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ESTIMATING THE COVARIANCE OF RANDOM MATRICES

PIERRE YOUSSEF

ABSTRACT. We extend to the matrix setting a recent result of Srivastava-Vershynin [21] about estimating the covariance matrix of a random vector. The result can be interpreted as a quantified version of the law of large numbers for positive semi-definite matrices which verify some regularity assumption. Beside giving examples, we discuss the notion of log-concave matrices and give estimates on the smallest and largest eigenvalues of a sum of such matrices.

1. INTRODUCTION

In recent years, interest in matrix valued random variables gained momentum. Many of the results dealing with real random variables and random vectors were extended to cover random matrices. Concentration inequalities like Bernstein, Hoeffding and others were obtained in the non-commutative setting ([5],[22],[14]). The methods used were mostly combination of methods from the real/vector case and some matrix inequalities like the Golden-Thompson inequality (see [8]).

Estimating the covariance matrix of a random vector has gained a lot of interest recently. Given a random vector X in \mathbb{R}^n , the question is to estimate $\Sigma = \mathbb{E}XX^t$. A natural way to do this is to take X_1, \dots, X_N independent copies of X and try to approximate Σ with the sample covariance matrix $\Sigma_N = \frac{1}{N} \sum_i X_i X_i^t$. The challenging problem is to find the minimal number of samples needed to estimate Σ . It is known using a result of Rudelson (see [19]) that for general distributions supported on the sphere of radius \sqrt{n} , it suffices to take $cn \log(n)$ samples. But for many distributions, a number proportional to n is sufficient. Using standard arguments, one can verify this for gaussian vectors. It was conjectured by Kannan- Lovasz- Simonovits [11] that the same result holds for log-concave distributions. This problem was solved by Adamczak et al ([3],[4]). Recently, Srivastava-Vershynin proved in [21] covariance estimate with a number of samples proportional to n , for a larger class of distributions covering the log-concave case. The method used was different from previous work on this field and the main idea was to randomize the sparsification theorem of Batson-Spielman-Srivastava [7].

Our aim in this paper is to adapt the work of Srivastava-Vershynin to the matrix setting replacing the vector X in the problem of the covariance matrix by an $n \times m$ random matrix A and try to estimate $\mathbb{E}AA^t$ by the same techniques. This will be possible since in the deterministic setting, the sparsification theorem of Batson-Spielman-Srivastava [7]

has been extended to a matrix setting by De Carli Silva-Harvey-Sato [9] who precisely proved the following:

Theorem 1.1. *Let B_1, \dots, B_m be positive semi-definite matrices of size $n \times n$ and arbitrary rank. Set $B := \sum_i B_i$. For any $\varepsilon \in (0, 1)$, there is a deterministic algorithm to construct a vector $y \in \mathbb{R}^m$ with $O(n/\varepsilon^2)$ nonzero entries such that $y \geq 0$ and*

$$B \preceq \sum_i y_i B_i \preceq (1 + \varepsilon)B.$$

For an $n \times n$ matrix A , denote by $\|A\|$ the operator norm of A seen as an operator on l_2^n . The main idea is to randomize the previous result using the techniques of Srivastava-Vershynin [21]. Our problem can be formulated as follows:

Take B a positive semi-definite random matrix of size $n \times n$. How many independent copies of B are needed to approximate $\mathbb{E}B$ i.e taking B_1, \dots, B_N independent copies of B , what is the minimal number of samples needed to make $\left\| \frac{1}{N} \sum_i B_i - \mathbb{E}B \right\|$ very small.

One can view this as an analogue to the covariance estimate of a random vector by taking for B the matrix AA^t where A is an $n \times m$ random matrix. With some regularity, we will be able to take N proportional to n . However, in the general case this is no longer true. In fact, take B uniformly distributed on $\{ne_i e_i^t\}_{i \leq n}$ where e_j denotes the canonical basis of \mathbb{R}^n . It is easy to verify that $\mathbb{E}B = I_n$ and $\frac{1}{N} \sum_i B_i$ is a diagonal matrix and its diagonal coefficients are distributed as

$$\frac{n}{N}(p_1, \dots, p_n),$$

where p_i denotes the number of times $e_i e_i^t$ is chosen. This problem is well- studied and it is known (see [12]) that we must take $N \geq cn \log(n)$. This example is essentially due to Aubrun [6]. More generally, if B is a positive semi-definite matrix such that $\mathbb{E}B = I_n$ and $\text{Tr}(B) \leq n$ almost surely, then by Rudelson's inequality in the non-commutative setting (see [15]) it is sufficient to take $cn \log(n)$ samples.

The method will work properly for a class of matrices satisfying a matrix strong regularity assumption which we denote by (MSR) and can be viewed as an analog to the property (SR) defined in [21].

Definition 1.2. *[Property (MSR)]*

Let B be an $n \times n$ positive semi-definite random matrix such that $\mathbb{E}B = I_n$. We will say that B satisfies (MSR) if for some $\eta > 0$ we have :

$$\mathbb{P}(\|PBP\| \geq t) \leq \frac{c}{t^{1+\eta}} \quad \forall t \geq c \cdot \text{rank}(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^n.$$

The main result of this paper is the following:

Theorem 1.3. *Let B be an $n \times n$ positive semi-definite random matrix verifying $\mathbb{E}B = I_n$ and (MSR) for some $\eta > 0$. Then for every $\varepsilon < 1$, taking $N = C_1(\eta) \frac{n}{\varepsilon^{2+\frac{2}{\eta}}}$ we have*

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^N B_i - I_n \right\| \leq \varepsilon \quad \text{where } B_1, \dots, B_N \text{ are independent copies of } B.$$

If X is an isotropic random vector of \mathbb{R}^n , put $B = XX^t$ then $\|PBP\| = \|PX\|_2^2$. Therefore if X verifies the property (SR) appearing in [21], then B verifies property (MSR) . So applying Theorem 1.3 to $B = XX^t$, we recover the covariance estimation as stated in [21].

In order to apply our result, beside some examples, we investigate the notion of log-concave matrices in relation with the definition of log-concave vectors. Moreover remarking some strong concentration inequalities satisfied by these matrices we are able, using the ideas developed in the proof of the main theorem, to have some results with high probability rather than only in expectation as is the case in the main result. This will be discussed in the last section of the paper.

The paper is organized as follows: in section 2, we show how to prove Theorem 1.3 using two other results (Theorem 2.1, Theorem 2.2) which we prove respectively in sections 3 and 4 using again two other results (Theorem 3.1, Theorem 4.1) whose proofs are given respectively in sections 5 and 6. In section 7, we give some applications and discuss the notion of log-concave matrices and prove some related results.

2. PROOF OF THEOREM 1.3

We first introduce a regularity assumption on the moments which we denote by (MWR) :

$$\exists p > 1 \text{ such that } \mathbb{E} \langle Bx, x \rangle^p \leq C_p \quad \forall x \in S^{n-1}.$$

Note that by a simple integration of tails, (MSR) (with P a rank one projection) implies (MWR) with $p < 1 + \eta$.

The proof of Theorem 1.3 is based on two theorems dealing with the smallest and largest eigenvalues of $\frac{1}{N} \sum_{i=1}^N B_i$.

Theorem 2.1. *Let B_i $n \times n$ independent positive semi-definite random matrices verifying $\mathbb{E}B_i = I_n$ and (MWR) .*

Let $\varepsilon < 1$, then for

$$N \geq 16 (16C_p)^{\frac{1}{p-1}} \frac{n}{\varepsilon^{\frac{2p-1}{p-1}}}$$

we get

$$\mathbb{E} \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) \geq 1 - \varepsilon$$

$$C_1(\eta) = (64c)^{1+\frac{2}{\eta}} (1 + \frac{1}{\eta})^{\frac{2}{\eta}} \vee 64(4c)^{\frac{1}{\eta}} (32 + \frac{32}{\eta})^{1+\frac{2}{\eta}} \vee 256(2c)^{\frac{3}{2}+\frac{2}{\eta}} (16 + \frac{16}{\eta})^{\frac{4}{\eta}}$$

Theorem 2.2. *Let B_i $n \times n$ independent positive semi-definite random matrices verifying $\mathbb{E}B_i = I_n$ and (MSR).*

Then for any N we have

$$\mathbb{E}\lambda_{\max}\left(\sum_{i=1}^N B_i\right) \leq C(\eta).(n + N)$$

Moreover, for $\varepsilon < 1$ and $N \geq C_2(\eta)\frac{n}{\varepsilon^{2+\frac{2}{\eta}}}$ we have

$$\mathbb{E}\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^N B_i\right) \leq 1 + \varepsilon$$

We will give the proof of these two theorems in sections 3 and 4 respectively. We need also a simple lemma:

Lemma 2.3. *Let $1 < r \leq 2$ and Z_1, \dots, Z_N be independent positive random variables with $\mathbb{E}Z_i = 1$ and satisfying $(\mathbb{E}Z_i^r)^{\frac{1}{r}} \leq M$ Then*

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N Z_i - 1\right| \leq \frac{2M}{N^{\frac{r-1}{r}}}.$$

Proof. Let $(\varepsilon_i)_{i \leq N}$ independent ± 1 Bernoulli variables. By symmetrization and Jensen's inequality we can write

$$\begin{aligned} \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N Z_i - 1\right| &\leq \frac{2}{N}\mathbb{E}\left|\sum_{i=1}^N \varepsilon_i Z_i\right| \leq \frac{2}{N}\mathbb{E}\left(\sum_{i=1}^N Z_i^2\right)^{\frac{1}{2}} \\ &\leq \frac{2}{N}\mathbb{E}\left(\sum_{i=1}^N Z_i^r\right)^{\frac{1}{r}} \leq \frac{2}{N}\left(\sum_{i=1}^N \mathbb{E}Z_i^r\right)^{\frac{1}{r}} \leq \frac{2M}{N^{\frac{r-1}{r}}} \end{aligned}$$

□

Proof of Theorem 1.3. Take $N \geq c(\eta)\frac{n}{\varepsilon^{2+\frac{2}{\eta}}}$ satisfying conditions of Theorem 2.1 (with $p = 1 + \frac{\eta}{2}$) and Theorem 2.2. Note that by the triangle inequality

$$\begin{aligned} \left\|\frac{1}{N}\sum_{i=1}^N B_i - I_n\right\| &\leq \left\|\frac{1}{N}\sum_{i=1}^N B_i - \frac{1}{n}\text{Tr}\left(\frac{1}{N}\sum_{i=1}^N B_i\right)I_n\right\| + \left\|\frac{1}{n}\text{Tr}\left(\frac{1}{N}\sum_{i=1}^N B_i\right)I_n - I_n\right\| \\ &:= \alpha + \beta \end{aligned}$$

$$C_2(\eta) = 16c^{\frac{1}{\eta}}\left(32 + \frac{32}{\eta}\right)^{1+\frac{3}{\eta}} \vee 16\sqrt{2}(4c)^{\frac{3}{2}+\frac{2}{\eta}}\left(8 + \frac{8}{\eta}\right)^{\frac{4}{\eta}}$$

Observe that

$$\begin{aligned}\alpha &= \max \left| \lambda \left(\frac{1}{N} \sum_{i=1}^N B_i - \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) I_n \right) \right| \\ &= \max \left[\lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) - \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^N B_i \right), \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) - \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) \right]\end{aligned}$$

Since the two terms in the max are non-negative, then one can bound the max by the sum of the two terms. More precisely, we get $\alpha \leq \lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) - \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N B_i \right)$ and by Theorem 2.1 and Theorem 2.2 we deduce that $\mathbb{E}\alpha \leq 2\varepsilon$.

Note that

$$\beta = \left| \frac{1}{N} \sum_{i=1}^N \frac{\text{Tr}(B_i)}{n} - 1 \right| = \left| \frac{1}{N} \sum_{i=1}^N Z_i - 1 \right|,$$

where $Z_i = \frac{\text{Tr}(B_i)}{n}$. Since B_i satisfies (MWR), then taking $r = \min(2, 1 + \frac{\eta}{2})$ we have

$$\forall i \leq N, \quad (\mathbb{E}Z_i^r)^{\frac{1}{r}} \leq \frac{1}{n} \sum_{j=1}^n (\mathbb{E} \langle B_i e_j, e_j \rangle^r)^{\frac{1}{r}} \leq c(\eta).$$

Therefore Z_i satisfy the conditions of Lemma 2.3 and we deduce that $\mathbb{E}\beta \leq \varepsilon$ by the choice of N .

As a conclusion

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^N B_i - I_n \right\| \leq \mathbb{E}\alpha + \mathbb{E}\beta \leq 3\varepsilon$$

□

3. PROOF OF THEOREM 2.1

Given A an $n \times n$ positive semi-definite matrix such that all eigenvalues of A are greater than a lower barrier $l_A = l$ i.e $A \succ l.I_n$, define the corresponding potential function to be $\phi_l(A) = \text{Tr}(A - l.I_n)^{-1}$.

The proof of Theorem 2.1 is based on the following result which will be proved in section 5:

Theorem 3.1. *Let $A \succ l.I_n$ and $\phi_l(A) \leq \phi$, B a positive semi-definite random matrix satisfying $\mathbb{E}B = I_n$ and Property (MWR) with some $p > 1$.*

Let $\varepsilon < 1$, if

$$\phi \leq \frac{1}{4(8C_p)^{\frac{1}{p-1}}} \varepsilon^{\frac{p}{p-1}}$$

then there exist l' a random variable such that

$$A + B \succ l'.I_n, \quad \phi_{l'}(A + B) \leq \phi_l(A) \text{ and } \mathbb{E}l' \geq l + 1 - \varepsilon.$$

Proof of Theorem 2.1. We start with $A_0 = 0$ and $l_0 = -\frac{n}{\phi}$ so that $\phi_{l_0}(A_0) = -\frac{n}{l_0} = \phi$. Applying Theorem 3.1, one can find l_1 such that

$$A_1 = A_0 + B_1 \succ l_1.I_n, \quad \phi_{l_1}(A_1) \leq \phi_{l_0}(A_0)$$

and

$$\mathbb{E}l_1 \geq l_0 + 1 - \varepsilon$$

Now apply Theorem 3.1 once again to find l_2 such that

$$A_2 = A_1 + B_2 \succ l_2.I_n, \quad \phi_{l_2}(A_2) \leq \phi_{l_1}(A_1)$$

and

$$\mathbb{E}l_2 \geq l_1 + 1 - \varepsilon \geq l_0 + 2(1 - \varepsilon)$$

After N steps, we get $\mathbb{E}\lambda_{\min}(A_N) \geq \mathbb{E}l_N \geq l_0 + N(1 - \varepsilon)$. Therefore,

$$\mathbb{E}\lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) \geq 1 - \varepsilon - \frac{n}{N\phi}$$

Taking $N = \frac{n}{\varepsilon\phi}$, we get $\mathbb{E}\lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N B_i \right) \geq 1 - 2\varepsilon$.

□

4. PROOF OF THEOREM 2.2

Given A an $n \times n$ positive semi-definite matrix such that all eigenvalues of A are less than an upper barrier $u_A = u$ i.e $A \prec u.I_n$, define the corresponding potential function to be $\psi_u(A) = \text{Tr}(u.I_n - A)^{-1}$.

The proof of Theorem 2.2 is based on the following result which will be proved in section 6:

Theorem 4.1. *Let $A \prec u.I_n$ and $\psi_u(A) \leq \psi$, B a positive semi-definite random matrix satisfying $\mathbb{E}B = I_n$ and Property (MSR).*

Let $\varepsilon < 1$, if

$$\psi \leq C_3(\eta)\varepsilon^{1+\frac{2}{\eta}}$$

there exists u' a random variable such that

$$A + B \prec u'.I_n, \quad \psi_{u'}(A + B) \leq \psi_u(A) \text{ and } \mathbb{E}u' \leq u + 1 + \varepsilon.$$

$$C_3(\eta) = \left[8(2c)^{\frac{1}{\eta}} \left(16 + \frac{16}{\eta} \right)^{1+\frac{3}{\eta}} \right]^{-1} \wedge \left[16(2c)^{\frac{3}{2}+\frac{2}{\eta}} \left(8 + \frac{8}{\eta} \right)^{\frac{4}{\eta}} \right]^{-1}$$

Proof of Theorem 2.2. We start with $A_0 = 0$, $u_0 = \frac{n}{\psi}$ so that $\psi_{u_0}(A_0) = \psi$. Applying Theorem 4.1, one can find u_1 such that

$$A_1 = A_0 + B_1 \prec u_1 \cdot I_n, \quad \psi_{u_1}(A_1) \leq \psi_{u_0}(A_0) \quad \text{and} \quad \mathbb{E}u_1 \leq u_0 + 1 + \varepsilon.$$

Now apply Theorem 4.1 once again to find u_2 such that

$$A_2 = A_1 + B_2 \prec u_2 \cdot I_n, \quad \psi_{u_2}(A_2) \leq \psi_{u_1}(A_1) \quad \text{and} \quad \mathbb{E}u_2 \leq u_1 + 1 + \varepsilon.$$

After N steps we get $\mathbb{E}\lambda_{\max}\left(\sum_{i=1}^N B_i\right) \leq \mathbb{E}u_N \leq u_0 + N(1 + \varepsilon)$.

Taking $N \geq \frac{n}{\varepsilon\psi} = c'(\eta)^{-1} \frac{n}{\varepsilon^2 + \frac{2}{\eta}}$, we deduce that

$$\mathbb{E}\lambda_{\max}\left(\frac{1}{N} \sum_{i=1}^N B_i\right) \leq 1 + 2\varepsilon$$

□

5. PROOF OF THEOREM 3.1

5.1. Notations. We are looking for a random variable l' of the form $l + \delta$ where δ is a positive random variable playing the role of the shift.

If in addition $A \succ (l + \delta) \cdot I_n$, we will note :

$$L_\delta = A - (l + \delta) \cdot I_n \succ 0 \quad \text{so that} \quad \text{Tr}\left(B^{\frac{1}{2}}(A - (l + \delta) \cdot I_n)^{-1}B^{\frac{1}{2}}\right) = \langle L_\delta^{-1}, B \rangle.$$

$\lambda_1, \dots, \lambda_n$ will denote the eigenvalues of A and v_1, \dots, v_n the corresponding eigenvectors. $(v_i)_{i \leq n}$ are also the eigenvectors of L_δ^{-1} corresponding to the eigenvalues $\frac{1}{\lambda_i - (l + \delta)}$.

5.2. Finding the shift. To find sufficient conditions for such δ exists, we need a matrix extension of Lemma 3.4 in [7] which, up to a minor change, is essentially contained in Lemma 20 in [9] and we formulate it here in Lemma 5.2. This method uses the Sherman-Morrison-Woodbury formula:

Lemma 5.1. *Let E be an $n \times n$ invertible matrix, C a $k \times k$ invertible matrix, U an $n \times k$ matrix and V a $k \times n$ matrix. Then we have:*

$$(E + UCV)^{-1} = E^{-1} - E^{-1}U(C^{-1} + VE^{-1}U)^{-1}VE^{-1}$$

Lemma 5.2. *Let A as above satisfying $A \succ l \cdot I_n$. Suppose that one can find $\delta > 0$ verifying $\delta \leq \frac{1}{\|L_0^{-1}\|}$ and*

$$\frac{\langle L_\delta^{-2}, B \rangle}{\phi_{l+\delta}(A) - \phi_l(A)} - \left\| B^{\frac{1}{2}} L_\delta^{-1} B^{\frac{1}{2}} \right\| \geq 1$$

Then

$$\lambda_{\min}(A + B) > l + \delta \quad \text{and} \quad \phi_{l+\delta}(A + B) \leq \phi_l(A).$$

Proof. First note that $\frac{1}{\|L_0^{-1}\|} = \lambda_{\min}(A) - l$, so the first condition on δ implies that $\lambda_{\min}(A) \geq l + \delta$.

Now using Sherman-Morrison-Woodbury formula with $E = L_\delta$, $U = V = B^{\frac{1}{2}}$, $C = I_n$ we get :

$$\begin{aligned} \phi_{l+\delta}(A+B) &= \text{Tr}(L_\delta + B)^{-1} \\ &= \phi_{l+\delta}(A) - \text{Tr}\left(L_\delta^{-1} B^{\frac{1}{2}} \left(I_n + B^{\frac{1}{2}} L_\delta^{-1} B^{\frac{1}{2}}\right)^{-1} B^{\frac{1}{2}} L_\delta^{-1}\right) \\ &\leq \phi_{l+\delta}(A) - \frac{\langle L_\delta^{-2}, B \rangle}{1 + \|B^{\frac{1}{2}} L_\delta^{-1} B^{\frac{1}{2}}\|} \end{aligned}$$

Rearranging the hypothesis, we get $\phi_{l+\delta}(A+B) \leq \phi_l(A)$. □

Since $\|L_0^{-1}\| \leq \text{Tr}(L_0^{-1}) = \phi_l(A)$ and $\|B^{\frac{1}{2}} L_\delta^{-1} B^{\frac{1}{2}}\| \leq \langle L_\delta^{-1}, B \rangle$ then in order to satisfy conditions of Lemma 5.2, we may search for δ satisfying:

$$(1) \quad \delta \leq \frac{1}{\phi_l(A)} \quad \text{and} \quad \frac{\langle L_\delta^{-2}, B \rangle}{\phi_{l+\delta}(A) - \phi_l(A)} - \langle L_\delta^{-1}, B \rangle \geq 1$$

For $t \leq \frac{1}{\phi}$, let us note :

$$q_1(t, B) = \langle L_t^{-1}, B \rangle = \text{Tr}(B(A - (l+t).I_n)^{-1})$$

and

$$q_2(t, B) = \frac{\langle L_t^{-2}, B \rangle}{\text{Tr}(L_t^{-2})} = \frac{\text{Tr}(B(A - (l+t).I_n)^{-2})}{\text{Tr}(A - (l+t).I_n)^{-2}}$$

We have already seen in Lemma 5.2 that if $t \leq \frac{1}{\phi} \leq \frac{1}{\|L_0^{-1}\|}$ then $A \succ (l+t).I_n$ so the definitions above make sense. Since we have :

$$\begin{aligned} \phi_{l+\delta}(A) - \phi_l(A) &= \text{Tr}(A - (l+\delta).I_n)^{-1} - \text{Tr}(A - l.I_n)^{-1} \\ &= \delta \text{Tr}((A - (l+\delta).I_n)^{-1} (A - l.I_n)^{-1}) \\ &\leq \delta \text{Tr}(A - (l+\delta).I_n)^{-2} \end{aligned}$$

In order to have (1), it will be sufficient to choose δ satisfying $\delta \leq \frac{1}{\phi}$ and

$$(2) \quad \frac{1}{\delta} q_2(\delta, B) - q_1(\delta, B) \geq 1$$

q_1 and q_2 can be expressed as follows:

$$q_1(t, B) = \sum_{i=1}^n \frac{\langle Bv_i, v_i \rangle}{\lambda_i - l - t} \quad \text{and} \quad q_2(t, B) = \frac{\sum_i \frac{\langle Bv_i, v_i \rangle}{(\lambda_i - l - t)^2}}{\sum_i (\lambda_i - l - t)^{-2}}$$

$\phi_l(A) = \sum_{i=1}^n (\lambda_i - l)^{-1} \leq \phi$ so that $(\lambda_i - l) \cdot \phi \geq 1$ for all i , then we have :

$$(1 - t \cdot \phi)(\lambda_i - l) = \lambda_i - l - t \cdot (\lambda_i - l) \cdot \phi \leq \lambda_i - l - t \leq \lambda_i - l$$

therefore

$$q_1(t, B) \leq (1 - t \cdot \phi)^{-1} q_1(0, B)$$

and

$$(1 - t \cdot \phi)^2 q_2(0, B) \leq q_2(t, B) \leq (1 - t \cdot \phi)^{-2} q_2(0, B)$$

Lemma 5.3. *Let $s \in (0, 1)$ and take $\delta = (1 - s)^3 q_2(0, B) \mathbf{1}_{\{q_1(0, B) \leq s\}} \mathbf{1}_{\{q_2(0, B) \leq \frac{s}{\phi}\}}$. Then $A + B \succ (l + \delta) \cdot I_n$ and $\phi_{l+\delta}(A + B) \leq \phi_l(A)$.*

Proof. As stated before, it is sufficient to check that $\delta \leq \frac{1}{\phi}$ and $\frac{1}{\delta} q_2(\delta, B) - q_1(\delta, B) \geq 1$. If $q_1(0, B) \geq s$ or $q_2(0, B) \geq \frac{s}{\phi}$ then $\delta = 0$ and there is nothing to prove since $\phi_l(A + B) \leq \phi_l(A)$.

In the other case i.e $q_1(0, B) \leq s$ and $q_2(0, B) \leq \frac{s}{\phi}$, we have $\delta = (1 - s)^3 q_2(0, B)$. So $\delta \leq (1 - s)^3 \frac{s}{\phi} \leq \frac{1}{\phi}$ and

$$\begin{aligned} \frac{1}{\delta} q_2(\delta, B) - q_1(\delta, B) &= \frac{1}{(1 - s)^3 q_2(0, B)} q_2(\delta, B) - q_1(\delta, B) \\ &\geq \frac{1}{(1 - s)^3 q_2(0, B)} (1 - \delta \phi)^2 q_2(0, B) - (1 - \delta \phi)^{-1} q_1(0, B) \\ &\geq \frac{(1 - s)^2}{(1 - s)^3} - \frac{s}{(1 - s)} = 1 \end{aligned}$$

□

5.3. Estimating the random shift. Now that we have found δ , we will estimate $\mathbb{E}\delta$ using the property (MWR). We will start by stating some basic facts about q_1 and q_2 .

Proposition 5.4. *Let as above $A \succ l \cdot I_n$ and $\phi_l(A) \leq \phi$, B satisfying (MWR). Then we have the following :*

- (1) $\mathbb{E}q_1(0, B) = \phi_l(A) \leq \phi$ and $\mathbb{E}q_1(0, B)^p \leq C_p \phi^p$.
- (2) $\mathbb{E}q_2(0, B) = 1$ and $\mathbb{E}q_2(0, B)^p \leq C_p$.
- (3) $\mathbb{P}(q_1(0, B) \geq u) \leq C_p \left(\frac{\phi}{u}\right)^p$ and $\mathbb{P}(q_2(0, B) \geq u) \leq \frac{C_p}{u^p}$.

Proof. Since $\mathbb{E}B = I_n$ then $\mathbb{E}q_1(0, B) = \phi_l(A)$ and $\mathbb{E}q_2(0, B) = 1$. Now using the triangle inequality and Property (MWR) we get :

$$(\mathbb{E}q_1(0, B)^p)^{\frac{1}{p}} = \left[\mathbb{E} \left(\sum_{i=1}^n \frac{\langle Bv_i, v_i \rangle}{\lambda_i - l} \right)^p \right]^{\frac{1}{p}} \leq \sum_{i=1}^n \frac{(\mathbb{E} \langle Bv_i, v_i \rangle^p)^{\frac{1}{p}}}{\lambda_i - l} \leq \sum_{i=1}^n \frac{C_p^{\frac{1}{p}}}{\lambda_i - l} \leq C_p^{\frac{1}{p}} \phi$$

With the same argument we prove that $\mathbb{E}q_2(0, B)^p \leq C_p$. The third part of the proposition follows by Markov's inequality. \square

Lemma 5.5. *If δ is as in Lemma 5.3. Then*

$$\mathbb{E}\delta \geq (1-s)^3 \left[1 - 2C_p \left(\frac{\phi}{s} \right)^{p-1} \right]$$

Proof. Using the above proposition and Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we get :

$$\begin{aligned} \mathbb{E}\delta &= \mathbb{E}(1-s)^3 q_2(0, B) \mathbf{1}_{\{q_1(0, B) \leq s\}} \mathbf{1}_{\{q_2(0, B) \leq \frac{s}{\phi}\}} \\ &= (1-s^3) \left[\mathbb{E}q_2(0, B) - \mathbb{E}q_2(0, B) \mathbf{1}_{\{q_1(0, B) > s \text{ or } q_2(0, B) > \frac{s}{\phi}\}} \right] \\ &\geq (1-s)^3 \left[1 - (\mathbb{E}q_2(0, B)^p)^{\frac{1}{p}} \cdot \left(\mathbb{P} \left\{ q_1(0, B) > s \text{ or } q_2(0, B) > \frac{s}{\phi} \right\} \right)^{\frac{1}{q}} \right] \\ &\geq (1-s)^3 \left[1 - C_p^{\frac{1}{p}} \left(C_p \left(\frac{\phi}{s} \right)^p + C_p \left(\frac{\phi}{s} \right)^p \right)^{\frac{1}{q}} \right] \\ &\geq (1-s)^3 \left[1 - 2C_p \left(\frac{\phi}{s} \right)^{p-1} \right] \end{aligned}$$

\square

Now it remains to make good choice of s and ϕ in order to finish the prove Theorem 3.1. Take $l' = l + \delta$, the choice of δ being as before with $s = \frac{\varepsilon}{4}$.

As we have seen, we get $A + B \succ l'.I_n$ and $\phi_{l'}(A + B) \leq \phi_l(A)$. Moreover,

$$\mathbb{E}l' = l + \mathbb{E}\delta \geq l + (1-s)^3 \left[1 - 2C_p \left(\frac{\phi}{s} \right)^{p-1} \right] \geq 1 - \varepsilon,$$

by the choice of ϕ . This ends the proof of Theorem 3.1.

6. PROOF OF THEOREM 4.1

6.1. Notations. We are looking for a random variable u' of the form $u + \Delta$ where Δ is a positive random variable playing the role of the shift.

We will note $U_t = (u+t).I_n - A$ so that $\text{Tr} \left(B^{\frac{1}{2}}((u+t).I_n - A)^{-1} B^{\frac{1}{2}} \right) = \langle U_t^{-1}, B \rangle$.

As before, $\lambda_1, \dots, \lambda_n$ will denote the eigenvalues of A and v_1, \dots, v_n the corresponding eigenvectors. $(v_i)_{i \leq n}$ are also the eigenvectors of U_t^{-1} corresponding to the eigenvalues $\frac{1}{u+t-\lambda_i}$.

6.2. Finding the shift. To find sufficient conditions for such Δ exists, we need a matrix extension of Lemma 3.3 in [7] which, up to a minor change, is essentially contained in Lemma 19 in [9]. For the sake of completeness, we include the proof.

Lemma 6.1. *Let A as above satisfying $A \prec u.I_n$. Suppose that one can find $\Delta > 0$ verifying*

$$(3) \quad \frac{\langle U_\Delta^{-2}, B \rangle}{\psi_u(A) - \psi_{u+\Delta}(A)} + \|B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}}\| \leq 1$$

Then

$$A + B \prec (u + \Delta).I_n \quad \text{and} \quad \psi_{u+\Delta}(A + B) \leq \psi_u(A).$$

Proof. Since $\langle U_\Delta^{-2}, B \rangle$ and $\psi_u(A) - \psi_{u+\Delta}(A)$ are positive, then by (3) we have

$$\|B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}}\| < 1 \quad \text{and} \quad \frac{\langle U_\Delta^{-2}, B \rangle}{1 - \|B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}}\|} \leq \psi_u(A) - \psi_{u+\Delta}(A)$$

First note that $\|B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}}\| = \|U_\Delta^{-\frac{1}{2}} B U_\Delta^{-\frac{1}{2}}\| < 1$, so $U_\Delta^{-\frac{1}{2}} B U_\Delta^{-\frac{1}{2}} \prec I_n$. Therefore we get $B \prec U_\Delta$ which means that $A + B \prec (u + \Delta).I_n$.

Now using the Sherman-Morrison-Woodbury (see Lemma 5.1) with $E = U_\Delta, U = V = B^{\frac{1}{2}}, C = I_n$ we get :

$$\begin{aligned} \psi_{u+\Delta}(A + B) &= \text{Tr} (U_\Delta - B)^{-1} \\ &= \psi_{u+\Delta}(A) + \text{Tr} \left(U_\Delta^{-1} B^{\frac{1}{2}} (I_n - B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}})^{-1} B^{\frac{1}{2}} U_\Delta^{-1} \right) \\ &\leq \psi_{u+\Delta}(A) + \frac{\langle U_\Delta^{-2}, B \rangle}{1 - \|U_\Delta^{-\frac{1}{2}} B U_\Delta^{-\frac{1}{2}}\|} \leq \psi_u(A) \end{aligned}$$

□

We may now find Δ satisfying (3). Let us note :

$$Q_1(t, B) = \|B^{\frac{1}{2}} U_t^{-1} B^{\frac{1}{2}}\| = \|B^{\frac{1}{2}} ((u + t).I_n - A)^{-1} B^{\frac{1}{2}}\|$$

and

$$Q_2(t, B) = \frac{\langle U_t^{-2}, B \rangle}{\psi_u(A) - \psi_{u+t}(A)} = \frac{\text{Tr} (B ((u + t).I_n - A)^{-2})}{\psi_u(A) - \psi_{u+t}(A)}$$

Since Q_1 and Q_2 are both decreasing in t , we work with each separately. Precisely fix $\theta \in (0, 1)$ and define Δ_1, Δ_2 as follows :

Δ_1 the smallest positive number such that $Q_1(\Delta_1, B) \leq \theta$

and

Δ_2 the smallest positive number such that $Q_2(\Delta_2, B) \leq 1 - \theta$

Now take $\Delta = \Delta_1 + \Delta_2$, then $Q_1(\Delta, B) + Q_2(\Delta, B) \leq \theta + 1 - \theta = 1$. So this choice of Δ satisfies (3) and it remains now to estimate Δ_1 and Δ_2 separately.

6.3. Estimating Δ_1 .

We may write $Q_1(\Delta_1, B) = \left\| \sum_{i=1}^n \frac{B^{\frac{1}{2}} v_i v_i^t B^{\frac{1}{2}}}{u + \Delta_1 - \lambda_i} \right\|$.

Put $\xi_i = B^{\frac{1}{2}} v_i v_i^t B^{\frac{1}{2}}$, $\mu_i = \psi(u - \lambda_i)$ and $\mu = \psi \Delta_1$. Denote P_S the orthogonal projection on $(v_i)_{i \in S}$, clearly $\text{rank}(P_S) = |S|$. Then we have :

$$\left\{ \begin{array}{l} \mathbb{E} \|\xi_i\| = 1 \\ \mathbb{P} \left(\left\| \sum_{i \in S} \xi_i \right\| \geq t \right) = \mathbb{P} (\|P_S B P_S\| \geq t) \leq \frac{c}{t^{1+\eta}} \quad \forall t \geq c|S| \\ \sum_{i=1}^n \frac{1}{\mu_i} = \frac{\psi_u(A)}{\psi} \leq 1 \\ \mu \text{ is the smallest positive number such that } \sum_{i=1}^n \frac{\xi_i}{\mu_i + \mu} \preceq \frac{\theta}{\psi} Id \end{array} \right.$$

We will need an analog of Lemma 3.5 appearing in [21]. We extend this lemma to a matrix setting:

Lemma 6.2. *Suppose $\{\xi_i\}_{i \leq n}$ are symmetric positive semi-definite random matrices with $\mathbb{E} \|\xi_i\| = 1$ and :*

$$\mathbb{P} \left(\left\| \sum_{i \in S} \xi_i \right\| \geq t \right) \leq \frac{c}{t^{1+\eta}} \quad \text{provided} \quad t > c|S| = c \sum_{i \in S} \mathbb{E} \|\xi_i\|.$$

for all subsets $S \subset [n]$ and some constants $c, \eta > 0$. Consider positive numbers μ_i such that

$$\sum_{i=1}^n \frac{1}{\mu_i} \leq 1.$$

Let μ be the minimal positive number such that

$$\sum_{i=1}^n \frac{\xi_i}{\mu_i + \mu} \preceq K \cdot Id,$$

for some $K \geq C = 4c$. Then $\mathbb{E} \mu \leq \frac{c(\eta)}{K^{1+\eta}}$.

We do not reproduce the proof here as it is a direct adaptation of the argument in [21] to the matrix setting. Applying Lemma 6.2 we get $\mathbb{E}\mu \leq c(\eta) \left(\frac{\psi}{\theta}\right)^{1+\eta}$, so that

$$(4) \quad \mathbb{E}\Delta_1 \leq c(\eta) \frac{\psi^\eta}{\theta^{1+\eta}}$$

6.4. Estimating Δ_2 .

Suppose $\theta \leq \frac{1}{2}$. Since $\psi_u(A) - \psi_{u+t}(A) = t \cdot \text{Tr} \left((u.I_n - A)^{-1} ((u+t).I_n - A)^{-1} \right)$ we can write

$$(5) \quad Q_2(t, B) = \frac{\sum_i \frac{\langle Bv_i, v_i \rangle}{(u+t-\lambda_i)^2}}{t \sum_i (u+t-\lambda_i)^{-1} (u-\lambda_i)^{-1}} \leq \frac{1}{t} \frac{\sum_i \frac{\langle Bv_i, v_i \rangle}{(u+t-\lambda_i)(u-\lambda_i)}}{\sum_i (u+t-\lambda_i)^{-1} (u-\lambda_i)^{-1}} \\ := \frac{1}{t} P_2(t, B)$$

First note that $P_2(t, B)$ can be written as $\sum_i \alpha_i(t) \langle Bv_i, v_i \rangle$ with $\sum_i \alpha_i = 1$. Having this in mind, one can easily check that $\mathbb{E}P_2(t, B) = 1$ and

$$(6) \quad \mathbb{E}P_2(t, B)^{1+\frac{3\eta}{4}} \leq c(\eta),$$

where for the last inequality, we used the fact that B satisfies (MWR) with $p = 1 + \frac{3\eta}{4}$. In order to estimate Δ_2 , we will divide it into two parts as follows:

$$\Delta_2 = \Delta_2 \mathbf{1}_{\{P_2(0, B) \leq \frac{\theta}{4\psi}\}} + \Delta_2 \mathbf{1}_{\{P_2(0, B) > \frac{\theta}{4\psi}\}} := H_1 + H_2$$

Let us start by estimating $\mathbb{E}H_1$. Suppose that $P_2(0, B) \leq \frac{\theta}{4\psi}$ and denote

$$x = (1 + 4\theta)P_2(0, B).$$

Since $\psi_u(A) \leq \psi$, we have $(u - \lambda_i) \cdot \psi \geq 1 \ \forall i$ and therefore $u + x - \lambda_i \leq (1 + x\psi)(u - \lambda_i)$. This implies that

$$P_2(x, B) \leq (1 + x\psi)P_2(0, B).$$

Now write

$$Q_2(x, B) \leq \frac{1}{x} P_2(x, B) \leq \frac{1 + x\psi}{x} P_2(0, B) \leq \frac{1 + (1 + 4\theta)\frac{\theta}{4}}{1 + 4\theta} \leq 1 - \theta,$$

which means that

$$\Delta_2 \mathbf{1}_{\{P_2(0, B) \leq \frac{\theta}{4\psi}\}} \leq (1 + 4\theta)P_2(0, B)$$

and therefore

$$(7) \quad \mathbb{E}H_1 = \mathbb{E}\Delta_2 \mathbf{1}_{\{P_2(0, B) \leq \frac{\theta}{4\psi}\}} \leq 1 + 4\theta$$

Now it remains to estimate $\mathbb{E}H_2$. For that we need to prove a moment estimate for Δ_2 . First observe that using (6) we have

$$\mathbb{P}\{\Delta_2 > t\} = \mathbb{P}\{Q_2(t, B) > 1 - \theta\} \leq \mathbb{P}\{P_2(t, B) > t \cdot (1 - \theta)\} \leq \frac{c(\eta)}{t^{1+\frac{3\eta}{4}}}$$

By integration, this implies

$$\mathbb{E}\Delta_2^{1+\frac{\eta}{2}} = \int_0^\infty \mathbb{P}\{\Delta_2 > t\} \left(1 + \frac{\eta}{2}\right) t^{\frac{\eta}{2}} dt \leq \int_0^1 \left(1 + \frac{\eta}{2}\right) t^{\frac{\eta}{2}} dt + \int_1^\infty \frac{c(\eta)}{t^{1+\frac{\eta}{4}}} dt \leq c(\eta)$$

Let $p' = 1 + \frac{\eta}{2}$, applying Hölder's inequality with $\frac{1}{p'} + \frac{1}{q'} = 1$ we have :

$$\begin{aligned} (8) \quad \mathbb{E}H_2 &= \mathbb{E}\Delta_2 \mathbf{1}_{\{P_2(0, B) > \frac{\theta}{4\psi}\}} \leq \left(\mathbb{E}\Delta_2^{p'}\right)^{\frac{1}{p'}} \left(\mathbb{P}\left\{P_2(0, B) > \frac{\theta}{4\psi}\right\}\right)^{\frac{1}{q'}} \\ &\leq c(\eta) \left(\left(\frac{\psi}{\theta}\right)^{1+\frac{\eta}{2}} \mathbb{E}P_2(0, B)^{1+\frac{\eta}{2}}\right)^{\frac{1}{q'}} \\ &\leq c(\eta) \left(\frac{\psi}{\theta}\right)^{\frac{\eta}{2}} \end{aligned}$$

Looking at (7) and (8) we have

$$\mathbb{E}\Delta_2 \leq 1 + 4\theta + c(\eta) \left(\frac{\psi}{\theta}\right)^{\frac{\eta}{2}}$$

Putting the estimates of Δ_1 and Δ_2 together we deduce

$$\mathbb{E}\Delta \leq 1 + 4\theta + c(\eta) \left(\frac{\psi}{\theta}\right)^{\frac{\eta}{2}} + c(\eta) \frac{\psi^\eta}{\theta^{1+\eta}}$$

We are now ready to finish the proof. Take $u' = u + \Delta$, Δ being chosen as before with $\theta = \frac{\varepsilon}{8}$. Then taking $\psi = c(\eta)\varepsilon^{1+\frac{2}{\eta}}$ with the constant depending on η properly chosen, we get $\mathbb{E}\Delta \leq 1 + \varepsilon$.

7. APPLICATIONS

In this section, we will show how to apply our main result. After giving examples, we will discuss in details the case of log-concave matrices for which we give results with high probability estimates. Let us first replace (MSR) with a stronger, but easier to manipulate, property which we denote by (MSR^*) . If B is an $n \times n$ positive semi-definite random matrix such that $\mathbb{E}B = Id$, we will say that B satisfies (MSR^*) if for some $\eta > 0$:

$$\mathbb{P}(\text{Tr}(PB) \geq t) \leq \frac{c}{t^{1+\eta}} \quad \forall t \geq c \cdot \text{rank}(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^n.$$

Note that since $\|PBP\| \leq \text{Tr}(PBP) = \text{Tr}(PB)$, then (MSR^*) is clearly stronger than (MSR) .

7.1. $(2 + \varepsilon)$ -moments for the spectrum. Looking carefully at (MSR) , one can see that it implies regularity assumptions on the eigenvalues of B . Putting some independence in the spectral decomposition of B , we will only need to use the regularity of the eigenvalues. To be more precise, we have the following:

Proposition 7.1. *Let $B = UDU^*$ be an $n \times n$ symmetric positive semi-definite random matrix. Denote $(\alpha_j)_{j \leq n}$ the diagonal entries of D . Suppose that U and D are independent and that $(\alpha_j)_{j \leq n}$ are independent and satisfy the following:*

$$\forall i \leq n, \mathbb{E}\alpha_i = 1 \quad \text{and} \quad (\mathbb{E}\alpha_i^p)^{\frac{1}{p}} \leq c,$$

for some $p > 2$. Then B satisfies (MSR^*) .

Proof. First note that since U and D are independent and $\mathbb{E}\alpha_i = 1$ then $\mathbb{E}B = Id$. Let $k > 0$ and P be an orthogonal projection of rank k on \mathbb{R}^n , then $Q = U^*PU$ is a random orthogonal projection of rank k independent of D . Note that $\text{Tr}(PB) = \sum_{i \leq n} q_{ii}\alpha_i$, and now using Markov's inequality we have for $t > k$,

$$\mathbb{P}\{\text{Tr}(PB) \geq t\} \leq \frac{1}{(t-k)^p} \mathbb{E} \left| \sum_{i \leq n} q_{ii}(\alpha_i - 1) \right|^p$$

Looking at the expectation with respect to D and using Rosenthal's inequality (see [18]) we get

$$\mathbb{E}_D \left| \sum_{i \leq n} q_{ii}(\alpha_i - 1) \right|^p \leq C(p) \max \left\{ \sum_{i \leq n} q_{ii}^p \mathbb{E}|\alpha_i - 1|^p, \left(\sum_{i \leq n} q_{ii}^2 \mathbb{E}|\alpha_i - 1|^2 \right)^{\frac{p}{2}} \right\}$$

Taking in account that $q_{ii} \leq 1$, which implies that for any $l \geq 1$, $\sum_i q_{ii}^l \leq k$, we deduce that

$$\mathbb{E} \left| \sum_{i \leq n} q_{ii}(\alpha_i - 1) \right|^p \leq C(p) k^{\frac{p}{2}}$$

Instead of Rosenthal's inequality, we could have used a symmetrization argument alongside Khintchine's inequality to get the estimate above.

One can easily conclude that B satisfies (MSR^*) with $\eta = \frac{p}{2} - 1$. \square

Applying Theorem 1.3, we can deduce the following proposition:

Proposition 7.2. *Let $B = UDU^*$ be an $n \times n$ symmetric positive semi-definite random matrix. Denote $(\alpha_j)_{j \leq n}$ the diagonal entries of D . Suppose that U and D are independent and that $(\alpha_j)_{j \leq n}$ are independent and satisfy the following:*

$$\forall i \leq n, \mathbb{E}\alpha_i = 1 \quad \text{and} \quad (\mathbb{E}\alpha_i^p)^{\frac{1}{p}} \leq c,$$

for some $p > 2$. Let $\varepsilon < 1$, then taking $N = C(p) \frac{n}{\varepsilon^{\frac{2p}{p-2}}}$ we have

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^N B_i - I_n \right\| \leq \varepsilon \quad \text{where } B_1, \dots, B_N \text{ are independent copies of } B.$$

7.2. From (SR) to (MSR) . A random vector X in \mathbb{R}^l is called isotropic if its covariance matrix is the identity i.e $\mathbb{E}XX^t = Id$. In [21], an isotropic random vector X in \mathbb{R}^l was said to satisfy (SR) if for some $\eta > 0$,

$$\mathbb{P} \left(\|PX\|_2^2 \geq t \right) \leq \frac{c}{t^{1+\eta}} \quad \forall t \geq c.rank(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^l.$$

We will show how to jump from this property dealing with vectors to the property (MSR^*) dealing with matrices.

Proposition 7.3. *Let A be an $n \times m$ random matrix and denote by $(C_i)_{i \leq m}$ its columns. Suppose that $A^t = \sqrt{m}(C_1^t, \dots, C_m^t)$ is an isotropic random vector in \mathbb{R}^{nm} which satisfies property (SR) . Then $B = AA^t$ verifies $\mathbb{E}B = I_n$ and Property (MSR^*) .*

Proof. For $l \leq nm$, one can write $l = (j-1)n + i$ with $1 \leq i \leq n$, $1 \leq j \leq m$. So that the coordinates of A' are given by $a'_l = \sqrt{m}a_{i,j}$, and since A' is isotropic we get $\mathbb{E}a_{i,j}a_{r,s} = \frac{1}{m}\delta_{(i,j),(r,s)}$.

The terms of B are given by $b_{i,j} = \sum_{s=1}^m a_{i,s}a_{j,s}$. We deduce that $\mathbb{E}b_{i,j} = \delta_{i,j}$ and therefore $\mathbb{E}B = I_n$.

Let P be an orthogonal projection of \mathbb{R}^n and put $P' = I_m \otimes P$. Clearly we have $\|P'A'\|_2^2 = m\text{Tr}(PB)$ and $rank(P') = m.rank(P)$.

Let $t \geq c.rank(P)$ then $mt \geq c.rank(P')$ and by property (SR) we have:

$$\mathbb{P} \left(\|P'A'\|_2^2 \geq mt \right) \leq \frac{c}{(mt)^{1+\eta}}$$

This means that

$$\mathbb{P} (\text{Tr}(PB) \geq t) \leq \frac{c}{(mt)^{1+\eta}}$$

and therefore B satisfies (MSR^*) . □

Remark 7.4. In [21], it was shown that an isotropic log-concave vector satisfies (SR) . We will discuss with details this notion for matrices in the next section.

In [1], it was shown that, with r properly chosen, an isotropic $(-\frac{1}{r})$ -concave random vector satisfies (SR) . Therefore, one can adapt the results of the next section to the case of $(-\frac{1}{r})$ -concave random matrices.

7.3. From log-concave vectors to matrices. In this section, we will discuss in details the case of log-concave matrices. After giving some tails estimate, we deduce results with high probability. The methods will use tools developed in the proof of the main theorem. Finally, we give an example of log-concave matrices and apply the results obtained to these matrices.

Definition 7.5. *Let A be an $n \times m$ random matrix and denote by $(C_i)_{i \leq m}$ its columns. We will say that A is an isotropic log-concave matrix if $A^t = \sqrt{m}(C_1^t, \dots, C_m^t)$ is an isotropic random vector in \mathbb{R}^{nm} with a log-concave density with respect to the Lebesgue measure in \mathbb{R}^{nm} .*

Remark 7.6. Let $(a_{i,j})$ the entries of A . Saying that A' is isotropic means that

$$\mathbb{E} a_{i,j} a_{k,l} = \frac{1}{m} \delta_{(i,j),(k,l)}$$

This implies that for any $n \times m$ matrix M we have :

$$\mathbb{E} \langle A, M \rangle A = \mathbb{E} \text{Tr} (A^t M) A = \frac{1}{m} M.$$

One can view this as an analogue to an isotropic condition in the vector case: in fact if $A = X$ is a vector (i.e an $n \times 1$ matrix), the above condition would be

$$\mathbb{E} \langle X, y \rangle X = y \quad \text{for all } y \in \mathbb{R}^n.$$

which means that X is isotropic in \mathbb{R}^n .

Proposition 7.7. Let A be an $n \times m$ isotropic log-concave matrix and denote $B = AA^t$. Then for every orthogonal projection P on \mathbb{R}^n we have a large deviation estimate for $\text{Tr}(PB)$

$$\mathbb{P} \{ \text{Tr}(PB) \geq c_1 t \} \leq \exp \left(-\sqrt{t.m} \right) \quad \forall t \geq \text{rank}(P).$$

and a small ball probability estimate

$$\mathbb{P} \{ \text{Tr}(PB) \geq c_2 \varepsilon . \text{rank}(P) \} \leq \varepsilon^{c_2 \sqrt{m . \text{rank}(P)}} \quad \forall \varepsilon \leq 1.$$

Proof. Let P an orthogonal projection on \mathbb{R}^n and denote $P' = I_m \otimes P$. As we have seen before $\text{Tr}(PB) = \|PA\|_{\text{HS}}^2 = \frac{1}{m} \|P' A'\|_2^2$ and $\text{rank}(P') = m . \text{rank}(P)$. Using Paouris result [16] for the isotropic log-concave vector A' , we have

$$\mathbb{P} \left\{ \|P' A'\|_2^2 \geq c_1 u \right\} \leq \exp \left(-\sqrt{u} \right) \quad \forall u \geq \text{rank}(P').$$

Let $t \geq \text{rank}(P)$ and write $u = t.m$. Since $u \geq m . \text{rank}(P) = \text{rank}(P')$ we have

$$\mathbb{P} \{ m . \text{Tr}(PB) \geq c_1 t . m \} \leq \exp \left(-\sqrt{t.m} \right)$$

which gives the large deviation estimate stated above.

For the small ball probability estimate, we use once again a result of Paouris [17] dealing with isotropic log-concave vector:

$$\mathbb{P} \left\{ \|P' A'\|_2^2 \geq c_2 \varepsilon . \text{rank}(P') \right\} \leq \varepsilon^{c_2 \sqrt{\text{rank}(P')}} \quad \forall \varepsilon \leq 1.$$

Writing this in terms of B and P , we easily get the conclusion. \square

Now we will apply Theorem 1.3 to this class of matrices. We get the following:

Proposition 7.8. Let A be an $n \times m$ isotropic log-concave matrix. Then $B = AA^t$ satisfies (MSR). Moreover $\forall \varepsilon > 0$, taking $N > c(\varepsilon)n$ independent copies of B we have

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^N B_i - I_n \right\|_2 \leq \varepsilon.$$

Proof. Note first that since A is isotropic in the sense of definition 7.5, then $B = AA^t$ satisfies $\mathbb{E}B = I_n$.

By proposition 7.7, B satisfies

$$\mathbb{P}(\text{Tr}(PB) \geq c_1 t) \leq \exp(-\sqrt{tm}) \quad \forall t \geq \text{rank}(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^n.$$

and therefore (MSR). Applying theorem 1.3 we deduce the result. \square

The probability estimate for these log-concave matrices are strong enough which allows us to obtain some results with high probability rather than in expectation as is the case above. Precisely, we can prove the following :

Proposition 7.9. *Let $\rho > 0$ and $N \leq n^\rho$, A an isotropic log-concave matrix and $B = AA^t$. If $m \geq [2(1 + \rho) \log 2n]^2$, then with probability $\geq 1 - \exp(-\frac{1}{2}\sqrt{m})$ we have*

$$\lambda_{\max} \left(\sum_{i=1}^N B_i \right) \leq 2c_1 \cdot (n + N).$$

Proof. The proof of proposition 7.9 follows the same idea as in the previous section. We will only need the following property satisfied by our matrix $B = AA^t$ (using Proposition 7.7 for a rank 1 projection):

$$\mathbb{P}(\langle Bx, x \rangle \geq c_1 t) \leq \exp(-\sqrt{tm}) \quad \forall t \geq 1 \quad \text{and} \quad \forall x \in \mathbb{S}^{n-1}.$$

Recall some notations :

$A_0 = 0$, $A_1 = B_1$, $A_2 = A_1 + B_1$, ..., $A_N = A_{N-1} + B_N = \sum_{i=1}^N B_i$. $u_0 = 2c_1 \cdot n$, $u_1 = u_0 + 2c_1$, $u_2 = u_1 + 2c_1$, ..., $u_N = u_{N-1} + 2c_1$ and $U_i = u_{i+1}I_n - A_i$, $\Delta = 2c_1$ plays the role of the shift which will no longer be random. Note $\psi_{u_i}(A_i) = \text{Tr}(u_i - A_i)^{-1}$ the corresponding potential function when $A_i \prec u_i \cdot I_n$ and $\psi = \psi_{u_0}(A_0) = \frac{1}{2c_1}$. Denote by \mathfrak{S}_i the event " $A_i \prec u_i \cdot I_n$ and $\psi_{u_i}(A_i) \leq \psi$ ".

Clearly $\mathbb{P}(\mathfrak{S}_0) = 1$. Suppose now that \mathfrak{S}_i is satisfied, as we have seen in Lemma 6.1 the following condition is sufficient for the occurrence of the event \mathfrak{S}_{i+1} :

$$Q_2(2c_1, B_{i+1}) + Q_1(2c_1, B_{i+1}) \leq 1$$

Note that $Q_2(2c_1, B_{i+1}) \leq \frac{1}{2c_1} P_2(2c_1, B_{i+1})$, where P_2 is defined in (5). Now denoting λ_j the eigenvalues of A_i and v_j the corresponding eigenvectors, taking the probability with respect to B_{i+1} one can write :

$$\begin{aligned}
\mathbb{P}(Q_2(2c_1, B_{i+1}) + Q_1(2c_1, B_{i+1}) > 1) &\leq \mathbb{P}\left(\frac{1}{2c_1}P_2(2c_1, B_{i+1}) + Q_1(2c_1, B_{i+1}) > 1\right) \\
&\leq \mathbb{P}\left(\frac{1}{2c_1}P_2(2c_1, B_{i+1}) > \frac{1}{2}\right) + \mathbb{P}\left(Q_1(2c_1, B_{i+1}) > \frac{1}{2}\right) \\
&\leq \mathbb{P}\left(\sum_{j=1}^n \frac{\langle B_{i+1}v_j, v_j \rangle}{(u_{i+1} - \lambda_j)(u_i - \lambda_i)} > c_1 \sum_{j=1}^n \frac{1}{(u_{i+1} - \lambda_j)(u_i - \lambda_i)}\right) \\
&\quad + \mathbb{P}\left(\sum_{j=1}^n \frac{\langle B_{i+1}v_j, v_j \rangle}{u_{i+1} - \lambda_j} > \frac{1}{2}\right) \\
&\leq \mathbb{P}(\exists j / \langle B_{i+1}v_j, v_j \rangle > c_1) + \mathbb{P}\left(\exists j / \langle B_{i+1}v_j, v_j \rangle > \frac{1}{2\psi}\right) \\
&\leq 2n \cdot \exp(-\sqrt{m})
\end{aligned}$$

So we have shown that $\mathbb{P}(\mathfrak{S}_{i+1}|\mathfrak{S}_i) \geq 1 - 2n \cdot \exp(-\sqrt{m})$. Since B_i are independent we have :

$$\begin{aligned}
\mathbb{P}\left(\lambda_{\max}\left(\sum_{i=1}^N B_i\right) \leq 2c_1 \cdot (n + N)\right) &\geq \mathbb{P}(\mathfrak{S}_N) \\
&\geq \mathbb{P}(\mathfrak{S}_N|\mathfrak{S}_{N-1}) \mathbb{P}(\mathfrak{S}_{N-1}|\mathfrak{S}_{N-2}) \dots \mathbb{P}(\mathfrak{S}_0) \\
&\geq 1 - 2Nn \cdot \exp(-\sqrt{m})
\end{aligned}$$

Proposition 7.9 follows by the choice of m . Note that the same method works whenever we have in (MSR) a probability less than $\frac{1}{2nN}$. \square

Now using the small ball probability estimate alongside the large deviation given by Proposition 7.7, we have also an estimate on the smallest eigenvalue.

Proposition 7.10. *Let $\rho > 0$ and $\frac{4c_1}{c_2}n < N \leq n^{1+\rho}$, A an $n \times m$ isotropic log-concave matrix and $B = AA^t$. If $m \geq \left[\frac{2}{c_2}(2 + \rho) \log(2n)\right]^2$, then with probability $\geq \exp(-\frac{c_2}{2}\sqrt{m})$ we have*

$$\lambda_{\min}\left(\sum_{i=1}^N B_i\right) \geq -c_1n + \frac{c_2}{4}N$$

Proof. Here we only need the following two properties satisfied by our matrix B :

$$\mathbb{P}(\langle Bx, x \rangle \geq c_1t) \leq \exp(-\sqrt{tm}) \quad \forall t \geq 1 \quad \text{and} \quad \forall x \in \mathbb{S}^{n-1}.$$

and taking $\varepsilon = \frac{1}{2}$ in Proposition 7.7

$$\mathbb{P}\left(\langle Bx, x \rangle \leq \frac{c_2}{2}t\right) \leq \exp(-c_2\sqrt{m}) \quad \forall x \in \mathbb{S}^{n-1}.$$

Recall some notations :

$A_0 = 0$, $A_1 = B_1$, $A_2 = A_1 + B_1$, ..., $A_N = A_{N-1} + B_N = \sum_{i=1}^N B_i$. $l_0 = -c_1.n$, $l_1 = l_0 + \frac{c_2}{4}$, $l_2 = l_1 + \frac{c_2}{4}$, ..., $l_N = l_{N-1} + \frac{c_2}{4}$ and $L_i = A_i - l_{i+1}I_n$, $\delta = \frac{c_2}{4}$ plays the role of the shift which will no longer be random. Note $\phi_{l_i}(A_i) = \text{Tr}(A_i - l_i)^{-1}$ the corresponding potential function when $A_i \succeq l_i.I_n$ and $\phi = \phi_{l_0}(A_0) = \frac{1}{c_1}$. Note also that $\delta \leq \phi$ since $c_1 > 1$ while $c_2 < 1$ for obvious reasons.

Denote by \mathfrak{S}_i the event " $A_i \succeq l_i.I_n$ and $\phi_i(A_i) \leq \phi$ ".

Clearly $\mathbb{P}(\mathfrak{S}_0) = 1$. Suppose now that \mathfrak{S}_i is satisfied, following what was done after Lemma 5.2, condition (2) was sufficient for the occurrence of the event \mathfrak{S}_{i+1} :

$$\frac{4}{c_2}q_2\left(\frac{c_2}{4}, B_{i+1}\right) - q_1\left(\frac{c_2}{4}, B_{i+1}\right) \geq 1$$

Denoting λ_j the eigenvalues of A_i and v_j the corresponding eigenvectors, taking the probability with respect to B_{i+1} one can write :

$$\begin{aligned} & \mathbb{P}\left(\frac{4}{c_2}q_2\left(\frac{c_2}{4}, B_{i+1}\right) - q_1\left(\frac{c_2}{4}, B_{i+1}\right) < 1\right) \leq \\ & \leq \mathbb{P}\left(\frac{4}{c_2}q_2\left(\frac{c_2}{4}, B_{i+1}\right) < 2\right) + \mathbb{P}\left(q_1\left(\frac{c_2}{4}, B_{i+1}\right) > 1\right) \\ & \leq \mathbb{P}\left(\sum_{j=1}^n \frac{\langle B_{i+1}v_j, v_j \rangle}{(\lambda_j - l_{i+1})^2} < \frac{c_2}{2} \sum_{j=1}^n \frac{1}{(\lambda_j - l_{i+1})^2}\right) + \mathbb{P}\left(\sum_{j=1}^n \frac{\langle B_{i+1}v_j, v_j \rangle}{\lambda_j - l_{i+1}} > 1\right) \\ & \leq \mathbb{P}\left(\exists j / \langle B_{i+1}v_j, v_j \rangle < \frac{c_2}{2}\right) + \mathbb{P}\left(\exists j / \langle B_{i+1}v_j, v_j \rangle > \frac{1}{\phi}\right) \\ & \leq 2n.exp(-c_2\sqrt{m}) \end{aligned}$$

So we have shown that $\mathbb{P}(\mathfrak{S}_{i+1}|\mathfrak{S}_i) \geq 1 - 2n.exp(-c_2\sqrt{m})$. Since B_i are independent we have :

$$\begin{aligned} \mathbb{P}\left(\lambda_{\min}\left(\sum_{i=1}^N B_i\right) \geq -c_1n + \frac{c_2}{4}N\right) & \geq \mathbb{P}(\mathfrak{S}_N) \\ & \geq \mathbb{P}(\mathfrak{S}_N|\mathfrak{S}_{N-1}) \mathbb{P}(\mathfrak{S}_{N-1}|\mathfrak{S}_{N-2}) \dots \mathbb{P}(\mathfrak{S}_0) \\ & \geq 1 - 2Nn.exp(-c_2\sqrt{m}) \end{aligned}$$

Proposition 7.10 follows by the choice of m .

□

Combining the two previous propositions we get the following:

Corollary 7.11. *Let A be an $n \times m$ isotropic log-concave matrix. There exists $c, C > 0$ universal constants such that if $m \geq [C \log(2n)]^2$, taking $N = Cn$ copies of A then*

with probability $\geq \exp(-c\sqrt{m})$ we have

$$c \leq \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N A_i A_i^t \right) \leq \lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^N A_i A_i^t \right) \leq C$$

7.3.1. Example of log-concave matrices.

For $x \in \mathbb{R}^k$, we denote by \hat{x} the vector with components $|x_i|$ arranged in nonincreasing order. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$, we say that f is absolutely symmetric if $f(x) = f(\hat{x})$ for all $x \in \mathbb{R}^k$. (For example, $\|\cdot\|_p$ is absolutely symmetric).

Define F a function on $\mathbb{M}_{n,m}$ by $F(A) = f(s_1(A), \dots, s_k(A))$ for $A \in \mathbb{M}_{n,m}$ and $k = \min(n, m)$. It was shown by Lewis [13] that f is absolutely symmetric if and only if F is unitary invariant and of this form. Moreover, f is convex if and only if F is convex.

Let A be an $n \times m$ random matrix whose density with respect to Lebesgue measure is given by $G(A) = \exp(-f(s_1(A), \dots, s_k(A)))$, where f is an absolutely symmetric convex function. By the remark above, G is log-concave. This covers the case of random matrices with density of the form $\exp(-\sum_i V(s_i(A)))$ where V is an increasing convex function on \mathbb{R}^+ . When $V(x) = x^2$, this would be the gaussian unitary ensemble GUE.

Let $(a_{i,j})$ the entries of A . By a good normalization of f we can suppose that A satisfies

$$\mathbb{E} a_{i,j} a_{k,l} = \frac{1}{m} \delta_{(i,j),(k,l)}$$

To see this, fix (i, j) and (k, l) two different indexes. Note $D_j = \text{diag}(1, \dots, -1, \dots, 1)$ the $m \times m$ diagonal matrix where the -1 is on the j^{th} term. Let $E_{(i,k)}$ is the $n \times n$ matrix obtained by swapping the i^{th} and k^{th} rows in the identity matrix. Note also $F_{(j,l)}$ the $m \times m$ matrix obtained by swapping the j^{th} and l^{th} rows in the identity matrix.

It is easy to see that AD_j change the j^{th} column of A to its opposite and keep the rest unchanged. Note that AD_j has the same singular values as A .

Similarly, $E_{(i,k)} A F_{(j,l)}$ permute $a_{i,j}$ with $a_{k,l}$ and keep the other terms unchanged. Note also that $E_{(i,k)} A F_{(j,l)}$ has the same singular values as A .

Finally note that these two transformations has a Jacobian equal to 1, and since f is absolutely symmetric these transformations which preserve the singular values don't affect the density.

If $j \neq l$, by a change of variables $M = AD_j$ the density is invariant and we have

$$\int a_{i,j} a_{k,l} G(A) dA = - \int a_{i,j} a_{k,l} G(A) dA$$

Doing the change of variables $M = D_i A$ when $i \neq k$, we can conclude that

$$\mathbb{E} a_{i,j} a_{k,l} = 0 \quad \text{if } (i, j) \neq (k, l)$$

Now by a change of variables $M = E_{(i,j)} A F_{(k,l)}$ the density is invariant and we have

$$\int a_{i,j}^2 G(A) dA = \int a_{k,l}^2 G(A) dA$$

This implies that

$$\int a_{i,j}^2 G(A) dA = \frac{1}{nm} \sum_{k \leq n, l \leq m} \int a_{k,l}^2 G(A) dA = \frac{1}{nm} \int \|A\|_{\text{HS}}^2 G(A) dA$$

Now we may normalize f in order to make the previous term equal $\frac{1}{m}$. Suppose that

$$\frac{1}{n} \int \|A\|_{\text{HS}}^2 G(A) dA = c$$

Define $\hat{f}(x) = f(\sqrt{c}x) - nm \log(\sqrt{c})$ and $\hat{G}(A) = \exp(-\hat{f}(s_1(A), \dots, s_k(A)))$.

Note that \hat{G} is a probability density. Indeed, by the change of variables $M = \sqrt{c}A$ we have

$$\begin{aligned} \int \hat{G}(A) dA &= \int \exp(-f(\sqrt{c}s_1(A), \dots, \sqrt{c}s_k(A))) (\sqrt{c})^{nm} dA \\ &= \int \exp(-f(s_1(M), \dots, s_k(M))) dM = 1 \end{aligned}$$

Note also that \hat{G} satisfies our isotropic condition. Indeed, by the same change of variables we can write

$$\frac{1}{n} \int \|A\|_{\text{HS}}^2 \hat{G}(A) dA = \frac{1}{cn} \int \|M\|_{\text{HS}}^2 G(M) dM = 1$$

As a conclusion, we can deduce that such matrices are isotropic log-concave. Moreover, since in this case $\mathbb{E} A^t A = \frac{n}{m} I_m$ then we also have that $\sqrt{\frac{m}{n}} A^t$ is an $m \times n$ isotropic log-concave matrix. We summarize this in the following proposition:

Proposition 7.12. *Let A be an $n \times m$ random matrix whose density with respect to Lebesgue is given by*

$$G(A) = \exp(-f(s_1(A), \dots, s_k(A))),$$

where f is an absolutely symmetric convex function, properly normalized as above and $k = \min(n, m)$. Then A is an isotropic log-concave matrix, and $\sqrt{\frac{n}{m}} A^t$ is an $m \times n$ isotropic log-concave matrix.

Remark 7.13. *In a similar way, one can prove that taking A an $n \times m$ random matrix whose density with respect to Lebesgue is given by*

$$G(A) = (f(s_1(A), \dots, s_k(A)))^{-(k+r)},$$

where f is an absolutely symmetric convex function, properly normalized, r chosen as in [1] and $k = \min(n, m)$, then A is an isotropic $(-\frac{1}{r})$ -concave matrix.

Applying Proposition 7.10 and Proposition 7.9 for A and A^t we get:

Proposition 7.14. *Let A be an $n \times m$ random matrix whose density with respect to Lebesgue is given by*

$$G(A) = \exp(-f(s_1(A), \dots, s_k(A))),$$

where f is an absolutely symmetric function, properly normalized as above and $k = \min(n, m)$.

Suppose that $n \geq [C \log(2m)]^2$ and $m \geq [C \log(2n)]^2$, taking $N = C \max(n, m)$ then with probability $\geq 1 - \exp(-c\sqrt{k})$ we have

$$c \leq \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N A_i A_i^t \right) \leq \lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^N A_i A_i^t \right) \leq C$$

and

$$c \frac{n}{m} \leq \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^N A_i^t A_i \right) \leq \lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^N A_i^t A_i \right) \leq C \frac{n}{m}$$

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REFERENCES

- [1] R. Adamczak, O. Guédon, R. Latała, K. Oleszkiewicz, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Moment estimates for convex measures. *Electronic Journal of Probability*, 17(101):1–19, 2012.
- [2] R. Adamczak, A. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Tail estimates for norms of sums of log-concave random vectors. arXiv:1107.4070v1, July 2011.
- [3] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles. *J. Amer. Math. Soc.*, 23(2):535–561, 2010.
- [4] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Sharp bounds on the rate of convergence of the empirical covariance matrix. *C. R. Math. Acad. Sci. Paris*, 349(3-4):195–200, 2011.
- [5] R. Ahlswede and A. Winter. Addendum to: “Strong converse for identification via quantum channels” [IEEE Trans. Inform. Theory **48** (2002), no. 3, 569–579; MR1889969 (2003d:94069)]. *IEEE Trans. Inform. Theory*, 49(1):346, 2003.
- [6] G. Aubrun. Sampling convex bodies: a random matrix approach. *Proc. Amer. Math. Soc.*, 135(5):1293–1303 (electronic), 2007.
- [7] J. D. Batson, D. A. Spielman, and N. Srivastava. Twice-Ramanujan sparsifiers. In *STOC’09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 255–262. ACM, New York, 2009.

- [8] R. Bhatia. *Matrix analysis*, volume 169 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.
- [9] M. De Carli Silva, N. Harvey, and c. Sato. Sparse Sums of Positive Semidefinite Matrices. arXiv:1107.0088v2, July 2011.
- [10] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.
- [11] R. Kannan, L. Lovász, and M. Simonovits. Random walks and an $O^*(n^5)$ volume algorithm for convex bodies. *Random Structures Algorithms*, 11(1):1–50, 1997.
- [12] V. F. Kolchin, B. A. Sevast'yanov, and V. P. Chistyakov. *Random allocations*. V. H. Winston & Sons, Washington, D.C., 1978. Translated from the Russian, Translation edited by A. V. Balakrishnan, Scripta Series in Mathematics.
- [13] A. S. Lewis. The convex analysis of unitarily invariant matrix functions. *J. Convex Anal.*, 2(1-2):173–183, 1995.
- [14] L. Mackey, M. Jordan, R. Chen, B. Farrell, and J. Tropp. Matrix concentration inequalities via the method of exchangeable pairs. Available at arXiv:1201.6002.
- [15] R. I. Oliveira. Sums of random Hermitian matrices and an inequality by Rudelson. *Electron. Commun. Probab.*, 15:203–212, 2010.
- [16] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006.
- [17] G. Paouris. Small ball probability estimates for log-concave measures. *Trans. Amer. Math. Soc.*, 364(1):287–308, 2012.
- [18] H. P. Rosenthal. On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.*, 8:273–303, 1970.
- [19] M. Rudelson. Random vectors in the isotropic position. *J. Funct. Anal.*, 164(1):60–72, 1999.
- [20] N. Srivastava. *Spectral Sparsification and Restricted Invertibility*. PhD thesis, Yale University, Mar. 2010.
- [21] N. Srivastava and R. Vershynin. Covariance Estimation for Distributions with $2+\epsilon$ Moments. to appear in *Annals of Probability*.
- [22] J. A. Tropp. User-friendly tail bounds for sums of random matrices. *Found. Comput. Math.*, 12(4):389–434, 2012.