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A Note on Efficient Computation of All Abelian Periods in a String

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Abstract

We derive a simple efficient algorithm for Abelian periods knowing all Abelian squares in a string. An efficient algorithm for the latter problem was given by Cummings and Smyth in 1997. By the way we show an alternative algorithm for Abelian squares. We also obtain a linear time algorithm finding all “long” Abelian periods. The aim of the paper is a (new) reduction of the problem of all Abelian periods to that of (already solved) all Abelian squares which provides new insight into both connected problems.

Keywords: algorithms, Abelian period, Abelian square

1. Introduction

We present an efficient reduction of the Abelian period problem to the Abelian square problem. For a string of length $n$ the latter problem was solved in $O(n^2)$ by Cummings and Smyth \cite{7}. The best previously known algorithms for the Abelian periods, see \cite{12}, worked in $O(n^2m)$ time (where $m$ is the alphabet size) which for large $m$ is $O(n^3)$. Our algorithm works in $O(n^2)$ time, independently of the alphabet size. As a by-product we obtain an alternative...
$O(n^2)$ time algorithm finding all Abelian squares and an $O(n)$ time algorithm finding a compact representation of all Abelian periods of length greater than $n/2$, in particular, the shortest such period.

Abelian squares were first studied by Erdős [11], who posed a question on the smallest alphabet size for which there exists an infinite Abelian-square-free string. An example of such a string over five-letter alphabet was given by Pleasants [16] and afterwards the best possible example over four-letter alphabet was shown by Keränen [13].

Quite recently there have been several results on Abelian complexity in words [1, 4, 8, 9, 10] and partial words [2, 3] and on Abelian pattern matching [5, 14, 15]. Abelian periods were first defined and studied by Constantinescu and Ilie [6].

We say that two strings are (commutatively) equivalent, and write $x \equiv y$, if one can be obtained from the other by permuting its symbols. In other words, the Parikh vectors $P(x), P(y)$ are equal, where the Parikh vector gives frequency of each symbol of the alphabet in a given string. Parikh vectors were introduced already in [6] for this problem.

A string $w$ is an Abelian $k$-power if $w = x_1 x_2 \ldots x_k$, where

$$x_1 \equiv x_2 \equiv \ldots \equiv x_k$$

The size of $x_1$ is called the base of the $k$-power. In particular $w$ is an Abelian square if and only if it is an Abelian 2-power.

A string $x$ is an Abelian factor of $y$ if $P(x) \leq P(y)$, that is, each element of $P(x)$ is smaller than the corresponding element of $P(y)$. The pair $(i, p)$ is an Abelian period of $w = w[1, n]$ if and only if $w[i+1, j]$ is an Abelian $k$-power with base $p$ (for some $k$) and $w[1, i]$ and $w[j+1, n]$ are Abelian factors of $w[i+1, i+p]$, see Fig. 1. Here $p$ is called the length of the period.

![Figure 1: A word of length 25 with an Abelian period $(i = 3, p = 6)$](image)

In Section 2 we introduce two auxiliary tables that we use in computing Abelian squares and powers. Next in Section 3 we show new $O(n^2)$ time algorithms for all Abelian squares and all Abelian periods in a string and a reduction between these problems.

Finally in Section 4 we present an $O(n)$ time algorithm finding a compact representation of all “long” Abelian periods. Define

$$\text{MinLong}(i) = \min \{ p > n/2 : (i, p) \text{ is an Abelian period of } w \}.$$ 

If no such $p$ exists, we set $\text{MinLong}(i) = \infty$. All long Abelian periods are of the form $(i, p)$ where $p \geq \text{MinLong}(i)$, the table $\text{MinLong}$ is a compact $O(n)$ space representation of potentially quadratic set of long Abelian periods.
2. Auxiliary tables

Let $w$ be a string of length $n$. Assume its positions are numbered from 1 to $n$, $w = w_1 w_2 \ldots w_n$. By $w[i, j]$ we denote the factor of $w$ of the form $w_i \ldots w_j$. Factors of the form $w[1, i]$ are called prefixes of $w$ and factors of the form $w[i, n]$ are called suffixes of $w$.

We introduce the following table:

$$\text{head}(i, j) = \text{minimum } k \text{ such that } \mathcal{P}(w[i, j]) \leq \mathcal{P}(w[j + 1, j + k]).$$

If no such $k$ exists, we set $\text{head}(i, j) = \infty$, and if $j < i$, we set $\text{head}(i, j) = 0$. In the algorithm below we actually compute a slightly modified table $\text{head}'(i, j) = j + \text{head}(i, j)$.

**Example 1.** For the infinite Fibonacci word $\mathcal{F} = \text{abaababaababaababaababaa} \ldots$
the first several values of the table $\text{head}(1, i)$ are:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}[i]$</td>
<td>$a$</td>
<td>$b$</td>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$a$</td>
<td>$b$</td>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$a$</td>
<td>\ldots</td>
</tr>
<tr>
<td>$\text{head}(1, i)$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>11</td>
<td>\ldots</td>
<td></td>
</tr>
</tbody>
</table>

We have here Abelian square prefixes of lengths 6, 10, 12, 16, 20, 22.

We show how to compute the $\text{head}'$ table in $O(n^2)$ time. The computation is performed in row-order of the table using the fact that it is non-decreasing:

**Observation 2.** For any $1 \leq i \leq j < n$, $\text{head}'(i, j) \leq \text{head}'(i, j + 1)$.

We assume that the alphabet of $w$ is $\Sigma = \{1, 2, \ldots, m\}$ where $m \leq n$. For a Parikh vector $Q$, by $Q[i]$ for $i = 1, 2, \ldots, m$ we denote the number of occurrences of the letter $i$. For two Parikh vectors $Q$ and $R$, we define their Parikh difference, denoted as $Q - R$, as a Parikh vector: $(Q - R)[i] = Q[i] - R[i]$.

In the algorithm we store the difference $\Delta_j = \mathcal{P}(y_j) - \mathcal{P}(x_j)$ of Parikh vectors of

$$x_j = w[i, j] \quad \text{and} \quad y_j = w[j + 1, k]$$

where $k = \text{head}'(i, j)$. Note that $\Delta_j[a] \geq 0$ for any $a = 1, 2, \ldots, m$.

Assume we have computed $\text{head}'(i, j - 1)$ and $\Delta_{j-1}$. When we proceed to $j$, we move the letter $w[j]$ from $y$ to $x$ and update $\Delta$ accordingly. Thus at most one element of $\Delta$ might have dropped below 0. If there is no such element, we conclude that $\text{head}'(i, j) = \text{head}'(i, j - 1)$ and that we have obtained $\Delta_j = \Delta$. Otherwise we keep extending $y$ to the right with new letters and updating $\Delta$ until all its elements become non-negative. We obtain the following algorithm Compute-$\text{head}$.

**Lemma 3.** The head table can be computed in $O(n^2)$ time.

**Proof.** The time complexity of the algorithm Compute-$\text{head}$ is $O(n^2)$. Indeed, the total number of steps of the while-loop for a fixed value of $i$ is $O(n)$, since each step increases the variable $k$. □

We also use the following tail table that is analogical to the head table:

$$\text{tail}(i, j) = \text{minimum } k \text{ such that } \mathcal{P}(w[i, j]) \leq \mathcal{P}(w[i - k, i - 1]).$$
3. Abelian squares and Abelian periods

In this section we show how Abelian periods can be inferred from Abelian squares in a string.

Define by $\maxpower(i, p)$ the maximal size of a prefix of $w[i, n]$ which is an Abelian $k$-power with base $p$ (for some $k$). Define $\square(i, p) = 1$ if and only if $\maxpower(i, p) \geq 2p$. Cummings and Smyth \[7\] compute an alternative table $\square'(i, p)$, such that $\square'(i, p) = 1$ if and only if $w[i - p + 1, i + p]$ is an Abelian square. These tables are clearly equivalent:

$$\square(i, p) = 1 \iff \square'(i + p - 1, p) = 1.$$ 

The $\maxpower(i, p)$ table can be computed from the $\square(i, p)$ table in linear time using a simple dynamic programming recurrence:

$$\maxpower(i, p) = \begin{cases} 0 & \text{if } n - i < p - 1 \\ p + \square(i, p) \cdot \maxpower(i + p, p) & \text{otherwise.} \end{cases}$$

(1)

An alternative $O(n^2)$ time algorithm for computing the table $\square(i, p)$ for a string $w$ of length $n$ is a consequence of the following observation, see also Example \[11\]

Observation 4. $\square(i, p) = 1 \iff \head(i, i + p - 1) = p$.

Theorem 5. All Abelian squares in a string of length $n$ can be computed in $O(n^2)$ time.

The following observation provides a constant-time condition for checking an Abelian period.
Observation 6. \((i, p)\) is an Abelian period of \(w\) if and only if
\[ p \geq \text{head}(1, i), \text{tail}(j, n) \]
where \(j = i + 1 + \text{maxpower}(i + 1, p)\).

We conclude with the following algorithm for computing Abelian periods. In the algorithm we use our alternative version of computing the table \textit{square} from \textit{head}, since the latter table is computed anyway (instead of that Cummings and Smyth’s algorithm can be used for Abelian squares).

\begin{algorithm}
\begin{itemize}
  \item Compute-Abelian-Periods
  \begin{itemize}
    \item Compute \textit{head}(i, j), \textit{tail}(i, j) using algorithm Compute-\textit{head};
    \item Initialize the table \textit{maxpower} to zero table;
    \item for \(p := 1\) to \(n\) do
      \begin{itemize}
        \item for \(i := n\) downto 1 do
          \begin{itemize}
            \item if \(i \leq n - p + 1\) then
              \begin{itemize}
                \item \textit{maxpower}(i, p) := \(p\);
                \item if \textit{head}(i, \(i + p - 1\)) = \(p\) then
                  \begin{itemize}
                    \item \textit{maxpower}(i, p) := \(p + \textit{maxpower}(i + p, p)\);
                  \end{itemize}
              \end{itemize}
          \end{itemize}
        \end{itemize}
    \end{itemize}
  \end{itemize}
\end{algorithm}

Theorem 7. All Abelian periods of a string of length \(n\) can be computed in \(O(n^2)\) time.

4. Long Abelian periods

In this section we show how to compute the table \(\text{MinLong}(i)\), see the example in the table below.

| \(i\) | \(0\) | \(1\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) | \(8\) | \(9\) | \(10\) | \(11\) | \(12\) | \(13\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(w[i]\) | \(c\) | \(a\) | \(a\) | \(b\) | \(b\) | \(c\) | \(a\) | \(b\) | \(b\) | \(c\) | \(a\) | \(a\) | \(a\) |
| \(\text{MinLong}(i)\) | 7 | 7 | 9 | 8 | 7 | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) |

For a non-decreasing function \(f : \{1, 2, \ldots, n + 1\} \to \{-\infty\} \cup \{1, 2, \ldots, n + 1\}\) define the function
\[ \hat{f}(i) = \min\{j : f(j) > i\}. \]
If the minimum is undefined then we set \(\hat{f}(i) = \infty\).
Observation 8. Let $f$ be a function non-decreasing and computable in constant time. Then all the values of $\hat{f}$ can be computed in linear time.

Theorem 9. A compact representation of all long Abelian periods can be computed in linear time.

Proof. Let us take $f(j) = j - \text{tail}(j,n)$. This function is non-decreasing, see also Observation 2. Then for $i < \frac{n}{2}$ we have:

$$\text{MinLong}(i) = \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor + 1, \text{head}(1,i), \hat{f}(i) - i - 1 \right\}$$

and otherwise $\text{MinLong}(i) = \infty$, see also Fig. 2.

Figure 2: A schematic view of a long Abelian period: $p > \frac{n}{2}$, $p \geq \text{head}(1,i)$, $\text{tail}(j,n)$.

Hence the computation of MinLong table is reduced to linear time algorithm for $f$ and the conclusion of the theorem follows from Observation 8.

References


