Metric properties of mean wiggly continua

Jacek Graczyk, Peter Jones, Nicolae Mihalache

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Jacek Graczyk
Univ. de Paris-Sud
Lab. de Mathématiques
91405 Orsay, France

Peter W. Jones
Dept. of Math.
Yale, New Haven
CT 06520, USA

Nicolae Mihalache
Univ. Paris-Est Créteil
LAMA
94 010 Créteil, France

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Abstract

We study lower and upper bounds of the Hausdorff dimension for sets which are wiggly at scales of positive density. The main technical ingredient is a construction, for every continuum K, of a Borel probabilistic measure μ with the property that on every ball $B(x,r)$, $x \in K$, the measure is bounded by a universal constant multiple of $r\exp(-g(x,r))$, where $g(x,r) \geq 0$ is an explicit function. The continuum K is mean wiggly at exactly those points $x \in K$ where $g(x,r)$ has a logarithmic growth to ∞ as $r \to 0$. The theory of mean wiggly continua leads, via the product formula for dimensions, to new estimates of the Hausdorff dimension for Cantor sets. We prove also that asymptotically flat sets are of Hausdorff dimension 1 and that asymptotically non-porous continua are of the maximal dimension. Another application of the theory is geometric Bowen's dichotomy for Topological Collet-Eckmann maps in rational dynamics. In particular, mean wiggly continua are dynamically natural as they occur as Julia sets of quadratic polynomials for parameters from a generic set on the boundary of the Mandelbrot set $\mathcal{M}$.

1 Introduction

1.1 Overwiew

An intuition about a compact connected set is that if it oscillates at every scale then its Hausdorff dimension is strictly bigger than 1. One way to quantify the concept of geometric oscillations is to use the theory of β-numbers of Bishop and Jones, see Definition 1.1. A set K is called wiggly at x and scale $r > 0$ if one can draw a triangle contained in the ball $B(x,r)$ with vertices in K so that after rescaling to the unit ball, the triangle belongs to a compact family of triangles in $\mathbb{R}^d$, with $d \geq 2$. An immediate consequence of this approach is that essentially, by the Pythagorean Theorem, the set K "accumulates" an additional length at scale r. The word "essentially" is needed here because a wiggly continuum can intersect $B(x,r)$ along disjoint intervals. Even in the case of the regular intersections, the accumulation of length at scale r remains true but the mechanism of a local length growth is more complicated and comes from the global hypothesis about the connectivity of K.

In [5], it is proven that every connected and compact planar set K which oscillates uniformly at every scale around every point of K has the Hausdorff dimension strictly bigger than 1. The dimension estimates of [5] are quantified in terms of β-numbers.

Definition 1.1. *Let K be a bounded set in $\mathbb{R}^d$ with $d \geq 2$, $x \in K$ and $r > 0$. We define $\beta_K(x,r)$ by*

$$\beta_K(x,r) := \inf_L \sup_{z \in K \cap B(x,r)} \frac{\operatorname{dist}(z,L)}{r} ,$$

where the infimum is taken over all lines L in $\mathbb{R}^d$.

A connected bounded set K is called *uniformly wiggly* if

$$\beta_\infty(K) := \inf_{x \in K} \inf_{r \leq \operatorname{diam} K} \beta(x,r) > 0 .$$

Theorem 1.1 of [5] states that if $K \subset \mathbb{C}$ is a continuum and $\beta_\infty(K) > 0$ then $\dim_{\mathrm{H}}(K) \geq 1 + c\beta_\infty^2(K)$, where c is a universal constant.

One of the main objectives of this paper is to introduce analytic tools based on corona type constructions [6, 10, 17] which can be useful in the area of complex dynamics. In the area of dynamical systems one cannot expect that generic systems are uniformly hyperbolic and that geometry of invariant fractals can be estimated at every scale as it is required in [5]. However, there are various results showing that non-uniformly hyperbolic systems are typical in ambient parameter spaces [15, 3, 25, 1, 13, 27]. Our results which rely only on mean estimates of wiggliness fit into a general scheme of studying metric properties of attractors for generic dynamical systems.

A direct outcome of the proposed methods is Theorem 1 which gives an integral and non-uniform version of the dimension results of [5]. Theorem 2 shows that the estimates of Theorem 1 are sharp. The main technical difficulty in proving Theorem 1 lies in controlling non-wiggly portions of a continuum K which can have a finite length. If non-wiggly portions of K have infinite length then the assertion that $\dim_{\mathrm{H}}(K) > 1$ is generally not true. As an example take a unit segment in the plane accumulated by a smooth curve winding infinitely many times around it. Admitting exception sets allows also for an immediate extension of the dimension estimates of Theorem 1 over those disconnected sets K which can be turned into a continuum $K \cup E'$ by adding a set E' of finite length.

In a different context, a conceptually similar approach was adopted by Koskela and Rohde in their proof that mean porosity replaces porosity for upper estimates of the Hausdorff dimension [18].

A standard observation about Hausdorff dimension is that only asymptotic properties should intervene in the estimates. Another point is that even if a connected set is not wiggly at every scale, the property to wiggle often enough, for example on a set of positive density of scales, should be sufficient to observe an exponential growth of length in every scale.

Theorem 1 generalizes Theorem 1.1 of [5] in three directions. Firstly, it allows for an exceptional set $E \subset K$ of finite 1-dimensional Hausdorff measure. Secondly, it is enough to assume that the continuum K oscillates at every point $x \in K \setminus E$ and every scale $r > 0$ with some parameter $\beta_K(x,r) > 0$ which depends both on x and $r > 0$ so that

$$\liminf_{r\to 0} \frac{\int_r^{\operatorname{diam} K} \beta_K^2(x,t)\frac{dt}{t}}{-\log r} \geq \beta_0^2 > 0 \ . \tag{1}$$

This means that the uniform hypothesis of [5] that there exists $\beta_0 > 0$ so that for every $x \in K$ and every $r < \operatorname{diam} K$,

$$\beta_K(x,r) \geq \beta_0$$

can replaced by the integral condition (1) without affecting the main estimate on $\dim_{\mathrm{H}}(K)$ of [4] that $\dim_{\mathrm{H}}(K) \geq 1 + c\beta_0^2$, where c is a universal constant. Theorem 1 is also valid in higher ambient dimension.

An additional feature of Theorem 1 is that the condition (1) can be further relaxed,

$$\liminf_{r\to 0} \frac{\int_r^{\operatorname{diam} K} \beta_K^2(x,r)\frac{dt}{t}}{-\log r} > 0 \tag{2}$$

and still obtain a non-uniform estimate that $\dim_{\mathrm{H}}(K) > 1$.

The main technical ingredient of the paper is Theorem 9 which claims that for every continuum K of diameter 1 there exists a Borel probabilistic measure μ supported on a wiggly subset Z of K such that for every ball $B(x,r)$, $x \in Z$,

$$\mu(B(x,r)) \ \leq \ \frac{c'r}{\operatorname{diam} Z} \exp\left(-c\int_r^{\operatorname{diam} K} \beta_K^2(x,t)\frac{dt}{t}\right) ,$$

where c, c' are universal constants. The construction of the measure μ is based on a combinatorial Proposition 1 and a geometric version of the corona type construction explained in [23].

The study of the distribution of the measure μ on Julia sets is of independent interest as the relations between μ and other natural measures in complex dynamics are not known. By Corollary 2.3, μ is absolutely continuous with respect to 1-dimensional Hausdorff measure $\mathcal{H}^1$ on connected Julia sets and thus μ is not atomic and $\dim_{\mathrm{H}}(\mu) \geq 1$.

Theorem 1. *Suppose a nontrivial compact connected $K \subset \mathbb{R}^d$ where $d \geq 2$ is the union of two subsets $K = W \cup E$, $\mathcal{H}^1(E) < \infty$ and $\mathcal{H}^1(W) > 0$.*

- *If there exists $\beta_0 > 0$ such that for all x in W*

  $$\liminf_{r\to 0} \frac{\int_r^{\operatorname{diam} K} \beta_K^2(x,t)\frac{dt}{t}}{-\log r} \geq \beta_0^2,$$

  then

  $$\dim_{\mathrm{H}}(K) \geq 1 + c\beta_0^2,$$

  where c is a universal constant.

- *If for all $x \in W$*

  $$\liminf_{r\to 0} \frac{\int_r^{\operatorname{diam} K} \beta_K^2(x,t)\frac{dt}{t}}{-\log r} > 0,$$

  then

  $$\dim_{\mathrm{H}}(K) > 1.$$

The condition $\mathcal{H}^1(E) < \infty$ can not be further relaxed as shows the following example (Warsaw sine): Let K be the closure of the graph G of $y = \sin(1/x)$, $x \in (0,1]$, in the Euclidean planar topology. The wiggly set W is a vertical segment $\{0\}\times[-1,1]$, the exceptional set E is the graph G which has an infinite length. Clearly, $\dim_{\mathrm{H}}(K) = 1$. Further examples showing that all hypotheses of Theorem 1 are essential are discussed in Section 1.6.

We have already observed that the hypothesis about connectivity of K can be slightly relaxed. In fact, the estimates of Theorem 1 can be also localized. If $D(x,R)$ is a ball such that $K' := \mathbb{D}(x,R)\cap K \cup \partial D(x,R)$ is connected we can apply Theorem 1 for a new continuum K' and a new exceptional set $E' = \mathbb{D}(x,R)\cap E\cup\partial\mathbb{D}(x,R)$. Clearly, $\dim_{\mathrm{H}}(K') = \dim_{\mathrm{H}}(K\cap \mathbb{D}(x,R))$.

On the other hand, there are obvious examples of totally disconnected and uniformly wiggly compacts $K\times K$, where $K\subset[0,1]$ is a Cantor set of bounded geometry with $\dim_{\mathrm{H}}(K) < \frac{1}{2}$.

Almost flat sets. We will show that the estimates of Theorem 1 are sharp. Recall that a set $N\subset\mathbb{C}$ is ε-porous at x at scale $r>0$ if there is $z\in\mathbb{D}(x,r)$ such that $\mathbb{D}(z,\varepsilon r)\subset\mathbb{D}(x,r)\setminus N$. The condition $\beta_N(x,r)\le\alpha$ implies that N is $(1-\alpha)/2$-porous at scale r at x.

If one assumes that for every x in a bounded set N,

$$\limsup_{r\to 0}\frac{\int_r^{\operatorname{diam} N}\beta_N^2(x,t)\frac{dt}{t}}{-\log r} = 0, \tag{3}$$

then $\dim_{\mathrm{H}}(N)\le 1$ follows from the dimension results of Belaev and Smirnov for mean porous sets, Corollary 1 in [2]. The condition $\beta_N(x,r) = 0$ is much stronger than the maximal porosity $1/2$ at x. It turns out that one can replace lim sup by lim inf in the inequality (3) and still obtain that $\dim_{\mathrm{H}}(N)\le 1$.

Theorem 2. *Let N be a set in $\mathbb{R}^d$ with $d\ge 2$ such that for all $x\in N$,*

$$\liminf_{r\to 0}\frac{\int_r^1\beta_N^2(x,t)\frac{dt}{t}}{-\log r}\le\beta_0^2,$$

then there is a universal constant $c>0$ such that

$$\dim_{\mathrm{H}}(N)\le 1+c\beta_0^2.$$

Proof. Without loss of generality we may assume that $\operatorname{diam} N = 1$. Fix $\epsilon>0$ and $n_0>1$. For every $n\ge n_0$, denote

$$X_n = \left\{x\in N : \int_{2^{-n}}^1\beta_N^2(x,t)\frac{dt}{t}\le(\epsilon+\beta_0^2)n\log 2\right\}.$$

By definition, $N=\bigcup_{n\ge n_0}X_n$. By Besicovitch's covering theorem, we can find M subcollections $\mathcal{G}_i$ such that every two balls from the same subcollection $\mathcal{G}_i$ are disjoint, every ball is of radius 2^{-n}, and X_n is covered by the balls from $\mathcal{G}(n) = \bigcup_{i \le M} \mathcal{G}_i$. Let Z_i be the set of the centers of the balls from $\mathcal{G}_i$. It is a finite set with the property that

$$\int_0^{\operatorname{diam} Z_i} \beta^2_{Z_i}(x,t)\frac{dt}{t} \le \int_{2^{-n}}^1 \beta^2_N(x,t)\frac{dt}{t} \le (\epsilon + \beta_0^2) n \log 2 \ .$$

By Theorem 11, for every $i \le M$, the set Z_i is contained in a curve Γ_i of length $\mathcal{H}^1(\Gamma_i) \le Ce^{C(\epsilon+\beta_0^2)n\log 2}$, where $C > 1$ is a universal constant. Using the fact that the balls from $\mathcal{G}_i$ are disjoint, we have that

$$\sum_{B \in \mathcal{G}_i} (\operatorname{diam} B)^{1+\alpha} \le 2^{(-n+1)\alpha}\, \mathcal{H}^1(\Gamma_i) \le C_1 2^{-n(\alpha - C(\epsilon+\beta_0^2))} \ .$$

Hence, the sum of the diameters to the power $1+\alpha$ of the balls from $\mathcal{G}(n)$ is smaller than

$$MC_1 2^{-n(\alpha - C(\epsilon+\beta_0^2))} \ .$$

Let $\mathcal{G} = \bigcup_{n \ge n_0} \mathcal{G}(n)$. Assuming that α is bigger than $2C(\epsilon + \beta_0^2)$, we obtain that

$$\sum_{B \in \mathcal{G}} (\operatorname{diam} B)^{1+\alpha} \le \sum_{n=n_0}^{\infty} MC_1 2^{-n(\alpha - C(\epsilon+\beta_0^2))} \le C' 2^{-n_0 C \beta_0^2} \ ,$$

where C' is a constant. Passing with n_0 to $+\infty$, we infer that $\mathcal{H}^{1+\alpha}(N) < +\infty$ which completes the proof.

□

Invariance property. Suppose that K is a continuum which satisfies the hypothesis of Theorem 1 with $E = \emptyset$. Let $\mathcal{S}_K$ be the set of all continuous functions $h : U \mapsto \mathbb{C}$, defined on some neighborhood U of K, conformal with the Jacobian different from zero on $K \setminus F \ne \emptyset$ and $\mathcal{H}^1(h(F)) = 0$. We recall that $f : U \mapsto \mathbb{C}$ is conformal at $z_0 \in U$ if the limit $(f(z) - f(z_0))/(z - z_0)$ exists and is different from 0. Then,

$$\inf_{h \in \mathcal{S}_K} \dim_{\mathrm{H}}(h(K)) \ge 1 + c\beta_0^2 \ .$$

Indeed, h does not decrease $\beta_K(z)$ at any point $z \in K \setminus F$. Since $h(K)$ is a continuum, the assumption that $\mathcal{H}^1(h(F)) = 0$ implies that $h(K \setminus F)$ is a wiggly part of $h(K)$, see Theorem 3, of positive length.

Generalizations are possible. A direction for possible generalizations can be adopted following [8] where a uniform non-flatness of compacts with respect to d-dimensional planes is studied. Under some additional topological assumptions which replace connectedness, it is proved that the Hausdorff dimension of uniformly non-flat compacts in $\mathbb{R}^n$ is strictly bigger than d.

1.2 Mean wiggly continua

The integral condition (1) has a discrete counterpart. Let K be a set in $\mathbb{R}^d$ of diameter 1 and fix $\lambda \in (0,1)$. For every $x \in K$ and $m \in \mathbb{N}$, if $\beta(x, \lambda^{-m}) > c_0$ then $\beta(x, \lambda^{m-1}) > \lambda c_0$. Hence, if $r \in A_m = [\lambda^{-m-1}, \lambda^{-m})$, $m \in \mathbb{N}$, then the limit

$$\liminf_{r\to 0} \frac{1}{-\log r} \int_r^1 \beta_K^2(x,t)\frac{dt}{t} \tag{4}$$

is equivalent to

$$\liminf_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \sup_{t\in A_i} \beta^2(x,t) \sim \liminf_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \beta^2(x, \lambda^{-i}) \; . \tag{5}$$

We will say that a set $K \subset \mathbb{R}^d$ is *mean wiggly* at a point $x \in K$ with parameters $\lambda, \kappa, \beta_0 \in (0,1)$ if there is $r(x) > 0$ such that for $\lambda^n < r(x)$ the number of wiggly scales $\lambda^m, m < n$, is greater than κn. Here, a wiggly scale is defined by the condition that $\beta(x, \lambda^m) \geq \beta_0$

In particular, the limit (5) is positive iff K is mean wiggly with some positive parameters at x. The hypothesis of Theorem 1 can be reformulated in terms of mean density of wiggly scales.

Theorem 3. *Suppose that $K \subset \mathbb{R}^d$ with $d \geq 2$ is a continuum of diameter 1 and $K = W \cup E$ where W and E satisfy the following:*

- *$\mathcal{H}^1(E) < \infty$ and $\mathcal{H}^1(W) > 0$.*
- *K is mean wiggly at every point $x \in W$.*

Then

$$\dim_{\mathrm{H}}(K) > 1 \; .$$

If additionally, the parameters of the wiggliness λ, κ, β_0 at $x \in W$ are uniform, that is do not depend on $x \in W$, then

$$\dim_{\mathrm{H}}(K) \geq 1 + c'\lambda^4\beta_0^2 \; \kappa,$$

where c' is a universal constant.

Proof. By Theorem 1, it is enough to estimate from below the limit (4) for all $x \in W$. For every integer $i \geq 0$, define $A_i = [\lambda^{i+1}, \lambda^i)$. Set $\chi_i = 1$ if K is β_0-wiggly at $x \in W$ at some scale from A_i. If $\chi_{i+1} = 1$ then

$$\int_{A_i} \beta_K^2(x,r)\frac{dt}{t} \ \geq \ \int_{\lambda^{i+1}}^{\lambda^i} \left(\beta_0 \frac{\lambda^{i+2}}{t}\right)^2 \frac{dt}{t} \ \geq \ \lambda^4 \beta_0^2 \log \lambda^{-1} \ .$$

Therefore, the lower bound of the limit (4) is given by

$$\lambda^4 \beta_0^2 \log \lambda^{-1} \ \liminf_{n\to\infty} \frac{\#\{\chi_i = 1 : i \in [0, n+1)\}}{(n+1)\log \lambda^{-1}} \geq \kappa \lambda^4 \beta_0^2 \ .$$

□

Almost flat sets. A set $K \subset \mathbb{R}^d$ is *almost flat* at a point $x \in K$ with parameters $\lambda, \kappa, \beta_0 \in (0,1)$ if there is a sequence of positive integers (N_i) such that for every $i \in \mathbb{N}$ the number of flat scales $\lambda^m, m < N_i$, is greater than κN_i. A flat scale is defined by the condition $\beta(x, \lambda^m) \leq \beta_0$.

Theorem 4. *Suppose that $K \subset \mathbb{R}^d$, $d \geq 2$, is a set of diameter 1 and K is almost flat at every point $x \in K$ with the parameters λ, κ, β_0 that do not depend on $x \in W$. Then,*

$$\dim_{\mathrm{H}}(K) \leq 1 + c' \lambda^{-4}(1 - \kappa + \beta_0^2 \ \kappa),$$

where c' is a universal constant.

Proof. The proof follows immediately from Theorem 2 and a short calculation very much the same as in the proof of Theorem 3. □

If for every $x \in K$, the sequence (N_i) from the hypotheses of Theorem 4 contains the set of almost all positive integers, then Corollary 1 in [2] (see also [26, 19])) implies that

$$\dim_{\mathrm{H}}(K) < d - \kappa + \frac{C}{|\log \beta_0|} \ ,$$

where C is a universal constant.

Non-porous continua. Sharp estimates of the Hausdorff dimension usually require that a set has a self-similar or at least a well-defined hierarchical structure. If this additional structure is present then a relative geometrical data (scalings) leads to useful dimension estimates, see for example [21] for applications of this technique in the complex dynamics. When the hierarchical structure of the set is missing, the dimension estimates become difficult. Our objective is to prove sharp lower bounds for the Hausdorff dimension under mild topological assumptions as connectivity, compactness, and relative density. In Section 1.3, we further discuss possible applications of our techniques in the case of compact sets.

Let $\epsilon \in (0, 1/2)$ and $\alpha, \lambda \in (0, 1)$. The set K is (ϵ, α)-dense in a ball $B(x, r)$ if there exists an one-sided cone C with apex at x such that

$$C \cap B \subseteq N(K \cap B, \varepsilon) \text{ and } m(C \cap B) \geq \alpha\, m(B),$$

where m is the Lebesgue measure and $N(A, \delta) = \cup_{x \in A} B(x, \delta)$ is the δ-neighborhood of the set A. The set K is (ϵ, α)-dense at $z \in K$ at scales of density $\kappa > 0$ if

$$\liminf_{m \to \infty} \frac{1}{m} \#\{n \in (0, m) : K \text{ is } (\epsilon, \alpha)\text{-dense in } B(z, \lambda^n)\} \geq \kappa\,.$$

Theorem 5. *Let K be a continuum in R^d, $d \geq 2$. Suppose that $K = W \cup E$ such that $\mathcal{H}^1(W) > 0$ and $\mathcal{H}^1(E) < \infty$. If there are positive numbers $\epsilon, \alpha, \lambda, \kappa \in (0, 1)$ such that for every $z \in W$ the set K is (ϵ, α)-dense at z at scales of density κ, then*

$$\dim_{\mathrm{H}}(K) \geq 1 + \kappa \left(d - 1 - \frac{C}{|\log \epsilon|} \right),$$

where $C > 0$ is a constant depending only on α, λ and d.

Example 1. Let us consider a modified Mandelbrot percolation process. Start with the square $Q_0 = [0, 1] \times [0, 1]$ and choose two sequences $(n_k)_{k \geq 0} \subseteq \mathbb{N}^*$ and $(s_k)_{k \geq 0} \subseteq (0, 1)$ such that $\lim_{k \to \infty} n_k = \infty$. Each square of generation k is divided in n_k^2 equal squares of generation $k + 1$, but only some of them survive, satisfying the following conditions:

- the closure of the union of all squares of generation k, denoted K_k, is connected
- if Q_k is a square of generation k and a ball $B(x, \delta) \subseteq Q_k \setminus K_{k+1}$, then

$$\delta \leq n_k^{s_k - 1} \text{diam}\,(Q_k).$$

A direct application of the previous theorem shows that

$$\dim_{\mathrm{H}}(K) \geq 2 - \limsup_{k\to\infty} s_k,$$

where $K = \cap_{k\geq 0} K_k$.

One can easily obtain corollaries of Theorem 5 concerning the convex density (see the definition in the following section), in the spirit of theorems 6 and 7, as illustrated by the following example. Without the assumption on the connectivity of K and in the absence of the exceptional set E, the lower bound of the Hausdorff dimension in Theorem 5 becomes

$$\dim_{\mathrm{H}}(K) \geq \kappa \left(d - \frac{C(d,\alpha,\lambda)}{|\log \epsilon|} \right).$$

If we remove the connectivity condition in the previous example, we obtain the following estimate

$$\dim_{\mathrm{H}}(K) \geq 2 \left(1 - \limsup_{k\to\infty} s_k \right).$$

Example 2. Let $C \subseteq [0,1]$ be a non-empty compact set having the following property. For any $x \in C$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 < r \leq \delta$ either $[x-r,x] \setminus C$ or $[x,x+r] \setminus C$ does not contain intervals of length larger than $\varepsilon\, r$. Then

$$\dim_{\mathrm{H}}(C) = 1.$$

In Section 2.3, we will provide examples showing that all hypotheses of Theorem 5 are essential.

1.3 Compact sets

The mass distribution principle is one of the basic techniques for Hausdorff dimension. The method is however not direct as one needs to construct a probability measure ν supported on K with suitable scaling properties to get lower bounds of $\dim_{\mathrm{H}}(K)$,

$$\nu(B(x,r)) \leq r^s \quad \text{for all } x \in K \text{ and all } r > 0 \Longrightarrow \dim_{\mathrm{H}}(K) \geq s\ .$$

For self-similar fractals, as one-third Cantor set, or more generally sets of "bounded geometry", the method leads to precise estimates of Hausdorff dimension. In general, for more complicated sets which show an "unbounded geometry", the mass distribution principle encounters difficulties as the geometry of a set is not controlled in every scale, as it is the case for self-similar fractals, but only in some and often scarcely distributed scales.

We propose a direct geometric method to produce universal lower bounds for Hausdorff dimension of compact sets. To this aim, we will define a notion of *convex density.*

Definition 1.2. *Let K be a subset of $\mathbb{R}^d$ with $d \geq 1$ and $x \in K$. We define a convex density $d_K(x,r)$ of K at x at the scale $r > 0$ as*

$$d(x,r) = \frac{I(x,r)}{2r} \ ,$$

where $I(x,r)$ is the diameter of the convex hull of $K \cap B(x,r)$.

Theorem 6. *Let $K \subset \mathbb{R}^d$ with $d \geq 1$ be a compact set. Suppose that for every $x \in K$ we have that*

$$\liminf_{r \to 0} \frac{\int_r^{\mathrm{diam}\ K} d_K^2(x,t) \frac{dt}{t}}{-\log r} > 0 \ .$$

Then

$$\dim_{\mathrm{H}}(K) > 0 \ .$$

If additionally, there is a constant $d_0 > 0$ such that for every $x \in K$,

$$\liminf_{r \to 0} \frac{\int_r^{\mathrm{diam}\ K} d_K^2(x,t) \frac{dt}{t}}{-\log r} \geq d_0^2 \ , \tag{6}$$

then

$$\dim_{\mathrm{H}}(K) \geq c d_0^2 \ ,$$

where c is a universal constant.

Proof. Without loss of generality, $K \subset [0,1]^d$. We build a continuum $K^* \subset \mathbb{R}^{d+1}$ by joining the point $S = (1, \ldots, 1) \in \mathbb{R}^{d+1}$ with every $x \in K$ by the segments I_x with the endpoints at S and x,

$$K^* = \bigcup_{x \in K} I_x \ .$$

Since every horizontal hyperplane $x_{d+1} = w$, $w \in (0,1)$, intersects K^* along an affine copy of K, the condition

$$\liminf_{r \to 0} \frac{\int_r^{\mathrm{diam}\, K} \beta_K^2(x,r) \frac{dt}{t}}{-\log r} \geq \beta_0^2$$

is satisfied for every $z \in K^* \setminus S$ and some $\beta_0 = c' d_0$ where c' is a universal constant. K^* is also wiggly at S but we rather refer to the obvious fact that $\mathcal{H}^1(S) = 0$, Theorem 1 implies that

$$\dim_{\mathrm{H}}(K^*) \geq 1 + c\beta_0^2 = 1 + cc'^2 d_0^2 \,.$$

By the product formula (Theorem 8.10 in [19]),

$$1 + cc' d_0^2 \leq \dim_{\mathrm{H}}(K^*) \leq \dim_{\mathrm{H}}(K) + \mathrm{MDim}([0,1]) = \dim_{\mathrm{H}}(K) + 1 \,. \quad (7)$$

□

In general, Theorem 6 admits only finite E as exceptional sets. Indeed, a countable compact $K = \{0\} \cup \bigcup_{n=1}^{\infty} \{\frac{1}{n}\}$ has positive convex density at all scales only at 0 and $\dim_{\mathrm{H}}(K) = 0$.

The estimates of Theorem 6 are sharp as shows the following theorem.

Theorem 7. *Let $N \subset \mathbb{R}^d$ with $d \geq 1$. Suppose that for every $x \in N$ we have that*

$$\liminf_{r \to 0} \frac{\int_r^1 d_N^2(x,t) \frac{dt}{t}}{-\log r} \leq d_0^2 \,.$$

Then

$$\dim_{\mathrm{H}}(N) \leq c d_0^2 \,.$$

where c is a universal constant.

Proof. Without loss of generality we may assume that $N \subset [0,1]^d$. Consider $K = N \times [0,1] \subset \mathbb{R}^{d+1}$. The set K satisfies the hypothesis of Theorem 2 with $\beta_0 = d_0$. Therefore, there exists a universal constant $c > 0$ such that

$$\dim_{\mathrm{H}}(K) \leq 1 + c\beta_0^2 \,.$$

By the product formula for Hausdorff dimension (Theorem 8.10 in [19]),

$$\dim_{\mathrm{H}}(N \times [0,1]) \geq \dim_{\mathrm{H}}(N) + \dim_{\mathrm{H}}([0,1]) = \dim_{\mathrm{H}}(N) + 1$$

and thus, $\dim_{\mathrm{H}}(N) \leq c d_0^2$. □

Invariance property. Suppose that K is a real compact which satisfies the hypothesis of Theorem 6. Let $\mathcal{D}_K$ be the set of all continuous real functions h defined on some neighborhood of K, with the property that there exist $\theta_1, \theta_2 > 1$ such that for every $x \in K$ and every interval I with the middle point at x and the lenght $|I|$ small enough,

$$\theta_1 |h(I)| \leq |h(2I)| \leq \theta_2 |h(I)|, \tag{8}$$

where $2I$ stands for the double of I and $|h(I)|$ denotes the length of the image $h(I)$. Then,

$$\inf_{h \in \mathcal{D}_K} \dim_{\mathrm{H}}(h(K)) \geq 1 + c(\theta_1, \theta_2) d_0^2 \ ,$$

where $c(\theta_1, \theta_2)$ a constant which depends only on θ_1 and θ_2. Morever, if θ_1 and θ_2 tend to 2 then $c(\theta_1, \theta_2)$ tends to a universal constant. The constant $c(\theta_1, \theta_2)$ can be easily estimated using Theorem 3 and the fact that the condition (8) prohibits, on one hand, too much expansion of non-wiggly scales and, on the other hand, too much contraction of wiggly scales (see also the proof of Theorem 8).

Every differentiable homeomorphism $h : U \mapsto \mathbb{R}$ with the derivative different from 0 belongs to $\mathcal{D}_K$. Another example of the class of maps in $\mathcal{D}_K$ are real quasi-regular functions.

Example. The $\frac{1}{3}$-Cantor set is obtained from the unit interval $X_0 = [0, 1]$ through an inductive procedure. The closed set X_n is obtained from the closed set X_{n-1} by removing the middle one-third of each component of X_{n-1}. The $\frac{1}{3}$-Cantor set is $X = \bigcap_{n=0}^{\infty} X_n$. The Hausdorff dimension of X is $\dim_{\mathrm{H}}(X) = \frac{\log 2}{\log 3}$. One can easily check that X is of positive convex density, that is for every $x \in X$, we have that $d_X(x, r) \geq d_0 = \frac{1}{4}$ provided r is small enough. Therefore, there exists a universal constant c such that the image of X by any real diffeomorphism is bigger than c.

1.4 Bowen's dichotomy

The property of uniform wiggliness was used in [5] to prove Bowen's dichotomy for a connected limit set of an analytically finite and not elementary Kleinian group. In polynomial dynamics, the corresponding result was proven earlier by Zdunik, [28]. The connected Julia set of a polynomial is either a segment/circle or its Hausdorff dimension is strictly bigger than 1. The strategy of the proof in [28] was different than that of Bishop and

Jones and was based on the study of the statistical properties of the unique measure μ of maximal entropy for f (in polynomial dynamics μ coincides with the harmonic measure with the pole at ∞). The main observation of [28] is that $\phi \circ f^n$, where $\phi := \dim_{\mathrm{H}}(\mu) \log |f'| - \log(\deg f)$, can be treated as a sequence of random variables. There are two possibilities, either ϕ is homologous to 0 in $L^2(J, \mu)$ or the law of iterated logarithm holds. The former case, by the"boot strapping" argument, leads to an analytic Julia set, while the latter implies that $\dim_{\mathrm{H}}(\mathcal{J}) > 1$.

Topological Collet-Eckmann rational maps. The definition of rational TCE maps (see [24]) is usually stated as follows. Let f be a rational map of the Riemann sphere $\hat{\mathbb{C}}$ of degree bigger than 1. For a given positive δ and L define $\mathcal{G}(z, \delta, L)$ to be the set of positive integers n such that

$$\#\{i : 0 \le i \le n, \mathrm{Crit}_f \cap \mathrm{Comp}_{f^i(z)} f^{-n+i}(B(f^n(z), \delta)) \ne \emptyset\} \le L,$$

where $\mathrm{Comp}_y(A)$ stands for the component of A which contains y.

Definition 1.3. *A rational map f satisfies TCE if there are positive δ, $L < \infty$ and κ so that for every point $z \in \hat{\mathbb{C}}$ (z belongs to the Julia set) we have that*

$$\inf_n \frac{\#\mathcal{G}(z, \delta, L) \cap [1, n]}{n} \ge \kappa \; .$$

We have the following geometric counterpart of Bowen's dichotomy.

Theorem 8. *Suppose that f is a rational TCE map of degree bigger than 1. If the Julia set $\mathcal{J} \ne \hat{\mathbb{C}}$ of f is connected then $\mathcal{J}$ is either an interval/circle or a mean wiggly continuum.*

1.5 Harmonic measure and mean wiggly Julia sets

Let E be a full compact in $\mathbb{C}$. The harmonic measure ω of E with a base point at ∞ can be described in terms of the Riemann map

$$\Psi : \mathbb{C} \setminus \overline{D(0,1)} \mapsto \mathbb{C} \setminus E$$

which is tangent to identity at ∞. Namely, Ψ extends radially almost everywhere on the unit circle with respect to the normalized 1-dimensional Lebesgue measure $d\theta$ and $\omega = \Psi_*(d\theta)$.

The Mandelbrot set $\mathcal{M}$ is the set of the parameters $c \in \mathbb{C}$ for which the corresponding Julia set $\mathcal{J}_c$ of the quadratic polynomial $f_c(z) = z^2 + c$

is connected. It is known that both the Mandelbrot set and its complement are connected. A parameter $c \in \partial\mathcal{M}$ is called *Collet-Eckmann* if $\liminf_{n\to\infty} \frac{\log|(f_c^n)'(c)|}{n} > 0$.

It was proven in [13, 27] that the Collet-Eckmann parameters in the boundary of the Mandelbrot set are of full harmonic measure. Since every Collet-Eckmann quadratic polynomial $f_c(z) = z^2 + c$ satisfies TCE property, we can invoke Theorem 8 to derive the following corollary.

Corollary 1.1. *For almost all $c \in \partial\mathcal{M}$, the corresponding Julia set is a mean wiggly continuum.*

Remark. The claim of Corollary 1.1 remains true for unicritical polynomials $z^d + c$, $d \geq 2$, with c from a generic set on the boundary of the connectedness locus $\mathcal{M}_d$, see [13, 27].

1.6 Various examples

Example 1. We describe an example where E has finite positive 1-measure (denoted by $\mathcal{H}^1(E)$) with $\dim_{\mathrm{H}}(W) = 1$ and $\mathcal{H}^1(W) = 0$.

This is a modified version of the four corners Cantor set. Let $S_0 = [0,1]^2$ be the unit square. Let $S_1^1, \ldots, S_1^4$ the disjoint sub-squares of side-length $0 < \frac{1}{4}a_1 < \frac{1}{2}$ which have each a common corner with S_0. We repeat the construction inside each square of n-th generation to obtain four squares of side length $|S_{n+1}| = \frac{1}{4}a_{n+1}|S_n|$. We set

$$W = \bigcap_{n\geq 1} \bigcup_{i=1}^{4^n} S_n^i.$$

Observe that $|S_n| = 4^{-n}\prod_{i=1}^n a_n$. We let $a_n \nearrow 1$ and show that $\dim_{\mathrm{H}}(W) = 1$. We may define a measure μ supported on W such that $\mu(S_n^i) = 4^{-n}$. Then

$$\lim_{n\to+\infty} \frac{\log\mu(S_n)}{\log|S_n|} = \liminf_{n\to+\infty} \frac{n\log 4}{n\log 4 + \log a_1 + \ldots + \log a_n} = 1.$$

As $\frac{|S_n|}{|S_{n+1}|}$ is bounded, Billingsley's lemma shows that $\dim_{\mathrm{H}}(W) = 1$. The same bound gives a lower bound of $\beta_W(x,r)$ for all $x \in W$ and $r > 0$.

The set W can be covered by the 4^n squares of n-th generation. Therefore it is enough to have $\lim_{n\to+\infty}\prod_{i=1}^n a_i = 0$ to obtain that $\mathcal{H}^1(W) = 0$.

Let E be the union of all diagonals of all squares. We may easily compute that

$$\mathcal{H}^1(E) = 2\sqrt{2}(1 + \sum_{n\geq 0} 4^n |S_n|).$$

We set $a_n = \frac{n^2}{(n+1)^2}$ so $\mathcal{H}^1(E) < +\infty$. Note that this sequence satisfies the previous conditions.

The set $K = W \cup E$ is a continuum with the desired properties. K is also locally connected.

Exemple 2. If in the previous example we set $a_n = 1$ for every $n \geq 0$ (the standard four corners Cantor set) then $\mathcal{H}^1(W)$ is finite, $\mathcal{H}^1(E) = \infty$, and $\dim_{\mathrm{H}}(W \cup E) = 1$. The continuum $K = W \cup E$ is locally connected.

Example 3. We give an example of a locally connected continuum such that $\dim_{\mathrm{H}}(K) = 1$ and $\mathcal{H}^1(W) > 0$ (but $\mathcal{H}^1(E) = +\infty$).

Let $W = [0, 1]$ and E the union of vertical segments of length 2^{-n}, with center $k2^{-n} \in W$, for all $n > 0$ and $0 < k < 2^n$, k odd. The continuum $K = W \cup E$ is locally connected and $\dim_{\mathrm{H}}(K) = 1$ as a countable union of segments. It is not hard to see that $\beta_K(x, r)$ has a uniform lower bound for all $x \in W$ and all $0 < r < 1$ (but $\beta_W(x, r) = 0$).

2 Density of wiggliness

We want to construct a probability measure μ which captures wiggliness of a continuum K and allows for effective lower bounds of $\dim_{\mathrm{H}}(K)$. Let F be the set of points of K which are not wiggly

$$F := \{x \in K : \int_0^1 \beta_K(x, t)^2 \frac{dt}{t} < +\infty\}.$$

We want to construct μ which vanishes on F. In this case, μ will be generally scale dependent as wiggly parts of K can be contained in a ball of arbitrary small radius.

The main ingredient of the proof of Theorem 1 is the following result.

Theorem 9. *Let $K \subset \mathbb{R}^d$ with $d \geq 2$ be a connected compact of diameter 1 and $F \subseteq E \subseteq K$ with $\mathcal{H}^1(E) < +\infty$ and $\mathcal{H}^1(K \setminus E) > 0$. There exist a universal constant $C' > 0$, a constant $C > 0$ depending only on d and a Radon probability measure μ supported on $B(y, R) \cap K$, $y \in K$ such that*

$\mu(E) = 0$ *and for all* $x \in K$ *and* $r > 0$

$$\mu(B(x,r)) \leq R^{-1} C r \exp\left(-C' \int_r^1 \beta_K^2(x,t) \frac{dt}{t}\right).$$

The proof of Theorem 9 is technically involving. We start by proving an important combinatorial result, Proposition 1, and then follow the construction from the proof of Theorem 45 in [23]. This construction was proposed by the second author and after some modifications made available in the written form by David in [7] around a decade ago. Proposition 1 replaces a *stopping time* argument, usually needed in corona type constructions, by a direct estimate of length of a crossing curve which is wiggly at many scales.

Planar continua. For the sake of simplicity, the proof of Theorem 9 is given in details only for planar sets K. We indicate what modifications are needed for the general case.

As a preparation, we need to state some results and prove a few facts about the length (or 1-Hausdorff measure) of wiggly continua in the plane.

Let $Q \subset \mathbb{C}$ be a square (with sides parallel to the axis). Unless specified otherwise, squares are considered to contain only the left and top edges. Let $|Q|$ denote the side length of Q. Let $\Delta(Q)$ denote the set of all dyadic sub-squares of Q and $\Delta_k(Q)$ those with side length $2^{-k}|Q|$. For any $\lambda > 0$, let λQ be the square with the same center as Q and with $|\lambda Q| = \lambda |Q|$. For a set S of squares, $|S| := \sum_{Q \in S} |Q|$ and $\#S$ is its cardinal.

For any $x \in Q_0$, where $Q_0 := [0,1] \times [-\frac{1}{2}, \frac{1}{2}]$, let

$$\Delta(x) = \{Q \in \Delta(Q_0) \mid x \in Q\}.$$

Let us also write Δ for $\Delta(Q_0)$ and Δ_k for $\Delta_k(Q_0)$.

Definition 2.1. *Let* K *be a compact set in the plane and* Q *a square such that* $Q \cap K \neq \emptyset$*. We define* $\beta_K(x,r)$ *by*

$$\beta_K(Q) := \inf_L \sup_{z \in K \cap 3Q} \frac{\operatorname{dist}(z, L)}{|Q|},$$

where the infimum is taken over all lines L in the plane.

For a compact set $K \subseteq Q_0$ is easy to check that

$$\beta_\infty^K(x) := \int_0^1 \beta_K(x,t)^2 \frac{dt}{t} \sim \sum_{Q \in \Delta(x)} \beta_K(Q)^2. \tag{9}$$

We will use the following fact, due to the second author (see [16] for a proof, [22] for a generalization in R^d). If $\gamma \subset Q_0$ is a rectifiable curve, then

$$\beta_\infty(\gamma) := \sum_{Q \in \Delta} \beta_\gamma(Q)^2 |Q| \lesssim \mathcal{H}^1(\gamma). \tag{10}$$

Proposition 1. *Let $\varepsilon \in [0,1)$ and $L > 1$. There exists $M > 0$ such that every curve $\gamma \subset \mathbb{C}$ joining 0 to 1 such that*

$$\beta_\infty^\gamma(x) \geq M$$

for all $x \in \gamma \setminus E$, $\mathcal{H}^1(E) \leq \varepsilon$, is of the length

$$\mathcal{H}^1(\gamma) > L.$$

Proof. By the inequality (10), it is sufficient to show that $\beta_\infty(\gamma)$ is large enough.

We may assume that $\gamma \subset Q_0 = [0,1] \times [-\frac{1}{2}, \frac{1}{2}]$, otherwise the proof is analogous in $[-L, L+1] \times [-L, L+1]$.

Let G_n be the set of maximal squares in

$$\{Q \in \Delta,\ |Q| \geq 2^{-n} \mid \mathcal{H}^1(\gamma \cap Q) > \frac{2L}{1-\varepsilon}|Q|\}.$$

The squares in G_n are disjoint. If $|G_n| > \frac{1-\varepsilon}{2}$ then $\mathcal{H}^1(\gamma) > L$.

We assume that $\beta_\infty(\gamma)$ is bounded and that $|G_n| \leq \frac{1-\varepsilon}{2}$ for all $n \geq 1$, $M > 0$, and prove the proposition by contradiction. Let

$$K_n = \bigcup_{Q \in G_n} Q.$$

K_n is an increasing sequence. Let $K = \cup_{n \geq 1} K_n$.

For $Q \in \Delta$, let

$$S(Q) = \sum_{Q \subseteq Q' \in \Delta} \beta_\gamma(Q')^2.$$

Observe that $S(\cdot)$ is decreasing.

By formula (9), there is $c > 0$ such that for any $x \in \gamma \setminus (K \cup E)$, there is $x \in Q \in \Delta$ with $S(Q) > cM$. Let $\mathcal{B}$ the cover of $\gamma \setminus (K \cup E)$ with such maximal squares. The squares in $\mathcal{B}$ are disjoint and are not contained in K.

Let us consider π the projection on the real line. $\mathcal{H}^1(\pi(K)) \leq \frac{1-\varepsilon}{2}$ and $\mathcal{H}^1(\pi(E)) \leq \varepsilon$. Therefore

$$|\mathcal{B}| \geq \frac{1-\varepsilon}{2}.$$

Let $S_n = \{Q \in \Delta_n \mid S(Q) > cM$ and $Q \cap \gamma \setminus (K \cup E) \neq \emptyset\}$. As $S(\cdot)$ is decreasing, we may also conclude that for n large enough

$$\#S_n \geq 2^{n-2}(1-\varepsilon). \tag{11}$$

Observe that squares in S_n are disjoint from K_n.

Lemma 2.1. *Let $Q \in \Delta$. Denote by $\#_k(Q)$ the cardinal of the set of squares in $\Delta_k(Q)$ which intersect γ. Then*

$$\mathcal{H}^1(\gamma \cap Q) \gtrsim (2^{-k}\#_k(Q) - 4)|Q|.$$

Proof. Let $Q' \in \Delta_k(Q)$ intersect γ and without common boundary with Q. Then $\mathcal{H}^1(\gamma \cap 3Q') \geq 2^{-k}|Q|$, as γ connects Q' to $\partial(3Q')$. We may extract a finite (universal) number of collections of such squares Q' such that in each collection, the squares $3Q'$ have disjoint interior. As we ignore $4 \cdot 2^k - 4$ squares in $\Delta_k(Q)$ with common boundary with Q, the conclusion follows by considering the collection of squares with maximal cardinal. $\square$

As a consequence of the previous lemma, for any $Q \in \Delta_n$ not contained in K_n, for all $k \geq 1$ we have

$$\#_k(Q) \lesssim 2^k.$$

As squares $Q' \in S_n$ are disjoint from K_n, for any $Q \in \Delta$ containing Q', we have

$$\#\{Q'' \in S_n \mid Q'' \subset Q\} \lesssim \frac{L}{1-\varepsilon}\frac{|Q|}{|Q'|}. \tag{12}$$

We may begin estimates. By the inequality (11), for n large enough, we have

$$2^{-n}\sum_{Q \in S_n} S(Q) \geq 2^{-n}2^{n-2}(1-\varepsilon)cM \gtrsim M.$$

We have assumed that $\beta_\infty(\gamma)$ is bounded, so for some $C > 0$

$$C \geq \sum_{\substack{Q\in\Delta \\ |Q| \geq 2^{-n}}} \beta_\gamma(Q)^2|Q| \geq \sum_{\substack{Q\in\Delta \\ \exists Q' \subseteq Q, Q' \in S_n}} \beta_\gamma(Q)^2|Q|$$

$$= \sum_{k=0}^{n} 2^{-k} \sum_{\substack{Q\in\Delta_k \\ \exists Q' \subseteq Q, Q' \in S_n}} \beta_\gamma(Q)^2 = 2^{-n}\sum_{k=0}^{n} 2^{n-k} \sum_{\substack{Q\in\Delta_k \\ \exists Q' \subseteq Q, Q' \in S_n}} \beta_\gamma(Q)^2$$

$$= 2^{-n}\sum_{k=0}^{n}\sum_{\substack{Q\in\Delta_k\\ \exists Q'\subseteq Q, Q'\in S_n}}\frac{|Q|}{|Q'|}\beta_\gamma(Q)^2 = 2^{-n}\sum_{\substack{Q\in\Delta\\ \exists Q'\subseteq Q, Q'\in S_n}}\frac{|Q|}{|Q'|}\beta_\gamma(Q)^2$$

$$\gtrsim 2^{-n}\frac{1-\varepsilon}{L}\sum_{Q\in\Delta}\#\{Q''\in S_n \mid Q''\subset Q\}\beta_\gamma(Q)^2 \gtrsim 2^{-n}\sum_{Q'\in S_n}S(Q')\gtrsim M,$$

a contradiction. □

Remark 1. Using the same notations and proof, the conclusion of Lemma 2.1 could be restated in $\mathbb{R}^d$ as follows

$$\mathcal{H}^1(\gamma\cap(1+2^{-k+1})Q)\gtrsim 2^{-k}\#_k(Q)|Q|.$$

In the proof of Proposition 1 we could define

$$K_n = \bigcup_{Q\in G_n} 2Q.$$

Observe that there is a sub-collection G_n' of G_n such that $\pi(2G_n')$ covers $\pi(\gamma\cap K_n)$ and each point is covered at most twice. Note also that $\gamma\cap K_n$ is increasing. Replacing the constant in the definition of G_n by $\frac{8L}{1-\varepsilon}$, the same proof shows that Proposition 1 generalizes to curves in R^d, while M is independent of the dimension d.

Let π be an orthogonal projection on the real axis in $\mathbb{C}$. We have used the fact that our set is a curve only in two instances: to obtain that $\beta_\infty(\gamma)\lesssim\mathcal{H}^1(\gamma)$ and to have $[0,1]\subseteq\pi(\gamma)$. Also, it is enough that $\mathcal{H}^1(\pi(E))\leq\varepsilon$. We could therefore relax the hypothesis using the following well known result (see for example [23]).

Theorem 10. *There exists a universal constant $C>1$ such that any continuum $K\subseteq\mathbb{R}^n$ with $\mathcal{H}^1(K)<+\infty$ is contained in a curve γ such that*

$$\mathcal{H}^1(K)\leq\mathcal{H}^1(\gamma)\leq C\mathcal{H}^1(K).$$

We obtain the following corollary.

Corollary 2.1. *Let $\varepsilon\in[0,1)$ and $L>1$. There exists $M>0$ such that every compact set $K\subseteq[0,1]\times[-\frac{1}{2},\frac{1}{2}]$ that satisfies the following conditions,*

1. *$K\cup\{0,1\}\times[-\frac{1}{2},\frac{1}{2}]$ is connected,*

2. $\beta_\infty^K(x) \geq M$ *for all* $x \in K \setminus E$*,*

3. $\mathcal{H}^1(\pi(E)) \leq \varepsilon$*,*

is of the length

$$\mathcal{H}^1(K) > L.$$

For a set $A \subseteq \mathbb{C}$, let $||A|| := \{|x| : x \in A\}$. We will need the following version of the previous corollary.

Corollary 2.2. *Let* $\varepsilon \in [0, 1)$ *and* $L > 1$*. There exists* $M > 0$ *such that every compact set* $K \subseteq \mathbb{D}(0, 1)$ *with the property that* $0 \in K$*,* $K \cup \partial\mathbb{D}(0, 1)$ *is connected, and*

$$\beta_\infty^K(x) \geq M$$

for all $x \in K \setminus E$*,* $\mathcal{H}^1(||E||) \leq \varepsilon$*, satisfies*

$$\mathcal{H}^1(K) > L.$$

Proof. We first unfold the annulus $\overline{\mathbb{A}(\frac{\varepsilon}{2}, 1)}$ to the rectangle $[\frac{\varepsilon}{2}, 1] \times [-\frac{1}{2}, \frac{1}{2}]$ and then map it linearly to $Q_0 = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}]$. The distortion and the dilatation of this map φ is bounded by a constant which depends only on ε. Therefore $\beta_K(x, t) \sim \beta_{\varphi(K)}(\varphi(x), |\varphi'(x)|t)$. We obtain that

$$\beta_\infty^{\varphi(K)}(x) > C(\varepsilon)M$$

for all $x \in \varphi(K) \setminus \varphi(E)$ with $\mathcal{H}^1(\pi(\varphi(E)))$ close to 0 in terms of ϵ only. All other hypothesis of the previous corollary are easy to check. □

Definition 2.2. *A set* $E \subset \mathbb{R}^d$ *with* $d \geq 2$ *is* Ahlfors regular *if there exists* $C > 1$ *such that for all* $x \in E$ *and* $0 < R < \operatorname{diam} E$*,*

$$C^{-1}R \leq \mathcal{H}^1(E \cap B(x, R)) \leq CR.$$

We will use the following result due to the second author and Bishop (Theorem 1 in [4]).

Theorem 11. *There exists* $C > 0$ *such that if* $K \subset \mathbb{R}^d$ *with* $d \geq 2$ *is a compact set of diameter* 1 *and if for all* $x \in K$*,*

$$\beta_\infty^K(x) \leq M,$$

then K *lies on a rectifiable curve* Γ *of length at most* Ce^{CM} *and which is Ahlfors regular with constant depending only on* M*.*

The fact that Γ is Ahlfors regular is not stated in [4] but is an immediate consequence of the bound for the length.

Proof. As a limit of rectifiable curves with uniform bounded length containing K, we may assume that Γ realizes the infimum of the length of such curves. We consider a square Q centered at $x \in K$ and apply the theorem to the set $|Q|^{-1}(\overline{Q} \cap K)$. We obtain a curve Γ' of length at most Ce^{CM}. We connect the endpoints of $|Q|\Gamma'$ to ∂Q and take the union with ∂Q to obtain γ'. By the choice of Γ and our construction,

$$\mathcal{H}^1(\Gamma \cap Q) \leq \mathcal{H}^1(\gamma') \leq |Q|(5 + Ce^{CM}).$$

$\square$

Proof of Theorem 9. We recall that F denotes the subset of K of points that are not wiggly. Take a Borel set $E \subset K$ containing F with $\mathcal{H}^1(E) < +\infty$. Assume that $\mathcal{H}^1(F) > 0$ and $\mathcal{H}^1(K \setminus F) > 0$. The density theorem for $\mathcal{H}^1$ (see Theorem 6.2 in [19]) implies that for almost all points $x \in K \setminus E$ with respect to $\mathcal{H}^1$,

$$\limsup_{r \to 0} \frac{\mathcal{H}^1(B(x,r) \cap E)}{2r} = 0 \,.$$

Therefore, there is a ball $B = \mathbb{D}(x, R)$, $x \in K$, such that

$$\mathcal{H}^1(E \cap B) < \varepsilon R.$$

We will construct a measure μ with the density properties as claimed in Theorem 9, supported on $B \cap K$, such that $\mu(E) = 0$. Let $B_0 := B$ and $\varepsilon < \frac{1}{100}$.

The proof has three steps.

1. For any ball $B = \mathbb{D}(x_B, R_B)$, $x_B \in K$ and such that $\mathcal{H}^1(B \cap E) < \frac{R_B}{100}$ we construct a measure μ_B.

2. We construct the probability measure μ on B_0.

3. We show that μ has the desired scaling properties and that $\mu(E) = 0$.

Step 1. Let $B = \mathbb{D}(x_B, R_B)$ with $R_B \leq 1$, $x_B \in K$, such that $\mathcal{H}^1(B \cap E) < \varepsilon R_B$ (with $0 < \varepsilon \leq \frac{1}{100}$). Let $K_B := K \cap \overline{B}$ and observe that $K_B \cup \partial B$ is a connected compact. For any $x \in K_B$, let

$$t_B(x) := \inf \left\{ r \in (0, R_B) : \int_r^{R_B} \beta^2_{K_B}(x,t) \frac{dt}{t} \leq M \right\},$$

where M is a large constant that will be specified later. Observe that $t_B(x) = 0$ if and only if $\int_0^{R_B} \beta^2_{K_B}(x,t)\frac{dt}{t} \le M$. Let

$$Z(B) := \overline{\{x \in K_B : t_B(x) = 0\}}.$$

One can easily check that for any $x \in Z(B)$,

$$\int_0^{R_B} \beta^2_{Z(B)}(x,t)\frac{dt}{t} \le \int_0^{R_B} \beta^2_{K_B}(x,t)\frac{dt}{t} \le 4M,$$

therefore $Z(B) \subseteq F \subseteq E$. We obtain that $\mathcal{H}^1(Z(B)) < \varepsilon R_B$.

We show that there exists $C_M > 0$, that depends only on M, and a measure μ_B such that

$$\begin{aligned} &1. \quad R_B \le \mu_B(\mathbb{C}) = \mu_B(B) \le C_M R_B, \\ &2. \quad \mu_B(\mathbb{D}(x,r)) \le C_M r, \text{ for all } x \in \mathbb{C} \text{ and } r > 0. \end{aligned} \tag{13}$$

Set $W(B) := K_B \setminus Z(B)$. By the standard covering lemma (see [14], page 2), there exists a countable set $X'(B) \subset W(B)$ such that

$$W(B) \subseteq \bigcup_{x \in X'(B)} \mathbb{D}(x, 10t_B(x)),$$

and the balls $\mathbb{D}(x, 2t_B(x))$, $x \in X'(B)$, are pairwise disjoint.

From now on we make a standing assumption (it is enough to take $M \ge 4\log 10$) that for all $x \in X'(B)$,

$$t_B(x) \le 10^{-4} R_B. \tag{14}$$

The following lemma provides a key estimate which allows to distribute the measure μ on the balls $\mathbb{D}(x, t_B(x))$ centered at $X(B)$ and prove the scaling properties of μ in Step 3.

Lemma 2.2. *If M is sufficiently large, then there exists $X(B) \subseteq X'(B)$ such that for each $x \in X(B)$, $\mathcal{H}^1(\mathbb{D}(x, t_B(x)) \cap E) < \frac{\varepsilon}{2}\, t_B(x)$,*

$$\sum_{x \in X(B)} t_B(x) \ge 10 R_B,$$

and

$$\sum_{x \in X(B)} \mathcal{H}^1(\mathbb{D}(x, t_B(x)) \cap E) \le \frac{\varepsilon}{2} R_B.$$

It is not hard to check that the set $Y = Z(B) \cup X'(B)$ is compact. Also, one can check that for every $x \in Y$, $\int_0^{R_B} \beta_Y^2(x,t)\frac{dt}{t} \leq 100M$. By Theorem 11, the set Y is contained in an Ahlfors regular curve Γ_B with the length comparable to R_B (and the constant depending only on M). As the balls $\mathbb{D}(x, 2t_B(x))$ from the the same $\mathcal{G}_i(B)$ are pairwise disjoint, we may assume that Γ_B contains a cross

$$G(x) := [x - t_B(x), x + t_B(x)] \cup [x - it_B(x), x + it_B(X)]$$

for every $x \in X'(B)$. We define

$$\mu_B := \mathcal{H}^1{}_{|G_B}, \text{ where } G_B := \bigcup_{x \in X(B)} G(x).$$

By the properties of Γ_B it is easy to check that μ_B satisfies the inequalities (13).

The following proof concludes Step 1.

Proof of Lemma 2.2. We use a geometric construction to show that if M is sufficiently large, then

$$\sum_{x \in X'(B)} t_B(x) \geq 23R_B, \tag{15}$$

As $t_B(x) \leq 10^{-4}R_B$ for all $x \in X'(B)$, there is a partition of $X'(B) = X_1(B) \cup X_2(B)$ such that for each $i \in \{1, 2\}$,

$$\sum_{x \in X_i(B)} t_B(x) \geq 11R_B.$$

As $\mathcal{H}^1(B \cap E) < \varepsilon R_B$, there is $i_0 \in \{1, 2\}$ such that

$$\sum_{x \in X_{i_0}(B)} \mathcal{H}^1(\mathbb{D}(x, t_B(x)) \cap E) \leq \frac{\varepsilon}{2} R_B.$$

Let $X(B) = \{x \in X_{i_0}(B) \ : \ \mathcal{H}^1(\mathbb{D}(x, t_B(x)) \cap E) < \frac{\varepsilon}{2}\, t_B(x)\}$ and suppose that

$$\sum_{x \in X_{i_0}(B) \setminus X(B)} t_B(x) \geq R_B.$$

Then

$$\sum_{x \in X_{i_0}(B) \setminus X(B)} \mathcal{H}^1(\mathbb{D}(x, t_B(x)) \cap E) \geq \frac{\varepsilon}{2}\, R_B,$$

which contradicts the choice of i_0, as balls $\mathbb{D}(x, t_B(x))$ with $x \in X_{i_0}(B)$ are disjoint.

In the sequel, we prove the inequality (15). For every $x \in X'(B)$, let

$$H(x) := \partial\mathbb{D}(x, 2t_B(x)) \cup \partial\mathbb{D}(x, 10t_B(x)) \cup [x - 10t_B(x), x + 10t_B(x)].$$

Then $H(x)$ is connected and $\mathcal{H}^1(H(x)) \leq 100t_B(x)$. Let

$$Z'(B) := Z(B) \setminus \bigcup_{x \in X'(B)} \mathbb{D}(x, 10t_B(x)).$$

Observe that $K_B \subset Z'(B) \cup \bigcup_{x \in X'(B)} \mathbb{D}(x, 10t_B(x))$. Let us define

$$S(B) := Z'(B) \cup \bigcup_{x \in X'(B)} H(x).$$

As $\mathcal{H}^1(Z'(B)) \leq 10^{-2}R_B$, it is enough to show that $\mathcal{H}^1(S(B)) \geq 2301R_B$ in order to prove the inequality (15). We may assume that for any $x \neq x' \in X'(B)$, $\mathbb{D}(x, 10t_B(x)) \not\subset \mathbb{D}(x', 10t_B(x'))$. The reader can therefore check that $S(B)$ is compact and $S(B) \cup \partial B$ is a connected compact. We will show that for every $x \in S(B) \setminus Z'(B)$,

$$\beta_\infty^{S(B)}(x) \geq 10^{-5}M, \tag{16}$$

provided M is large enough. Having established the estimate (16), we can conclude the proof of the inequality (15) by applying Corollary 2.2 to $S(B)$ with $Z'(B)$ as an exceptional set, $\varepsilon = 10^{-2}, L = 2301$ and M large enough.

We still have to prove the inequality (16). For this, let $y \in S(B) \setminus Z'(B)$. We can find $x_0 \in X'(B)$ such that

$$t_B(x_0) = \sup\{t_B(x) : x \in X'(B), |x - y| < 20t_B(x)\}.$$

We will show that for $t \geq 10^3 t_B(x_0)$,

$$\beta_{S(B)}(y, t) \geq 10^{-2}\beta_{K_B}(x_0, 10^{-2}t). \tag{17}$$

To this end, observe that by the choice of x_0 and t, we have $\mathbb{D}(x_0, 10^{-2}t) \subset \mathbb{D}(y, t)$, and therefore

$$\beta_{K_B}(y, t) \geq 10^{-2}\beta_{K_B}(x_0, 10^{-2}t).$$

It is enough to prove that

$$\beta_{S(B)}(y, t) \geq \beta_{K_B}(y, t). \tag{18}$$

Let L be a line minimizing $\beta_{S(B)}(y,t)$ and take a point $z \in \mathbb{D}(y,t) \cap (K_B \setminus S(B))$. In particular, $z \in K_B \setminus Z'(B)$ so there is $x \in X'(B)$ such that $z \in \mathbb{D}(x, 10t_B(x))$. By the choice of x_0, $\mathbb{D}(y,t) \not\subset \mathbb{D}(x, 10t_B(x))$. As $\partial\mathbb{D}(x, 10t_B(x)) \subset S(B)$, we have that $\partial\mathbb{D}(x, 10t_B(x)) \cap \mathbb{D}(y,t)$ is contained in a $t\beta_{S(B)}(y,t)$ neighborhood of L, and so is z. The inequality (18) is now established.

For each $y \in S(B) \setminus Z'(B)$, we integrate the inequality (17) and obtain

$$\begin{aligned} \int_{10^3 t_B(x_0)}^{R_B} \beta_{Y(B)}^2(y,t)\frac{dt}{t} &\geq 10^{-4}\int_{10^3 t_B(x_0)}^{R_B} \beta_{K_B}^2(x_0,t)\frac{dt}{t} \\ &\geq 10^{-4}M - 3\log 10 \\ &\geq 10^{-5}M, \end{aligned}$$

if M is large enough. This proves the inequality (16). $\square$

Step 2. We define a sequence $(\mu_n)_{n\geq 0}$ of probability measures supported respectively on a decreasing sequence of compact neighborhoods of K, having K as their intersection. The measure μ is then a weak limit of this sequence.

For each $n \geq 0$, we define a collection of disjoint balls $\mathcal{F}_n$ as follows. Let $\mathcal{F}_0 := \{B_0\}$ and

$$\mathcal{F}_{n+1} := \{\mathbb{D}(x, t_B(x)) : x \in X(B), B \in \mathcal{F}_n\}.$$

Let $B_1, B_2 \in \mathcal{F}_n$, $B_1', B_2' \in \mathcal{F}_{n+1}$ such that $B_1' = \mathbb{D}(x, t_{B_1}(x))$ with $x \in X(B_1)$ and respectively $B_2' = \mathbb{D}(y, t_{B_2}(y))$ with $y \in X(B_2)$. As $2B_1$ and $2B_2$ are disjoint and $t_B(x) \leq 10^{-3}R_{B_1}$, $t_B(y) \leq 10^{-3}R_{B_2}$ we obtain that

$$B_1' \cap B_2' = \emptyset, B_1 \cap B_2' = \emptyset \text{ and } B_1' \cap B_2 = \emptyset.$$

We will use the measures μ_B constructed at the previous step to define inductively the sequence $(\mu_n)_{n\geq 0}$. Let

$$\mu_0 := \frac{\mu_{B_0}}{\mu_{B_0}(\mathbb{C})}.$$

Assume that $\mu_0, \ldots, \mu_n$ have been defined with the following properties.

(P1) $\operatorname{supp}\mu_n$ is contained in the disjoint union of balls in $\mathcal{F}_n$ which is contained in a 10^{-n}-neighborhood of K. Also $\mathcal{H}^1(E \cap \operatorname{supp}\mu_n) < \frac{2^{-n}}{100}R$ by Lemma 2.2, where R is the radius of B_0.

(P2) $\mu_n(B) \leq 10^{-k}R^{-1}R_B$ if $B \in \mathcal{F}_k$ for some $k \leq n$ (R_B is the radius of B).

We define μ_{n+1} in the following way. For any ball $B \in \mathcal{F}_{n+1}$,

$$\mu_{n+1|B} := \frac{\mu_n(B)}{\mu_B(B)}\mu_B. \tag{19}$$

Observe that $\mu_{n+1}(B) = \mu_n(B)$ and that by Lemma 2.2, for every $x \in X(B)$ we have

$$\frac{\mu_{n+1}(\mathbb{D}(x, t_B(x)))}{\mu_{n+1}(B)} = \frac{t_B(x)}{\sum_{y\in X(B)} t_B(y)} \leq \frac{t_B(x)}{10R_B}.$$

Properties (P1) and (P2) for μ_{n+1} are direct consequences of the inequality (14) and Lemma 2.2 and the choice of $X(B) \subseteq X'(B)$ for any ball B.

Step 3. Observe that if $B \in \mathcal{F}_n$ is a ball of radius R_B, then

$$\mu(B) \leq 10^{-n}R^{-1}R_B. \tag{20}$$

We want to show that for every $x \in K$ and $r > 0$,

$$\mu(\mathbb{D}(x,r)) \leq CR^{-1}r \exp\left(-C' \int_r^1 \beta_K^2(x,t)\frac{dt}{t}\right), \tag{21}$$

where $C, C' > 0$ are universal constants.

Let us note that it is enough to prove this bound for $x \in \operatorname{supp}\mu$. Otherwise, we have either $\operatorname{supp}\mu \cap \mathbb{D}(x,r) = \emptyset$, so $\mu(\mathbb{D}(x,r)) = 0$, or for some $y \in \operatorname{supp}\mu \cap \mathbb{D}(x,r)$, $\mathbb{D}(x,r) \subset \mathbb{D}(y,2r)$, so

$$\mu(\mathbb{D}(x,r)) \leq \mu(\mathbb{D}(y,2r)) \leq 2CR^{-1}r \exp\left(-C' \int_{2r}^1 \beta_K(y,t)^2\frac{dt}{t}\right).$$

As $|x-y| < r$, a simple computation shows that

$$1 + \int_r^1 \beta_K^2(x,t)\frac{dt}{t} \sim 1 + \int_{2r}^1 \beta_K(y,t)^2\frac{dt}{t},$$

with universal constants.

Fix $x \in \operatorname{supp}\mu$ and $r > 0$. If $r \gtrsim R$ there is nothing to prove. We have $\{x\} = \cap_{n\geq 0} B_n$, where $B_n \in \mathcal{F}_n$ for all $n > 0$. Let

$$N := \max\{n : \mathbb{D}(x,r) \subseteq 2B_n\}.$$

A direct computation leads to

$$\int_r^1 \beta_K^2(x,t)\frac{dt}{t} \lesssim (N+4)M. \tag{22}$$

On the other hand,

$$\mu(2B_N) \le 10^{-N} R^{-1} R_{B_N}.$$

If $r \sim R_{B_N}$, the last two inequalities and the properties (13) imply the conclusion (21).

Suppose that $r < 10^{-6} R_{B_N}$. For any $B \in \mathcal{F}_{N+1}$ which intersects $\mathbb{D}(x,r)$, we have $R_B \le 4r$. Otherwise $\mathbb{D}(x,r) \subseteq 2B$ which contradicts the choice of N. Thus $B \subseteq \mathbb{D}(x,10r)$. Denote $\mathcal{K} := \{B \in \mathcal{F}_{N+1} : B \cap \mathbb{D}(x,r) \ne \emptyset\}$. Using the inequality (20) and the fact that Γ_{B_N} constructed in Step 1 is Ahlfors regular, we can estimate

$$\begin{array}{rcl} \mu(\mathbb{D}(x,r)) & \le & \sum_{B\in\mathcal{K}} \mu(B) \le 10^{-N-1} R^{-1} \sum_{B\in\mathcal{K}} R_B \\ & \le & c 10^{-N-1} R^{-1} \mathcal{H}^1(\Gamma_{B_N} \cap \mathbb{D}(x,10r)) \\ & \le & c' 10^{-N} R^{-1} r, \end{array}$$

where $c, c' > 0$ are universal constants. Combined with the inequality (22), this proves the conclusion (21).

By (P1), $\mathcal{H}^1(E \cap \operatorname{supp}\mu) = 0$. The inequality (21) implies that

$$\mu(\mathbb{D}(x,r)) \le C R^{-1} r$$

for every ball $\mathbb{D}(x,r)$, $x \in K$, $r \le R$ and thus the measure μ vanishes on $E \supset F$. $\square$

2.1 Proof of Theorem 1

For simplicity, suppose that $\operatorname{diam} K = 1$. Theorem 9 supplies a Radon probability measure μ with $\mu(W) = 1$.

For every $x \in W$ we define a measurable function $\beta(x) : W \mapsto [0, +\infty)$ by

$$\liminf_{r\to 0} \frac{\int_r^1 \beta_K^2(x,t)\frac{dt}{t}}{-\log r} = \beta^2(x)\ .$$

For every $x \in W$ we have that

$$\begin{array}{rcl} \liminf_{r\to 0} \dfrac{\log \mu(\mathbb{D}(x,r)}{\log r} & \ge & \liminf_{r\to 0} \dfrac{-\log r + C' \int_r^{\operatorname{diam} K} \beta_K^2(x,t)\frac{dt}{t}}{-\log r} \\ & = & 1 + C' \liminf_{r\to 0} \dfrac{\int_r^1 \beta_K^2(x,t)\frac{dt}{t}}{-\log r} \ = \ 1 + C'\beta^2(x)\ . \end{array}$$

The mass distribution principle implies that

$$\dim_{\mathrm{H}}(K) \ge \dim_{\mathrm{H}}(W') \ge 1 + C' \mathrm{essup}_\mu \beta^2(x)\ ,$$

where $\mathrm{essup}_\mu \beta^2(x)$ is an essential supremum of $\beta^2(x)$ with respect to μ.

The proof of the theorem follows as $\mathrm{essup}_\mu \beta(x) > 0$ if $\beta(x) > 0$ and $\mathrm{essup}_\mu \beta(x) \geq \beta_0^2$ if $\beta(x) \geq \beta_0$ for every $x \in W$.

2.2 Proof of Theorem 5

As the constant C may be large, it is enough to prove the dimension estimate when ε is asymptotically close to 0. We will construct a measure on K following the inductive strategy of the proof of Theorem 9. Let M be defined as in the proof of Theorem 9. We assume $\varepsilon \ll \alpha\lambda^2 e^{-2M}$ and define

$$\varepsilon' = \frac{2e^{2M}\varepsilon}{\lambda^2}.$$

If K is not (ε', α)-dense in $2B$, we define μ_B as in Step 1. of the proof of Theorem 9. Otherwise, we set $t_B(x) = \varepsilon' R_B$ for all $x \in K_B$. $X(B)$ has the property that for all $x, x' \in X(B)$, $\mathcal{H}^1(E \cap B(x, \varepsilon' R_B)) < \frac{\varepsilon' R_B}{100}$, the balls $B(x, 2\varepsilon' R_B)$ and $B(x', 2\varepsilon' R_B)$ are disjoint, $\frac{10}{11}|X'(B)| \leq |X(B)|$, and

$$K \subseteq \bigcup_{x \in X'(B)} B(x, 10\varepsilon' R_B).$$

As K is (ε', α)-dense in $2B$, we obtain that

$$\alpha\, m(B) \leq m\left(\bigcup_{x \in X'(B)} B(x, 11\varepsilon' R_B)\right).$$

As balls $B(x, 2\varepsilon' R_B)$ with $x \in X(B)$ are disjoint, using a volume argument, we obtain

$$10 \cdot 11^{-d-1} \alpha\, \varepsilon'^{-d} < |X(B)| < \varepsilon'^{-d}.$$

We define the measure μ_B in the same way as in the proof of Theorem 9. We observe that for $x \in X(B)$,

$$\frac{\mu_B(B(x, t_B(x)))}{\mu_B(B)} = \frac{t_B(x)}{\sum_{y \in X(B)} t_B(y)} \leq \frac{\varepsilon' R_B}{\varepsilon' R_B |X(B)|} < \frac{11^{d+1}}{10} \frac{t_B(x)}{R_B} \alpha^{-1} \varepsilon'^{d-1}.$$

We obtain a new form of the inequality (20). For $B \in \mathcal{F}_n$, let $n = k_1 + k_2$, where k_2 is the number of steps m at which K is (ε', α)-dense in $2B_m$, where $B_m \in \mathcal{F}_m$ with $B \subseteq B_m$. By the previous inequality and Lemma 2.2 we obtain

$$\mu(B) \leq 10^{-k_1} \left(\frac{11^{d+1}}{10} \alpha^{-1} \varepsilon'^{d-1}\right)^{k_2} R^{-1} R_B.$$

The new measure μ_B that we have constructed does not satisfy the inequalities (13), because the constant C_M is replaced by $C(d,\varepsilon') = \varepsilon'^{1-d}$. Repeating the argument from Step 3. of Theorem 9, we obtain for all $x \in K$ and $0 < r < \operatorname{diam} K$

$$\mu(B(x,r)) \leq C'\varepsilon'^{-d}\left(11^{d+1}\alpha^{-1}\varepsilon'^{d-1}\right)^{k_2} R^{-1}r, \tag{23}$$

where $C' > 0$ is a universal constant.

Observe that as a consequence of (P1), we obtain that $\mathcal{H}^1(\operatorname{supp}\mu \cap E) = 0$ therefore $\mu(E) = 0$ and $\mu(W) = 1$. We now fix $x \in \operatorname{supp}\mu \cap W$, $\delta > 0$ and $r = \lambda^N$ for some large $N \in \mathbb{N}^*$ such that

$$\#P_\lambda(N) > N\left(\kappa - \frac{\delta}{2}\right),$$

where

$$P_\lambda(N) = \{m \in \{[-\log R]+1,\ldots,N\} \ : \ K \text{ is } (\varepsilon',\alpha)\text{-dense in } B(x,\lambda^m)\},$$

where we denote by $[x]$ the integer part of x, and R is the radius of B_0.

For each $n \geq 0$, let $B_n \in \mathcal{F}_n$ containing x and $R_n = R_{B_n}$. By construction, $\{x\} = \cap_{n>0} B_n$. Let $n_0 = n_0(r)$ be the first $n > 0$ such that $2B_n \subseteq B(x,r)$. We construct a map

$$\chi : P_\lambda(N) \to P_{\mathcal{F}}(n_0)$$

that is at most s to 1, where $s = \frac{\log\varepsilon}{\log\lambda}$ and

$$P_{\mathcal{F}}(n_0) = \{n \in \{0,\ldots,n_0\} \ : \ K \text{ is } (\varepsilon',\alpha)\text{-dense in } 2B_n\}.$$

As $k_2 = \#P_{\mathcal{F}}(n_0)$, if N is large enough, we obtain

$$k_2 \geq \frac{N}{s}(\kappa - \delta). \tag{24}$$

Let $m \in P_\lambda(N)$ and $n = n_0(\lambda^m) - 1$. We define

$$\chi(m) = \begin{cases} n & \text{if } n \in P_{\mathcal{F}}(n_0), \\ n+1 & \text{otherwise.} \end{cases}$$

If $n \in P_{\mathcal{F}}(n_0)$ then $R_{n+1} = \varepsilon' R_n$, thus there are at most $\left[\frac{\log\varepsilon' - \log 2}{\log\lambda}\right] + 1$ values in $P_\lambda(N)$ mapped to n by the first branch of f. If $n \notin P_{\mathcal{F}}(n_0)$, then by construction

$$M = \int_{R_{n+1}}^{R_n} \beta_K(x,t)^2 \frac{dt}{t} \geq \int_{R_{n+1}}^{\lambda^m/4} \beta_K(x,t)^2 \frac{dt}{t}.$$

Because K is (ε, α)-dense in $B(x, \lambda^m)$, it is (ε', α)-dense in $B(x, t)$ for every $t \in [e^{-2M}\lambda^m, \lambda^m]$. Therefore

$$\beta_K(x,t) \geq (1-\varepsilon'), \text{ for all } t \in [e^{-2M}\lambda^m, \lambda^m].$$

We can therefore estimate

$$\int_{e^{-2M}\lambda^m}^{\lambda^m/4} \beta_K(x,t)^2 \frac{dt}{t} \geq (1-\varepsilon')^2(2M - \log 4) > M,$$

as ε' is small and M is large. We conclude that $R_n > e^{-2M}\lambda^m$ so there are at most $\left[\frac{-2M}{\log\lambda}\right] + 1$ values in $P_\lambda(N)$ mapped to $n+1$ by the second branch of f. This completes the proof that χ is at most s to 1.

Combining the inequalities (23) and (24), we can compute that

$$\frac{-\log\mu(B(x,r))}{-\log r} \geq \frac{C(d,\varepsilon',R)}{N\log\lambda} + 1 + k_2\frac{(d+1)\log 11 + (d-1)\log\varepsilon' - \log\alpha}{N\log\lambda}$$

$$\geq \frac{C(d,\varepsilon',R)}{N\log\lambda} + 1 + (d-1)(\kappa-\delta)\frac{\log\varepsilon' - \frac{\log\alpha}{d-1}}{\log\varepsilon} + \frac{(d+1)(\kappa-\delta)\log 11}{\log\varepsilon}.$$

By the definition of ε',

$$\frac{\log\varepsilon'}{\log\varepsilon} = 1 + \frac{2\log\lambda - 2M - \log 2}{|\log\varepsilon|}.$$

Therefore,

$$\liminf_{N\to\infty} \frac{\log\mu(B(x,r))}{\log r} \geq 1 + \kappa\left(d - 1 - \frac{C(d,\alpha,\lambda)}{|\log\varepsilon|}\right),$$

which, by the mass distribution principle, yields the desired lower bound for $\dim_{\mathrm{H}}(K)$.

2.3 About the hypotheses of Theorem 5.

A prototype example of the unit circle $S^1 \subset \mathbb{C}$ together with two smooth curves γ_- and γ_+ winding infinitely many times around it both from inside and outside, shows that the hypothesis that $\mathcal{H}^1(E) < \infty$ can not be dropped. Indeed, assume that a point traveling along any of these two smooth curves is approching S^1 slowly enough so that for every $\varepsilon > 0$, S^1 is not ε-porous at any $z \in S^1$ at all scales small enough. Clearly, $\dim_{\mathrm{H}}(S^1 \cup \gamma_- \cup \gamma_+) = 1$ and $\mathcal{H}^1(\gamma_- \cup \gamma_+) = \infty$.

The following examples show that the hypotheses about connectivity and $\mathcal{H}^1(W) > 0$ are also essential.

Example 1. We recall that if $S_i \subset \mathbb{C}$, $i = 1, 2$, then $S_1 + S_2$ stands for a set of all $z \in \mathbb{C}$ such that $z = v_1 + v_2$, where $v_1 \in S_1$ and $v_2 \in S_2$.

Let $[0, 1] \supset K_m = [0, 2^{-4m}] + \{k2^{-2m} \ : \ k = 1, \ldots, 2^{2m} - 1\}$. In the formula below, $i^2 = -1$.

$$K = [0, 1] \cup \bigcup_{m \geq 1} \bigcup_{k=0}^{2^m - 1} \left(\pm i(2^{-m} + k2^{-2m}) + K_m \right).$$

One can easily check that $\mathcal{H}^1(K) < \infty$ and that for any $\varepsilon > 0$, K is not ε-porous at scales of density 1 at every $z \in (0, 1)$.

Example 2. Let $\alpha \in (0, 1)$. We consider a standard α-Cantor set $K_\alpha \subset [0, 1]$ defined as an invariant set for the map $g : [0, 1] \mapsto \mathbb{R}$,

$$g(x) = \begin{cases} x/\alpha & \text{if} \quad x \in [0, 1/2] \\ (1 - x)/\alpha & \text{if} \quad x \in [1/2, 1] \end{cases}$$

Geometrically, K_α is an intersection of the union of 2^n closed intervals J_j^n, $j = 1, \ldots 2^n$, each of the length $\left(\frac{1-\alpha}{2}\right)^n$. Every J_j^n is a connected component of $g^{-n}[0, 1]$. Let Q_j^n be a square with the base equal to J_j^n contained in the half-plane $\Im z \geq 0$. In the formula below, i stands for the imaginary number. Put R_j^n to be the union of $\partial Q_j^n \cup \{2J_j^n + i\frac{k}{n}|J_j^n| : k = 1, \ldots, n\}$ and its reflection with respect to the real axis. The set

$$R_\alpha := [0, 1] \cup \bigcup_{n \geq 1} \bigcup_{j=1}^{2^n} R_j^n$$

is connected and every point of K_α is not ε-porous for any $\varepsilon > 0$ at any scale small enough. One can easily check that $\mathcal{H}^1(R_\alpha)$ is bounded by a constant which depends only on α. Suppose that α_n tends to 0 and denote the corresponding R_α by R_n and K_α by K_n. The union

$$R = [0, 1] \cup \bigcup_{k \geq 1} \left(\beta_k R_k + \frac{1}{k} \right),$$

where β_k are chosen so that $\beta_k \mathcal{H}^1(R_k) \leq 4^{-k}$ is a continuum with the property that every point $z \in \bigcup_{n \geq 1} K_n$ is not ϵ-porous for any $\varepsilon > 0$ and any scale small enough. Therefore, $\dim_{\mathrm{H}}(W) \geq \sup_{n \geq 1} \dim_{\mathrm{H}}(K_n) = 1$. Nevertheless, $\mathcal{H}^1(R) < \infty$.

2.4 Universal version of μ

If we do not require that the measure μ from Theorem 9 has the support disjoint from F then the construction can be modified to obtain a uniform scaling property of μ.

Theorem 12. *Let $K \subset \mathbb{R}^d$ with $d \geq 2$ be a connected compact of diameter 1. There exist a universal constants $C' > 0$, a constant $C > 0$ depending only on d, and a Radon probability measure μ supported on K such that for all $x \in K$ and $r > 0$*

$$\mu(B(x,r)) \leq Cr \exp\left(-C' \int_r^1 \beta_K^2(x,t)\frac{dt}{t}\right).$$

Proof. The proof of Theorem 12 is very similar to the proof of Theorem 9 and is based on Proposition 1 and the constructions from the proof of Theorem 45 in [23], compare [7]. As in the proof of Theorem 9 we have three steps.

1. For any ball B centered on K we construct a measure μ_B.

2. We construct the probability measure μ on K.

3. We show that μ has the desired scaling properties.

The estimates from the third step are very much the same as in the proof of Theorem 9 (with B_0 replaced by K, R replaced by 1, and the claims about $\mathcal{H}^1(E \cup \operatorname{supp} \mu_j)$ and $\mathcal{H}^1(E \cap \operatorname{supp} \mu)$ dropped). The first and the second steps are slightly different as they must account for the existance of non-wiggly parts of K. We use the notation from the proof of Theorem 9.

Step 1. Let $B = \mathbb{D}(x_B, R_B)$ be a ball centered on K with $R_B \leq 1$. Let $K_B := K \cap \overline{B}$ and observe that $K_B \cup \partial B$ is a connected compact. We say that $B \in \mathcal{G}$ (respectively that $B \in \mathcal{B}$) if $\mathcal{H}^1(Z(B)) \geq \frac{R_B}{100}$ (respectively if $\mathcal{H}^1(Z(B)) < \frac{R_B}{100}$).

Assume first that $B \in \mathcal{G}$. Since $\int_0^{R_B} \beta_{Z(B)}^2(x,t)\frac{dt}{t} \leq 4M$ for any $x \in Z(B)$, by Theorem 11, the set $Z(B)$ is contained in an Ahlfors regular curve Γ_B whose regularity constant depends only on M and of length comparable to R_B. We set

$$\mu_B := \mathcal{H}^1{}_{|Z(B)}.$$

By the properties of Γ_B, there exists a constant $C_M > 0$ that depends only on M such that the conditions (13) are satisfied.

If $B \in \mathcal{B}$ then we can repeat the construction of μ_B from the first step of Theorem 9. The only modification is that we define μ on $G'_B :=$

$\bigcup_{x\in X'(B)} G(x)$ rather than on $G_B := \bigcup_{x\in X(B)} G(x)$, as it was the case in the proof of the Theorem 9. We put

$$\mu_B := \mathcal{H}^1{}_{|G'_B}$$

By the properties of Γ_B it is easy to check that μ_B satisfies the inequalities (13).

Step 2. We construct a sequence $(\mu_n)_{n\geq 0}$ of probability measures supported respectively on a decreasing sequence of compact neighborhoods of K, having K as their intersection. The measure μ is a weak limit of this sequence.

For each $n \geq 0$, we define a collection of disjoint balls $\mathcal{F}_n$ as follows. Let $x_0, y_0 \in K$ be such that $|x_0 - y_0| = 1 = \operatorname{diam} K$. Let $\mathcal{F}_0 := \{\mathbb{D}(x_0, 1)\}$ and

$$\mathcal{F}_{n+1} := \{\mathbb{D}(x, t_B(x)) : x \in X(B), B \in \mathcal{F}_n \cap \mathcal{B}\}.$$

Remarks. 1) If $\mathcal{F}_n \subseteq \mathcal{G}$ then $\mathcal{F}_{n+1} = \emptyset$.
2) Let $B_1, B_2 \in \mathcal{F}_n$, $B'_1, B'_2 \in \mathcal{F}_{n+1}$. As in the second step of the proof of Theorem 9 we obtain that

$$B'_1 \cap B'_2 = \emptyset, B_1 \cap B'_2 = \emptyset \text{ and } B'_1 \cap B_2 = \emptyset.$$

Let $\mathcal{G}_n := \mathcal{F}_n \cap \mathcal{G}$ and $\mathcal{B}_n := \mathcal{F}_n \cap B$. We will use the measures μ_B constructed in the previous step to define inductively the sequence $(\mu_n)_{n\geq 0}$. Let

$$\mu_0 := \frac{\mu_B}{\mu_B(\mathbb{C})},$$

where $B = \mathbb{D}(x_0, 1)$. Assume that $\mu_0, \ldots, \mu_n$ have been defined with the following properties.

(P1) $\operatorname{supp} \mu_n$ is contained in the disjoint union of balls

$$\mathcal{F}_n \cup \bigcup_{k=0}^{n-1} \mathcal{G}_k\,,$$

which is contained in a 10^{-n}-neighborhood of K.

(P2) $\mu_n(B) \leq 10^{-k} R_B$ if $B \in \mathcal{F}_k$ for some $k \leq n$ (R_B is the radius of B).

Define μ_{n+1} in the following way. For any ball $B \in \mathcal{G}_k, k \leq n$,

$$\mu_{n+1|B} := \mu_{n|B}.$$

For any ball $B \in \mathcal{F}_{n+1}$,

$$\mu_{n+1|B} := \frac{\mu_n(B)}{\mu_B(B)}\mu_B .$$

Properties (P1) and (P2) for μ_{n+1} are direct consequences of the inequalities (15) and (14), and the fact that $Z(B) \subset K$ for any ball B.

$\square$

Corollary 2.3. *Let μ be supplied either by Theorem 9 or Theorem 12. Then μ is absolutely continuous with respect to 1-dimensional Hausdorff measure $\mathcal{H}^1$.*

Proof. In both cases, for each measure μ, there exists a constant $M > 0$ (which is universal if μ is provided by Theorem 12) such that for every ball $B(x, r)$, $x \in K$, $r < \operatorname{diam} K$,

$$\mu(B(x, r)) < Mr . \tag{25}$$

A standard argument shows that $\mathcal{H}^1(A) = 0 \Longrightarrow \mu(A) = 0$. $\square$

3 Topological Collet-Eckmann rational maps

TCE property (see the definition in Section 1.4) implies the so called *exponential shrinking* which states that there exists a number $\xi < 1$ (depending on δ, κ, and L but not on z) so that

$$\operatorname{diam} \operatorname{Comp}_z f^{-n}(B_\delta(f^n(z)) \leq \xi^n$$

for every $n \in \mathbb{N}$ and $z \in \mathcal{J}$.

Proof of Theorem 8. The proof is by contradiction. We may assume, by decreasing slightly δ, that every component $\operatorname{Comp}_z f^{-n}(B_\delta(f^n(z))$ satisfies the inclusions,

$$B(z, r_n) \subset \operatorname{Comp}_z f^{-n}(B_\delta(f^n(z)) \subset B(z, \alpha r_n) ,$$

where $\alpha > 1$ is a constant which depends only on f and L. Additionally, $f^n(B(z, r_n))$ contains a ball of radius comparable to δ.

Let $\lambda = 1/2$. Put $\chi_i = 1$ if there is n such that $r_n \in A_i$, where $A_i = [\lambda^{i+1}, \lambda^i)$, i is non-negative integer, and $\chi_i = 0$ otherwise. We want to show that there exists $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \chi_i}{n} \geq \varepsilon .$$

It is enough to prove that the sequence $[\log 1/r_n]$ is of positive density amongst integers, where $[\cdot]$ stands for an integer part of a given number. Enumerate by n_k all consecutive passages to the scale δ with bounded criticality L. By the exponential shrinking property, for every $x \in \mathcal{J}$ diam $\mathrm{Comp}_{f^{n_k-i}(x)} f^{-i}(B(f^{n_k}(x),\delta)$ is smaller than $\delta/2$ if only i is large enough. Therefore, without loss of generality, the modulus of the annulus $(B(f^{n_k}(x),\delta) \setminus \mathrm{Comp}_{f^{n_k}(x)} f^{-n_{k+1}+n_k}(B(f^{n_{k+1}}(x),\delta)$ is bigger than $\log 2$. By the definition of n_k, and Teichmüller's module theorem, there exists a constant $P > 0$ such that at most P consecutive numbers $\log \frac{1}{r_n}$ can be counted as the same integer.

Let $M = \sup_{z\in\mathcal{J}} |f'(z)|$. Then

$$M^{-n} \leq \text{diam } \mathrm{Comp}_z f^{-n}(B_\delta(f^n(z))$$

and

$$\begin{aligned} \frac{1}{N}\#(\{[\log \frac{1}{r_k}] : k \geq 0\} \cap [0,N]) &\gtrsim \frac{1}{NP}\#\{k \geq 0 : 2k\kappa^{-1}\log M < N\} \\ &\geq \frac{\kappa}{2P\log M} := \kappa_0 \end{aligned} \tag{26}$$

provided N is large enough.

Suppose that the Julia set $\mathcal{J}$ of f is not mean wiggly. Hence, for every $\beta > 0$ and $\rho \in (0,1)$ there is a point $z_\beta \in \mathcal{J}$ such that $B(z_\beta, \lambda^n) \cap \mathcal{J}$ is contained in a $\beta\lambda^n$ neighborhood of a line minimizing $\beta_{\mathcal{J}}(z_\beta, \lambda^n)$ and this property holds for the set Z_β of integers $n \in \mathbb{N}$ of the density bigger than ρ. Let us choose ρ so that $\rho + \kappa_0 > 1$, where κ_0 is defined in the estimate (26).

For every $n \in Z_\beta$,

(i) $f(B(z_\beta, \alpha r_n)) \supset B_n = B(f^n(z), \delta')$ and $\delta' \sim \delta$,

(ii) the degree of f^n on $B(z_\beta, r_n)$ is bounded by L.

This means that there is $\beta' > 0$ (which does not depend on n) such that the Julia set $\mathcal{J}$ in B_n is contained in a β'-neighborhood of a finite union of analytic Jordan arcs. Also, β' tends to 0 when β does so. Passing to the limit with $\beta \to 0$, we obtain that the Julia set in some ball B of radius δ' is a finite union of analytic Jordan arcs. By the eventually onto property (B is mapped over $\mathcal{J}$ by an iterate of f), the whole $\mathcal{J}$ is a finite union of analytic Jordan arcs.

Since every rational function f can have either 1, 2 or an infinite number of Fatou components, the piecewise analyticity of $\mathcal{J}$ implies that f has either

1 or 2 Fatou components. By connectivity of $\mathcal{J}$, every Fatou component $\mathcal{F}$ is simply connected. Without loss of generality, we can assume that every Fatou component is totally invariant by f. Also, $\mathcal{J}$ coincides with the boundary of every Fatou component. For every $z \in \mathcal{J}$, we define the set of angles of accesses $\{\theta_i(z)\}_{\mathcal{F}}$ from inside of every invariant Fatou component $\mathcal{F}$. Observe that if $\{\theta_i(z)\}_{\mathcal{F}} \setminus \{\pi, 2\pi\} \neq \emptyset$ then the same is true for every preimage $y \in f^{-1}(z)$. Since the set of preimages of a given $z \in \mathcal{J}$ is infinite and $\mathcal{J}$ is a union of finitely many Jordan analytic arcs, we infer that for every $z \in \mathcal{J}$, $\{\theta_i(z)\}_{\mathcal{F}} \subset \{\pi, 2\pi\}$. As a result, $\mathcal{J}$ is either an analytic Jordan arc or an analytic circle. By the Fatou theorem [9], the Julia set coincides with a geometric circle or a segment.

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