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Abstract. This work deals with the semilinear equation $-\Delta u + u - u^p = 0$ in $\mathbb{R}^N$, $2 \leq p < \frac{N+2}{N-2}$. We consider the positive solutions which are $\frac{2\pi}{\varepsilon}$-periodic in x_1 and decreasing to 0 in the other variables, uniformly in x_1. Let a periodic configuration of points be given on the x_1-axis, which repel each other as the period tends to infinity. If there exists a solution which has these points as peaks, we prove that the points must be asymptotically uniformly distributed on the x_1-axis. Then, for ε small enough, we prove the uniqueness up to a translation of the positive solution with some peaks on the x_1-axis, for a given minimal period in x_1.

1 Introduction.

We consider the equation

$$-\Delta u + u - u_+^p = 0 \text{ in } \frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1} \tag{1.1}$$

where $u_+ = \max(u, 0)$.
By $\frac{S^1}{\varepsilon}$, we mean that

$$u(x_1 + \frac{2\pi}{\varepsilon}, x') = u(x_1, x')$$

and that

$$\frac{\partial u}{\partial x_1}(x_1 + \frac{2\pi}{\varepsilon}, x') = \frac{\partial u}{\partial x_1}(x_1, x').$$

We suppose that

$$u(x_1, x') \to 0, \quad \text{as } |x'| \to 0, \text{ uniformly in } x_1.$$

If $u > 0$, we know that u is radial and decreasing in x' ([7], [2]). We consider the subcritical case

$$2 \leq p < \frac{N+2}{N-2} \quad \text{for } N \geq 3, \quad p \geq 2 \quad \text{for } N = 2.$$

We assume that $p \geq 2$ instead of $p > 1$ for some technical reasons.
Let U be the groundstate solution in $\mathbb{R}^N$. It verifies

$$-\Delta U + U - U^p = 0 \quad \text{in } \mathbb{R}^N.$$

It is known that U is positive, radial and tending to 0 at infinity. Moreover the behavior at infinity is

$$|x|^{\frac{N-1}{2}} e^{|x|} U(x) \to L_0 \quad \text{as } |x| \to +\infty$$

and

$$|x|^{\frac{N-1}{2}} e^{|x|} \frac{\partial U}{\partial x_1}(x) \to L_1 \quad \text{as } |x| \to +\infty,\ x_1 > 0,$$

for some positive limits L_0 and L_1. (see [9].)
Several recent articles deal with the construction of positive solutions for the equation

$$-\Delta u + u - u^p = 0 \quad \text{in } \mathbb{R}^N.$$

Let us refer to [5], [10], [6].
Let us call the Dancer solution the positive solution of (1.1) which is $\frac{2\pi}{\varepsilon}$-periodic in x_1, tending to 0 as $|x'| \to +\infty$, even in x_1 and decreasing in x_1 in $[0, \frac{\pi}{\varepsilon}]$. This solution, that we call u_D was constructed in [5] by a bifurcation from the ground-state solution in $\mathbb{R}^{N-1}$. The Dancer solution exists when $0 < \varepsilon < \varepsilon^\star$, where $\varepsilon^\star$ is a known threshold. We have

$$\|u_D - U\|_{L^\infty(]-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}[\times \mathbb{R}^{N-1})} \to 0 \text{ as } \varepsilon \to 0. \tag{1.2}$$

For all x' the fonction $x_1 \mapsto u_D(x_1, x')$ reaches its maximum value at the points $\frac{l2\pi}{\varepsilon}$, $l \in \mathbb{Z}$ and reaches its minimum value at the points $\frac{l\pi}{\varepsilon}$.

Now, for any $\varepsilon > 0$, for any $k \geq 2$, let a_ε^i, $i = 1,...,k$, be k points of $[-\pi, \pi[$ which are such that

$$\frac{a_\varepsilon^{i+1} - a_\varepsilon^i}{\varepsilon} \to +\infty \quad \text{as } \varepsilon \to 0,\ i = 0, ..., k+1. \tag{1.3}$$

where we denote $a_\varepsilon^0 = a_\varepsilon^k - 2\pi$ and $a_\varepsilon^{k+1} = a_\varepsilon^0 + 2\pi$.
Let us denote

$$U_i(x_1, x') = U(x_1 - \frac{a_\varepsilon^i}{\varepsilon}, x').$$

Let us give the following

Definition 1.1 *The solution u of (1.1) admits the points $\frac{a_\varepsilon^1}{\varepsilon}, ..., \frac{a_\varepsilon^k}{\varepsilon}$ as peaks if $\frac{a_\varepsilon^1}{\varepsilon}, ..., \frac{a_\varepsilon^k}{\varepsilon}$ are k points of $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}[$ verifying (1.3) and if*

$$\|u - \sum_{i=1}^{k} U_i\|_{L^\infty(]-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}[\times \mathbb{R}^{N-1})} \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.4}$$

Let us remark that by the Maximum Principle, any solution of (1.1) verifying (1.4) needs to be positive.
We can ask whether for any configuration of points in a period which repel each other in the sense of (1.3), there exists a solution having these points as peaks. We give a negative answer. In particular it is not possible to consider peaks which repel each other with an infinitely small speed wrt the period.
Our main result is the following uniqueness result for the small values of ε.

Theorem 1.1 *Let u be a solution of (1.1) that admits the points $\frac{a_\varepsilon^1}{\varepsilon}, ..., \frac{a_\varepsilon^k}{\varepsilon}$ in $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}[$ as peaks in the sense of the definition 1.1. Then, for ε small enough there exists $\alpha_\varepsilon \to 0$ such that*

$$u(x_1, x') = u_D(x_1 - \frac{a_\varepsilon^1}{\varepsilon} - \alpha_\varepsilon, x')$$

where u_D *is the Dancer solution of period* $\frac{2\pi}{k\varepsilon}$.
We can write u_D *as*

$$u_D(x) = \sum_{l\in\mathbb{Z}} U(x_1 + \frac{2l\pi}{k\varepsilon}, x') + \psi(x) \tag{1.5}$$

and if we define

$$d_x = dist(x, \cup_{l\in\mathbb{Z}}\{(\frac{2l\pi}{k\varepsilon}, 0)\}),$$

then for every $0 < \eta' \leq \min\{2-\eta, 2(p-1-\eta)\}$, *there exists* C *independent of* ε *such that*

$$(|\psi| + |\nabla\psi|)(x) \leq Ce^{-\eta d_x} e^{-\frac{\eta'\pi}{k\varepsilon}} (\frac{\pi}{k\varepsilon})^{\frac{1-N}{2}}. \tag{1.6}$$

The most involved part of the proof of Theorem 1.1 is to prove that the peaks are asymptotically uniformly distributed. More precisely, we will begin by the proof of the following

Proposition 1.1 *Let* u *be a solution of (1.1) admetting the points* $\frac{a^1_\varepsilon}{\varepsilon}, ..., \frac{a^k_\varepsilon}{\varepsilon}$ *in* $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}[$ *as peaks.*
Then we have necessarily

$$\frac{a^{i+1}_\varepsilon - a^i_\varepsilon}{\varepsilon} - \frac{2\pi}{k\varepsilon} \to 0, \quad i = 0, ..., k+1. \tag{1.7}$$

In [10], part 3, Malchiodi gives a construction of a periodic solution with one peak, using a Lyapunov-Schmitt method.

Let us quote the following

Proposition 1.2 *(Malchiodi, [10], Corollary 3.2.) For* $1 < p < \frac{N+2}{N-2}$, *there exists a solution of (1.1), even in* x_1, *of the form*

$$v = \sum_{i\in\mathbb{Z}} U(x_1 + i\frac{2\pi}{\varepsilon}, x') + \overline{w} \tag{1.8}$$

where

$$\|\overline{w}\|_{H^1(]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1})} \to 0$$

and

$$|\overline{w}(x)| + |\nabla\overline{w}(x)| \leq Ce^{-\frac{\pi}{\varepsilon}(1+\xi_0)} e^{-\eta_0 \, dist(x, \cup_{l\in\mathbb{Z}}\{(\frac{2l\pi}{\varepsilon}, 0)\})} \tag{1.9}$$

for some $\xi_0 > 0$ *and* $\eta_0 > 0$.

This solution is the Dancer solution, in consideration of the uniqueness of the even $\frac{2\pi}{\varepsilon}$-periodic solution which verifies (1.2) (see [4], p. 969). In that previous work, the functions are assumed to be even in x_1. In ours, we have to overcome some difficulties arising from the lack of evenness. Finally, we prove that the solution is even.

In the course of the proof of Theorem 1.1, we will consider an approximate solution of (1.1).

Let us denote

$$U_{i,l} = U(x_1 - \frac{a_\varepsilon^i + 2\pi l}{\varepsilon}, x'), \quad i = 1, ..., k, \quad l \in \mathbb{Z},$$

then

$$U_{i,0} = U_i.$$

Let us define

$$v_i = \sum_{l \in \mathbb{Z}} U_{i,l} \quad \text{and} \quad \overline{u}_\varepsilon = \sum_{i=1}^{k} v_i.$$

We will study the linearized operator about this approximate solution, namely

$$\mathbb{L} = -\Delta + 1 - p\overline{u}_\varepsilon^{p-1}.$$

We will prove that the linearized operator $\mathbb{L}$ has no zero eigenvalue and we will give an estimate of the eigenvalues which tend to 0.
The operator $(-\Delta + I)^{-1}\mathbb{L}$ is an operator of $H^1(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})$ into itself of the form $id - \mathcal{K}$, where $\mathcal{K}$ is a compact operator. So $(-\Delta + I)^{-1}\mathbb{L}$ is a Fredholm operator of index 0.
We consider the eigenvalues of the operator $\mathbb{L}$, in the following sense

there exists $\xi \in H^1(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})$, $\xi \neq 0$, verifying $\mathbb{L}\xi = \lambda(-\Delta + 1)\xi$.

The operator $\mathbb{L}$ has a countably infinite discrete set of eigenvalues, λ_i, $i = 1, 2....$ If we designate by V_i the eigenspace corresponding to λ_i, by H^1 the space $H^1(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})$ and by L^2 the space $L^2(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})$, then

$$\lambda_i = \inf\{\frac{< \mathbb{L}u, u >_{L^2}}{< u, u >_{H^1}}, u \neq 0, < u, v >_{H^1} = 0, \forall v \in V_1 \oplus ... \oplus V_{i-1}\}, \quad \text{for } i \geq 2, \quad (1.10)$$

and

$$\lambda_1 = \inf\{\frac{< \mathbb{L}u, u >_{L^2}}{< u, u >_{H^1}}, u \neq 0\},$$

(see [8]). Let us quote the following result concerning the eigenvalues of the operator $-\Delta + 1 - pU^{p-1}$ (with the definition above, with $\mathbb{R}^N$ instead of $\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1}$ and when $k = 1$ and $a_\varepsilon^1 = 0$)

Theorem 1.2 *The first eigenvalue of $-\Delta + 1 - pU^{p-1}$ in $\mathbb{R}^N$ is $1 - p$. The eigenspace associated with the eigenvalue 0 is spanned by the eigenvectors $\frac{\partial U}{\partial x_j}$, $j = 1, ..., N$.*

This theorem follows from [1].
Let us define

$$\sigma_i = \frac{1}{2}\text{dist}(\frac{a_\varepsilon^i}{\varepsilon}, \cup_{j \neq i, j=0,...,k+1}\{\frac{a_\varepsilon^j}{\varepsilon}\}) \quad i = 1, ..., k. \quad (1.11)$$

and

$$\underline{\sigma} = \min_{i=1}^{k} \sigma_i.$$

Let us summarize the properties of the eigenvalues and of the eigenvectors of $\mathbb{L}$ in the following

Theorem 1.3 *(i)The eigenvalues of $\mathbb{L}$ are less than 1. There exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ such that each eigenvalue of $\mathbb{L}$ tends either to 1 or to an eigenvalue of $-\Delta + 1 - pU^{p-1}$ as $\varepsilon_m \to 0$.*

(ii) Let F be the vector space associated with the eigenvalues tending to 0. Then the dimension of F is k and F is spanned by k eigenvectors φ_i, $i = 1, ..., k$ such that there exist k real numbers $\alpha_i \neq 0$, independent of ε, verifying

$$< \varphi_i, \varphi_j >_{H^1(\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1})} = 0 \quad i \neq j \quad ; \quad \|\varphi_i\|_\infty = 1 \tag{1.12}$$

and

$$\|\varphi_i - \alpha_i \frac{\partial v_i}{\partial x_1}\|_{L^q(\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1})} + \|\nabla(\varphi_i - \alpha_i \frac{\partial v_i}{\partial x_1})\|_{L^q(\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1})} \to 0 \tag{1.13}$$

for all $1 \leq q \leq \infty$.

(iii) If $\lambda_i(\varepsilon_m) \to 0$, then $\lambda_i(\varepsilon_m) \neq 0$ and

$$\lambda_i(\varepsilon_m) e^{-2\sigma_i} \sigma_i^{\frac{1-N}{2}} \to H \tag{1.14}$$

where $H \neq 0$.

The paper is organized as follows. In section 2, we study the eigenvalues of the operator $\mathbb{L}$ which tend to 0 and the associated eigenvectors. We give the proof of Theorem 1.3. In section 3, we use a Lyapunov-Schmitt method to give the proof of Proposition 1.1. In section 4, we conclude the proof of Theorem 1.1.

In sections 2 and 3, we will refer to some technical results, which are reported in the appendix (section 5).

2 An analysis of the eigenvalues.

In this part, we prove the theorem 1.3.

Proof of (i).

Let φ be such that

$$\mathbb{L}\varphi = \lambda(-\Delta\varphi + \varphi) \quad \text{in } \frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1}.$$

We suppose that there exists c such that $\varphi(c) = 1$ and that $\|\varphi\|_\infty = 1$. We denote

$$\phi(x) = \varphi(x + c).$$

By standard elliptic estimates, there exists a subsequence such that $\phi \to \overline{\phi}$ uniformly on the compact sets of $\mathbb{R}^N$. Let us suppose that $\lambda \not\to 1$. First, if $|c - (\frac{a_\varepsilon^i}{\varepsilon}, 0)| \to +\infty$ for all i, then

$$-\Delta\overline{\phi} + \overline{\phi} = 0 \quad \text{in } \mathbb{R}^N \quad ; \quad \|\overline{\phi}\|_\infty = \overline{\phi}(0) = 1.$$

This is in contradiction with the maximum principle, so this case does not occur. So there exists $\overline{c}$ and i such that $(c - (\frac{a^i_\varepsilon}{\varepsilon}, 0)) \to \overline{c}$. Then,

$$(-\Delta + 1 - pU^{p-1}(x + \overline{c}))\overline{\phi} = \overline{\lambda}(-\Delta\overline{\phi} + \overline{\phi}) \quad \text{in } \mathbb{R}^N \tag{2.15}$$

and $\overline{\phi}$ is non zero and even in x'. Thus $\overline{\lambda}$ is an eigenvalue of $-\Delta + 1 - pU^{p-1}$.
By a diagonal process, we can construct a subsequence (ε_m) such that any eigenvalue of $\mathbb{L}$ which does not tend to 1 converges to an eigenvalue of $-\Delta + 1 - pU^{p-1}$.

Proof of (ii).

We divide the proof into three parts.
Firstly, let us prove that if $\varphi \in F\backslash\{0\}$, then there exists $I \subset \{1, ..., k\}$ and some real numbers $\alpha_i \neq 0$ and independent of ε such that

$$\|\varphi - \sum_{i \in I} \alpha_i \frac{\partial v_i}{\partial x_1}\|_\infty + \|\nabla(\varphi - \sum_{i \in I} \alpha_i \frac{\partial v_i}{\partial x_1})\|_\infty \to 0 \tag{2.16}$$

We follow the proof of (i) to get (2.15) with $\overline{\lambda} = 0$. Then we can denote $\overline{c}$ instead of $(\overline{c}, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and there exists some real number $\alpha \neq 0$ such that

$$\overline{\phi}(x) = \alpha \frac{\partial U}{\partial x_1}(x_1 + \overline{c}, x').$$

We get

$$\varphi(x + c) - \alpha \frac{\partial U}{\partial x_1}(x + \overline{c}) \to 0 \quad \text{uniformly on the compact sets,}$$

that is

$$\varphi(x + c) - \alpha \frac{\partial U}{\partial x_1}(x + c - \frac{a^i_\varepsilon}{\varepsilon}) \to 0 \quad \text{uniformly on the compact sets,}$$

that leads to

$$\varphi(x) - \alpha \frac{\partial U_i}{\partial x_1}(x) \to 0 \quad \text{uniformly for } x \text{ such that } (x_1 - \tfrac{a^i_\varepsilon}{\varepsilon}) \text{ is bounded.}$$

Finally for each i, either there exists $\alpha_i \neq 0$ such that

$$(\varphi - \alpha_i \frac{\partial v_i}{\partial x_1}) \to 0 \quad \text{uniformly for } x \text{ such that } (x_1 - \tfrac{a^i_\varepsilon}{\varepsilon}) \text{ is bounded}$$

or

$$\varphi \to 0 \quad \text{uniformly for } x \text{ such that } (x_1 - \tfrac{a^i_\varepsilon}{\varepsilon}) \text{ is bounded.}$$

Moreover, the first case occurs for at least one i. By the beginning of the present proof,

$$\varphi(x) \to 0 \quad \text{if } |x - (\tfrac{a^i_\varepsilon}{\varepsilon}, 0)| \to +\infty \ \forall i.$$

Thus there exists $J \subset \{1, ..., k\}$ and $\alpha_i \neq 0$ and independent of ε such that

$$\|\varphi - \sum_{i \in J} \alpha_i \frac{\partial v_i}{\partial x_1}\|_\infty \to 0. \tag{2.17}$$

We deduce that

$$\|\nabla(\varphi - \sum_{i \in J} \alpha_i \frac{\partial v_i}{\partial x_1})\|_\infty \to 0 \tag{2.18}$$

by standard elliptic arguments. Since the functions $\frac{\partial v_i}{\partial x_1}$ are linearly independent, we deduce that

$$\dim F \leq k.$$

Secondly, let us assume that $\varphi_1 \in F$ and $\varphi_2 \in F$ are such that

$$< \varphi_1, \varphi_2 >_{H^1(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})} = 0 \quad \text{and} \quad \|\varphi_1\|_\infty = \|\varphi_2\|_\infty = 1.$$

We write

$$\varphi_1 = \sum_{i \in J_1} \alpha_i \frac{\partial v_i}{\partial x_1} + o(1) \quad \text{and} \quad \varphi_2 = \sum_{i \in J_2} \beta_i \frac{\partial v_i}{\partial x_1} + o(1)$$

in the sense of (2.17) and (2.18). Taking the scalar product in H^1 we obtain

$$0 = \sum_{i \in J_1, j \in J_2} \alpha_i \beta_j < \frac{\partial v_i}{\partial x_1}, \frac{\partial v_j}{\partial x_1} >_{H^1} + < o(1), \sum_{i \in J_1} \alpha_i \frac{\partial v_i}{\partial x_1} >_{H^1} + < o(1), \sum_{i \in J_2} \beta_i \frac{\partial v_i}{\partial x_1} >_{H^1}$$

$$+ < o(1), o(1) >_{H^1}.$$

In view of Proposition 5.7, the Lebesgue Theorem leads to

$$0 = \sum_{i \in J_1 \cap J_2} \alpha_i \beta_i \|\frac{\partial U}{\partial x_1}\|^2_{H^1(\mathbb{R}^N)}. \tag{2.19}$$

We deduce that $J_1 \cap J_2 = \emptyset$.

Thirdly, let us assume that $F \neq \{0\}$. We define a finite set J and eigenvalues λ_j, $j \in J$ such that $\lambda_j(\varepsilon_m) \to 0$. Let φ_j be an eigenvector associated with λ_j. Let us assume that

$$< \varphi_i, \varphi_j >_{H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})} = 0 \quad i \neq j \quad ; \quad \|\varphi_i\|_{L^\infty([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})} = 1.$$

We write

$$\frac{\partial v_i}{\partial x_1} = \sum_{j \in J} c_j \varphi_j + \xi \quad ; \quad < \xi, \varphi_j >= 0 \text{ for } j \in J.$$

We have

$$\mathbb{L}\xi = \mathbb{L}\frac{\partial v_i}{\partial x_1} - \sum_j c_j \lambda_j (-\Delta \varphi_j + \varphi_j). \tag{2.20}$$

In view of (5.61), we deduce that

$$\|\xi\|_\infty \leq C\|\mathbb{L}\xi\|_\infty$$

and consequently that

$$\|\xi\|_\infty \to 0.$$

We conclude that there exists at least one j such that $c_j \not\to 0$ and such that $\varphi_j((\frac{a_\varepsilon^i}{\varepsilon},0)) \not\to 0$. By the second step, this j is unique. Let us call it i and then we have that

$$\frac{\partial v_i}{\partial x_1} - c_i\varphi_i \to 0 \quad \text{if } |x_1 - \frac{a_{\varepsilon m}^i}{\varepsilon_m}| \text{ is bounded.}$$

Now, since $\|\frac{\partial v_i}{\partial x_1}\|_\infty \not\to 0$, we deduce that $F \neq \{0\}$ and that $\dim F \geq k$. Consequently

$$\dim F = k$$

and we define $(\varphi_1, ..., \varphi_k)$ a basis of F verifying (1.12) and such that

$$\|\varphi_i - \alpha_i\frac{\partial v_i}{\partial x_1}\|_\infty + \|\nabla(\varphi_i - \alpha_i\frac{\partial v_i}{\partial x_1})\|_\infty \to 0$$

with $\alpha_i \neq 0$, independent of ε.
Finally we write

$$\frac{\partial v_i}{\partial x_1} = \sum_{j=1}^{k} c_j\varphi_j + \xi \quad ; \quad <\xi, \varphi_j>_{H^1} = 0 \text{ for all } j \tag{2.21}$$

and $c_i \not\to 0$ and $\|\xi\|_\infty \to 0$. For each $j \neq i$ we have, by our convention, $\varphi_j((\frac{a_\varepsilon^j, 0}{\varepsilon})) \not\to 0$. Since $\frac{\partial v_i}{\partial x_1}$ and ξ tend to 0 at $\frac{a_\varepsilon^j}{\varepsilon}$, we infer that

$$c_j \to 0 \quad \text{for all } j \neq i.$$

Moreover, by Proposition 5.7, φ_i is bounded in L^q and $\|\xi\|_{L^q} \to 0$ for all $q \geq 1$, that gives (1.13).

Proof of (iii).

Let us adopt the case where $k \geq 2$ and where $|\frac{\pi}{\varepsilon} - \sigma_i| \to +\infty$. Otherwise, Proposition 1.1 is irrelevant, and the estimate of λ_i is true, but must be done for the period $\frac{4\pi}{\varepsilon}$ instead of $\frac{2\pi}{\varepsilon}$.

We have by (2.21)

$$\int \mathbb{L}\frac{\partial v_i}{\partial x_1}\varphi_i dx = c_i\lambda_i\|\varphi_i\|_{H^1}^2$$

and $\frac{1}{|c_i|\|\varphi_i\|_{H^1}^2}$ is bounded from below, in view of (5.77). We write

$$\int \mathbb{L}\frac{\partial v_i}{\partial x_1}\varphi_i dx = \frac{1}{c_i}\int \mathbb{L}\frac{\partial v_i}{\partial x_1}\frac{\partial v_i}{\partial x_1}dx - \int \mathbb{L}\frac{\partial v_i}{\partial x_1}(\frac{1}{c_i}\frac{\partial v_i}{\partial x_1} - \varphi_i)dx$$

Let us define, for $j = 0, ..., k+1$

$$\Omega_j = \{x \in [-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}; \mathrm{dist}(x, \cup_{l=0}^{k+1}\{(\frac{a_\varepsilon^l}{\varepsilon}, 0)\} = |x - (\frac{a_\varepsilon^j}{\varepsilon}, 0)|\}. \tag{2.22}$$

We have

$$[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1} = \bigcup_{i=0}^{k+1} \Omega_i.$$

We write

$$< \mathbb{L}\frac{\partial v_i}{\partial x_1}, \frac{\partial v_i}{\partial x_1} >_{L^2} = p\sum_l \int_{\Omega_i} (U_{i,l}^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_{i,l}}{\partial x_1}dx \tag{2.23}$$

$$+\sum_{j=0_{j\neq i}}^{k+1} p\sum_l \int_{\Omega_j} (U_{i,l}^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_{i,l}}{\partial x_1}dx.$$

We are going to prove on one hand that

$$\int_{\Omega_i} (U_i^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx = (1-p)\int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j)(\frac{\partial U_i}{\partial x_1})^2 dx + o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}) \tag{2.24}$$

and on the other hand that

$$\sum_{j=0_{j\neq i}}^{k+1}\sum_l |\int_{\Omega_j} (U_{i,l}^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_{i,l}}{\partial x_1}dx| + \sum_{l\neq 0}|\int_{\Omega_i} (U_{i,l}^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_{i,l}}{\partial x_1}dx| \tag{2.25}$$

$$= o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}).$$

Using Lemma 5.2, we write, if $p > 2$,

$$\int_{\Omega_i} (U_i^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx = (1-p)\int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j + \sum_{l\neq 0} U_{i,l})\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx$$

$$+O(\int_{\Omega_i}(\sum_{j\neq i} v_j + \sum_{l\neq 0} U_{i,l})^{\min\{p-1,2\}}\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx)$$

while, if $p = 2$

$$\int_{\Omega_i} (U_i^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx = (1-p)\int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j + \sum_{l\neq 0} U_{i,l})\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx.$$

Now, we use Proposition 5.9 to get, when $p - 1 > 1$

$$\int_{\Omega_i}(\sum_{j\neq i} v_j + \sum_{l\neq 0} U_{i,l})^{\min\{p-1,2\}}\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx = O(e^{-2\sigma_i\min\{p-1,2\}}\sigma_i^{\frac{(1-N)\min\{p-1,2\}}{2}}).$$

Since $\frac{\pi}{\varepsilon} - \sigma_i \to +\infty$, we have

$$e^{-\frac{2\pi}{\varepsilon}} = o(e^{-2\sigma_i}).$$

Using Proposition 5.9 again, we obtain for all $p \geq 2$

$$\int_{\Omega_i} U_i^{p-2}(\sum_{l\neq 0} U_{i,l})\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx = o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}})$$

and

$$\int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j)\frac{\partial v_i}{\partial x_1}\frac{\partial U_i}{\partial x_1}dx = \int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j)(\frac{\partial U_i}{\partial x_1})^2dx + o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}).$$

We have proved (2.24).

Let us turn now to the proof of (2.25).

We estimate, for $l \neq 0$ and using Proposition 5.9

$$\int_{\Omega_i} (U_{i,l}^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_{i,l}}{\partial x_1}dx = o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}).$$

Now, let $0 < \beta < 1$ be given. We have for all l and for $j = 0, ..., k+1$, $j \neq i$

$$|\int_{\Omega_j} (U_{i,l}^{p-1} - \overline{u}_\varepsilon^{p-1})\frac{\partial v_i}{\partial x_1}\frac{\partial U_{i,l}}{\partial x_1}dx| \leq \int_{\Omega_j} (U_{i,l}^{p-1} + \overline{u}_\varepsilon^{p-1})|\frac{\partial v_i}{\partial x_1}|^\beta|\frac{\partial U_{i,l}}{\partial x_1}|dx\|\frac{\partial v_i}{\partial x_1}\|_{L^\infty(\Omega_j)}^{1-\beta}.$$

But

$$|\frac{a_\varepsilon^i}{\varepsilon} + \frac{2\pi l}{\varepsilon} - \frac{a_\varepsilon^j}{\varepsilon}| \geq 2\sigma_i.$$

Choosing β such that $p - 1 + \beta > 1$, we get, using Proposition 5.9

$$\int_{\Omega_j} (U_{i,l}^{p-1} + \overline{u}_\varepsilon^{p-1}))|\frac{\partial v_i}{\partial x_1}|^\beta|\frac{\partial U_{i,l}}{\partial x_1}|dx = O(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}).$$

Since

$$\|\frac{\partial v_i}{\partial x_1}\|_{L^\infty(|x-(\frac{a_\varepsilon^j}{\varepsilon},0)|<\sigma_j)}^{1-\beta} \to 0$$

we deduce (2.25).

Now (2.24) and (2.25) give

$$< \mathbb{L}\frac{\partial v_i}{\partial x_1}, \frac{\partial v_i}{\partial x_1} >_{L^2} = \int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j)(\frac{\partial U_i}{\partial x_1})^2dx + o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}). \qquad (2.26)$$

Using Corollary 5.1, we get

$$e^{2\sigma_i}\sigma_i^{\frac{-1+N}{2}}\int_{\Omega_i} U_i^{p-2}(\sum_{j\neq i} v_j)(\frac{\partial U_i}{\partial x_1})^2dx \to H_i$$

where $H_i \neq 0$ is a real number.

It remains now to prove that

$$\int \mathbb{L}\frac{\partial v_i}{\partial x_1}(\frac{1}{c_i}\frac{\partial v_i}{\partial x_1} - \varphi_i)dx = o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}). \tag{2.27}$$

We write

$$|\int_{\Omega_i} \mathbb{L}\frac{\partial v_i}{\partial x_1}(\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1})dx| = (1-p)|\int_{\Omega_i} U_i^{p-2}\sum_{j\neq i} v_j \frac{\partial U_i}{\partial x_1}(\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1})dx|$$

$$+o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}})$$

$$\leq \int_{\Omega_i}\sum_{j\neq i} e^{-|x-(\frac{a_\varepsilon^j}{\varepsilon},0)|}|x-(\frac{a_\varepsilon^j}{\varepsilon},0)|^{\frac{1-N}{2}}e^{-|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|}|\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1}|dx + o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}})$$

$$= O(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}})\int_{\Omega_i}|\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1}|dx + o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}) = o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}).$$

For $j \neq i$, we write

$$|\int_{\Omega_j} \mathbb{L}\frac{\partial v_i}{\partial x_1}(\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1})dx| \leq C\sum_l \int_{\Omega_j}|(U_{i,l}^{p-1} + U_j^{p-1})\frac{\partial U_{i,l}}{\partial x_1}(\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1})|dx$$

$$\leq C\int_{\Omega_j} e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}}|\varphi_i - \frac{1}{c_i}\frac{\partial v_i}{\partial x_1}|dx = o(e^{-2\sigma_i}\sigma_i^{\frac{1-N}{2}})$$

We obtain (2.27) and consequently, we have proved (1.14).

3 The Lyapunov-Schmitt reduction.

In this part, we prove the proposition 1.1.

Let us define

$$\mathcal{M}(u) = -\Delta u + u - u_+^p.$$

To begin with, we have

Lemma 3.1

$$\|\mathcal{M}(\overline{u}_\varepsilon)\|_\infty \leq Ce^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}. \tag{3.28}$$

Proof.

$$\mathcal{M}(\overline{u}_\varepsilon) = \sum_{i,l} U_{i,l}^p - (\sum_{i,l} U_{i,l})^p. \tag{3.29}$$

We have for all i

$$\sum_{j\neq i; l\in\mathbb{Z}} U_{j,l} + \sum_{l\in\mathbb{Z}^\star} U_{i,l} \leq Ce^{-|\frac{a_\varepsilon^i - a_\varepsilon^j}{\varepsilon}|}|\frac{a_\varepsilon^i - a_\varepsilon^j}{2\varepsilon}|^{\frac{1-N}{2}}e^{|x-\frac{a_\varepsilon^i}{\varepsilon}|} \quad \text{in } \Omega_i.$$

In Ω_i, we write by Lemma 5.2

$$\mathcal{M}(\overline{u}_\varepsilon) = -pU_i^{p-1}(\sum_{j\neq i;l\in\mathbb{Z}} U_{j,l} + \sum_{l\in\mathbb{Z}^\star} U_{i,l}) + 0(\sum_{j\neq i;l\in\mathbb{Z}} U_{j,l} + \sum_{l\in\mathbb{Z}^\star} U_{i,l})^2 + \sum_{j\neq i;l\in\mathbb{Z}} U_{j,l}^p$$

while $p-1 \geq 1$. We easily deduce the proof of the Lemma.

Let k real numbers δ_1,...,δ_k and $v \in H^1(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})$ be given. Let us suppose that

$$u = \overline{u}_\varepsilon + v + \sum_{i=1}^k \delta_i\varphi_i; \quad < v, \varphi_i >_{H^1} = 0, \quad i = 1, ..., k \tag{3.30}$$

is a solution of (1.1) and that

$$\|v\|_\infty + \sum_{i=1}^k |\delta_i| \to 0.$$

We define

$$h = -\mathcal{M}(\overline{u}_\varepsilon + v + \sum_{i=1}^k \delta_i\varphi_i) + \mathbb{L}(v + \sum_{i=1}^k \delta_i\varphi_i). \tag{3.31}$$

Then v and δ_1,...,δ_k are such that

$$\mathbb{L}(v + \sum_{i=1}^k \delta_i\varphi_i) = h.$$

We denote

$$h = h^\perp + h^\top \quad , \quad h^\top \in \text{Vect}\{\varphi_1, ..., \varphi_k\} \quad , \quad h^\perp \in (\text{Vect}\{\varphi_1, ..., \varphi_k\})^\perp,$$

relatively to the Hilbert space $H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$. First, v is a $\frac{2\pi}{\varepsilon}$-periodic solution of the equation

$$\begin{cases} \mathbb{L}v = h^\perp \\ < v, \varphi_i >_{H^1} = 0, \quad i = 1, ..., k. \end{cases} \tag{3.32}$$

Then $(\delta_1, ..., \delta_k)$ verifies

$$\mathbb{L}(\sum_{i=1}^k \delta_i\varphi_i) = h^\top.$$

We have the following

Proposition 3.3 *Let v be a solution of (3.32). Then there exists C independent of ε such that*

$$\textit{if } p > 2 \quad \|v\|_{H^1} \leq C(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}} + \sum_{i=1}^k |\delta_i|^2) \tag{3.33}$$

$$\textit{if } p = 2 \ \forall\eta \in]0,1[, \quad \|v\|_{H^1} \leq C(e^{-2\eta\underline{\sigma}} + \sum_{i=1}^k |\delta_i|^2)$$

and for all p

$$\|v\|_\infty + \|\nabla v\|_\infty \leq C(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}} + \sum_{i=1}^k |\delta_i|^2). \tag{3.34}$$

Proof.
We write

$$h = \Delta \overline{u}_\varepsilon - \overline{u}_\varepsilon + (\overline{u}_\varepsilon + v + \sum \delta_i \varphi_i)_+^p - p\overline{u}_\varepsilon^{p-1}(v + \sum \delta_i \varphi_i)$$

and Lemma 5.2 gives

$$|(\overline{u}_\varepsilon + v + \sum \delta_i \varphi_i)_+^p - \overline{u}_\varepsilon^p - p\overline{u}_\varepsilon^{p-1}(v + \sum \delta_i \varphi_i)| \leq C|v + \sum \delta_i \varphi_i|^2.$$

We deduce that

$$|h + \mathcal{M}(\overline{u}_\varepsilon)| \leq C|v + \sum \delta_i \varphi_i|^2. \tag{3.35}$$

Since

$$h^\perp = h - \sum_{i=1}^{k} \frac{< h, \varphi_i >}{\|\varphi_i\|_{H^1}^2}(-\Delta \varphi_i + \varphi_i),$$

then

$$\|h^\perp\|_{L^2} \leq C\|h\|_{L^2} \quad \text{and} \quad \|h^\perp\|_\infty \leq C\|h\|_\infty.$$

Now, we use (5.61) to obtain

$$\|v\|_\infty \leq C\|h^\perp\|_\infty.$$

Using (3.28), we deduce the estimate

$$\|v\|_\infty \leq C(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}} + \sum_{i=1}^{k} |\delta_i|^2) \tag{3.36}$$

and the estimate (3.34) follows in the standard way.
We have also by (5.64)

$$\|v\|_{H^1} \leq C\|h^\perp\|_{L^2}.$$

We write

$$\|h\|_{L^2}^2 \leq C(\|\mathcal{M}(\overline{u}_\varepsilon)\|_{L^2}^2 + \|v\|_{L^4}^4 + \sum_j |\delta_j|^4).$$

Using (3.36), we deduce, for ε small enough

$$\|v\|_{H^1} \leq C(\|\mathcal{M}(\overline{u}_\varepsilon)\|_{L^2} + \sum_j |\delta_j|^2).$$

Now we use (3.29) and Lemma 5.2 to obtain

$$\|\mathcal{M}(\overline{u}_\varepsilon)\|_{L^2}^2 \leq \sum_{i=0}^{k+1} \int_{\Omega_i} pU_i^{2(p-1)}(\sum_{j\neq i; l \in \mathbb{Z}} U_{j,l} + \sum_{l \in \mathbb{Z}^\star} U_{i,l})^2 + C\int_{\Omega_i}(\sum_{j\neq i; l \in \mathbb{Z}} U_{j,l} + \sum_{l \in \mathbb{Z}^\star} U_{i,l})^4.$$

Proposition 5.9 gives

$$\|\mathcal{M}(\overline{u}_\varepsilon)\|_{L^2} \leq C(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}) \quad \text{if } p > 2 \tag{3.37}$$

and for all $\eta \in]0,1[$

$$\|\mathcal{M}(\overline{u}_\varepsilon)\|_{L^2} \leq C(e^{-2\eta\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}) \quad \text{if } p=2. \tag{3.38}$$

We deduce (3.33).
We have proved the proposition.

Now let d_i be defined by

$$h^\top = \sum_{i=1}^{k} d_i(-\Delta\varphi_i + \varphi_i).$$

We have the following

Proposition 3.4 *For $i = 1, ..., k$*

$$d_i = \frac{p}{\|\varphi_i\|_{H^1}^2}\int_{\Omega_i} U_i^{p-1}\sum_{j\neq i} v_j \frac{\partial U_i}{\partial x_1}dx + O(\sum_{j=1}^{k}\delta_j^2) + o(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}). \tag{3.39}$$

Proof. We have

$$d_i = \frac{1}{\|\varphi_i\|_{H^1}^2}\int_{\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1}}(h\varphi_i)dx.$$

$$d_i = (\int_{\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1}}(h+\mathcal{M}(\overline{u}_\varepsilon))\varphi_i dx - \int_{\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1}}\mathcal{M}(\overline{u}_\varepsilon)\varphi_i dx)\frac{1}{\|\varphi_i\|_{H^1}^2} \tag{3.40}$$

The coefficient $\|\varphi_i\|_{H^1}$ does not matter, thanks to (5.77). We deduce from (3.34) and (3.35) that

$$\int_{\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1}}(h+\mathcal{M}(\overline{u}_\varepsilon))\varphi_i dx = O(e^{-4\eta\underline{\sigma}}\underline{\sigma}^{1-N} + \sum_j^{k}\delta_j^2). \tag{3.41}$$

Now let us estimate the second integral, for $i = 1, ..., k$.
Without loss of generality, we let $i = 1$. We write

$$\int_{]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}}\mathcal{M}(\overline{u}_\varepsilon)\varphi_1 dx = \int_{]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}}\mathcal{M}(\overline{u}_\varepsilon)(\varphi_1 - \alpha_1\frac{\partial v_1}{\partial x_1})dx + \alpha_1\int_{]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}}\mathcal{M}(\overline{u}_\varepsilon)\frac{\partial v_1}{\partial x_1}dx.$$

The estimate (3.28) gives directly

$$\int_{]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}}\mathcal{M}(\overline{u}_\varepsilon)(\varphi_1 - \alpha_1\frac{\partial v_1}{\partial x_1})dx = o(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}), \tag{3.42}$$

since

$$\|\varphi_1 - \alpha_1\frac{\partial v_1}{\partial x_1}\|_{L^1} \to 0.$$

Now, as in the proof of Theorem 1.3, (iii), we write

$$\int_{]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}}\mathcal{M}(\overline{u}_\varepsilon)\frac{\partial v_1}{\partial x_1}dx = \int_{\Omega_1}\mathcal{M}(\overline{u}_\varepsilon)\frac{\partial v_1}{\partial x_1}dx + +\sum_{j=0}^{k+1}{}_{j\neq 1}\, p\int_{\Omega_j}\mathcal{M}(\overline{u}_\varepsilon)\frac{\partial v_1}{\partial x_1}dx$$

We have

$$\mathcal{M}(\overline{u}_\varepsilon) = \sum_{i,l} U_{i,l}^p - (\sum_{i,l} U_{i,l})^p.$$

By Lemma 5.2 we have, in Ω_j,

$$\mathcal{M}(\overline{u}_\varepsilon) = -pU_j^{p-1}(\sum_{i\neq j, l\in\mathbb{Z}} U_{i,l} + \sum_{l\neq 0} U_{j,l}) + O(\sum_{i\neq j, l\in\mathbb{Z}} U_{i,l} + \sum_{l\neq 0} U_{j,l})^2$$

We get, for $j \neq 1$

$$\int_{\Omega_j} \mathcal{M}(\overline{u}_\varepsilon)\frac{\partial v_1}{\partial x_1}dx = O(\|\frac{\partial v_1}{\partial x_1}\|^{\frac{1}{2}}_{L^\infty(\Omega_j)})\int_{\Omega_j} |\frac{\partial v_1}{\partial x_1}|^{\frac{1}{2}} U_j^{p-1}(\sum_{i\neq j,l} U_{i,l} + \sum_{l\neq 0} U_{j,l} + O(e^{-2\sigma_j}\sigma_j^{1-N}))dx$$

$$= o(e^{-2\sigma_j}\sigma_j^{\frac{1-N}{2}}) \tag{3.43}$$

by Proposition 5.9.

Now we write

$$\int_{\Omega_1} \mathcal{M}(\overline{u}_\varepsilon)\frac{\partial v_1}{\partial x_1}dx = o(e^{-2\sigma_1}\sigma_1^{\frac{1-N}{2}}) - p\int_{\Omega_1} U_1^{p-1}(\sum_{j\neq 1} v_j + \sum_{l\neq 0} U_{1,l})\frac{\partial v_1}{\partial x_1}dx. \tag{3.44}$$

Moreover, since $k \geq 2$, we have

$$\frac{\pi}{\varepsilon} - \sigma_1 \to +\infty$$

and consequently

$$e^{\frac{-2\pi}{\varepsilon}} = o(e^{-2\sigma_1}).$$

So, we deduce from Proposition 5.9 that

$$\int_{\Omega_1} U_1^{p-1}\sum_{l\neq 0} U_{1,l}\sum_l \frac{\partial U_{1,l}}{\partial x_1}dx = o(e^{-2\sigma_1}\sigma_1^{\frac{1-N}{2}})$$

and

$$\int_{\Omega_1} U_1^{p-1}\sum_{j\neq 1} v_j \sum_{l\neq 0}\frac{\partial U_{1,l}}{\partial x_1}dx = o(e^{-2\sigma_1}\sigma_1^{\frac{1-N}{2}}).$$

Finally

$$\int_{\Omega_1} U_1^{p-1}(\sum_{j\neq 1} v_j + \sum_{l\neq 0} U_{1,l})\frac{\partial v_1}{\partial x_1}dx = \int_{\Omega_1} U_1^{p-1}\sum_{j\neq 1} v_j\frac{\partial U_1}{\partial x_1}dx + o(e^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}). \tag{3.45}$$

Now (3.40), (3.42), (3.43) and (3.44) give the proof of the proposition.

Proposition 3.5 *Let u be given as in Proposition 1.1 and let δ_1,....,δ_k be defined in (3.30). We can possibly replace the k given points $\frac{a_\varepsilon^1}{\varepsilon}$,....,$\frac{a_\varepsilon^k}{\varepsilon}$ by k points $\frac{b_\varepsilon^1}{\varepsilon}$,....,$\frac{b_\varepsilon^k}{\varepsilon}$ verifying*

$$\frac{a_\varepsilon^i}{\varepsilon} - \frac{b_\varepsilon^i}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0$$

in order to have

$$\delta_i = 0, \quad i = 1, ..., k.$$

Proof. Let

$$u = \overline{u}_\varepsilon + \sum_{j=1}^{k} \delta_j \varphi_j + v$$

be the given solution of (1.1).
Let us give $(\alpha_1, ..., \alpha_k)$ depending on ε, such that $(\alpha_1, ..., \alpha_k) \to 0$. We can replace the points $\frac{a_\varepsilon^i}{\varepsilon}$ by the points $\frac{a_\varepsilon^i}{\varepsilon} + \alpha_i$. In other words, we write

$$u = \tilde{u}_\varepsilon + \sum_{j=1}^{k} \tilde{\delta}_j \tilde{\varphi}_j + \tilde{v}$$

where

$$\tilde{u}_\varepsilon(x) = \sum_{j=1}^{k} \sum_{l \in \mathbb{Z}} U_{j,l}(x_1 - \alpha_j, x')$$

$$\text{and} \quad \tilde{\varphi}_j, \quad j = 1, ..., k$$

are the eigenfunctions corresponding to the eigenvalues tending to 0, for the configuration of points $\frac{a_\varepsilon^j}{\varepsilon} + \alpha_j$, and

$$< \tilde{v}, \tilde{\varphi}_j >= 0, \quad j = 1, ..., k.$$

Soustraying the expressions of u and performing the scalar product in H^1 by φ_i, we get

$$\tilde{\delta}_i \|\varphi_i\|_{H^1}^2 + \sum_j \tilde{\delta}_j < \tilde{\varphi}_j - \varphi_j, \varphi_i >=< \overline{u}_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > + \delta_i \|\varphi_i\|_{H^1}^2 + < v - \tilde{v}, \varphi_i > . \quad (3.46)$$

First we remark that we have

$$< v - \tilde{v}, \varphi_i >=< \tilde{v}, \tilde{\varphi}_i - \varphi_i >$$

while by (3.33)

$$\|\tilde{v}\|_{H^1} \leq C(e^{-2\eta\underline{\sigma}} \underline{\sigma}^{\frac{1-N}{2}} + \sum_j \tilde{\delta}_j^2),$$

with $\eta = 1$, for $p > 2$, thus

$$| < v - \tilde{v}, \varphi_i > | \leq C(e^{-2\eta\underline{\sigma}} \underline{\sigma}^{\frac{1-N}{2}} + \sum_j \tilde{\delta}_j^2). \quad (3.47)$$

Moreover

$$| < \overline{u}_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > | \leq C \sum_j |\alpha_j|$$

and, as a consequence of (1.13)

$$\|\tilde{\varphi}_j - \varphi_j\|_{H^1} \to 0.$$

Thanks to (3.46), we deduce that

$$\sum_i |\tilde{\delta}_i| \leq C(\sum_j |\delta_j| + \sum_j |\alpha_j| + e^{-2\eta\underline{\sigma}} \underline{\sigma}^{\frac{1-N}{2}}). \quad (3.48)$$

and consequently

$$\|\tilde{v}\|_{H^1} \le C(e^{-2\eta\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}} + \sum_j |\delta_j|^2 + \sum_j |\alpha_j|^2),$$

thus

$$| < v - \tilde{v}, \varphi_i > | \le C(e^{-2\eta\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}} + \sum_j |\delta_j|^2 + \sum_j |\alpha_j|^2) \tag{3.49}$$

for some C independent of ε .
Now let us prove that we can choose $(\alpha_1, ..., \alpha_k)$ such that

$$< \overline{u}_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > + \delta_i \|\varphi_i\|^2 + < v - \tilde{v}, \varphi_i > = 0.$$

We define

$$\mathcal{F}(\alpha_1, ..., \alpha_k) = (< \overline{u}_\varepsilon - \tilde{u}_\varepsilon, \varphi_i >)_{i=1,...,k}.$$

This definition gives, for i and $j = 1, ..., k$

$$\frac{\partial \mathcal{F}_i}{\partial \alpha_j} = \sum_{l\in\mathbb{Z}} \int_{[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}]\times\mathbb{R}^{N-1}} \frac{\partial U_{j,l}}{\partial x_1}(x_1 - \alpha_j, x')\varphi_i(x)dx + \sum_{l\in\mathbb{Z}} \int_{[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}]\times\mathbb{R}^{N-1}} \nabla\frac{\partial U_{j,l}}{\partial x_1}(x_1 - \alpha_j, x').\nabla\varphi_i(x)dx.$$

We deduce that, as $\varepsilon \to 0$

$$\frac{\partial \mathcal{F}_i}{\partial \alpha_i}(0) \to \|\frac{\partial U}{\partial x_1}\|^2_{H^1} \quad \text{and} \quad \frac{\partial \mathcal{F}_i}{\partial \alpha_j}(0) \to 0 \quad \text{for } j \ne i.$$

Thus $d\mathcal{F}(0)$ is an isomorphism, for ε small enough.
Let us define $\alpha = (\alpha_1, ..., \alpha_k)$. We have to solve

$$\mathcal{F}(\alpha) + (\delta_i \|\varphi_i\|^2 + < v - \tilde{v}, \varphi_i >)_{i=1,...,k} = 0.$$

We define

$$Q(\alpha) = \mathcal{F}(\alpha) - \mathcal{F}(0) - d\mathcal{F}(0)(\alpha)$$

and

$$\mathcal{G}(\alpha) = (-d\mathcal{F}(0))^{-1}(Q(\alpha) + (\delta_i \|\varphi_i\|^2 + < v - \tilde{v}, \varphi_i >)_{i,...,k}).$$

Since we have together

$$|Q(\alpha)| = O(|\alpha|^2)$$

and (3.49), we can use the Brouwer fixed point Theorem in a standard way. We find a real number R,

$$R \le C(e^{-2\eta\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}} + \sum_j |\delta_j|)$$

such that

$$(|\alpha| \le R) \Rightarrow |\mathcal{G}(\alpha)| \le R).$$

So we find α, $|\alpha| \le R$, such that $\mathcal{G}(\alpha) = \alpha$, that is

$$< \overline{u}_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > + \delta_i \|\varphi_i\|^2 + < v - \tilde{v}, \varphi_i > = 0.$$

Returning to (3.46), we deduce that $\tilde{\delta}_i = 0$, $i = 1, ..., k$.

Proof of Proposition 1.1. The points are asymptotically uniformly distributed.

From now on, we suppose that $\delta_i = 0$, $i = 1, ..., k$.

Let $i_0 \in \{1, ..., k\}$ be such that

$$\sigma_{i_0} - \underline{\sigma} \to 0$$

(we know that there exists at least one i such that $\sigma_i = \underline{\sigma}$). We have

$$\mathbb{L}(\delta_1\varphi_1 + ... + \delta_k\varphi_k) = h^\top,$$

that is

$$\sum_{i=1}^{k} \delta_i\lambda_i(-\Delta\varphi_i + \varphi_i) = \sum_{i=1}^{k} d_i(-\Delta\varphi_i + \varphi_i)$$

thus

$$\delta_i\lambda_i = d_i \quad \text{for all } i.$$

In particular

$$d_{i_0} = 0.$$

Since, by Theorem 1.3 we have

$$|\lambda_{i_0}| \geq He^{-2\underline{\sigma}}\underline{\sigma}^{\frac{1-N}{2}}$$

we deduce from (3.39) that

$$\int_{\Omega_{i_0}} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx = o(\lambda_{i_0}).$$

We deduce that

$$\int_{\Omega_{i_0};x_1 > \frac{a_\varepsilon^{i_0}}{\varepsilon}} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx$$

$$= -\int_{\Omega_{i_0};x_1 < \frac{a_\varepsilon^{i_0}}{\varepsilon}} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx + o(\lambda_{i_0}).$$

But let us suppose that

$$2\underline{\sigma} = \frac{a_\varepsilon^{i_0+1}}{\varepsilon} - \frac{a_\varepsilon^{i_0}}{\varepsilon}.$$

We use Corollary 5.1 to get a positive real number D_0 such that

$$e^{2\underline{\sigma}}\underline{\sigma}^{\frac{N-1}{2}} \int_{\Omega_{i_0};x_1 > \frac{a_\varepsilon^{i_0}}{\varepsilon}} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx \to D_0$$

and

$$-e^{|\frac{a_\varepsilon^{i_0} - a_\varepsilon^{i_0-1}}{\varepsilon}|}|\frac{a_\varepsilon^{i_0} - a_\varepsilon^{i_0-1}}{\varepsilon}|^{\frac{N-1}{2}} \int_{\Omega_{i_0};x_1 < \frac{a_\varepsilon^{i_0}}{\varepsilon}} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx \to D_0.$$

and consequently

$$2\underline{\sigma} - (\frac{a_\varepsilon^{i_0}}{\varepsilon} - \frac{a_\varepsilon^{i_0-1}}{\varepsilon}) \to 0.$$

This property is valid for all exponent i such that $\sigma_i - \underline{\sigma} \to 0$ instead of i_0. Thus we have (1.7).

4 The proof of Theorem 1.1 completed.

The uniqueness.

Now, we have $\frac{a_\varepsilon^i}{\varepsilon} - (\frac{a_\varepsilon^1}{\varepsilon} + \frac{i2\pi}{k\varepsilon}) \to 0$. Replacing the points $\frac{a_\varepsilon^i}{\varepsilon}$ by $\frac{a_\varepsilon^1}{\varepsilon} + \frac{i2\pi}{k\varepsilon}$, $i = 1, ..., k$, we can write u as

$$u = \sum_{l\in\mathbb{Z}} U(x_1 - \frac{a_\varepsilon^1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + \sum_{i=1}^{k} \tilde{\delta}_i \tilde{\varphi}_i + \tilde{v}, \quad < \tilde{v}, \tilde{\varphi}_i >= 0, \quad i = 1, ..., k.$$

By the definition of $\tilde{\varphi}_i$ given in section 2 (analogue to that of φ_i), $\tilde{\varphi}_i$ is $\frac{2\pi}{\varepsilon}$-periodic in x_1. But now, the corresponding operator $\mathbb{L}$ is of minimal period $\frac{2\pi}{k\varepsilon}$, since now $\overline{u}_\varepsilon$ is replaced by $\sum_{l\in\mathbb{Z}} U(x_1 - \frac{a_\varepsilon^1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x')$. So we have

$$\tilde{\varphi}_i(x) = \tilde{\varphi}_1(x_1 + \frac{2i\pi}{k\varepsilon}, x')$$

and $\tilde{\varphi}_1$ is $\frac{2\pi}{k\varepsilon}$-periodic. Let us denote $\varphi_1 = \tilde{\varphi}$.
Now we recall that

$$\mathbb{L}v = h^\perp$$

with $h = -\mathcal{M}(\overline{u}_\varepsilon) + O(v^2 + (\sum_i \tilde{\delta}_i \tilde{\varphi}_i)^2)$. We can use the Banach fixed point theorem in L^∞ to deduce that v is of minimal period $\frac{2\pi}{k\varepsilon}$.
Consequently, u is $\frac{2\pi}{k\varepsilon}$-periodic and in the space $H^1(\frac{S^1}{k\varepsilon} \times \mathbb{R}^{N-1})$ we write

$$u = \sum_{l\in\mathbb{Z}} U(x_1 - \frac{a_\varepsilon^1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + \tilde{\delta}\tilde{\varphi} + \tilde{v}, \quad < \tilde{v}, \tilde{\varphi} >= 0.$$

Thanks to Proposition 3.5, we can perform a translation in x_1 to get $\tilde{\delta} = 0$. We get some $\frac{a_\varepsilon}{\varepsilon} \to 0$ such that

$$u = \sum_{l\in\mathbb{Z}} U(x_1 - \frac{a_\varepsilon^1}{\varepsilon} - \frac{a_\varepsilon}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + v, \quad < v, \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}) >= 0.$$

Let u_D be the Dancer solution of period $\frac{2\pi}{k\varepsilon}$. Then

$$u_D(x_1 - \frac{a_\varepsilon^1}{\varepsilon}, x') = \sum_{l\in\mathbb{Z}} U(x_1 - \frac{a_\varepsilon^1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + \delta\tilde{\varphi} + v$$

for some v such that $< v, \tilde{\varphi} >= 0$ and some $\delta \to 0$.
Exactly as for u, we find some point $\frac{b_\varepsilon}{\varepsilon} \to 0$ such that

$$u_D(x_1 - \frac{a_\varepsilon^1}{\varepsilon}, x') = \sum_{l\in\mathbb{Z}} U(x_1 - \frac{a_\varepsilon^1}{\varepsilon} - \frac{b_\varepsilon}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + v_D, \quad < v_D, \tilde{\varphi}(x_1 - \frac{b_\varepsilon}{\varepsilon}) >= 0.$$

Now we can prove that for ε small enough,

$$u = u_D(x_1 + \frac{b_\varepsilon - a_\varepsilon - a_\varepsilon^1}{\varepsilon}, x').$$

The proof is the same as for the case of a solution which is even in x_1. Let us write it for the sake of completeness.
Without loss of generality, let $a_\varepsilon^1 = 0$.
We define

$$\overline{u}_D(x) = u_D(x_1 + \frac{b_\varepsilon - a_\varepsilon}{\varepsilon}, x') \quad \text{and} \quad w = u - \overline{u}_D.$$

Let us suppose that $w \neq 0$, at least for a sequence $\varepsilon \to 0$. Then $\|w\|_\infty$ is attained at a point $c = (c_1, c')$, with c' obviously bounded independently of ε and $c_1 \in] - \frac{\pi}{k\varepsilon}, \frac{\pi}{k\varepsilon}]$. Now c_1 is bounded. To see that, we write

$$-\Delta w + w(1 - \frac{u^p - v^p}{u - v}) = 0. \tag{4.50}$$

If $w(c) > 0$, then

$$\frac{u^p - \overline{u}_D^p}{u - \overline{u}_D}(c) \leq pu^{p-1}$$

thus

$$\Delta w(c) \geq w(c)(1 - pu^{p-1}(c)).$$

But if $|c_1| \to +\infty$, we have $u^{p-1}(c) \to 0$, thus

$$1 - pu^{p-1}(c) > 0$$

for ε small enough, that is in contradiction with the Maximum Principle. So we may extract a subsequence such that $c \to \overline{c}$ for some $\overline{c}$.
Let us define

$$z(x) = \frac{w}{\|w\|_\infty}.$$

It verifies (4.50). By standard arguments $z \to \overline{z}$ uniformly on the compact sets. Moreover

$$\overline{z}(\overline{c}) = 1$$

and

$$p\overline{u}_D^{p-1} \leq \frac{u^p - \overline{u}_D^p}{u - \overline{u}_D} \leq pu^{p-1} \quad \text{if } u > \overline{u}_D$$

and we have the reverse inequality if $u < \overline{u}_D$. More,

$$\lim u = \lim \overline{u}_D = U(x).$$

So

$$\lim \frac{u^p - \overline{u}_D^p}{u - \overline{u}_D} = pU.$$

Thus

$$-\Delta \overline{z} + \overline{z}(1 - pU^{p-1}) = 0.$$

We deduce that $\overline{z} = \alpha \frac{\partial U}{\partial x_1}$, for some $\alpha \neq 0$.

We have

$$< z, \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}, x') >_{H^1} = \frac{1}{\|u - \overline{u}_D\|_\infty} < v - v_D(x_1 + \frac{b_\varepsilon - a_\varepsilon}{\varepsilon}, x'), \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}, x') >_{H^1} = 0.$$

Moreover

$$|z| \leq 1, \quad \tilde{\varphi} \to \frac{\partial U}{\partial x_1},$$

and by Proposition 5.7,

$$|\nabla \tilde{\varphi}(x)| + |\tilde{\varphi}(x)| \leq Ce^{-\eta|x|} \quad \text{in} \quad [-\frac{\pi}{k\varepsilon}, \frac{\pi}{k\varepsilon}].$$

We use the Lebesgue Theorem to infer that

$$< z, \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}, x') >_{H^1} \to \alpha \| \frac{\partial U}{\partial x_1} \|^2_{H^1(\mathbb{R}^N)}.$$

So we are led to a contradiction. We conclude that $u = \overline{u}_D$, for ε small enough.

The proof of the estimate 1.6.

Without loss of generality, we let $k = 1$. We write

$$u_D = \sum_l U_l + v$$

where v is even in x_1 and verifies

$$\mathbb{L}v = h$$

and $\overline{u}_\varepsilon$ is replaced by $\sum_l U_l$ in the definition of $\mathbb{L}$. The restriction of $\mathbb{L}$ to the even functions has no eigenvalue tending to 0. The same proof as for (5.61) gives

$$\|v\|_\infty + \|\nabla v\|_\infty \leq C\|h\|_\infty$$

and consequently the same proof as for (3.34) gives

$$\forall \eta \in]0, 1[\quad \|v\|_\infty + \|\nabla v\|_\infty \leq Ce^{-\frac{2\pi}{\varepsilon}} (\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}. \tag{4.51}$$

Let $R_0 > 0$ be given. It remains to estimate, for all $\eta \in]0, 1[$

$$(|v(y)| + |\nabla v(y)|)e^{\eta d_y}$$

when $d_y > R_0$.

Let β be a positive real number, independent of ε, which will be chosen later. Let $y \in]-\frac{\pi}{\varepsilon}+\beta, \frac{\pi}{\varepsilon}-\beta[\times\mathbb{R}^{N-1}$. We follow the course of the proof of Proposition 5.6 from (5.65), ξ being replaced by v. With the notations of that proof, we perform the truncation around 0, using the truncature function θ. So we drop the index i. By (5.67), we have

$$(\theta v)(y) = \int_{\mathbb{R}^N} G(y-x)(p\theta\overline{u}_\varepsilon^{p-1}v + \theta h - \Delta\theta v - 2\nabla\theta.\nabla v)(x)dx$$

Now

$$h = -\mathcal{M}(\overline{u}_\varepsilon) + O(v^2).$$

We have

$$|v(y)| \leq C(\int_{\mathbb{R}^N} vG(y-x)p\theta e^{(-p+1)|x|}dx + \int_{\mathbb{R}^N} G(y-x)\theta|\mathcal{M}(\overline{u}_\varepsilon)|dx + \int_{\mathbb{R}^N} G(y-x)\theta v^2 dx$$

$$+ \int_{\mathbb{R}^N; \frac{\pi}{\varepsilon}-\beta<x_1<\frac{\pi}{\varepsilon}} G(y-x)(|v|+|\nabla v|)dx).$$

Now In Suppθ,

$$d_x = |x| \quad \text{and} \quad U \leq Ce^{-|x|}|x|^{\frac{1-N}{2}}.$$

Let us recall that

$$\mathcal{M}(\overline{u}_\varepsilon) = pU^{p-1}(\sum_{l\neq 0} U_l) + O((\sum_{l\neq 0} U_l)^2,$$

We deduce that for all $\eta \in]0,1[$

$$\int_{\mathbb{R}^N} G(y-x)\theta|\mathcal{M}(\overline{u}_\varepsilon)|dx \leq C\sum_{l\in\mathbb{Z}^\star}\int_{[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}} e^{-\eta|y|}e^{\eta|x-y|}G(y-x)e^{\eta|x|}(U^{p-1}U_l+U_l^2)\theta dx. \tag{4.52}$$

Using Proposition 5.9, we obtain for all η' such that $0<\eta'<p-1-\eta$

$$\sum_{l\in\mathbb{Z}^\star}\int_{[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}} e^{-\eta|y|}e^{\eta|x-y|}G(y-x)e^{\eta|x|}U^{p-1}U_l\theta dx \leq Ce^{-\eta d_y}e^{-\eta'\frac{2\pi}{\varepsilon}}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}.$$

Moreover

$$\sum_{l\in\mathbb{Z}^\star}\int_{[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1}} e^{-\eta|y|}e^{\eta|x-y|}G(y-x)e^{\eta|x|}U_l^2\theta dx \leq Ce^{-\eta d_y}e^{(-2+\eta)\frac{\pi}{\varepsilon}}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}.$$

Now Proposition 5.9 gives also, since $p-1+\eta>1$

$$\int_{\mathbb{R}^N}|v|G(y-x)p\theta e^{(-p+1)|x|}dx \leq \|ve^{\eta d_x}\|_\infty \int_{\mathbb{R}^N} G(y-x)p\theta e^{(-p+1-\eta)|x|}dx \tag{4.53}$$

$$\leq \|ve^{\eta d_x}\|_\infty e^{-|y|}|y|^{\frac{1-N}{2}} \leq \|ve^{\eta d_x}\|_\infty e^{-\eta d_y}e^{-\eta_1 R_0},$$

where we define $\eta_1 = 1-\eta$, and

$$\int_{\mathbb{R}^N} G(y-x)v^2\theta dx \leq \|ve^{\eta d_x}\|_\infty^2 \int_{\mathbb{R}^N} G(y-x)\theta e^{-2\eta|x|}dx \tag{4.54}$$

$$\leq C\|ve^{\eta d_x}\|_\infty^2 e^{-\eta d_y}\int_{\mathbb{R}^N} G(y-x)\theta e^{\eta|x-y|}e^{-\eta|x|}dx \leq C\|ve^{\eta d_x}\|_\infty^2 e^{-\eta d_y}.$$

Now

$$\int_{\mathbb{R}^N;\frac{\pi}{\varepsilon}-\beta<x_1<\frac{\pi}{\varepsilon}} G(y-x)(|v|+|\nabla v|)dx \leq (\|ve^{\eta d_x}\|_\infty+\|(\nabla v)e^{\eta d_x}\|_\infty)e^{-\eta d_y}\int_{\mathbb{R}^N} e^{\eta|x-y|}G(y-x)dx. \tag{4.55}$$

Finally, we obtain, for $y\in]-\frac{\pi}{\varepsilon}+\beta,\frac{\pi}{\varepsilon}-\beta[\times\mathbb{R}^{N-1}$ and for all $0<\eta'\leq\min\{2-\eta,2(p-1-\eta)\}$

$$|v(y)e^{\eta|y|}|\leq C_1 e^{-\frac{\eta'\pi}{\varepsilon}}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}+\frac{C_2}{\beta}(\|ve^{\eta d_x}\|_\infty+\|(\nabla v)e^{\eta d_x}\|_\infty)+C_1(\|ve^{\eta d_x}\|_\infty^2+\|ve^{\eta d_x}\|_\infty e^{-\eta_1 R_0})$$

where the constants are independent of β and of R_0.
We obtain the same estimate for $|(\nabla v)(y)e^{\eta|y|}|$, using the proof of (5.73).
We terminate by a barrier function argument, as in the proof of (5.62). Let us define

$$\overline{\phi}=C_1 e^{-\frac{\eta'\pi}{\varepsilon}}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}+\frac{C_2}{\beta}(\|ve^{\eta d_x}\|_\infty+\|(\nabla v)e^{\eta d_x}\|_\infty)+C_1(\|ve^{\eta d_x}\|_\infty^2+\|ve^{\eta d_x}\|_\infty e^{-\eta_1 R_0}).$$

We have

$$-\Delta(v-\overline{\phi})+(1-p\overline{u}_\varepsilon^{p-1})(v-\overline{\phi})=-(1-p\overline{u}_\varepsilon^{p-1})\overline{\phi}+h$$

If $x\in[\frac{\pi}{\varepsilon}-\beta,\frac{\pi}{\varepsilon}]\times\mathbb{R}^{N-1}$

$$|h+\mathcal{M}(\overline{u}_\varepsilon)|\leq Cv^2\leq C\|ve^{\eta d_x}\|_\infty^2 e^{-2\eta(\frac{\pi}{\varepsilon}-\beta)}$$

and

$$|\mathcal{M}(\overline{u}_\varepsilon)|\leq Ce^{-(p-1)|x|}e^{-|x-(\frac{2\pi}{\varepsilon},0)|}|x-(\frac{2\pi}{\varepsilon},0)|^{\frac{1-N}{2}}\leq C(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}e^{-\frac{2\pi}{\varepsilon}}e^{(-p+2)|x|}\quad .$$

Choosing if necessary C_1 large enough, we deduce that

$$-(1-p\overline{u}_\varepsilon^{p-1})\overline{\phi}+h<0\quad\text{for } y\in[\tfrac{\pi}{\varepsilon}-\beta,\tfrac{\pi}{\varepsilon}]\times\mathbb{R}^{N-1}.$$

The same proof gives

$$-(1-p\overline{u}_\varepsilon^{p-1})\overline{\phi}+h<0\quad\text{for } y\in[0,\tfrac{\pi}{\varepsilon}+\beta]\times\mathbb{R}^{N-1}.$$

Finally, the Maximum Principle gives

$$|v(y)|-\overline{\phi}\leq 0\quad\text{for } y\in[\tfrac{\pi}{\varepsilon}-\beta,\tfrac{\pi}{\varepsilon}+\beta]\times\mathbb{R}^{N-1}.$$

Thus, in $[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}]\times\mathbb{R}^{N-1}$, we have when $d_y\geq R_0$

$$(|\nabla v(y)|+|v(y)|)e^{\eta d_y}\leq Ce^{-\frac{\eta'\pi}{\varepsilon}}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}+\frac{C}{\beta}(\|ve^{\eta d_x}\|_\infty+\|(\nabla v)e^{\eta d_x}\|_\infty)+C\|ve^{\eta d_x}\|_\infty^2$$
$$+C\|ve^{\eta d_x}\|_\infty e^{-\eta_1 R_0}$$

and when $d_y\leq R_0$

$$(|\nabla v(y)|+|v(y)|)e^{\eta d_y}\leq Ce^{-\frac{\eta'\pi}{\varepsilon}}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}e^{\eta R_0}.$$

We choose β and R_0 large enough to obtain (1.6).

5 Appendix.

Let $\eta > 0$ be given. Let h be a function, defined on $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}$, which has the property

$$\text{the fonction } he^{\eta \text{dist}(x, \cup_{j=0}^{k+1}\{(\frac{a_\varepsilon^j}{\varepsilon}, 0)\})} \text{ is bounded in } L^\infty([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}) \tag{5.56}$$

independently of ε.

Then h belongs to $H^{-1}([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$(the dual space of $H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$) in the following sense

$$< h, \psi >_{H^{-1}, H^1} = \int_{[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}} h\psi dx \quad \text{for } \psi \in H^1.$$

We will denote H^1 in place of $H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$.
By the Lax-Milgram Theorem, there exists $u \in H^1$ such that

$$-\Delta u + u = h. \tag{5.57}$$

It is classical that $u \in L^\infty$ (see [8], Theorem 9.13 and use the Sobolev embedding Theorem).

As a consequence of the maximum principle,

$$\|u\|_{L^\infty} \leq \|h\|_{L^\infty}. \tag{5.58}$$

More we have the following

Proposition 5.6 *Let $h \in H^{-1}([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$. There exists a unique $\xi \in H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$ which verifies*

$$\mathbb{L}\xi = h. \tag{5.59}$$

Let us suppose that $h \in L^\infty([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$ and that

$$< h, \varphi_i >_{H^{-1}, H^1} = 0, \quad i = 1, ..., k. \tag{5.60}$$

Then

$$\|\xi\|_\infty + \|\nabla \xi\|_\infty \leq C\|h\|_\infty \tag{5.61}$$

where C is independent of ε.
Let $\eta \in]0, 1[$ be given. Let us suppose that h verifies (5.60) and has the additional property (5.56) for all $\eta \in]0, 1[$.
Then for all $\eta \in]0, 1[$ there exists C independent of ε and dependent of η such that

$$\|\xi e^{\eta \, dist(x, \cup_{j=0}^{k+1}\{(\frac{a_\varepsilon^j}{\varepsilon}, 0)\})}\|_\infty + \|\nabla \xi e^{\eta \, dist(x, \cup_{j=0}^{k+1}\{(\frac{a_\varepsilon^j}{\varepsilon}, 0)\})}\|_\infty \tag{5.62}$$

$$\leq C\|he^{\eta \, dist(x, \cup_{j=0}^{k+1}\{(\frac{a_\varepsilon^j}{\varepsilon}, 0)\})}\|_\infty.$$

Proof Let $\xi \in H^1(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})$. First, we deduce from (1.14) that

$$(\mathbb{L}\xi = 0) \Rightarrow (\xi = 0).$$

Moreover, the operator $\mathbb{L}$ is a Fredholm operator, so we have the existence of a unique solution ξ of (5.59) when $h \in H^{-1}$. The property (5.60) for h implies the existence of ξ, without knowing the property (1.14). In this case, ξ verifies

$$\int_{\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1}} \xi(-\Delta + 1)(\varphi_i)dx = 0, \quad \text{i=1,...,k.} \tag{5.63}$$

Moreover we have in this case

$$\|\xi\|_{H^1} \leq C\|u\|_{H^1} \tag{5.64}$$

where u is defined in (5.57) and C is independent of ε. Indeed, this can be proved using the expansion of ξ on a basis of eigenvectors of the operator $(-\Delta + 1)^{-1}\mathbb{L}$.
Now, let $h \in L^\infty$ verifying (5.56) and (5.60). Let us prove (5.61).
Let us assume that $\|h\|_\infty \to 0$ and that $\|\xi\|_\infty = 1$. There exists c such that $\xi(c) = 1$. Let $\tilde{\xi}(x) = \xi(x + c)$. By the standard elliptic estimates, $\tilde{\xi}$ tends to a limit $\overline{\xi}$, uniformly on the compact sets of $\mathbb{R}^N$ and we have either

$$(-\Delta + 1)\overline{\xi} = 0 \quad \text{in } \mathbb{R}^N \text{ if } |c - \tfrac{a_\varepsilon^i}{\varepsilon}| \to +\infty \text{ for all } i$$

or

$$(-\Delta + 1 - pU^{p-1}(x + \overline{c}))\overline{\xi} = 0 \quad \text{in } \mathbb{R}^N \text{ if there exists } i \text{ and } \overline{c} \text{ such that } (c - \tfrac{a_\varepsilon^i}{\varepsilon}) \to \overline{c}.$$

The first case is in contradiction with the maximum principle, so it does not occur. In the second case, we have that

$$\xi(x + c) \to \frac{\partial U}{\partial x_1}(x_1 + \overline{c}, x') \quad \text{uniformly on the compact sets.}$$

We use (5.63). Since $(1 - \lambda_i)(-\Delta + 1)\varphi_i = p\overline{u}_\varepsilon^{p-1}\varphi_i$, we use the Lebesgue Theorem to get a contradiction.

We have proved that

$$\|\xi\|_\infty \leq C\|h\|_\infty.$$

The inequality for $\|\nabla\xi\|_\infty$ follows from standard elliptic estimates ([8],Theorem 9.13). So we have proved (5.61).

In what follows, we denote

$$d_x = \text{dist}(x, \cup_{j=0}^{k+1}\{(\frac{a_\varepsilon^j}{\varepsilon}, 0)\}).$$

We define

$$\tilde{\sigma}_i = \frac{a_\varepsilon^{i+1} - a_\varepsilon^i}{2\varepsilon} \quad \text{for } i = 0, ..., k.$$

Let β be a positive real number, independent of ε. Let $i = 1, ..., k$ and let us consider the domain $\mathcal{D}$ defined by

$$\mathcal{D} = \{y; -\tilde{\sigma}_{i-1} + \beta \leq y_1 - \frac{a_\varepsilon^i}{\varepsilon} \leq \tilde{\sigma}_i - \beta\}.$$

The number β will be chosen later.

Let $R_0 > 0$ be given. We are going to estimate

$$|\xi(y)e^{\eta d_y}| + |\nabla\xi(y)e^{\eta d_y}|$$

when $y \in \mathcal{D}$ is such that $d_y > R_0$.

For $i = 1, ..., k$, let θ_i be a function which is $\frac{2\pi}{\varepsilon}$-periodic in x_1 and which verifies in $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}$

$$\theta_i(x) = 1 \quad \text{for} \quad -\tilde{\sigma}_{i-1} + \beta \leq x_1 - \frac{a_\varepsilon^i}{\varepsilon} \leq \tilde{\sigma}_i - \beta \tag{5.65}$$

$$\theta_i(x) = 0 \quad \text{for} \qquad x_1 - \frac{a_\varepsilon^i}{\varepsilon} \geq \tilde{\sigma}_i \quad \text{or} \quad x_1 - \frac{a_\varepsilon^i}{\varepsilon} \leq -\tilde{\sigma}_{i-1}.$$

Moreover, we suppose that θ_i is $\mathcal{C}^2$ in x_1. More precisely, we build θ_i from the function $\tilde{\theta}$ defined in $[0, \beta]$, $\tilde{\theta}(x_1) = -\frac{6}{\beta^5}x_1^5 + \frac{15}{\beta^4}x_1^4 - \frac{10}{\beta^3}x_1^3 + 1$, by $\theta_i(x) = \tilde{\theta}(x_1 - \frac{a_\varepsilon^i}{\varepsilon} - \tilde{\sigma}_i + \beta)$, if $\tilde{\sigma}_i - \beta \leq x_1 - \frac{a_\varepsilon^i}{\varepsilon} \leq \tilde{\sigma}_i$ and $\theta_i(x) = \tilde{\theta}(\frac{a_\varepsilon^i}{\varepsilon} - \tilde{\sigma}_{i-1} + \beta - x_1)$, if $-\tilde{\sigma}_{i-1} \leq x_1 - \frac{a_\varepsilon^i}{\varepsilon} \leq -\tilde{\sigma}_{i-1} + \beta$. Thus we have for all x and for M independent of i, of β and of ε

$$|\theta_i(x)| \leq M, \quad |\nabla\theta_i(x)| \leq \frac{M}{\beta}, \quad |\Delta\theta_i(x)| \leq \frac{M}{\beta^2}.$$

Let G be the Green function of the operator

$$-\Delta + 1 \qquad \text{on } \mathbb{R}^N.$$

We have

$$0 < G(x) \leq C\frac{e^{-|x|}}{|x|^{N-2}}(1 + |x|)^{\frac{N-3}{2}} \quad \text{if } N \geq 2. \tag{5.66}$$

We write

$$(\theta_i\xi)(y) = \int_{\mathbb{R}^N} G(y - x)(p\theta_i\overline{u}_\varepsilon^{p-1}\xi + \theta_i h - \Delta\theta_i\xi - 2\nabla\theta_i.\nabla\xi)(x)dx, \tag{5.67}$$

Let $y \in \mathcal{D}$, we have $(\theta_i\xi)(y) = \xi(y)$.
We consider (5.67). Firstly, we have for all $\eta \in]0, 1[$

$$|\int_{\mathbb{R}^N} G(y - x)(p\theta_i\overline{u}_\varepsilon^{p-1}\xi + \theta_i h)(x)dx| \leq C\|he^{\eta d_x}\|_\infty \int_{\mathbb{R}^N} G(y - x)\theta_i(x)e^{-\eta d_x}dx \tag{5.68}$$

$$+C\|\xi e^{\eta d_x}\|_\infty \int_{\mathbb{R}^N} G(y - x)\theta_i(x)e^{-\eta d_x}\overline{u}_\varepsilon^{p-1}(x)dx.$$

For x in $\mathrm{Supp}\theta_i$ we write

$$d_x \geq d_y - |x-y|$$

while

$$\forall \eta \in]0,1[\exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 \quad \overline{u}_\varepsilon(x) \leq Ce^{-\eta d_x}.$$

So we have

$$|\int_{\mathbb{R}^N} G(y-x)\theta_i(x)e^{-\eta d_x}dx| \tag{5.69}$$

$$\leq \int_{\mathbb{R}^N} G(y-x)e^{\eta|y-x|}|\theta_i(x)|e^{-\eta d_y}dx \leq Ce^{-\eta d_y}$$

and, for any η' such that $\eta < \eta' < 1$

$$|\int_{\mathbb{R}^N} G(y-x)\theta_i(x)e^{-\eta d_x}\overline{u}_\varepsilon^{p-1}(x)dx| \tag{5.70}$$

$$\leq Ce^{-\eta' d_y}\int_{\mathbb{R}^N} G(y-x)e^{\eta'|x-y|}e^{-((p-1)\eta+\eta-\eta')d_x}|\theta_i(x)|dx \leq Ce^{-\eta' d_y}.$$

Secondly, we get for all $\eta \in]0,1[$

$$|\int_{\mathbb{R}^N} G(y-x)(-\Delta\theta_i\xi - 2\nabla\theta_i.\nabla\xi)(x)dx| \leq \frac{C}{\beta}(\|\xi e^{\eta d_x}\|_\infty \tag{5.71}$$

$$+\|\nabla\xi e^{\eta d_x}\|_\infty)\int_{x\in\mathrm{Supp}\theta_i} G(y-x)e^{-\eta d_x}dx.$$

But, as above

$$|\int_{x\in\mathrm{Supp}\theta_i} G(y-x)e^{-\eta d_x}dx| \leq Ce^{-\eta d_y}.$$

Then (5.67)-(5.71) give for all $\eta \in]0,1[$ and for a constant C independent of β and for some $\eta_1 > 0$

$$|\xi(y)e^{\eta d_y}| \leq C(\|he^{\eta d_x}\|_\infty + \frac{1}{\beta}(\|\nabla\xi e^{\eta d_x}\|_\infty + \|\xi e^{\eta d_x}\|_\infty) + Ce^{-\eta_1 R_0}\|\xi e^{\eta d_x}\|_\infty \quad \text{for all } y \in \mathcal{D}. \tag{5.72}$$

We have to prove that

$$|\nabla\xi(y)e^{\eta d_y}| \leq C(\|he^{\eta d_x}\|_\infty + \frac{1}{\beta}\|\nabla\xi e^{\eta d_x}\|_\infty + \|\xi e^{\eta d_x}\|_\infty) + Ce^{-\eta_1 R_0}\|\xi e^{\eta d_x}\|_\infty \quad \text{for all } y \in \mathcal{D}. \tag{5.73}$$

Let $u = \xi e^{\eta|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|}$. When $d_x = |x-(\frac{a_\varepsilon^i}{\varepsilon},0)| \geq R_0$, u satisfies the equation

$$-\Delta u + u(1-\eta^2 - p\overline{u}_\varepsilon^{p-1}) + 2\eta\nabla|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|.\nabla u = e^{\eta d_x}h.$$

Without loss of generality, we suppose that there exists C independent of ε such that

$$\text{for all } x \text{ such that } |x-y| \leq 1, \quad |d_x - |x-(\frac{a_\varepsilon^i}{\varepsilon},0)|| \leq C.$$

We use Theorem 9.13 of [8], with $\Omega' = \{x, |x-y| \le \frac{1}{2}\}$ and $\Omega = \{x, |x-y| < 1\}$. We get

$$\|\nabla u\|_{L^\infty(\Omega')} \le C(\|u\|_{L^\infty(\Omega)} + \|e^{\eta d_x} h\|_{L^\infty(\Omega)}$$

that gives

$$|\nabla(\xi e^{\eta d_y})(y)| \le C(\|\xi e^{\eta d_x}\|_\infty + \|h e^{\eta d_x}\|_\infty). \tag{5.74}$$

Then (5.72) and (5.74) lead to (5.73). Now (5.73) and (5.72) give

$$(|\xi(y)| + |\nabla \xi(y)|) e^{\eta d_y} \le C_1 \|h e^{d_x}\|_\infty + \frac{C}{\beta}(\|\xi e^{d_x}\|_\infty + \|\nabla \xi e^{d_x}\|_\infty) + C e^{-\eta_1 R_0} \|\xi e^{\eta d_x}\|_\infty \quad \text{for } y \in \mathcal{D}. \tag{5.75}$$

It remains to prove (5.75) for y in $\Omega_i \backslash \mathcal{D}$. Let us denote

$$\tilde{\mathcal{D}} = \{y; \frac{a_\varepsilon^i}{\varepsilon} + \tilde{\sigma}_i - \beta < y_1 < \frac{a_\varepsilon^{i+1}}{\varepsilon} - \tilde{\sigma}_i + \beta\}$$

By (5.75), we have on $\partial \tilde{\mathcal{D}}$

$$|\xi(y)| \le C_1 e^{-\eta(\tilde{\sigma}_i - \beta)} \|h e^{\eta d_x}\|_\infty + \frac{C}{\beta}(\|\xi e^{d_x}\|_\infty + \|\nabla \xi e^{d_x}\|_\infty) + C e^{-\eta_1 R_0} \|\xi e^{\eta d_x}\|_\infty.$$

Let us denote

$$\overline{\phi}(y) = C_1 e^{-\eta(\tilde{\sigma}_i - \beta)} \|h e^{\eta d_x}\|_\infty + \frac{C}{\beta}(\|\xi e^{d_x}\|_\infty + \|\nabla \xi e^{d_x}\|_\infty) + C e^{-\eta_1 R_0} \|\xi e^{\eta d_x}\|_\infty.$$

We can suppose that $C_1 > 1$.
We have

$$-\Delta(\xi - \overline{\phi}) + (\xi - \overline{\phi})(1 - p \overline{u}_\varepsilon^{p-1}) = -\overline{\phi}(1 - p \overline{u}_\varepsilon^{p-1}) + h.$$

In $\tilde{\mathcal{D}}$, we have $d_x \ge \tilde{\sigma}_i - \beta$, thus

$$|h| \le e^{-\eta(\tilde{\sigma}_i - \beta)} \|h e^{\eta d_x}\|_\infty \quad \text{in} \quad \tilde{\mathcal{D}}.$$

We deduce that for ε small enough, we have together

$$-\overline{\phi}(1 - p \overline{u}_\varepsilon^{p-1}) + h \le 0 \quad \text{and} \quad 1 - p \overline{u}_\varepsilon^{p-1} > 0 \quad \text{in} \quad \tilde{\mathcal{D}}.$$

Then the Maximum Principle gives

$$\xi - \overline{\phi} \le 0 \quad \text{in} \quad \tilde{\mathcal{D}}.$$

and the same proof gives

$$-\xi - \overline{\phi} \le 0 \quad \text{in} \quad \tilde{\mathcal{D}}.$$

Moreover, we have

$$d_x \le \tilde{\sigma}_i \quad \text{in} \quad \tilde{\mathcal{D}}.$$

Finally

$$|\xi(x) e^{\eta d_x}| \le C_1 e^{\eta \beta} \|h e^{\eta d_x}\|_\infty + \frac{C}{\beta}(\|\xi e^{d_x}\|_\infty + \|\nabla \xi e^{d_x}\|_\infty) + C e^{-\eta_1 R_0} \|\xi e^{\eta d_x}\|_\infty \quad \text{in} \quad \tilde{\mathcal{D}}.$$

The same proof as for (5.73) gives

$$|\nabla\xi(x)e^{\eta d_x}| \leq C_1 e^{\eta\beta}\|he^{\eta d_x}\|_\infty + \frac{C}{\beta}(\|\xi e^{d_x}\|_\infty + \|\nabla\xi e^{d_x}\|_\infty) + Ce^{-\eta_1 R_0}\|\xi e^{\eta d_x}\|_\infty \quad \text{in} \quad \tilde{\mathcal{D}}.$$

We have proved

$$|\xi(x)e^{\eta d_x}|+|\nabla\xi(x)e^{\eta d_x}| \leq C_1 e^{\eta\beta}\|he^{\eta d_x}\|_\infty+\frac{C}{\beta}(\|\xi e^{d_x}\|_\infty+\|\nabla\xi e^{d_x}\|_\infty)+Ce^{-\eta_1 R_0}\|\xi e^{\eta d_x}\|_\infty \quad \text{in } \Omega_i,$$

when $d_x \geq R_0$,
and

$$|\xi(x)e^{\eta d_x}| + |\nabla\xi(x)e^{\eta d_x}| \leq C_1 e^{\eta\beta}\|he^{\eta d_x}\|_\infty e^{-\eta R_0} \quad \text{in } \Omega_i, \text{ when } d_x \leq R_0,$$

for all i.
Now we choose β and R_0 large enough to get (5.62). We have proved the proposition.

Proposition 5.7 *Let φ be an eigenfunction of $\mathbb{L}$, associated with an eigenvalue λ which does not tend to 1. Let us suppose that $\|\varphi\|_{L^\infty(]-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}[\times\mathbb{R}^{N-1})} = 1$. Then for all $\eta \in]0,1[$ there exists $C > 0$, independent of ε, such that*

$$|\varphi(x)| + |\nabla\varphi(x)| \leq Ce^{-\eta dist(x,\cup_{i=0}^{k+1}\{(\frac{a_\varepsilon^i}{\varepsilon},0)\})}, \tag{5.76}$$

where we use the notation : $a_\varepsilon^0 = a_\varepsilon^k - 2\pi$ and $a_\varepsilon^{k+1} = a_\varepsilon^1 + 2\pi$.
Moreover

$$C_1 \leq \|\varphi\|_{H^1} \leq C_2 \tag{5.77}$$

where C_1 and C_2 are some positive real numbers independent of ε.
Let ξ be defined in (2.21). Then

$$|\xi(x)| + |\nabla\xi(x)| \leq Ce^{-\frac{\sigma}{\varepsilon}}e^{-\eta dist(x,\cup_{i=0}^{k+1}\{(\frac{a_\varepsilon^i}{\varepsilon},0)\})}. \tag{5.78}$$

Proof To prove (5.76), we follow the proof of (5.62) in Proposition 5.6, with $h = 0$
We find, for $y \in \Omega_i$ such that $d_y \geq R_0$

$$|\varphi(y)e^{\eta d_y}| + |\nabla\varphi(y)e^{\eta d_y}| \leq \frac{C}{\beta}(\|\varphi e^{d_x}\|_\infty + \|\nabla\varphi e^{d_x}\|_\infty) + Ce^{-\eta_1 R_0}\|\varphi e^{\eta d_x}\|_\infty$$

while for $y \in \Omega_i$ such that $d_y \leq R_0$

$$|\varphi(y)e^{\eta d_y}| + |\nabla\varphi(y)e^{\eta d_y}| \leq Ce^{\eta R_0}$$

where the constant C is independent of β and of R_0. We choose R_0 and β large enough to obtain(5.76).

Let us prove (5.77).
We have

$$(1-\lambda)\|\varphi\|_{H^1}^2 = p\int_{\frac{S^1}{\varepsilon}\times\mathbb{R}^{N-1}} \overline{u}_\varepsilon^{p-1}\varphi^2 dx.$$

In view of (5.76) and of (2.16), we may use the Lebesgue Theorem to obtain that

$$\|\varphi\|_{H^1} \not\to 0, \quad i = 0, ..., k.$$

Now we have

$$\mathbb{L}\xi = \mathbb{L}\frac{\partial v_i}{\partial x_1} - \sum_{j=1}^{k} c_j \lambda_j (-\Delta\varphi_j + \varphi_j) = \sum_{l\in\mathbb{Z}} (\overline{u}_\varepsilon^{p-1} - U_{i,l}) \frac{\partial U_{i,l}}{\partial x_1} + p \sum_{j=1}^{k} c_j \frac{\lambda_j}{1-\lambda_j} \overline{u}_\varepsilon^{p-1} \varphi_j.$$

If we write $\mathbb{L}\xi = h$, then, for all $\eta \in]0, 1[$

$$|h(x)| \le C_\eta e^{-\frac{\sigma}{\varepsilon}} e^{-\eta d_x}.$$

We use Proposition 5.6 to obtain (5.78).
We have proved the proposition.

Proposition 5.8 *Let C and A be positive real numbers. Let f and g be functions which verify the following property, for $|x| > A$*

$$|f(x)| \le Ce^{-|x|} \quad , \quad |g(x)| \le Ce^{-|x|}|x|^{\frac{1-N}{2}}.$$

Let $a > b > 0$. Let y_0 be such that $|y_0| \to +\infty$ and $\alpha = \frac{|y_0|}{2}$. Then

$$|\int_{\alpha^{\frac{1}{2}}<|x|<\alpha} f^a(x) g^b(x - (y_0, 0)) dx| = o(|y_0|^{b\frac{1-N}{2}} e^{-b|y_0|}). \tag{5.79}$$

Proof We easily see that if $|x| < \alpha$,

$$|x - (y_0, 0)| = |y_0| |\frac{x}{y_0} - (1, 0)| \ge \frac{1}{2}|y_0|.$$

Thus

$$|\int_{\alpha^{\frac{1}{2}}<|x|<\alpha} f^a(x) g^b(x - (y_0, 0)) dx| \le C|y_0|^{b\frac{1-N}{2}} \int_{\alpha^{\frac{1}{2}}<|x|<\alpha} e^{-a|x| - b|x-(y_0,0)|} dx$$

$$\le C|y_0|^{b\frac{1-N}{2}} \int_{\alpha^{\frac{1}{2}}<r<\alpha} r^{N-1} \int_{S^{N-1}} e^{-ar} e^{-b\sqrt{(rz_1-y_0)^2 + r^2\sum_{i=2}^{N} z_i^2}} dr d\mu(z)$$

$$\le C|y_0|^{b\frac{1-N}{2}} \int_{\alpha^{\frac{1}{2}}<r<\alpha} r^{N-1} \int_{S^{N-1}} e^{-ar} e^{-b\sqrt{r^2 - 2rz_1y_0 + y_0^2}} dr d\mu(z)$$

$$\le C|y_0|^{b\frac{1-N}{2}} \int_{\alpha^{\frac{1}{2}}<r<\alpha} r^{N-1} \int_{S^{N-1}} e^{-ar} e^{-b|y_0-r|} dr d\mu(z)$$

$$\le C|y_0|^{b\frac{1-N}{2}} e^{-b|y_0|} e^{(b-a)\alpha^{\frac{1}{2}}} \alpha^N$$

and $e^{(b-a)\alpha^{\frac{1}{2}}} \alpha^N \to 0$.

Proposition 5.9 *Let f and g be smooth functions and C, C_1, C_2 and A be positive real numbers which verify, for $|x| > A$*

$$0 \leq f(x) \leq Ce^{-|x|} \quad ; \quad C_1 e^{-|x|}|x|^{\frac{1-N}{2}} \leq g(x) \leq C_2 e^{-|x|}|x|^{\frac{1-N}{2}}.$$

Let us define

$$\Omega_i = \{x \in [-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}; dist(x, \cup_{l=0}^{k+1}\{(\frac{a_\varepsilon^l}{\varepsilon}, 0)\} = |x - (\frac{a_\varepsilon^i}{\varepsilon}, 0)|\}.$$

Let $a > b > 0$. Let $i \neq j$ and let $y_0 = \frac{a_\varepsilon^j - a_\varepsilon^i}{\varepsilon}$.
If $f(x) \geq 0$ and $f \neq 0$, then there exist positive real numbers C_1 and C_2 such that, for $i = 0, ..., k+1$

$$(1+o(1))C_1^b C_0 e^{-b|y_0|}|y_0|^{b\frac{1-N}{2}} \leq \int_{\Omega_i} f^a(x-(\frac{a_\varepsilon^i}{\varepsilon}, 0))g^b(x-(\frac{a_\varepsilon^j}{\varepsilon}, 0))dx \leq C_2^b C_0 e^{-b|y_0|}|y_0|^{b\frac{1-N}{2}}(1+o(1)), \tag{5.80}$$

where

$$C_0 = \int_{\mathbb{R}^N} f^a(x)dx.$$

The estimate (5.80) holds true if we replace Ω_i by the set

$$\Omega_i^+ = \{x \in \Omega_i; \quad x_1 > \frac{a_\varepsilon^i}{\varepsilon}\}$$

while C_0 is replaced by

$$\int_{\mathbb{R}^N, x_1 > 0} f^a(x)dx.$$

Proof. Let

$$\alpha = \frac{|y_0|}{2}.$$

For x such that $|x - (\frac{a_\varepsilon^i}{\varepsilon}, 0)| < \alpha^{\frac{1}{2}}$, we have

$$|y_0| - |x - (\frac{a_\varepsilon^i}{\varepsilon}, 0)| \leq |x - (\frac{a_\varepsilon^j}{\varepsilon}, 0)| \leq |y_0| + |x - (\frac{a_\varepsilon^i}{\varepsilon}, 0)|,$$

thus

$$|x - (\frac{a_\varepsilon^j}{\varepsilon}, 0)| = (1 + o(1))|y_0|.$$

We write

$$\int_{|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|<\alpha^{\frac{1}{2}}} f^a(x - (\frac{a_\varepsilon^i}{\varepsilon}, 0))g^b(x - (\frac{a_\varepsilon^j}{\varepsilon}, 0))dx$$

$$\geq C_1^b(1 + o(1))|y_0|^{b\frac{(1-N)}{2}} e^{-b|y_0|} \int_{|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|<\alpha^{\frac{1}{2}}} f^a(x - (\frac{a_\varepsilon^i}{\varepsilon}, 0))dx$$

$$\geq C_1^b C_0(1 + o(1))|y_0|^{b\frac{(1-N)}{2}} e^{-b|y_0|}$$

and

$$\int_{|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|<\alpha^{\frac12}} f^a(x-(\frac{a_\varepsilon^i}{\varepsilon},0))g^b(x-(\frac{a_\varepsilon^j}{\varepsilon},0))dx$$

$$\leq C_2^b(1+o(1))|y_0|^{b\frac{(1-N)}{2}}e^{-b|y_0|}\int_{|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|<\alpha^{\frac12}} f^a(x-(\frac{a_\varepsilon^i}{\varepsilon},0))dx$$

$$\leq C_2^bC_0(1+o(1))|y_0|^{b\frac{(1-N)}{2}}e^{-b|y_0|}.$$

Moreover, in view of Proposition 5.8, we have

$$\int_{x\in\Omega_i;\alpha^{\frac12}\leq|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|<\alpha} f^a(x-(\frac{a_\varepsilon^i}{\varepsilon},0))g^b(x-(\frac{a_\varepsilon^j}{\varepsilon},0))dx = o(1)|y_0|^{b\frac{(1-N)}{2}}e^{-b|y_0|}.$$

Last, we have

$$\int_{x\in\Omega_i;|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|>\alpha} f^a(x-(\frac{a_\varepsilon^i}{\varepsilon},0))g^b(x-(\frac{a_\varepsilon^j}{\varepsilon},0))dx$$

$$\leq C_2^bC^ae^{-b|y_0|}|y_0|^{b\frac{1-N}{2}}\int_{x\in\Omega_i;|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|>\alpha} e^{b|x-(\frac{a_\varepsilon^i}{\varepsilon},0)|}f^a(x-(\frac{a_\varepsilon^i}{\varepsilon},0))dx$$

$$= o(e^{-b|y_0|}|y_0|^{b\frac{1-N}{2}}),$$

since $a>b$.
We have proved the proposition.

Corollary 5.1 *If*

$$\lim_{|x|\to+\infty} g(x)e^{|x|}|x|^{\frac{N-1}{2}} = L$$

then

$$\lim_{|x|\to+\infty}\int_{\Omega_i} f^a(x-(\frac{a_\varepsilon^i}{\varepsilon},0))g^b(x-(\frac{a_\varepsilon^j}{\varepsilon},0))dx e^{b|y_0|}|y_0|^{b\frac{N-1}{2}} = LC_0$$

where C_0 is defined in Proposition 5.9. Moreover, we can replace Ω_i by Ω_i^+.

Lemma 5.2 *Let k be a positive integer. Let a and b be real numbers, with $a>0$. If $p>1$ is given and $k=[p]$. There exists C independent of a and b such that*

$$|-(a+b)_+^p + a^p + ... + \frac{p....(p-k+1)}{k!}a^{p-k}b^k| \leq C|b|^p.$$

Proof First, let us suppose that $b<0$. If $a+2b<0$, then $a<2|b|$, and the claim is true. If $a+2b>0$, then $a+b>0$ and we write

$$|-(a+b)^p + a^p + ... + \frac{p....(p-k+1)}{k!}a^{p-k}b^k| = \frac{p....(p-k)}{(k+1)!}(a+\alpha b)^{p-k-1}b^{k+1} \quad (5.81)$$

where $\alpha\in]0,1[$. But $a+\alpha b > -2b+\alpha b$, thus

$$|-(a+b)^p + a^p + ... + \frac{p....(p-k+1)}{k!}a^{p-k}b^k| \leq \frac{p....(p-k)}{(k+1)!}(2-\alpha)^{p-k-1}|b|^p$$

that gives the claim.
Secondly, let us suppose that $b > 0$. If $a < b$, the claim is true. If $a > b$, we use (5.81) again and we obtain the claim.

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