{p,q}(Y)$ satisfy the required equations. It is certainly worth to mention that this model may also be seen as the image of the triangle model $\{s_1+s_2+s_3=1,\ s_i\geq 0\}$ through the transformation $X=s_1s_2+s_2s_3+s_3s_1,\ Y=s_1s_2s_3$.

For $p_1=1$, one may consider the following model: we consider (X_1,X_2,X_3) in $\mathbb{R}^{3d}$, where $d=q_1+5$, and the 3×3 symmetric Gram matrix $M_{ij}=(X_i\cdot X_j)-1/3\mathrm{Id}$. Then, the restriction of this matrix to the unit sphere in $\mathbb{R}^{3d}$ has trace 0, and one considers it's characteristic polynomial $P(\lambda)=\det(M-\lambda\mathrm{Id})$. Write $P(\lambda)=\lambda^3-\lambda X/3-2Y/3^3$. Then, the image of the operator $\Delta_{\mathbb{S}^{3d-1}}$ is $\mathbf{L}_{p,q}$. Note that the same construction carries immediately to $p\times p$ trace 0 symmetric matrices in $\mathbb{R}^{pd}$, using the technique described in Section 4.7. Then, the associated operator on $P(\lambda)$ is defined through

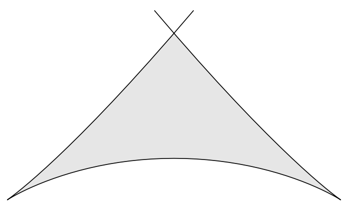
$$\begin{aligned}\mathbf{L}_{p,d}(P)= \quad & 2p(2-2p-pd)P+2P'\big((pd+\frac{1}{p}(p-2)(2p+1))x+\frac{2}{p}(p-1)(p+2)\big)\\ & -4P''\big(x^2+\frac{x}{p}(p+2)+\frac{1}{p^2}(p+1)\big)\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Gamma}(P(x),P(y))= \quad & \frac{4}{x-y}\Big((y+\frac{1}{p})P'(y)P(x)-(x+\frac{1}{p})P'(x)P(y)\Big)\\ & -4(pP(x)-(x+\frac{1}{p})P'(x))(pP(y)-(y+\frac{1}{p})P'(y)).\end{aligned}$$

It is likely (but we did not check), that the same construction on complex or quaternionic matrices provides models for the cases $p_1=2,4$.

4.11 The swallow tail



This is a degree 4 algebraic curve, whose, up to affine transfromations, we may chose the equation to be

$$4\,X^2-27\,X^4+16\,Y-128\,Y^2-144\,X^2Y+256\,Y^3=0.$$

This is the discriminant in T of the polynomial T^4-T^2+XT+Y. Once again, the metric is unique up to scaling, and we have

$$G=\begin{pmatrix}2-8Y-9X^2 & -X(12Y+1)\\ -X(12Y+1) & \frac{3}{2}X^2-16Y^2+4Y\end{pmatrix}.$$

The boundary being degree 4, the measure density $\det(G)^{-1/2}$ is an admissible solution, and for this measure, the corresponding Laplace operator has constant scalar curvature 2, and therefore the operator may be represented on the unit sphere $\mathbb{S}^2$.

For the general density measure $\rho = \det(G)^p$, we have

$$\mathbf{L}_p(X) = -6(5+6p)X,\ \mathbf{L}_p(Y) = -4(11+12p)Y + 3 + 4p.$$

For the Laplace Beltrami case

$$\mathbf{L}_{-1/2}(X) = -12X,\ \mathbf{L}_{-1/2}Y = 1 - 20Y,$$

which corresponds for X to be an eigenvector of degree 3 and $Y-1$ to be an eigenvector of degree 4.

Taking in account that the boundary is a discriminant, we should look for

$$-X = t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2,\ Y = t_1t_2t_3t_4,$$

with

$$t_1 + t_2 + t_3 + t_4 = 0,\ \sum_{i<j} t_it_j = -1,$$

which is the intersection of a sphere $\mathbb{S}^3$ with radius $\sqrt{2}$ with the hyperplane $\{\sum_i t_i = 0\}$, which is again a sphere with radius $\sqrt{2}$.

From his, we may see that if we chose $X = 2\sqrt{2}(x^1+x^2)(x^2+x^3)(x^3+x^1)$ and $Y = -4x^1x^2x^3(x^1+x^2+x^3)$, on the 2-dimensional sphere $\Sigma = \{(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2 = 1,\ x^1+x^2+x^3+x^4 = 0\}$. For the Laplace operator Δ_Σ, then the required relations on $\Delta(X)$, $\Delta(Y)$, $\mathbf{\Gamma}(X,X)$, $\mathbf{\Gamma}(X,Y)$ and $\mathbf{\Gamma}(Y,Y)$ are satisfied and we may see therefore L as an image of this Laplace operator through (X,Y). One has to be careful here, since the linear forms x^i are not orthogonal on the Euclidean space $\sum_{i=1}^4 x^i = 0$. Indeed, one has for the dual Euclidean structure, $|x^i|^2 = 3/4$ and $x^i \cdot x^j = -1/4$ when $i \neq j$

Once again, this is the spherical Laplace operator acting on functions which are invariant under the Weil group of symmetries exchanging (x^1,x^2,x^3,x^4).

For other values of p, the discriminant form of the boundary suggests that one looks at symmetric 4×4 matrices M with vanishing trace, restricted on the sphere $M^*M = \mathrm{Id}$, embedded with the induced spherical structure, in the real, complex and quaternionic cases, and look at the induced operator on the characteristic polynomial $P(\lambda) = \lambda^4 - \lambda^2/2 + \lambda X + Y$, to obtain an operator with parameter $(p-1)/2 = 2+6q$, with $q = 1,2,4$ corresponding to the real, complex and quaternionic case. This is left to the reader as an exercise.

4.12 The deltoid

In this case, up to affine transformation, we may choose the boundary equation to be

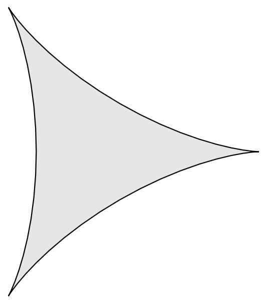
$$(X^2+Y^2)^2 + 18(X^2+Y^2) - 8X^3 + 24XY^2 - 27 = 0.$$

There is a unique metric g up to scaling, which is

$$G = \begin{pmatrix} 9+6X+Y^2-3X^2 & -2Y(2X+3) \\ -2Y(2X+3) & 9-6X+X^2-3Y^2 \end{pmatrix}.$$

For the density measure $\rho = \det(G)^p$, we have

$$\mathbf{L}_p(X) = -2(5+6p)X,\ \mathbf{L}_p(Y) = -2(5+6p)Y.$$



The operator looks simpler in complex variables: setting $Z = X + iY$, one gets

$$\mathbf{\Gamma}(Z,Z) = 12\bar{Z} - 4Z^2, \mathbf{\Gamma}(Z,\bar{Z}) = 18 - 2Z\bar{Z},\ \mathbf{\Gamma}(\bar{Z},\bar{Z}) = 12Z - 4\bar{Z}^2,$$

and

$$\mathbf{L}_p(Z) = -2(5+6p)Z, \mathbf{L}_p(\bar{Z}) = -2(5+6p)\bar{Z}.$$

Under this form, it is easier to check the eigenvalues for $\mathbf{L}_p$, since the actions of the operator on the highest degree term of a polynomial is diagonal, and we se that, for any degree p, the highest degree part of any eigenvector is a monomial, say $Z^q\bar{Z}^r$, for which the eigenvalue is -and the same use of complex variables gives that the eigenvalue corresponding to a polynomial whose highest degree term is $Z^p\bar{Z}^q$ is $-3(q+r)(q+r+4p+2) - (q-r)^2$.

Once again, as it is the case whenever the boundary is degree 4, the density measure $\det(G)^{-1/2}$ is an admissible solution, and this corresponds to a Laplace operator for a metric which has 0 scalar curvature. We may represent this operator from a Euclidean Laplacian in dimension 2. For this choice of the measure, one has $\mathbf{L}_{-1/2}(Z) = -4Z$, and if we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, one may represent this using

$$Z = e^{2i(1\cdot z)} + e^{2i(j\cdot z)} + e^{2i(\bar{j}\cdot z)},$$

where $z = x + iy \in \mathbb{C}$, $1, j, \bar{j}$ are the three third unit roots (solution of $z^3 = 1$) and $z_1 \cdot z_2$ is the euclidean scalar product $\Re(z_1\bar{z}_2)$. One may directly check the $\mathbf{L}_{-1/2}$ is the image of $\Delta_{\mathbb{R}^2}$ through Z. (The interior of the deltoid is indeed the image of $\mathbb{R}^2$ through Z.)

Moreover, the function Z is invariant under the symmetries with respect to the lines

$$D_1 = \{\Im(z) = 0\},\ D_2 = \{te^{i\pi/3}, t \in \mathbb{R}\},\ D_3 = \{ae^{i\pi/6} + te^{2i\pi/3}\},$$

with $a = \pi/\sqrt{3}$. Those three lines determine a equilateral triangle (ABC) in the plane , and any function which have those symmetries is also invariant under all the symmetries with respect to the lines of the triangular network generated by A, B, C (that is all the lines parallel to D_1, D_2, D_3 which are distant from ka, $k \in \mathbb{N}$). (This group of symmetries is the Weyl group associated with the root system A_2.

The deltoid is then the image of the boundary of the triangle (ABC) through Z, and it is not hard to see that the restriction of Z to (ABC) is injective. Then, the functions of the form $F(Z)$ are nothing else than the functions which are invariant under the symmetries of the triangular network, and $\mathbf{L}_{-1/2}$ is just $\Delta_{-1/2}$ acting on functions invariant under the Weyl group associated with A_2.

As usual, this model extends to rank 2 symmetric spaces with restricted root system A_2. For a study of this case, see also Koornwinder [32, 33]. Indeed, it is perhaps worth to notice that the boundary equation is the discriminant of the polynomial $T^3 - ZT^2 + \bar{Z}T + 1$, putting forward the interest of representing Z as $\lambda_1 + \lambda_2 + \lambda_3$ where $|\lambda_i| = 1$ and $\lambda_1\lambda_2\lambda_3 = 1$. In particular, one may consider the Casimir operator on $SU(3)$. Indeed, if Z denotes the trace of a $SU(3)$ matrix, due to the fact that eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ satisfy $|\lambda_i| = 1$, $\lambda_1\lambda_2\lambda_3 = 1$, one sees that the characteristic polynomial $P(\lambda)$ writes $P(\lambda) = -\lambda^3 + Z\lambda^2 - \lambda\bar{Z} + 1$. Then, using formulae (4.40) and (4.41), one sees that

$$\Delta_{SU(3)} Z = -\frac{16}{3} Z, \Delta_{SU(3)} \bar{Z} = -\frac{16}{3} \bar{Z},$$

and

$$\begin{aligned} &\mathbf{\Gamma}_{SU(3)}(Z, Z) = 4(\bar{Z} - \tfrac{Z^2}{3}), \quad \Gamma_{SU(3)}(Z, Z) = 4(Z - \frac{\bar{Z}^2}{3}), \\ &\Gamma_{SU(3)}(Z, \bar{Z})) = 2(\tfrac{Z\bar{Z}}{3} - 3), \end{aligned}$$

which shows that $\mathbf{L}_{1/2}$ is the image of $3\Delta_{SU(3)}$ through Z.

5 The full $\mathbb{R}^2$ case

In this Section, we consider the case where $\Omega = \mathbb{R}^2$, and concentrate on the SDOP problem. From the one dimensional models and the tensorization procedure, we already know that Gaussian measures provide such orthogonal polynomials, with the two-dimensional Ornstein-Uhlenbeck operator as associated diffusion operator. We shall prove in this section that in any case that the only admissible measure are Gaussian measures, although there may be other diffusion operators having as eigenvectors orthogonal polynomials for these measures.

In this situation, we do not have boundary equation to restrict the analysis of the (co-)metric (G^{ij}). We therefore look for 3 polynomials G^{11}, G^{12}, G^{22} of degree at most 2 in the variables (X, Y) with $\Delta = G^{11}G^{22} - (G^{12})^2 > 0$ on $\mathbb{R}^2$ and for a function h such that for the measure $d\rho = e^h dX dY$, the polynomials of (X, Y) are dense in $\mathcal{L}^2(\rho, \mathbb{R}^2)$. From (2.16), there exists L_X and L_Y affine forms in $\mathbb{R}^2$ s.t.

$$\partial_X h = \frac{G^{22}L_X - G^{12}L_Y}{\Delta}, \ \partial_Y h = \frac{-G^{12}L_X + G^{11}L_Y}{\Delta} \tag{5.43}$$

Let us first show that Δ has degree at most 2: we denote by $R^2 = X^2 + Y^2$. If Δ is of degree 4, and since $\Delta > 0$ on $\mathbb{R}^2$, there is at least a cone in which, for some constant c, $\Delta \geq cR^4$ at infinity and $d\rho$ cannot integrate any polynomial. Hence Δ is of degree less than 3 and since it is positive on $\mathbb{R}^2$, Δ is of degree 2 or zero.

5.1 Δ irreducible, with degree 2

From (2.13) we know that h writes

$$h = \log \rho = P + \sum_p \alpha_p \log |\Delta_p| + \beta_p \arctan(\frac{\mathcal{I}_p}{\mathcal{R}_p}), \tag{5.44}$$

where Δ_p are the real irreducible factors of Δ and each Δ_p writes $\Delta_p = (\mathcal{R}_p + i\mathcal{I}_p)(\mathcal{R}_p - i\mathcal{I}_p)$. In this section, the degree of Δ is 2 hence the degree of the polynomial P is at most $4 - deg(\Delta) = 2$. The terms of higher degrees has to be a negative non–degenerated quadratic form in (X, Y). Let us write $\partial_X P = P_X + p_X$ and $\partial_Y P = P_Y + p_Y$ the partial derivatives of P in both variables where P_Y and P_Y are homogeneous of degree 1 and p_X and p_Y are constant. In particular, P_X and P_Y are linearly independent. We also denote by

$$G^{11} = A_2 + A_1 + A_0\,, \quad G^{12} = B_2 + B_1 + B_0 \text{ and } G^{22} = C_2 + C_1 + C_0 \tag{5.45}$$

where the terms A_i, B_i, C_i are homogeneous of degree i. Now combining (5.43) and (5.44), one gets

$$\begin{pmatrix} G^{22} & G^{12} \\ G^{12} & G^{22} \end{pmatrix} \begin{pmatrix} \Delta\partial_X P + \alpha\partial_X\Delta \\ \Delta\partial_Y P + \alpha\partial_Y\Delta \end{pmatrix} = \begin{pmatrix} \Delta L_X \\ \Delta L_Y \end{pmatrix} \tag{5.46}$$

Since $\Delta = G^{11}G^{22} - (G^{12})^2$ is of degree 2 and positive on $\mathbb{R}^2$, then $A_2C_2 = B_2^2$. If B_2 is non zero and irreducible, it divides A_2 and C_2. This leads to $B_2 = \lambda A_2$ and $C_2 = \lambda^2 A_2$ ($\lambda \neq 0$) and then to $P_X = -\lambda P_Y$, which is excluded. The conclusion is the same if $B_2 = \hat{b}^2$ where $B_2 = \lambda A_2$, $C_2 = \lambda^2 A_2$, and also if $B_2 = 0$. It remains only the case where $B_2 = \hat{A}\hat{C}$, where $\hat{A}$ and $\hat{C}$ are linearly independent (hence non zero) and $A_2 = \hat{A}^2$, $C_2 = \hat{C}^2$. In the general case, we may reduce to $B_1 = 0$, hence $A_1 = C_1 = 0$ and $\alpha_p = \beta_p = 0$ for every p. Therefore $\partial_X P = \mu\hat{C}, \partial_Y P = -\mu\hat{A}$, and $\partial_X\hat{A} = \partial_Y\hat{C}$. The general solution is then as follows

$$G^{11} = (\alpha X + \lambda Y)^2 + A_0, G^{12} = -(X + \lambda Y)(Y + \gamma X) + B_0, G^{22} = (\alpha Y + \gamma X)^2 + C_0\,,$$

where $A_0 > 0$ and $A_0C_0 > B_0^2$. This corresponds to a Gaussian measure. By affine transformation, we may reduce to the standard Gaussian measure. Therefore the only generic example (up to affine transformations) is

$$G^{11} = Y^2 + A_0, G^{12} = -XY + B_0, G^{22} = X^2 + C_0$$

with $A_0 > 0$ and $A_0C_0 > B_0^2$, and

$$h = -X^2/2 - Y^2/2 + \text{cste}$$

and

$$\begin{pmatrix} Y^2 + A_0 & -XY + B_0 \\ -XY + B_0 & X^2 + C_0 \end{pmatrix} \begin{pmatrix} -X \\ -Y \end{pmatrix} = \begin{pmatrix} -A_0X - B_0Y \\ -B_0X - C_0Y \end{pmatrix}$$

with $\Delta = A_0X^2 + C_0yY^2 + 2B_0XY + A_0C_0 - B_0^2$. The generator is then

$$\begin{aligned} L(f) = \quad & (Y^2 + A_0)\partial_X^2 f - 2(XY - B_0)\partial_{xXY}^2 f + (X^2 + C_0)\partial_Y^2 f \\ & -\Big((A_0 + 1)X + B_0Y)\Big)\partial_X f - \Big((C_0 + 1)Y + B_0X)\Big)\partial_Y f, \end{aligned}$$

and its degenerated limit (classical Ornstein-Uhlenbeck)

$$L_{A_0,B_0,C_0}(f) = a_0\partial_X^2 f + 2B_0\partial_{XY}^2 f + C_0\partial_Y^2 f - \Big(A_0X + B_0Y\Big)\partial_X f - \Big(C_0Y + B_0X\Big)\partial_Y f,$$

so that the generic operator is

$$L_{A_0,B_0,C_0} + (Y\partial_X - X\partial_Y)^2,$$

which is the sum of an Ornstein-Uhlenbeck operator and the square of a planar rotation.

5.2 $\Delta = L^2$ and L is a non constant affine form.

In this case, the operator is not elliptic. It appears as an example in the previous case with $A_0 = 1$, $C_0 = B_0 = 0$ but such cases are excluded, although interesting per se.

5.3 Δ is constant (non zero)

We can boil down to $\Delta = 1$. Then $G^{11}G^{22} = (G^{12} - i)(G^{12} + i)$. If G^{11} or G^{22} is irreducible, then $G^{11} = \lambda(G^{12} \pm i)$, which is not possible since G^{11} is real. Therefore, $G^{11} = l_1 l_2$, where l_i are affine forms. Since $G^{11} > 0$ on $\mathbb{R}^2$, the only solution is $G^{11} = l\bar{l}$, where l is an affine (complex) form non zero on $\mathbb{R}^2$. Therefore, up to a constant, $l = l_a + \alpha$, where l_a is real linear form, and α in a non real complex. Similarly, $G^{22} = (l_c + \gamma)(l_c + \bar{\gamma})$, where l_c is real linear form, and γ is a non real complex. Hence

$$G^{12} + i = (l_a + \alpha)(l_c + \gamma),\ G^{12} - i = (l_a + \bar{\alpha})(l_c + \bar{\gamma}).$$

If one of l_a or l_c is non zero, then by identification, we get

$$l_c = \nu l_a,\ \gamma = \nu\bar{\alpha} + \frac{1}{\Im(\alpha)},$$

Hence $G^{11} = (l_a + \alpha)(l_a + \bar{\alpha})$, $G^{12} = (l_a + \alpha)(l_c + \gamma) - i$, $G^{22} = (l_c + \gamma)(l_c + \bar{\gamma})$ et $l_c = \nu l_a$ $(\nu \neq 0)$, $\gamma = \nu\bar{\alpha} + 1/\Im(\alpha)$.

Up to a change of variables, and still whenever $l_a \neq 0$, one may reduce to the case where $G^{11} = X^2 + 1$. Then, it is easily seen from equation (5.43) that there cannot exist any measure which is a solution and for which all polynomials are integrable.

We are then reduced to the case of constant G^{ij}, which correspond to Ornstein-Uhlenbeck operators.

6 Non compact cases with boundaries.

In this section, we again consider the SDOP problem, which is perhaps not enough to describe all the possible solutions of the general DOP problem (although we have no example of solution of the latter beyond the cases described here).

We describe all the possible models, but we do not give any geometric interpretation, and do not detail for which values of the parameters appearing in the measure the polynomials are dense in $\mathcal{L}^2(\mu)$. However, in all the cases described below, it is indeed the case for at least some values of these parameters. Moreover, one could give a geometric construction for many models as images of Ornstein-Uhlenbeck operators in some Euclidean space, associated to Gaussian measures.

Following the results of Section 2, we reduce to the cases where every factor Δ_p appearing in the boundary satisfies the fundamental equations (2.12). We also need, for the measure $d\mu = e^h dx$, that $\mathcal{L}^2(\mu)$ contains any polynomial. Hence in any case, we have to look for the existence of such a measure, which will turn out to be the main restriction. We indeed require more, namely that polynomials are dense in $\mathcal{L}^2(\mu)$. We know from Proposition 2.13 the general form of the measure. In addition to the boundary terms, there appear in h a polynomial term P which will be crucial when integrating the polynomials on the domain (see previous section) . This constraint will help us to restrict the number of cases for the metric (G). Moreover, if the determinant Δ of (G) has no multiple factors and the domain contains an open cone, the degree of Δ is at most 3. When there are multiple factors, the same kind of analysis can be undergone.

The algebraic analysis undergone in Section 3 still holds, and produces the following list of possible boundaries.

(1) $\partial\Omega = \{Y^2 - X^3 = 0\}$. In this case, the general metric is given by

$$G := \begin{pmatrix} \frac{4}{3}\alpha X^2 + \beta X + \gamma Y & \frac{3}{2}\gamma X^2 + 2\alpha XY + \frac{3}{2}\beta Y \\ \frac{3}{2}\gamma X^2 + 2\alpha XY + \frac{3}{2}\beta Y & \frac{9}{4}\beta X^2 + \frac{9}{4}\gamma XY + 3\alpha Y^2 \end{pmatrix}.$$

Here, the determinant Δ is $\frac{3}{4}(Y^2 - X^3))\big((3\gamma^2 - 4\alpha\beta)X + 4\alpha\gamma Y - 3\beta^2\big)$. Since $\deg\Delta \leq 3$, then $\alpha = \gamma = 0$, and by homogeneity we may restrict to $\beta = 1$. Since $\Delta > 0$ in the interior of the domain, Ω, one sees that the domain must be $X^3 > Y^2$. This leads to measures of the form $d\mu_{a,b} = C_{a,b}(Y^2 - X^3)^a \exp(-bX)dXdY$, $a > -1, b > 0$, on the domain $X^3 > Y^2$.

(2) $\partial\Omega = \{Y - X^2 = 0\}$. Then $\Delta = (Y - X^2)\Delta_1$ with either Δ_1 proportional to $(Y - X^2)$ or $\deg\Delta_1 \leq 1$.

(2i) $\Delta = c(Y - X^2)^2$. In this case, the general metric has form

$$\begin{pmatrix} \alpha X^2 + \beta X + \gamma & \beta X^2 + 2\alpha XY + 2\gamma X + \beta Y \\ \beta X^2 + 2\alpha XY + 2\gamma X + \beta Y & 4\gamma X^2 + 4\beta XY + 4\alpha Y^2 \end{pmatrix}.$$

In this case, we may show that there is no measure solution for the problem.

(2ii) $\deg\Delta = 3$. The metric for which there exist a measure solution for the problem may be written as

$$G = \begin{pmatrix} Y - X^2 + \mu & \lambda(Y - X^2) + 2\mu X \\ \lambda(Y - X^2) + 2\mu X & \lambda^2(Y - X^2) + 4\mu Y \end{pmatrix},$$

By a change of coordinates $X \mapsto cX, Y \mapsto c^2Y$, we may reduce to $\mu = \pm 1$ or $\mu = 0$ and $\lambda = 1$. The latter case is excluded since then $\Delta = 0$. In the first case, the existence of a finite measure solution imposes $\mu = 1$, $\Omega = \{Y > X^2\}$ and measures to be $d\mu = (Y - X^2)^a \exp\big(-b(Y - \lambda X)\big)dXdY$, $a > -1, b > 0$.

(2iii) Δ has degree 2. The only solutions are, up to a constant,

$$\begin{pmatrix} 1 & 2X \\ 2X & \mu(Y - X^2) + 4Y \end{pmatrix}.$$

This is a limit case of the previous one (after multiplication by a constant).

(3) $\partial\Omega = \{X = 0\}$. In this case, G^{11} and G^{12} are multiple of X. If Δ has no multiple factors, then $deg\Delta = 2$, and Δ has form $\alpha X(X+\gamma)$. If Δ has multiple factors, it has form $\Delta = Xl_1^2 l_2$ where l_1 and l_2 are two linear forms. Since it is an elliptic operator, l_1 and l_2 are of type $rX + d$. Then for he measure to be finite when integrating the polynomials on $\mathbb{R}$ for the Y variable, we need to have $deg\Delta = 2$. Since G^{11} is positive on $\{X > 0\}$, it has form X or $X(\alpha X + \beta)$ with $\alpha > 0$ or $\beta > 0$ (by homothety or dilatation, we could reduce to $G^{11} = X$, $G^{11} = X(X + 1)$ or $G^{11} = X^2$). We denote by $G^{12} = Xl_b$ where l_b is a linear form.

(3i) Case $\Delta = X^2$. There are two cases:

(a) $G^{11} = \alpha X^2$, $G^{12} = xl_b$, $G^{22} = (l_b^2+1)/\alpha$, $\alpha \neq 0$. To have measure solutions, we must have $l_b = \beta X$ or $l_b = \beta X + \frac{\alpha}{2}Y + \beta_0$.

(b) $G^{11} = \alpha X$, $G^{12} = \beta X$, $G^{22} = \gamma X$, $\alpha\gamma = \beta^2+1$: there are no measure solutions in this case.

(3ii) $\Delta = X(X+1)$. With $G11 = Xl_a$, $G^{12} = Xl_b$ and $G^{22} = \gamma_2 X^2 + \gamma_1 X + c_0$, and l_b depending only on X. The general metric admitting measure solutions is, up to a constant and up to dilatation on Y,

$$\begin{pmatrix} X(X+1) & \lambda X(X+1) \\ \lambda X(X+1) & \lambda^2 X(X+1)+1 \end{pmatrix}.$$

The curvature is zero. The change of coordinate $Y = Y_1 + X$ reduces to $\begin{pmatrix} X(X+1) & 0 \\ 0 & G^{22} \end{pmatrix}$, and in this case three is no measure solution.

(3iii) $\Delta = X$. Once more, $G^{11} = Xl_a$, $G^{12} = Xl_b$ and G^{22} depends only on X, and we can reduce to $G^{11} = \alpha X(X+1)$, or $G^{11} = \alpha X$. Then $G^{22} = Xl_c + 1/\alpha$, and we have 2 subcases:

(a) $G^{11} = \alpha X$, $G^{12} = \beta X$, $G^{22} = (\beta^2 X + 1)/\alpha$. We can reduce by a change of variable on Y to $\beta = 0$ which is the classical case of a product of Laguerre and Hermite polynomials.

(b) $a = \alpha X(X+1)$, then either $l_b = 1, l_c = 0$ or $l_b = -1, l_c = 0$, or $l_b = -1+\alpha\mu(X+1), l_c = \mu(-2+\alpha\mu(X+1))$ or $l_b = 1+\alpha\mu(X+1), l_c = \mu(2+\alpha\mu(X+1))$. In any of these cases, there is no measure solution.

(4) $\partial\Omega = \{XY = 0\}$. Then Δ is of degree at most 3. The boundary equations imply G^{11} and G^{12} are multiple of X while G^{12} and G^{22} are multiple of Y. Hence the general form of the metric is

$$G = \begin{pmatrix} Xl_a & \beta XY \\ \beta XY & Yl_c \end{pmatrix}.$$

By homothety, symmetry in (X, Y) and dilatation on X and Y, we may reduce to one of the following cases:

(4i) $G = \begin{pmatrix} X(\alpha_1 X + \alpha_0) & \alpha_1\beta XY \\ \alpha_1\beta XY & Y(\alpha_1\beta^2 Y + c_0) \end{pmatrix}$ From the ellipticity condition, we have therefore

(a) $\alpha_1\alpha_0\beta c_0 \neq 0$: no integrable measure solution.

(b) $\alpha_1 = 0$: then α_0 and c_0 are non zero.It is a product of Laguerre polynomials. The only mesure that are solutions are product of exponential terms with the boundary terms.

(c) $\alpha_0 = 0$ and $\alpha_1\beta c_0 \neq 0$: no measure solution.

(d) $\beta = 0$ and $\alpha_1\alpha_0 c_0 \neq 0$: no measure solution.

(e) $c_0 = 0$ and $\alpha_1\alpha_0\beta \neq 0$: no measure solution.

(f) $\alpha_0 = \beta = 0$ and $\alpha_1 c_0 \neq 0$: no measure solution.

(4ii) $G = \begin{pmatrix} X(\alpha_2 Y + \alpha_0) & \beta\alpha_2 XY \\ \beta\alpha_2 XY & Y(\alpha_2\beta^2 X + c_0) \end{pmatrix}$, with $\alpha_2 \neq 0$. We can chose $\alpha_2 = 1$. Up to symmetry on X and Y, we have the following subcases:

(a) $\alpha_0\beta c_0 \neq 0$. There is a measure solution $\exp(-\mu X - \lambda Y)$ if $\beta < 0$.

(b) $\alpha_0 = 0$, $\beta c_0 \neq 0$. There is an exponential measure solution if $\beta < 0$ (a limit case of the previous one). Then $\Delta = XY^2$. In this generic case, the curvature is non constant. By dilatation, we can boil down to $\alpha_0 = 1$, but not simultaneously to $c_0 = 1$.

(c) $\beta = 0$, $\alpha_0 c_0 \neq 0$: no measure solution.

(d) $\alpha_0 = \beta = 0$, $c_0 \neq 0$: no measure solution.

(5) $\partial\Omega = \{X(1-X) = 0\}$. The (co)-metric solution is (up to homothety and affine change)

$$G = \begin{pmatrix} X(1-X) & \beta X(1-X) \\ \beta X(1-X) & P(X,Y) \end{pmatrix}$$

and P is any polynomial of degree 2, positive on the domain. Hence Δ is at most of degree 2 and therefore $P(X,Y) = \gamma + \beta^2 X(1-X)$ and the admissible metrics are finally

$$G = X(1-X)\begin{pmatrix} 1 & \beta \\ \beta & \beta^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}.$$

Up to a dilatation on Y we can assume $\beta = 0$ or $\beta = 1$. In this case, the curvature is zero and measures have form:

$$X^r(1-X)^s \exp(\lambda X + \mu Y + \delta(Y^2 + \beta^2 X^2) - XY(\lambda\mu + 2\beta\delta).$$

We can reduce to a simpler form with an affine change of variable on $Y \to Y + dX + e$, and it is a standard form of r$\beta = 0$. For $\beta \neq 0$, the form is more complicated. Moreover, on the case $\beta = 0$, there are non trivial terms depending on x in the polynomial part, which were not expected.

(6) $XY(1-X) = 0$. The metric solution is (up to homothety and affine change)

$$G = \begin{pmatrix} X(1-X) & 0 \\ 0 & Y(\alpha X + \beta Y + \gamma) \end{pmatrix}.$$

Except in the case $\alpha = 0$, the curvature is non constant, and the additional factor in Δ: $\alpha X + \beta Y + \gamma$ does not satisfy the boundary equation. The only case when there is a measure solution on the domain is $\alpha = \beta = 0$, which is a product of Jacobi and Laguerre polynomials.

7 Two folds covers

For many examples in dimension 2, with domain Ω described by the equation $P(X,Y) \geq 0$, one may look at models in dimension 3 given by the equation $Z^2 \leq P(X,Y)$. It turns out that, in every case where no cusp or double tangent appears in $\partial\Omega$, this provides a new domain in dimension 3 which is again a solution of the problem. This is therefore the case for the circle, the triangle, the double parabola and the double point cubic.

Those new three dimensional models present the same pathology than the circle and triangle models in dimension 2: the metric is not in general unique up to scaling, the curvature is not constant (except for specific values of the parameters). In fact, in those models, the boundary of the domain has degree at most 4, whereas the maximal degree of the boundary in general is 6. The Laplace operator associated with the metric does not in general belong the admissible operators.

For example, if one starts with the double point cubic described in section 4.8, one gets for the metric, up to scaling,

$$G = \begin{pmatrix} 4X(1-X) & 2Y(2-3X) & 2Z(2-3X) \\ 2Y(2-3X) & 4X-3X^2-9Y^2-(9+A)Z^2 & AYZ \\ 2Z(2-3X) & AYZ & 4X-3X^2-(9+A)Y^2-9Z^2 \end{pmatrix}.$$

For the double cover of the triangle, however, one gets a unique metric up to scaling, which is

$$G = \begin{pmatrix} 4X(1-X) & -4YX & 2Z(1-3X) \\ -4YX & 4Y(1-Y) & 2Z(1-3Y) \\ 2Z(1-3X) & 2Z(1-3Y) & X+Y-X^2-XY-Y^2-9Z^2 \end{pmatrix},$$

which has no constant curvature. We did not try to push the analysis of these models any further, but this shows that one may construct in higher dimension some models which are not direct extensions of the 2 dimensional models, and that the higher dimension analysis of the problem seems much more complex.

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