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► **To cite this version:**

| Pierre Youssef. A note on column subset selection. 2012. hal-00811804

HAL Id: hal-00811804

<https://hal-upec-upem.archives-ouvertes.fr/hal-00811804>

Preprint submitted on 11 Apr 2013

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A NOTE ON COLUMN SUBSET SELECTION

PIERRE YOUSSEF

ABSTRACT. Given a matrix U , using a deterministic method, we extract a "large" submatrix of \tilde{U} (whose columns are obtained by normalizing those of U) and estimate its smallest and largest singular value. We apply this result to the study of contact points of the unit ball with its maximal volume ellipsoid. We consider also the paving problem and give a deterministic algorithm to partition a matrix into almost isometric blocks recovering previous results of Bourgain-Tzafriri and Tropp. Finally, we partially answer a question raised by Naor about finding an algorithm in the spirit of Batson-Spielman-Srivastava's work to extract a "large" square submatrix of "small" norm.

1. INTRODUCTION

Let U a $n \times m$ matrix, the stable rank of U is given by $srank(U) = \frac{\|U\|_{\text{HS}}^2}{\|U\|^2}$, where $\|U\|_{\text{HS}}^2 = \text{Tr}(UU^t)$ denotes the Hilbert-Schmidt norm of U and $\|U\|^2$ the operator norm of U seen as an operator from l_2^m to l_2^n . Denoting \tilde{U} the matrix whose columns are obtained by normalizing those of U , our aim is to extract almost $srank(U)$ number of linearly independent columns of \tilde{U} and estimate the smallest and the largest singular value of the restricted matrix.

This problem is closely related to the restricted invertibility where only an estimate on the smallest singular value is needed. The restricted invertibility was first studied by Bourgain-Tzafriri [2] who proved the following:

Theorem A. *Given an $n \times n$ matrix T whose columns are of norm one, there exists $\sigma \subset \{1, \dots, n\}$ with $|\sigma| \geq d \frac{n}{\|T\|_2^2}$ such that $\|T_\sigma x\|_2 \geq c \|x\|_2$ for all $x \in \mathbb{R}^\sigma$, where $d, c > 0$ are absolute constants and T_σ denotes the restriction of T onto the columns in σ .*

In [17], Vershynin extended this result for any decomposition of the identity, whereas the previous result was valid for the canonical decomposition. Moreover the size of the restriction depended on the Hilbert-Schmidt norm of the operator. Precisely, Vershynin proved the following:

Theorem B. *Let $Id = \sum_{j \leq m} x_j x_j^t$ and let T be a linear operator on ℓ_2^n . For any $\varepsilon \in (0, 1)$ one can find $\sigma \subset \{1, \dots, m\}$ with*

$$|\sigma| \geq (1 - \varepsilon) \frac{\|T\|_{\text{HS}}^2}{\|T\|_2^2}$$

such that

$$c(\varepsilon) \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \in \sigma} a_j \frac{T x_j}{\|T x_j\|_2} \right\|_2 \leq C(\varepsilon) \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}}$$

for all scalars (a_j) .

Vershynin applied this result to the study of contact points. The normalization on the vectors $T x_j$ is crucial for the applications and the dependence on ε plays an important role. Spielman-Srivastava [13] generalized the restricted invertibility principle (Theorem A) for any decomposition of the identity without the normalizing factors appearing in Vershynin's result and improved the dependence on ε . In [18], we unified the two previous results obtaining a good dependence on ε for any normalizing factors. However this result deals only with the lower

bound and doesn't give any information on the norm of the restricted matrix. Our aim here, is to improve Vershynin's result obtaining simultaneously a restricted invertibility principle and an estimate on the norm of the restricted matrix. Our proof uses ideas of the method of Batson-Spielman-Srivastava ([1], [14], see also [18] for related topics).

The main result of this paper is the following:

Theorem 1.1. *Let U be an $n \times m$ matrix and denote by \tilde{U} the matrix whose columns are the columns of U normalized. For all $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, m\}$ of size*

$$|\sigma| \geq (1 - \varepsilon)^2 \frac{\|U\|_{\text{HS}}^2}{\|U\|^2}$$

such that

$$\frac{\varepsilon}{2 - \varepsilon} \leq s_{\min}(\tilde{U}_\sigma) \leq s_{\max}(\tilde{U}_\sigma) \leq \frac{2 - \varepsilon}{\varepsilon}$$

In other terms, for all $(a_j)_{j \in \sigma}$

$$\frac{\varepsilon}{2 - \varepsilon} \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \in \sigma} a_j \frac{U e_j}{\|U e_j\|_2} \right\|_2 \leq \frac{2 - \varepsilon}{\varepsilon} \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}}.$$

Note that the lower bound problem is the restricted invertibility problem treated in [13] and [18] while the upper bound problem is related to the sparsification method and the Kashin-Tzafriri column selection theorem [8] treated respectively in [1] and [18]. Our idea is to merge the two algorithms together to get the two conclusions simultaneously. The heart of these methods is the study of the evolution of the eigenvalues of a matrix when perturbed by a rank one matrix.

In the regime where ε is close to one, the previous result yields the following:

Corollary 1.2. *Let U be an $n \times m$ matrix and denote by \tilde{U} the matrix whose columns are the columns of U normalized. For all $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, m\}$ of size*

$$|\sigma| \geq \frac{\varepsilon^2}{9} \cdot \frac{\|U\|_{\text{HS}}^2}{\|U\|^2}$$

such that

$$1 - \varepsilon \leq s_{\min}(\tilde{U}_\sigma) \leq s_{\max}(\tilde{U}_\sigma) \leq 1 + \varepsilon$$

In other terms, for all $(a_j)_{j \in \sigma}$

$$(1 - \varepsilon) \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \in \sigma} a_j \frac{U e_j}{\|U e_j\|_2} \right\|_2 \leq (1 + \varepsilon) \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}}$$

Where $(e_j)_{j \leq m}$ denotes the canonical basis of \mathbb{R}^m .

This result is also related to the problem of column paving that is partitioning the columns into sets such that each of the corresponding restrictions has "good" bounds on the singular values, in particular such that the singular values are close to one. We will show how our Theorem allows us to recover a result of Tropp [16] (and of Bourgain-Tzafriri [2]) dealing with this problem, using our deterministic method instead of the probabilistic methods used previously.

In a survey [11] on Batson-Spielman-Srivastava's sparsification theorem, Naor asked about giving a proof of another theorem of Bourgain-Tzafriri [2], which is stronger than the restricted

invertibility, using tools from Batson-Spielman-Srivastava's method. The theorem in question is the following:

Theorem C (Bourgain-Tzafriri). *There is a universal constant $c > 0$ such that for every $\varepsilon < 1$ and $n \in \mathbb{N}$ if an operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $\langle Te_i, e_i \rangle = 0$ for all $i \in \{1, \dots, n\}$ then there exists a subset $\sigma \subseteq \{1, \dots, n\}$ satisfying $|\sigma| \geq c\varepsilon^2 n$ and $\|R_\sigma T R_\sigma^*\| \leq \varepsilon \|T\|$.*

Using our result, we will be able to give a deterministic algorithm to solve this problem for symmetric matrices.

The paper is organized as follows: in section 2, we prove our main result, in section 3 we give applications to the local theory of Banach spaces and compare it with Vershynin's result. In section 4, we prove column paving results and finally in section 5 we answer Naor's question.

2. PROOF OF THEOREM 1.1

Note $k = |\sigma| = (1 - \varepsilon)^2 \frac{\|U\|_{\text{HS}}^2}{\|U\|^2}$ and

$$A_k = \sum_{j \in \sigma} s_j (\tilde{U} e_j) \cdot (\tilde{U} e_j)^t,$$

where s_j are positive numbers which will be determined later. Since our aim is to find σ such that the smallest singular value of \tilde{U}_σ is bounded away from zero and its largest one is upper bounded, it is equivalent to try to construct the matrix A_k such that A_k has k eigenvalues bounded away from zero and bounded from above and to estimate the weights s_j . Our construction will be done step by step starting from $A_0 = 0$. So at the beginning, all the eigenvalues of A_0 are zero. At the first step we will try to find a vector v among the columns of \tilde{U} and a weight s such that $A_1 = A_0 + s v v^t$ has one nonzero eigenvalue which have a lower and upper bound. Of course the first step is trivial, since for whatever column we choose the matrix A_1 will have one eigenvalue equal to s . At the second step, we will try to find a vector v among the columns of \tilde{U} and a weight s such that $A_2 = A_1 + s v v^t$ has two nonzero eigenvalues for which we can update the lower and upper bound found in the first step. We will continue this procedure until we construct the matrix A_k .

For a symmetric matrix A such that $b < \lambda_{\min}(A) \leq \lambda_{\max}(A) < u$, we define:

$$\phi(A, b) = \text{Tr} \left(U^t (A - b \cdot \text{Id})^{-1} U \right) \quad \text{and} \quad \psi(A, u) = \text{Tr} \left(U^t (u \cdot \text{Id} - A)^{-1} U \right)$$

For $l \leq k$, we denote by b_l the lower bound of the l nonzero eigenvalues of A_l and by u_l the upper bound i.e $A_l \prec u_l \cdot \text{Id}$ and has l eigenvalues $> b_l$. We also note

$$\phi = \phi(A_0, b_0) = -\frac{\|U\|_{\text{HS}}^2}{b_0} \quad \text{and} \quad \psi = \psi(A_0, u_0) = \frac{\|U\|_{\text{HS}}^2}{u_0},$$

where b_0 and u_0 will be determined later.

As we said before, we want to control the evolution of the eigenvalues so we will make sure to choose a "good" vector so that our bounds b_l and u_l do not move too far. Precisely, we will fix this amount of change and denote it by δ for the lower bound and Δ for the upper bound i.e at the next step the lower bound will be $b_{l+1} = b_l - \delta$ and the upper one $u_{l+1} = u_l + \Delta$. We will choose these two quantities as follows:

$$\delta = (1 - \varepsilon) \frac{b_0}{k} = \frac{b_0 \|U\|^2}{(1 - \varepsilon) \|U\|_{\text{HS}}^2} \quad \text{and} \quad \Delta = (1 - \varepsilon) \frac{u_0}{k} = \frac{u_0 \|U\|^2}{(1 - \varepsilon) \|U\|_{\text{HS}}^2}$$

Our choice of δ is motivated by the fact that after k steps we want the updated lower bound to remain positive but not too small due to some obstructions in the proof. In this case the final lower bound will be

$$b_k = b_{k-1} - \delta = \dots = b_0 - k\delta = \varepsilon b_0$$

The choice of Δ is motivated by the fact that we don't want the upper bound to move too far from the initial one and as for δ we have some obstruction on taking its value too small. The final upper bound will be

$$u_k = u_{k-1} + \Delta = \dots = u_0 + k\Delta = (2 - \varepsilon)u_0$$

Definition 2.1. *We will say that a positive semidefinite matrix A satisfies the l -requirement if the following properties are verified:*

- $A \prec u_l \cdot Id$.
- A has l eigenvalues $> b_l$.
- $\phi(A, b_l) \leq \phi$.
- $\psi(A, u_l) \leq \psi$.

In order to construct A_{l+1} which has $l + 1$ nonzero eigenvalues larger than b_{l+1} and such that $\phi(A_{l+1}, b_{l+1}) \leq \phi(A_l, b_l)$, we may look at the algorithm used for the restricted invertibility problem ([13], [18]) and more precisely, at the condition needed on the vector v to be chosen:

Lemma 2.2. *If A_l has l nonzero eigenvalues greater than b_l and if for some vector v and some positive scalar s we have*

$$(1) \quad G_l(v) := -\frac{v^t (A_l - b_{l+1} \cdot Id)^{-2} v}{\phi(A_l, b_l) - \phi(A_l, b_{l+1})} \cdot \|U\|^2 - v^t (A_l - b_{l+1} \cdot Id)^{-1} v \geq \frac{1}{s}.$$

Then $A_{l+1} = A_l + svv^t$ has $l + 1$ nonzero eigenvalues all greater than b_{l+1} and $\phi(A_{l+1}, b_{l+1}) \leq \phi(A_l, b_l)$.

Now, in order to construct A_{l+1} which has all its eigenvalues smaller than u_{l+1} and such that $\psi(A_{l+1}, u_{l+1}) \leq \psi(A_l, u_l)$ we may look at the algorithm used for the sparsification theorem [1] or the Kashin-Tzafriri column selection theorem (see Theorem 4.2 [18]):

Lemma 2.3. *If $A_l \prec u_l \cdot Id$ and if for some vector v and some positive scalar s we have*

$$(2) \quad F_l(v) := \frac{v^t (u_{l+1} \cdot Id - A_l)^{-2} v}{\psi(A_l, u_l) - \psi(A_l, u_{l+1})} \cdot \|U\|^2 + v^t (u_{l+1} \cdot Id - A_l)^{-1} v \leq \frac{1}{s}.$$

Then denoting $A_{l+1} = A_l + svv^t$, we have $A_{l+1} \prec u_{l+1} \cdot Id$ and $\psi(A_{l+1}, u_{l+1}) \leq \psi(A_l, u_l)$.

The proof of these two lemmas makes use of the Sherman-Morrison formula.

For our problem, we will need to find a vector v satisfying (1) and (2) simultaneously. For that we need to merge these two conditions in one equation:

Lemma 2.4. *If $A_l \prec u_l \cdot Id$ has l eigenvalues greater than b_l and if for some vector v we have*

$$(3) \quad F_l(v) \leq G_l(v)$$

Then taking any s such that $F_l(v) \leq \frac{1}{s} \leq G_l(v)$, then $A_{l+1} = A_l + svv^t$ satisfies the $(l + 1)$ -requirement.

Remark 2.5. *Since A_{l+1} has $l + 1$ nonzero eigenvalues while A_l had only l nonzero eigenvalues, then the vector v chosen is linearly independent with the eigenvectors of A_l . Therefore one can see that $\text{Ker}(A_{l+1}) \subset \text{Ker}(A_l)$ and $\text{Dim}[\text{Ker}(A_{l+1})] = \text{Dim}[\text{Ker}(A_l)] - 1$.*

Proposition 2.6. *Let A_l satisfying the l -requirement. If b_0 and u_0 satisfy*

$$(4) \quad b_0 \leq \frac{\varepsilon u_0}{2 - \varepsilon}$$

then there exists $i \leq m$ and a positive number s_i such that $A_{l+1} = A_l + s_i (\tilde{U}e_i) \cdot (\tilde{U}e_i)^t$ satisfies the $(l+1)$ -requirement.

Proof. According to Lemma 2.4, it is sufficient to find $i \leq m$ such that $F_l(\tilde{U}e_i) \leq G_l(\tilde{U}e_i)$ and then take s_j such that

$$(5) \quad F_l(\tilde{U}e_i) \leq \frac{1}{s_j} \leq G_l(\tilde{U}e_i)$$

Since F_l and G_l are quadratic forms, it is equivalent to find $i \leq m$ such that $F_l(Ue_i) \leq G_l(Ue_i)$. For that, it is sufficient to prove

$$(6) \quad \sum_{j \leq m} F_l(Ue_j) \leq \sum_{j \leq m} G_l(Ue_j)$$

Before estimating $\sum_{j \leq m} F_l(Ue_j)$, let us note that

$$\begin{aligned} \psi(A_l, u_l) - \psi(A_l, u_{l+1}) &= \text{Tr} \left[U^t (u_l \cdot \text{Id} - A_l)^{-1} U \right] - \text{Tr} \left[U^t (u_{l+1} \cdot \text{Id} - A_l)^{-1} U \right] \\ &= \Delta \text{Tr} \left[U^t (u_l \cdot \text{Id} - A_l)^{-1} (u_{l+1} \cdot \text{Id} - A_l)^{-1} U \right] \\ &\geq \Delta \text{Tr} \left[U^t (u_{l+1} \cdot \text{Id} - A_l)^{-2} U \right] \end{aligned}$$

Replacing this in F_l we get

$$\begin{aligned} \sum_{j \leq m} F_l(Ue_j) &= \frac{\sum_{j \leq m} e_j^t U^t (u_{l+1} \cdot \text{Id} - A_l)^{-2} Ue_j}{\psi(A_l, u_l) - \psi(A_l, u_{l+1})} \cdot \|U\|^2 + \sum_{j \leq m} e_j^t U^t (u_{l+1} \cdot \text{Id} - A_l)^{-1} Ue_j \\ &= \frac{\text{Tr} \left[U^t (u_{l+1} \cdot \text{Id} - A_l)^{-2} U \right]}{\psi(A_l, u_l) - \psi(A_l, u_{l+1})} \cdot \|U\|^2 + \text{Tr} \left[U^t (u_{l+1} \cdot \text{Id} - A_l)^{-1} U \right] \\ &\leq \frac{\|U\|^2}{\Delta} + \psi \end{aligned}$$

Now we may estimate $\sum_{j \leq m} G_l(Ue_j)$. We denote by P_l the orthogonal projection onto the image of A_l and Q_l the orthogonal projection onto the kernel of A_l . Note that $\forall l \leq k$ we have the following

$$(7) \quad b_l \leq \delta \frac{\|Q_l U\|_{\text{HS}}^2}{\|U\|^2}$$

Since $Q_0 = \text{Id}$, this fact is true at the beginning by our choice of δ . Taking in account Remark 2.5, at each step $\|Q_l U\|_{\text{HS}}^2$ decreases by at most $\|U\|^2$ so that the right hand side of (7) decreases by at most δ . Since at each step we replace b_l by b_{l+1} , (7) remains true.

Since $\text{Id} = P_l + Q_l$ and P_l, Q_l, A_l commute we can write

$$\begin{aligned} \text{Tr} \left[U^t (A_l - b_{l+1} \cdot \text{Id})^{-2} U \right] &= \text{Tr} \left[U^t P_l (A_l - b_{l+1} \cdot \text{Id})^{-2} P_l U \right] + \text{Tr} \left[U^t Q_l (A_l - b_{l+1} \cdot \text{Id})^{-2} Q_l U \right] \\ &= \text{Tr} \left[U^t P_l (A_l - b_{l+1} \cdot \text{Id})^{-2} P_l U \right] + \frac{\|Q_l U\|_{\text{HS}}^2}{b_{l+1}^2} \end{aligned}$$

Doing the same decomposition for $\phi_l(A_l)$ we get

$$\begin{aligned}\phi(A_l, b_l) &= \text{Tr} \left[U^t P_l (A_l - b_l \cdot \text{Id})^{-1} P_l U \right] + \text{Tr} \left[U^t Q_l (A_l - b_l \cdot \text{Id})^{-1} Q_l U \right] \\ &= \text{Tr} \left[U^t P_l (A_l - b_l \cdot \text{Id})^{-1} P_l U \right] - \frac{\|Q_l U\|_{\text{HS}}^2}{b_l} \\ &:= \phi^P(A_l, b_l) + \phi^Q(A_l, b_l)\end{aligned}$$

Denote $\Lambda_l = \phi(A_l, b_l) - \phi(A_l, b_{l+1})$ and Λ_l^P, Λ_l^Q the corresponding decompositions onto the image part and the kernel part as above. As we did before, we have $\Lambda_l = \Lambda_l^P + \Lambda_l^Q$ and

$$\begin{aligned}\Lambda_l^P &= \text{Tr} \left[U^t P_l (A_l - b_l \cdot \text{Id})^{-1} P_l U \right] - \text{Tr} \left[U^t P_l (A_l - b_{l+1} \cdot \text{Id})^{-1} P_l U \right] \\ &= \delta \text{Tr} \left[U^t P_l (A_l - b_l \cdot \text{Id})^{-1} (A_l - b_{l+1} \cdot \text{Id})^{-1} P_l U \right] \\ &\geq \delta \text{Tr} \left[U^t P_l (A_l - b_{l+1} \cdot \text{Id})^{-2} P_l U \right]\end{aligned}$$

Using (7) we have

$$\Lambda_l^Q = -\frac{\|Q_l U\|_{\text{HS}}^2}{b_l} + \frac{\|Q_l U\|_{\text{HS}}^2}{b_{l+1}} = \frac{\delta \|Q_l U\|_{\text{HS}}^2}{b_l b_{l+1}} \geq \frac{\|U\|^2}{b_{l+1}}$$

Looking at the previous information, we can write

$$\begin{aligned}\sum_{j \leq m} G_l(Ue_j) &= -\frac{\text{Tr} \left[U^t (A_l - b_{l+1} \cdot \text{Id})^{-2} U \right]}{\Lambda_l} \cdot \|U\|^2 - \text{Tr} \left[U^t (A_l - b_{l+1} \cdot \text{Id})^{-1} U \right] \\ &= -\frac{\text{Tr} \left[U^t P_l (A_l - b_{l+1} \cdot \text{Id})^{-2} P_l U \right] + \frac{\|Q_l U\|_{\text{HS}}^2}{b_{l+1}^2}}{\Lambda_l} \cdot \|U\|^2 - \phi_{l+1}(A_l) \\ &\geq -\frac{\frac{\Lambda_l^P}{\delta} + \frac{\delta \|Q_l U\|_{\text{HS}}^2}{b_l b_{l+1}} \left[\frac{b_l}{\delta b_{l+1}} \right]}{\Lambda_l} \cdot \|U\|^2 + \Lambda_l - \phi_l(A_l) \\ &\geq -\frac{\frac{\Lambda_l^P}{\delta} + \Lambda_l^Q \left[\frac{1}{\delta} + \frac{1}{b_{l+1}} \right]}{\Lambda_l} \cdot \|U\|^2 + \Lambda_l^Q - \phi \\ &\geq -\frac{\|U\|^2}{\delta} - \frac{\|U\|^2}{b_{l+1}} + \Lambda_l^Q - \phi \\ &\geq -\frac{\|U\|^2}{\delta} - \phi\end{aligned}$$

Until now we have proven that

$$\sum_{j \leq m} G_l(Ue_j) \geq -\frac{\|U\|^2}{\delta} - \phi \quad \text{and} \quad \sum_{j \leq m} F_l(Ue_j) \leq \frac{\|U\|^2}{\Delta} + \psi$$

So in order to prove (6), it will be sufficient to verify

$$(8) \quad \frac{\|U\|^2}{\Delta} + \psi \leq -\frac{\|U\|^2}{\delta} - \phi$$

Replacing in (8) the values of the corresponding parameters as chosen at the beginning, it is sufficient to prove

$$\frac{(2 - \varepsilon) \|U\|_{\text{HS}}^2}{u_0} \leq \frac{\varepsilon \|U\|_{\text{HS}}^2}{b_0}$$

which is after rearrangement condition (4). □

Keeping in mind that $k = (1 - \varepsilon)^2 \frac{\|U\|_{HS}^2}{\|U\|^2}$, we are ready to finish the construction of A_k . We may iterate Proposition 2.6 starting with $A_0 = 0$. Of course, A_0 satisfies the 0-requirement so by the proposition we can find a column vector and a corresponding scalar to form A_1 satisfying the 1-requirement. Once again we use the proposition to construct A_2 satisfying the 2-requirement. We can continue with this procedure as long as the corresponding lower bound b_l is positive (which is the case for b_k). So after k steps we have constructed $A_k = \sum_{j \in \sigma} s_j (\tilde{U}e_j) \cdot (\tilde{U}e_j)^t$ satisfying the k -requirement which means that

$$A_k \prec u_k Id = (1 - 2\varepsilon)u_0 \text{ and } A_k \text{ has } k \text{ eigenvalues bigger than } b_k = \varepsilon b_0.$$

Now it remains to estimate the weights s_j chosen. This will be done by a trivial calculation:

Lemma 2.7. *For any $l \leq k$ and any unit vector v we have*

$$G_l(v) \leq \frac{1}{\varepsilon b_0} \quad \text{and} \quad F_l(v) \geq \frac{1}{(2 - \varepsilon)u_0}$$

Proof. Write again $Id = P_l + Q_l$ and notice that $\frac{v^t(A_l - b_{l+1}.Id)^{-2}v}{\phi(A_l, b_l) - \phi(A_l, b_{l+1})}$ and $v^t P_l (A_l - b_{l+1}.Id)^{-1} P_l v$ are positive then we have

$$G_l(v) \leq -v^t (A_l - b_{l+1}.Id)^{-1} v \leq -v^t Q_l (A_l - b_{l+1}.Id)^{-1} Q_l v \leq \frac{\|Q_l v\|_2^2}{b_{l+1}} \leq \frac{\|v\|_2^2}{b_k} \leq \frac{1}{\varepsilon b_0}$$

Now since $\frac{v^t(u_{l+1}.Id - A_l)^{-2}v}{\psi(A_l, u_l) - \psi(A_l, u_{l+1})} \geq 0$ then

$$F_l(v) \geq v^t (u_{l+1}.Id - A_l)^{-1} v \geq v^t (u_k.Id)^{-1} v \geq \frac{1}{(2 - \varepsilon)u_0}$$

□

The weights s_j that we have chosen satisfied (5) and therefore by the previous lemma

$$\forall i \leq k, \quad \varepsilon b_0 \leq s_j \leq (2 - \varepsilon)u_0$$

Back to our problem note that

$$\tilde{U}_\sigma \tilde{U}_\sigma^t = \sum_{j \in \sigma} (\tilde{U}e_j) \cdot (\tilde{U}e_j)^t$$

and therefore

$$\frac{1}{(2 - \varepsilon)u_0} A_k = \frac{1}{(2 - \varepsilon)u_0} \sum_{j \in \sigma} s_j (\tilde{U}e_j) \cdot (\tilde{U}e_j)^t \preceq \tilde{U}_\sigma \tilde{U}_\sigma^t \preceq \frac{1}{\varepsilon b_0} \sum_{j \in \sigma} s_j (\tilde{U}e_j) \cdot (\tilde{U}e_j)^t = \frac{1}{\varepsilon b_0} A_k$$

Transferring the properties of A_k , we deduce that $\tilde{U}_\sigma \tilde{U}_\sigma^t \preceq \frac{(2 - \varepsilon)u_0}{\varepsilon b_0} Id$ and $\tilde{U}_\sigma \tilde{U}_\sigma^t$ has k eigenvalues greater than $\frac{\varepsilon b_0}{(2 - \varepsilon)u_0}$.

This means that

$$\frac{\varepsilon b_0}{(2 - \varepsilon)u_0} Id \preceq \tilde{U}_\sigma^t \tilde{U}_\sigma \preceq \frac{(2 - \varepsilon)u_0}{\varepsilon b_0} Id$$

Taking $b_0 = \frac{\varepsilon u_0}{2 - \varepsilon}$ in order to satisfy (4) we finish the proof of Theorem 1.1.

3. APPLICATION TO THE LOCAL THEORY OF BANACH SPACES

As for the restricted invertibility principle where one can interpret the result as the invertibility of an operator on a decomposition of the identity, we will write the result in terms of a decomposition of the identity. This will be useful for applications to the local theory of Banach spaces since by John's theorem [7] one can have a decomposition of the identity formed by contact points of the unit ball with its maximal volume ellipsoid.

Proposition 3.1. *Let $Id = \sum_{i \leq m} y_i y_i^t$ be a decomposition of the identity in \mathbb{R}^n and T be a linear operator on l_2^n . For $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, m\}$ such that*

$$|\sigma| \geq (1 - \varepsilon)^2 \frac{\|T\|_{\text{HS}}^2}{\|T\|^2}$$

and for all $(a_j)_{j \leq k}$,

$$\frac{\varepsilon}{2 - \varepsilon} \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \leq k} a_j \frac{T y_j}{\|T y_j\|_2} \right\|_2 \leq \frac{2 - \varepsilon}{\varepsilon} \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}}.$$

Proof. Let U be the $n \times m$ matrix whose columns are $T y_j$. Therefore we can write

$$UU^t = \sum_{j \leq m} (T y_j) \cdot (T y_j)^t = TT^t$$

We deduce that $\|U\|_{\text{HS}} = \|T\|_{\text{HS}}$ and $\|U\| = \|T\|$. Applying Theorem 1.1 to U , we find $\sigma \subset \{1, \dots, m\}$ such that

$$|\sigma| \geq (1 - \varepsilon)^2 \frac{\|T\|_{\text{HS}}^2}{\|T\|^2}$$

and for all $(a_j)_{j \in \sigma}$,

$$\frac{\varepsilon}{2 - \varepsilon} \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \in \sigma} a_j \frac{U e_j}{\|U e_j\|_2} \right\|_2 \leq \frac{2 - \varepsilon}{\varepsilon} \left(\sum_{j \in \sigma} a_j^2 \right)^{\frac{1}{2}}$$

Noting that $\frac{U e_j}{\|U e_j\|_2} = \frac{T y_j}{\|T y_j\|_2}$, we finish the proof. □

This result improves the dependence on ε in comparison with Vershynin's result [17]. While Vershynin proved that $(T y_j)_{j \in \sigma}$ is $c(\varepsilon)$ -equivalent to an orthogonal basis, the value of $c(\varepsilon)$ was of the order of $\varepsilon^{-c \log(\varepsilon)}$. Here our sequence is $(4\varepsilon^{-2})$ -equivalent to an orthogonal basis.

In the regime where ε is close to one, the previous proposition yields the following:

Corollary 3.2. *Let $Id = \sum_{i \leq m} y_i y_i^t$ a decomposition of the identity in \mathbb{R}^n and T a linear operator on l_2^n . For $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, m\}$ such that*

$$|\sigma| \geq \frac{\varepsilon^2 \|T\|_{\text{HS}}^2}{9 \|T\|^2}$$

and for all $(a_j)_{j \leq k}$,

$$(1 - \varepsilon) \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \leq k} a_j \frac{T y_j}{\|T y_j\|_2} \right\|_2 \leq (1 + \varepsilon) \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}}$$

The two previous results can be written in terms of contact points, let us for instance write the case of $T = Id$. If $X = (\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is a norm on \mathbb{R}^n such that B_2^n is the ellipsoid of maximal volume contained in B_X , then by John's theorem one can get an identity decomposition formed by contact points of B_X with B_2^n . Applying Proposition 3.1 we get the following:

Proposition 3.3. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is a norm on \mathbb{R}^n such that B_2^n is the ellipsoid of maximal volume contained in B_X . For $\varepsilon < 1$, there exists x_1, \dots, x_k contact points of B_X with B_2^n such that*

$$k \geq (1 - \varepsilon)^2 n$$

and for all $(a_j)_{j \leq k}$,

$$\frac{\varepsilon}{2 - \varepsilon} \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \leq k} a_j x_j \right\|_2 \leq \frac{2 - \varepsilon}{\varepsilon} \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}}.$$

In other terms, we can find a system of almost n contact points which is $(4\varepsilon^{-2})$ -equivalent to an orthonormal basis. Looking at the lower bound, this gives a proportional Dvoretzky-Rogers factorization [6] with the best known dependence on ε .

If we are willing to give up on the fact of extracting a large number of contact points, we can have a system of contact points which is $(1 + \varepsilon)$ -equivalent to an orthonormal basis. For that we write the previous proposition in the regime where ε is close to 1.

Corollary 3.4. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is a norm on \mathbb{R}^n such that B_2^n is the ellipsoid of maximal volume contained in B_X . For $\varepsilon < 1$, there exists x_1, \dots, x_k contact points of B_X with B_2^n such that*

$$k \geq \frac{\varepsilon^2 n}{9}$$

and for all $(a_j)_{j \leq k}$,

$$(1 - \varepsilon) \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j \leq k} a_j x_j \right\|_2 \leq (1 + \varepsilon) \left(\sum_{j \leq k} a_j^2 \right)^{\frac{1}{2}}$$

Applying proposition 3.3, we get the following corollary:

Corollary 3.5. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is a norm on \mathbb{R}^n such that B_2^n is the ellipsoid of maximal volume contained in B_X . For $\varepsilon < 1$, there exists x_1, \dots, x_k linearly independent contact points of B_X with B_2^n such that*

$$k \geq (\sqrt{2} - 1)^4 \varepsilon^2 n$$

and for any $i \leq k$,

$$\sum_{j \neq i} \langle x_j, x_i \rangle^2 \leq \varepsilon$$

Proof. Let $\varepsilon < 1$ and denote $\alpha = (\sqrt{2} - 1)^2$. Apply proposition 3.3 with $(1 - \alpha\varepsilon)$ in order to find a system of linearly independent contact points $(x_j)_{j \leq k}$ such that

$$A_k = \sum_{j \leq k} x_j x_j^t \preceq \left(\frac{1 + \alpha\varepsilon}{1 - \alpha\varepsilon} \right)^2 Id$$

For $i \leq k$,

$$\langle A_k x_i, x_i \rangle = 1 + \sum_{j \neq i} \langle x_j, x_i \rangle^2 \leq \left(\frac{1 + \alpha\varepsilon}{1 - \alpha\varepsilon} \right)^2$$

The conclusion follows by a trivial calculation. \square

4. COLUMN PAVING

Extracting a large column submatrix reveals to be useful since the extracted matrix may have better properties. First results in this direction were given by Kashin in [9], and others followed improving or dealing with different properties (see [1], [2], [8], [10],[16]). One can also be interested in partitioning the matrix into disjoint sets of columns such that each block has "good" properties. Obtaining a constant number of blocks (independent of the dimension) turns out to be a difficult problem and many conjectures concerning this were given previously (see [4]).

The previous algorithms for extraction used probabilistic arguments and Grothendieck's factorization theorem. Here we propose a deterministic algorithm to achieve the extraction, we apply our main result iteratively in order to partition the matrix into blocks on each of them we have good estimates on the singular values.

Definition 4.1. *Let U a $n \times m$ matrix. We will say that U is standardized if all its columns are of norm 1.*

Note that when U is standardized we have $\|U\|_{\text{HS}}^2 = m$ and $\|U\| \geq 1$. Applying Theorem 1.1 to a standardized matrix, we get the following proposition:

Proposition 4.2. *Let U a $n \times m$ standardized matrix. For $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, m\}$ with*

$$|\sigma| \geq \frac{(1 - \varepsilon)^2 m}{\|U\|^2}$$

such that

$$\frac{\varepsilon}{2 - \varepsilon} \leq s_{\min}(U_\sigma) \leq s_{\max}(U_\sigma) \leq \frac{2 - \varepsilon}{\varepsilon}$$

In the regime where ε is close to one, the previous proposition yields an almost isometric estimation:

Corollary 4.3. *Let U a $n \times m$ standardized matrix. For $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, m\}$ with*

$$|\sigma| \geq \frac{\varepsilon^2 m}{9\|U\|^2}$$

such that

$$1 - \varepsilon \leq s_{\min}(U_\sigma) \leq s_{\max}(U_\sigma) \leq 1 + \varepsilon$$

Proposition 4.4. *Let U a $n \times m$ standardized matrix. For $\varepsilon < 1$, there exists a partition of $\{1, \dots, m\}$ into p sets $\sigma_1, \dots, \sigma_p$ such that*

$$p \leq \frac{\|U\|^2 \log(m)}{(1 - \varepsilon)^2}$$

and for any $i \leq p$,

$$\frac{\varepsilon}{2 - \varepsilon} \leq s_{\min}(U_{\sigma_i}) \leq s_{\max}(U_{\sigma_i}) \leq \frac{2 - \varepsilon}{\varepsilon}$$

Proof. Apply Proposition 4.2 to U in order to get σ_1 verifying

$$|\sigma_1| \geq \frac{(1 - \varepsilon)^2}{\|U\|^2} m$$

such that

$$\frac{\varepsilon}{2 - \varepsilon} \leq s_{\min}(U_{\sigma_1}) \leq s_{\max}(U_{\sigma_1}) \leq \frac{2 - \varepsilon}{\varepsilon}$$

Now note that $U_{\sigma_1^c}$ is a $n \times |\sigma_1^c|$ standardized matrix and $\|U_{\sigma_1^c}\| \leq \|U\|$ and apply Proposition 4.2 to $U_{\sigma_1^c}$ in order to get $\sigma_2 \subset \sigma_1^c$ verifying

$$|\sigma_2| = \frac{(1 - \varepsilon)^2}{\|U\|^2} |\sigma_1^c| = \frac{(1 - \varepsilon)^2}{\|U\|^2} \left[1 - \frac{(1 - \varepsilon)^2}{\|U\|^2} \right] m$$

Doing this procedure p times, the number of remaining columns is

$$\left(1 - \frac{(1 - \varepsilon)^2}{\|U\|^2} \right)^p m$$

So in order to cover all the columns, we need to take p such that

$$\left(1 - \frac{(1 - \varepsilon)^2}{\|U\|^2}\right)^p m < 1$$

By a trivial calculation, it is sufficient to take $\frac{\|U\|^2 \log(m)}{(1 - \varepsilon)^2}$ blocks. \square

In the regime where ε is close to one, the previous proposition yields a column partition with almost isometric blocks. This recovers a result of Tropp (see Theorem 1.2 in [16]), which follows results of Bourgain-Tzafriri [2], with a deterministic method.

Corollary 4.5. *Let U a $n \times m$ standardized matrix. For $\varepsilon < 1$, there exists a partition of $\{1, \dots, m\}$ into p sets $\sigma_1, \dots, \sigma_p$ such that*

$$p \leq \frac{9\|U\|^2 \log(m)}{\varepsilon^2}$$

and for any $i \leq p$,

$$1 - \varepsilon \leq s_{\min}(U_{\sigma_i}) \leq s_{\max}(U_{\sigma_i}) \leq 1 + \varepsilon$$

The number of blocks here depends on the dimension. The challenging problem is to partition into a number of blocks which does not depend on the dimension. This would give a positive solution to the paving conjecture (see [4] for related problems).

5. EXTRACTING SQUARE SUBMATRIX WITH SMALL NORM

In this section, we will show how using our main result we can answer Naor's question [11]: find an algorithm, using the Batson-Spielman-Srivastava's method [1], to prove Theorem C [2]. However, we will be able to do this only for symmetric matrices.

Proposition 5.1. *Let T a $n \times n$ symmetric matrix with 0 diagonal. For any $\varepsilon < 1$, there exists $\sigma \subset \{1, \dots, n\}$ of size*

$$|\sigma| \geq \frac{(\sqrt{2} - 1)^4 \varepsilon^2 n}{2}$$

such that

$$\|P_\sigma T P_\sigma^*\| \leq \varepsilon \|T\|$$

Proof. Denote $A = T + \|T\|.Id$, then A is a positive semidefinite symmetric matrix so we may take $U = A^{\frac{1}{2}}$. First note that since T has 0 diagonal then

$$\|Ue_i\|_2^2 = \langle Ue_i, Ue_i \rangle = \langle Ae_i, e_i \rangle = \|T\|$$

Therefore $\tilde{U} = \frac{U}{\|T\|^{\frac{1}{2}}}$ is a standardized matrix. Moreover $\|\tilde{U}\|^2 = 2$.

Denote $\alpha = (\sqrt{2} - 1)^2$ and apply Proposition 4.2 with $1 - \alpha\varepsilon$ to find $\sigma \subset \{1, \dots, n\}$ of size $\frac{\alpha^2 \varepsilon^2 n}{2}$ such that

$$\frac{1 - \alpha\varepsilon}{1 + \alpha\varepsilon} \leq s_{\min}(\tilde{U}_\sigma) \leq s_{\max}(\tilde{U}_\sigma) \leq \frac{1 + \alpha\varepsilon}{1 - \alpha\varepsilon}$$

This means that

$$\left(\frac{1 - \alpha\varepsilon}{1 + \alpha\varepsilon}\right)^2 .Id_\sigma \preceq (\tilde{U}_\sigma)^* \cdot (\tilde{U}_\sigma) \preceq \left(\frac{1 + \alpha\varepsilon}{1 - \alpha\varepsilon}\right)^2 .Id_\sigma$$

Recall that $\tilde{U}_\sigma = \tilde{U}P_\sigma^*$ and $\tilde{U}^* \cdot \tilde{U} = \frac{A}{\|T\|}$. Therefore by the choice of α

$$(1 - \varepsilon)\|T\|.Id_\sigma \preceq P_\sigma A P_\sigma^* \preceq (1 + \varepsilon)\|T\|.Id_\sigma$$

which after rearrangement gives

$$-\varepsilon\|T\| \preceq P_\sigma T P_\sigma^* \preceq \varepsilon\|T\|$$

and finishes the proof. \square

Iterating the previous result, we obtain by a deterministic method the strongest result on the paving problem which is due to Bourgain-Tzafriri ([2], see also [15]) that is every zero-diagonal matrix of size $n \times n$ can be paved with at most $O(\log(n))$ blocks. Once again, we are able to achieve this for symmetric matrices.

Proposition 5.2. *Let T a $n \times n$ symmetric matrix with 0 diagonal. For any $0 < \varepsilon < 1$, there exists a partition of $\{1, \dots, n\}$ into k subsets $\sigma_1, \dots, \sigma_k$ such that*

$$k \leq \frac{2 \log(n)}{(\sqrt{2} - 1)^4 \varepsilon^2}$$

and for any $i \leq k$,

$$\|P_{\sigma_i} T P_{\sigma_i}^*\| \leq \varepsilon \|T\|$$

Proof. As before denote $A = T + \|T\| \cdot Id$ and $U = A^{\frac{1}{2}}$. Note $\tilde{U} = \frac{U}{\|T\|^{\frac{1}{2}}}$ the standardized matrix. Applying Corollary 4.5, we have a column partition for which we do on each block as we did in the previous proposition. The result follows easily. \square

Acknowledgement : I am grateful to my PhD advisor Olivier Guédon for his constant encouragement and his careful review of this manuscript.

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