Topology-preserving conditions for 2D digital images under rigid transformations

Phuc Ngo, Nicolas Passat, Yukiko Kenmochi, Hugues Talbot

Noname manuscript No.
(will be inserted by the editor)

Topological properties of 2D digital images under rigid transformations

Phuc Ngo · Yukiko Kenmochi · Nicolas Passat · Hugues Talbot

the date of receipt and acceptance should be inserted later

Abstract In the continuous domain $\mathbb{R}^n$, rigid transformations (*i.e.*, compositions of reflections, rotations and translations) are topology-preserving operations. Due to digitization, this is not the case when considering digital images, *i.e.*, images defined on $\mathbb{Z}^n$. In this article, we begin to investigate this problem by studying conditions for digital images to preserve their topological properties under arbitrary rigid transformations on $\mathbb{Z}^2$. Based on (*i*) the recently introduced notion of DRT graph, and (*ii*) the notion of simple point, we propose an algorithm for evaluating digital images topological invariance in quasi-linear time.

Keywords Rigid transformation · 2D digital image · discrete geometry · discrete topology · simple point · DRT graph

1 Introduction

1.1 Context and motivation

Proper rigid transformations (*i.e.*, rotations composed with translations) are involved in numerous 2D and 3D image processing and analysis tasks, for instance in registration [**?**] and tracking [**?**]. In such applications, the images are necessarily digital, and can then be considered as functions $I : \mathbb{S} \to \mathbb{V}$ from a finite subset $\mathbb{S} \subset \mathbb{Z}^n$ to a value space $\mathbb{V}$.

Continuous rigid transformations are topology-preserving in $\mathbb{R}^n$. Unfortunately, this property –which is highly desirable in image analysis and processing– is generally lost in $\mathbb{Z}^n$. In particular, digital rigid transformations (*i.e.*, rigid transformations followed by a digitization process) may not preserve the topological properties of digital images, such as the homotopy-type, as exemplified in Fig. **??**. This is due to the sampling effects induced by the mandatory digitization process from $\mathbb{R}^n$ to $\mathbb{Z}^n$.

Phuc Ngo (corresponding author), Yukiko Kenmochi, Hugues Talbot
Université Paris-Est, LIGM, UPEMLV-ESIEE-CNRS, France
Tel.: +33-145926737
Fax: +33-145926699
E-mail: ngoh@esiee.fr

Nicolas Passat
Université de Reims Champagne-Ardenne, CReSTIC, EA 3804, France

Fig. 1 Left: binary digital image and the grid modeling its discrete structure. Middle: a rigid transformation applied on this grid. Right: the resulting transformed image, which is not topologically identical to the initial image (for instance, if considered 8-connected, the black component was split).

In this work, we present an extension of previous work presented in [**?**]. We initiate the study of this specific problem. More precisely, we focus on the 2D case, and on defining some conditions under which 2D digital images preserve their topological properties under arbitrary rigid transformation. To reach this goal, we consider (*i*) the notion of DRT graph, recently introduced in [**?**]. This graph defines a combinatorial model of all the rigid transformations of a 2D digital image; and (*ii*) the classical notion of simple points, which can be used to guarantee the preservation of homotopy-type, and has been extended to several categories (binary, grey-level, labeled) of digital images.

By combining these two notions, we provide a combinatorial analysis of the notion of digital image topological invariance under a considered transformation. This analysis leads us to a methodology for evaluating the homotopy-type preservation of a 2D digital image under arbitrary rigid transformations.

1.2 Contributions and structure of the article

The state of the art in both digital rigid transformation and topology preservation is exposed in Section **??**. Basic definitions and notations are provided in Section **??**. In Section **??**, we introduce the main issues related to topology alterations induced by the embedding of rigid transformations into digital spaces. Our contribution is exposed in Sections **??**–**??**. Specifically, in Section **??**, we introduce the DRT graph [**?**] as a tool for studying the behaviour of rigid transformations on digital images from a topological point of view. In particular, we propose a first algorithm for assessing the topological invariance of a digital image under all possible rigid transformations, with a superlinear time and space complexity, corresponding to that of the associated DRT graph. In Section **??**, we refine this first approach, by spatially decomposing the image analysis process, leading to an equivalent algorithm that presents a quasi-linear complexity with respect to the image size. In Section **??**, this method is experimentally assessed in terms of complexity and correctness. Section **??** finally summarizes the contributions and proposes some future work. Appendix A provides a full description of the notion of DRT graph used in Sections **??** and **??**.

2 State of the art

2.1 Rigid transformations on digital images

The study of rigid transformations is widely motivated by practical considerations in digital image processing, as mentioned in Section **??**. In this article, we focus on the study of *proper* rigid transformations, *i.e.*, combinations of only translations and rotations, and excluding reflections. The reason for this is twofold. First of all any reflection can be trivially decomposed into a reflection across a given axis (to swap the handedness of the image), combined with two rotations with the same center and opposite angles. We can then choose an axis of reflection that best suits our purpose, for instance an axis aligned with the grid, which results in a topologically invariant reflection. Second, reflections are not as generally useful in matching and tracking applications, and may needlessly complicate the search space. By an abuse of language, we will continue to refer to combinations of only translations and rotations as rigid transformations.

To reach our stated goal of finding conditions for topology preservation, it is necessary to compare the properties of the initial and transformed images. However studying the problem in the continuous domain is unfeasible due to the infinite number of possible transformations in this domain, and so a discrete method must be devised, if possible involving only integer operations for exactness.

Over the last two decades, several methods were proposed to study transformations on digital images as fully discrete processes. Decompositions of rotations using quasi-shears [**?**,**?**] were introduced to preserve bijectivity. A drawback of these approaches is that the result obtained by quasi-shear composition is not always identical to the discretized result of the initial transform. Moreover not all rotations for a given finite subspace of $\mathbb{Z}^n$ can be decomposed in this way. An alternative discrete formulation for rotations based on hinge angles was proposed in [**?**,**?**,**?**,**?**]. Informally, for any given discrete finite set, rotations around a fixed center and with a small enough angle will not result into any change. To the contrary, some large enough angles will indeed result into some pixels changing positions. The notion of hinge angles formalizes this intuitive property of digital image rotations. In particular, hinge angles (represented by integer triples) give sufficient information for incrementally generating and performing all rotations. It is proved in [**?**,**?**] that the total number of hinge angles associated to an image of size $N \times N$ is $O(N^3)$. In a similar context, different studies in combinatorial analysis for the problem of 2D pattern matching under different classes of geometric transformations have been considered, in particular for: rotations [**?**,**?**]; scalings [**?**,**?**]; combined scalings and rotations [**?**]; affine transformations [**?**,**?**]; projective and linear transformations [**?**].

To the best of our knowledge, fully discrete approaches devoted to rigid transformations are in quite limited numbers. Following the idea of rotations by hinge angles and inspired by the discretization technique of the problem of 2D pattern matching, we have recently studied in [**?**] the combinatorial aspects and properties of the class of rigid transformations, by simultaneously considering the parameter space for both translations and rotations. Our approach discretizes the induced parameter space of rigid transformations on 2D digital images, and models this space by a combinatorial structure, namely a graph. This structure presents a space complexity of $O(N^9)$ for any subset of $\mathbb{Z}^2$ of size $N \times N$. Moreover, an algorithm to build this graph with *exact computation* (*i.e.*, using only integers), in linear time with respect to its space complexity is proposed in [**?**].

2.2 Topology-preserving digital image transformations

The study of discrete transforms involving topological alteration rely mostly –but not exclusively [**?**]– on the notion of simple point, which provides conditions for the preservation of strong topological properties, and in particular the homotopy-type.

Simple points were initially defined for binary images on $\mathbb{Z}^2$ [**?**]. This notion was later formulated in the framework of digital topology [**?**], and was recently shown [**?**,**?**] to extend to richer discrete frameworks that explicitly describe cubic grids as topological spaces [**?**,**?**]. Several extensions have then been proposed during the following forty years, in terms of dimensions (3D [**?**] and 4D [**?**] simple points); of cardinality (deletable sets [**?**], P-simple points [**?**], minimal simple sets [**?**]); and in terms of image value spaces (grey-level images [**?**], label images [**?**,**?**,**?**]).

Practically, simple points have been intensively involved in the development of segmentation methods, for instance in the field of medical image analysis [**?**]. In particular, many such segmentation methods have been defined in monotonic transformation paradigms [**?**] (*i.e.*, reduction, skeletonisation, region-growing) or in (continuous [**?**] or discrete [**?**]) deformation model paradigms.

Only a few works have involved topology preservation notions combined with geometric transformations, for instance in the field of (digital) image warping [**?**,**?**] based on (continuous topology-preserving) deformation fields obtained from registration procedures. In particular, the question of digital topology preservation in the case of rigid transformations has –to the best of our knowledge– never been considered until now.

3 Background notions

3.1 (Digital) images

In the continuous domain, a (2D) image can be formalised as a function $\mathcal{I} : \mathbb{R}^2 \to \mathbb{V}$, where $\mathbb{V}$ is a value space. In particular:

- if $|\mathbb{V}| = 2$, then I is a binary image;
- if $|\mathbb{V}| \geq 3$ and is equipped with a total order, then I is a grey-level image;
- if $|\mathbb{V}| \geq 3$ and is not equipped with a total order, then I is a label image.

We assume that $\mathbb{V}$ contains at least two elements, including one, noted $\perp$, corresponding to the "background" value, usually with little semantic content.

A (2D) *digital* image associated to $\mathcal{I}$ can be defined as $I : \mathbb{Z}^2 \to \mathbb{V}$, by sampling $\mathcal{I}$ on the discrete space $\mathbb{Z}^2$. In other words, we have $I = \mathcal{I}_{|\mathbb{Z}^2}$, and for each $\mathbf{p} \in \mathbb{Z}^2$, the value $I(\mathbf{p})$ of the digital image models the value of $\mathcal{I}$ on the associated pixel $\mathbf{p} + [-\frac{1}{2}, \frac{1}{2}]^2$, namely the Voronoi cell of $\mathbb{R}^2$ induced by $\mathbb{Z}^2$ around $\mathbf{p}$. This paradigm relies on the digitization function D defined as

$$\left| \begin{array}{lcl} D : \mathbb{R}^2 & \longrightarrow & \mathbb{Z}^2 \\ \quad (x, y) & \longmapsto & ([x], [y]) \end{array} \right. \tag{1}$$

where $[\,\cdot\,]$ is the standard rounding function. We assume that a digital image is actually defined on a subset of $\mathbb{Z}^2$, namely $I^{-1}(\mathbb{V} \setminus \{\perp\})$, which is finite. Then, it is plain that $I^{-1}(\mathbb{V} \setminus \{\perp\}) \subseteq \mathbb{S} = [0, N-1]^2 \cap \mathbb{Z}^2$, for a given $N \in \mathbb{N}$. The set $\mathbb{S}$ is called the support of I and $N \times N$ is the size of I. By abuse of notation –and without loss of generality– we will sometimes note a digital image as $I : \mathbb{S} \to \mathbb{V}$ instead of $I : \mathbb{Z}^2 \to \mathbb{V}$.

3.2 (Digital) rigid transformations

In the continuous framework, a rigid transformation (only compositions of translations and rotations are considered) is expressed as a bijection $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined, for any $\mathbf{x} = (x, y) \in \mathbb{R}^2$ by

$$\mathcal{T}(\mathbf{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \tag{2}$$

where $a, b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$. Such a transformation is unambiguously modeled by the triple of parameters (a, b, θ), and will sometimes be noted $\mathcal{T}_{ab\theta}$.

It is not possible to apply directly $\mathcal{T}$ on a digital image $I : \mathbb{S} \to \mathbb{V}$, since there is no guarantee that $\mathcal{T}(\mathbf{x}) \in \mathbb{Z}^2$, for any $\mathbf{x} \in \mathbb{Z}^2$. The correct handling of *digital* rigid transformations (a)(b)(c)

Fig. 2 (a) Forwards and backwards transformation models in $\mathbb{R}^2$. (b) Lagrangian and (c) Eulerian transformation models in $\mathbb{Z}^2$.

then requires to define a digital analogue $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ of $\mathcal{T}$. By considering the digitization paradigm proposed in Equation (**??**), this can be conveniently performed by setting

$$T = D \circ \mathcal{T}_{|\mathbb{S}}. \tag{3}$$

In other words, the transformation T is obtained by applying $\mathcal{T}$ and then digitising the result by the function D, as illustrated in the diagram below.

$$\begin{array}{ccc} \mathbb{S} \subseteq \mathbb{Z}^2 & \xrightarrow{T = D \circ \mathcal{T}_{|\mathbb{S}}} & T(\mathbb{S}) \subseteq \mathbb{Z}^2 \\ \Big\downarrow Id & & \Big\uparrow D \\ \mathbb{S} \subseteq \mathbb{R}^2 & \xrightarrow{\mathcal{T}} & \mathcal{T}(\mathbb{S}) \subseteq \mathbb{R}^2 \end{array} \tag{4}$$

The function $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ is then explicitly defined, for any $\mathbf{p} = (p, q) \in \mathbb{Z}^2$, by

$$T(\mathbf{p}) = D \circ \mathcal{T}(\mathbf{p}) = \begin{pmatrix} [p\cos\theta - q\sin\theta + a] \\ [p\sin\theta + q\cos\theta + b] \end{pmatrix} \tag{5}$$

In $\mathbb{R}^2$, the transformation $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective. Consequently, determining $\mathbf{y} \in \mathbb{R}^2$ such that $\mathcal{T}(\mathbf{x}) = \mathbf{y}$, and determining $\mathbf{x} \in \mathbb{R}^2$ such that $\mathcal{T}^{-1}(\mathbf{y}) = \mathbf{x}$, are equivalent questions. The first issue corresponds to the forwards model for image transformation, while the second issue corresponds to the backwards model (Figure **??**(a)).

In general, the bijective hypothesis is no longer verified in the digital case, for $T = D \circ \mathcal{T}_{|\mathbb{S}} : \mathbb{Z}^2 \to \mathbb{Z}^2$. In such a context, the forwards model (namely the Lagrangian model, and illustrated in Figure **??**(b)) can be correctly handled, but not the backwards version (namely the Eulerian model). However, by setting $T^{-1} = D \circ \mathcal{T}^{-1}_{|\mathbb{Z}^2} : \mathbb{Z}^2 \to \mathbb{Z}^2$, we can define a transformed digital image $I \circ T^{-1} : \mathbb{Z}^2 \to \mathbb{V}$ with respect to T, that conveniently enables to handle the Eulerian model (Figure **??**(c)). (Note that T^{-1} is not the inverse function of T in general.)

In the sequel, we only focus on the Eulerian model (the justification of this choice will be discussed in Section **??**). From this point on –for the sake of readability and without loss of correctness– we will note T instead of T^{-1}, due to the bijectivity of $\mathcal{T}$ and $\mathcal{T}^{-1}$.

3.3 The topology of digital images

In the context of digital image transformations, it is often required to guarantee the preservation of the image topology, that is the preservation of given topological invariants. Among these topological invariants, the homotopy-type is generally considered.

Homotopy-type preservation can be conveniently handled thanks to the notion of simple point, already presented in Section **??**. In particular, the following property is verified in all the works devoted to define variants of simple points in various kinds of images.

Property 1 *Let $I : \mathbb{Z}^n \to \mathbb{V}$ be a digital image. Let $\mathbf{p} \in \mathbb{Z}^n$ be a simple point of I. Then, the modified image $I_{\mathbf{p}}$, obtained from I by modifying the value of I at the point $\mathbf{p}$ into a licit value (depending on $\mathbb{V}$ and I) has the same homotopy-type as I.*

(a) (b) (c)
$N_4(\mathbf{p})N_8(\mathbf{p})N_{20}(\mathbf{p})$

Fig. 3 The neighborhoods N_4, N_8 and N_{20} of a point $\mathbf{p}$.

Remark 2 *The notion of simplicity can be extended to sets of (successively) simple points between the initial image I and a final image I', that still preserves the homotopy-type between I and I'. This leads to a notion of* simple-equivalence *[?] between images.*

Independently from the kind of topological structure [**?**,**?**,**?**] mapped on $\mathbb{Z}^n$, and from the value space $\mathbb{V}$, the notion of simple point has the virtue of being characterizable by considering the immediate neighborhood of a point in the image.

Property 3 *Let $I : \mathbb{Z}^n \to \mathbb{V}$ (with $n \leq 4$) be a digital image. Let $\mathbf{p} \in \mathbb{Z}^n$ be a point of I. The characterisation of $\mathbf{p}$ as a simple point can be computed locally,* i.e.*, by only considering the $3^n - 1$ points $\mathbf{q} \in \mathbb{Z}^n$ such that $||\mathbf{p} - \mathbf{q}||_\infty \leq 1$.*

Corollary 4 *A simple point can be characterised in constant time, up to the dimension* 4.

Based on these considerations, the concepts developed in the sequel of this article require only the following two hypotheses related to the considered images $I : \mathbb{Z}^2 \to \mathbb{V}$:

(H1) $\mathbb{Z}^2$ is equipped with a standard topological structure [**?**,**?**,**?**]; and
(H2) for this topological structure and the value space $\mathbb{V}$, a notion of simple point is available (this is, for instance, the case for binary, grey-level or label images).

For the sake of readability –but without loss of generality– we will hereafter focus on the case of binary images endowed with the digital topology [**?**]. In this framework, the topological notions derive from a graph structure induced by two dual adjacency (*i.e.*, irreflexive and symmetric) relations, namely the 4- and 8-adjacencies, which are defined as follows.

Definition 5 ([?]) *Given a point $\mathbf{p} = (p_1, p_2) \in \mathbb{Z}^2$, we consider the two neighborhoods N_4 and N_8, which are defined for the point $\mathbf{p}$ as sets of points $\mathbf{q} = (q_1, q_2) \in \mathbb{Z}^2$ such that:*

$$N_4(\mathbf{p}) = \{\mathbf{q} \in \mathbb{Z}^2 \mid ||\mathbf{p} - \mathbf{q}||_1 = \sum_{i=1}^{2} |p_i - q_i| \leq 1\} \tag{6}$$

$$N_8(\mathbf{p}) = \{\mathbf{q} \in \mathbb{Z}^2 \mid ||\mathbf{p} - \mathbf{q}||_\infty = \max_{i=1}^{2} |p_i - q_i| \leq 1\} \tag{7}$$

We say that the point $\mathbf{q}$ is 4- (resp. 8-) adjacent *to $\mathbf{p}$ if $\mathbf{q} \in N_4(\mathbf{p}) \setminus \{\mathbf{p}\}$ (resp. $\mathbf{q} \in N_8(\mathbf{p}) \setminus \{\mathbf{p}\}$).*

Remark 6 *For technical reasons that will be justified in Section* **??***, we also introduce a third neighborhood for point $\mathbf{p}$, namely N_{20}, as well as the induced adjacency relation: the* 20*-adjacency. It is defined by*

$$N_{20}(\mathbf{p}) = \Big\{\mathbf{q} \in \mathbb{Z}^2 \mid ||\mathbf{p} - \mathbf{q}||_2 = \Big(\sum_{i=1}^{2}(p_i - q_i)^2\Big)^{1/2} < 2\sqrt{2}\Big\} \tag{8}$$

From the adjacency relations induced by these neighborhoods (illustrated in Figure **??**), we can define the notion of paths and then derive important topological concepts from connectedness to fundamental groups. In this framework, the characterisation of simple points $\mathbf{p}$ (which are either 4- or 8-simple, according to the chosen adjacency for the point value) can be made by only considering $N_8(\mathbf{p})$ (see Property **??**). Some examples and counter-examples of simple points are provided in Figure **??**.

Fig. 4 Examples of simple $(\mathbf{x}, \mathbf{y})$ and non-simple $(\mathbf{z}, \mathbf{t})$ points in a binary image. Modifying the value of $\mathbf{z}$ would merge two black connected components, while modifying the value of $\mathbf{t}$ would create a white connected component. In both cases, the homotopy-type of the image would be modified.

(a) (b)

(c) (d)

Fig. 5 Examples of non-preservation of distances and angles induced by digital rigid transformations. Left column ($\mathbb{S}$): two points (in red) are considered for distance alterations; a third point is considered for angle alteration. Middle ($\mathcal{T}(\mathbb{S})$): rigid transformation of the grid defined in the left column. Right ($T(\mathbb{S}) = D \circ \mathcal{T}(\mathbb{S})$): result of the induced digital rigid transformation. The distances between the red points change from 1 in (a,b) and $\sqrt{2}$ (c,d) to $\sqrt{2}$ in (a), 0 in (b), 1 in (c) and 2 in (d). In particular, the two red points which were 4-adjacent in (a,b) and 8-adjacent in (c,d) become 8-adjacent in (a), are fused in (b), become 4-adjacent in (c), and are neither 4- nor 8-adjacent in (d) respectively. Angle alterations also happen between the triplets of points in (a,b,c) where a difference of $\pi/4$ is observed before and after the digital rigid transformation.

4 Digital rigid transformations: Topological issues

As stated in Section **??**, going from rigid transformations in $\mathbb{R}^2$ (Equation (**??**)) to digital rigid transformations in $\mathbb{Z}^2$ (Equation (**??**)) requires considering a digitization function (Equation (**??**)) that discretizes both the space and the transformation. This process may modify some properties between the input image in the transformed output (see, *e.g.*, Figure **??**). In this section, we investigate such digitization effects on the topological properties of digital images during rigid transformations.

4.1 Non-preservation of geometric properties

In $\mathbb{R}^2$, it is well-known that rigid transformations preserve the distances (and in particular the Euclidean one) between any pairs of points, as well as the angles induced by any triplet of (distinct) points. However, when rigid transformations are digitised from $\mathbb{R}^2$ to $\mathbb{Z}^2$, these properties are often lost.

Indeed, let us consider a point $\mathbf{p} \in \mathbb{Z}^2$ and a point $\mathbf{q} \in N_8(\mathbf{p}) \setminus \{\mathbf{p}\}$. From Equation (**??**), it is plain that the Euclidean distance d_2 between $\mathbf{p}$ and $\mathbf{q}$ verifies $d_2(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|_2 \in \{1, \sqrt{2}\}$. Let us now consider the points $\mathbf{p'}$ and $\mathbf{q'}$, obtained from a (digital) rigid transformation of $\mathbf{p}$ and $\mathbf{q}$, respectively.

From the expression of the digitization function D (Equation (**??**)), one can easily prove that:

$$d_2(\mathbf{p}, \mathbf{q}) = 1 \Longrightarrow d_2(\mathbf{p'}, \mathbf{q'}) \in \{0, 1, \sqrt{2}\} \tag{9}$$

$$d_2(\mathbf{p}, \mathbf{q}) = \sqrt{2} \Longrightarrow d_2(\mathbf{p'}, \mathbf{q'}) \in \{1, \sqrt{2}, 2\} \tag{10}$$

This shows that digital rigid transformations generally do not preserve distances.

Remark 7 *The fact that we may have* $d_2(\mathbf{p'}, \mathbf{q'}) = 0$ *when* $d_2(\mathbf{p}, \mathbf{q}) = 1$ *also implies that digital rigid transformations are non-injective in general. Due to the discrete nature of* $\mathbb{Z}^2$*, this also implies that such transformations are non-surjective.*

Similar alterations related to the angles between points can be straightforwardly derived. Both distance and angle alterations are exemplified in Figure **??**, for a 3×3 sample of $\mathbb{Z}^2$.

Fig. 6 Left: a digital image support and the grid modeling its discrete structure. Right: examples of a null pixel (in green), a single pixel (in blue) and a double pixel (in red) with respect to a digital rigid transformation.

Beyond geometrical alterations, these examples also illustrate that digital rigid transformations can lead to the alteration of adjacency relations between points of the transformed space, with respect to the original one.

We show, in the remainder of this section, how such alterations can raise topological issues in the transformed spaces. To this end, we will first study some properties of pixels related to the influence of digital rigid transformations on their neighborhoods.

4.2 Pixel status during digital rigid transformations

As explained above, a (continuous) rigid transformation $\mathcal{T}$ establishes a bijection from $\mathbb{R}^2$ to itself. In contrast, a digital rigid transformation T is generally not a bijection from $\mathbb{Z}^2$ to itself (see Remark **??**).

It is plain that for any three distinct points $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \in \mathbb{Z}^2$, we have $\max_{i,j \in \{1,2,3\}} \{d_2(\mathbf{p_i}, \mathbf{p_j})\} \geq \sqrt{2}$. From Equation (**??**), we derive that the three points $\mathbf{p'_1}, \mathbf{p'_2}, \mathbf{p'_3}$ obtained by a digital rigid transformation T of $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ can not be mapped into the same pixel by the associated rigid transformation $\mathcal{T}$.

Let us define a set $P_T(\mathbf{p})$ of points $\mathbf{q} \in \mathbb{Z}^2$ associated to a point $\mathbf{p} \in \mathbb{Z}^2$ with respect to a digital rigid transformation T such that $P_T(\mathbf{p}) = \{\mathbf{q} \in \mathbb{Z}^2 \mid T(\mathbf{q}) = \mathbf{p}\}$. In other words, $P_T(\mathbf{p})$ contains all points $\mathbf{q} \in \mathbb{Z}^2$ such whose images by T is $\mathbf{p}$. It is possible to characterise the *status* of $\mathbf{p}$ with respect to T by using this set $P_T(\mathbf{p})$. In particular, there exist only three possibilities, illustrated in Figure **??**.

Definition 8 *Let us consider a point* $\mathbf{p} \in \mathbb{Z}^2$*, and a digital rigid transformation* T.

- *If* $|P_T(\mathbf{p})| = 0$*, then* $\mathbf{p}$ *is called a* null *point.*
- *If* $|P_T(\mathbf{p})| = 1$*, then* $\mathbf{p}$ *is called a* single *point.*
- *If* $|P_T(\mathbf{p})| = 2$*, then* $\mathbf{p}$ *is called a* double *point.*

A double point appears when a (digital) rigid transformation maps two points into a single point, while a null point appears when a transformation maps no point into this point. This is a well-known issue, which has already been identified in the literature dealing with rotations in discrete spaces, for instance [**?**,**?**,**?**,**?**,**?**]. Some examples are provided in Figure **??**.

From Definition **??**, we can derive the following property, related to null and double points.

Property 9 *A (digital) rigid transformation generates the same number of null and double points.*

In [**?**,**?**], Nouvel *et al.* investigated the effects of rotations on the neighbouring relations in digital images. The following result, proved for digital rotations in their study, remains valid in the more general case of digital rigid transformations.

Property 10 ([?]) *Let* $\mathbf{p} \in \mathbb{Z}^2$ *be a double (resp. null) point, with respect to a given digital rigid transformation. Then, for any* $\mathbf{q} \in \mathbb{Z}^2$ *being* 4*-adjacent to* $\mathbf{p}$*,* $\mathbf{q}$ *is not a double (resp. null) point for this transformation.*

In other words, two 4-adjacent points cannot be either both double or both null. It is however possible that they form a pair of double and null points (see, *e.g.*, Figure **??**(b,c)).

(a) $\theta = \arcsin(\frac{3}{5})$ (b) $\theta = \pi/12$

(c) $\theta = \pi/5$ (d) $\theta = \pi/4$

Fig. 7 Some digital rotations by angles θ of a white square of size 100×100. Double points are depicted in red, and null points in grey.

Fig. 8 The effects of double (left) and null (right) points are exemplified in the context of digital rigid transformations (see text).

(a) (b)

(c) (d)

Fig. 9 (a) A 2×2 set $\mathbb{S}$ (each pixel is identified by a label: a, b, c, d). (b) Local configurations of $\mathbb{Z}^2$ (up to rotations and symmetries) leading to the sample $\mathbb{S}$ depicted in (a), when applying a digital rigid transformation. (c) Examples of digital rigid transformations in which binary samples (induced by a binary image I defined on $\mathbb{Z}^2$) preserve their topology. (d) Examples of digital rigid transformations in which binary samples are subject to topological alterations.

4.3 Topological alterations due to digitization

A digital rigid transformation T behaves like a bijection for single points, while the possible existence of null and double ones generally forbids T to be a surjection and an injection, as already evoked in Remark **??** and illustrated in Figure **??**.

The existence of null and double points is a first cause of potential topological alterations. Indeed, some connected components may be lost when applying a digital rigid transformation, in particular those composed of exactly one pixel. An example of this phenomenon is exemplified in the third configuration depicted in Figure **??**(d).

In addition to such "cardinality-based" issues, introduced in Section **??**, some "adjacency-based" issues are derived from the geometric alterations, evoked in Section **??**. Indeed, it has been observed that the non-preservation of distances between points, when applying a digital rigid transformation, had a direct interpretation in terms of modification of the adjacency relations between such points. The adjacency relations between points may change from 4- to 8-adjacency or *vice versa* (see Figure **??**(a,c)), or could even lead to a loss of adjacency between points initially 8-adjacent (see Figure **??**(d)). In such situations, some connected components may be either split or merged. Examples of such phenomena are illustrated in the first and second configurations depicted in Figure **??**(d).

Remark 11 *As illustrated by Figure* **??***(c), some topological alterations of the discrete structure of a subset* $\mathbb{S}$ *of* $\mathbb{Z}^2$ *do not necessarily lead to topological modifications of an image I defined on* $\mathbb{S}$*. Consequently, the study the potential topological alterations induced by digital rigid transformation must be considered not only as a transformation-dependent problem, but also as an image-dependent one.*

5 DRT graphs and image topology

In this section, we briefly recall the notion of DRT graph, previously proposed in [**?**]. This combinatorial structure (more extensively described in Appendix A) is used to model the subdivision of the parameter space (a, b, θ) of rigid transformations. Then, we discuss how to use this structure as a topology analysis tool for rigidly transformed images.

5.1 A brief presentation of the DRT graph

Contrarily to rigid transformations in $\mathbb{R}^2$ (see Equation (**??**)), *digital rigid transformations* are not continuously defined with respect to the parameters a, b (that control the "translation" part of the transformations) and θ (that controls their "rotation" part). More precisely, the parameter space $\mathbb{R}^3$ of (a, b, θ) is divided into 3D open cells, in which the transformations $T_{ab\theta}$ are equal, while the 2D surfaces bounding these open cells correspond to discontinuities of the digital rigid transformations, induced by the digitization process (see Equation (**??**)).

From a theoretical point of view, each 3D open cell of the parameter space (a, b, θ) can be seen as an equivalence class of rigid transformations $\mathcal{T}$ of $\mathbb{R}^2$ that lead to a same transformation $T = D \circ \mathcal{T}$ in $\mathbb{Z}^2$. Such an equivalence class is called a *discrete rigid transformation (DRT)* [**?**]. (Note that the term *digital* refers to the digitization process of numeric images and transformations for such images, while the term *discrete* refers to the non-continuous structure of these transformations.)

Each 3D open cell can also be considered as the resulting digital transformed space generated by any (digital) rigid transformation of the associated DRT. Moreover, the existence of a 2D surface between two cells indicates that the two associated transformed images differ by exactly one pixel. By mapping any 3D cell onto a 0D point, and any 2D surface onto a 1D edge, the combinatorial structure of the parameter space can be modeled, in a dual way, as a (connected) graph, namely a *DRT graph*.

Definition 12 (DRT graph [?]) *Let $G = (V, E)$ be the graph defined such that:*

- *any vertex $v \in V$ models a 3D open cell associated to a DRT;*
- *any edge $e = (v, w) \in E$ models a 2D surface between two distinct vertices $v, w \in V$.*

The graph $G = (V, E)$ is called a DRT graph.

A DRT graph models the subdivision of the whole rigid transformations parameter space; therefore, it models *all* the possible rigid transformations of a given set $\mathbb{S}$. Despite the fact that the space of these transformations is infinite, the DRT graph is actually defined as a *finite* structure, up to integer translation periodicity. Based on these considerations, the space complexity of the DRT graph for any set $\mathbb{S}$ of size $N \times N$ has been proved to be polynomial. An exact computation algorithm also exists to build this graph in linear time with respect to the size of the graph [**?**].

Property 13 ([?]) *The DRT graph associated to a set $\mathbb{S}$ of size $N \times N$ has a space complexity of $O(N^9)$.*

DRT graphs do not depend on the values that are assigned to the pixels of $\mathbb{S}$. In other words, its structure is invariant for any image defined on a same support $\mathbb{S}$. In the sequel, we will however consider –without loss of generality– a DRT graph with respect to a given image I defined on $\mathbb{S}$. In this context, any edge $e = (v, w) \in E$ of the DRT graph can

Fig. 10 Left: part of a DRT graph in which each vertex is a DRT representing a digital transformed image and each edge indicates the *only* value modification of a pixel between two connected vertices. More precisely, if an edge $e = (v, w, (\mathbf{p}, \mathbf{p}'))$ connects two vertices v and w, then the associated images I_v and I_w of v and w respectively differ at the *single* pixel $\mathbf{p}'$, and $\mathbf{p}$ is the pixel corresponding to $\mathbf{p}$ in the original image (see text). Right: the transformed images associated to the vertices of the DRT graph G and their relations according to the edges in G. The images from left to right and from top to bottom correspond to the vertices ordered in the graph, in which the first image corresponds to the original image.

be "enriched" as $e = (v, w, (\mathbf{p}, \mathbf{p}'))$, where $\mathbf{p}'$ is the only pixel where the transformed spaces differ with respect to the DRTs v and w, respectively, while $\mathbf{p}$ is the pixel corresponding to $\mathbf{p}'$ in the initial image I. (In this enriched framework, an edge (v, w) must be considered twice: as (v, w), and as (w, v), due to the non-symmetric definition of $(\mathbf{p}, \mathbf{p}')$; this only remains a formal detail, and the graph G is –of course– still non-directed.)

The DRT graph relies on geometric information provided by (a, b, θ). However, it does not explicitly model such geometric information. Indeed, it only provides structural information, that models the relationship between any "neighbouring" transformed images. In particular, the label $(\mathbf{p}, \mathbf{p}')$ of each edge e is implicitly associated to a function indicating the value modification of the pixel $\mathbf{p}'$ that differs between the transformed images corresponding to the DRTs v and w. More precisely, the rigid transformation associated to the 2D surface of the edge e modifies only the pixel value of $\mathbf{p}'$ (which is initially equal to the value of $\mathbf{p}$), such that $\mathbf{p}'$ will get its value from one of the 4-neighbouring pixels of $\mathbf{p}$. (This is induced by the notion of tipping surfaces introduced in Appendix A.) This property is exemplified in Fig. **??**. Practically, let I_v and I_w be the transformed images corresponding to the vertices v and w respectively. The value of $\mathbf{p}'$ at the vertex v is defined by $I_v(\mathbf{p}') = I(\mathbf{p})$ where $I : \mathbb{S} \to \mathbb{V}$ is the original image function. After the elementary modification at the edge e, we obtain a new transformed image I_w by simply changing the pixel value at $\mathbf{p}'$ as $I_w(\mathbf{p}') = I(\mathbf{p} + \boldsymbol{\delta})$ where $\boldsymbol{\delta} = (\pm 1, 0)$ or $(0, \pm 1)$. In this way, one can generate all the transformed images of I by incrementally and exhaustively scanning the associated DRT graph.

Remark 14 *Let $G = (V, E)$ be a DRT graph associated to a given image $I : \mathbb{S} \to \mathbb{V}$. For each edge $e = (v, w, (\mathbf{p}, \mathbf{p}')) \in E$, two cases can occur:*

(*i*) $I_v(\mathbf{p}') = I_w(\mathbf{p}')$, i.e., *the transformed images of I by the DRTs v and w are equivalent ($I_v = I_w$);*
(*ii*) $I_v(\mathbf{p}') \neq I_w(\mathbf{p}')$, i.e., $I_v \neq I_w$.

In the case of binary images, the value of $\mathbf{p}'$ may then be flipped from white to black or *vice versa*, and this may constitute the only modification between the images of I by two consecutive DRTs. Such a change of value may consequently alter the topological property of the binary images. In the sequel, we will show how to verify whether this actually occurs, for any arbitrary transformations, using a DRT graph.

5.2 DRT graph as a topological analysis tool

On the one hand, we would like to know if a given image I defined on $\mathbb{S}$ preserves its topological properties under any digital/discrete rigid transformations. Let us first formalise this preservation, via the notion of *topological invariance*.

Algorithm 1: Generation of simple-equivalent images and topological invariance verification.

Input: A DRT graph $G = (V, E)$ associated to an image I.
Input: An initial vertex $u \in V$ corresponding to the image I.
Output: A partial sub-graph $G'' = (V'', E'')$ of G such that, for any $v \in V''$, the images I_v are simple-equivalent to I.
Output: A Boolean B that indicates if I is a topologically invariant image.

$(V'', E'') \leftarrow (\{u\}, \emptyset)$
$S \leftarrow \{u\}$
while $S \neq \emptyset$ **do**
  Let $v \in S$
  $S \leftarrow S \setminus \{v\}$
  foreach $e = (v, w, (\mathbf{p}, \mathbf{p}')) \in E$, *such that* $w \notin V''$ **do**
    if $(I_v(\mathbf{p}') = I_w(\mathbf{p}'))$ *or* $((I_v(\mathbf{p}') \neq I_w(\mathbf{p}'))$ *and* $(\mathbf{p}'$ *is a simple point in* $I_v))$ **then**
      $(V'', E'') \leftarrow (V'' \cup \{w\}, E'' \cup \{e\})$
      $S \leftarrow S \cup \{w\}$
$B \leftarrow (V = V'')$

Definition 15 *We say that a digital image I is* topologically invariant *if all its transformed images have the same homotopy-type as I.*

On the other hand, as mentioned in Section **??** the DRT graph allows us to generate exhaustively *all* the transformed images of I. From the definition of the DRT graph and from Remark **??**, this can be achieved by incrementally modifying (at most) one pixel value between two successive transformed images. Moreover, we know from Property **??** that the notion of simple point can be used to preserve the homotopy-type between two images that differ in exactly one point. We also know from Remark **??** that this preservation of the homotopy-type is also guaranteed via the notion of simple-equivalence, that consists of considering successively simple points.

The local notion of simple point and the incremental notion of simple equivalence are therefore compatible with an incremental exploration of the DRT graph of image I, in order toevaluate its topological invariance.

Practically, the edges of the DRT graph $G = (V, E)$ of I can be classified in two categories: those that do not modify the topology of the transformed images and those that do. The first category contains the edges that correspond to the case (*i*) in Remark **??** as well as those that correspond to the case (*ii*) for which $\mathbf{p}'$ is a simple point; and the second one contains the edges that correspond to the case (*ii*) in Remark **??** for which $\mathbf{p}'$ is not simple.

Based on this binary classification, we can straightforwardly create a partial graph $G' = (V, E')$ of G by preserving in $E' \subseteq E$ only the edges of the first category. In particular, if G' is connected, it is plain that I is topologically invariant. Otherwise, I is not topologically invariant, and every connected component in G' corresponds to a set of simple-equivalent transformed images.

Such an approach presents an algorithmic complexity that is linear, in every case, with the (polynomial) space complexity of the DRT graph. It is however possible to reach a better (mean) complexity by using a standard spanning-tree algorithm (see Algorithm **??**), that provides two outputs: a Boolean evaluating the topological invariance of I, and a (non-necessarily maximal) set of simple-equivalent transformed images with respect to the image associated to the initial vertex u in the DRT graph (*e.g.*, I or any other transformed image of I). In Algorithm **??**, the graph G'' providing the set of simple-equivalent images is in fact a partial subgraph of G' (and of G as well). Nevertheless, the high algorithmic complexity of this approach practically forbids the generation the whole graph for large images, and therefore to consequently verify topological invariance. In the next section, we show that this problem can however be decomposed spatially, thus leading to a much lower complexity algorithm.

6 A local approach for analyzing topological invariance under DRTs

In the previous section, we have proposed to explore the whole DRT graph of a given image I in order to evaluate its topological invariance for all DRTs. More precisely, for each edge $e = (v, w, (\mathbf{p}, \mathbf{p'}))$ of the DRT graph, this exploration consists of verifying that $\mathbf{p'}$ is a simple point between the transformed images I_v and I_w with respect to the DRTs v and w, if $I_v \neq I_w$. From Property **??**, we know that this verification can be carried out locally, more precisely in the neighborhood $N_8(\mathbf{p'})$ of the transformed image space(s). We now propose to take advantage of the local nature of these tests to develop a space decomposition strategy that leads to a *local* version of the previously proposed *global* method.

6.1 From global to local DRTs

On the one hand, it is plain that the set of all DRTs defined on a subset of size $N \times N$ of $\mathbb{Z}^2$, does not depend on the way to locate this subset into $\mathbb{Z}^2$. In other words –provided that we choose a set $\mathbb{S} \subset \mathbb{Z}^2$ "sufficiently large" to include the informative part of I– the DRT graph $G = (V, E)$ associated to an image $I : \mathbb{S} \to \mathbb{V}$ is isomorphic to the DRT graph of any translated image of I (this isomorphism actually concerns the vertices/edges that involve at least one point with a value distinct from $\perp$). In particular, for a given $\mathbf{o_1} \in \mathbb{Z}^2$ let us consider the image $I_{\mathbf{o_1}}$ such that for any $\mathbf{q} \in \mathbb{Z}^2$, $I_{\mathbf{o_1}}(\mathbf{q}) = I(\mathbf{q}-\mathbf{o_1})$. Then, any edge $e = (v, w, (\mathbf{p}, \mathbf{p'}))$ of the DRT graph G of I is equivalent to the edge $e' = (v', w', (\mathbf{o_1}, \mathbf{p'}))$ of the DRT graph $G_{\mathbf{o_1}}$ of $I_{\mathbf{o_1}}$and v', w' are the DRTs corresponding to v, w, respectively, up to the translation of vector $-\mathbf{o_1}$.

On the other hand, let us consider an edge $e = (v, w, (\mathbf{p}, \mathbf{p'}))$ of the DRT graph G of I. For a given $\mathbf{o_2} \in \mathbb{Z}^2$, we can have two images $I_{v'}$ and $I_{w'}$ with respect to I_v and I_w such that, for any $\mathbf{q} \in \mathbb{Z}^2$, $I_{v'}(\mathbf{q}) = I_v(\mathbf{q}-\mathbf{o_2})$ and $I_{w'}(\mathbf{q}) = I_w(\mathbf{q}-\mathbf{o_2})$. Therefore, any edge $e = (v, w, (\mathbf{p}, \mathbf{p'}))$ is considered to be equivalent to the edge $e' = (v', w', (\mathbf{p}, \mathbf{o_2}))$, where v', w' correspond to v, w, respectively, up to the translation of vector $-\mathbf{o_2}$.

From the two above paragraphs, we derive the following statement.

Remark 16 *The study of any edge of label $(\mathbf{p}, \mathbf{p'})$ in the DRT graph $G = (V, E)$ associated to an image $I : \mathbb{S} \to \mathbb{V}$ can be carried out by considering the edge of label $(\mathbf{o_1}, \mathbf{o_2})$ in the –equivalent– DRT graph $G_{\mathbf{o_1}} = (V_{\mathbf{o_1}}, E_{\mathbf{o_1}})$ associated to a translated image $I_{\mathbf{o_1}}$ of I.*

In order to establish our local strategy, we now state some lemmas related to the behaviour of DRTs with respect to the 8-neighborhoods. Our first lemma, derived from Definition **??** and Equation (**??**), deals with the extension of a 8-neighborhood induced by digital rigid transformations.

Lemma 17 *Let $\mathbf{p} \in \mathbb{Z}^2$ and $\mathbf{q} \in N_8(\mathbf{p})$. For any digital rigid transformation $T : \mathbb{Z}^2 \to \mathbb{Z}^2$, we have $T(\mathbf{q}) \in N_{20}(T(\mathbf{p}))$.*

From the result of Remark**??** and the local characterisation of simple points (Property **??**), we then derive the following lemma where we consider T_v as the digital rigid transformation associated to a DRT v. Our next lemma states that it is sufficient to consider a local neighborhood to evaluate simple points under rigid transformations.

Lemma 18 *Let $I : \mathbb{S} \to \mathbb{V}$ be a digital image. Let $I' : N_{20}(\mathbf{o_1}) \to \mathbb{V}$ be the restriction of I to $N_{20}(\mathbf{o_1})$. Let v, w (resp. v', w') be two adjacent vertices of the DRT graph associated to I (resp. I') such that the DRTs T_v, T_w (resp. $T_{v'}, T_{w'}$) differ only in $\mathbf{o_2}$ and $T_v(\mathbf{o_2}) = \mathbf{o_1}$ (resp. $T_{v'}(\mathbf{o_2}) = \mathbf{o_1}$). Let $I_v, I_w : \mathbb{S} \to \mathbb{V}$ (resp. $I'_{v'}, I'_{w'} : N_{20}(\mathbf{o_1}) \to \mathbb{V}$) be the transformed images of I (resp. I') with respect to v, w (resp. v', w'), according to the Eulerian model. Then $\mathbf{o_2}$ is a simple point in I_v (and I_w) if and only if $\mathbf{o_2}$ is a simple point in $I'_{v'}$ (and $I'_{w'}$).*

In the DRT graph G of an image I, we can define an *equivalence relation* between the edges of G as follows.

Definition 19 *Let $G = (V, E)$ be the DRT graph associated to an image I, and $E_{(\mathbf{o_1},\mathbf{o_2})} \subset E$ be the set of edges with $(\mathbf{o_1}, \mathbf{o_2})$ as their label. Two edges $e_1 = (v_1, w_1, (\mathbf{o_1}, \mathbf{o_2}))$ and $e_2 = (v_2, w_2, (\mathbf{o_1}, \mathbf{o_2}))$ in $E_{(\mathbf{o_1},\mathbf{o_2})}$ are* equivalent*, and denoted by $e_1 \sim e_2$, iff $T_{v_1 \mid N_8(\mathbf{o_2})} = T_{v_2 \mid N_8(\mathbf{o_2})}$ (and $T_{w_1 \mid N_8(\mathbf{o_2})} = T_{w_2 \mid N_8(\mathbf{o_2})}$).*

In other words, an equivalence class of any $e = (v, w, (\mathbf{o_1}, \mathbf{o_2})) \in E_{(\mathbf{o_1},\mathbf{o_2})}$ under $\sim$, denoted by $[(v, w, (\mathbf{o_1}, \mathbf{o_2}))]_\sim$, contains a set of T_v that provide the same transformed image in the restriction of I to $N_{20}(\mathbf{o_1})$. Let us consider the DRT graph G' associated to the 20-neighborhood of $\mathbf{o_1}$ in the initial space. According to the Eulerian model and Lemma **??**, this DRT graph G' contains edges $(v', w', (\mathbf{o_1}, \mathbf{o_2}))$ that "summarize" the edges $(v, w, (\mathbf{o_1}, \mathbf{o_2}))$ of the DRT graph G associated to I.

Proposition 20 *Let $E_{(\mathbf{o_1},\mathbf{o_2})}$ (resp. $E'_{(\mathbf{o_1},\mathbf{o_2})}$) be the set of edges of the graph $G = (V, E)$ (resp. $G' = (V', E')$) associated to I (resp. I', the restriction of I to $N_{20}(\mathbf{o_1})$). We have $E_{(\mathbf{o_1},\mathbf{o_2})}/\sim$ equivalent to $E'_{(\mathbf{o_1},\mathbf{o_2})}$, by associating each equivalence class $[(v, w, (\mathbf{o_1}, \mathbf{o_2}))]_\sim \in E$ to the edge $(v', w', (\mathbf{o_1}, \mathbf{o_2})) \in E'$ such that $T_{v \mid N_8(\mathbf{o_2})} = T_{v'}$ and $T_{w \mid N_8(\mathbf{o_2})} = T_{w'}$.*

Without loss of generality, we can consider the case where $\mathbf{o_1}$ and $\mathbf{o_2}$ are the origin $\mathbf{0}$ of the images I and I' respectively, the relations between I, G, $E_{(\mathbf{0},\mathbf{0})}$ and I', G', $E'_{(\mathbf{0},\mathbf{0})}$ are illustrated in Figures **??** and **??**.

Based on Lemma **??** and Proposition **??**, it follows that the "topological" behaviour of any edge of $[(v, w, (\mathbf{o_1}, \mathbf{o_2}))]_\sim$ in the DRT graph G associated to image I can be determined from the edges $(v', w', (\mathbf{o_1}, \mathbf{o_2}))$ in the DRT graph G'. In other words, the study of the *local* DRT graph G' associated to the partial images of I defined on $N_{20}(\mathbf{o_1})$ directly provides access to a subset of the required *global* knowledge related to the topological invariance of I under any DRTs.

In particular, from Remark **??**, Lemmas **??**, **??**, and Proposition **??**, it becomes possible to develop a local approach for the topologically invariance verification of digital images under arbitrary rigid transformation.

(a)(b)

Fig. 11 (a) A cross-section in the plane (a, θ) of the 2D surfaces bounding the DRTs (see Section **??**) associated to the image I, and inducing the DRT graph $G = (V, E)$. (b) A cross-section in the plane (a, θ) of the 2D surfaces bounding DRTs associated to $I' = I_{|N_{20}(\mathbf{0})}$, and inducing the DRT graph $G' = (V', E')$ (see text). In both figures, the red segment corresponds to the edges of label $(\mathbf{0}, \mathbf{0})$ (*i.e.*, $E_{(\mathbf{0},\mathbf{0})}$ and $E'_{(\mathbf{0},\mathbf{0})}$), while the blue ones are the edges in E' and the black ones are the edges in $E \setminus E'$.

(a) (b)

(c) (d)

Fig. 12 (a,b) Zoom in the curves of Figure **??**. (c,d) Illustration of the dual space of (a,b) for the part of the DRT graph corresponding to the edges with label $(\mathbf{0}, \mathbf{0})$. By Definition **??**, the green edges in (c) form an equivalence class $[(v, w, (\mathbf{0}, \mathbf{0}))]_\sim \in E$. From Proposition **??**, the equivalence class $[(v, w, (\mathbf{0}, \mathbf{0}))]_\sim$ can be associated to the blue edge $(v', w', (\mathbf{0}, \mathbf{0}))$ in (d).

Algorithm 2: Generation of LUT for topologically invariance verification.

Input: The DRT graph $G'_\mathbf{0} = (V'_\mathbf{0}, E'_\mathbf{0})$ of $N_{20}(\mathbf{0})$ (computed from Algorithm **??**).
Input: The set C of all different images $I : N_{20}(\mathbf{0}) \to \mathbb{V}$ (computed in a greedy fashion).
Output: The set $P \subseteq C$ of topologically preserving samples for the center point $\mathbf{0}$.

$P \leftarrow \emptyset$;
foreach $I \in C$ **do**
  $B \leftarrow true$
  $S \leftarrow E_\mathbf{0}$
  while $(S \neq \emptyset) \wedge (B = true)$ **do**
    Let $e = (v, w, (\mathbf{p}, \mathbf{p'})) \in S$
    $S \leftarrow S \setminus \{e\}$
    if $\mathbf{p} = \mathbf{0}$ **then**
      if *(($I_v(\mathbf{p'}) \neq I_w(\mathbf{p'})$) and ($\mathbf{p'}$ is not a simple point in I_v))* **then**
        $B \leftarrow false$
  if $B = true$ **then**
    $P \leftarrow P \cup \{I\}$;

Algorithm 3: Local verification of the topological invariance of a digital image.

Input: A digital image $I : \mathbb{S} \to \mathbb{V}$.
Input: The set P (computed from Algorithm **??**).
Output: A Boolean value B evaluating the topological invariance of I.

$B \leftarrow true$
$S \leftarrow \mathbb{S}$
while $(S \neq \emptyset) \wedge (B = true)$ **do**
  Let $p \in S$
  $S \leftarrow S \setminus \{p\}$
  $B \leftarrow (I_{|N_{20}(\mathbf{p})} \in P)$ (up to a translation of $-\mathbf{p}$)

6.2 LUT-based algorithm

Practically, an image I is topologically invariant with respect to any DRTs if all its transformed images share the same homotopy-type, and in particular if they are simple-equivalent (Remark **??**). This simple-equivalence can be locally determined using to the notion of simple point (Properties **??** and **??**). In particular, any elementary modification between transformed images is encoded in an edge in the DRT graph G of I, and such an edge models the modification of exactly one point between two transformed images. This point can in particular be characterised as simple or not. Consequently, by analysing the edges of the whole DRT graph G, the topological invariance of I can be determined. This is the strategy developed in Algorithm **??**, that processes these edges in a (quasi) exhaustive fashion, leading to a computational cost directly linked to the size of the DRT graph.

(a) (b)

Fig. 13 (a) A 20-neighborhood image, centered on $\mathbf{p}$ (in blue), that belongs to the LUT P, and an (overlapped) image, centered on $\mathbf{q}$ (in red), that does not belong to P. (b) Two 20-neighborhood images, centered on $\mathbf{p}$ and $\mathbf{q}$, respectively, that both belong to the LUT P. (See Remark **??**.)

In the previous section, it was observed that any edge of the DRT graph G of $I : \mathbb{S} \to \mathbb{V}$ is equivalent to an edge in a smaller DRT graph $G'_{\mathbf{p}}$, associated to the restriction of I in the 20-neighborhood of a given point $\mathbf{p} \in \mathbb{S}$ (Remark **??** and Proposition **??**). In particular, the characterisation of this edge as topologically preserving is algorithmically the same in G and in $G'_{\mathbf{p}}$ (Lemma **??**).

From these facts, we deduce that the topological invariance of I can be equivalently analyzed from G or from the set $\{G'_{\mathbf{p}}\}_{\mathbf{p}\in\mathbb{S}}$ of all the local DRT graphs in the 20-neighborhoods of the points $\mathbf{p} \in \mathbb{S}$. In particular, in any of these local DRT graphs $G'_{\mathbf{p}}$, it is sufficient to focus on a (strict) subset of edges, namely those that involve $\mathbf{p}$.

Moreover, since only a finite number of images can be defined on a 20-neighborhood, this topological analysis can be performed exhaustively just once for all the images defined on a 20-neighbourhood; these images can then be used to characterize the topological invariance of I. This pre-computation, formalised in Algorithm **??**, leads to the definition of a look-up table (LUT) P that contains all the 20-neighborhood images that authorize topological invariance in a larger image, for a given value space $\mathbb{V}$, and a given topology.

Remark 21 *The LUT P obtained from Algorithm* **??** *potentially constitutes a* strict superset *of the actual set of the* 20*-neighbourhood images that authorize in the LUT some patterns that necessarily imply the existence of neighboring patterns that are not themselves in the LUT. Algorithm* **??** *can then be optimised by a post-processing that removes from P some non-relevant configurations. Such a post-processing, that leads to a smaller LUT, presents a time complexity* $O(|P|^3)$.

We discuss in more details experimental results obtained with this LUT in the case of binary images in Section **??**. Once the LUT P has been computed, any image $I : \mathbb{S} \to \mathbb{V}$ can be characterized by a simple pixelwise process, that checks, for every $\mathbf{p} \in \mathbb{S}$, that the restriction of I to $N_{20}(\mathbf{p})$ belongs to P. This LUT-based approach is formalised in Algorithm **??**.

6.3 Parametrisation of the approach

The proposed approach for evaluating the topological invariance of digital images I defined on $\mathbb{S}$, under any DRTs, has been presented –for the sake of readability– in the classical framework of digital topology [**?**], *i.e.*, by considering binary images ($|\mathbb{V}| = 2$), equipped with a standard pair of dual $(8, 4)$- or $(4, 8)$-adjacencies. Nevertheless, the nature (and thus the cardinality) of $\mathbb{V}$, such as the topological space used to equip $\mathbb{S}$ with respect to $\mathbb{V}$, can be conveniently modified without loss of generality, making the proposed approach parametric from both structural and spectral points of view.

Indeed, on the one hand, the proposed algorithms (and in particular Algorithms **??** and **??**) rely on the notion of DRT graph, that defines explicitly the structure of the transformed spaces, but neither the transformed images nor their associated value space $\mathbb{V}$, which are implicitly handled.

On the other hand, the topological space that is mapped on $\mathbb{S}$ (and more generally on $\mathbb{Z}^2$) with respect to $\mathbb{V}$, is only considered via the notion of simple point. More precisely, the only constraint related to the choice of the topology is the necessity to characterise locally the preservation of homotopy-type, with respect to the images of $\mathbb{S} \to \mathbb{V}$.

Consequently, the proposed approach can be parametrized by a couple composed of (*i*) a value space $\mathbb{V}$, and (*ii*) a notion of simple point for the space of the images of $\mathbb{S} \to \mathbb{V}$.

Based on this assertion, several topological frameworks that provide different notions of simple point can be considered, including the following:

- binary images, equipped with the digital topology [**?**], deriving from the dual $(4, 8)$- or $(8, 4)$-adjacencies (proposed in this article);
- binary images, defined as well-composed sets [**?**], deriving from the $(4, 4)$-adjacencies;
- label images, defined as well-composed sets [**?**,**?**], deriving from the $(4, 4)$-adjacencies;
- label images, equipped with the notion of digitally simple Xels [**?**] defined from the topology of cubical complexes;
- label images, equipped with the notion of simple point in covering images [**?**];
- grey-level images, equipped with the notion of λ-destructible point [**?**].

7 Analysis and experiments

In this section, we first analyse the complexity of the proposed algorithm(s). We then provide experiments devoted to validate the behaviour of the developed approach.

7.1 Theoretical complexity analysis

Given a digital image I of size $N \times N$, the first algorithm (Algorithm **??**, in Section **??**) relies on the DRT graph of I, and scans it entirely in the worst case. Consequently, both space and time complexities of this algorithm are $O(N^9)$, due to the space complexity of the DRT graph (Property **??**).

The second algorithm (Algorithm **??**, in Section **??**) relies on (*i*) a LUT P of topology-preserving 20-neighborhood images; and (*ii*) the verification of the compliance of I with P for any point of I. The generation of P (Algorithm **??**, in Section **??**), for a given value space $\mathbb{V}$ and a given adjacency, has a time complexity $O(5^9\ |\mathbb{V}|^{20}) = O(|\mathbb{V}|^{20})$, since the complexity for generating the DRT graph for an image defined on a 20-neighborhood is $O(5^9)$ [**?**], and any image $I : N_{20}(\mathbf{0}) \to \mathbb{V}$ has to be processed via its DRT graph. Note however that this process has to be carried out only once, if P –that has a space complexity of $O(|\mathbb{V}|^{20})$– is stored. The topological invariance verification (Algorithm **??**) then presents a quasi-linear time complexity with respect to the size $N \times N$ of image $O(N^2\ \log_2(|P|)) = O(N^2\ |\mathbb{V}|)$, since the LUT P can be ordered and processed as a tree structure. One may notice that this algorithm can trivially be parallelized, leading in particular to a quasi constant time complexity $O(|\mathbb{V}|)$, when processed as N^2 subtasks.

7.2 Computational and space cost: The binary case

In this section, we experimentally assess the actual cost of the algorithm, previously discussed from a theoretical point of view. To this end, we consider the case of binary images, *i.e.*, images defined on a set of values $\mathbb{V}$ such that $|\mathbb{V}| = 2$.

Let C be the set of all the binary images defined on $N_{20}(\mathbf{0})$, that is used to build P via Algorithm **??**. We have, in particular, $|C| = 2^{20}$. However, from Remark **??**, we only have

(a) P in $(4, 8)$-adjacency (samples).

(b) P in $(8, 4)$-adjacency (samples).

Fig. 14 Some samples defined on 20-neghbourhoods which are topology preserving, computed from Algorithm **??**, in the case of $(4, 8)$-adjacency (a), and $(8, 4)$-adjacency (b). The foreground pixels are depicted in black, while the background pixels ($\perp$) are depicted in white.

to consider the images such that at least one point in the 4-neighborhood of $\mathbf{0}$ has a distinct (binary) value from the one of $\mathbf{0}$. By using this fact, plus considerations related to invariance up to rotations and symmetries, the set C can be reduced, without loss of completeness to a subset $C' \subset C$ such that $|C'| = 124\,260 \ll |C|$.

Using Algorithm **??** on this set C', we obtain some sets P of 10 643 and 19 446 elements, in the $(4, 8)$- and $(8, 4)$-adjacency, respectively. Figure **??** provides some samples of P in both cases.

7.3 Experiments: The binary case

We now propose some experiments to illustrate the behaviour of the algorithms on images representing different kinds of objects using the $(4, 8)$-adjacent relations. We first consider basic geometric primitives, namely half-planes and disks, the evolution of which is (in theory) predictable with respect to rigid transformations. Then, we consider more generally, arbitrary shapes, whose topological invariance is not easily predictable.

7.3.1 Topological (in)variance of geometric primitives

We define a discrete half-plane as the set of all discrete points on one side of a digital straight line. The number of digital line segments was studied in [**?**]. In particular, it is known that there exist 14 digital segments of length 5 inside a pattern of 5×5 (see Figure **??**(a)). From this knowledge, we can generate all the possible half-planes in a 20-neighborhood, as illustrated in Figure **??**(b). By using Algorithm **??** to study the properties if these patterns, we find that all of them are topologically invariant. Therefore, we can conclude that any discrete half-plane preserves homotopy-type during digital/discrete rigid transformations. Some examples of rigidly transformed half-planes are illustrated in Figure **??**.

The digital disks, defined on $\mathbb{Z}^2$ and studied, *e.g.*, in [**?**], can be defined as the sets of all discrete points lying inside a real disc (defined on $\mathbb{R}^2$). It is plain that the digitization of a disc depends on its size (*i.e.*, its radius) but also on its position (*i.e.*, the position of its centre) with respect to the discrete grid. Some examples of digital disks with the same radius are

(a)

(b)

Fig. 15 (a) The 14 samples of digital lines of length 5, and (b) the half-planes generated by these lines in a 20-neighborhood.

Fig. 16 Some examples of half-planes rigidly transformed from an image of size 20×20. All transformed half-planes images are simple-equivalent.

(a)(b)

Fig. 17 (a) Some disks of radius 5, generated in a image of size 20×20. Some of them are topologically invariant (in black frames), while the others (in red frame) have been characterised as not topologically invariant by Algorithm **??**. (b) Concerning the four topologically-variant disks in (a), the pixels detected by our algorithm as those that alter the topology of the four disks are colored in red and blue in the first row; in the rows below the non-simple-equivalent transformed disks associated to the four above disks are illustrated.

(a)

(b)

Fig. 18 (a) A topologically-invariant 5×5 image (in blue frame), and (some of) its transformations (in black frames). (b) A topologically-variant 5×5 image (in blue frame), and (some of) its simple-equivalent (in black frames) and non-simple-equivalent transformations (in red frames).

shown in Figure **??**. In the continuous domain, the real disks are –of course– topologically invariant under rigid transformations. Surprisingly, this property is lost in the digital case. Indeed, Algorithm **??**, performed on the images of Figure **??**, has detected that some of them are not topologically invariant. This emphasizes the influence of position the centre of the disk for this property.

7.3.2 Topological (in)variance of arbitrary shapes

To complete these experiments, we finally exhibit some examples of arbitrary binary images that have been characterised by Algorithm **??** as being topologivally invariant (Figures **??**(a) and **??**), or topologivally variant (Figures **??**(b) and **??**).

(a)(b)(c)(d)

Fig. 19 (a–d) Four examples of topologically invariant images.

8 Conclusion

In this article, we have considered geometrical and topological concepts, to propose an approach for studying the topological behaviour of rigid transformations in $\mathbb{Z}^2$. More precisely, (a)(b)(c)(d)

(e)(f)(g)(h)

Fig. 20 (a–d) Four examples of topologically variant images. (e–h) Non-simple-equivalent transformed images associated to the above images (a–d).

Fig. 21 The effects of double (left) and null (right) pixels in the Lagrangian transformation model.

we combined the notion of simple point with the notion of DRT graph, leading to algorithmic processes that can characterise the topological invariance of digital images under any rigid transformations. In particular, by taking advantage of the respective strengths of both notions, it has been possible to develop an efficient algorithm, able to evaluate this topological invariance in a quasi-linear time with respect to the image size.

Beyond its theoretical aspects, this work may contribute to the better understanding of the relationships between geometry and topology in the framework of digital imaging, where both notions are less strongly linked than in the continuous space. In particular, the proposed algorithm may provide an efficient tool for further studying the notion of regularity [**?**,**?**,**?**], that is currently used to assess the preservation of topological properties during the digitization of an image from $\mathbb{R}^2$ to $\mathbb{Z}^2$. In particular, a *discrete* notion of regularity may be derived from the continuous one, in order to assess the topological behaviour of image transformations in a fully discrete framework.

In this article, we have considered the specific case of the Eulerian transformation model (see Section **??**). Further work may consider the case of the Lagrangian transformation model. In the context of topological alterations induced by rigid transformations of digital images, this latter model comes with some additional difficulties. Indeed, while in the Eulerian model, a double (resp. null) point transfers its value to two (resp. no) point(s) in the transformed image (Section **??** and Figure **??**), in the Lagrangian case, a double (resp. null) point in the transformed image will receive two (resp. no) values; this leads to a result being both incomplete and ambiguous (see Figure **??**). In order to deal with these supplementary issues, it may be necessary to study more deeply the relations that exist between the digital images, defined on $\mathbb{Z}^2$, and the continuous ones, defined on $\mathbb{R}^2$, as they are linked via the digitization processes.

Acknowledgements

The research leading to these results has received partial funding from the French *Agence Nationale de la Recherche* (Grant Agreement ANR-10-BLAN-0205 03).

Appendix A – A description of DRT graphs [**?**]

Observing Equation (**??**), for any $\mathbf{x}$, an infinitesimal variation of the values a, b, θ leads to an infinitesimal variation of the point $\mathcal{T}(\mathbf{x})$. More formally, the function $(a, b, \theta) \mapsto \mathcal{T}_{ab\theta}$ is continuous. Contrarily, in Equation (**??**), an infinitesimal variation of a, b, θ may lead to a variation of $T(\mathbf{x})$ from a point of $\mathbb{Z}^2$ to a different one. In particular, the parameter space $\mathbb{R}^3$ of (a, b, θ) (more precisely, the finite subspace of this parameter space that relevantly enables to handle transformations on $\mathbb{S}$) is divided into 3D open cells where the function

$(a,b,\theta) \mapsto T_{ab\theta} = D \circ \mathcal{T}_{ab\theta|\mathbb{S}}$ is piecewise constant, bounded by 2D closed sets where it is discontinuous. The transformations $\mathcal{T}_{ab\theta}$, which lead to such discontinuities, are called *critical transformations* (see Figure **??**).

Definition 22 (Critical transformation) *Let $(a,b,\theta) \in \mathbb{R}^3$, and $\mathcal{T}_{ab\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ be its associated rigid transformation. We say that $\mathcal{T}_{ab\theta}$ is a* critical transformation *if there exists $\mathbf{p} \in \mathbb{S}$ such that $\mathcal{T}_{ab\theta}(\mathbf{p}) \in \mathcal{H}$, where $\mathcal{H}$ is the half-grid defined by*

$$\mathcal{H} = \left[\mathbb{R} \times \left(\mathbb{Z} + \frac{1}{2}\right)\right] \cup \left[\left(\mathbb{Z} + \frac{1}{2}\right) \times \mathbb{R}\right]. \tag{11}$$

The critical transformations can be explicitly expressed, for any $\mathbf{p} = (p,q) \in \mathbb{S}$ that is mapped onto a half-grid point which can be either a horizontal one $\mathbf{p}_\Phi = (k + \frac{1}{2}, \lambda) \in \mathcal{H}$ or a vertical one $\mathbf{p}_\Psi = (\lambda, l + \frac{1}{2}) \in \mathcal{H}$ (with $k, l \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$), by the following equations:

$$\mathbf{p}_\Phi = \begin{pmatrix} k + \frac{1}{2} \\ \lambda \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \tag{12}$$

$$\mathbf{p}_\Psi = \begin{pmatrix} \lambda \\ l + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \tag{13}$$

The 2D surfaces associated to these critical transformations, in the 3D parameter space, are

(a)(b)

Fig. 22 Examples of horizontal and vertical critical transformations $\mathcal{T}_{ab\theta}$, which map at least one integer value onto a horizontal (a) and vertical (b) half-grid point respectively.

called *tipping surfaces* (see Figure **??**(a)), and are expressed by

$$\left| \begin{array}{l} \Phi_{pqk} : \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \qquad (b,\theta) \longmapsto a = k + \frac{1}{2} + q\sin\theta - p\cos\theta \end{array} \right. \tag{14}$$

$$\left| \begin{array}{l} \Psi_{pql} : \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \qquad (a,\theta) \longmapsto b = l + \frac{1}{2} - p\sin\theta - q\cos\theta \end{array} \right. \tag{15}$$

Their projection ϕ_{pqk} (resp. ψ_{pql}) on the 2D plane (a,θ) (resp. (b,θ)) are called *tipping curves* (see Figure **??**(b)), and defined by

$$\left| \begin{array}{l} \phi_{pqk} : \mathbb{R} \longrightarrow \mathbb{R} \\ \qquad \theta \longmapsto a = k + \frac{1}{2} + q\sin\theta - p\cos\theta \end{array} \right. \tag{16}$$

$$\left| \begin{array}{l} \psi_{pql} : \mathbb{R} \longrightarrow \mathbb{R} \\ \qquad \theta \longmapsto b = l + \frac{1}{2} - p\sin\theta - q\cos\theta \end{array} \right. \tag{17}$$

In particular, the set of all the non-critical transformations is partitioned into equivalence classes induced by the equivalence relation $\sim$ defined by

$$(\mathcal{T}_{ab\theta} \sim \mathcal{T}_{a'b'\theta'}) \Longleftrightarrow (T_{ab\theta} = T_{a'b'\theta'}) \tag{18}$$

In this isomorphic framework, each of these equivalence classes is called a *discrete rigid transformation (DRT)*, and is modeled by one of the 3D open cells defined above, bounded by 2D tipping surfaces.

(a) (b)
Tip-Tip-
pingping
sur-curves
faces

Fig. 23 (a) Tipping surfaces in the 3D parameter space (a, b, θ) and (b) their cross-sections, namely tipping curves, in the 2D planes (a, θ) and (b, θ).

(a) (b)

Fig. 24 A part of the parameter space subdivided by four tipping surfaces (a), and the associated (part of the) DRT graph (b).

In [**?**], it was shown that this parameter space subdivision can be modeled as a dual combinatorial structure, namely a graph. In particular, each 3D open cell (*i.e.*, each DRT) is associated to a vertex, and each tipping surface segment (linked to a critical transformation) shared by two adjacent 3D open cells, is associated to an edge. More precisely, this structure models a kind of "neighbouring" relationship between DRTs. Indeed, the existence of an edge between two vertices indicates that the associated transformations differ by one pixel among the N^2 ones of $\mathbb{S}$. The resulting graph is called a *DRT graph* (see Fig. **??**). This structure presents a space complexity of $O(N^9)$, where $N \times N$ is the size of $\mathbb{S}$. An exact computation algorithm was also proposed to build this graph in linear time with respect to this space complexity. Supplementary details may be found in [**?**,**?**].

References

1. B. Zitová, J. Flusser, Image registration methods: A survey, Image and Vision Computing 21 (11) (2003) 977–1000.
2. A. Yilmaz, O. Javed, M. Shah, Object tracking: A survey, ACM Computing Surveys 38 (4) (2006) 1–45.
3. P. Ngo, Y. Kenmochi, N. Passat, H. Talbot, Sufficient conditions for topological invariance of 2D digital images under rigid transformations, in: DGCI, Proceedings, Lecture Notes in Computer Science, Springer, 2013 (To Appear).
4. P. Ngo, Y. Kenmochi, N. Passat, H. Talbot, Combinatorial structure of rigid transformations in 2D digital images, Computer Vision and Image Understanding (In Press), `DOI: 10.1016/j.cviu.2012.08.014`.
5. E. Andres, The quasi-shear rotation, in: DGCI, Proceedings, Vol. 1176 of Lecture Notes in Computer Science, Springer, 1996, pp. 307–314.
6. J.-P. Reveillès, Géométrie discrète, calcul en nombres entiers et algorithmique, Thèse d'État, Université Strasbourg 1 (1991).
7. M. S. Richman, Understanding discrete rotations, in: ICASSP, Proceedings, Vol. 3, IEEE, 1997, pp. 2057–2060.
8. B. Nouvel, Rotations discrètes et automates cellulaires, Ph.D. thesis, École Normale Supérieure de Lyon (2006).
9. B. Nouvel, E. Rémila, Incremental and transitive discrete rotations, in: IWCIA, Proceedings, Vol. 4040 of Lecture Notes in Computer Science, Springer, 2006, pp. 199–213.
10. Y. Thibault, Y. Kenmochi, A. Sugimoto, Computing upper and lower bounds of rotation angles from digital images, Pattern Recognition 42 (8) (2009) 1708–1717.
11. M.-A. Jacob, E. Andres, On discrete rotations, in: DGCI, Proceedings, 1995, pp. 161–174.
12. A. Amir, O. Kapah, D. Tsur, Faster two-dimensional pattern matching with rotations, Theoretical Computer Science 368 (3) (2006) 196–204.
13. A. Amir, G. M. Landau, U. Vishkin, Efficient pattern matching with scaling, Journal of Algorithms 13 (1) (1992) 2–32.
14. A. Amir, A. Butman, M. Lewenstein, E. Porat, Real two dimensional scaled matching, Algorithmica 53 (3) (2009) 314–336.

15. C. Hundt, M. Liśkiewicz, N. Ragnar, A combinatorial geometrical approach to two-dimensional robust pattern matching with scaling and rotation, Theoretical Computer Science 410 (51) (2009) 5317–5333.
16. C. Hundt, M. Liśkiewicz, On the complexity of affine image matching, in: STACS, Proceedings, Vol. 4393 of Lecture Notes in Computer Science, Springer, 2007, pp. 284–295.
17. C. Hundt, Affine image matching is uniform TC^0-complete, in: CPM, Proceedings, Vol. 6129 of Lecture Notes in Computer Science, Springer, 2010, pp. 13–25.
18. C. Hundt, M. Liśkiewicz, Combinatorial bounds and algorithmic aspects of image matching under projective transformations, in: MFCS, Proceedings, Vol. 5162 of Lecture Notes in Computer Science, Springer, 2008, pp. 395–406.
19. G. Bertrand, On critical kernels, Comptes Rendus de l'Académie des Sciences—Série Mathématiques I (345) (2007) 363–367.
20. T. Y. Kong, A. Rosenfeld, Digital topology: Introduction and survey, Computer Vision Graphics & Image Processing 48 (3) (1989) 357–393.
21. L. Mazo, N. Passat, M. Couprie, C. Ronse, Paths, homotopy and reduction in digital images, Acta Applicandae Mathematicae 113 (2) (2011) 167–193.
22. L. Mazo, N. Passat, M. Couprie, C. Ronse, Digital imaging: A unified topological framework, Journal of Mathematical Imaging and Vision 44 (1) (2012) 19–37.
23. E. Khalimsky, Topological structures in computer science, Journal of Applied Mathematics and Simulation 1 (1) (1987) 25–40.
24. V. A. Kovalevsky, Finite topology as applied to image analysis, Computer Vision, Graphics & Image Processing 46 (2) (1989) 141–161.
25. A. Rosenfeld, Connectivity in digital pictures, Journal of the ACM 17 (1) (1970) 146–160.
26. G. Bertrand, G. Malandain, A new characterization of three-dimensional simple points, Pattern Recognition Letters 15 (2) (1994) 169–175.
27. M. Couprie, G. Bertrand, New characterizations of simple points in 2D, 3D, and 4D discrete spaces, IEEE Transactions on Pattern Analysis and Machine Intelligence 31 (4) (2009) 637–648.
28. C. Ronse, A topological characterization of thinning, Theoretical Computer Science 43 (1) (2007) 31–41.
29. G. Bertrand, On P-simple points, Comptes Rendus de l'Académie des Sciences—Série Mathématiques I (321) (1995) 1077–1084.
30. N. Passat, L. Mazo, An introduction to simple sets, Pattern Recognition Letters 30 (15) (2009) 1366–1377.
31. M. Couprie, F. N. Bezerra, G. Bertrand, Topological operators for grayscale image processing, Journal of Electronic Imaging 10 (4) (2001) 1003–1015.
32. L. J. Latecki, Multicolor well-composed pictures, Pattern Recognition Letters 16 (4) (1997) 425–431.
33. G. Damiand, A. Dupas, J.-O. Lachaud, Fully deformable 3D digital partition model with topological control, Pattern Recognition Letters 32 (9) (2011) 1374–1383.
34. L. Mazo, N. Passat, M. Couprie, C. Ronse, Topology on digital label images, Journal of Mathematical Imaging and Vision 44 (3) (2012) 254–281.
35. D. Pham, P.-L. Bazin, J. Prince, Digital topology in brain imaging, IEEE Signal Processing Magazine 27 (4) (2010) 51–59.
36. J.-F. Mangin, V. Frouin, I. Bloch, J. Régis, J. López-Krahe, From 3D magnetic resonance images to structural representations of the cortex topography using topology preserving deformations, Journal of Mathematical Imaging and Vision 5 (4) (1995) 297–318.
37. X. Han, C. Xu, J. L. Prince, A topology preserving level set method for geometric deformable models, IEEE Transactions on Pattern Analysis and Machine Intelligence 25 (6) (2003) 755–768.
38. P.-L. Bazin, L. M. Ellingsen, D. L. Pham, Digital homeomorphisms in deformable registration, in: IPMI, Proceedings, Vol. 4584 of Lecture Notes in Computer Science, Springer, 2007, pp. 211–222.
39. S. Faisan, N. Passat, N. Noblet, R. Chabrier, C. Meyer, Topology preserving warping of 3-D binary images according to continuous one-to-one mappings, IEEE Transactions on Image Processing 20 (8) (2011) 2135–2145.
40. G. Bertrand, M. Couprie, N. Passat, A note on 3-D simple points and simple-equivalence, Information Processing Letters 109 (13) (2009) 700–704.
41. B. Nouvel, E. Rémila, Configurations induced by discrete rotations: Periodicity and quasi-periodicity properties, Discrete Applied Mathematics 147 (2–3) (2005) 325–343.
42. Y. Thibault, Rotations in 2D and 3D discrete spaces, Ph.D. thesis, University Paris-Est (2010).
43. L. J. Latecki, U. Eckhardt, A. Rosenfeld, Well-composed sets, Computer Vision and Image Understanding 61 (1) (1995) 70–83.
44. L. Mazo, A framework for label images, in: CTIC, Proceedings, Vol. 7309 of Lecture Notes in Computer Science, Springer, 2012, pp. 1–10.
45. C. Berenstein, D. Lavine, On the number of digital straight line segments, IEEE Transactions on Pattern Analysis and Machine Intelligence 10 (6) (1988) 880–887.

46. B. Nagy, An algorithm to find the number of the digitizations of discs with a fixed radius, Electronic Notes in Discrete Mathematics 20 (2005) 607–622.
47. J. Serra, Image Analysis and Mathematical Morphology, Academic Press, Inc., 1983.
48. H. J. A. M. Heijmans, Discretization of morphological operators, Journal of Visual Communication and Image Representation 3 (2) (1992) 182–193.
49. L. J. Latecki, C. Conrad, A. Gross, Preserving topology by a digitization process, Journal of Mathematical Imaging and Vision 8 (2) (1998) 131–159.
50. P. Ngo, Y. Kenmochi, N. Passat, H. Talbot, Combinatorial properties of 2D discrete rigid transformations under pixel-invariance constraints, in: IWCIA, Proceedings, Vol. 7655 of Lecture Notes in Computer Science, Springer, 2012, pp. 234–248.