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Abstract: In rational dynamics, we prove the existence of a polynomial that satisfies the *Topological Collet-Eckmann condition*, but which has a recurrent critical orbit that is not Collet-Eckmann. This shows that the converse of the main theorem in [12] does not hold.

In interval dynamics, we show that the Collet-Eckmann property for recurrent critical orbits is not a topological invariant for real polynomials with negative Schwarzian derivative. This contradicts a conjecture of Świątek [25].

1. Introduction

In one-dimensional real and complex dynamics, there are several conditions which guarantee some form of non-uniform hyperbolicity, which in turn gives a reasonable understanding on statistical and geometric properties of the dynamics. Classical examples include the *Misiurewicz condition* [13], *semi-hyperbolicity* [5] and the *Collet-Eckmann condition* (CE) [7, 11, 2, 14, 16, 15, 19, 20, 10, 21]. More recent examples include the *Topological Collet-Eckmann condition* (TCE) [15, 21, 22], summability conditions [17, 9, 3] and the *Collet-Eckmann condition for recurrent critical orbits* (RCE) [25, 12].

In the setting of rational dynamics, CE implies TCE [10, 22], but there are TCE polynomials which do not verify the CE condition [22]. Graczyk asked whether RCE is equivalent to TCE. In a previous paper, the author showed that RCE implies TCE [12] for rational maps. Here, we show that the converse is not true.

Theorem A. *There exists a TCE rational map that is not RCE.*

The example constructed in the proof is a degree 3 real polynomial which is a bimodal map of the unit interval with negative Schwarzian derivative (S-bimodal). On the interval, TCE is equivalent to CE in the S-unimodal setting

[16,15], thus CE is a topological invariant. However, with more than one critical point this is no longer true. Świątek conjectured that RCE is topologically invariant for multimodal analytic maps with negative Schwarzian derivative [25]. We show this conjecture to be false.

Theorem B. *In interval dynamics, the RCE condition for analytic S-multimodal maps is not topologically invariant.*

1.1. History. In one-dimensional dynamics, the orbits of critical points (zeros of the derivative in the smooth case) play a special role. Conditions on critical orbits and their consequences for the dynamics were first studied in the context of interval dynamics.

Two important results in this direction were published in 1981. Misiurewicz showed that S-unimodal maps with non-recurrent critical orbit have an absolutely continuous invariant probability measure [13], see Definitions 9 and 18. Jakobson proved that real quadratic maps which have such invariant measures are abundant, that is, the set of their parameters is of positive measure [11]. Collet and Eckmann introduced the CE condition and showed that S-unimodal CE maps have absolutely continuous invariant probability measures [7]. Benedicks and Carleson showed that CE maps are abundant in the quadratic family [2].

Nowicki and Sands showed that CE is equivalent to several non-uniform hyperbolicity conditions for S-unimodal maps [14,16]. One of them is *Uniform Hyperbolicity on repelling Periodic orbits* (UHP), see Definition 5. Shortly after, it was noticed by Nowicki and Przytycki that CE is also equivalent to TCE in this setting [15], see Definition 7. Therefore CE becomes a topological invariant for S-unimodal maps. This is no longer true for S-multimodal maps [15,25]. Świątek conjectured that RCE (see Definition 10) is topologically invariant for multimodal analytic maps with negative Schwarzian derivative, see Conjecture 1 in [25]. We provide a counterexample to this conjecture, Theorem B. Results on the attractors and ergodic invariant measures of CE S-multimodal maps were recently obtained by Cedervall [6].

A generalization of the Misiurewicz condition, semi-hyperbolicity, was studied in complex polynomial dynamics by Carleson, Jones and Yoccoz [5], see Definition 2. They show that semi-hyperbolicity is equivalent to TCE with $P = 1$ (see Definition 7), but also to John regularity of Fatou components. They also prove that such polynomials satisfy the *Exponential Shrinking of components condition* (*ExpShrink*), using a telescopic construction, see Definition 4.

CE rational maps were initially studied by Przytycki [19,20]. Later, he showed that TCE implies CE if the Julia set contains only one critical point (unicritical case) [21]. Graczyk and Smirnov show that CE implies the *backward or second Collet-Eckmann condition* ($CE2(z_0)$), which in turn implies the Hölder regularity of Fatou components [10], see Definition 6. The first implication is obtained using a telescopic construction along the backward orbit of z_0. Przytycki and Rohde proved that CE implies TCE [23]. Therefore, in the unicritical case they are equivalent and CE is topologically invariant. Rees proved that rational maps which have an invariant probability measure equivalent to Lebesgue measure are abundent [24]. Recently, Aspenberg showed that CE rational maps are also abundant [1].

Przytycki, Rivera-Letelier and Smirnov establish the equivalence of several non-uniform hyperbolicity conditions, as TCE, $CE2(z_0)$, UHP, $ExpShrink$ and the existence of a positive lower bound for the Lyapunov exponent of invariant measures [22]. A semi-hyperbolic counterexample shows that TCE does not imply CE. Another counterexample, involving semi-hyperbolic maps, shows that CE is not a quasi-conformal invariant.

In an attempt to characterize TCE in terms of properties of critical orbits, the author studied RCE for rational maps (see Definition 3). In [12] it is shown that RCE implies TCE. We provide a counterexample to the converse, Theorem A.

1.2. A short overview. In the following section we present some definitions and basic results and lemmas necessary for our study.

In Section 3, we describe a technique of building real polynomials with prescribed topological and analytical properties by specifying their combinatorial properties. The critical orbits of the polynomials we shall construct will be on the interval $[0, 1]$. Therefore, we can restrict our attention to the dynamics on the unit interval. We shall make use of the theory of *kneading sequences* to construct our maps.

A kneading sequence is a sequence of symbols associated to the points of a critical orbit. The critical points of an interval map define a partition of the interval. To each element of the partition we associate a symbol. The orbit of a critical point will thus generate a symbol sequence which describes the itinerary of the point through the various partition elements. Knowledge of the kneading sequences is enough to fully describe the combinatorics of a map. Moreover, in the absence of *homtervals* (Definition 14), two maps with the same kneading sequences are topologically conjugate.

We shall consider one-parameter families of bimodal maps (i.e. with two critical points) of the interval, see Definition 9. While the theory of multimodal maps and kneading sequences is generally well understood [8], for the most part it is related to topological properties of the dynamics. We develop new tools to obtain a prescribed growth (or lack thereof) of the derivative on the critical orbits.

In Section 4 we prove Theorem A, constructing an $ExpShrink$ polynomial (thus TCE) which is not RCE. In the vicinity of critical points the diameter of a small domain decreases at most in the power rate, while the derivative can approach 0 as fast as one wants (in comparison with the diameter of the domain). This important difference in the behaviour of derivative and diameter is the main idea of the (rather technical) proof.

In Section 5 we prove Theorem B, a counterexample to the conjecture of Świątek. Using careful estimates of the derivative on the critical orbits we construct two polynomials with negative Schwarzian derivative with the same combinatorics, thus topologically conjugate on the interval, such that only one is RCE. This situation is in sharp contrast with the unimodal case, where the Collet-Eckmann condition is topologically invariant [16,15]. An important feature of our counterexample is that the corresponding critical points of this two polynomials are of different degree. One should be aware that considered as maps of the complex plane they are not conjugate.

1.3. A construction. Our counterexamples are bimodal real polynomials with a preperiodic critical point. Let $c_1 < c_2$ be the critical points of such a polynomial f. Then $f(c_1) = 1$ and $0, 1$ and $r \in (c_1, c_2)$ are its fixed points, all repelling. The orbit of c_2 accumulates on c_1 and c_2 and spends most of the time near r. As r is repelling, we observe exponential growth of the derivative on this orbit most of the time. The tools from Section 3 let us make this orbit pass near c_2, thus making c_2 recurrent by the inductive construction, without losing too much growth of the derivative.

We use passages near c_1 to control the growth of the derivative. In this case we have the freedom to approach c_1 as close as needed. Moreover, it is easy to do estimates of the loss of derivative by the distance of the image to $1 = f(c_1)$. This distance is directly related to the time spent by the orbit near 1. This time translates immediately in the number of symbols I_3 (corresponding to the interval $(c_2, 1]$) that follow I_1 (corresponding to the interval $[0, c_1)$) in the kneading sequence. Near c_1, we have explicit expression of the map and its derivative. After spending some time in I_3, the orbit comes back near r, where it stays much longer. This relates to the number of consecutive symbols I_2 we prescribe for the kneading sequence.

We repeat the inductive construction and obtain a polynomial with the desired properties. In practice, to show that such a polynomial exists, we exhibit a family of polynomials and we prove that it contains the polynomial. Every inductive step corresponds to the restriction to a subfamily and their intersection contains exactly one polynomial, the desired counterexample.

While the explicit construction of the families may not be needed to prove Theorem A, its necessity becomes clear in the proof of Theorem B.

1.4. Ideas of the proofs.

Theorem A. In [12], *RCE* rational maps are shown to be *ExpShrink* (thus *TCE*). A backward telescope construction of pullbacks of balls (the pullbacks are the *tubes* of the telescope) is used. When those pullbacks contain Collet-Eckmann critical orbits, the growth of the derivative is recovered in terms of diameters using distortion bounds. Koebe's Theorem is such a distortion result for univalent maps. Non-recurrent critical points cannot appear twice inside such a tube, provided the diameters of its pullbacks are small. Therefore the degree is bounded inside a tube that does not contain Collet-Eckmann critical points. Again by distortion results and compactness of the Julia set, contraction is obtained for such tubes. A bootstrap strategy is used to show the needed bound for the diameters using the aforementioned contraction results. In turn, this bound combined with the contraction results show *ExpShrink*.

As the polynomial we construct in the proof of Theorem A is not *RCE*, some of the tools of [12] break down. On the critical orbit (of c_2) we have exponential expansion most of the time, except when it approaches c_1. Here, the derivative is very close to 0 but the diameter of a pullback can only grow in the power rate (square root). This is compensated by the long periods of exponential expansion. We cannot use the bootstrap strategy to show *Backward Stability* (see Definition Definition 46). We obtain however a weaker version by the proximity of an *ExpShrink* polynomial (see Proposition 48).

Theorem B. We will see that the loss of growth at the passage of the critical orbit near c_1 is controlled by the degree of c_1, the time spent in I_3 afterwards and by the multiplier at 1. The general expansion is controlled by the multiplier at r and the time spent in I_2.

Two conjugated bimodal maps have the same kneading sequence, therefore equal times spent by their critical orbits in corresponding areas. We fix the degrees of the critical points of both polynomials and start the inductive construction. Their kneading sequence has the same structure as the aforementioned basic construction. We may vary the time spent near 1 and r. The loss of derivative on the second critical orbit as it passes near c_1 should be large enough in one case and small enough in the other so only one of the two maps is RCE.

We observe that in order to find such a delicate balance, multipliers at 1 and r of both maps should satisfy an inequality that also depends of the degrees of the critical points c_1. That is inequality (60) and the only way we managed to prove it for suitable families of bimodal polynomials was their explicit construction.

2. Preliminaries

Let R be a rational map, J its Julia set and Crit the set of critical points.

Definition 1. *We say that* $c \in \mathrm{Crit}$ *satisfies the* Collet-Eckmann condition ($c \in CE$) *if* $|(R^n)'(R(c))| > C\lambda^n$ *for all* $n > 0$ *and some constants* $C > 0, \lambda > 1$. *We say that* R *is* Collet-Eckmann *if all critical points in* J *are* CE.

Definition 2. *Given* $c \in \mathrm{Crit}$ *we say that it is* non-recurrent ($c \in NR$) *if* $c \notin \omega(c)$, *where* $\omega(c)$ *is the* ω-limit set*, the set of accumulation points of the orbit* $(R^n(c))_{n>0}$. *We call* R semi-hyperbolic *if all critical points in* J *are non-recurrent and* R *has no parabolic periodic orbits.*

Recurrent Collet-Eckmann condition is weaker than Collet-Eckmann or semi-hyperbolicity alone. Indeed, in [10] it is shown that a Collet-Eckmann rational map cannot have parabolic cycles.

Definition 3. *We say that* R *satisfies the* Recurrent Collet-Eckmann (RCE) *condition if every recurrent critical point in the Julia set is Collet-Eckmann and* R *has no parabolic periodic orbits.*

Let us remark that a RCE rational map may have critical points in J that are Collet-Eckmann and non-recurrent in the same time. Moreover any critical orbit may accumulate on other critical points.

Several weak hyperbolicity standard conditions are shown to be equivalent in [22]. Among these conditions we recall *Topological Collet-Eckmann condition* (TCE), *Uniform Hyperbolicity on Periodic orbits* (UHP), *Exponential Shrinking of components* (*ExpShrink*) and *Backward Collet-Eckmann condition at some* $z_0 \in \overline{\mathbb{C}}$ ($CE2(z_0)$).

Let us define these conditions.

Definition 4. R *satisfies the* Exponential Shrinking of components condition (*ExpShrink*) *if there are* $\lambda > 1,\ r > 0$ *such that for all* $z \in J,\ n > 0$ *and every connected component* W *of* $R^{-n}(B(z,r))$

$$\operatorname{diam} W < \lambda^{-n}.$$

Here we use the spherical distance as ∞ may be contained in the Julia set.

Definition 5. *R satisfies* Uniform Hyperbolicity on Periodic orbits (UHP) *if there is* $\lambda > 1$ *such that for all periodic points* $z \in J$ *with* $R^n(z) = z$ *for some* $n > 0$

$$|(R^n)'(z)| > \lambda^n.$$

Definition 6. *R satisfies the* Backward Collet-Eckmann condition at some $z_0 \in \overline{\mathbb{C}}$ $(CE2(z_0))$ *if there are* $\lambda > 1, C > 0$ *and* $z_0 \in \overline{\mathbb{C}}$ *such that for any preimage* $z \in \overline{\mathbb{C}}$ *of* z_0 *with* $R^n(z) = z_0$ *for some* $n > 0$

$$|(R^n)'(z)| > C\lambda^n.$$

The definition of TCE is technical but it is stated exclusively in topological terms, therefore it is invariant under topological conjugacy.

Definition 7. *R satisfies the* Topological Collet-Eckmann condition (TCE) *if there are* $M \geq 0, P \geq 1$ *and* $r > 0$ *such that for all* $z \in J$ *there exists a strictly increasing sequence of integers* $(n_j)_{j \geq 1}$ *such that for all* $j \geq 1$, $n_j \leq P \cdot j$ *and*

$$\#\left\{i \ : \ 0 \leq i < n_j, \mathrm{Comp}_{R^i(z)} R^{-(n_j - i)} B(R^{n_j}(z), r) \cap \mathrm{Crit} \neq \emptyset \right\} \leq M,$$

where Comp_y *means the connected component containing* y.

Using the equivalence of these conditions, we may formulate the main result in [12] as follows.

Theorem 8. *The* RCE *condition implies* TCE *for rational maps.*

To produce our counterexamples we restrict to dynamics of real polynomials on the interval with all critical points real. This is a particular case of multimodal dynamics.

In the sequel all distances and derivatives are considered with respect to the Euclidean metric if not specified otherwise.

Let us define multimodal maps and state some classical results about their dynamics.

Definition 9. *Let* I *be the compact interval* $[0, 1]$ *and* $f : I \to I$ *a* piecewise strictly monotone *continuous map. This means that* f *has a finite number of turning points* $0 < c_1 < \ldots < c_l < 1$, *points where* f *has a local extremum, and* f *is strictly monotone on each of the* $l + 1$ *intervals* $I_1 = [0, c_1), I_2 = (c_1, c_2), \ldots, I_{l+1} = (c_l, 1]$. *Such a map is called* l-modal *if* $f(\partial I) \subseteq \partial I$. *If* $l = 1$ *then* f *is called* unimodal. *If* f *is* C^{1+r} *with* $r \geq 0$ *it is called a smooth* l*-modal map if* f' *has no zeros outside* $\{c_1, \ldots, c_l\}$.

If f is a l-modal map, let us denote by Crit_f the set of turning points - or critical points

$$\mathrm{Crit}_f = \{c_1, \ldots, c_l\}.$$

Let us define the *Recurrent Collet-Eckmann condition* (RCE) in the context of multimodal dynamics. Remark that it is similar to Definition 3.

Definition 10. *We say that f satisfies RCE if every recurrent critical point $c \in \mathrm{Crit}_f$, $c \in \omega(c)$ is Collet-Eckmann, that is, there exist $C > 0, \lambda > 1$ such that for all $n \geq 0$*

$$|(f^n)'(f(c))| > C\lambda^n.$$

Our counterexamples are polynomials which have all critical points in $I = [0,1]$ which is included in the Julia set, as they do not have attracting or neutral periodic orbits and I is forward invariant. Therefore in this case the previous definition is equivalent to Definition 3. Analogously, semi-hyperbolicity, UHP, $CE2(x)$ and TCE admit very similar definitions to the rational case.

For all $x \in I$ we denote by $O(x)$ or $O^+(x)$ its forward orbit

$$O(x) = (f^n(x))_{n \geq 0}.$$

Analogously, let $O^-(x) = \{y \in f^{-n}(x) \ : \ n \geq 0\}$ and $O^{\pm}(x) = \{y \in f^n(x) \ : \ n \in \mathbb{Z}\}$. We also extend these notations to orbits of sets. For $S \subseteq I$ let $O^+(S) = \{f^n(x) \ : \ x \in S, n \geq 0\}$, $O^-(S) = \{y \in f^{-n}(x) \ : \ x \in S, n \geq 0\}$ and $O^{\pm}(S) = O^+(S) \cup O^-(S)$.

One of the most important questions in all areas of dynamics is when two systems have similar underlaying dynamics. A natural equivalence relation for multimodal maps is topological conjugacy.

Definition 11. *We say that two multimodal maps $f, g : I \to I$ are* topologically conjugate *or simply* conjugate *if there is a homeomorphism $h : I \to I$ such that*

$$h \circ f = g \circ h.$$

One may remark that if f and g are conjugate by h then $h(f^n(x)) = g^n(h(x))$ for all $x \in I$ and $n \geq 0$ so h maps orbits of f onto orbits of g. It is easy to check that h is a monotone bijection form the critical set of f to the critical set of g. We may also consider combinatorial properties of orbits and use the order of the points of critical orbits to define another equivalence relation between multimodal maps. Theorem II.3.1 in [8] shows that it is enough to consider only the forward orbit of the critical set.

Theorem 12. *Let f, g be two l-modal maps with turning points $c_1 < \ldots < c_l$ respectively $\tilde{c}_1 < \ldots < \tilde{c}_l$. The following properties are equivalent.*

1. *There exists an order preserving bijection h from $O^+(\mathrm{Crit}_f)$ to $O^+(\mathrm{Crit}_g)$ such that*
$$h(f(x)) = g(h(x)) \text{ for all } x \in O^+(\mathrm{Crit}_f).$$
2. *There exists an order preserving bijection $\tilde{h}$ from $O^{\pm}(\mathrm{Crit}_f)$ to $O^{\pm}(\mathrm{Crit}_g)$ such that*
$$\tilde{h}(f(x)) = g(\tilde{h}(x)) \text{ for all } x \in O^{\pm}(\mathrm{Crit}_f).$$

If f and g satisfy the properties of the previous theorem we say that they are *combinatorially equivalent*. Note that if f and g are conjugate by an order preserving homeomorphism h then the restriction of h to $O^+(\mathrm{Crit}_f)$ is an order preserving bijection onto $O^+(\mathrm{Crit}_g)$ so f and g are combinatorially equivalent. The converse is true only in the absence of homtervals. It is the case of all the examples in this chapter. There is a very convenient way to describe the combinatorial type of a multimodal map using symbolic dynamics. We associate to every point $x \in I$ a sequence of symbols $\underline{i}(x)$ that we call the *itinerary* of x. The itineraries $\underline{k}_1, \ldots, \underline{k}_l$ of the critical values $f(c_1), \ldots, f(c_l)$ are called the kneading sequences of f and the ordered set of kneading sequences the kneading invariant. Combinatorially equivalent multimodal maps have the same kneading invariants but the converse is true only in the absence of homtervals. We use the kneading invariant to describe the dynamics of multimodal maps in one-dimensional families. We build sequences $(\mathcal{F}_n)_{n\geq 0}$ of compact families of C^1 multimodal maps with $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ for all $n \geq 0$ and obtain our examples as the intersection of such sequences.

When not specified otherwise, we assume f to be a multimodal map.

Definition 13. *Let $O(p)$ be a periodic orbit of f. This orbit is called* attracting *if its* basin

$$B(p) = \left\{x \in I \ : \ f^k(x) \to O(p) \ \textit{as} \ k \to \infty\right\}$$

contains an open set. The immediate basin $B_0(p)$ *of $O(p)$ is the union of connected components of $B(p)$ which contain points from $O(p)$. If $B_0(p)$ is a neighborhood of $O(p)$ then this orbit is called a* two-sided attractor *and otherwise a* one-sided attractor. *Suppose f is C^1 and let $m(p) = |(f^n)'(p)|$ where n is the period of p. If $m(p) < 1$ we say that $O(p)$ is attracting respectively super-attracting if $m(p) = 0$. We call $O(p)$* neutral *if $m(p) = 1$ and we say it is* repelling *if $m(p) > 1$.*

Let us denote by $B(f)$ the union of the basins of periodic attracting orbits and by $B_0(f)$ the union of immediate basins of periodic attractors. The basins of attracting periodic contain intervals on which all iterates of f are monotone. Such intervals do not intersect $O^-(\mathrm{Crit}_f)$ and they do not carry too much combinatorial information.

Definition 14. *Let us define a* homterval *to be an interval on which f^n is monotone for all $n \geq 0$.*

Homtervals are related to *wandering intervals* and they play an important role in the study of the relation between conjugacy and combinatorial equivalence.

Definition 15. *An interval $J \subseteq I$ is* wandering *if all its iterates $J, f(J), f^2(J), \ldots$ are disjoint and if $(f^n(J))_{n\geq 0}$ does not tend to a periodic orbit.*

Homtervals have simple dynamics described by the following lemma, Lemma II.3.1 in [8].

Lemma 16. *Let J be a homterval of f. Then there are two possibilities:*

1. *J is a wandering interval;*
2. *$J \subseteq B(f)$ and some iterate of J is mapped into an interval L such that f^p maps L monotonically into itself for some $p \geq 0$.*

Multimodal maps satisfying some regularity conditions have no wandering intervals. Let us say that f is *non-flat* at a critical point c if there exists a C^2 diffeomorphism $\phi : \mathbb{R} \to I$ with $\phi(0) = c$ such that $f \circ \phi$ is a polynomial near the origin.

The following theorem is Theorem II.6.2 in [8].

Theorem 17. *Let f be a C^2 map that is non-flat at each critical point. Then f has no wandering intervals.*

Guckenheimer proved this theorem in 1979 for unimodal maps with *negative Schwarzian derivative* with *non-degenerate* critical point, that is with $|f''(c)| \neq 0$. The Schwarzian derivative was first used by Singer to study the dynamics of quadratic unimodal maps $x \to ax(1-x)$ with $a \in [0,4]$. He observed that this property is preserved under iteration and that is has important consequences in unimodal and multimodal dynamics.

Definition 18. *Let $f : I \to I$ be a C^3 l-modal map. The* Schwarzian derivative *of f at x is defined as*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f(x)}\right)^2,$$

for all $x \in I \setminus \{c_1, \ldots, c_l\}$.

We may compute the Schwarzian derivate of a composition

$$S(g \circ f)(x) = Sg(f(x)) \cdot |f'(x)|^2 + Sf(x), \tag{1}$$

therefore if $Sf < 0$ and $Sg < 0$ then $S(f \circ g) < 0$ so negative Schwarzian derivative is preserved under iteration. Let us state an important consequence of this property for C^3 maps of the interval proved by Singer (see Theorem II.6.1 in [8]).

Theorem 19 (Singer). *If $f : I \to I$ is a C^3 map with negative Schwarzian derivative then*

1. *the immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval I;*
2. *each neutral periodic point is attracting;*
3. *there are no intervals of periodic points.*

Combining this result with Theorem 17 and Lemma 16 we obtain the following

Corollary 20. *If f is C^3 multimodal map with negative Schwarzian derivative that is non-flat at each critical point and which has no attracting periodic orbits then it has no homterval. Therefore $O^-(\mathrm{Crit}_f)$ is dense in I.*

The following corollary is a particular case of the corollary of Theorem II.3.1 in [8].

Corollary 21. *Let f, g and h be as in Theorem 12. If f and g have no homtervals then they are topologically conjugate.*

All our examples of multimodal maps in this chapter are polynomials with negative Schwarzian derivative and without attracting periodic orbits. We prefer however to use slightly more general classes of multimodal maps, as suggested by the previous two corollaries. As combinatorially equivalent multimodal maps have the same monotonicity type we only use maps that are increasing on the leftmost lap I_1, that is exactly the multimodal maps f with $f(0) = 0$. Let us define some classes of multimodal maps

$$\mathcal{S}_l = \left\{ f : I \to I \ : \ f \text{ is a } C^1 l\text{-modal map with } f(0) = 0 \right\},$$

$$\mathcal{S}_l' = \left\{ f \in \mathcal{S}_l \ : \ f \text{ is } C^3 \text{ and } Sf < 0 \right\},$$

$$\mathcal{P}_l = \{ f \in \mathcal{S}_l' \ : \ f \text{ non-flat at each critical point} \} \text{ and}$$

$$\mathcal{P}_l' = \{ f \in \mathcal{P}_l \ : \ \text{all periodic points of } f \text{ are repelling} \}.$$

We have seen that in the absence of homtervals combinatorially equivalent multimodal maps are topologically conjugate. Using symbolic dynamics is a more convenient way to describe the combinatorial properties of forward critical orbits. Let $\mathcal{A}_I = \{I_1, \ldots, I_{l+1}\}$ and $\mathcal{A}_c = \{c_1, \ldots, c_l\}$ be two alphabets and $\mathcal{A} = \mathcal{A}_I \cup \mathcal{A}_c$. Let

$$\Sigma = \mathcal{A}_I^{\mathbb{N}} \cup \bigcup_{n \geq 0} (\mathcal{A}_I^n \times \mathcal{A}_c)$$

be the space of sequences of symbols of $\mathcal{A}$ with the following property. If $\underline{i} \in \Sigma$ and $m = |\underline{i}| \in \overline{\mathbb{N}}$ is its length then $m = \infty$ if and only if $\underline{i}$ consists only of symbols of $\mathcal{A}_I$. Moreover, if $m < \infty$ then $\underline{i}$ contains exactly one symbol of $\mathcal{A}_c$ on the rightmost position. Let $\Sigma' = \Sigma \setminus \mathcal{A}_c$ be the space of sequences $\underline{i} \in \Sigma$ with $|\underline{i}| > 1$. Let us define the shift transformation $\sigma : \Sigma' \to \Sigma$ by

$$\sigma(i_0 i_1 \ldots) = i_1 i_2 \ldots.$$

If $f \in \mathcal{S}_l$ let $\underline{i} : I \to \Sigma$ be defined by $\underline{i}(x) = i_0(x) i_1(x) \ldots$ where $i_n(x) = I_k$ if $f^n(x) \in I_k$ and $i_n(x) = c_k$ if $f^n(x) = c_k$ for all $n \geq 0$. The map $\underline{i}$ relates the dynamics of f on $I \setminus \{c_1, \ldots, c_l\}$ with the shift transformation σ on Σ'

$$\underline{i}(f(x)) = \sigma(\underline{i}(x)) \text{ for all } x \in I \setminus \{c_1, \ldots, c_l\}.$$

Moreover, we may define a *signed lexicographic ordering* on Σ that makes $\underline{i}$ increasing. It becomes strictly increasing in the absence of homtervals.

Definition 22. *A* signed lexicographic ordering $\prec$ *on* Σ *is defined as follows. Let us define a sign* $\epsilon : \mathcal{A} \to \{-1, 0, 1\}$ *where* $\epsilon(I_j) = (-1)^{j+1}$ *for all* $j = 1, \ldots, l+1$ *and* $\epsilon(c_j) = 0$ *for all* $j = 1, \ldots, l$. *Using the natural ordering on* $\mathcal{A}$ *we say that* $\underline{x} \prec \underline{y}$ *if there exists* $n \geq 0$ *such that* $x_i = y_i$ *for all* $i = 0, \ldots, n-1$ *and*

$$x_n \cdot \prod_{i=0}^{n-1} \epsilon(x_i) < y_n \cdot \prod_{i=0}^{n-1} \epsilon(y_i).$$

Let us observe that $\prec$ is a complete ordering and that $\epsilon \cdot f' > 0$ on $I \setminus \{c_1, \ldots, c_l\}$, that is ϵ represents the monotonicity of f. The product $\prod_{i=0}^{n-1} \epsilon(x_i)$ represents therefore the monotonicity of f^n. This is the main reason for the monotonicity of $\underline{i}$ with respect to $\prec$.

Proposition 23. *Let* $f \in \mathcal{S}_l$ *for some* $l \geq 0$.

1. *If* $x < y$ *then* $\underline{i}(x) \preceq \underline{i}(y)$.
2. *If* $\underline{i}(x) \prec \underline{i}(y)$ *then* $x < y$.
3. *If* $f \in \mathcal{P}_l'$ *then* $x < y$ *if and only if* $\underline{i}(x) \prec \underline{i}(y)$.

Proof. The first two points are Lemma II.3.1 in [8]. If $f \in \mathcal{P}'_l$ then by Corollary 20 $O^-(\mathrm{Crit}_f)$ is dense in I. Let us note that

$$O^-(\mathrm{Crit}_f) = \{x \in I \ : \ |\underline{i}(x)| < \infty\} .$$

Moreover, $O^-(\mathrm{Crit}_f)$ is countable as $f^{-1}(x)$ is finite for all $x \in I$, therefore $\underline{i}$ is strictly increasing. □

Let us define the kneading sequences of $f \in \mathcal{S}_l$ by $\underline{k}_i = \underline{i}(f(c_i))$ for $i = 1, \ldots, l$, the itineraries of the critical values. The kneading invariant of f is $\underline{K}(f) = (\underline{k}_1, \ldots, \underline{k}_l)$. The last point of the previous lemma shows that if $f, g \in \mathcal{P}'_l$ and $\underline{K}(f) = \underline{K}(g)$ then there is an order preserving bijection $h : O^+(\mathrm{Crit}_f) \to O^+(\mathrm{Crit}_g)$. Therefore, by Corollaries 20 and 21, f and g are topologically conjugate.

Let us define one-dimensional smooth families of multimodal maps. They are the central object of this paper.

Definition 24. *We say that $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ is a* family of l-modal maps *if $\mathcal{F}$ is continuous with respect to the C^1 topology of $\mathcal{S}_l$.*

Note that we do not assume the continuity of critical points in such a family - as in the general definition of a family of multimodal maps in [8] - as it is a direct consequence of the smoothness conditions we impose.

When not stated otherwise we suppose $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ is a family of l-modal maps and denote $f_\gamma = \mathcal{F}(\gamma)$.

Lemma 25. *The critical points $c_i : [\alpha, \beta] \to I$ of f_γ are continuous maps for all $i = 1, \ldots, l$.*

Proof. Fix $\gamma_0 \in [\alpha, \beta]$ and

$$0 < \varepsilon < \frac{1}{2} \min_{i \neq j} |c_i(\gamma_0) - c_j(\gamma_0)|.$$

Let $A = \{x \in [0, 1] \ : \ \varepsilon \leq \min_i |x - c_i(\gamma_0)|\}$, a finite union of compact intervals and

$$\theta = \min_{x \in A} |f'_{\gamma_0}(x)| > 0$$

by Definition 9. Then the monotonicity of f_{γ_0} alternates on the connected components of A. Let $\delta > 0$ be such that $||f_\gamma - f_{\gamma_0}||_{C^1} < \frac{\theta}{2}$ for all $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$. Therefore the critical points $c_i(\gamma)$ satisfy

$$|c_i(\gamma) - c_i(\gamma_0)| < \varepsilon$$

for all $i = 1, \ldots, l$ and $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ as $f'_\gamma(x) \cdot f'_{\gamma_0}(x) > 0$ for all $x \in A$. □

Let us remark that the C^1 continuity of families of multimodal maps is preserved under iteration.

Lemma 26. *Let $G, H : [a, b] \to C^1(I, I)$ be continuous. Then the map*

$$c \to G(c) \circ H(c) \ \textit{is continuous on} \ [a, b].$$

This lemma admits a straightforward proof by compactness. Remark that by iteration $\gamma \to f_\gamma^n$ from $[a,b]$ to $C^1(I,I)$ is continuous for all $n \geq 1$.

The following proposition shows that pullbacks of given combinatorial type of continuous maps are continuous in a family of multimodal maps.

Proposition 27. *Let* $y : [\alpha,\beta] \to \overset{\circ}{I}$ *be continuous and* $S \in \mathcal{A}_I^n$*, where* $\overset{\circ}{I}$ *denotes the interior of* I*. A maximal connected domain of definition* D *of the map* $\gamma \to x_\gamma$ *such that*

$$\begin{aligned} &f_\gamma^n(x_\gamma) = y(\gamma) \text{ and} \\ &\underline{i}(x_\gamma) \in S \times \Sigma \end{aligned}$$

is open in $[\alpha,\beta]$ *and* $\gamma \to x_\gamma$ *is unique and continuous on* D*.*

Proof. Suppose that for some γ there are $x_1 < x_2 \in \overset{\circ}{I}$ with $f_\gamma^n(x_1) = f_\gamma^n(x_2) = y(\gamma)$ and such that $\underline{i}(x_1) = \underline{i}(x_2) = S\underline{i}(y(\gamma))$ for some $\gamma \in [\alpha,\beta]$. But $S \in \mathcal{A}_I^n$ so f^n is strictly monotone on $[x_1,x_2]$, which contradicts $f_\gamma^n(x_1) = f_\gamma^n(x_2)$ so $\gamma \to x_\gamma$ is unique.

Let x_{γ_0} be as in the hypothesis and $\varepsilon > 0$ such that $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon) \subseteq \overset{\circ}{I}$. We show that there exists $\delta > 0$ such that $\gamma \to x_\gamma$ is defined on $(\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha,\beta]$ and takes values in $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$. Let

$$\theta = (f_{\gamma_0}^n)'(x_{\gamma_0}) \neq 0$$

and by eventually diminishing ε we may suppose that

$$|(f_{\gamma_0}^n)'(x) - \theta| < \frac{\theta}{4} \text{ for all } x \in (x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon).$$

Let $\delta_1 > 0$ be such that

$$||f_\gamma^n - f_{\gamma_0}^n||_{C^1} < \frac{\theta\varepsilon}{4} < \frac{\theta}{4} \text{ for all } \gamma \in (\gamma_0 - \delta_1, \gamma_0 + \delta_1) \cap [\alpha,\beta].$$

Let also $\delta_2 > 0$ be such that

$$|y(\gamma) - y(\gamma_0)| < \frac{\theta\varepsilon}{4} \text{ for all } \gamma \in (\gamma_0 - \delta_2, \gamma_0 + \delta_2) \cap [\alpha,\beta].$$

We choose $\delta = \min(\delta_1, \delta_2)$ and show that

$$y(\gamma) \in f_\gamma^n((x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)) \text{ for all } \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha,\beta].$$

Indeed, f_γ^n is monotone on $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$ and

$$|f_\gamma^n(x_{\gamma_0} \pm \varepsilon) - y(\gamma_0)| > \frac{\theta\varepsilon}{4}$$

for all $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha,\beta]$ as $|f_\gamma^n(x_{\gamma_0} \pm \varepsilon) - y(\gamma_0)| = |f_\gamma^n(x_{\gamma_0} \pm \varepsilon) - f_{\gamma_0}^n(x_{\gamma_0} \pm \varepsilon) + f_{\gamma_0}^n(x_{\gamma_0} \pm \varepsilon) - f_{\gamma_0}^n(x_{\gamma_0})|$ and $|f_{\gamma_0}^n(x_{\gamma_0} \pm \varepsilon) - f_{\gamma_0}^n(x_{\gamma_0})| > \frac{3}{4}\theta\varepsilon$. □

As an immediate consequence of the previous proposition and Lemma 25 we obtain the following corollary.

Corollary 28. *If $\mathcal{F}$ realizes a finite itinerary sequence $\underline{i}_0 \in \Sigma$, that is for all $\gamma \in [\alpha, \beta]$ there is $x(\underline{i}_0)(\gamma) \in I$ such that*

$$\underline{i}(x(\underline{i}_0)(\gamma)) = \underline{i}_0,$$

then $x(\underline{i}_0) : [\alpha, \beta] \to I$ is unique and continuous.

One may observe that if $x, y : [\alpha, \beta] \to I$ are continuous and for some $k \geq 0$

$$\left(f_\alpha^k(x(\alpha)) - y(\alpha)\right) \cdot \left(f_\beta^k(x(\beta)) - y(\beta)\right) < 0$$

then there exists $\gamma \in [\alpha, \beta]$ such that

$$f_\gamma^k(x(\gamma)) = y(\gamma). \tag{2}$$

Therefore if $\underline{i}(x(\alpha)) \neq \underline{i}(x(\beta))$ then there exists $\gamma \in [\alpha, \beta]$ such that $\underline{i}(x(\gamma))$ is finite. Let $m = \min\{k \geq 0 \ : \ \exists \gamma \in [\alpha, \beta]$ such that $\underline{i}(x(\alpha))(k) \neq \underline{i}(x(\gamma))(k)\}$ then the itinerary $\sigma^m \underline{i}(x(\gamma)) = \underline{i}(f_\gamma^m(x(\gamma)))$ changes the first symbol on $[\alpha, \beta]$. Without loss of generality we may assume that $\sigma^m \underline{i}(x(\alpha)) \prec \sigma^m \underline{i}(x(\beta))$. Therefore there exists $i \in \{1, \ldots, l\}$ such that $f_\gamma^m(x(\alpha)) \leq c_i(\alpha)$ and $f_\gamma^m(x(\beta)) \geq c_i(\alpha)$, which yields γ using the previous remark.

A simplified version of the proof of Proposition 27 shows that if $F : [\alpha, \beta] \to C^1(I)$ is continuous, $r_0 \in I$ is a root of $F(\gamma_0)$ and $(F(\gamma_0))'(r_0) \neq 0$ then there are $J \subseteq [\alpha, \beta]$ a neighborhood of γ_0 and $r : J \to I$ continuous such that $F(\gamma)(r(\gamma)) = 0$ for all $\gamma \in J$. For $F(\gamma)(x) = f_\gamma^n(x) - x$ we obtain the following corollary.

Corollary 29. *Let r_0 be a periodic point of f_{γ_0} of period $n \geq 1$ that is not neutral. There exists a connected neighborhood $J \subseteq [\alpha, \beta]$ of γ_0 and $r : J \to I$ continuous such that $r(\gamma)$ is a non-neutral periodic point of f_γ of period n. Moreover, provided $r(\gamma)$ is not super-attracting for any $\gamma \in J$, the itinerary $\underline{i}(r(\gamma))$ is constant.*

Proof. As a periodic point, $r(\gamma)$ exists and is continuous on a connected neighborhood J_0 of γ_0, using the previous remark. As $|(f_{\gamma_0}^n)'(r_0)| \neq 1$, there is a connected neighborhood J_1 of γ_0 such that

$$|(f_\gamma^n)'(r(\gamma))| \neq 1 \text{ for all } \gamma \in J_1.$$

Let $J = J_0 \cap J_1$ so $r(\gamma)$ is a non-neutral periodic point of period n for all $\gamma \in J$. Suppose that its itinerary $\underline{i}(r(\gamma))$ is not constant, then there is $\gamma_1 \in J$ such that $\underline{i}(r(\gamma_1))$ is finite so the orbit of $r(\gamma_1)$ contains a critical point thus it is super-attracting. □

Let us define the *asymptotic kneading sequences* $\underline{k}_j^-(\gamma)$ and $\underline{k}_j^+(\gamma)$ for all $\gamma \in [\alpha, \beta]$ and $j = 1, \ldots, l$. When they exist, the asymptotic kneading sequences capture important information about the local variation of the kneading sequences.

Definition 30. *Let $j \in \{1, \ldots, l\}$ and $\gamma \in [\alpha, \beta]$. If $\gamma > \alpha$ and for all $n \geq 0$ there exists $\delta > 0$ such that $\underline{k}_j(\gamma - \theta) \in S_n \times \Sigma$ with $S_n \in \mathcal{A}_I^n$ for all $\theta \in (0, \delta)$ then we set $\underline{k}_j^-(\gamma)(k) = S_n(k)$ for all $0 \leq k < n$. Analogously, if $\gamma < \beta$ and for all $n \geq 0$ there exists $\delta > 0$ such that $\underline{k}_j(\gamma + \theta) \in S_n' \times \Sigma$ with $S_n' \in \mathcal{A}_I^n$ for all $\theta \in (0, \delta)$ then we set $\underline{k}_j^+(\gamma)(k) = S_n'(k)$ for all $0 \leq k < n$.*

Let us define a sufficient condition for the existence of the asymptotic kneading sequences for all $\gamma \in [\alpha, \beta]$.

Definition 31. *We call a family* $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ *of* l*-modal maps* natural *if for all* $j = 1, \ldots, l$ *the set*

$$\underline{k}_j^{-1}(\underline{i}) = \left\{\gamma \in [\alpha, \beta] \ : \ \underline{k}_j(\gamma) = \underline{i}\right\} \text{ is finite for all } \underline{i} \in \Sigma \text{ finite.}$$

This property does not hold in general for C^1 families of multimodal maps, even polynomial, as such a family could be reparametrized to have intervals of constancy in the parameter space. It is however generally true for analytic families such as the quadratic family $a \to ax(1-x)$ with $a \in [0, 4]$.

The following proposition shows that this property guarantees the existence of all asymptotic kneading sequences.

Proposition 32. *Let* $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ *be a natural family of* l*-modal maps and* $j \in \{1, \ldots, l\}$. *Then* $\underline{k}_j^-(\gamma)$ *exists for all* $\gamma \in (\alpha, \beta]$ *and* $\underline{k}_j^+(\gamma)$ *exists for all* $\gamma \in [\alpha, \beta)$. *Moreover, if* $\underline{k}_j(\gamma) \in \mathcal{A}_I^\infty$ *for some* $\gamma \in (\alpha, \beta)$ *then* $\underline{k}_j^-(\gamma) = \underline{k}_j(\gamma) = \underline{k}_j^+(\gamma)$. *If* $\underline{k}_j(\gamma) = Sc_i$ *with* $S \in \mathcal{A}_I^n$ *for some* $n \geq 0$ *and* $i \in \{1, \ldots, l\}$ *then* $\underline{k}_j^-(\gamma) = Sl_1l_2\ldots$ *and* $\underline{k}_j^+(\gamma) = Sr_1r_2\ldots$ *with* $l_1, r_1 \in \{I_i, I_{i+1}\}$.

Proof. If $\mathcal{F}$ is natural then the set of all $\gamma \in [\alpha, \beta]$ that have at least one kneading sequence of length at most n for some $n > 0$

$$K_n = \bigcup_{j=1}^{l} \{\gamma \in [\alpha, \beta] \ : \ |\underline{k}_j(\gamma)| \leq n\}$$

is finite. This is sufficient for the existence of all asymptotic kneading sequences.

If $\underline{k}_j(\gamma_0) \in S \times \Sigma$ with $S \in \mathcal{A}_I^n$ and $n \geq 0, j \in \{1, \ldots, l\}$ then by the continuity of $\gamma \to f_\gamma^m(c_j)$ and of $\gamma \to c_i$ for all $m = 0, \ldots, n-1$ and $i = 1, \ldots, l$ there exists $\delta > 0$ such that

$$\underline{k}_j(\gamma) \in S \times \Sigma \text{ for all } \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta].$$

Therefore if $\underline{k}_j(\gamma) \in \mathcal{A}_I^\infty$ then $\underline{k}_j^-(\gamma) = \underline{k}_j(\gamma) = \underline{k}_j^+(\gamma)$. If $\underline{k}_j(\gamma) = Sc_i$ for some $i \in \{1, \ldots, l\}$ then $\underline{k}_j^-(\gamma) = Sl_1l_2\ldots$ and $\underline{k}_j^+(\gamma) = Sr_1r_2\ldots$. Again by the continuity of $\gamma \to f_\gamma^n(c_j)$ and of $\gamma \to c_k$ for all $k = 1, \ldots, l$

$$l_1, r_1 \in \{I_i, I_{i+1}\}.$$

□

Note that we may omit the parameter γ whenever there is no danger of confusion but c_j, $\underline{i}$ and $\underline{k}_j$ for some $j \in \{1, \ldots, l\}$ should always be understood in the context of some f_γ. However, the symbols of the itineraries of Σ are $I_1, \ldots, I_{l+1}, c_1, \ldots, c_l$ and do not depend on γ.

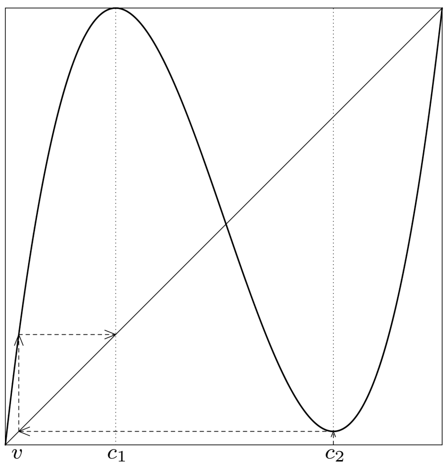

Fig. 1. bimodal map with $\underline{k}_2 = I_1 c_1$.

3. One-parameter families of bimodal maps

In this section we consider a natural family $\mathcal{G} : [\alpha, \beta] \to \mathcal{P}_2$ of bimodal polynomials with negative Schwarzian derivative satisfying the following conditions

$$0, 1 \in \partial I \text{ are fixed and repelling for } g_\alpha, \tag{3}$$

$$g_\gamma(c_1) = 1 \text{ for all } \gamma \in [\alpha, \beta], \tag{4}$$

$$g_\gamma(c_2) = 0 \text{ if an only if } \gamma = \alpha. \tag{5}$$

Let us denote by $v_n = g_\gamma^{n+1}(c_2)$ for $n \geq 0$ the points of the second critical orbit and let $\underline{k} = \underline{k}_2(\gamma) = k_0 k_1 \ldots$. If $S \in \mathcal{A}_I^k, k \geq 1$ and $n \geq 1$ we write S^n for $SS \ldots S \in \mathcal{A}_I^{kn}$ repeated n times and S^∞ for $SS \ldots \in \mathcal{A}_I^\infty$.

Proposition 32 shows the existence of $\underline{k}^+(\alpha) = \underline{k}(\alpha) = I_1^\infty$ therefore, there is $\delta_0 > 0$ such that

$$\underline{k} \in I_1^2 \times \Sigma \tag{6}$$

for all $\gamma \in [\alpha, \alpha + \delta_0]$. Figure 1 represents the graph of a bimodal map with the second kneading sequence $I_1 c_1 \succ \underline{k}(\gamma)$ for all $\gamma \in [\alpha, \alpha + \delta_0]$.

Let us observe that $O^+(\mathrm{Crit}_{g_\alpha}) = \{0, c_1, c_2, 1\}$ and that by Singer's Theorem 19, g_α has no homtervals. Therefore by Corollary 21, if $\mathcal{H} : [\alpha', \beta'] \to \mathcal{P}_2$ is a natural family satisfying conditions (3) to (5) then g_α and $h_{\alpha'}$ are topologically conjugate. Moreover, g_α is conjugate to the second Chebyshev polynomial (on $[-2, 2]$) and topological properties of its dynamics are universal. Let us study this dynamics and extend by continuity some of its properties to some neighborhood of α in the parameter space.

We have seen that g_α has no homtervals and that all its periodic points are repelling. Proposition 23 shows that the map

$$\underline{i}(g_\alpha) : I \to \Sigma \text{ is strictly increasing.}$$

Let us denote by $\sigma^-(\underline{i})$ the set of all preimages of $\underline{i}$ by some shift

$$\sigma^-(\underline{i}) = \left\{\underline{i}' \in \Sigma \ : \ \exists k \geq 0 \text{ such that } \sigma^k(\underline{i}') = \underline{i}\right\}.$$

As $(0,1) = \overset{\circ}{I} \subseteq g_\alpha(I_j)$ for $j = 1,2,3$, $g_\alpha(c_1) = 1$, $g_\alpha(c_2) = 0$, $\underline{i}(g_\alpha)(0) = I_1^\infty$ and $\underline{i}(g_\alpha)(1) = I_3^\infty$

$$\underline{i}(g_\alpha)(\overset{\circ}{I}) = \Sigma \setminus \left(\sigma^-(I_1^\infty) \cup \sigma^-(I_3^\infty)\right).$$

Let us denote by $\Sigma_0 = \underline{i}(g_\alpha)(I) = \underline{i}(g_\alpha)(\overset{\circ}{I}) \cup \{I_1^\infty, I_3^\infty\}$. Then

$$\underline{i}(g_\alpha) : I \to \Sigma_0 \text{ is an order preserving bijection.} \tag{7}$$

Remark also that Σ_0 is the space of all itinerary sequences of I under a bimodal map.

As g_α is decreasing on I_2, $g_\alpha(c_1) > c_1$ and $g_\alpha(c_2) < c_2$ it has exactly one fixed point $r \in I_2$ and it is repelling. Moreover, g_α has no fixed points in I_1 or I_3 other than 0 and 1 as this would contradict the injectivity of $\underline{i}(g_\alpha)$. As 0 and 1 are repelling fixed points $g_\alpha(x) > x$ for all $x \in (0, c_1)$ and $g_\alpha(x) < x$ for all $x \in (c_2, 1)$. Then by the C^1 continuity of $\mathcal{G}$ and Corollary 29 we obtain the following lemma.

Lemma 33. *There is $\delta_1 > 0$ such that g_γ has exactly one fixed point $r(\gamma)$ in $(0,1)$ and all its fixed points $0,1$ and $r(\gamma)$ are repelling for all $\gamma \in [\alpha, \alpha+\delta_1]$. Moreover, the map $\gamma \to r(\gamma)$ is continuous and $\underline{i}(r) = I_2^\infty$.*

Let p be a periodic point of period 2 of g_α. Then $\underline{i}(p)$ is periodic of period 2 and infinite. So $\underline{i}(p) \in \{(I_jI_k)^\infty \ : \ j,k = 1,2,3\}$. But $\underline{i}(g_\alpha)$ is injective, $\underline{i}(g_\alpha)(0) = I_1^\infty$, $\underline{i}(g_\alpha)(r) = I_2^\infty$ and $\underline{i}(g_\alpha)(1) = I_3^\infty$ so

$$\underline{i}(p) \in \{(I_jI_k)^\infty \ : \ j \neq k \text{ and } j,k = 1,2,3\} \subseteq \Sigma_0.$$

Therefore g_α has exactly 3 periodic orbits of period 2 with itinerary sequences $(I_1I_2)^\infty$, $(I_1I_3)^\infty$, $(I_2I_3)^\infty$ and their shifts. Figure 2 illustrates the periodic orbits of period 2 of g_α. By continuity of $\gamma \to g_\gamma^2$ and Corollary 29 we obtain the following lemma.

Lemma 34. *There is $\delta_2 > 0$ such that g_γ has exactly 3 periodic orbits of period 2 with itinerary sequences $(I_1I_2)^\infty$, $(I_1I_3)^\infty$, $(I_2I_3)^\infty$ for all $\gamma \in [\alpha, \alpha+\delta_2]$. Moreover, the periodic orbits of period 2 are repelling and continuous with respect to γ on $[\alpha, \alpha+\delta_2]$.*

Let us define

$$\beta' = \alpha + \min\{\delta_0, \delta_1, \delta_2\} \tag{8}$$

so that $\mathcal{G}$ satisfies equality (6), Lemma 33 and the previous lemma for all $\gamma \in [\alpha, \beta']$.

Let us consider the dynamics of all maps g_γ with $\gamma \in [\alpha, \beta']$ from the combinatorial point of view. We observe that if $x \geq v = g_\gamma(c_2)$ then $g_\gamma^n(x) \geq v$ for all $n \geq 0$. This means that any itinerary of g_γ is of the form $\underline{i}_\gamma = I_1^k a \ldots \in \Sigma_0$ with $k \geq 0$, $a \neq I_1$ and such that $\sigma^{k+p}\underline{i}_\gamma \succeq \underline{k}$ for all $p \geq 0$. Let $\Sigma(\underline{k})$ denote the set of itineraries satisfying this condition. We observe that $(v,1) \subseteq g_\gamma(I_j)$ for $j = 1,2,3$ and $c_1, c_2 \in (v,1)$ for all $\gamma \in [\alpha, \beta']$ by relation (6) so we obtain the following lemma. The continuity is an immediate consequence of Proposition 27.

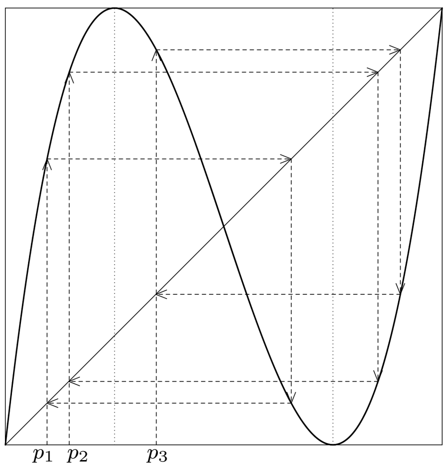

Fig. 2. g_α and its periodic orbits of period 2, p_1 with $\underline{i}(p_1) = (I_1 I_2)^\infty$, p_2 with $\underline{i}(p_2) = (I_1 I_3)^\infty$ and p_3 with $\underline{i}(p_3) = (I_2 I_3)^\infty$.

Lemma 35. *Let $\gamma_0 \in [\alpha, \beta']$ and $\underline{k} = \underline{k}_2(\gamma_0)$. Then every finite itinerary*

$$\underline{i}_0 \in \{\underline{i} \in \Sigma(\underline{k}) \ : \ |\underline{i}| < \infty\}$$

is realized by a unique point $x(\underline{i}) \in I$ and $\gamma \to x(\underline{i})$ is continuous on a neighborhood of γ_0.

A kneading sequence $\underline{k} \in \Sigma(\underline{k})$ satisfies the following property.

Definition 36. *We call $\underline{m} \in \Sigma_0$* minimal *if*

$$\underline{m} \preceq \sigma^k \underline{m} \text{ for all } 0 \leq k < |m|.$$

The following proposition shows that the minimality is an almost sufficient condition for an itinerary to be realized as the second kneading sequence in the family $\mathcal{G}$. This is very similar to the realization of maximal kneading sequences in unimodal families but the proof involves some particularities of our family $\mathcal{G}$. For the convenience of the reader, we include a complete proof.

Proposition 37. *Let $\alpha \leq \alpha_0 < \beta_0 \leq \beta'$ and $\underline{m}$ be a minimal itinerary such that*

$$\underline{k}(\alpha_0) \prec \underline{m} \prec \underline{k}(\beta_0).$$

Then there exists $\gamma \in (\alpha_0, \beta_0)$ such that

$$\underline{k}(\gamma) = \underline{m}.$$

Proof. Suppose that $\underline{k}(\gamma) \neq \underline{m}$ for all $\gamma \in (\alpha_0, \beta_0)$. Let $\gamma_0 = \sup\{\gamma \in [\alpha_0, \beta_0] \ : \ \underline{k}(\gamma) \preceq \underline{m}\}$ and $n = \min\{j \geq 0 \ : \ \underline{k}(\gamma_0)(j) \neq \underline{m}(j)\} < \infty$. Then, using the continuity of g_γ^n, c_1 and c_2 one may check that

$$k_n = \underline{k}(\gamma_0)(n) \in \mathcal{A}_c = \{c_1, c_2\},$$

otherwise the maximality of γ_0 is contradicted as $\underline{k}(0),\ldots,\underline{k}(n-1)$ and $\underline{k}(n)$ would be constant on an open interval that contains γ_0. There are two possibilities

1. $k_n = c_1$ so $g^n_{\gamma_0}(c_2) = c_1$ therefore c_2 is preperiodic.
2. $k_n = c_2$ so $g^n_{\gamma_0}(c_2) = c_2$ therefore c_2 is super-attracting.

Therefore $\gamma_0 > \alpha$ and $\gamma_0 \leq \beta' < \beta$. Let us recall that $\mathcal{G}$ is a natural family so the asymptotic kneading sequences $\underline{k}^-(\gamma_0)$ and $\underline{k}^+(\gamma_0)$ do exist and are infinite. Then the definition of γ_0 shows that

$$\min(\underline{k}(\gamma_0),\underline{k}^-(\gamma_0)) \preceq \underline{m} \preceq \underline{k}^+(\gamma_0). \tag{9}$$

Let $\underline{m} = m_0m_1\ldots m_n\ldots$ and $S = m_0\ldots m_{n-1} \in \mathcal{A}_I^n$ be the maximal common prefix of $\underline{k}(\gamma_0)$ and $\underline{m}$, so $\underline{k}(\gamma_0) = Sc_j$ with $j \in \{1,2\}$. Therefore, using Proposition 32, $m_n \in \{I_j, I_{j+1}\}$.

Suppose $k_n = c_1$ so $g^n_{\gamma_0}(c_2) = c_1$. Lemma 35 and property (6) show that the sequences $I_1I_3^kc_2$ and $I_2I_3^kc_2$ are realized as itineraries by all g_γ with $\gamma \in [\alpha,\beta']$ for all $k \geq 0$. Moreover $x(I_1I_3^kc_2)$ is strictly increasing in k for all $\gamma \in [\alpha,\beta']$ and it is continuous in γ. Analogously, $x(I_2I_3^kc_2)$ is strictly decreasing in k for all $\gamma \in [\alpha,\beta']$ and it is continuous in γ. Then by compactness and by the continuity of $\gamma\to g^n_\gamma$ and of $\gamma\to c_1$

$$\underline{k}^-(\gamma_0),\underline{k}^+(\gamma_0) \in S \times \{I_1,I_2\} \times I_3^\infty.$$

Therefore inequality (9) shows that

$$\min(c_1, I_1I_3^\infty) = I_1I_3^\infty \preceq \sigma^n\underline{m} \preceq I_2I_3^\infty = \max(c_1, I_2I_3^\infty).$$

But $\underline{m} \in \Sigma_0$ so

$$I_1I_3^\infty \prec \sigma^n\underline{m} \prec I_2I_3^\infty$$

therefore $m_n = c_1$ as $I_1I_3^\infty = \max(I_1 \times \Sigma)$ and $I_2I_3^\infty = \min(I_2 \times \Sigma)$, a contradiction.

Consequently $\underline{k}(\gamma_0) = Sc_2$ so $c_2(\gamma_0)$ is super-attracting. Then by Corollary 29 there is a neighborhood J of γ_0 such that $a(\gamma)$ is a periodic attracting point of period n for all $\gamma \in J$, $\gamma\to a(\gamma)$ is continuous and $a(\gamma_0) = c_2(\gamma_0)$. By Singer's Theorem 19, c_2 is contained in the immediate basin of attraction $B_0(a(\gamma))$ for all $\gamma \in J$, which is disjoint from c_1. Therefore, considering the local dynamics of g^n_γ on a neighborhood of $a(\gamma)$, $\underline{k}(\gamma) = \underline{i}(g_\gamma(a))$ is also periodic of period n or finite of length n for all $\gamma \in J$. As the family $\mathcal{G}$ is natural, there exists $\varepsilon > 0$ such that c_2 is not periodic for all $\gamma \in (\gamma_0-\varepsilon,\gamma_0+\varepsilon)\backslash\{\gamma_0\}$. Again by Corollary 29, $\underline{k}(\gamma) = \underline{k}^-(\gamma_0)$ for all $\gamma \in (\gamma_0-\varepsilon,\gamma_0)$ and $\underline{k}(\gamma) = \underline{k}^+(\gamma_0)$ for all $\gamma \in (\gamma_0,\gamma_0+\varepsilon)$. Then Proposition 32 shows that

$$\underline{k}^-(\gamma_0),\underline{k}^+(\gamma_0) \in \{(SI_2)^\infty,(SI_3)^\infty\}.$$

Let $\underline{m}_1 = \min((SI_2)^\infty,(SI_3)^\infty)$ and $\underline{m}_2 = \max((SI_2)^\infty,(SI_3)^\infty)$ and

$$K = \{\underline{i} \in \Sigma \;:\; \underline{i} \text{ minimal and } \underline{m}_1 \prec \underline{i} \prec \underline{m}_2\}\,.$$

As the sequences Sc_2, $\underline{k}^-(\gamma_0)$ and $\underline{k}^+(\gamma_0)$ are all realized as a kneading sequence $\underline{k}(\gamma)$ with $\gamma \in [\alpha,\beta']$, using inequality (9) it is enough to show that

$$K = \{Sc_2\}.$$

Let $\underline{i} \in K \backslash \{Sc_2\}$ so
$$\underline{i} \in S \times \{I_2, I_3\} \times \Sigma.$$

Suppose $\epsilon(S) = 1$ so $\underline{m}_1 = (SI_2)^\infty$ and $\underline{m}_2 = (SI_3)^\infty$. Suppose $\underline{i}(n) = I_2$, then as $\epsilon(SI_2) = -1$ and $\underline{i}$ is minimal

$$\underline{i} \preceq \sigma^n(\underline{i}) \prec (SI_2)^\infty = \sigma^n(\underline{m}_1) = \underline{m}_1,$$

a contradiction.

Analogously, suppose $\underline{i}(n) = I_3$, then for all $k \geq 1$

$$\underline{i} \preceq \sigma^{kn}(\underline{i}) \prec (SI_3)^\infty = \sigma^{kn}(\underline{m}_2),$$

so, by induction, $\underline{i} = (SI_3)^\infty = \underline{m}_2 \notin K$.

The case $\epsilon(S) = -1$ is symmetric so we may conclude that $K = \{Sc_2\}$ which contradicts our initial supposition. □

Let us prove a complementary combinatorial property.

Lemma 38. *Let $S \in \mathcal{A}_I^n$ with $\underline{k}(\alpha) \preceq SI_2^\infty \preceq \underline{k}(\beta')$ and such that SI_2^∞ is minimal. If $i_1 i_2 \ldots \in \Sigma$ and $i_1, i_2, \ldots \in \mathcal{A} \backslash \{I_1\}$ then*

$$SI_2^k i_1 i_2 \ldots \in \Sigma \textit{ is minimal for all } k \geq |S|.$$

Proof. Let $\underline{i} = SI_2^k i_1 i_2 \ldots \in \Sigma$, $n = |S|$ and $k \geq n$. Suppose there exists $j > 0$ such that
$$\sigma^j(\underline{i}) \prec \underline{i}.$$
As $SI_2^\infty \preceq \underline{k}(\beta') = I_1 \ldots$
$$\underline{i} \in I_1 \times \Sigma.$$
Then $j < n$ and we set $m = \min\left\{p \geq 0 \ : \ \sigma^j(\underline{i})(p) \neq \underline{i}(p)\right\}$. Therefore $m \leq n-1$ so
$$\sigma^j(SI_2^\infty) \prec SI_2^\infty$$
as $\underline{i}$ coincides with SI_2^∞ on the first $2n$ symbols, a contradiction. □

Using relation (6), $\underline{k}(\gamma) = I_1 \ldots$ so $I_2^k c_j \in \Sigma(\underline{k}(\gamma))$ for all $k \geq 0$, $j = 1, 2$ and $\gamma \in [\alpha, \beta']$. Then by Lemma 35 the maps

$$\gamma \to p_k(\gamma) = x(I_2^k c_1)(\gamma) \text{ and } \gamma \to q_k(\gamma) = x(I_2^k c_2)(\gamma)$$

are uniquely defined and continuous on $[\alpha, \beta']$ for all $k \geq 0$. Let us recall that g_γ is decreasing on I_2 so

$$c_1 \prec I_2 c_2 \prec I_2^2 c_1 \prec I_2^3 c_2 \prec \ldots \prec I_2^\infty \prec \ldots \prec I_2^3 c_1 \prec I_2^2 c_2 \prec I_2 c_1 \prec c_2,$$

therefore

$$c_1 = p_0 < q_1 < p_2 < q_3 < \ldots < r < \ldots < p_3 < q_2 < p_1 < q_0 = c_2$$

for all $\gamma \in [\alpha, \beta']$.

Let us show that $p_k \to r$ and $q_k \to r$ as $k \to \infty$ for all $\gamma \in [\alpha, \beta']$. Let

$$r^- = \lim_{k\to\infty} p_{2k} = \lim_{k\to\infty} q_{2k+1} \text{ and}$$
$$r^+ = \lim_{k\to\infty} q_{2k} = \lim_{k\to\infty} p_{2k+1}.$$

Suppose that $r^- < r^+$ then by continuity $g_\gamma(r^-) = r^+$ and $g_\gamma(r^+) = r^-$, as $g_\gamma(p_{k+1}) = p_k$ and $g_\gamma(q_{k+1}) = q_k$ for all $k \geq 0$. Then r^- and r^+ are periodic points of period 2 and with itinerary sequence I_2^∞, which contradicts Lemma 34. By compactness

$$p_k, q_k \to r \text{ uniformly as } k \to \infty. \tag{10}$$

The following proposition shows that these convergences have a counterpart in the parameter space.

Proposition 39. *Let $S \in \mathcal{A}_I^n$ for some $n \geq 0$ be such that SI_2^∞ is minimal and $\underline{k}^{-1}(SI_2^\infty)$ is finite. Let $\alpha \leq \alpha_0 < \beta_0 \leq \beta'$ be such that $\underline{k}(\alpha_0) \prec SI_2^\infty \prec \underline{k}(\beta_0)$ and $S' = SI_2^{k+1}$ with $k \geq 0$ and such that $\epsilon(S') = 1$. If $\underline{i}_1 = S'c_1$, $\underline{i}_2 = S'c_2$ and k is sufficiently large then we may define*

$$\begin{aligned} \gamma_1 = \gamma_1(k) &= \max\left(\underline{k}^{-1}(\underline{i}_1) \cap (\alpha_0, \beta_0)\right) \text{ and} \\ \gamma_2 = \gamma_2(k) &= \min\left(\underline{k}^{-1}(\underline{i}_2) \cap (\gamma_1, \beta_0)\right) \end{aligned} \tag{11}$$

and then

$$\lim_{k\to\infty} (\gamma_2 - \gamma_1) = 0.$$

Proof. First let us remark that the condition $\epsilon(S') = 1$ guarantees that

$$\underline{i}_1 \prec SI_2^\infty \prec \underline{i}_2.$$

Using for example convergences (10) and the bijective map $\underline{i}(g_\alpha)$ defined by (7) there exists $N_0 > 0$ such that for all $k \geq N_0$, $\underline{k}(\alpha_0) \prec \underline{i}_1 \prec \underline{i}_2 \prec \underline{k}(\beta_0)$. Moreover, if $k \geq n$ then $\underline{i}_1$ and $\underline{i}_2$ are minimal, using Lemma 38.

Therefore for $k \geq \max(N_0, n)$ we may apply Proposition 37 to show that there exist $\gamma_1 \in \underline{k}^{-1}(\underline{i}_1) \cap (\alpha_0, \beta_0)$ and $\gamma_2 \in \underline{k}^{-1}(\underline{i}_2) \cap (\gamma_1, \beta_0)$. As $\underline{i}_1$ and $\underline{i}_2$ are finite and the family $\mathcal{G}$ is natural, $\underline{k}^{-1}(\underline{i}_1)$ and $\underline{k}^{-1}(\underline{i}_2)$ are finite.

We may apply again Proposition 37 to see that γ_1 is increasing to a limit γ^- as $k \to \infty$. Again by Proposition 37 and by the finiteness of $\underline{k}^{-1}(SI_2^\infty)$ there exists

$$\gamma_\infty = \max\left(\underline{k}^{-1}(SI_2^\infty) \cap (\alpha_0, \beta_0)\right) < \beta_0 \text{ and } \gamma^- \leq \gamma_\infty.$$

For the same reasons there is $N > 0$ such that $\gamma_2 > \gamma_\infty$ for all $k \geq N$, therefore γ_2 becomes decreasing and converges to some $\gamma^+ \geq \gamma_\infty$.

Suppose that the statement does not hold, that is

$$\gamma^- < \gamma^+.$$

The map $\underline{i}(g_\alpha) : I \to \Sigma_0$ is bijective and order preserving and $p_i \to r$, $q_i \to r$ as $i \to \infty$ therefore

$$\{\underline{i} \in \Sigma_0 \ : \ \underline{i}_1 \preceq \underline{i} \preceq \underline{i}_2 \text{ for all } k > 0\} = \{SI_2^\infty\}.$$

Then the definitions of γ^- and γ^+ imply that

$$\underline{k}(\gamma) = SI_2^\infty \text{ for all } \gamma \in [\gamma^-, \gamma^+],$$

which contradicts the hypothesis. □

From the previous proof we may also retain the following Corollary.

Corollary 40. *Assume the hypothesis of the previous proposition. Then*

$$\lim_{k\to\infty} \gamma_1(k) = \lim_{k\to\infty} \gamma_2(k) = \gamma_\infty$$

and $\underline{k}(\gamma_\infty) = SI_2^\infty$.

We may also control the growth of the derivative on the second critical orbit in the setting of the last proposition. In fact, letting $k\to\infty$, the second critical orbit spends most of its time very close to the fixed repelling point r. Therefore the growth of the derivative along this orbit is exponential.

Let us also compute some bounds for the derivative along two types of orbits.

Lemma 41. *Let* $[\gamma_1, \gamma_2] \subseteq [\alpha, \beta']$, $n \geq 0$, $S \in \mathcal{A}_I^n$ *and* $\underline{i}_1, \underline{i}_2 \in S \times \Sigma$ *with* $\underline{i}_1 \prec \underline{i}_2$ *be finite or equal to* I_1^∞, I_2^∞ *or* I_3^∞. *If* $\underline{i}_1$, $\underline{i}_2$ *are realized on* $[\gamma_1, \gamma_2]$ *then there exists* $\theta > 0$ *such that*

$$\theta < \left| \left(g_\gamma^j\right)'(x) \right| < \theta^{-1}$$

for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in [x(\underline{i}_1), x(\underline{i}_2)]$ *and* $j = 1, \ldots, n$.

Proof. Let us remark that $\underline{i}(x) \in S \times \Sigma$ therefore $\left(g_\gamma^j\right)'(x) \neq 0$ for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in [x(\underline{i}_1), x(\underline{i}_2)]$ and $j = 1, \ldots, n$. As $x(\underline{i}_1)$ and $x(\underline{i}_2)$ are continuous by Lemmas 33 and 35, the set

$$\left\{(\gamma, x) \in \mathbb{R}^2 \ : \ \gamma \in [\gamma_1, \gamma_2], x \in [x(\underline{i}_1), x(\underline{i}_2)]\right\}$$

is compact. Therefore the continuity of $(\gamma, x) \to \left(g_\gamma^j\right)'(x)$ for all $j = 1, \ldots, n$ implies the existence of θ. □

The previous lemma helps us estimate the derivative of $g_\gamma^n(x)$ on a compact interval of parameters if $\underline{i}(x) \in I_j^n \times \Sigma$ and n is sufficiently large. Let us denote

$$I_j(n)(\gamma) = \left\{x \in I_j \ : \ g_\gamma^k(x) \in I_j \text{ for all } k = 1, \ldots, n\right\}$$

for j=1,2,3, the interval of points of I_j that stay in I_j under n iterations. Let also s_j be the unique fixed point in I_j. Let us remark that a point in $I_j(n)$ spends a lot of time near the fixed point s_j and the derivative along the orbit exhibits uniform exponential growth.

Lemma 42. *Let* $[\gamma_1, \gamma_2] \subseteq [\alpha, \beta']$, $j \in \{1, 2, 3\}$ *and* $\varepsilon > 0$. *Let also*

$$\lambda_1 = \lambda_1(j) = \min_{\gamma \in [\gamma_1, \gamma_2]} \left| g_\gamma'(s_j) \right|,$$

$$\lambda_2 = \lambda_2(j) = \max_{\gamma \in [\gamma_1, \gamma_2]} \left| g_\gamma'(s_j) \right|.$$

There exists $N > 0$ *such that for all* $k > 0$, $\gamma \in [\gamma_1, \gamma_2]$ *and* $x \in I_j(\max(k, N))(\gamma)$

$$\lambda_1^{k(1-\varepsilon)} < \left| \left(g_\gamma^k\right)'(x) \right| < \lambda_2^{k(1+\varepsilon)}.$$

This statement admits an elementary proof, using the previous lemma and a compactness argument.

Let us denote $m = \max(k, N)$ and remark that if we assume the hypothesis of the previous lemma then g_γ^k is monotone on $I_j(m)(\gamma)$ therefore for all $\gamma \in [\gamma_1, \gamma_2]$

$$\lambda_2(j)^{-k(1+\varepsilon)} < |I_j(m)(\gamma)| < \lambda_1(j)^{-k(1-\varepsilon)}. \tag{12}$$

Let $d_n : [\alpha, \beta'] \to \mathbb{R}_+$ be defined by

$$d_n(\gamma) = \left| \left(g_\gamma^n\right)' (v) \right|,$$

where $v = g_\gamma(c_2)$ the second critical value. As $\gamma \to v$ and $\gamma \to g_\gamma^n$ are continuous, d_n is continuous. The family $\mathcal{G}$ is natural so d_n has finitely many zeros for all $n \geq 0$.

Corollary 43. *Assume the hypothesis of Proposition 39 and let* $\lambda_0 = |g'_{\gamma_\infty}(r)| > 1$*. For all* $0 < \varepsilon < 1$ *there exists* $N > 0$ *such that if* $k \geq N$ *then*

$$\lambda_0^{(n+k)(1-\varepsilon)} < d_{n+k}(\gamma) < \lambda_0^{(n+k)(1+\varepsilon)} \textit{ for all } \gamma \in [\gamma_1(k), \gamma_2(k)],$$

where $n, k, \gamma_1(k)$ *and* $\gamma_2(k)$ *are as in Proposition 39.*

Proof. Let us remark that $|\underline{k}(\gamma)| > n$ for all $\gamma \in [\gamma_1(k), \gamma_2(k)]$ therefore there exists $\theta > 0$ such that

$$\theta < d_n(\gamma) < \theta^{-1} \text{ for all } \gamma \in [\gamma_1(k), \gamma_2(k)].$$

Using the previous lemma and Corollary 40, there exists $N_0 > 0$ such that if $k \geq N_0$ then

$$\lambda_0^{k\left(1-\frac{\varepsilon}{2}\right)} < \left| \left(g_\gamma^k\right)' (v_n) \right| < \lambda_0^{k\left(1+\frac{\varepsilon}{2}\right)} \text{ for all } \gamma \in [\gamma_1(k), \gamma_2(k)].$$

Therefore it is enough to choose $N \geq N_0$ such that

$$\lambda_0^{N\frac{\varepsilon}{2}} > \theta^{-1} \lambda_0^{n(1-\varepsilon)}.$$

□

4. TCE does not imply RCE

In this section we consider a family $\mathcal{G} : [\alpha, \beta] \to \mathcal{P}_2$ (see the definition of $\mathcal{P}_2$ at page 10) satisfying all properties (3) to (6) and Lemmas 33 and 34 for all $\gamma \in [\alpha, \beta]$. By restriction, we build a decreasing sequence of families $\mathcal{G}_n : [\alpha_n, \beta_n] \to \mathcal{P}_2$ with $\mathcal{G}_0 = \mathcal{G}$, $\alpha_n \nearrow \overline{\gamma}$ and $\beta_n \searrow \overline{\gamma}$ as $n \to \infty$. This means that $\mathcal{G}_n(\gamma) = \mathcal{G}(\gamma)$ for all $n \geq 0$ and $\gamma \in [\alpha_n, \beta_n]$. We obtain our counterexample as a limit $g_{\overline{\gamma}} = \mathcal{G}(\overline{\gamma}) = \mathcal{G}_n(\overline{\gamma})$ for all $n \geq 0$. In fact, every restriction corresponds to additional conditions on the maps in the previous family. We show that there is only one map in the intersection of all families which automatically satisfies all inductively defined properties.

For all $n \geq 0$ we choose two finite minimal itinerary sequences $\underline{i}_1(n+1)$ and $\underline{i}_2(n+1)$ as in Proposition 39 such that

$$\underline{k}_2(\alpha_n) \prec \underline{i}_1(n+1) \prec \underline{i}_2(n+1) \prec \underline{k}_2(\beta_n).$$

We set $\alpha_{n+1} = \gamma_1$ and $\beta_{n+1} = \gamma_2$. Choosing sufficient long sequences $\underline{i}_1(n+1)$ and $\underline{i}_2(n+1)$ we obtain the convergences $\alpha_n \to \overline{\gamma}$ and $\beta_n \to \overline{\gamma}$ as $n \to \infty$.

Let $T_2(x) = x^3 - 3x$ be the second Chebyshev polynomial. Observe that -2, 0 and 2 are fixed and that the critical points $c_1 = -1$ and $c_2 = 1$ are sent to 2 respectively -2. Its Schwarzian derivative $S(T_2)(x) = -\frac{4x^2+1}{(x^2-1)^2}$ is negative on $\mathbb{R} \setminus \{c_1, c_2\}$. Let $h > 0$ small and for each $\gamma \in [0, h]$ two order preserving linear maps $P_\gamma(x) = x(4+\gamma) - 2 - \gamma$ and $Q_\gamma(y) = \frac{y - T_2(-2-\gamma)}{2 - T_2(-2-\gamma)}$ that map $[0, 1]$ onto $[-2-\gamma, 2]$ respectively $[T_2(-2-\gamma), T_2(2)]$ onto $[0, 1]$. Let then

$$g_\gamma = Q_\gamma \circ T_2 \circ P_\gamma \tag{13}$$

be a bimodal degree 3 polynomial. As $S(P_\gamma) = S(Q_\gamma) = 0$ for all $\gamma \in [0, h]$, using equality (1), one may check that

$$S(g_\gamma) < 0 \text{ on } I \setminus \{c_1(\gamma), c_2(\gamma)\} \text{ for all } \gamma \in [0, h].$$

If we write

$$g_\gamma(x) = \sum_{k=0}^{3} a_k(\gamma) x^k \tag{14}$$

it is not hard to check that $\gamma \to a_k(\gamma)$ is continuous on $[0, h]$ for $k = 0, \ldots, 3$ therefore $\gamma \to g_\gamma$ is continuous with respect to the C^1 topology on I. By the definition of $\mathcal{P}_2$ (see page 10), as $g_\gamma(0) = 0$ for all $\gamma \in [0, h]$, $\mathcal{G} : [0, h] \to \mathcal{P}_2$ with $\mathcal{G}(\gamma) = g_\gamma$ for all $\gamma \in [0, h]$ is a family of bimodal maps with negative Schwarzian derivative. Observe that 0 and 1 are fixed points for all $\gamma \in [0, h]$ and that they are repelling for g_0, with $g_0'(0) = g_0'(1) = 9$, which is condition (3). Moreover, $g_\gamma(c_1) = 1$ for all $\gamma \in [0, h]$ thus $\mathcal{G}$ satisfies also (4). Observe that if $\gamma \in [0, h]$ then $Q_\gamma(-2) = 0$ if and only if $\gamma = 0$, therefore condition (5) is also satisfied by $\mathcal{G}$. We show that $\mathcal{G}$ is also natural and that any minimal sequence SI_2^∞ with $S \in \mathcal{A}_I^n$ and $n \geq 0$ equals the second kneading sequence $\underline{k}(\gamma)$ for at most finitely many $\gamma \in [0, h]$. This allows us to use all the results of the previous section for the family $\mathcal{G}$.

Let $G : [0, h] \times [0, 1] \to \mathbb{R}$ be defined by

$$G(\gamma, x) = g_\gamma(x) \text{ for all } \gamma \in [0, h] \text{ and } x \in [0, 1].$$

Then

$$G(\gamma, x) = \frac{P_1(\gamma, x)}{P_2(\gamma)},$$

where P_1 and P_2 are polynomials. Using definition (13), we may compute P_2 easily

$$P_2(\gamma) = 2 - T_2(-2-\gamma) = (\gamma+1)^2(\gamma+4).$$

We may therefore extend G analytically on a neighborhood $\Omega \subseteq \mathbb{R}^2$ of $[0, h] \times [0, 1]$. The critical points c_1 and c_2 are continuously defined on $[0, h]$ by Lemma 35. They are also analytic in γ as a consequence of the Implicit Functions Theorem for real analytic maps applied to $\frac{\partial G}{\partial x}$. Therefore for all $n \geq 0$ the map $g_\gamma^n(c_2)$ is analytic on a neighborhood of $[0, h]$ so

$$c_j(\gamma) - g_\gamma^n(c_2) \text{ has finitely many zeros in } [0, h]$$

for all $j \in \{1, 2\}$ and $n \geq 0$ as $g_0^n(c_2) = 0$ and $c_1(\gamma), c_2(\gamma) \in (0, 1)$ for all $\gamma \in [0, h]$. The family $\mathcal{G}$ is therefore natural so by eventually shrinking h we may also suppose that $\mathcal{G}$ satisfies property (6) and Lemmas 33 and 34 for all $\gamma \in [0, h]$. Then the repelling fixed point r is continuously defined on $[0, h]$ and again by the Implicit Functions Theorem applied to $G(\gamma, x) - x$, it is analytic on a neighborhood of $[0, h]$. Then

$$r(\gamma) - g_\gamma^n(c_2) \text{ has finitely many zeros in } [0, h]$$

for all $n \geq 0$ as $r(0) - g_0^n(c_2) = \frac{1}{2}$.

Let then $\mathcal{G}_0 = \mathcal{G}$ so $\alpha_0 = 0$ and $\beta_0 = h$. Our counterexample $g_{\overline{\gamma}}$ should be TCE but not RCE (see Definitions 3 and 5). Its first critical point is non-recurrent as $g_\gamma(c_1) = 1$ and 1 is fixed for all $\gamma \in [\alpha_0, \beta_0]$. Therefore the second critical point c_2 should be recurrent and not Collet-Eckmann. We let c_2 accumulate on c_1 also to control the growth of the derivative along its orbit. We build $g_{\overline{\gamma}}$ such that its second critical orbit spends most of the time near r or 1 so its derivative accumulates sufficient expansion. We show that $g_{\overline{\gamma}}$ is $ExpShrink$ (thus TCE) using a telescopic construction, in an analogous way to the proof of Theorem 8.

4.1. A construction. The construction of the sequence $(\mathcal{G}_n)_{n \geq 0}$ is realized by imposing at the n-th step the behavior of the second critical orbit for a time span $t_{n-1} + 1, t_{n-1}, \ldots, t_n$. This is achieved specifying the second kneading sequence and using Proposition 39. We set $t_0 = 0$.

We have seen that $\underline{k}^+(0) = I_1^\infty$ and that $g_\gamma(x) > x$ for all $x \in (0, c_1)$ and all $\gamma \in [0, h]$ as 0 is repelling and g_γ has no fixed point in $(0, c_1)$. Therefore the backward orbit of c_1 in I_1 converges to 0 and by compactness the convergence is uniform. Then

$$\underline{k}^{-1}\left(I_1^k c_1\right) \to 0 \text{ as } k \to \infty,$$

using Proposition 37 for their existence. Then for any $\varepsilon_0 > 0$ there is $k_0 > 0$ such that $I_1^{k_0} c_1 \prec \underline{k}(\beta_0)$ and $||g_0 - g_\gamma||_{C^1} < \varepsilon_0$ for all $\gamma \in [0, \underline{k}^{-1}(I_1^{k_0} c_1)]$. In particular, if

$$1 < \lambda < \lambda' < |g_0'(r)| = 3 < |g_0'(0)| = |g_0'(1)| = 9$$

then for ε_0 sufficiently small

$$\lambda' < \left|g_\gamma'(r)\right|, \lambda' < \left|g_\gamma'(0)\right| \text{ and } \lambda' < \left|g_\gamma'(1)\right| \tag{15}$$

for all $\gamma \in [0, \underline{k}^{-1}(I_1^{k_0} c_1)]$. Let $S_0 = I_1^{k_0+1} \in \mathcal{A}_I^{k_0+1}$ so $\underline{i} \prec I_1^{k_0} c_1$ for all $\underline{i} \in S_0 \times \Sigma$. Moreover, $S_0 I_2^\infty$ is minimal. Using Proposition 39 we find $\alpha_0 < \gamma_1 < \gamma_2 < \beta_0$ such that

$$\underline{k}(\alpha_0) \prec \underline{k}(\gamma_1) \prec S_0 I_2^\infty \prec \underline{k}(\gamma_2) \prec \underline{k}(\beta_0)$$

with $\underline{k}(\gamma_1), \underline{k}(\gamma_2) \in S_0 I_2 \times \Sigma$ and

$$|\gamma_2 - \gamma_1| < 2^{-1}.$$

We set $\alpha_1 = \gamma_1$ and $\beta_1 = \gamma_2$ and define $\mathcal{G}_1 : [\alpha_1, \beta_1] \to \mathcal{P}_2$ by $\mathcal{G}_1(\gamma) = \mathcal{G}(\gamma) = g_\gamma$ for all $\gamma \in [\alpha_1, \beta_1]$. Moreover, let $t_1 = k + |S_0|$ and $S_1 = S_0 I_2^k$, where k is specified by Proposition 39, then

$$\underline{k}(\gamma) \in S_1 I_2 \times \Sigma,$$

for all $\gamma \in [\alpha_1, \beta_1]$. Using Corollary 43 we may also assume that

$$d_m(\gamma) > \lambda^m, \tag{16}$$

for all $\gamma \in [\alpha_1, \beta_1]$, where $m = t_1 = |S_1|$. Let us recall that $d_n(\gamma) = \left|\left(g_\gamma^n\right)'(v)\right|$ and $v = g_\gamma(c_2)$.

Then we build inductively the decreasing sequence of families $(\mathcal{G}_n)_{n\geq 0}$ such that for all $n \geq 1$, $\mathcal{G}_n$ satisfies

$$\underline{k}(\gamma) \in S_n I_2 \times \Sigma, \tag{17}$$

$$|\underline{k}(\alpha_n)|, |\underline{k}(\beta_n)| < \infty, \tag{18}$$

$$|\beta_n - \alpha_n| < 2^{-n}, \tag{19}$$

$$\underline{k}(\alpha_n) \prec S_n I_2^\infty \prec \underline{k}(\beta_n), \tag{20}$$

and conditions (15) and (16) for all $\gamma \in [\alpha_n, \beta_n]$, for some $S_n \in \mathcal{A}_I^m$ with $S_n I_2^\infty$ minimal, where $m = t_n$. As the sequence $(\mathcal{G}_n)_{n\geq 0}$ is decreasing, inequality (15) is satisfied by all $\mathcal{G}_n$ with $n \geq 1$. For transparency we denote $v_n = g_\gamma^n(v)$ and

$$d_{n,p}(\gamma) = \left|\left(g_\gamma^p\right)'(v_n)\right|,$$

which also equals $d_{n+p}(\gamma) d_n^{-1}(\gamma)$, whenever $|\underline{k}(\gamma)| > n$ so $d_n(\gamma) \neq 0$.

Let us describe two types of steps, one that takes the second critical orbit near c_1 to control the growth of the derivative and the other that takes it near c_2 to make the second critical point c_2 recurrent. We alternate the two types of steps in the construction of the sequence $(\mathcal{G}_n)_{n\geq 0}$ to obtain our counterexample.

The following proposition describes the passage near c_1.

Proposition 44. *Let the family $\mathcal{G}_n$ with $n \geq 1$ satisfy conditions (15) to (20) and*

$$0 < \lambda_1 < \lambda_2 < \lambda.$$

Then there exists a subfamily $\mathcal{G}_{n+1}$ of $\mathcal{G}_n$ satisfying the same conditions and such that there exists $2t_n < p < t_{n+1}$ with the following properties

1. $\max_{\gamma \in [\alpha_{n+1}, \beta_{n+1}]} \left|\log\left|g'_\gamma(r)\right| - \frac{1}{p-1}\log d_{p-1}(\gamma)\right| < \log\lambda_2 - \log\lambda_1.$
2. $\lambda_1^p < d_p(\gamma) < \lambda_2^p$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.
3. $d_{t_n,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \ldots, p-1-t_n$.
4. $d_{p,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \ldots, t_{n+1} - p$.
5. $d_{t_n, t_{n+1}-t_n}(\gamma) > \lambda^{t_{n+1}-t_n}$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.

Proof. This proof follows a very simple idea, to define the family $\mathcal{G}_{n+1}$ with

$$S_{n+1} = S_n I_2^{k_1+1} I_3^{k_2} I_2^{k_3},$$

as described by properties (17) and (20). For k_1 and k_3 sufficiently large there exist k_2 such that the conclusion is satisfied for $p = t_n + k_1 + 1$.

Let us apply Proposition 39 to S_n, α_n and β_n. Let $k_1 = k + 1$, $\lambda_0 = \left|g'_{\gamma_\infty}(r)\right|$ and $\lambda_3 = \left|g'_{\gamma_\infty}(1)\right|$. By inequality (15)

$$0 < \lambda_1 < \lambda_2 < \lambda < \lambda_0$$

therefore there exists $\varepsilon_0 \in (0, 1)$ such that

$$\frac{(1 + \varepsilon_0) \log \lambda_0 - \log \lambda_2}{(1 - \varepsilon_0) \log \lambda_3} < \frac{(1 - \varepsilon_0) \log \lambda_0 - \log \lambda_1}{(1 + \varepsilon_0) \log \lambda_3}.$$

We choose $0 < \varepsilon < \varepsilon_0$ such that

$$\varepsilon < \frac{\log \lambda_2 - \log \lambda_1}{8 \log \lambda_0}.$$

Let us recall that

$$\underline{k}(\gamma_1) = S_n I_2^{k_1} c_1 \prec S_{n+1} \times \Sigma \prec \underline{k}(\gamma_2) = S_n I_2^{k_1} c_2. \tag{21}$$

Using Lemma 42 and Corollaries 40 and 43 there exists N_0 such that if $k_1 > N_0$ then the first and the third conclusions are satisfied provided $[\alpha_{n+1}, \beta_{n+1}] \subseteq [\gamma_1, \gamma_2]$.

Let $y(\gamma) \in I$ with $\underline{i}(y) \in I_2 I_3^{k_2} I_2 \times \Sigma$ and $y' = g_\gamma(y)$. By Corollary 40, Lemma 42 and inequality (12) there exist $N_1, N'_0 > 0$ such that if $k_1 > N_1$ and $k_2 > N'_0$ then for all $\gamma \in [\gamma_1, \gamma_2]$

$$\lambda_3^{-k_2(1+\varepsilon)} < |1 - y'| < \lambda_3^{-(k_2-1)(1-\varepsilon)}, \tag{22}$$

as $y \in I_3(k_2 - 1) \backslash I_3(k_2)$.

Let us recall that $g_\gamma(x) = \sum_{k=0}^{3} a_k(\gamma) x^k$ with a_i continuous and $g'_\gamma(c_1) = 0$, $g''_\gamma(c_1) \neq 0$ for all $\gamma \in [\alpha, \beta']$ and c_1 is continuous. Therefore there exist constants $M > 1$, $\delta > 0$ and $N_2 > 0$ such that if $k_1 > N_2$ and $\gamma \in [\gamma_1, \gamma_2]$ then

$$\begin{array}{lcl} M^{-1}(x - c_1)^2 < |1 - g_\gamma(x)| < M(x - c_1)^2 \text{ and} \\ M^{-1}(x - c_1) \;<\; \left|g'_\gamma(x)\right| \;<\; M(x - c_1) \end{array} \tag{23}$$

for all $x \in (c_1 - \delta, c_1 + \delta)$. Using inequality (22) there exists N'_1 such that if $k_2 > N'_1$ then $|1 - y'| = |1 - g_\gamma(y)| < M^{-1}\delta^2$ so $|y - c_1| < \delta$, therefore

$$M^{-\frac{3}{2}} \lambda_3^{-\frac{k_2}{2}(1+\varepsilon)} < \left|g'_\gamma(y)\right| < M^{\frac{3}{2}} \lambda_3^{-\frac{k_2-1}{2}(1-\varepsilon)}.$$

Let $k_1 > \max(t_n, N_0, N_1, N_2)$ and $k_2 > \max(N'_0, N'_1)$. Lemma 38 shows that $S_{n+1} I_2^\infty$ is minimal. We may therefore apply Proposition 39 with $S = S_n I_2^{k_1+1} I_3^{k_2}$, using inequality (21). Let $k_3 = k$ and α_{n+1} and β_{n+1} be the new bounds for γ provided by Proposition 39. Let us recall that $p = t_n + k_1 + 1$

and $v_n = g_\gamma^{n+1}(c_2)$ for all $n \geq 0$, therefore $\underline{i}(v_{p-1}) \in I_2 I_3^{k_2} \times \Sigma$ so we may set $y = v_{p-1}$ and $y' = v_p$. Let us remark that

$$d_p(\gamma) = d_{p-1}(\gamma) \cdot \left|g'_\gamma(y)\right| \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}].$$

By Corollary 43, if k_1 is sufficiently large, then for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$

$$M^{-\frac{3}{2}} \lambda_0^{(p-1)(1-\varepsilon)} \lambda_3^{-\frac{k_2}{2}(1+\varepsilon)} < d_p(\gamma) < M^{\frac{3}{2}} \lambda_0^{(p-1)(1+\varepsilon)} \lambda_3^{-\frac{k_2-1}{2}(1-\varepsilon)}.$$

Therefore the second conclusion is satisfied if

$$p \log \lambda_1 < -\frac{3}{2} \log M + (p-1)(1-\varepsilon) \log \lambda_0 - \frac{k_2}{2}(1+\varepsilon) \log \lambda_3$$

and

$$p \log \lambda_2 > \frac{3}{2} \log M + (p-1)(1+\varepsilon) \log \lambda_0 - \frac{k_2 - 1}{2}(1-\varepsilon) \log \lambda_3.$$

We may let $p \to \infty$ and $\frac{k_2}{2p} \to \eta$ so it is enough to find $\eta > 0$ such that

$$\begin{aligned} &\log \lambda_1 < (1-\varepsilon) \log \lambda_0 - \eta(1+\varepsilon) \log \lambda_3 \text{ and} \\ &\log \lambda_2 > (1+\varepsilon) \log \lambda_0 - \eta(1-\varepsilon) \log \lambda_3. \end{aligned}$$

The existence of η is guaranteed by the choice of $\varepsilon < \varepsilon_0$.

Again by inequality (15), Lemma 42 and Corollary 43, if k_2 and k_3 are sufficiently large then the last two conclusions are satisfied. If k_3 is sufficiently large then by Corollary 40 inequality (19) is also satisfied. □

The following proposition describes the passage of the second critical orbit near c_2.

Proposition 45. *Let the family $\mathcal{G}_n$ with $n \geq 1$ satisfy conditions (15) to (20) and*

$$\Delta > 0.$$

Then there exists a subfamily $\mathcal{G}_{n+1}$ of $\mathcal{G}_n$ satisfying the same conditions and such that there exists $t_n < p < t_{n+1}$ with the following properties

1. $\left|g_\gamma^p(c_2) - c_2\right| < \Delta$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.
2. $d_{t_n,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \ldots, t_{n+1} - t_n$.
3. $d_{p-1, t_{n+1}-p+1}(\gamma) > \lambda^{t_{n+1}-p+1}$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.

Proof. Once again, we build the family $\mathcal{G}_{n+1}$ using the prefix of the kneading sequence

$$S_{n+1} = S_n I_2^{k_1} S_n I_2^{k_2+1} I_3 I_2^{k_3},$$

and show that we may choose k_2 such that if k_1 and k_3 are sufficiently large then the conclusion is satisfied for $p = t_n + k_1$.

We apply Proposition 39 to S_n, α_n and β_n. Let $k_1 = k+2$, $\lambda_0 = |g_{\gamma_\infty}{}'(r)| > \lambda'$ and

$$S' = S_n I_2^{k_2+1} I_3.$$

In the sequel k_2 is chosen such that $\epsilon(S') = 1$ therefore $\underline{k}(\gamma_\infty) = S_n I_2^\infty \prec S' \ldots$, so

$$S_{n+1} I_2^\infty \text{ is minimal if } k_1 - 1 > k_2 > t_n.$$

Indeed, suppose that there exists $j > 0$ such that $\sigma^j \left(S_{n+1} I_2^\infty\right) \prec S_{n+1} I_2^\infty$. Let us recall that $t_n = |S_n|$ and $S_n I_2^\infty$ is minimal, using property (20) of $\mathcal{G}_n$. A similar reasoning to the proof of Lemma 38 shows that j can only be equal to $t_n + k_1$ so

$$S' \ldots \prec S_n I_2^{k_1} \ldots$$

which contradicts $S_n I_2^\infty \prec S' \ldots$, as $k_1 \geq k_2 + 2$. Moreover,

$$\underline{k}(\gamma_1) = S_n I_2^{k_1 - 1} c_1 \prec S_{n+1} I_2^\infty \prec \underline{k}(\gamma_2) = S_n I_2^{k_1 - 1} c_2,$$

and $\underline{i}' = I_2^2 S' c_1 \prec I_2^2 S' c_2 = \underline{i}'' \prec c_2$ are realized for all $\gamma \in [\gamma_1, \gamma_2]$, using Lemma 35. Let us remark that g_{γ_∞} has no homterval as $v_{t_n} = r$, using Singer's Theorem 19. Therefore

$$\lim_{k_2 \to \infty} g_{\gamma_\infty}(x(\underline{i}'')) = c_2$$

as $g_{\gamma_\infty}(x(\underline{i}'')) = x(\sigma \underline{i}'') < c_2$ is increasing with respect to k_2 and

$$\{\underline{i} \in \Sigma_0 \ : \ I_2 S' c_2 \prec \underline{i} \prec c_2 \text{ for all } k_2 > 0\} = \emptyset.$$

Let k_2 be such that $|c_2(\gamma_\infty) - x(\sigma \underline{i}'')(\gamma_\infty)| < \Delta$. Using Corollary 40 and the continuity of c_2 and of $x(\sigma \underline{i}'') < x(\sigma \underline{i}') < c_2$ there exists $N_0 > 0$ such that if $k_1 > N_0$ then

$$|c_2 - x| < \Delta, \tag{24}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in [x(\sigma \underline{i}''), x(\sigma \underline{i}')]$. Lemma 41 applied to $\underline{i}'$ and $\underline{i}''$ yields $\theta > 0$ such that if $l = t_n + k_2 + 4$ then

$$\theta < \left| \left(g_\gamma^j\right)'(x) \right| < \theta^{-1}, \tag{25}$$

for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in [x(\underline{i}'), x(\underline{i}'')]$ and $j = 1, \ldots, l$. Lemma 42 provides $N_1 > 0$ such that if $k_1 > N_1$ then

$$(\lambda')^j < d_{t_n, j}(\gamma) \text{ for all } \gamma \in [\gamma_1, \gamma_2] \text{ for all } j = 1, \ldots, k_1 - 2. \tag{26}$$

As $\lambda' > \lambda$ there exists also $N_2 > 0$ such that

$$\theta^{-1} \lambda^{N_2 - 2 + l} < (\lambda')^{N_2 - 2}. \tag{27}$$

Let $k_1 > \max(k_2 + 1, N_0, N_1, N_2)$ and $S'' = S_n I_2^{k_1} S'$. Let us remark that $S'' I_2^\infty = S_{n+1} I_2^\infty$ thus we may apply Proposition 39 to S'', γ_1 and γ_2. Let α_{n+1} and β_{n+1} be the new bounds for γ provided by Proposition 39 and $k_3 = k$.

As $g_\gamma^p(c_2) = v_{p-1}$ and $\sigma^{p-1} \left(S'' I_2^{k_3} \ldots\right) = I_2 S' I_2 \ldots$

$$g_\gamma^p(c_2) \in [x(\sigma \underline{i}''), x(\sigma \underline{i}')],$$

for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ thus, by inequality (24), the first conclusion is satisfied. Moreover, using inequalities (25), (26) and (27)

$$\lambda^j < d_{t_n, j}(\gamma) \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}] \text{ and } j = 1, \ldots, |S''| = k_1 - 2 + l.$$

Using Lemma 42 and Corollary 43, for k_3 sufficiently large the last two conclusions are satisfied. If k_3 is sufficiently large then by Corollary 40 inequality (19) is also satisfied by α_{n+1} and β_{n+1}. $\square$

4.2. Some properties of polynomial dynamics. Let us introduce some notations. For any set $E \subseteq \overline{\mathbb{C}}$ and $\alpha > 0$, we define the α-neighborhood of E by

$$E_{+\alpha} = B(E, \alpha) = \{x \in \overline{\mathbb{C}} | \operatorname{dist}(x, E) < \alpha\}.$$

One may easily check that if $f, g : \Omega \to \overline{\mathbb{C}}$ with $\Omega \subseteq \overline{\mathbb{C}}$ and $\delta > ||f - g||_\infty$ then for all $B \subseteq \overline{\mathbb{C}}$

$$g^{-1}(B) \subseteq f^{-1}(B_{+\delta}). \tag{28}$$

Using this simple observation we show that in a neighborhood of an *ExpShrink* polynomial some weaker version of Backward Stability is satisfied, see Proposition 48.

Definition 46. *We say that a rational map f has* Backward Stability *if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in J$, the Julia set of f, all $n \geq 0$ and every connected component W of $f^{-n}(B(z, \delta))$*

$$\operatorname{diam} W < \varepsilon.$$

Let us first show that the Julia set is *continuous* in the sense of Lemma 47. For transparency we introduce additional notations. We denote by $\mathbb{C}_d[z]$ the space of complex polynomials of degree d. If $f(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{C}_d[z]$ let us also denote

$$|f| = \max_{0 \leq i \leq d} |a_i|.$$

By convention, when $f \in \mathbb{C}_d[z]$ and we compare it to another polynomial g writing $|f - g|$ we also assume that $g \in \mathbb{C}_d[z]$.

Let us observe that the coefficients of $f^n = f \circ f \circ \ldots \circ f$, the n-th iterate of f, are continuous with respect to $(a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$ for all $n > 0$. Therefore given $f \in \mathbb{C}_d[z]$, $m > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

$$\left|f^i - g^i\right| < \varepsilon \text{ for all } i = 1, \ldots, m.$$

Given a compact $K \subseteq \mathbb{C}$, the map $\mathbb{R}^{d+1} \ni (a_0, a_1, \ldots, a_d) \to f \in \mathbb{C}_d[z]$ is continuous with respect to the topology of $C(K, \mathbb{C})$. Therefore for all $f \in \mathbb{C}_d[z]$, $\varepsilon > 0$ and $m > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

$$\left|\left|f^i - g^i\right|\right|_{\infty, K} < \varepsilon \text{ for all } i = 1, \ldots, m. \tag{29}$$

Lemma 47. *Let $f \in \mathbb{C}_d[z]$ with $d \geq 2$ and such that its Fatou set is connected and let J be its Julia set. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then*

$$J_g \subseteq J_{+\varepsilon}.$$

Proof. The Fatou set of f is the basin of attraction of ∞ and J is compact and invariant. Let $|J| = \max_{z \in J} |z|$, then for all $M \geq |J|$

$$J = \{z \in \mathbb{C} \ : \ |f^n(z)| \leq M \text{ for all } n \geq 0\}.$$

Let $f(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{C}_d[z]$. There exists $R > 1$ such that if $|f - g| < \frac{1}{2}|a_d|$ then

$$|J_g| \le R.$$

Indeed, it is enough to choose

$$R > 4d + 2|a_d|^{-1}\left(1 + \sum_{i=0}^{d-1}|a_i|\right),$$

and check that if $|z| > R$ then $|g(z)| > |z| + 1$.

Let $T = \left\{z \in \overline{\mathbb{C}} \, : \, \mathrm{dist}(z, J) \ge \varepsilon\right\}$. As T is compact in $\overline{\mathbb{C}}$ and contained in the basin of attraction of ∞, there is $m > 0$ such that

$$|f^m(z)| > R + 1 \text{ for all } z \in T.$$

Let $K = \overline{B(0, R+1)}$ a compact such that $J_{+\varepsilon}, J_g \subseteq K$ if $|f - g| < \frac{1}{2}|a_d|$. Inequality (29) yields $0 < \delta < \frac{1}{2}|a_d|$ such that if $|f - g| < \delta$ then

$$\left|\left|f^i - g^i\right|\right|_{\infty,K} < 1 \text{ for all } i = 1, \ldots, m.$$

Therefore by the definitions of R and m, if $|f - g| < \delta$ then

$$|g^m(z)| > R \text{ for all } z \in T,$$

thus $J_g \cap T = \emptyset$. $\square$

Remark. *The hypothesis f polynomial and its Fatou set connected are somewhat artificial, introduced for the elegance of the proof. It may be easily generalized to rational maps with attracting periodic orbits but without parabolic periodic orbits nor rotation domains.*

Proposition 48. *Let f be an ExpShrink polynomial satisfying the hypothesis of Lemma 47. There exists $\delta > 0$ such that for all $0 < r < \delta$ there exist $N > 0$ and $d > 0$ such that for all g with $|f - g| < d$ and $z \in J_g$*

$$\mathrm{diam}\,\mathrm{Comp}\, g^{-N}(B(z, \delta)) < r.$$

We use the notation $\mathrm{Comp}\, A$ for connected components of the set A. The previous statement means that the inequality holds for any such component.

Proof. Let us denote J the Julia set of f. Let $r_0 > 0$ and $\lambda_0 > 1$ be provided by Definition 4 such that for all $z \in J$

$$\mathrm{diam}\,\mathrm{Comp}\, f^{-n}\left(B(z, r_0)\right) < \lambda_0^{-n} \text{ for all } n \ge 0.$$

Let $\delta = \frac{r_0}{4}$ and choose $N \ge 1$ such that

$$\lambda_0^{-N} < r.$$

Inequality (29) provides d_0 such that if $|f - g| < d_0$ then

$$\left|f^N(z) - g^N(z)\right| < \delta \text{ for all } z \in \overline{J_{+r_0}}.$$

Lemma 47 yields $d_1 > 0$ such that if $|f - g| < d_1$ and $z \in J_g$ then there exists $z' \in J$ such that $|z - z'| < 2\delta$ therefore

$$B(z, 2\delta) \subseteq B(z', r_0).$$

We choose $d = \min(d_0, d_1)$ and $g \in \mathbb{C}_d[z]$ with $|f - g| < d$. Using inequality (28)

$$\operatorname{diam} \operatorname{Comp} g^{-N}(B(z,\delta)) < \lambda_0^{-N} < r \text{ for all } z \in J_g.$$

□

Corollary 49. *Let f satisfy the hypothesis of Proposition 48 and $\varepsilon > 0$. There exist $d, \delta > 0$ such that if $|f - g| < d$ then for all $z \in J_g$ and $n \geq 0$*

$$\operatorname{diam} \operatorname{Comp} g^{-n}(B(z,\delta)) < \varepsilon.$$

Proof. Let us use the notations defined by the proof of Proposition 48. It is straightforward to check that f has Backward Stability and that, by eventually decreasing r_0, we may also suppose

$$\operatorname{diam} \operatorname{Comp} f^{-n}(B(z, r_0)) < \varepsilon \text{ for all } z \in J \text{ and } n \geq 0.$$

Let $m \geq 1$ such that

$$\lambda_0^{-m} < \delta.$$

Inequality (29) provides d_0 such that if $|f - g| < d_0$ then

$$\left|f^i(z) - g^i(z)\right| < \delta \text{ for all } z \in \overline{J_{+r_0}} \text{ and } i = 1, \ldots, m.$$

Let d_1, d and g be as in the proof of Proposition 48. By inequality (28), for all $z \in J_g$

$$\operatorname{diam} \operatorname{Comp} g^{-m}(B(z,\delta)) < \delta$$

and

$$\operatorname{diam} \operatorname{Comp} g^{-i}(B(z,\delta)) < \varepsilon \text{ for all } i = 0, \ldots, m.$$

For some $z \in J_g$, let $W \in \operatorname{Comp} g^{-m}(B(z,\delta))$ and $z_1 \in W \cap J_g$. Then

$$W \subseteq B(z_1, \delta)$$

and the proof is completed by induction. □

Let us show that the hypothesis of Lemma 47 is easy to check for polynomials in $\mathcal{G}_0$.

Lemma 50. *If $g_\gamma \in \mathcal{G}_0$ and its second critical orbit $(v_n)_{n\geq 0}$ accumulates on a repelling periodic orbit then g_γ satisfies the hypothesis of Lemma 47. Moreover, if $(v_n)_{n\geq 0}$ is preperiodic then g_γ has ExpShrink.*

Proof. By Theorems III.2.2 and III.2.3 in [4], the immediate basin of attraction of an attracting or parabolic periodic point contains a critical point. But c_1 is strictly preperiodic and $(v_n)_{n\geq 0}$ accumulates on a repelling periodic orbit thus it cannot converge to some attracting or parabolic periodic point. Using Theorem V.1.1 in [4] we rule out Siegel disks and Herman rings as their boundary should be contained in the closure of the critical orbits which is contained in $[0,1]$ for all $g_\gamma \in \mathcal{G}_0$. Using Sullivan's classification of Fatou components, Theorem IV.2.1 in [4], the Fatou set equals the basin of attraction of infinity which is connected for all polynomials by the maximum principle.

If $(v_n)_{n\geq 0}$ is preperiodic then g_γ is semi-hyperbolic therefore by the main result in [5] or by Theorem 8 it has *ExpShrink*. $\square$

Let us recall some general distortion properties of rational maps. The following result is a classical distortion estimate due to Koebe, see for example Corollary 1.4 and Theorem 1.3 in [18].

Theorem 51 (Koebe). *Let $g : B \to \mathbb{C}$ be a univalent map from the unit disk into the complex plane. Then the image $g(B)$ contains the ball $B\left(g(0), \frac{1}{4}|g'(0)|\right)$. Moreover, for all $z \in B$ we have that*

$$\frac{(1-|z|)}{(1+|z|)^3} \leq \frac{|g'(z)|}{|g'(0)|} \leq \frac{(1+|z|)}{(1-|z|)^3},$$

and

$$|g(z)-g(0)| \leq |g'(z)|\,\frac{|z|(1+|z|)}{1-|z|}.$$

For the remainder of this section, let f be any rational map and Crit the set of critical points of f.

Distortion. This is a reformulation of the previous theorem. For all $D > 1$ there exists $\varepsilon > 0$ such that if the open W satisfies

$$\operatorname{diam} W \leq \varepsilon \operatorname{dist}(W, \mathrm{Crit}), \tag{30}$$

then the distortion of f in $\overline{W}$ is bounded by D.

Pullback. Once a sufficiently small $r > 0$ is fixed, there exists $M \geq 1$ such that for any open U with $\operatorname{diam} U \leq r$ and for all $W \in \operatorname{Comp} f^{-1}(U)$ and all $z \in \overline{W}$

$$\operatorname{diam} W \leq M|f'(z)|^{-1} \operatorname{diam} U. \tag{31}$$

We shall use this estimate for W close to Crit.

4.3. A counterexample. Using Propositions 44 and 45 we build a sequence of families $(\mathcal{G}_n)_{n\geq 1}$ which converge to a bimodal polynomial g that has *ExpShrink*. Its first critical point c_1 is non-recurrent as $g(c_1) = 1$ and 1 is a repelling fixed point. The second critical point c_2 is recurrent and it does not satisfy the Collet-Eckmann condition. Therefore g does not satisfy *RCE*.

We obtain the following theorem which states that the converse of Theorem 8 does not hold. We use the equivalence of *TCE* and *ExpShrink* [22].

Theorem A. *There exists a TCE rational map that is not RCE.*

The proof that g has *ExpShrink* is analogous to that of Theorem 8. This paper contains a complete proof of Theorem A. However, remarks about the proof of Theorem 8 are present for the convenience of the reader.

As g is not *RCE* we have to modify some of the tools like Propositions 9, 10 and 11 in [12]. The polynomial g_0 is Collet-Eckmann and semi-hyperbolic thus *RCE*. By the main result of [5] or by Theorem 8, g_0 has *ExpShrink*. Choosing the family $\mathcal{G}_1$ in a sufficiently small neighborhood of g_0 we show two contraction results similar to Propositions 9 and 10 in [12] that hold on $\mathcal{G}_1$, Corollary 52 and Proposition 56 below. As $g \in \mathcal{G}_1$ we may choose constants μ, θ, ε, R and N_0 - as described in the sequel - that do not depend on g.

The main idea of the proof of Theorem A is that in inequality (31) the right term may be much larger than the left term, see also Lemma 55. This means that when pulling back a ball B to B^{-1} near a degree two critical point, the diameter of B^{-1} is comparable to the square root of the radius of B but $|f'(z)|^{-1}$ may be as large as one wants for some $z \in B^{-1}$. This is the main difference between growth conditions in terms of the derivative or in terms of the diameter of pullbacks.

Corollary 53, an immediate consequence of Corollary 49, replaces Proposition 11 in [12] in the proof of Theorem A.

Corollary 52. *There exists $\delta > 0$ such that for all $0 < r < R \leq \delta$ there exist $\beta > \alpha_0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta]$ and $z \in J$ the Julia set of g_γ*

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-N}(B(z, R)) < r.$$

Proof. Using Lemma 50, g_0 satisfies the hypothesis of Proposition 48. Using the continuity of coefficients of g_γ (14) there exists $\beta > \alpha_0$ such that

$$|g_0 - g_\gamma| < d \text{ for all } \gamma \in [\alpha_0, \beta].$$

□

The following consequence of Corollary 49 is a weaker version of uniform Backward Stability. The proof is analogous to the proof of the previous proposition.

Corollary 53. *For all $\varepsilon > 0$ there exist $\beta > \alpha_0$ and $\delta > 0$ such that for all $\gamma \in [\alpha_0, \beta]$ and $z \in J$ the Julia set of g_γ*

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-n}(B(z, \delta)) < \varepsilon \textit{ for all } n \geq 0.$$

Let us compute an estimate of the diameter of a pullback far from critical points.

Lemma 54. *Let $h : B(z, 2R) \to \mathbb{C}$ be an analytic univalent map and $U \ni z$ a connected open set with* $\operatorname{diam} U \leq R$. *If*

$$\sup_{x,y \in B(z,2R)} \left| \frac{h'(x)}{h'(y)} \right| \leq D$$

then

$$\operatorname{diam} U \leq D \, |h'(z)|^{-1} \operatorname{diam} h(U).$$

Proof. Let $x, y \in \partial U$ such that $|x-y| = \operatorname{diam} U$. Let $a = h(x)$, $b = h(y)$ and consider the pullback of the line segment $[a, b]$ that starts at x.

There are two cases. Either $[a,b] \subseteq W := h(B(z, 2R))$ or there exists a largest $t' \in [0,1)$ such that $[a, (1-t')a + t'b) \subseteq W$. If $[a,b] \subseteq W$, let $t_0 = 1$. Otherwise, let $c = (1-t')a + t'b$. Remark that $c \in \partial W$ so $h^{-1}([a,c))$ connects ∂U to $\partial B(z, 2R)$. Therefore $\operatorname{diam} h^{-1}([a,c)) > R \geq \operatorname{diam} U$. Then there exists $t_0 \in (0, t')$ such that

$$[a, (1-t_0)a + t_0 b] \subseteq h\left(B(z, 2R)\right)$$

and such that the length of $h^{-1}\left([a, (1-t_0)a + t_0 b]\right)$ is at least $\operatorname{diam} U$, in both cases.

We may also notice that

$$\left|\left(h^{-1}\right)'\left((1-t)a + tb\right)\right| \leq D\left|h'(z)\right|^{-1} \text{ for all } t \in [0, t_0],$$

which completes the proof as $|(1-t_0)a + t_0 b - a| = t_0 |b-a| \leq \operatorname{diam} h(U)$. □

Proposition 10 in [12] relies on inequalities (30) and (31). We remark that they are satisfied uniformly on a neighborhood of g_0. By Koebe's Lemma 51, the definition (30) of ε does not depend on f. Let us prove the uniform version of inequality (31) in $\mathcal{G}$.

Lemma 55. *There exist $M > 1$, $\beta_M > \alpha_0$ and $r_M > 0$ such that for all $\gamma \in [\alpha_0, \beta_M]$ if W is a connected open with* $\operatorname{diam} W < r_M$*, W^{-1} a connected component of $g_\gamma^{-1}(W)$ and $x \in W^{-1}$ then*

$$\operatorname{diam} W^{-1} < M\left|g_\gamma'(x)\right|^{-1} \operatorname{diam} W.$$

Proof. Let $\gamma \in [\alpha_0, \beta_1]$, $x \in W^{-1}$ and suppose

$$3 \operatorname{diam} W^{-1} \leq \operatorname{dist}\left(W^{-1}, \mathrm{Crit}\right),$$

where we denote by Crit the set of critical points $\{c_1, c_2\}$. Then by Koebe's Lemma 51 the distortion is bounded by an universal constant $M_1 \geq 1$ on the ball $B\left(x, 2 \operatorname{diam} W^{-1}\right)$. Using Lemma 54

$$\operatorname{diam} W^{-1} \leq M_1 \left|g_\gamma'(x)\right|^{-1} \operatorname{diam} W. \tag{32}$$

Let us remark some properties of the map $f_b : \mathbb{C} \to \mathbb{C}$ defined by $f_b(z) = bz^2$ for all $z \in \mathbb{C}$ and $b > 0$. Let U be a connected open and $V = f_b(U)$. If $3 \operatorname{diam} U > \operatorname{dist}(U, 0)$ then there exist universal constants $M_2, M_3 > 1$ such that

$$\begin{aligned} bM_2^{-1} \operatorname{diam} U < &\sup_{z \in U} |f_b'(z)| < bM_2 \operatorname{diam} U, \\ bM_3^{-1} (\operatorname{diam} U)^2 < &\operatorname{diam} V \quad < bM_3 (\operatorname{diam} U)^2. \end{aligned}$$

Let us also remark that using equality (14) if $\gamma \in [\alpha_0, \beta_1]$ and $c \in \mathrm{Crit}$ then

$$g_\gamma(x) = g_\gamma(c) + \frac{g_\gamma''(c)}{2}(x-c)^2 + \frac{g_\gamma'''(c)}{6}(x-c)^3.$$

As $g_0''(c) \neq 0$ and $g_\gamma(c)$, $g_\gamma''(c)$ and $g_\gamma'''(c)$ are continuous there exist $r_M > 0$, $\beta_M > \alpha_0$ and $M_4 > 1$ such that if $\gamma \in [\alpha_0, \beta_M]$, $\operatorname{diam} W < r_M$ and

$$3 \operatorname{diam} W^{-1} > \operatorname{dist}\left(W^{-1}, \operatorname{Crit}\right),$$

then

$$\begin{aligned} M_4^{-1} \operatorname{diam} W^{-1} < \sup_{x \in W^{-1}} \left|g_\gamma'(x)\right| < M_4 \operatorname{diam} W^{-1}, \\ M_4^{-1} \left(\operatorname{diam} W^{-1}\right)^2 < \operatorname{diam} W < M_4 \left(\operatorname{diam} W^{-1}\right)^2. \end{aligned} \tag{33}$$

The previous inequality together with inequality (32) complete the proof. □

We may now prove a uniform contraction result on a neighborhood of g_0 in $\mathcal{G}$. It replaces Proposition 10 in [12] in the proof of Theorem A.

Proposition 56. *For any $1 < \lambda_0 < \lambda$ and $\theta < 1$ there exist $\beta > \alpha_0$, $\delta > 0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta]$, $0 < R \leq \delta$, $n \geq N$ and $z \in J_\gamma$, the Julia set of g_γ, if $W \in \operatorname{Comp} g_\gamma^{-n}(B(z,R))$ and there exists $x \in \overline{W}$ such that $\left|\left(g_\gamma^n\right)'(x)\right| > \lambda^n$ then*

$$\operatorname{diam} W < \theta R \lambda_0^{-n}. \tag{34}$$

Proof. Let us fix $D \in (1, \lambda/\lambda_0)$. Let $\varepsilon \in (0,1)$ be provided by inequality (30). Let also $r_M > 0$ be small and $M > 1$ provided by the Lemma 55. Let $l \geq 1$ such that

$$2M^{j/l} D^j \lambda^{-j} \leq \theta \lambda_0^{-j} \text{ for all } j \geq l. \tag{35}$$

Let $N = 2l$. There exists $r_1 < r_M$ such that for all $i = 1, 2$, $k = 1, \ldots, N$ and any connected component W of $g_0^{-k}(B(c_i, 4r_1))$

$$\operatorname{diam} W \leq 2\varepsilon \operatorname{dist}(W, \operatorname{Crit}).$$

An argument similar to the proof of Proposition 48 and the continuity of the critical points and of the coefficients (14) of g_γ show that there exists $b_0 > \alpha_0$ such that for all $\gamma \in [\alpha_0, b_0]$, $i = 1, 2$ and $k = 1, \ldots, N$

$$g_\gamma^{-k}(B(c_i, 2r_1)) \subseteq g_0^{-k}(B(c_i, 4r_1)).$$

There are only a finite number of connected components of $g_0^{-k}(B(c_i, 4r_1))$ for all $i = 1, 2$ and $k = 1, \ldots, N$. Therefore by the continuity of the critical points there exists $b_1 > \alpha_0$ such that for all $\gamma \in [\alpha_0, b_1]$, $i = 1, 2$ and $k = 1, \ldots, N$ all connected components of $g_\gamma^{-k}(B(c_i, 2r_1))$ satisfy inequality (30).

Corollary 53 provides $b_2 > \alpha_0$ and $\delta > 0$ such that for all $\gamma \in [\alpha_0, b_2]$, $z \in J_\gamma$ and $k \geq 0$

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-k}(B(z, \delta)) < \varepsilon r_1.$$

Let us define $\beta = \min(\beta_M, b_0, b_1, b_2)$ and fix $\gamma \in [\alpha_0, \beta]$, $z \in J_\gamma$ and $n > N$. Then

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-k}\left(B(z,R)^{-k}\right) < \varepsilon r_1 < r_M \text{ for all } 0 \leq k \leq n.$$

Let us also fix W and x as in the hypothesis. Denote $x_k = g_\gamma^{n-k}(x) \in W_k = g_\gamma^{n-k}(W)$ for all $k = 0, \ldots, n$.

Let $0 < k_1 < \ldots < k_t \leq N$ be all the integers $0 \leq k \leq n$ such that W_k does not satisfy the inequality (30). As $\varepsilon r_1 \geq \operatorname{diam} W_{k_i}$

$$r_1 > \operatorname{dist}(W_{k_i}, \mathrm{Crit}) \text{ for all } 1 \leq i \leq t.$$

Then for all $1 \leq i \leq t$ there exists $c \in \{c_1, c_2\}$ such that $W_{k_i} \subseteq B(c, 2r_1)$. By the definition of r_1

$$k_{i+1} - k_i > N \text{ for all } 1 \leq i < t. \tag{36}$$

We may begin estimates. For all $0 < j \leq n$ with $j \neq k_i$ for all $1 \leq i \leq t$, W_j satisfies the inequality (30), so the distortion on W_j is bounded by D. Thus by Lemma 54

$$\operatorname{diam} W_j \leq D|g'_\gamma(x_j)|^{-1} \operatorname{diam} W_{j-1}. \tag{37}$$

If $j = k_i$ for some $1 \leq i \leq t$ we use Lemma 55 to obtain

$$\operatorname{diam} W_j \leq M|g'_\gamma(x_j)|^{-1} \operatorname{diam} W_{j-1}. \tag{38}$$

Let us recall that $x_n = x$ with $\left|\left(g_\gamma^n\right)'(x)\right| > \lambda^n$ and that $W_0 = B(z, R)$ so $\operatorname{diam} W_0 = 2R$. If $t \geq 2$ inequality (36) yields $lt \leq 2l(t-1) = N(t-1) < n$. Consequently, as $n > 2l = N$,

$$t < \frac{n}{l}.$$

Multiplying all the relations (37) and (38) for all $0 < j \leq n$ we obtain

$$\begin{aligned}\operatorname{diam} W_n &\leq M^t D^{n-t} \left|\left(g_\gamma^n\right)'(x_n)\right|^{-1} \operatorname{diam} W_0 \\ &< 2M^{n/l} D^n \lambda^{-n} R \\ &\leq \theta R \lambda_0^{-n}.\end{aligned}$$

The last inequality is inequality (35). □

As a direct consequence of inequality (36) we obtain the following corollary.

Corollary 57. *Assume the hypothesis of Proposition 56. If there exist*

$$-1 \leq k_1 < k_2 < n$$

such that $v \in \overline{g_\gamma^{k_1+1}(W)}$ *and* $\overline{g_\gamma^{k_2}(W)} \cap \{c_1, c_2\} \neq \emptyset$ *then* $k_2 - k_1 > N$ *therefore condition* $n \geq N$ *is superfluous.*

Let us compute a diameter estimate similar to (12).

Lemma 58. *There exist* $\delta > 0$ *and* $N > 0$ *such that for all* $\gamma \in [\alpha_0, \beta_1]$, $k \geq 1$ *and* $x \in I_3(N)$ *with* $\underline{i}(x) = I_3^k I_* \ldots$ *where* $I_* \in \{I_2, I_3\}$, *the following statement holds. If* $x \in W \subseteq \mathbb{C}$ *is a connected open such that* $\operatorname{diam} g_\gamma^i(W) < \delta$ *for all* $i = 0, \ldots, k-1$ *then*

$$\operatorname{diam} W < \lambda^{-k} \operatorname{diam} g_\gamma^k(W).$$

Proof. Let us denote $x_i = g_\gamma^i(x)$ and $W_i = g_\gamma^i(W)$ for all $i = 0,\ldots,k$. Using Lemma 42, inequalities (15) and Lemma 41 for $\underline{i}_1 = I_3c_1$, $\underline{i}_2 = I_3c_2$ if $I_* = I_2$ and $\underline{i}_1 = I_3c_2$, $\underline{i}_2 = I_3^\infty$ if $I_* = I_3$ there exists $N_0 > 0$ that does not depend on γ such that if $N \geq N_0$ then

$$\left|\left(g_\gamma^k\right)'(x)\right| > (\lambda')^k.$$

Let $D \in \left(1, \frac{\lambda'}{\lambda}\right)$ and $\varepsilon > 0$ given by inequality (30). Using Lemma 54 it is enough to show that $B(x_i, 2\delta)$ satisfies inequality (30) for all $i = 0,\ldots,k-1$.

Let us recall that $g_\gamma(y) < y$ for all $y \in (c_2, 1) = I_3\backslash\{1\}$. Therefore for all $i = 0,\ldots,k-1$

$$\operatorname{dist}(x_i, \{c_1, c_2\}) \geq \operatorname{dist}(x_{k-1}, \{c_1, c_2\}) > \operatorname{dist}(x(I_3c_1), \{c_1, c_2\}).$$

Let

$$d = \min_{\gamma\in[\alpha_0,\beta_1]} \operatorname{dist}(x(I_3c_1), \{c_1, c_2\})$$

and recall that ε does not depend on γ. Therefore there exists

$$\delta = \frac{d}{2(1+2\varepsilon^{-1})} > 0$$

such that if $\operatorname{dist}(y, \{c_1, c_2\}) \geq d$ then $B(y, 2\delta)$ satisfies inequality (30). □

The following corollary admits a very similar proof.

Corollary 59. *There exist $\delta > 0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta_1]$, $k \geq 1$ and $x \in I_2(\max(k, N)+1)$, the following statement holds. If $x \in W\subseteq\mathbb{C}$, a connected open such that* $\operatorname{diam} g_\gamma^i(W) < \delta$ *for all $i = 0,\ldots,k-1$, then*

$$\operatorname{diam} W < \lambda^{-k} \operatorname{diam} g_\gamma^k(W).$$

Let us recall that all distances and diameters are considered with respect to the Euclidean metric, as we deal exclusively with polynomial dynamics. Let us state Lemma 3 in [12] in this setting.

Lemma 60. *Let f be a polynomial, $z \in \mathbb{C}$ and $0 < r < R$. Let $W \in \operatorname{Comp} f^{-1}(B(z, R))$ and $W' \in \operatorname{Comp} f^{-1}(B(z, r))$ with $W' \subseteq W$. If $\deg_W(f) \leq \mu$ then*

$$\frac{\operatorname{diam} W'}{\operatorname{diam} W} < 32\left(\frac{r}{R}\right)^{\frac{1}{\mu}}.$$

Let us set some constants that define the telescope construction used in the proof of Theorem A. For convenience, we use the same notations as in the proof of Theorem 8 in [12]. Let $\mu = 2$ and $\theta = \frac{1}{2}32^{-\mu}$. Let $\delta_0 > 0$ be provided by Corollary 52 and $\beta_0' > \alpha_0$, $\delta_1 > 0$, $N_1 > 0$ be provided by Proposition 56 applied to $\lambda^{\frac{1}{2}}$. Let $\delta' > 0$, $N_2 > 0$ be provided by Lemma 58, $\delta'' > 0$, $N_3 > 0$ be provided by Corollary 59 and $\beta_M > \alpha_0$, $r_M > 0$ and $M > 1$ defined by Lemma 55.

Let us observe that

$$I_1^\infty \prec I_1c_2 \prec c_1 \prec I_2c_2 \prec I_2^\infty \prec I_2c_1 \prec c_2 \prec I_3c_1 \prec I_3^\infty$$

and that all these sequences are continuously realized on $[\alpha_0, \beta_1]$. Let us define

$$\varepsilon_0 = \min_{\gamma \in [\alpha_0, \beta_1]} \left(|x(I_1 c_2) - c_1|, |x(I_2 c_2) - c_1|, |x(I_2 c_1) - c_2|, |x(I_3 c_1) - c_2| \right)$$

therefore $\varepsilon_0 > 0$ is smaller than $|c_1 - c_2|$, $|c_1|$ and $|1 - c_2|$ for all $\gamma \in [\alpha_0, \beta_1]$. We set

$$\varepsilon = \min(\varepsilon_0, \delta', \delta'', r_M). \tag{39}$$

Corollary 53 provides $\beta_1' > \alpha_0$ and $\delta_2 > 0$ such that for all $\gamma \in [\alpha_0, \beta_1']$ the diameter of any pullback of a ball of radius at most δ_2 centered on J_γ is smaller than ε. Let $\beta_2' = \min(\beta_1, \beta_0', \beta_1', \beta_M)$ and $\delta_3 = \min(\delta_0, \delta_1, \delta_2)$, such that Proposition 56 applies for balls centered on J_γ of radius at most δ_3, for all $\gamma \in [\alpha_0, \beta_2']$. Moreover, Lemma 58 and Corollary 59 apply and inequalities (32) and (33) hold on all pullbacks of such balls.

Corollary 52 provides δ_4 such that for

$$r = \theta R < R = \min(\delta_3, \delta_4),$$

there exist $\beta_3' > \alpha_0$ and $N_0 > 0$ the time span needed to contract the pullback of balls of radius R into components of diameter smaller than θR for all $\gamma \in [\alpha_0, \beta_3']$. We define

$$\beta = \min(\beta_2', \beta_3').$$

Let f be a rational map and Crit its critical set. If $W \subseteq \overline{\mathbb{C}}$ is an open set and $\overline{f^k(W)}$ contains at most one critical point for all $0 \le k < n$, let us define

$$\deg_{\overline{W}}(f^n) = \prod_{\substack{c \in \overline{f^k(W)} \cap \mathrm{Crit} \\ 0 \le k < n}} \deg(c),$$

counted with multiplicities.

The following fact provides the hypothesis of Corollary 57.

Corollary 61. *For all $\gamma \in [\alpha_0, \beta]$, $z \in J_\gamma$, $0 < r \le R$ and $(W_k)_{k \ge 0}$ a backward orbit of $B(z, r) = W_0$, if $\deg_{\overline{W_k}} g_\gamma^k > \mu$ then there exist $0 < k_1 < k_2 \le n$ such that $\overline{W_{k_1}} \cap \{c_1, c_2\} \ne \emptyset$ and $c_2 \in \overline{W_{k_2}}$.*

Proof. By the definition of R, $\operatorname{diam} W_k < \varepsilon \le \varepsilon_0 < |c_1 - c_2|$ therefore $\overline{W_k}$ contains at most one critical point for all $k \ge 0$. As $\mu = \mu_{c_1} = \mu_{c_2}$ there exist $0 < k_1 < k_2 \le n$ such that $\overline{W_{k_1}}$ and $\overline{W_{k_2}}$ contain exactly one critical point each. Suppose $c_1 \in \overline{W_{k_2}}$ therefore $1 \in \overline{W_k}$ for all $0 \le k < k_2$ which contradicts $\operatorname{diam} W_{k_1} < \varepsilon \le \varepsilon_0$. □

Let us prove the main result of this section.

Proof (Proof of Theorem A). This proof has two parts. The first part describes the construction of a convergent sequence of families $(\mathcal{G}_n)_{n \ge 0}$ of bimodal polynomials with negative Schwarzian derivative. Its limit g does not satisfy *RCE*. The second part shows that g has *ExpShrink* and it is similar to the proof of Theorem 8.

Let us recall the construction of the family $\mathcal{G}_1$. It is described by the common prefix S_1 of its kneading sequences $\underline{k}(\gamma)$ for all $\gamma \in [\alpha_1, \beta_1]$. We defined $S_1 =$

$I_1^{k_0+1} I_2^{k_1}$ so $\beta_1 < \underline{k}^{-1}\left(I_1^{k_0} c_1\right)$ which converges to $\alpha_0 = 0$ as $k_0 \to \infty$. Using this convergence, inequalities (15), Lemma 42 applied to $v = g_\gamma(c_2)$ and Lemma 41 applied to $\underline{i}_1 = I_1 c_1$, $\underline{i}_2 = I_1 c_2$ to bound $\left|g'_\gamma(v_{k_0})\right|$ there exists $k_0 > 0$ such that the following inequalities hold

$$\beta_1 < \max \underline{k}^{-1}\left(I_1^{k_0} c_1\right) < \beta,$$

$$d_\gamma(k) > \lambda^k \text{ for all } \gamma \in [\alpha_0, \beta_1] \text{ and } k = 1, \ldots, k_0 + 1.$$

Again by Lemma 42, property (17) and inequalities (15), if k_1 is sufficiently large then

$$d_\gamma(k) > \lambda^k \text{ for all } \gamma \in [\alpha_1, \beta_1] \text{ and } k = 1, \ldots, t_1, \tag{40}$$

where $t_1 = k_0 + 1 + k_1 = |S_1|$. Let us choose k_1 such that the previous inequality holds and such that $t_1 > N_1$ and

$$\max\left(\varepsilon R^{-1}, 2M_4 (\theta R)^{-1}, \varepsilon^2 (\theta R)^{-2}, 2M_1\right) < \lambda^{t_1 - 1}, \tag{41}$$

where M_1 and M_4 are defined by inequalities (32) respectively (33). This achieves the construction of the family $\mathcal{G}_1$.

For all $k \geq 1$ we construct $\mathcal{G}_{2k}$ using Proposition 44 with

$$\lambda^{-1} < \lambda_1 < \lambda_2 < 1,$$

and $\mathcal{G}_{2k+1}$ using Proposition 45 with

$$\Delta_k = 2^{-k}.$$

Using inequality (19) the sequence $(\mathcal{G}_n)_{n \geq 1}$ converges to a map $g = g_{\overline{\gamma}}$. Let us denote $d(n) = d_n(\overline{\gamma}) = \left|(g^n)'(v)\right|$ and $d(n,p) = d_{n,p}(\overline{\gamma}) = \left|(g^p)'(v_n)\right|$ for all $n, p \geq 0$, where v is the second critical value and $v_n = g^n(v)$. For all $n \geq 2$ let $p_n = p$ be provided by Proposition 44 or Proposition 45 used to construct $\mathcal{G}_n$. Therefore for all $n \geq 1$

$$t_n < p_{n+1} < t_{n+1},$$

where $t_n = |S_n|$ the length of the common prefix S_n of kneading sequences in $\mathcal{G}_n$. As $\overline{\gamma} \in [\alpha_n, \beta_n]$ for all $n \geq 1$,

$$\underline{k} = \underline{k}(\overline{\gamma}) \in S_n \times \Sigma \text{ for all } n \geq 1.$$

Let us also recall that for all $k \geq 1$

$$S_{2k} = S_{2k-1} I_2^{k_1+1} I_3^{k_2} I_2^{k_3},$$

and that we may choose k_1, k_2 and k_3 as large as we need. We impose therefore for all $k \geq 1$

$$k_1 > N_3, k_2 > N_2 \text{ and } k_3 > N_3. \tag{42}$$

Let us remark that $g(c_1) = 1$, $g(1) = 1$ and $|g'(1)| > 1$ therefore $c_1 \in J$ the Julia set of g and c_1 is non-recurrent and Collet-Eckmann. Let us remark that $\Delta_k \to 0$ as $k \to \infty$ and $\overline{\gamma} \in [\alpha_{2k+1}, \beta_{2k+1}]$ for all $k \geq 1$ therefore the second critical orbit is recurrent. By construction and inequality (12) the second critical orbit accumulates on r and on 1. Therefore $c_2 \in J$ using for example a similar argument to the proof of Lemma 50. Let us show that c_2 is not Collet-Eckmann. Indeed, by Proposition 44 for all $k \geq 1$

$$d(p_{2k}) < \lambda_2^{p_{2k}} < 1,$$

and $p_{2k} \to \infty$ as $k \to \infty$. Therefore by Definition 3

$$g \text{ is not } RCE.$$

Combining inequalities (40) and (16), the third claim of Proposition 44 and the second claim of Proposition 45

$$d(n) > \lambda^n \text{ for all } n \in \bigcup_{k \geq 0} \{t_{2k}, \ldots, p_{2k+2} - 1\}. \tag{43}$$

Let us check that for all $m > 0$ such that $|g^m(c_2) - c_2| < \varepsilon$

$$d(m) > \lambda^m. \tag{44}$$

Let us recall that $\varepsilon \leq \varepsilon_0$ by its definition (39) so $|g^m(c_2) - c_2| < \varepsilon$ implies that $v_m = g^{m+1}(c_2) \in I_1$ therefore $\underline{k}(m) = I_1$ so there exists $k \geq 1$ such that

$$t_{2k} < m < t_{2k+1},$$

as Proposition 44 extends S_{2n-1} to S_{2n} using only the symbols I_2 and I_3 for all $n \geq 1$. Therefore $m \in \{t_{2k}, \ldots, p_{2k+2} - 1\}$ thus inequality (44) is a direct consequence of inequality (43).

Let us show that g has *ExpShrink*. We use a telescope that is very similar to the one used in the proof of Theorem 8. Loosely speaking, the strategy is the following. We consider pullbacks of a small ball centered on the Julia set J of g. We show that after some time, the pullbacks are contracted. We include such a component in another ball centered on J. The construction is achieved inductively and contractions at each step are used to show uniform contraction, thus *ExpShrink*. We call blocks of the telescope the sequences of pullbacks associated to each step. In order to deal with various configurations of critical points inside the telescope, we need three types of blocks.

Let us introduce additional notation and rigorously define the telescope.

For $B \subseteq \overline{\mathbb{C}}$ connected and $n \geq 0$, we write B^{-n} or $g^{-n}(B)$ for some connected component of $g^{-n}(B)$. When $z \in B$ and some backward orbit $z_n \in g^{-n}(z)$ are fixed, B^{-n} is the connected component of $g^{-n}(B)$ that contains z_n.

We consider a pullback of an arbitrary ball $B(z, R)$ with $z \in J$, of length $N > 0$. We show that there are constants $C_1 > 0$ and $\lambda_3 > 1$ that do not depend on z nor on N such that

$$\operatorname{diam} B(z, R)^{-N} \leq C_1 \lambda_3^{-n}.$$

It is easy to check that the previous inequality for all $z \in J$ and $N > 0$ implies the *ExpShrink* condition.

Let $(z_n)_{n \geq 0} \subseteq J$ be a backward orbit of z, that is $z_0 = z$ and $g(z_{n+1}) = z_n$ for all $n \geq 0$. We consider preimages of $B(z, R'_0) := B(z, R)$ up to time N. We show that there is some moment N'_0 when the pullback $B(z, R'_0)^{-N'_0}$ observes a strong contraction. Then $B(z, R'_0)^{-N'_0}$ can be nested inside some ball $B(z_{N'_0}, R'_1)$ where

$R_1' \leq R$. This new ball is pulled back and the construction is achieved inductively. The pullbacks $B(z, R_0')$, $B(z, R_0')^{-1} \dots B(z, R_0')^{-N_0'}$ form the first block of the telescope. The pullbacks $B(z_{N_0'}, R_1')$, $B(z_{N_0'}, R_1')^{-1} \dots B(z_{N_0'}, R_1')^{-N_1'}$ form the second block and so on. Lemma 60 is essential to manage the passage between two such consecutive telescope blocks. We show contraction for every block using either Corollary 53 or Proposition 56. This leads to a classification of blocks depending on the presence and on the type of critical points inside them.

Let R' be the radius of the initial ball of some block and N' be its length. We introduce a new parameter $r' < R$ for each block, a lower bound for R'. It is an upper bound of the diameter of the last pullback of the previous block. This choice guarantees that consecutive blocks are nested. A block that starts at time n is defined by the choice of R' with $r' \leq R' \leq R$ and of N' with $1 \leq N' \leq N-n$. It is the pullback of length N' of $B(z_n, R')$.

For all $n, t \geq 0$ and $r > 0$ we denote

$$\begin{aligned} d(n,r,t) &= \deg_{B(z_n,r)^{-t}}(g^t) \ \text{ and} \\ \overline{d}(n,r,t) &= \deg_{\overline{B(z_n,r)^{-t}}}(g^t)\,. \end{aligned}$$

Fix $n \geq 0$ and $t \geq 1$ and consider the maps d and $\overline{d}$ defined on $[r', R]$. They are increasing and $d \leq \overline{d}$. Moreover, for all $n \geq 0$, $r > 0$, $t \geq 0$ and $s > 0$,

$$\overline{d}(n,r,t) \leq d(n, r+s, t).$$

The set $\left\{r \in [r', R] \mid d(n,r,t) < \overline{d}(n,r,t)\right\}$ is the common set of discontinuities of d and $\overline{d}$. Note also that d is lower semi-continuous and $\overline{d}$ is upper semi-continuous.

For transparency, we also denote

$$W_k = B(z_n, R')^{-k}.$$

Let us define the three types of blocks. For convenience, we keep the same notations as in the proof of Theorem 8 in [12].

Type 1 Blocks with $R' = r'$ and N' such that $\overline{d}(n, R', N') > 1$ and $c_2 \in \overline{W_{N'+1}}$.
Type 2 Blocks with $R' = R$, $N' = \min(N_0, N-n)$ and $d(n, R, N-n) \leq \mu$.
Type 3 Blocks with $\overline{d}(n, R', N') > 1$, $c_2 \in \overline{W_{N'+1}}$ and $d(n, R', N-n) \leq \mu$.

Let us define r'. It is the diameter of the last pullback of the previous block of type 1 or 3. It is $r = \theta R$ if the previous block is of type 2 and $r' = R$ for the first block.

Let us first show that for all $z \in J$ and $N > 0$ we may define a telescope using the three types of blocks. The construction is inductive and we show that given $0 \leq n < N$ and $0 < r' \leq R$ we can find $R' \in [r', R]$ and $0 < N' \leq N-n$ that define a block of one of the three types. We also show contraction along every block so r' defined as above is smaller than R, thus completing the proof of the existence of the telescope.

If $\overline{d}(n, r', N-n+1) > \mu = 2$ then by Corollary 61 there is $1 \leq N' \leq N-n$ that defines a type 1 pullback for $R' = r'$. If $\overline{d}(n, R, N-n) \leq \mu$ then we define a type 2 block, as $d \leq \overline{d}$. Note that the first block of the telescope is already constructed as $r' = R$. In all other cases $r' < R$. If $\overline{d}(n, R, N-n+1) > \mu$ there is a smallest R', with $r' < R' \leq R$, such that $\overline{d}(n, R', N-n+1) > \mu$. Thus

R' is a point of discontinuity of $\overline{d}$ so $d(n, R', N-n+1) < \overline{d}(n, R', N-n+1)$, therefore $d(n, R', N-n) \leq d(n, R', N-n+1) \leq \mu$. Then by Corollary 61 there is $1 \leq N' \leq N-n$ that defines a type 3 pullback.

Let us be more precise with our notations. We denote by n'_i, N'_i, r'_i and R'_i the parameters n, N', r' and R' of the i-th block. Let also $W_{i,k}$ be W_k in the context of the i-th block with $i \in \{0, \ldots, b\}$, where $b+1$ is the number of blocks of the telescope. So $n'_0 = 0$, $r'_0 = R$ and $n'_1 = N'_0$. In the general case $i > 0$, we have

$$n'_i = n'_{i-1} + N'_{i-1} \text{ and}$$
$$r'_i \geq \operatorname{diam} W_{i-1,N'_{i-1}}.$$

Let us also denote by $T_i \in \{1, 2, 2', 3\}$ the type of the i-th block. The type $2'$ is a particular case of the second type, when $N' < N_0$. This could only happen for the last block, when $N - n'_b < N_0$. So $T_i \in \{1, 2, 3\}$ for all $0 \leq i < b$. We may code our telescope by the type of its blocks, from right to left

$$T_b \ldots T_2 T_1 T_0.$$

Our construction shows that

$$\operatorname{diam} W_{i-1,N'_{i-1}} \leq r'_i < R \text{ for all } 0 < i < b \tag{45}$$

is a sufficient condition for the existence of the telescope that contains the pullback of $B(z, R)$ of length N.

If $T_i = 2$ we apply Corollary 53 so

$$\operatorname{diam} W_{i,N'_i} < r'_{i+1} = \theta R < R. \tag{46}$$

If $T_i \in \{1, 3\}$ we show that there exists $\lambda_0 > 1$ such that

$$\operatorname{diam} W_{i,N'_i} < \theta R'_i \lambda_0^{-N'_i} < R, \tag{47}$$

as $\theta < \frac{1}{2}$, $R'_i \leq R$ and $\lambda_0^{-N'_i} < 1$. This inequality completes the proof of the existence of the telescope.

Let us fix $i \geq 0$ such that $T_i \in \{1, 3\}$. Suppose that $i > 0$ and $T_{i-1} \in \{1, 3\}$ also, therefore

$$c_2 \in \overline{W_{i-1,N'_{i-1}+1}} \subseteq \overline{W_{i,1}} = \overline{B(z_{n_i}, R'_i)^{-1}} = \overline{g^{N'_i}\left(W_{i,N'_i+1}\right)} \subseteq \overline{B(z_{n_i}, R)^{-1}}.$$

But $c_2 \in \overline{W_{i,N'_i+1}}$ also and $\operatorname{diam} B(z_{n_i}, R)^{-1} < \varepsilon$ by the definition of R. Therefore by inequality (44)

$$d(N'_i) > \lambda^{N'_i}$$

so by Corollary 57 we may apply Proposition 56 to obtain

$$\operatorname{diam} W_{i,N'_i} < \theta R'_i \lambda^{-\frac{N'_i}{2}}.$$

We have proved that for all $i > 0$ with $T_{i-1} \in \{1, 3\}$ inequality (47) holds for all $\lambda_0 \leq \lambda^{\frac{1}{2}}$. If $i = 0$ or $T_{i-1} = 2$ then $R'_i \in [\theta R, R]$. Therefore it is enough to show that there exists $\lambda_0 > 1$ such that for all $z \in J$, $\tau \in [\theta R, R]$, $n > 0$ and W a connected component of $g^{-n}(B(z,\tau))$ the following statement holds. If $v \in \overline{W}$ and there exist $0 \le m < n$ such that $\overline{g^m(W)} \cap \mathrm{Crit} \ne \emptyset$ then

$$\operatorname{diam} W < \theta \tau \lambda_0^{-n}. \tag{48}$$

Again, if $d(n) > \lambda^n$ using Corollary 57 and Proposition 56 the previous inequality is satisfied for all $1 < \lambda_0 \le \lambda^{\frac{1}{2}}$. Therefore using inequality (43) we may suppose that there exist $k' \ge 1$ such that

$$p_{2k'} \le n < t_{2k'}.$$

Let us denote $p = p_{2k'}$, $t = t_{2k'-1}$ and $W^k = g^k(W)$ for all $k = 0, \ldots, n$. By the definition of p in Proposition 44

$$2t < p. \tag{49}$$

Using Corollary 59, inequalities (42) and (41)

$$\operatorname{diam} W^t < \lambda^{-(p-1-t)} \operatorname{diam} W^{p-1} < \lambda^{-(p-1-t)} \varepsilon < R.$$

As $t_1 > N_1$, inequality (16) lets us apply Proposition 56 to $B(v_t, \operatorname{diam} W^t)$ which combined to the previous inequality shows that

$$\operatorname{diam} W < \theta \lambda^{-\left(p-1-\frac{t}{2}\right)} \operatorname{diam} W^{p-1}. \tag{50}$$

By inequalities (42), using Lemma 58 and eventually Corollary 59 if $v_n \in I_2$

$$\operatorname{diam} W^p < \lambda^{-(n-p)} \operatorname{diam} W^n = 2\lambda^{-(n-p)} \tau. \tag{51}$$

Therefore the only missing link is an estimate of $\operatorname{diam} W^{p-1}$ with respect to $\operatorname{diam} W^p$. We distinguish the following two cases.

1. $\operatorname{dist}\left(W^{p-1}, c_1\right) < 3 \operatorname{diam} W^{p-1}$.
2. $\operatorname{dist}\left(W^{p-1}, c_1\right) \ge 3 \operatorname{diam} W^{p-1}$.

Suppose the first case. The by the definition (39) of ε we may use inequality (33) therefore

$$\begin{aligned} \operatorname{diam} W^{p-1} &< \left(M_4 \operatorname{diam} W^p\right)^{\frac{1}{2}} \\ &< \left(2M_4\tau\right)^{\frac{1}{2}} \lambda^{-\frac{n-p}{2}} \end{aligned}$$

using inequality (51). Recall that $\tau \ge \theta R$ and $t \ge t_1$. Therefore inequalities (50), (49) and (41) imply that

$$\begin{aligned} \operatorname{diam} W &< \theta \lambda^{-\frac{n}{2}} \tau \left(2\lambda M_4 r^{-1}\right)^{\frac{1}{2}} \lambda^{-\frac{t}{2}} \\ &< \theta \lambda^{-\frac{n}{2}} \tau. \end{aligned}$$

Therefore in the first case it is enough to choose $\lambda_0 \le \lambda^{\frac{1}{2}}$.

Suppose the second case. Using inequalities (50), (49) and (41) we may compute

$$\begin{aligned} \operatorname{diam} W &< \theta \lambda^{-\left(p-1-\frac{t}{2}\right)} \varepsilon \\ &< \theta \lambda^{-\frac{p}{2}} \theta R \le \theta \lambda^{-\frac{p}{2}} \tau \\ &= \theta \lambda^{-n\left(\frac{p}{2n}\right)} \tau. \end{aligned} \tag{52}$$

This is not enough as λ_0 should depend only on g. We may remark that we are in position to use inequality (32) for W^p therefore

$$\operatorname{diam} W^{p-1} < M_1 \left|g'(v_{p-1})\right|^{-1} \operatorname{diam} W^p.$$

Let us compute an upper bound for $\left|g'(v_{p-1})\right|^{-1} = d(p-1,1)^{-1}$. Using the first two claims of Proposition 44

$$d(p)^{-1} = d(p-1)^{-1} d(p-1,1)^{-1} < \lambda_1^{-p} < \lambda^p,$$

and

$$d(p-1) < \lambda_r^{p-1} \lambda^{p-1},$$

where we denote $\lambda_r = |g'(r(\overline{\gamma}))|$. Let $\nu = \frac{\log \lambda_r}{\log \lambda}$. Combining the previous inequalities

$$d(p-1,1)^{-1} < \lambda^{p(\nu+2)},$$

therefore using inequalities (50), (51), (49) and (41)

$$\begin{aligned}\operatorname{diam} W &< 2M_1 \theta \lambda^{-\left(p-1-\frac{t}{2}\right)} \lambda^{p(\nu+2)} \lambda^{-(n-p)} \tau \\ &< \theta(2M_1) \lambda^{\nu p + 2p + 1 + \frac{t}{2}} \lambda^{-n} \tau \\ &< \theta \lambda^{-n+p(\nu+3)} \tau.\end{aligned}$$

If $n > 2p(\nu+3)$ then inequality (48) is satisfied for all $\lambda_0 \leq \lambda^{\frac{1}{2}}$. If $n \leq 2p(\nu+3)$ then using inequality (52), inequality (48) is satisfied for all

$$\lambda_0 \leq \lambda^{\frac{1}{4(\nu+3)}} \leq \lambda^{\frac{p}{2n}},$$

which completes the proof of inequality (47) and therefore of the existence of the telescope.

The remainder of the proof shows the global exponential contraction of the diameter of pullbacks. It is identical to the second part of the proof of Theorem 8 in [12]. We reproduce it here for convenience.

Note that if $T_i = 1$ then we may rewrite inequality (47) as follows

$$r'_{i+1} < \theta r'_i \lambda_0^{-N'_i} < r'_i \lambda_0^{-N'_i}. \tag{53}$$

Recall also that if there are $\lambda_3 > 1$ and $C_1 > 0$ such that

$$\operatorname{diam} B(z,R)^{-N} = \operatorname{diam} W_{0,N} < C_1 \lambda_3^{-N}, \tag{54}$$

then the theorem holds. We may already set

$$\lambda_3 = \min\left(2^{\frac{1}{\mu N_0}}, \lambda_0^{\frac{1}{\mu}}\right). \tag{55}$$

As inequality (53) provides an easy way to deal with the first type of block, we compute estimates only for sequences of blocks of types $1\ldots1$, $1\ldots12$ and $1\ldots13$, as the sequence $T_b \ldots T_2 T_1 T_0$ can be decomposed in such sequences. Sequences with only one block of type 2 or 3 are allowed as long as the following block is not of type 1. For a sequence of blocks $T_{i+p} \ldots T_i$, let

$$N'_{i,p} = N'_{i+p} + \ldots + N'_i$$

be its length.

A sequence $1\dots1$ may only occur as the first sequence of blocks, thus $i=0$. As $r'_0 = R$, iterating inequality (53) for such a sequence we obtain

$$\begin{aligned} r'_{p+1} &< \theta^{p+1} R \lambda_0^{-N'_{0,p}} \\ &< 2\theta R'_0 \lambda_3^{-\mu N'_{0,p}}. \end{aligned} \tag{56}$$

Combining inequalities (53), (46) and the definition (55) of λ_3, for a sequence $1\dots12$

$$\begin{aligned} r'_{i+p+1} &< r'_{i+1}\lambda_0^{-N'_{i+1,p-1}} \\ &< 2\theta R 2^{-1} \lambda_0^{-N'_{i+1,p-1}} \\ &\le 2\theta R'_i \lambda_3^{-\mu N'_{i,p}}, \end{aligned} \tag{57}$$

as $N'_i = N_0$, $N'_{i,p} = N'_i + N'_{i+1,p-1}$ and $R'_i = R$.

For a sequence $1\dots13$, inequalities (53) and (47) yield

$$\begin{aligned} r'_{i+p+1} &< r'_{i+1}\lambda_0^{-N'_{i+1,p-1}} \\ &< \theta R'_i \lambda_0^{-N'_{i,p}} \\ &< 2\theta R'_i \lambda_3^{-\mu N'_{i,p}}. \end{aligned} \tag{58}$$

We also find a bound for r'_{b+1} in the case $T_b = 2'$. Using notations introduced in Section 4.2 we define $K = ||g'||_{\infty, J_{+\varepsilon}}$. We compute

$$\begin{aligned} r'_{b+1} &< R'_b \mu K^{N'_b} \\ &= \mu (K\lambda_3)^{N'_b} R'_b \lambda_3^{-N'_b}. \end{aligned} \tag{59}$$

We decompose the telescope into $m+1$ sequences $1\dots1$, $1\dots12$, $1\dots13$ and eventually $2'$ on the leftmost position

$$S_m \dots S_2 S_1 S_0.$$

Consider a sequence of blocks

$$S_j = T_{i+p} \dots T_i.$$

Denote $n''_j = n'_i$, $N''_j = N'_{i,p}$, $r''_j = r'_i$ and $R''_j = R'_i$. Let also

$$\Delta_j = \operatorname{diam} W_{i, N-n'_i}$$

be the diameter of the pullback of the first block of the sequence up to time $-N$.

With the eventual exception of S_m, inequalities (56), (57) and (58) provide good contraction estimates for each sequence S_j

$$r''_{j+1} < 2\theta R''_j \lambda_3^{-\mu N''_j}.$$

If $T_b = 2'$ then inequality (59) yields a constant $\mu(K\lambda_3)^{N'_b} < C_1 = \mu(K\lambda_3)^{N'_0}$ such that

$$r''_{m+1} < C_1 R''_m \lambda_3^{-N''_m},$$

as $R''_m = R'_b$ and $N''_m = N'_b$. Note that the previous inequality also holds if $T_b \in \{1,2,3\}$. We cannot simply multiply these inequalities as $R''_j > r''_j$ for all $0 < j \le m$.

By the definitions of types 2 and 3, the degree $d(n''_j, R''_j, N - n''_j)$ is bounded by μ in all cases. So Lemma 60 provides a bound for the distortion of pullbacks up to time $-N$

$$\begin{aligned}\frac{\Delta_{j-1}}{\Delta_j} &< 32\left(\frac{r''_j}{R''_j}\right)^{\frac{1}{\mu}}\\ &< 32\left(2\theta\lambda_3^{-\mu N''_{j-1}}\frac{R''_{j-1}}{R''_j}\right)^{\frac{1}{\mu}}\\ &= \lambda_3^{-N''_{j-1}}\left(\frac{R''_{j-1}}{R''_j}\right)^{\frac{1}{\mu}},\end{aligned}$$

for all $0 < j \le m$. Therefore

$$\frac{\Delta_0}{\Delta_m} < \lambda_3^{-N+N''_m}\left(\frac{R''_0}{R''_m}\right)^{\frac{1}{\mu}}.$$

Recall that $R''_j \le R < 1$ for all $0 \le j \le m$ and $\Delta_m = r''_{m+1}$, so

$$\begin{aligned}\Delta_0 &< \lambda_3^{-N+N''_m} C_1 R''_m \lambda_3^{-N''_m}\left(\frac{R}{R''_m}\right)^{\frac{1}{\mu}}\\ &< \lambda_3^{-N} C_1 (R''_m)^{1-\frac{1}{\mu}}\\ &< C_1\lambda_3^{-N}.\end{aligned}$$

By definition $\Delta_0 = \operatorname{diam} W_{0,N}$, therefore the previous inequality combined with inequality (54) completes the proof of the theorem. □

5. RCE is not a topological invariant for real polynomials with negative Schwarzian derivative

Let $\mathcal{H} : [0,h] \to \mathcal{P}_2$ (see the definition of $\mathcal{P}_2$ at page 10) be equal to the family $\mathcal{G}$ defined in the previous section. Let us define another family of bimodal maps $\tilde{\mathcal{H}} : [0,h'] \to \mathcal{P}_2$ in an analogous fashion. Let $T \in \mathbb{R}_7[x]$ be a degree 7 polynomial such that $T(0) = 0$ and such that $T'(x) = (x+1)^3(x-1)^3$. Therefore T has two critical points -1 and 1 of degree 4 and $T(-x) = -T(x)$ for all $x \in \mathbb{R}$. Let $y_0 = T(-1)$ and $x_0 > 1$ such that $T(x_0) = y_0$. Let $h' > 0$ be small and for each $\gamma \in [0,h']$ two order preserving linear maps $R_{\gamma'}(x) = x(2x_0+\gamma') - x_0 - \gamma'$ and $S_{\gamma'}(y) = \frac{y - T(-x_0-\gamma')}{y_0 - T(-x_0-\gamma')}$ that map $[0,1]$ onto $[-x_0-\gamma', x_0]$ respectively $[T(-x_0-\gamma'), T(x_0)]$ onto $[0,1]$. One may show by direct computation that if a real polynomial P is such that all the roots of P' are real then P has negative Schwarzian derivative. Therefore

$$\tilde{h}_{\gamma'} = S_{\gamma'} \circ T \circ R_{\gamma'} \in \mathcal{P}_2 \text{ for all } \gamma' \in [0,h'].$$

We define $\tilde{\mathcal{H}}(\gamma') = \tilde{h}_{\gamma'}$ for all $\gamma' \in [0,h']$. Let us remark that $y_0 \in (0,1)$ and $x_0 \in \left(\frac{3}{2}, 2\right)$ therefore all three fixed points of $\tilde{h}_0$ are repelling. Let $\tilde{r}(\gamma')$ be the only fixed point of $\tilde{h}_{\gamma'}$ in $(0,1)$ and $\tilde{c}_1 < \tilde{c}_2$ its critical points. The proofs that for $h' > 0$ sufficiently small $\tilde{\mathcal{H}}$ satisfies properties (3) to (6), Lemmas 33 and 34, that it is natural, that $\tilde{r}$, $\tilde{c}_1$ and $\tilde{c}_2$ are continuous and that for all $n > 1$

$$\tilde{r}(\gamma') - \tilde{h}^n_{\gamma'}(\tilde{c}_2) \text{ has finitely many zeros in } [0, h'],$$

go exactly the same way as for $\mathcal{G}$. As $h_0'(r(0)) = -3$, $h_0'(1) = 9$, $y_0 = \frac{16}{35}$, $\frac{3}{2} < x_0 < 2$, $\tilde{r}(0) = S_0(0)$ and $|h_0'(\tilde{r}(0))| = \frac{x_0}{y_0}$ one may compute that

$$\frac{1}{2}\frac{\log|h_0'(1)|}{\log|h_0'(r(0))|} = 1 < \frac{3}{4}\frac{\log\left|\tilde{h}_0'(1)\right|}{\log\left|\tilde{h}_0'(\tilde{r}(0))\right|}.$$

We may also suppose $h > 0$ and $h' > 0$ sufficiently small such that there exist $1 < \lambda < \lambda'$, $1 < \tilde{\lambda} < \tilde{\lambda}'$ and $\theta_1 < \theta_2$ such that for all $\gamma \in [0, h]$ and $\gamma' \in [0, h']$

$$\lambda' < \min\left(\left|h_\gamma'(0)\right|, \left|h_\gamma'(r)\right|, \left|h_\gamma'(1)\right|\right) \text{ and}$$
$$\tilde{\lambda}' < \min\left(\left|\tilde{h}_{\gamma'}'(0)\right|, \left|\tilde{h}_{\gamma'}'(\tilde{r})\right|, \left|\tilde{h}_{\gamma'}'(1)\right|\right)$$

and

$$\frac{1}{2}\frac{\log\left|h_\gamma'(1)\right|}{\log\left|h_\gamma'(r(\gamma))\right|} < \theta_1 < \theta_2 < \frac{3}{4}\frac{\log\left|\tilde{h}_{\gamma'}'(1)\right|}{\log\left|\tilde{h}_{\gamma'}'(\tilde{r}(\gamma'))\right|}. \tag{60}$$

Let us denote $\underline{k}(\gamma)$ the second kneading sequence of h_γ and $\underline{\tilde{k}}(\gamma')$ the second kneading sequence of $\tilde{h}_{\gamma'}$. We construct two decreasing sequences of families of bimodal maps $(\mathcal{H}_n)_{n\geq 1}$ and $(\tilde{\mathcal{H}})_{n\geq 1}$. Let $\mathcal{H}_n : [\alpha_n, \beta_n] \to \mathcal{P}_2$ with $\mathcal{H}_n(\gamma) = \mathcal{H}(\gamma)$ for all $\gamma \in [\alpha_n, \beta_n]$ and $\tilde{\mathcal{H}}_n : [\alpha_n', \beta_n'] \to \mathcal{P}_2$ with $\tilde{\mathcal{H}}_n(\gamma) = \tilde{\mathcal{H}}(\gamma)$ for all $\gamma' \in [\alpha_n', \beta_n']$. By construction we choose that for all $n \geq 1$

$$\underline{k}(\alpha_n) = \underline{\tilde{k}}(\alpha_n') \text{ and } \underline{k}(\beta_n) = \underline{\tilde{k}}(\beta_n').$$

Let us denote $v = h_\gamma(c_2)$, $\tilde{v} = \tilde{h}_{\gamma'}(\tilde{c}_2)$ and $v_n = h^n_\gamma(v)$, $\tilde{v}_n = \tilde{h}^n_{\gamma'}(\tilde{v})$ for all $n \geq 0$, $\gamma \in [0, h]$ and $\gamma' \in [0, h']$. Let also $d_n(\gamma) = \left|\left(h^n_\gamma\right)'(v)\right|$, $\tilde{d}_n(\gamma') = \left|\left(\tilde{h}^n_{\gamma'}\right)'(\tilde{v})\right|$, $d_{n,p}(\gamma) = \left|\left(h^p_\gamma\right)'(v_n)\right|$ and $\tilde{d}_{n,p}(\gamma') = \left|\left(\tilde{h}^p_{\gamma'}\right)'(\tilde{v}_n)\right|$ for all $n, p \geq 0$, $\gamma \in [0, h]$ and $\gamma' \in [0, h']$. The basic construction tool is again Proposition 39 and we build the sequences $(\mathcal{H}_n)_{n\geq 1}$ and $(\tilde{\mathcal{H}})_{n\geq 1}$ by specifying the common prefix S_n of the kneading sequences in $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_n$ for all $n \geq 1$. We also reuse the notation $t_n = |S_n|$ for all $n \geq 1$. In an analogous way to the construction of the family $\mathcal{G}_1$, see inequality (40), we choose

$$S_1 = I_1^{k_0+1} I_2^{k_1}$$

such that

$$d_k(\gamma) > \lambda^k \text{ and } \tilde{d}_k(\gamma') > \tilde{\lambda}^k \tag{61}$$

for all $\gamma \in [\alpha_1, \beta_1]$, $\gamma' \in [\alpha_1', \beta_1']$ and $k = 1, \ldots, t_1$ and

$$\beta_1 < h \text{ and } \beta_1' < h'.$$

Let us describe the construction of the sequences $(\mathcal{H}_n)_{n\geq 1}$ and $(\tilde{\mathcal{H}})_{n\geq 1}$ which satisfy properties (16) to (20) and

$$\tilde{d}_{t_n}(\gamma') > \tilde{\lambda}^{t_n} \tag{62}$$

for all $n \geq 1$.

Let us recall that Proposition 44 employs twice Proposition 39 to construct a subfamily $\mathcal{G}_{n+1}$ of $\mathcal{G}_n$ with

$$S_{n+1} = S_n I_2^{k_1+1} I_3^{k_2} I_2^{k_3}.$$

Let γ_∞ and γ'_∞ be provided by Proposition 39 such that $\underline{k}(\overline{\gamma}) = \underline{\tilde{k}}(\overline{\gamma}') = S_n I_2^\infty$. We use the same strategy as in the proof of Proposition 44 to define both $\mathcal{H}_{n+1}$ and $\tilde{\mathcal{H}}_{n+1}$ with the same combinatorics. Taking k_1, k_2 and k_3 sufficiently large we may control the growth of $d_m(\gamma)$ and $\tilde{d}_m(\gamma')$ uniformly for all $t_n < m \leq t_{n+1}$. We let

$$\frac{k_1}{k_2} \rightarrow \eta > 0,$$

$p = t_n + k_1 + 1$ and compute some bounds for $d_p(\gamma)$ and $\tilde{d}_p(\gamma')$. For transparency, let us denote $\lambda_0 = \left|h'_{\gamma_\infty}(r)\right|$, $\tilde{\lambda}_0 = \left|\tilde{h}'_{\gamma'_\infty}(\tilde{r})\right|$, $\lambda_3 = \left|h'_{\gamma_\infty}(1)\right|$ and $\tilde{\lambda}_3 = \left|\tilde{h}'_{\gamma'_\infty}(1)\right|$. As in the proof of Proposition 44 we obtain

$$\lim_{k_1\to\infty} \frac{1}{k_1} \log d_p(\gamma) = \log \lambda_0 - \frac{1}{2\eta} \log \lambda_3 \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}]. \tag{63}$$

We may observe that inequalities (23) hold exactly when c_1 is a second degree critical point. We may however write similar bounds for $\tilde{\mathcal{H}}_{n+1}$. By the same arguments there exist constants $\tilde{M} > 1$, $\tilde{\delta} > 0$ and $\tilde{N}_2 > 0$ such that if $k_1 > \tilde{N}_2$ and $\gamma' \in [\gamma'_1, \gamma'_2]$ then

$$\begin{aligned} \tilde{M}^{-1}(x - \tilde{c}_1)^4 &< |1 - \tilde{h}_{\gamma'}(x)| < \tilde{M}(x - \tilde{c}_1)^4 \text{ and} \\ \tilde{M}^{-1}(x - \tilde{c}_1)^3 &< \left|\tilde{h}'_{\gamma'}(x)\right| < \tilde{M}(x - \tilde{c}_1)^3 \end{aligned}$$

for all $x \in (\tilde{c}_1 - \tilde{\delta}, \tilde{c}_1 + \tilde{\delta})$, where γ'_1, γ'_2 are the bounds for γ' provided by Proposition 39 applied to S_n and $\tilde{\mathcal{H}}_n$. Therefore we obtain

$$\lim_{k_1\to\infty} \frac{1}{k_1} \log \tilde{d}_p(\gamma') = \log \tilde{\lambda}_0 - \frac{3}{4\eta} \log \tilde{\lambda}_3 \text{ for all } \gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]. \tag{64}$$

Using inequalities (60) and the limits (63) and (64) it is enough to choose

$$\theta_1 < \eta < \theta_2$$

to obtain the following corollary of Proposition 44.

Corollary 62. *There exist*

$$0 < \lambda_1 < 1 < \lambda_2 < \min\left(\lambda, \tilde{\lambda}\right)$$

that depend only on $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_1$ such that if $\mathcal{H}_n$ is a subfamily of $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_n$ is a subfamily of $\tilde{\mathcal{H}}_1$ both satisfying conditions (16) to (20) and (62) then there exist $\mathcal{H}_{n+1}$ a subfamily of $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_{n+1}$ a subfamily of $\tilde{\mathcal{H}}_n$ satisfying the same condition and $2t_n < p < t_{n+1}$ with the following properties

1. $d_p(\gamma) > \lambda_2^p$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.
2. $\tilde{d}_p(\gamma') < \lambda_1^p$ *for all* $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$.
3. $d_{t_n,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \dots, p-1-t_n$.
4. $\tilde{d}_{t_n,l}(\gamma') > \tilde{\lambda}^l$ *for all* $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$ *and* $l = 1, \dots, p-1-t_n$.
5. $d_{p,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \dots, t_{n+1}-p$.
6. $\tilde{d}_{p,l}(\gamma') > \tilde{\lambda}^l$ *for all* $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$ *and* $l = 1, \dots, t_{n+1}-p$.

Proposition 45 has an immediate corollary for the families $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

Corollary 63. *Let the subfamilies* $\mathcal{H}_n$ *and* $\tilde{\mathcal{H}}_n$ *of* $\mathcal{H}_1$ *respectively* $\tilde{\mathcal{H}}_1$ *with* $n \geq 1$ *satisfy conditions (16) to (20) and (62) and*

$$\Delta > 0.$$

Then there exist subfamilies $\mathcal{H}_{n+1}$ *of* $\mathcal{H}_n$ *and* $\tilde{\mathcal{H}}_{n+1}$ *of* $\tilde{\mathcal{H}}_n$ *satisfying the same conditions and such that there exists* $t_n < p < t_{n+1}$ *with the following properties*

1. $\left|h_\gamma^p(c_2) - c_2\right| < \Delta$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.
2. $\left|\tilde{h}_{\gamma'}^p(\tilde{c}_2) - \tilde{c}_2\right| < \Delta$ *for all* $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$.
3. $d_{t_n,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \dots, t_{n+1}-t_n$.
4. $\tilde{d}_{t_n,l}(\gamma') > \tilde{\lambda}^l$ *for all* $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$ *and* $l = 1, \dots, t_{n+1}-t_n$.

For all $k \geq 1$ we define $\mathcal{H}_{2k}$ and $\tilde{\mathcal{H}}_{2k}$ using Corollary 62 and $\mathcal{H}_{2k+1}$ and $\tilde{\mathcal{H}}_{2k+1}$ using Corollary 63 with $\Delta = 2^{-k}$. Let h be the limit of $(\mathcal{H}_n)_{n\geq 1}$ and $\tilde{h}$ be the limit of $(\tilde{\mathcal{H}}_n)_{n\geq 1}$. Then h is CE therefore RCE and the second critical point $\tilde{c}_2$ of $\tilde{h}$ is recurrent but not CE therefore $\tilde{h}$ is not RCE. Both h and $\tilde{h}$ have negative Schwarzian derivative and their second critical orbits accumulate on r and 1 respectively on $\tilde{r}$ and 1. Moreover, using Lemma 50, h and $\tilde{h}$ do not have attracting or neutral periodic points on $[0, 1]$. We may therefore apply Corollaries 20 and 21 to obtain the following theorem that contradicts Conjecture 1 in [25].

Theorem B. *The* RCE *condition for analytic S-multimodal maps is not topologically invariant.*

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