From Pitman's theorem to crystals

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Abstract.

We describe an extension of Pitman's theorem on Brownian motion and the three dimensional Bessel process to several dimensions. We show how this extension is suggested by considering a random walk on a noncommutative space, and is connected with crystals and Littelmann paths.

§1. Introduction

Two of the most famous results on Brownian motion, Lévy's and Pitman's theorems, which we recall in the following sections, involve transformations of the Brownian paths which are closely related to transformations of paths occuring in a completely different area of mathematics, the Littelmann path model for representations of semisimple Lie algebras. Recently some generalisations of Pitman's theorem to multidimensional Brownian motion have been discovered, which throw some light on this coincidence. These generalisations involve Brownian motion on hermitian matrices, and the motion of its eigenvalues, which takes place in a cone, the Weyl chamber of the Cartan algebra of the group $SU(N)$. In this paper I will consider a random walk on the dual of the group $SU(2)$, considered as a noncommutative space, which I introduced and studied some years ago, and show that, by deforming it to a random walk on the dual of the quantum group $SU_q(2)$, and letting q go to 0, one can reproduce the proof of Pitman's theorem. This construction immediately suggests how to extend Pitman's theorem to several dimensions, and this extension will be described in the following sections, using the Pitman operators, which are path transformations generalizing straightforwardly the transformation occuring in the original Pitman's theorem. These Pitman operators satisfy braid relations, and can be used to define path transformations associated to elements of an arbitrary Coxeter group. Considering the longest element of a finite Coxeter groups, one obtains a transformation which maps usual

Brownian motion to the Brownian motion conditionned to stay in a fundamental domain of the Coxeter group.

§2. Paul Lévy's theorem

Let $(B_t)_{t\geq 0}$ be a real Brownian motion, starting from 0, Paul Lévy showed [L] that the stochastic process

$$X_t := B_t - \inf_{0\leq s\leq t} B_s$$

is distributed as the absolute value of a real Brownian motion. The theorem is illustrated in the following figure, where a typical Brownian path, as well as its transform, are depicted.

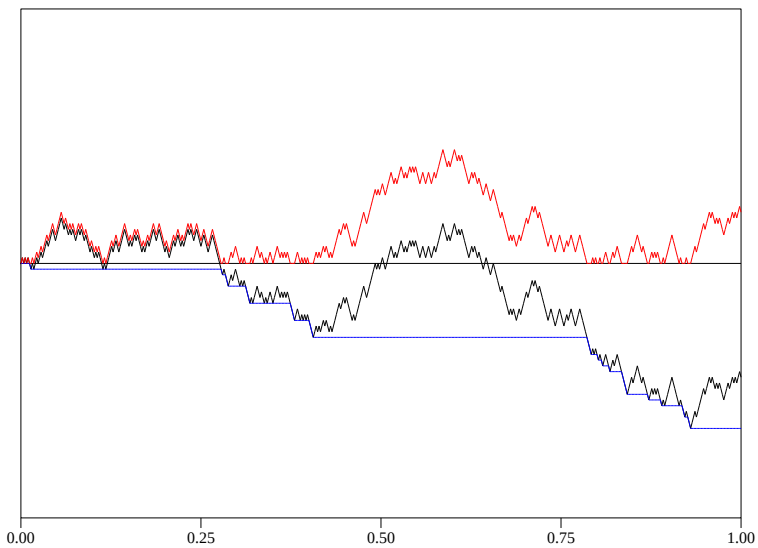

Fig. 1

Furthermore, Lévy observed that the original Brownian motion B_t can be recovered from X_t by

$$B_t = X_t - L_t^0 \tag{2.1}$$

where L_t^0 is the local time at zero of X, defined by

$$L_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{X_s \in [-\varepsilon, \varepsilon]} ds.$$

This is a purely probabilistic statement since, in general, given a non-negative continuous function f, there exists many continuous functions g satisfying $f(t) = g(t) - \inf_{0 \le s \le t} g(s)$, therefore the equation (2.1) holds only almost surely.

§3. Pitman's theorem

Many years after Lévy's theorem, Jim Pitman [P] obtained the remarkable result that the stochastic process

$$R_t := B_t - 2 \inf_{0 \le s \le t} B_s$$

is a Markov process, actually it is distributed as the norm of a three dimensional Brownian motion, also called a three dimension Bessel process. See the figure below.

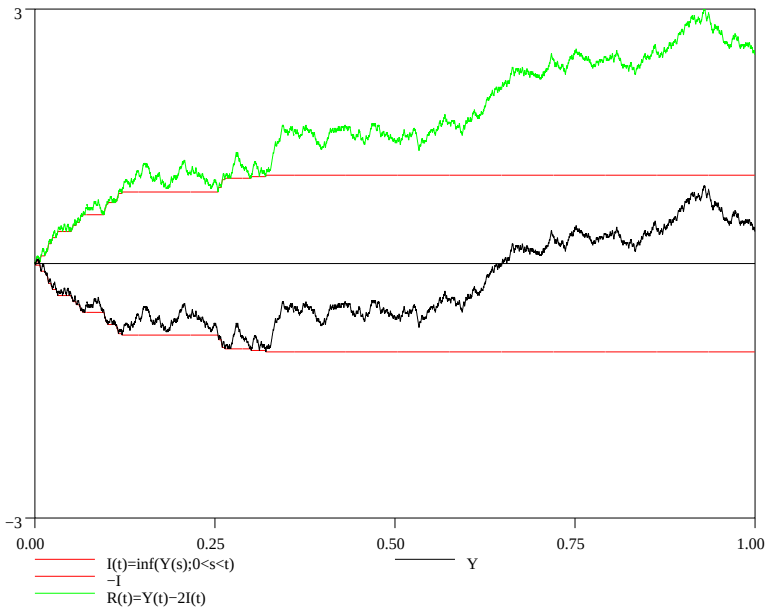

Fig. 2

This time the original Brownian motion cannot be recovered from the knowledge of the process $(R_s; s \leq t)$ up to time t, however there is an inversion formula for $t = \infty$, namely one has $R_t \to \infty$ as $t \to \infty$ and for all $t \geq 0$, one has

$$B_t = R_t - 2 \inf_{s \geq t} R_s.$$

In fact, given a nonnegative function $f : [0, T] \to \mathbf{R}$ with $f(0) = 0$, and a real number $x \in [0, f(T)]$ there exists exactly one function g such that

$$f(t) = g(t) - 2 \inf_{0 \leq s \leq t} g(s) \qquad 0 \leq t \leq T$$

and

$$g(T) = f(T) - 2x.$$

The function g can be recovered by

$$g(t) = f(t) - 2 \inf(x, \inf_{t \leq s \leq T} f(s)). \tag{3.1}$$

It is therefore enough to give the value of the random variable $\inf_{s \geq t} R_s$, in order to recover $(B_s)_{0 \leq s \leq t}$ from $(R_s)_{0 \leq s \leq t}$.

There are many proofs of Pitman's theorem, but the original one by Pitman is probably the simplest, it relies on a discrete time approximation, by a Bernoulli random walk. One considers such a random walk

$$S_n = X_1 + \ldots + X_n \tag{3.2}$$

where the X_i are a sequence of iid random variables, with distribution $P(X_i = \pm 1) = 1/2$. The two dimensional discrete time stochastic process

$$(S_n - 2 \inf_{k \leq n} S_k, S_n)_{n \geq 0}$$

is a Markov chain on the set

$$\{(t, k) \in \mathbb{N}^* \times \mathbb{Z} \,|\, k \in (-t, -t+2, \ldots, t-2, t)\},$$

with probability transitions

$$\begin{array}{rl} p((t,k),(t-1,k-1)) = \frac{1}{2}, & p((t,k),(t+1,k+1)) = \frac{1}{2} \text{ if } t > k > -t \\ p((t,-t),(t+1,-t+1)) = \frac{1}{2}, & p((t,-t),(t+1,-t-1)) = \frac{1}{2} \end{array}$$

It is easy to infer from this that the discrete time process

$$Z_n = S_n - 2 \inf_{k \leq n} S_k \tag{3.3}$$

is a Markov process on the nonnegative integers, with probability transitions given by

$$p(k,k+1) = \frac{k+2}{2(k+1)}; \qquad p(k,k-1) = \frac{k}{2(k+1)}. \tag{3.4}$$

By space-time scaling, one has a convergence in distribution of the processes

$$\lambda^{-1/2} S_{[\lambda t]} \to_{\lambda\to\infty} B_t \qquad \lambda^{-1/2} Z_{[\lambda t]} \to_{\lambda\to\infty} R_t$$

and Pitman's theorem follows.

§4. Brownian motion on hermitian matrices

Consider a 2×2 matrix valued Brownian motion of the form

$$\begin{pmatrix} X_t & Y_t + iZ_t \\ Y_t - iZ_t & -X_t \end{pmatrix}$$

where X, Y, Z are three independent real Brownian motions. The eigenvalues of this matrix are $\pm\lambda_t$ where

$$\lambda_t = \sqrt{X_t^2 + Y_t^2 + Z_t^2}$$

therefore the process $(\lambda_t)_{t\geq 0}$ is a Bessel process of dimension three, which can also be described as "Brownian motion conditionned to stay positive". In this last statement the conditionning has to be done in Doob's sense [D], namely we consider Brownian motion on the positive real line, killed at the first hitting time of 0. It is well known that this is a Markov process whose probability transition densities can be computed using the reflection principle, and are given by

$$p_t^0(x,y) = p_t(x,y) - p_t(x,-y) \qquad x,y > 0$$

where

$$p_t(x,y) = e^{-(x-y)^2/2t}/(2\pi t)^{1/2}$$

is the usual gaussian kernel. There exists a unique positive harmonic function for this kernel, which is $h(x) = x$, and the transition probabilities of the three dimensional Bessel process are given, by Doob's h-transform, as

$$q_t(x,y) = \frac{h(y)}{h(x)} p_t^0(x,y) \qquad x,y > 0$$

This situation generalises to the case of hermitian matrices of arbitrary size. Let $(M_t)_{t\geq 0}$ be a brownian motion with values in the space of traceless hermitian matrices of size $n \times n$. Thus M is a gaussian process with covariance

$$E[Tr(AM_t)Tr(BM_s)] = (s \wedge t)\, Tr(AB)$$

for all traceless hermitian matrices A, B, and times s, t. Then the eigenvalues $\lambda_1(t) \leq \ldots \leq \lambda_n(t)$ perform a Markov process in the cone

$$C = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \,|\, x_1 + \ldots + x_n = 0, x_1 \leq x_2 \leq \ldots \leq x_n\}$$

The transition probabilities of Brownian motion killed at the exit of this cone are given by the reflection principle as

$$p_t^0(x, y) = \sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) p_t(x, \sigma(y)) \tag{4.1}$$

where $\mathcal{S}_n$ is the symmetric group acting by permutation of coordinates, and there exists a unique positive harmonic function for this kernel, given by the Vandermonde determinant

$$h(x) = \prod_{i>j} (x_i - x_j)$$

The transition densities for the process $(\lambda_1(t), \ldots, \lambda_n(t))$ are then equal to

$$q_t(x, y) = \frac{h(y)}{h(x)} p_t^0(x, y) \qquad x, y \in C \tag{4.2}$$

The generalisation of Pitman's theorem that we are going to discuss constructs a process distributed as $(\lambda_1(t), \ldots, \lambda_n(t))$ by a path transformation, starting from an ordinary Brownian motion on $\mathbf{R}^{n-1}$. Such a result was first given in [OY], for the case we discuss above, then generalized in [BJ], with a completely different proof, to processes with values in the Lie algebra of a compact semisimple Lie group. Finally in [BBO] we gave still another generalisation, with different techniques, to the case of Brownian motion in the fundamental cone associated with a finite Coxeter group.

§5. Random walks on noncommutative spaces

The two Markov chains S_n and Z_n appearing in the proof of Pitman's theorem ((3.2) and (3.3)) can be realized naturally using a random

walk on a noncommutative space. Consider the following construction, which was introduced in [B1],[B2]. Let $\mathcal{A}(SU(2))$ be the (reduced) group C^*-algebra of the compact group $SU(2)$. There exists, for each integer n, a unique (up to equivalence) irreducible representation of dimension n of $SU(2)$. By Peter-Weyl theorem, this implies that $\mathcal{A}(SU(2))$ is the C^* direct sum of matrix algebras $\oplus_{n=1}^{\infty} M_n(\mathbf{C})$. Each element of $SU(2)$ defines a unitary element λ_g in the multiplier algebra of $\mathcal{A}(SU(2))$. The C^* algebra $\mathcal{A}(SU(2))$ is a C^*-Hopf algebra for the coproduct induced by

$$\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$$

Let ρ be the state on $\mathcal{A}(SU(2))$ given by the normalized trace of the fundamental representation, i.e. one has

$$\rho(a) = \frac{1}{2} Tr(a_2)$$

if $a = (a_n)_{n\geq 1} \in \oplus_{n=1}^{\infty} M_n(\mathbf{C})$. The map $P : \mathcal{A}(SU(2)) \to \mathcal{A}(SU(2))$ given by

$$P = (\rho \otimes Id) \circ \Delta$$

is the composition of a morphism and the partial evalutation with respect to a state, it is therefore completely positive, and unit preserving. In general a completely positive, unit preserving map on a C^* algebra is a noncommutative analogue of a Markov kernel, and it is possible to construct noncommutative stochastic processes having this map as transition probabilities, see e.g. [B4] for an introduction to these topics. If the C^*-algebra B is commutative, thus isomorphic to $C_0(X)$ for some locally compact space X, then a completely positive, unit preserving map $Q : C_0(X) \to C_0(X)$ (more precisely, preserving the unit of the multiplier algebra) is given by a Markov kernel $q(x, dy)$ on X, by the formula

$$Qf(x) = \int_X f(y) q(x, dy)$$

Consider now the following two commutative subalgebras of $\mathcal{A}(SU(2))$. The first one is $\mathcal{A}(T)$, the C^*-algebra generated by the maximal torus T of $SU(2)$ consisting of diagonal matrices. The Gelfand spectrum of this algebra is the dual group of T, isomorphic to $\mathbf{Z}$. The completely positive map P leaves this algebra invariant and its restriction to $\mathcal{A}(T)$ is given by a Markov kernel on $\mathbf{Z}$, which we easily identify with

$$q(n, dm) = \frac{1}{2}(\delta_{n-1} + \delta_{n+1})$$

The corresponding Markov chain on the integers is the usual symmetric Bernoulli random walk, as in (3.2). The second subalgebra is the center of $\mathcal{A}(SU(2))$. Its spectrum is the set of irreducible representations, identified with $\mathbf{N}$ (for convenience one identifies the representation of dimension n with $n-1 \in \mathbf{N}$). This algebra again is invariant by P, and the corresponding Markov kernel can be computed using the Clebsch Gordan formula giving the decomposition of tensor product of representations

$$\rho_2 \otimes \rho_n = \rho_{n-1} + \rho_{n+1}$$

ρ_n being the irreducible representation of dimension n. This yields the transition probabilities of the Markov kernel

$$p(n, n+1) = \frac{n+2}{2(n+1)}, \quad p(n, n-1) = \frac{n}{2(n+1)}$$

This time we find the same transition probabilities (3.4) as the Markov chain (3.3) of Pitman's theorem. We now consider the maximal abelian subalgebra of $\mathcal{A}(SU(2))$ generated by $\mathcal{A}(T)$ and by the center. This can be identified with the subalgebra

$$\oplus_{n=1}^{\infty} D_n \subset \oplus_{n=1}^{\infty} M_n(\mathbf{C})$$

consisting of diagonal matrices. The spectrum is naturally the set of pairs of integers (r, k) with $r \geq 0$ and $k \in \{-r, -r+2, \ldots, r\}$. This abelian algebra is invariant by P, and the restriction of P gives the Markov kernel on the spectrum

$$\begin{aligned}
p((r,k),(r+1,k+1)) &= \frac{r+k+2}{4(r+1)} \\
p((r,k),(r+1,k-1)) &= \frac{r-k+2}{4(r+1)} \\
p((r,k),(r-1,k+1)) &= \frac{r-k}{4(r+1)} \\
p((r,k),(r-1,k-1)) &= \frac{r+k}{4(r+1)}.
\end{aligned}$$

This Markov kernel is different from the one appearing in the proof of Pitman's theorem. However, we can consider the deformed algebra of the quantum group $SU_q(2)$ (in the sense of Woronowicz). As a C^*-algebra it is isomorphic to $\mathcal{A}(SU(2))$, but the coproduct is deformed into a new coproduct Δ_q by a parameter $q \in]0,1]$. Let us define a unit preserving completely positive map on $\mathcal{A}(SU_q(2))$ by the formula $P^{(q)} = (\rho \otimes Id) \circ \Delta_q$ We find in this algebra the same maximal abelian

subalgebra, which again is invariant by $P^{(q)}$. Identifying its spectrum as before, we find, using the q-analogue of the Clebsch-Gordan formula, the following transition probabilities

$$\begin{aligned}
p((r,k),(r+1,k+1)) &= q^{(r-k)/2}\frac{\left[\frac{r+k+2}{2}\right]_q}{2[r+1]_q} = \frac{q^{r+1}-q^{-k-1}}{2(q^{r+1}-q^{-r-1})}\\
p((r,k),(r+1,k-1)) &= q^{-(r+k)/2}\frac{\left[\frac{r-k+2}{2}\right]_q}{2[r+1]_q} = \frac{q^{-k+1}-q^{-r-1}}{2(q^{r+1}-q^{-r-1})}\\
p((r,k),(r-1,k+1)) &= q^{-(r+k+2)/2}\frac{\left[\frac{r-k}{2}\right]_q}{2[r+1]_q} = \frac{q^{-k-1}-q^{-r-1}}{2(q^{r+1}-q^{-r-1})}\\
p((r,k),(r-1,k-1)) &= q^{(r-k+2)/2}\frac{\left[\frac{r+k}{2}\right]_q}{2[r+1]_q} = \frac{q^{r+1}-q^{-k+1}}{2(q^{r+1}-q^{-r-1})}
\end{aligned}$$

Letting q go to zero then yields essentially the same Markov chain as in Pitman's theorem. Details are in [B3]. Once we have recognized the role of crystallisation (i.e. letting q go to zero) in obtaining Pitman's theorem we can look for a multidimensional generalisation, by considering compact semisimple Lie groups of higher rank than $SU(2)$, and try to emulate the same kind of construction as above. In fact we shall provide in the next sections a generalisation of Pitman's theorem in which the three dimensional Bessel process is replaced by a Brownian motion conditionned to stay in a cone associated with a finite Coxeter group (not necessarily a Weyl group).

§6. Pitman operators

Let V be a real vector space V, with dual $V^\vee$. Let $\alpha \in V$ and $\alpha^\vee \in V^\vee$ be such that $\alpha^\vee(\alpha) = 2$, then one defines the Pitman operator $\mathcal{P}_\alpha$, acting on continuous path π with values in V, and such that $\pi(0) = 0$, by the formula

$$\mathcal{P}_\alpha \pi(t) = \pi(t) - \inf_{t \geq s \geq 0} \alpha^\vee(\pi(s))\alpha, \qquad T \geq t \geq 0.$$

It is immediate to check that

$$\alpha^\vee(\mathcal{P}_\alpha \pi(t)) \geq 0 \quad \text{for all } t$$

and

$$\mathcal{P}_\alpha \mathcal{P}_\alpha = \mathcal{P}_\alpha.$$

Let $\alpha, \beta \in V$ and $\alpha^\vee, \beta^\vee \in V^\vee$ be such that $\alpha^\vee(\beta) = \beta^\vee(\alpha) = -2\rho < 0$. Since the Pitman operators are idempotent, in order to understand the monoid generated by $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$, it is enough to be able to compute the products of the form $\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \ldots$, where $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ alternate. Plugging the definitions of $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ into the product $\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \ldots$ yields

a complicated expression involving minima of quantities in which signs alternates. It is therefore remarquable that this expression simplifies to give the following. Let n be a positive integer, such that $\rho \geq \cos\frac{\pi}{n}$. Define

$$Y_\alpha^\beta(s_0, s_1, \ldots, s_k) = \\ T_0(\rho)\alpha^\vee(\pi(s_0)) + T_1(\rho)\beta^\vee(\pi(s_1)) + \ldots + T_{n-1}(\rho)\gamma^\vee(\pi(s_{n-1}))$$

where $\gamma = \alpha$ if n is odd, and $\gamma = \beta$ if n is even. Then

$$\underbrace{(\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\ldots)}_{n \text{ terms}}\pi(t) = \quad \pi(t) - \inf_{t\geq s_0\geq s_1\geq\ldots\geq s_{n-1}\geq 0} Y_\alpha^\beta(s_0,\ldots,s_{n-1})\alpha \\ - \inf_{t\geq s_0\geq s_1\geq\ldots\geq s_{n-2}\geq 0} Y_\beta^\alpha(s_0,\ldots,s_{n-2})\beta$$

see Theorem 2.5 in [BBO]. The T_k are the Tchebycheff polynomials defined by

$$T_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}.$$

In particular, under the assumptions on ρ and n, one has $T_k(\rho) \geq 0$ for all $k \leq n-1$.

The braid relations for Pitman operators follow: if $\rho = \cos\frac{\pi}{n}$ then (Theorem 2.4 in [BBO])

$$\underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\ldots}_{n \text{ terms}} = \underbrace{\mathcal{P}_\beta\mathcal{P}_\alpha\mathcal{P}_\beta\ldots}_{n \text{ terms}}$$

where there are n terms in the product.

Let (W, S) be a Coxeter system (see e.g. [BO], [H]). We choose a root and coroot system $(\alpha_s, \alpha_s^\vee)_{s\in S}$ for (W, S), such that for each $s \in S$, and for all $x \in V$, one has

$$s(x) = x - \alpha^\vee(x)\alpha$$

Denote by $\mathcal{P}_s$ the Pitman transform associated with the pair $(\alpha_s, \alpha_s^\vee)$. Let H_s be the closed half space

$$H_s = \{v \in V | \alpha_s^\vee(v) \geq 0\}.$$

Let $w \in W$ and let

$$w = s_1 \ldots s_l$$

be a reduced decomposition of w, where $l = l(w)$ is the length of w. Using the braid relations for Pitman operators, and a fundamental result of Matsumoto ([BO] Ch. IV, n° 1.5, Proposition 5) the operator

$$\mathcal{P}_w := \mathcal{P}_{s_1} \ldots \mathcal{P}_{s_l}$$

depends only on w, and not on the chosen reduced decomposition. If W is a finite Coxeter group, and w_0 its longest element, then one has

$$\mathcal{P}_s \mathcal{P}_{w_0} = \mathcal{P}_{w_0}.$$

In particular, for any path π the path $\mathcal{P}_{w_0}\pi$ takes values in the fundamental chamber

$$C = \cap_{s\in S} H_s$$

Let π be a path with values in the fundamental chamber, then the set of paths η such that $\mathcal{P}_{w_0}\eta = \pi$ is a continuous (as opposed to discrete) analogue, for arbitrary finite Coxeter groups, of the Littelmann module, and the Pitman operator is the analogue of the Kashiwara map on a crystal, see [BBO].

We have seen that, under certain conditions on n, the product of n terms

$$\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \dots$$

has a nice expression, given by minimizing over some sequences of times. It is an interesting, but certainly difficult problem, to try to find similar expressions for an arbitrary product

$$\mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_2} \mathcal{P}_{\alpha_3} \dots$$

where $\alpha_i, \alpha_i^\vee$ are arbitrary elements with $\alpha_i^\vee(\alpha_i) = 2$, and associated reflections s_i. A necessary condition for this product to have a nice expression is that the images $s_k s_{k-1} \dots s_1 C$ of the fundamental cone $C = \cap_i(\alpha_i^\vee > 0)$ do not overlap. When the reflections s_i belong to a Weyl group, such an expression was found in [BBO], Theorem 3.12.

§7. Brownian motion in cones

We assume that the Coxeter group W is finite, and that the space V is euclidean, with W acting by orthogonal reflections. We choose a fundamental chamber C and let h be the product of the positive coroots, i.e.

$$h(x) = \prod_{\beta\in R^+} \beta^\vee(x)$$

where R^+ is the set of positive roots, then the function h is still the only (up to a multiplicative constant) positive harmonic function vanishing on the boundary of C, and formula (4.1) with $\mathcal{S}_n$ replaced by W gives the semigroup of Brownian motion killed at the boundary of the cone. The Brownian motion conditionned to stay in C is then defined as the Markov process with transition probabilities given by (4.2). The following result has been proved in [BBO], Theorem 5.6.

Theorem 7.1. *Let $(B_t)_{t\geq 0}$ be a Brownian motion in V, then the stochastic process $(\mathcal{P}_{w_0}B(t))_{t\geq 0}$ is a Brownian motion conditionned to stay in the cone C.*

The proof of the theorem uses ideas from queuing theory.

§8. Connection with the Duistermaat-Heckman measure

We have seen that the information lost in passing from a path π to $\mathcal{P}_\alpha\pi$ is a real number in the interval $[0, \mathcal{P}_\alpha\pi(T)]$, since one can recover π from $\mathcal{P}_\alpha\pi$ and this number by (3.1), therefore the information needed to recover π from $\mathcal{P}_{w_0}\pi$ is a family of $l(w_0)$ nonnegative real numbers (depending on the chosen reduced decomposition of w_0). One can show that, for a fixed path η with values in the cone C, and for some reduced decomposition

$$w_0 = s_1 \dots s_l$$

the set of all numbers arising from paths π such that $\mathcal{P}_{w_0}\pi = \eta$, is a convex polytope which depends only on $\eta(T)$ and not on the rest of the path. Furthermore, when $\pi = B$ is chosen randomly as a Brownian path, the conditional distribution of the point in the polytope, knowing $\mathcal{P}_{w_0}B(T)$, is the normalized Lebesgue measure, and the conditional distribution of $B(T)$, knowing $\mathcal{P}_{w_0}B(T)$, is a probability distribution on the convex hull of the points $w(B(T)), w \in W$, which is the image of the Lebesgue measure on the polytope by the affine map

$$(x_1, \dots, x_l) \to \mathcal{P}_{w_0}B(T) - \sum_{j=1}^{l} x_i\alpha_{s_i}$$

When W is a Weyl group, and the root system corresponds to the root system of a semi-simple Lie algebra, this probability measure is the (normalized) Duistermaat-Heckman measure, and the polytopes defined above have been studied by Berenstein and Zelevinsky [BZ]. These topics will be discussed further in [BBO2].

References

[B1] Biane, Philippe Marches de Bernoulli quantiques. Sinaire de Probabilit, XXIV, 1988/89, 329–344, Lecture Notes in Math., 1426, Springer, Berlin, 1990.

[B2] P. Biane, Quantum random walk on the dual of SU(n). Probab. Theory Related Fields 89 (1991), no. 1, 117–129.

[B3] P. Biane, Le thre de Pitman, le groupe quantique $\mathrm{SU}_q(2)$, et une question de P. A. Meyer. In memoriam Paul-AndrMeyer: Séminaire de Probabilités XXXIX, 61–75, Lecture Notes in Math., 1874, Springer, Berlin, 2006.

[B4] P. Biane, Introduction to random walks on noncommutative spaces. Notes of a course given in Greifswald. http://www.math-inf.uni-greifswald.de/algebra/qpt/

[BBO] P. Biane; P. Bougerol, N. O'Connell, Littelmann paths and Brownian paths. Duke Math. J. 130 (2005), no. 1, 127–167.

[BBO2] P. Biane; P. Bougerol, N. O'Connell, Continuous crystals and Duistermaat-Heckman measure for Coxeter groups, arXiv:0804.2356.

[BJ] P. Bougerol and T. Jeulin. Paths in Weyl chambers and random matrices. Probab. Theory Related Fields 124 (2002), no. 4, 517–543.

[BO] N. Bourbaki. *Groupes et algèbres de Lie*, Ch. IV-VI, Hermann, Paris, 1968.

[BZ] A. Berenstein, A. Zelevinsky, Tensor product multiplicities, canonical bases and totally positive varieties. Invent. Math. 143 (2001), no. 1, 77–128.

[D] J.L. Doob, *Classical potential theory and its probabilistic counterpart.* Grundlehren der Mathematischen Wissenschaften, 262. Springer-Verlag, New York, 1984.

[H] J. E. Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge Studies in Advanced Mathematics 29. Cambridge University Press, 1990.

[L] P. Lévy, *Processus stochastiques et mouvement brownien.* Gauthier-Villars, Paris 1965

[OY] N. O'Connell and M. Yor. A representation for non-colliding random walks. Elect. Commun. Probab. 7 (2002) 1-12.

[P] J.W. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. Adv. Appl. Probab. 7 (1975) 511-526.

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