SYMMETRIC ITINERARY SETS

Michael Barnsley, Nicolae Mihalache

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MICHAEL F. BARNSLEY AND NICOLAE MIHALACHE

Abstract. We consider a one parameter family of dynamical systems $W : [0,1] \to [0,1]$ constructed from a pair of monotone increasing diffeomorphisms W_i, such that $W_i^{-1} : [0,1] \to [0,1]$, $(i = 0,1)$. We characterize the set of symbolic itineraries of W using an attractor $\overline{\Omega}$ of an iterated closed relation, in the terminology of McGehee, and prove that there is a member of the family for which $\overline{\Omega}$ is symmetrical.

1. Introduction

Let $W_0 : [0,a] \to [0,1]$ and $W_1 : [1-b,1] \to [0,1]$ be continuous and differentiable, and such that $a+b > 1$, $W_0(0) = W_1(1-b) = 0$, $W_0(a) = W_1(1) = 1$. Let the derivatives $W_i'(x)$ $(i = 0,1)$ be uniformly bounded below by $d > 1$.

For $\rho \in [1-b, a]$ we define $W : [0,1] \to [0,1]$ by

$$[0,1] \ni x \mapsto \begin{cases} W_0(x) \text{ if } x \in [0,\rho] \\ W_1(x) \text{ otherwise.} \end{cases}$$

See Figure 1. Similarly, we define $W_+ : [0,1] \to [0,1]$ by replacing $[0,\rho]$ by $[0,\rho)$.

Let $I = \{0,1\}$. Let $I^\infty = \{0,1\} \times \{0,1\} \times \cdots$ have the product topology induced from the discrete topology on I. For $\sigma \in I^\infty$ write $\sigma = \sigma_0\sigma_1\sigma_2\ldots$, where $\sigma_k \in I$ for all $k \in \mathbb{N}$. The product topology on I^∞ is the same as the topology induced by the metric $d(\omega,\sigma) = 2^{-k}$ where k is the least index such that $\omega_k \neq \sigma_k$. It is well known that (I^∞, d) is a compact metric space. We define a total order relation $\preceq$ on I^∞, and on I^n for any $n \in \mathbb{N}$, by $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\sigma_k < \omega_k$ where k is the least index such that $\sigma_k \neq \omega_k$. For $\sigma \in I^\infty$ and $n \in \mathbb{N}$ we write $\sigma|_n = \sigma_0\sigma_1\sigma_2 ... \sigma_n$. I^∞ is the appropriate space in which to embed and study the itineraries of the family of discontinuous dynamical systems $W : [0,1] \to [0,1]$.

For $W_{(+)} \in \{W, W_+\}$ let $W_{(+)}^k$ denote $W_{(+)}$ composed with itself k times, for $k \in \mathbb{N}$, and let $W_{(+)}^{-k} = (W_{(+)}^k)^{-1}$. We define a map $\tau : [0,1] \to I^\infty$, using all of the orbits of W, by

$$\tau(x) = \sigma_0\sigma_1\sigma_2\ldots$$

where σ_k equals 0, or 1, according as $W^k(x) \in [0,\rho]$, or $(\rho,1]$, respectively. We call $\tau(x)$ the *itinerary* of x under W, or an *address* of x, and we call $\Omega = \tau([0,1])$ an *address space* for $[0,1]$. Similarly, we define $\tau^+ : [0,1] \to I^\infty$ so that $\tau^+(x)_k$ equals 0, or 1, according as $W_+^k(x) \in [0,\rho)$, or $[\rho,1]$, respectively; and we define $\Omega_+ = \tau^+([0,1])$. Note that W, W_+, Ω, Ω_+, τ, and τ^+ all depend on ρ.

The main goals of this paper are to characterise $\overline{\Omega}$ and to show that there exists a value of ρ such that $\overline{\Omega}$ is symmetric.

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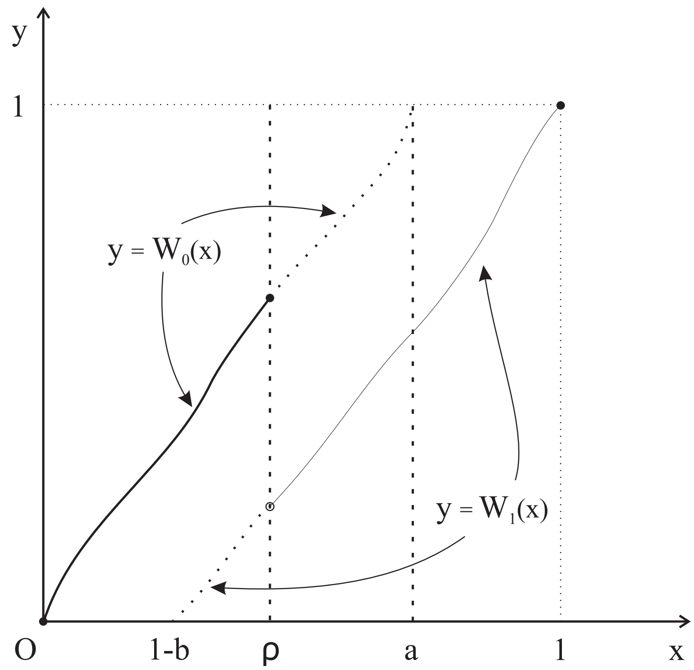

FIGURE 1. The piecewise continuous dynamical system W:[0,1]→[0,1] is defined in terms of two monotone strictly increasing differentiable functions W_0(x) and W_1(x), and a real parameter ρ.

Theorem 1. *Let an iterated closed relation $r \subset I^\infty \times I^\infty$ be defined by*

$$r := \{(\sigma, 0\sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(\sigma, 1\sigma) \in I^\infty \times I^\infty : \sigma \succeq \beta\}$$

where $\alpha = \tau(W_0(\rho))$ and $\beta = \tau^+(W_1(\rho))$. The only attractors of r are $\{\overline{0}\}$, $\{\overline{1}\}, \{\overline{0}, \overline{1}\}$, and $\overline{\Omega}$. The corresponding dual repellers are $\{\sigma \in I^\infty : \beta \preceq \sigma\}, \{\sigma \in I^\infty : \sigma \preceq \alpha\}$, $\{\sigma \in I^\infty : \beta \preceq \sigma\} \cup \{\sigma \in I^\infty : \sigma \preceq \alpha\}$, and the empty set, respectively. The chain recurrent set for r is $\{\overline{0}, \overline{1}\} \cup \{\sigma \in \overline{\Omega} : \beta \preceq \sigma \preceq \alpha\}$.

We write $\overline{E}$ to denote the closure of a set E. But we write $\overline{0} = 000..., \overline{1} = 111... \in I^\infty$. For $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in I^\infty$ we write 0σ to mean $0\sigma_0\sigma_1\sigma_2 \ldots \in I^\infty$ and $1\sigma = 0\sigma_0\sigma_1\sigma_2 \ldots \in I^\infty$.

Define a symmetry function $^* : I^\infty \to I^\infty$ by $\sigma^* = \omega$ where $\omega_k = 1 - \sigma_k$ for all k.

Theorem 2. *There exists a unique $\rho \in [1-b, a]$ such that $\overline{\Omega}^* = \overline{\Omega}$.*

Theorem 1 tells us that $\overline{\Omega}$ is fixed by itineraries of two inverse images of the critical point ρ, and provides the basis for a stable algorithm to determine $\overline{\Omega}$. It relates the address spaces of dynamical systems of the form of W to the beautiful theory of iterated closed relations on compact Hausdorff spaces [3], and hence to the work of Charles Conley.

Theorem 2 is interesting in its own right and also because it has applications in digital imaging, as explained and demonstrated, in the special case of affine maps, in [1]. It enables the construction of parameterized families of nondifferentiable homeomorphisms on $[0,1]$, using pairs of overlapping iterated function systems, see Proposition 4. Theorem 2 generalizes results in [1] to nonlinear W_i's. The proof uses symbolic dynamics in place of the geometrical construction outlined in [1]. The approach and results open up the mathematics underlying [1] and [2].

To tie the present work into [1], note that τ is a section, as defined in [1], for the hyperbolic iterated function system

$$\mathcal{F} := ([0,1]; W_0^{-1}, W_1^{-1}).$$

Our observations interrelate to, but are more specialized than, the work of Parry [5]. Our point of view is topological rather than measure-theoretic, and our main results appear to be new.

2. Basic properties of τ

The following list of properties is relatively easy to check. Below the list we elaborate on points 1, 2, and 3.

(1) W^n is piecewise differentiable and its derivative is uniformly bounded below by d^n; each, except the leftmost branch of W^n, is defined on an interval of the form $(r,s]$. W_+^n is piecewise differentiable and its derivative is uniformly bounded below by d^n; each, except for the rightmost branch of W_+^n, is defined on an interval of the form $[r,s)$.

(2) If (r,s) is the interior of the definition domain of a branch of W^n (and of W_+^n) then $\tau(x)|_n$ is constant on $(r,s]$, $\tau^+(x)|_n$ is constant on $[r,s)$, and $\tau(x)|_n = \tau^+(x)|_n$ for all $x \in (r,s)$.

(3) The boundary of the definition domain of a branch of W^n is contained in $\{0,1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho)$; by (1), the length of such a domain is at most d^{-n}.

(4) The set $\bigcup_{k\in\mathbb{N}} W^{-k}(\rho)$ is dense in $[0,1]$. This follows from (3).

(5) $\tau(x) = \tau^+(x)$ unless $x \in \bigcup_{k\in\mathbb{N}} W^{-k}(\rho)$ in which case $\tau(x) \prec \tau^+(x)$.

(6) Both $\tau(x)$ and $\tau^+(x)$ are strictly increasing functions of $x \in [0,1]$ and $\tau(x) \preceq \tau^+(x)$. This follows from (4) and (5).

(7) For all $x \in [0,1]$, $\tau(x)$ is continuous from the left, $\tau^+(x)$ is continuous from the right. Moreover, for all $x \in (0,1)$,

$$\tau(x) = \lim_{\varepsilon\to 0^+} \tau^+(x-\varepsilon) \text{ and } \tau^+(x) = \lim_{\varepsilon\to 0^+} \tau(x+\varepsilon).$$

These assertions follow from (2), (3) and (4).

(8) Each $x \in W^{-n}(\rho)$, such that $\tau(x)|_n$ is constant, moves continuously with respect to ρ with positive velocity bounded above by d^{-n}. This follows from (1).

(9) For $x \in (0,1) \setminus \bigcup_{k=0}^{n} W^{-k}(\rho)$, $\tau(x)|_n = \tau^+(x)|_n$ is locally constant with respect to ρ; moreover, this holds if x depends continuously on ρ. This follows from (2), (3) and (6).

(10) The symmetry function $^* : I^\infty \to I^\infty$ is strictly decreasing and continuous.

(11) For any $\sigma|_n \in I^n$, $n \in \mathbb{N}$, the set

$$\mathcal{I}(\sigma|_n) := \{x \in [0,1] : \tau(x)|_n = \sigma|_n \text{ or } \tau^+(x)|_n = \sigma|_n\},$$

is either empty or a non-degenerate compact interval of length at most d^{-n}. This follows from (2), (3) and (6).

(12) The projection $\hat{\pi} : I^{\infty} \to [0,1]$ is well-defined by

$$\hat{\pi}(\sigma) = \sup\{x \in [0,1] : \tau^{+}(x) \preceq \sigma\} = \inf\{x \in [0,1] : \tau(x) \succeq \sigma\}.$$

This follows from (6).

(13) The projection $\hat{\pi} : I^{\infty} \to [0,1]$ is increasing, by (6); continuous, by (11); and, by (7),

$$\hat{\pi}(\tau(x)) = \hat{\pi}(\tau^{+}(x)) = x \text{ for all } x \in [0,1],$$
$$\tau(\hat{\pi}(\sigma)) \preceq \sigma \preceq \tau^{+}(\hat{\pi}(\sigma)) \text{ for all } \sigma \in I^{\infty}.$$

(14) Let $S : I^{\infty} \to I^{\infty}$ denote the left-shift map $\sigma_0\sigma_1\sigma_2... \mapsto \sigma_1\sigma_2\sigma_3....$ For all $\sigma \in I^{\infty}$ such that $\sigma \preceq \tau(\rho)$ or $\sigma \geq \tau^{+}(\rho)$,

$$\hat{\pi}(S(\sigma)) = W(\hat{\pi}(\sigma)).$$

Also $\hat{\pi}(\tau^{+}(\rho)) = \rho$ and $\hat{\pi}(S(\tau^{+}(\rho))) = W_1(\rho)$. These statements follow from (7).

Here we elaborate on points (1), (2) and (3). Consider the piecewise continuous function $W^k(x)$, for $k \in \{1,2,...\}$. Its discontinuities are at ρ and, for $k > 1$, other points in $(0,1)$, each of which can be written in the form $W_{\sigma_0}^{-1} \circ W_{\sigma_2}^{-1} \circ ...W_{\sigma_{l-1}}^{-1}(\rho)$ for some $\sigma_0\sigma_1...\sigma_{l-1} \in \{0,1\}^l$ for some $l \in \{1,2,...k-1\}$. We denote these discontinuities, together with the points 0 and 1, by

$$D_{k,0} := 0 < D_{k,1} < D_{k,2} < .... < D_{k,D(k)-1} < 1 =: D_{k,D(k)},$$

where $D(1) = 3, D(2) = 5 < D(3) < D(4)...$. For each $k \geq 1$, one of the $D_{k,j}$'s is equal to ρ. For $k \geq 1$ we have $W^k(x) = W_0^k(x)$ for $x \in [D_{k,0}, D_{k,1}]$ and $W_+^k(x) = W_0^k(x)$ for $x \in [D_{k,0}, D_{k,1})$. Similarly $W^k(x) = W_1^k(x)$ for all $x \in (D_{k,D(k)-1}, D_{k,D(k)}]$ and $W_+^k(x) = W_1^k(x)$ for $x \in [D_{D(k)-1}, D_{D(k)}]$.

For all $x \in (D_{k,l}, D_{k,l+1})$ $(l = 0,1,...,D(k)-1)$, $W^k(x) = W_+^k(x) = W_{\theta_k} \circ W_{\theta_{k-1}} \circ ...W_{\theta_1}(x)$ for some fixed $\theta_1\theta_2...\theta_k \in \{0,1\}^k$. We refer to $\theta_1\theta_2...\theta_k$ as the *address of the interval* $(D_{k,l}, D_{k,l+1})$, we say $(D_{k,l}, D_{k,l+1})$ that "*has address* $\theta_1\theta_2...\theta_k$", and we write, by slight abuse of notation, $\tau((D_{k,l}, D_{k,l+1})) = \theta_1\theta_2...\theta_k$.

Let $k > 1$. Consider two adjacent intervals,$(D_{k,m-1}, D_{k,m}]$ and $(D_{k,m}, D_{k,m+1}]$ for $m \in \{1,2,...,D(k)-1\}$ and $k > 1$. Let the one on the right have address $\theta_0\theta_1...\theta_{k-1}$ and the one on the left have address $\eta_0\eta_1...\ \eta_{k-1}$. Then $\eta_0\eta_1...\ \eta_{k-1} \prec \theta_0\theta_1...\theta_{k-1}$ and we have

$$\tau(x)|_{k-1} = \eta_0\eta_1...\eta_{k-1} \text{ for all } x \in (D_{k,m-1}, D_{k,m}],$$
$$\tau^{+}(x)|_{k-1} = \eta_0\eta_1...\eta_{k-1} \text{ for all } x \in [D_{k,m-1}, D_{k,m}),$$
$$\tau(x)|_{k-1} = \theta_0\theta_1...\theta_{k-1} \text{ for all } x \in (D_{k,m}, D_{k,m+1}],$$
$$\tau^{+}(x)|_{k-1} = \theta_0\theta_1...\theta_{k-1} \text{ for all } x \in [D_{k,m}, D_{k,m+1}).$$

In particular, $\tau(x)|_{k-1}$ and $\tau^{+}(x)|_{k-1}$ are constant and equal on each of the open intervals $(D_{k,m-1}, D_{k,m})$ and have distinct values at the discontinuity points $\{D_{k,m}\}_{m=1}^{D(k)-1}$.

3. The structures of Ω, Ω_+ and $\overline{\Omega}$.

In this section we characterize Ω and Ω_+ as certain inverse limits, and we characterize $\overline{\Omega}$ as an attractor of an iterated closed relation on I^∞. These inverse limits are natural and they clarify the structures of Ω and Ω_+. They are implied by the shift invariance of Ω and Ω_+. Recall that $S : I^\infty \to I^\infty$ denotes the left-shift map $\sigma_0\sigma_1\sigma_2... \mapsto \sigma_1\sigma_2\sigma_3....$

Proposition 1. *(i)* $\tau(W(x)) = S(\tau(x))$ *and* $\tau^+(W_+(x)) = S(\tau^+(x))$ *for all* $x \in [0, 1]$.

(ii) $S(\Omega) = \Omega$ *and* $S(\Omega_+) = \Omega_+$.

Proof. (i) This follows at once from the definitions of τ and τ^+. (ii) This follows from (i) together with $W([0, 1]) = W_+([0, 1]) = [0, 1]$. □

We say that $\Lambda \subset I^\infty$ is *closed from the left* if, whenever $\{x_n\}_{n=0}^\infty$ is an non-decreasing sequence of points in Λ, $\lim x_n \in \Lambda$. We say that $\Lambda \subset I^\infty$ is *closed from the right* if, whenever $\{x_n\}_{n=0}^\infty$ is non-increasing sequence in Λ, $\lim x_n \in \Lambda$. For $S \subset X$, where $X = I^\infty$ or $[0, 1]$, we write $L(S) = \{\sigma \in X :$ *there is a non-decreasing sequence* $\{z_n\}_{n=0}^\infty \subset S$ *with* $\sigma = \lim z_n\}$ to denote the closure of S from the left. Analogously, we define $R(S)$ for the closure of S from the right.

Proposition 2. *(i)* Ω *is closed from the left and* Ω_+ *is closed from the right;*

(ii) $\overline{\Omega} = \overline{\Omega_+} = \Omega \cup \Omega_+ = \overline{\Omega \cap \Omega_+}$

Proof. Proof of (i): By (6) $\tau : [0, 1] \to I^\infty$ is monotone strictly increasing. By (7) τ is continuous from the left. Let $\{z_n\}_{n=0}^\infty$ be a non-decreasing sequence of points in Ω. Let $y_n = \tau^{-1}(z_n)$. Let $y = \lim y_n \in [0, 1]$. Since τ is continuous from the left, $\Omega \ni \tau(y) = \tau(\lim y_n) = \lim \tau(y_n) = \lim z_n$. It follows that Ω is closed from the left. Similarly, Ω_+ is closed from the right.

Proof of (ii): Let $Q = \{x \in [0, 1] : \tau(x) = \tau^+(x)\}$. Then by (4) $\overline{Q} = [0, 1]$. Also, by (5),

$$\Omega \cap \Omega_+ = \tau([0, 1]) \cap \tau^+([0, 1]) = \tau(Q) = \tau^+(Q).$$

Hence

$$\overline{\Omega \cap \Omega_+} = \overline{\tau(Q)} = \overline{\tau^+(Q)} = \overline{\Omega} = \overline{\Omega_+}.$$

Finally, $\Omega \cup \Omega_+ = L(\tau(Q)) \cup R(\tau^+(Q)) = L(\tau(Q)) \cup R(\tau(Q)) = \overline{\tau(Q)} = \overline{\Omega}$. □

We define $s_i : I^\infty \to I^\infty$ by $s_i(\sigma) = i\sigma$ $(i = 0, 1)$. Note that both s_0, and s_1, are contractions with contractivity 1/2. We write 2^{I^∞} to denote the set of all subsets of I^∞. For $\sigma, \omega \in I^\infty$ we define

$$[\sigma, \omega] := \{\zeta \in I^\infty : \sigma \preceq \zeta \preceq \omega\},$$
$$(\sigma, \omega) := \{\zeta \in I^\infty : \sigma \prec \zeta \prec \omega\},$$
$$(\sigma, \omega] := \{\zeta \in I^\infty : \sigma \prec \zeta \preceq \omega\},$$
$$[\sigma, \omega) := \{\zeta \in I^\infty : \sigma \preceq \zeta \prec \omega\}.$$

Proposition 3. *Let* $\alpha = S(\tau(\rho))$ *and* $\beta = S(\tau^+(\rho))$.

(i) $\Omega = \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0}, \overline{1}])$ *where* $\Psi : 2^{I^\infty} \to 2^{I^\infty}$ *is defined by*

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0}, \alpha]) \cup s_1(\Lambda \cap (\beta, \overline{1}]).$$

(ii) $\Omega_+ = \bigcap_{k\in\mathbb{N}} \Psi_+^k([\overline{0},\overline{1}])$ *where* $\Psi_+ : 2^{I^\infty} \to 2^{I^\infty}$ *is defined by*

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0},\alpha)) \cup s_1(\Lambda \cap [\beta,\overline{1}]).$$

(iii) $\overline{\Omega} = \overline{\Omega_+} = \bigcap_{k\in\mathbb{N}} \overline{\Psi}^k([\overline{0},\overline{1}])$ *where* $\overline{\Psi} : 2^{I^\infty} \to 2^{I^\infty}$ *is defined by*

$$2^{I^\infty} \ni \Lambda \mapsto s_0(\Lambda \cap [\overline{0},\alpha]) \cup s_1(\Lambda \cap [\beta,\overline{1}]).$$

Proof. Proof of (i): Let $S|_\Omega : \Omega \to \Omega$ denote the domain and range restricted shift map. It is readily found that the branches of $S|_\Omega^{-1} : \Omega \to \Omega$ are $s_0|_\Omega : [\overline{0},\alpha] \cap \Omega \to \Omega$ where

$$s_0|_\Omega(\sigma) = s_0(\sigma) = 0\sigma \text{ for all } \sigma \in [\overline{0},\alpha] \cap \Omega,$$

and $s_1|_\Omega : (\beta,\overline{1}] \cap \Omega \to \Omega$ where

$$s_1|_\Omega(\sigma) = s_1(\sigma) = 1\sigma \text{ for all } \sigma \in (\beta,\overline{1}] \cap \Omega.$$

(Note that $\alpha_0 = 1$, $\beta_0 = 0$ and $\beta \prec \alpha$.) It follows that

$$S|_\Omega^{-1}(\Lambda) = s_0(\Lambda \cap [\overline{0},\alpha]) \cup s_1(\Lambda \cap (\beta,\overline{1}]) = \Psi(\Lambda)$$

for all $\Lambda \subset \Omega$. Since $\Omega \subset [\overline{0},\overline{1}]$ it follows that

$$\Omega = S|_\Omega^{-1}(\Omega) = \Psi(\Omega) \subset \Psi([\overline{0},\overline{1}]).$$

Also, since $\Psi([\overline{0},\overline{1}]) \subset [\overline{0},\overline{1}]$ it follows that $\{\Psi^k([\overline{0},\overline{1}])\}$ is a decreasing (nested) sequence of sets, each of which contains Ω; hence

$$\Omega \subset \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0},\overline{1}]).$$

It remains to prove that $\Omega \supset \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0},\overline{1}])$. We note that $s_0([\overline{0},\alpha]) = [\overline{0},\tau(\rho)]$ and $s_1((\beta,\overline{1}]) = (\tau^+(\rho),\overline{1}]$, from which it follows that

$$\bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0},\overline{1}]) = \bigcap_{k\in\mathbb{N}} \{\sigma \in I^\infty : S^k(\sigma) \in [\overline{0},\tau(\rho)] \cup (\tau^+(\rho),\overline{1}]\}. \tag{3.1}$$

Let $\omega \in \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0},\overline{1}])$. Suppose $\omega \notin \Omega$. Let

$$\omega_- = \sup\{\sigma \in \Omega : \sigma \preceq \omega\} \text{ and } \omega_+ = \inf\{\sigma \in \Omega : \omega \preceq \sigma\},$$

so that

$$\omega_- \preceq \omega \preceq \omega_+.$$

But $\omega_- \in \Omega$ (since Ω is closed from the left), so

$$\omega_- \prec \omega \preceq \omega_+.$$

Note that, since $\inf\{\sigma \in \Omega : \omega \preceq \sigma\} = \inf\{\sigma \in \Omega_+ : \omega \preceq \sigma\}$, and Ω_+ is closed from the right, we have $\omega_+ \in \Omega_+$. Let $K = \min\{k \in \mathbb{N} : (\omega_-)_k \neq (\omega_+)_k\}$. Then $S^K(\omega_-) \prec S^K(\omega) \preceq S^K(\omega_+)$ and we must have $S^K(\omega_-) = \tau(\rho)$ and $S^K(\omega_+) = \tau^+(\rho)$. So

$$\tau(\rho) \prec S^K(\omega) \preceq \tau^+(\rho),$$

therefore $\omega \notin \{\sigma \in I^\infty : S^K(\sigma) \in [\overline{0},\tau(\rho)] \cup (\tau^+(\rho),\overline{1}]\}$ which, because of (3.1), contradicts our assumption that $\omega \in \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0},\overline{1}])$. Hence $\omega \in \Omega$ and we have

$$\Omega \supset \bigcap_{k\in\mathbb{N}} \Psi^k([\overline{0},\overline{1}]).$$

This completes the proof of (i).

Proof of (ii): similar to the proof of (i), with the role of $[\overline{0},\tau(\rho)]$ played by $[\overline{0},\tau(\rho))$ and the role of $(\tau^{+}(\rho),\overline{1}]$ played by $[\tau^{+}(\rho),\overline{1}]$.

Proof of (iii): similar to the proofs of (i) and (ii). □

It is helpful to note that the addresses α and β in Proposition 3 obey

$$\alpha=\tau(W_0(\rho)),\beta=\tau(W_1(\rho)),$$

$$\tau(\rho)=0\alpha=01\alpha_1\alpha_2... \text{ and } \tau^{+}(\rho)=0\beta=10\beta_1\beta_2....$$

Let $M>0$ be such that $D_{k,M+1}=\rho$. It follows from the discussion at the end of Section 2 that $\tau((D_{k,M},\rho))=\tau^{+}((D_{k,M},\rho))=01\alpha_1\alpha_2..\alpha_{k-2}$ and $\tau((\rho,D_{k,M+2}))=\tau^{+}((\rho,D_{k,M+2}))=10\beta_1\beta_2...\beta_{k-2}$.

Corollary 1. *Let $k\geq 1$, $\alpha=\tau(W_0(\rho))$, $\beta=\tau(W_1(\rho))$, and let $M>0$ be such that $D_{k,M+1}=\rho$. The set of addresses $\{\tau((D_{k,l},D_{k,l+1}))\}_{l=0}^{D(k)-1}$ is uniquely determined by $\alpha|_{k-1}$ and $\beta|_{k-1}$. For some n_1,n_2 such that $0\leq n_1<M<n_2\leq D(k)-1$, $\tau((D_{k,n_1},D_{k,n_1+1}))=\beta_0\beta_1...\beta_{k-2}\beta_{k-1}$ and $\tau((D_{k,n_2},D_{k,n_2+1}))=\alpha_0\alpha_1...\alpha_{k-2}\alpha_{k-1}$. The set of addresses $\{\tau((D_{k,l},D_{k,l+1})):l\in\{0,1,...,D(k)-1\},l\neq n_1,l\neq n\}$ are uniquely determined by $\alpha|_{k-2}$ and $\beta|_{k-2}$; for example, $\tau((D_{k,M},\rho))=0\alpha_0\alpha_1...\alpha_{k-2}$, and $\tau((\rho,D_{k,M+2}))=1\beta_0\beta_1...\beta_{k-2}$.*

Proof. It follows from Proposition 3 that the set of addresses at level k, namely $\{\tau((D_{k,l},D_{k,l+1}))\}_{l=0}^{D(k)-1}$, is invariant under the following operation: put a "0" in front of each address that is less than or equal to α, then truncate back to length k; take the union of the resulting set of addresses with the set of addresses obtained by: put a "1" in front of each address that is greater than or equal to β, and drop the last digit. □

4. Symmetry of $\overline{\Omega}$ and a consequent homeomorphism of $[0,1]$

Lemma 1. $\overline{\Omega}=\{\sigma\in I^{\infty}:$ *for all $k\in\mathbb{N}$, $\sigma_k=0\Rightarrow S^k(\sigma)\preceq\tau(\rho)$ and $\sigma_0=1\Rightarrow\tau^{+}(\rho)\preceq S^k(\sigma)\}$.*

Proof. This is an immediate consequence of Proposition 3. □

Corollary 2. *$\overline{\Omega}$ is symmetric if and only if $\alpha=\beta^*$ (or equivalently $\tau(\rho)=(\tau^{+}(\rho))^*$).*

Lemma 2. *The maps $\tau(\rho)$ and $\tau^{+}(\rho)$ are strictly increasing as functions of $\rho\in[a,b]$ to I^{∞}.*

Proof. Note that $\tau(\rho)$ depends both implicitly and explicitly on ρ. Let $1-b\leq\rho<\rho'\leq a$ be such that $\tau(\rho)\succeq\tau(\rho')$. Observe that $\tau(\rho)|_0=\tau(\rho')|_0$.

Assume first that there is a largest $n>0$ such that $\tau(\rho)|_n=\tau(\rho')|_n:=\theta_0\theta_1...\theta_n$. Then $\tau(\rho)=\theta_0\theta_1...\theta_n1...$ and $\tau^{+}(\rho)=\theta_0\theta_1...\theta_n0...$, which implies

$$W_{\rho}^{n}(\rho)\geq\rho \text{ and } W_{\rho'}^{n}(\rho')\leq\rho'. \quad (4.1)$$

(We write $W=W_\rho$ when we want to note the dependence on ρ.) We may assume that $\tau(\rho)|_n$ is constant on $[\rho,\rho']$ for otherwise we can restrict to a smaller interval with a strictly smaller value of n. As a consequence, at every iteration, we apply the same branch W_0 or W_1 to W_ξ to compute compute $g(\xi):=W_{\xi}^{n}(\xi)$ for all $\xi\in[\rho,\rho']$. Therefore g is continuous with derivative at least $d^n>1$, which contradicts (4.1).

The only remaining possibility is that $\tau(\rho) = \tau(\rho')$. We may assume that $\tau(\rho)$ is constant on $[\rho, \rho']$, otherwise we can reduce the problem to the previous case. This would mean that for arbitrarily large n, the image of the interval $[\rho, \rho']$ under g is at an interval of size least $d^n(\rho' - \rho)$, a contradiction.

Essentially the same argument, with the role of τ played by τ^+ and the role of played by W_+, proves that $\tau^+(\rho)$ is strictly increasing as a function of $\rho \in [1-b, a]$ to I^∞. □

Corollary 3. *The map $\rho \mapsto \tau(\rho)$ is left continuous and the map $\rho \mapsto \tau^+(\rho)$ is right continuous.*

Proof. Fix a parameter ρ_0 and let $\varepsilon > 0$. Then by (7) there is $x < \rho_0$ which is not a preimage of ρ_0 for any order and such that

$$d(\tau^+_{\rho_0}(x), \tau_{\rho_0}(\rho_0)) < \frac{\varepsilon}{2}.$$

By (9), for any $n \in \mathbb{N}$ there exists $\delta > 0$ such that the prefix $\tau^+_\rho(x)|_n$ is constant when $\rho \in (\rho_0 - \delta, \rho_0 + \delta)$. Let n be such that $2^{-n} < \varepsilon$, and let $\rho > x$ and $\rho \in (\rho_0 - \delta, \rho_0)$. We have that $\tau^+_\rho(x) \prec \tau^+_\rho(\rho)$ and

$$d(\tau^+_\rho(x), \tau^+_{\rho_0}(x)) < \frac{\varepsilon}{2}.$$

Combining the two inequalities we obtain

$$d(\tau^+_\rho(x), \tau_{\rho_0}(\rho_0)) < \varepsilon,$$

and by Lemma 2 we also have

$$\tau^+_\rho(x) \prec \tau_\rho(\rho) \prec \tau_{\rho_0}(\rho_0).$$

The distance d has the property that if $\sigma \prec \zeta \prec \sigma'$ then $d(\sigma, \zeta) \le d(\sigma, \sigma')$ and $d(\zeta, \sigma') \le d(\sigma, \sigma')$. This shows that $\rho \mapsto \tau(\rho)$ is left continuous. The right continuity of $\rho \mapsto \tau^+(\rho)$ admits an analagous proof. □

As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of ρ for which $\overline{\Omega}$ is symmetric.As a consequence of Corollary 2, Lemma 2 and (10), we obtain the unicity of ρ for which $\overline{\Omega}$ is symmetric.

Corollary 4. *There is at most one $\rho \in [1-b, a]$ such that $\overline{\Omega} = \overline{\Omega}^*$.*

Proof of Theorem 2. By Lemma 2 and (10), we may define

$$\rho_0 := \sup\{\rho \in [1-b, a] : \tau(\rho) \preceq \tau^+(\rho)^*\} = \inf\{\rho \in [1-b, a] : \tau(\rho)^* \preceq \tau^+(\rho)\}.$$

Assume $\tau(\rho_0) \prec \tau^+(\rho_0)^*$. It is straighfoward to check $1 - b < \rho_0 < a$.

There is a largest $n \ge 2$ such that $\tau(\rho_0)|_n = \tau^+(\rho_0)^*|_n =: \eta = 01...$.

Observe that $\tau(\rho_0) = 0\tau(W_0(\rho_0))$ and $\tau^+(\rho_0) = 1\tau^+(W_1(\rho_0))$. If neither $W_0(\rho_0)$ nor $W_1(\rho_0)$ belongs to $\{0,1\} \cup \bigcup_{k=0}^{n-1} W^{-k}(\rho_0)$, then by (9) both $\tau(\rho)|_{n+1}$ and $\tau^+(\rho)|_{n+1}$ are constant on a neighborhood of ρ_0 which contradicts the definition of ρ_0.

Let us consider the projection $\hat{\pi}(\tau^+(W_1(\rho_0))^*)$. If $\hat{\pi}(\tau^+(W_1(\rho_0))^*) > W_0(\rho_0)$ then by the continuity of W_0, of $\hat{\pi}$ by (13), of $\rho \mapsto \tau^+_\rho(\rho)$ (Corollary 3) there is a $\rho > \rho_0$ such that $\hat{\pi}_\rho(\tau^+_\rho(W_1(\rho_0))^*) > W_0(\rho)$. By (6) and (13) this implies $\tau_\rho(\rho) \prec \tau^+_\rho(\rho)^*$, which again contradicts the definition of ρ_0.

As $\hat{\pi}$ is increasing, (13) and $\tau(W_0(\rho_0)) \prec \tau^+(W_1(\rho_0))^*$, we have $\hat{\pi}(\tau^+(W_1(\rho_0))^*) = W_0(\rho_0)$. Let $0 < m < n$ be minimal such that $W^m \circ W_0(\rho_0) = \rho_0$ or $W^m \circ W_1(\rho_0) = \rho_0$. We may apply (14) m times and obtain

$$W^m \circ W_0(\rho_0) = \hat{\pi}(S^m(\tau^+(W_1(\rho_0))^*)) = \hat{\pi}(\tau^+(W^m \circ W_1(\rho_0))^*). \tag{4.2}$$

As $\tau^+(\rho_0) = 1...$, if $W^m \circ W_1(\rho_0) = \rho_0$ then we have

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* = \tau^+(W^m \circ W_1(\rho_0))^* \prec \tau^+(\rho_0),$$

which by (6) and equation (4.2) implies $W^m \circ W_0(\rho_0) = \rho_0$. Therefore $\tau(\rho_0) = \tau^+(\rho_0)^*$ as both are periodic of period $m+1$ and have the same prefix of length $n > m$, a contradiction.

If $W^m \circ W_1(\rho_0) \neq \rho_0$ then $W^m \circ W_0(\rho_0) = \rho_0$ thus by (13), (10) and equality (4.2) we obtain

$$\tau(\rho_0) \prec \tau^+(\rho_0)^* \preceq \tau^+(W^m \circ W_1(\rho_0)) := \sigma'.$$

By (6), this means that $\rho_0 \leq W^m \circ W_1(\rho_0)$ so in fact

$$\rho_0 < W^m \circ W_1(\rho_0).$$

As $W^{m+1}(\rho_0) = \rho_0$, $\tau(\rho_0) = \kappa\kappa\kappa... := \kappa^\infty$ where $\kappa = \tau(\rho_0)|_{m+1} = \tau^+(\rho_0)^*|_{m+1}$, as $m+1 \leq n$. We can write $\tau^+(\rho_0) = \kappa^*\sigma'$ therefore $\kappa^*\sigma' \prec \sigma'$ by (6) and the previous inequality. By induction we get $\kappa^{*\infty} \prec \sigma'$ so

$$\tau^+(\rho_0)^* = \kappa(\sigma'^*) \prec \kappa^\infty = \tau(\rho_0),$$

a contradiction.

The case $\tau(\rho_0) \succ \tau^+(\rho_0)^*$ is analagous by the symmetric definition of ρ_0, therefore $\overline{\Omega}_{\rho_0}$ is symmetric. □

Proposition 4. *If $\overline{\Omega} = \overline{\Omega}^*$ then the map $h : [0,1] \to [0,1]$ defined by $h(x) = \hat{\pi}(\tau(x)^*)$ is a homeomorphism and $h \circ \hat{\pi} = \hat{\pi} \circ^*$ on I^∞.*

Proof. First by Corollary 2, we have $\tau(\rho) = \tau^+(\rho)^*$ and points x for which $\tau(x) \neq \tau^+(x)$ are exactly preimages of ρ. In this case, there is $n \geq 0$ such that $\tau(x)$ and $\tau^+(x)$ have the same initial prefix $\kappa := \tau(x)|_n = \tau^+(x)|_n$, and $\tau(x) = \kappa\tau(\rho)$, $\tau^+(x) = \kappa\tau^+(\rho)$. Therefore, by (13), for all $x \in [0,1]$, we have

$$\tau(h(x)) = \tau^+(x)^* \text{ and } \tau^+(h(x)) = \tau(x)^*,$$

thus $h \circ h(x) = x$. By (6), (10) and (13), h is also decreasing. Therefore $h : [0,1] \to [0,1]$ is a homeomorphism.

Let $\sigma \in I^\infty$ and $x = \hat{\pi}(\sigma)$. By (13) we have $\tau(x) \preceq \sigma \preceq \tau^+(x)$. As $\overline{\Omega} = \overline{\Omega}^*$, by Proposition 2, we obtain that there exists $y \in [0,1]$ such that $\tau(x)^* = \tau^+(y)$. By Lemma 1 and Corollary 2 we also have that $\tau^+(x)^* = \tau(y)$. We may compute $h \circ \hat{\pi}(\sigma) = h(x) = \hat{\pi}(\tau(x)^*) = \hat{\pi}(\tau(y)) = y$, which is also equal to $\hat{\pi}(\sigma^*)$ as $\tau(y) \preceq \sigma^* \preceq \tau^+(y)$.

□

5. Iterated Closed Relations and Conley Decomposition for Itineraries of W

Theorem 1 follows from Proposition 3, but some extra language is needed. In explaining this language we describe the Conley-McGehee-Wiandt decomposition theorem, [3, Theorem 13.1].

For X a compact Hausdorff space, let 2^X be the subsets of X. A *relation* r on X is simply a subset of $X \times X$. A relation r on X is called a *closed relation* if r is a closed of $X \times X$. For example the set $r \subset I^\infty \times I^\infty$ defined in Theorem 1, namely

$$r = \{(0\sigma, \sigma) \in I^\infty \times I^\infty : \sigma \preceq \alpha\} \cup \{(1\sigma, \sigma) \in I^\infty \times I^\infty : \beta \preceq \sigma\},$$

is a closed relation. Following [3], a relation $r \in 2^X$ provides a mapping $r : 2^X \to 2^X$ defined by

$$r(C) = \{y \in X : (x, y) \in r \text{ for some } x \in C\}.$$

Notice that the image of a nonempty set may be empty. Iterated relations are defined by $r^0 = X \times X$ and, for all $k \in \mathbb{N}$,

$$r^{k+1} = r \circ r^k = \{(x, z) : (x, y) \in r, (y, z) \in r^k \textit{ for some } y \in X\}.$$

The *omega limit set* of $C \subset X$ under a closed relation $r \subset X \times X$ is

$$\omega(C) = \cap\mathfrak{K}(C)$$

where

$$\mathfrak{K}(C) = \{D \textit{ is a closed subset of } X : r(D) \cup r^n(C) \subset D \textit{ for some } n \in \mathbb{N}\}.$$

By definition, an *attractor* of a closed relation r is a closed set A such that the following two conditions hold:

(i) $r(A) = A$;

(ii) there is a closed neighborhood $\overline{\mathcal{N}}(A)$ of A such that $\omega(C) \subset A$ for all $C \subset \overline{\mathcal{N}}(A)$.

The basin $\mathcal{B}(A)$ of an attractor A for a closed relation r on a compact Hausdorff space X is the union of all open sets $O \subset X$ such that $\omega(C) \subset A$ for all $C \subset O$.

Given an attractor A for a closed relation r on a compact Hausdorff space X, there exists a corresponding *attractor block*, namely a closed set $E \subset X$ such that E contains both A and $r(E)$ in its interior, and $A = \omega(E)$. Also, there exists a unique *dual repeller* $A^* = X \backslash \mathcal{B}(A)$. This repeller is an attractor for the transpose relation $r^* = \{(y, x) : (x, y) \in r\}$. The set of connecting orbits associated with the attractor/repeller pair A, A^* is $\mathcal{C}(A) = X \backslash (A \cup A^*)$.

If r is a closed relation on a compact Hausdorff space X, then $x \in X$ is called *chain-recurrent* for r if for every closed neighborhood f of r, x is periodic for f (i.e. there exists a finite sequence of points $\{x_n\}_{n=0}^{p-1} \subset X$ such that $x_0 = x$, $(x_{p-1}, x_0) \in f$ and $(x_{n-1}, x_n) \in f$ for $n = 1, 2, ..., p-1$). The chain recurrent set $\mathcal{R}$ for r is the union of all the points that are chain recurrent for r. A transitive component of $\mathcal{R}$ is a member of the equivalence class on $\mathcal{R}$ defined by $x \sim y$ when for every closed neighborhood f of r there is an orbit from x to y under f (i.e. there exists a finite sequence of points $\{x_n\}_{n=0}^{p-1} \subset \mathcal{R}$ such that $x_0 = x$, $x_{p-1} = y$, and $(x_n, x_{n+1}) \in f$ for all $n \in \{0, 1, ...p-1\}$.)

Theorem 3 (Conley-McGehee-Wiandt)**.** *If r is a closed relation on a compact Hausdorff space X, then*

$$\mathcal{R} = \bigcup_{A \in \mathcal{U}} \mathcal{C}(A)$$

where $\mathcal{R}$ is the chain-recurrent set and $\mathcal{U}$ is the set of attractors.

Proof of Theorem 1. This follows at once from Proposition 3 together with Theorem 3, but see [3]. □

We note the following. $\overline{\Omega}$ can be embedded in $[0,1] \subset \mathbb{R}$ using the (continuous and surjective) coding map $\pi : I^\infty \to [0,1]$ associated with the iterated function system $([0,1]; x \mapsto x/2, x \mapsto (1+x)/2)$. This coding map π is defined, for all σ, by

$$\pi(\sigma) = \sum_{k \in \mathbb{N}} \frac{\sigma_k}{2^{k+1}}.$$

π provides a homeomorphism between $\overline{\Omega}$ and $\pi(\overline{\Omega})$. The point $\sigma \in \overline{\Omega}$ is uniquely and unambiguously represented by the binary real number $0.\sigma$. In the representation provided by π, the map $\overline{\Psi} : 2^{I^\infty} \to 2^{I^\infty}$ becomes the action of the iterated closed relation $\widetilde{r} \subset [0,1] \times [0,1] \subset \mathbb{R}^2$ defined by

$$\widetilde{r} := \{(x, x/2) : x \in [0, \pi(\alpha)]\} \cup \{(x, (x+1)/2) : x \in [\pi(\beta), 1]\}$$

on subsets of $[0,1]$. It follows from Proposition 3 (iii) that $\pi(\overline{\Omega})$ is the maximal attractor, as defined in [3], of $\widetilde{r}$. The corresponding dual repeller is the empty set. It is also easy to see that $\{0\}$ and $\{1\}$ are the only other attractors, with corresponding dual repellers $[\pi(\alpha), 1]$ and $[0, \pi(\beta)]$ respectively. It follows from Theorem 3 that the chain recurrent set of $\widetilde{r}$ is $\{0,1\} \cup (\pi(\overline{\Omega}) \cap (\pi(\beta), \pi(\alpha)))$.

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References

[1] M. F. Barnsley, B. Harding and K. Igudesman, How to filter and transform images using iterated function systems, SIAM Journal of Imaging Science, 4 (2011) 1001–1024.

[2] K. B. Igudesman, A certain family of self-similar sets, Russian Mathematics (Iz. Vuz) 55 (2011) 26-38.

[3] Richard P. McGehee and T. Wiandt, Conley decomposition for closed relations, Difference Equations and Applications, 12 (2006) 1-47.

[4] John E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.

[5] William Parry, Symbolic dynamics and transformations of the unit interval, Trans. Am. Math. Soc, 122 (1966) 368-378.

Department of Mathematics, Australian National University, Canberra, ACT, Australia

E-mail address: michael.barnsley@maths.anu.edu.au

Université Paris-Est Créteil, LAMA, 94 010 Créteil, France

E-mail address: nicolae.mihalache@u-pec.fr